Electromagnetic fields and biological tissues: effects and medical applications

Please **initialize** individual items of the declaration, and **sign** it at bottom.

Upon my word of honor, and aware of the consequences of a false declaration under the Italian law, as well as those deriving from unfair conduct at Politecnico,

I, the undersigned Tong Lin
ID n. (matricola) S287649
Hereby declare (*dichiarazione sostitutiva di atto notorio*) that the home assignment n. 1
Has been carried out in a strictly individual manner from beginning to end; in particular,

TL I have not obtained help from any classmate or external person to carry out in part or whole the assignment;

TL I have not employed any paper or electronic material directly related to the assignment; (note: textbooks are indirectly related only)

TL I have not employed scripts, computer programs or any other such procedures that have not been entirely developed by myself, or provided as course material (by the Instructor and/or the Teaching Assistant), and that are not commercial, or cannot be referenced in the open literature or internet; please note that *all employed software not personally and individually developed must be referenced in the submitted papers*. In particular, I have not employed any script, programs etc. developed by my classmates, and that the employed scripts, programs etc. have not been developed in cooperation with my classmates.

TL I have discussed this assignment with the following persons: (enter "none" if appropriate):

Tong Lin
(Complete name, please print)

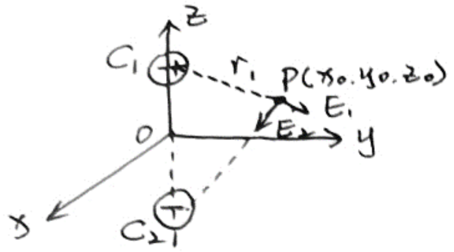
Tong Lin
signature

Torino, 2022 / 3 / 15 (date)

Note: Use of commercial software, of free-ware or shareware, or otherwise publicly available software (e.g. via Internet) is allowed, but usage of all software not developed personally and individually by the student, or provided as course material, MUST be clearly stated and precisely referenced in the submitted paper.

Problem No.1

A. Static Electric Field:



The static electric field $E(P)$ can be expressed as the sum of two components,

$$\underline{E}(\underline{P}) = \underline{E}_1 + \underline{E}_2$$

where $\underline{E}_1$ is the electric field generated by the positive charge at C_1, and $\underline{E}_2$ is generated by the negative charge at C_2.

$$\underline{E}_1 = kq \frac{1}{r_1^2} \hat{r}_1$$

$$\underline{E}_2 = -kq \frac{1}{r_2^2} \hat{r}_2$$

$\underline{r}_1$ and $\underline{r}_2$ are the vector separately between P and C_1, and P and C_2. Taken $\underline{r}_1$ as an example with the calculation,

$$\underline{r}_1 = \underline{P} - \underline{C}_1 = x_0 \hat{x} + y_0 \hat{y} + (z_0 - h) \hat{z}$$

so that,

$$r_1 = |\underline{P}\underline{C}_1| = \sqrt{x_0^2 + y_0^2 + (z_0 - h)^2}$$

$$\hat{r}_1 = \frac{\underline{r}_1}{r} = \frac{x_0 \hat{x} + y_0 \hat{y} + (z_0 - h) \hat{z}}{\sqrt{x_0^2 + y_0^2 + (z_0 - h)^2}}$$

Taking the equations above into the function of $\underline{E}_1$.

$$\underline{E}_1 = kq \frac{x_0 \hat{x} + y_0 \hat{y} + (z_0 - h) \hat{z}}{[x_0^2 + y_0^2 + (z_0 - h)^2]^{\frac{3}{2}}}$$

Similarly,

$$\underline{E}_2 = -kq \frac{x_0 \hat{x} + y_0 \hat{y} + (z_0 + h) \hat{z}}{[x_0^2 + y_0^2 + (z_0 + h)^2]^{\frac{3}{2}}}$$

The static electric field can be finally obtained,

$$\underline{E}(\underline{P}) = \underline{E}_1 + \underline{E}_2$$

$$= kq \left\{ \frac{x_0\hat{x} + y_0\hat{y} + (z_0-h)\hat{z}}{[x_0^2 + y_0^2 + (z_0-h)^2]^{\frac{3}{2}}} - \frac{x_0\hat{x} + y_0\hat{y} + (z_0+h)\hat{z}}{[x_0^2 + y_0^2 + (z_0+h)^2]^{\frac{3}{2}}} \right\}$$

B. Case of $z=0,\ y=0$:

Applying $z_0=0,\ y_0=0$ into the general electric field function above,

$$\underline{E} = kq \left\{ \frac{x_0\hat{x} - h\hat{z}}{[x_0^2 + h^2]^{\frac{3}{2}}} - \frac{x_0\hat{x} + h\hat{z}}{[x_0 + h^2]^{\frac{3}{2}}} \right\}$$

$$= kq \frac{-2h}{(x_0^2 + h^2)^{\frac{3}{2}}} \hat{z}$$

The field magnitude is,

$$|\underline{E}(\underline{P})| = \frac{-2kqh}{(x_0^2 + h^2)^{\frac{3}{2}}}$$

The plot of the field magnitude $|\underline{E}(\underline{P})|$ with the variation of x is as shown in Figure 1.1.

C. Case of $z=0,\ x=0$

Similarly as in part B, the electric field magnitude can be obtained as,

$$|\underline{E}(\underline{P})| = \frac{-2kqh}{(y_0^2 + h^2)^{\frac{3}{2}}}$$

As the symmetry of the field, the plot of with the variation of y_0 is exactly the same pattern as x_0, which is shown in Figure 1.2.

D. Case of $x = 0$, $y = 0$:

The function of electric field in this case can be written as,

$$\underline{E} = kq \left\{ \frac{(z_0 - h)\hat{z}}{[(z_0 - h)^2]^{\frac{3}{2}}} - \frac{(z_0 + h)\hat{z}}{[(z_0 + h)^2]^{\frac{3}{2}}} \right\}$$

$$= kq \left\{ \frac{z_0 - h}{[(z_0 - h)^2]^{\frac{3}{2}}} - \frac{z_0 + h}{[(z_0 + h)^2]^{\frac{3}{2}}} \right\} \hat{z}$$

The magnitude of the field is then,

$$|\underline{E}(\underline{P})| = kq \left\{ \frac{(z_0 - h)}{[(z_0 - h)^2]^{\frac{3}{2}}} - \frac{(z_0 + h)}{[(z_0 + h)^2]^{\frac{3}{2}}} \right\}$$

The plot of the field magnitude as with variation of z_0 is as shown in Figure 1.3.

E. Arrow plot of xz plane:

The electric field of xz plane, which setting $\hat{y} = 0$, can be written as,

$$\underline{E}(x, 0, z) = kq \left\{ \frac{x_0 \hat{x} + (z_0 - h)\hat{z}}{[x_0^2 + (z_0 - h)^2]^{\frac{3}{2}}} - \frac{x_0 \hat{x} + (z_0 + h)\hat{z}}{[x_0^2 + (z_0 + h)^2]^{\frac{3}{2}}} \right\}$$

Using the form of different directions, which,

$$\underline{E}(x, 0, z) = e_x \hat{x} + e_z \cdot \hat{z}$$

e_x and e_z can be respectively expressed as,

$$e_x = kq \left\{ \frac{1}{[x_0^2 + (z_0 - h)^2]^{\frac{3}{2}}} - \frac{1}{[x_0^2 + (z_0 + h)^2]^{\frac{3}{2}}} \right\} \cdot x$$

$$e_z = kq \left\{ \frac{(z_0 - h)}{[x_0^2 + (z_0 - h)^2]^{\frac{3}{2}}} - \frac{(z_0 + h)}{[x_0^2 + (z_0 + h)^2]^{\frac{3}{2}}} \right\}$$

Using the MATLAB function "quiver(x, z, e_x, e_z)", the plot of electric field magnitude and direction is as shown in Figure 1.4 and 1.5.

The Plots of Problem 1:

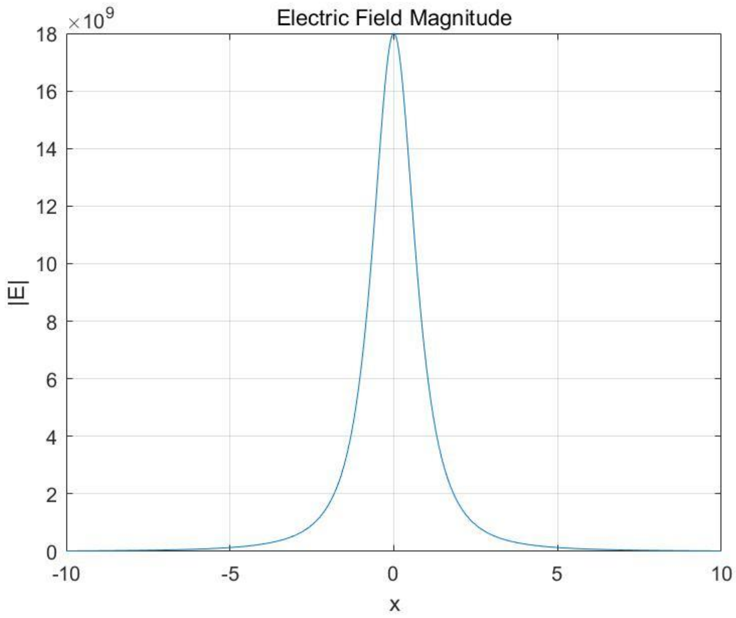


Figure 1.1: The Field Magnitude with case z=0, y=0

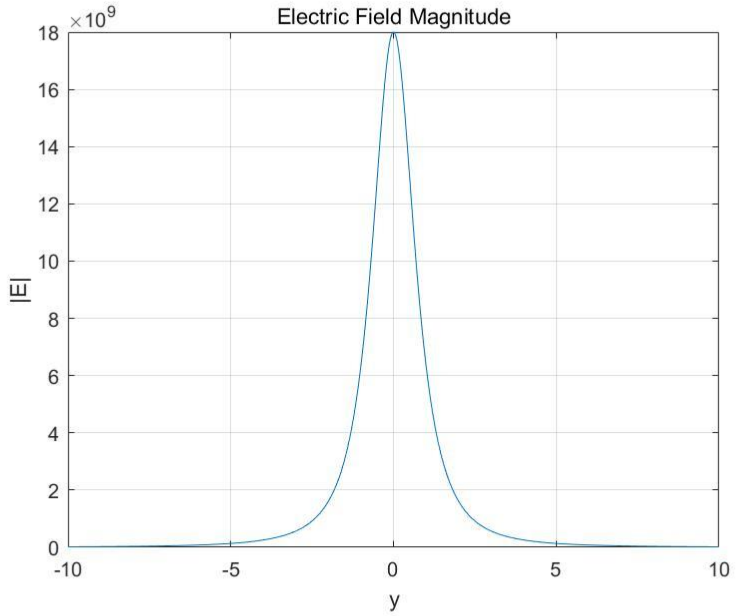


Figure 1.2: The Field Magnitude with case z=0, x=0

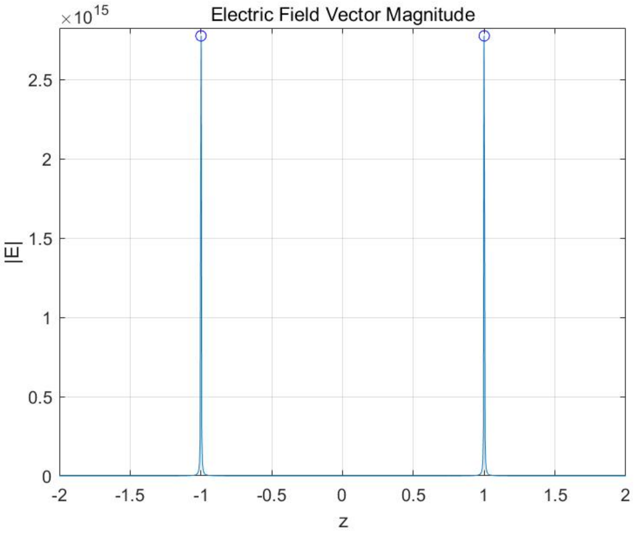


Figure 1.3: The Field Magnitude with case x=0, y=0

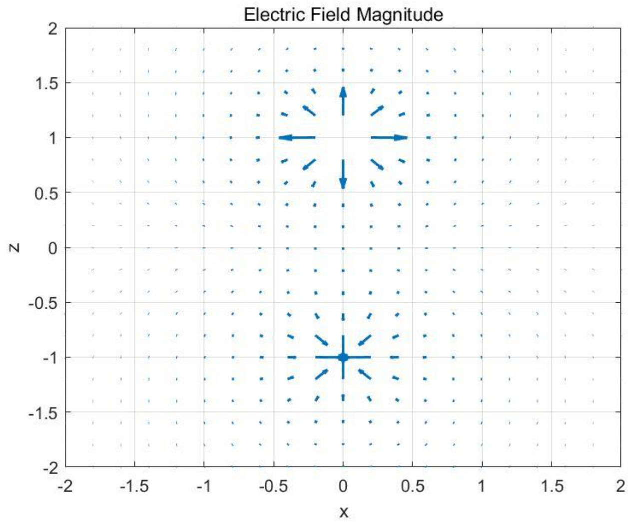


Figure 1.4: The Electric Field Magnitude Arrow Plot in (x, z) Plane

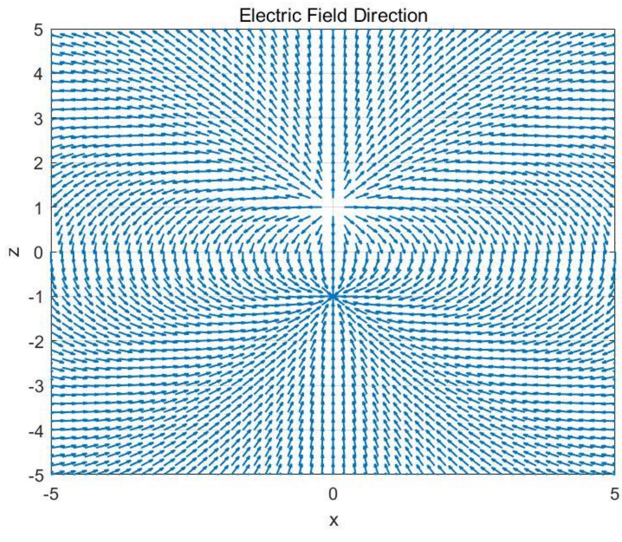
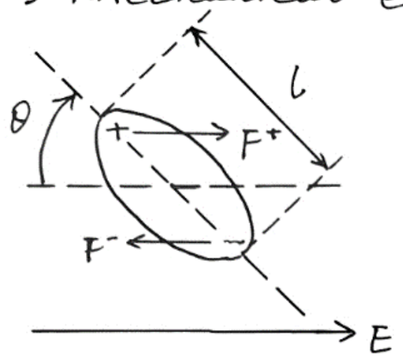


Figure 1.5: The Electric Field Arrow Plot in (x, z) Plane

Problem No. 2

a) Mechanical equations:



The mechanical equation of this system can be analyzed ~~as~~ as ~~the~~ two parts of the positive point and the negative point.

The torque of the positive point can be expressed as,

$$\underline{M}^+ = F^+ \cdot \frac{l}{2} \cdot \sin\theta$$

In this function, the force F^+ is,

$$F^+ = \frac{E}{q}$$

So that,

$$\underline{M}^+ = \frac{El}{2q} \cdot \sin\theta$$

Simlary, the torque of the negative point can be written as,

$$\underline{M}^- = -\frac{El}{2q} \cdot \sin\theta$$

The total torque is then,

$$\underline{M}_s = \underline{M}^+ - \underline{M}^- = \frac{El}{2q} \cdot \sin\theta$$

Given the known inerota of I, the length of the system l, can be expressed by,

$$I = 2m\left(\frac{l}{2}\right)^2 = \frac{1}{2}ml^2$$

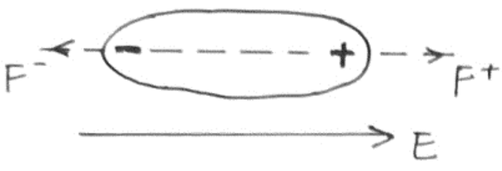
$$l = \sqrt{\frac{2I}{m}}$$

Then the mechanical equation as the total torque can be further written as,

$$\underline{M}_s = \frac{E}{2q} \cdot \sqrt{\frac{2I}{m}} \cdot \sin\theta$$

The moving direction is along clockwise.

b) Rest position :



The rest position is as shown in the figure on the left, where the system is balanced by the equal two forces pointing to different directions respectively on left and right.

Problem No. 3

The initial time-harmonic vector can be written as,

$$\underline{\mathcal{E}}_0(t) = A\left[-\sin(\omega t)\,\hat{x} + 2\cos(\omega t - \tfrac{\pi}{3})\,\hat{y}\right].$$
$$= -A\sin(\omega t)\,\hat{x} + 2A\cos(\omega t - \tfrac{\pi}{3})\,\hat{y}$$

which can then expressed as two seperate components,

$$\mathcal{E}_x(t) = -A\sin(\omega t)$$
$$\mathcal{E}_y(t) = 2A\cos(\omega t - \tfrac{\pi}{3}).$$

Accroding to the function of the complex vector $\underline{E}_0$,

$$\underline{E}_0 = \underline{E}_0' + j\,\underline{E}_0''$$
$$\underline{E}_0' = E_x'\,\hat{x} + E_y'\,\hat{y}$$
$$\underline{E}_0'' = E_x''\,\hat{x} + E_y''\,\hat{y}$$

where,

$$E_x' = \mathcal{E}_x(t=0) = 0$$
$$E_y' = \mathcal{E}_y(t=0) = 2A\cdot\cos(-\tfrac{\pi}{3}) = -A$$
$$E_x'' = -\mathcal{E}_x(t=\tfrac{T}{4}) = A$$
$$E_y'' = -\mathcal{E}_y(t=\tfrac{T}{4}) = -2A\cos(\tfrac{\pi}{6}) = -\sqrt{3}A.$$

So that the real part of the phasor can be written as follows, which indicates the in-phase vector,

$$\underline{E}_0' = -A\hat{y}$$

The imaginary part as follows indicates the quadrature vector,

$$\underline{E}_0'' = A\hat{x} - \sqrt{3}A\hat{y}$$

Problem No. 4

With the expression of phasor,

$$\underline{A} = (4 + j)\,\hat{x} + 2\hat{z}$$
$$= (4\hat{x} + 2\hat{z}) + j\hat{x}$$

The real and imaginary parts are seperately,

$$\underline{A}' = 4\hat{x} + 2\hat{z}$$
$$\underline{A}'' = \hat{x}$$

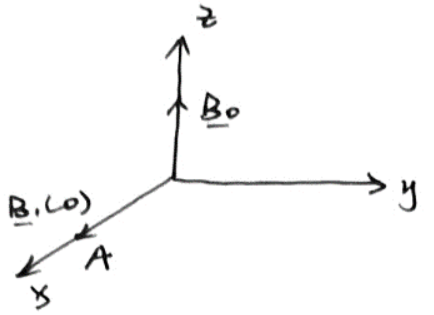
According to the function of the time-harmonic vector, the components can be expressed as, neglecting the direction of $\hat{y}$,

$$A_x(t) = 4\cos\omega t - \sin\omega t$$
$$A_z(t) = 2\cos\omega t$$

Therefore, the time-harmonic vector is,

$$\underline{A}(t) = (4\cos\omega t - \sin\omega t)\,\hat{x} + 2\cos\omega t \cdot \hat{z}$$

Problem No. 5



The initial expression of the time-varying vector can be written as,

$$\underline{B}_1(t) = \underline{B}_1' \cos \omega t - \underline{B}_1'' \sin \omega t$$

As the RF magnetic field of MRI is circularly polarized, so that,

$$\underline{B}_1' \perp \underline{B}_1''$$

$$|\underline{B}_1'|^2 = |\underline{B}_1''|^2$$

Also, the RF magnetic field is perpendicular to the static magnetic field $\underline{B}_0 = B_0 \hat{z}$, the real and imaginary part of the phasor of the RF field can therefore be further expressed as follows, applying the equal and constant amplitude of A, and with $\underline{B}_1(0)$ along $\hat{x}$ direction,

$$\underline{B}_1' = A\hat{x}$$

$$\underline{B}_1'' = A\hat{y}$$

The expression of the phasor can be finally written as,

$$\underline{B}_1 = A\hat{x} + jA\hat{y}$$

The time-varying expression can be written as,

$$\underline{B}_1(t) = A\cos\omega t\,\hat{x} - A\sin\omega t\,\hat{y}$$

where ω can chose, $\omega_0 = 2.67 \times 10^8$ rad/s, which indicates the hydrogen nucleus placed in a static magnetic field with $B_0 = 1T$.

Problem No. 6

1. Magnetic field $\underline{H}(P)$:

According to the Maxwell functions, the magnetic field can be demonstrated with the exact function of the electric field, which is expressed as,

$$-\nabla \times \underline{E}(P) = j w_0 \mu_0 \underline{H}(P)$$

where w_0 and μ_0 are coefficients, so that the magnetic field can be expressed as,

$$\underline{H}(P) = -\frac{1}{j w_0 \mu_0}\left[\nabla \times \underline{E}(P)\right]$$

The electric field function can be derived as follows,

$$\underline{E}(P) = \underline{E}_0 f(P)$$

$$= (E_{0x}\hat{x} + E_{0y}\hat{y}) \cdot \exp(-j\underline{k}\underline{r})$$

in this function, $\underline{r} = P - \underline{O}$, where P is a generic point, can be expressed as $\underline{P}(x_P, y_P, z_P)$, so,

$$\underline{r} = x_P\hat{x} + y_P\hat{y} + z_P\hat{z}$$

Also, $\underline{k} = k_0\hat{z} = \frac{w}{c}\hat{z}$, therefore,

$$\underline{k}\cdot\underline{r} = \frac{w}{c}\cdot z_P.$$

So that,

$$\underline{E}(P) = \exp\left(-j\frac{w}{c}\cdot z_P\right)(E_{0x}\hat{x} + E_{0y}\hat{y}).$$

The cross product $\nabla \times \underline{E}$ is defined as,

$$\nabla \times \underline{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{x} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{z}$$

Neglecting the term on the $\hat{z}$ direction according to the question, the calculation here becomes,

$$\nabla \times \underline{E}(P) = \exp\left(-j\frac{w}{c}\cdot z_P\right)\left[-\frac{\partial E_{0y}}{\partial z}\hat{x} + \frac{\partial E_{0x}}{\partial z}\hat{y} + \left(\frac{\partial E_{0y}}{\partial x} - \frac{\partial E_{0x}}{\partial y}\right)\hat{z}\right]$$

The magnetic field can finally be expressed as,

$$\underline{H}(\underline{P}) = -\frac{\exp(-j\frac{w}{c}z_p)}{jw_0\mu_0}\left[-\frac{\partial E_{0y}}{\partial z}\hat{x} + \frac{\partial E_{0x}}{\partial z}\hat{y} + \left(\frac{\partial E_{0y}}{\partial x} - \frac{\partial E_{0x}}{\partial y}\right)\hat{z}\right].$$

2. Physical Unit:

With the initial function of the magnetic field,

$$\underline{H}(\underline{P}) = -\frac{\nabla \times \underline{E}(\underline{P})}{jw_0\mu_0}$$

The units are,

$$\underline{H}(\underline{P}) = \frac{[V/m^2]}{[\Omega/m]} = [A/m].$$

3. Explicit expression of $\underline{\mathcal{E}}(P;t) = \underline{\mathcal{E}}(z;t)$:

As already shown in question 1,

$$\underline{E}(\underline{P}) = \underline{E_0}\exp(-j\underline{k}\cdot\underline{r}).$$

Using Euler's formular,

$$\exp(-j\underline{k}\cdot\underline{r}) = \cos(\underline{k}\cdot\underline{r}) - j\sin(\underline{k}\cdot\underline{r})$$

Therefore,

$$\underline{E}(\underline{P}) = \underline{E_0}\cos(\underline{k}\cdot\underline{r}) - j\,\underline{E_0}\cdot\sin(\underline{k}\cdot\underline{r}).$$

Seperating the components of the above expression, applying $\underline{E} = \underline{E}' + j\underline{E}''$,

$$\underline{E}' = \underline{E_0}\cos(\underline{k}\cdot\underline{r})$$

$$\underline{E}'' = -\underline{E_0}\cdot\sin(\underline{k}\cdot\underline{r})$$

The time-varying vector is therefore,

$$\begin{aligned}\underline{\mathcal{E}}(\underline{P};t) &= \underline{E}'\cos wt - \underline{E}''\sin wt \\ &= \underline{E_0}\cos(\underline{k}\cdot\underline{r})\cos wt + \underline{E_0}\sin(\underline{k}\cdot\underline{r})\sin wt \\ &= \underline{E_0}\cos(\underline{k}\cdot\underline{r} + wt) \\ &= \underline{E_0}\cos(wt + \frac{w}{c}\cdot z_p)\end{aligned}$$

(where z_p stands for the position of the point on z direction)

4. Situation for general medium:

From part 3, the general function of the time-varying vector of the electric field can be expressed as,

$$\mathcal{E}(P;t) = \underline{E_0}\cos(\underline{k}\cdot\underline{r} + \omega t)$$

Within a general medium, the phase can be written as,

$$\underline{k}\cdot\underline{r} = \frac{\omega}{c}\cdot(b - ja)(\alpha\cdot x + \beta y + \gamma z)$$

where
$$\underline{k} = \frac{\omega}{c}(b - ja)(\alpha\hat{x} + \beta\hat{y} + \gamma\hat{z})$$
$$\underline{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

Then the amplitude can be expressed as,

$$|\underline{E_0}| = \sqrt{E_{0x}^2 + E_{0y}^2}$$

a) For a constant-phase surface, which $(\underline{k}\cdot\underline{r})$ is constant, is a wavefront surface. When $(\alpha x + \beta y + \gamma z) =$ constant, where $\alpha^2 + \beta^2 + \gamma^2 = 1$, the surface becomes a sphere surface.

b) For a constant-amplitude surface, which $|\underline{E_0}|$ is constant, can be consider as a surface far from the source, which is in far-field, as in the far-field, the amplitude will not decrease noticeably with the increase of the distance due to the propagation scale.
Otherwise, it can be a surface in free space, which can be perfectly considered as a linear system.

Appendix: MATLAB codes

<u>**Problem 1:**</u>

```
clear all;
close all;
clc

h = 1;              %normalized
k = 9e9;
q = 1;             %[C],normalized

%B
x = linspace(-10,10,1000);
E1 = sqrt((-2*h*k*q./(x.^2+h^2).^1.5).^2);
figure
plot(x,E1);
xlabel('x');
ylabel('|E|');
title('Electric Field Magnitude');
grid on
%C
y = linspace(-10,10,1000);
E2 = sqrt((-2*k*q*h./(y.^2+h^2).^1.5).^2);
figure
plot(y,E2);
xlabel('y');
ylabel('|E|');
title('Electric Field Magnitude');
grid on
%D
z = linspace(-10,10,5000);
in = (z-h)./((z-h).^2).^1.5+(z+h)./((z+h).^2).^1.5;
E3 = sqrt((k*q*in).^2);
figure
plot(z,E3);
xlabel('z');
ylabel('|E|');
title('Electric Field Magnitude');
grid on
hold on
plot(1,max(E3),'bo');
plot(-1,max(E3),'bo');
axis([-2 2 0 max(E3)+5e13]);
```

```matlab
%E
[x,z] = meshgrid(-10:0.2:10,-10:0.2:10);
X = x.*(1./(x.^2+(z-h).^2).^1.5-1./(x.^2+(z+h).^2).^1.5);
Z = (z-h)./(x.^2+(z-h).^2).^1.5-(z+h)./(x.^2+(z+h).^2).^1.5;
ex = k*q*X;
ez = k*q*Z;
figure
quiver(x,z,ex,ez,'LineWidth',1.5);
xlabel('x');
ylabel('z');
title('Electric Field Magnitude');
axis([-2 2 -2 2]);
grid on

exv = ex./sqrt(ex.^2+ez.^2);
ezv = ez./sqrt(ex.^2+ez.^2);
figure
quiver(x,z,exv,ezv,'LineWidth',1);
xlabel('x');
ylabel('z');
title('Electric Field Direction');
axis([-5 5 -5 5]);
grid on
```