Capital Allocation to Risky Assets

Source: chapter 6, Bodie, Kane and Marcus, Investments, 12 ed., McGraw-Hill, 2021

Real and Nominal Rates of Interest

- A **nominal interest rate** is the growth rate of your money
- A **real interest rate** is the growth rate of your purchasing power

$r_{nom} =$ Nominal Interest Rate

$r_{real} =$ Real Interest Rate

$i =$ Inflation Rate

$$r_{real} = \frac{r_{nom} - i}{1 + i}$$

$$Note: r_{real} \approx r_{nom} - i$$

Interest Rates and Inflation

- We expect higher nominal interest rates when inflation is higher

- If $E(i)$ denotes current expectations of inflation, the Fisher hypothesis is

$$r_{nom} = r_{real} + E\left(i\right)$$

Effective Annual Rate (EAR) and Annual Percentage Rate (APR)

- Effective annual rates (EAR) explicitly account for compound interest

$$1 + \text{EAR} = \left(1 + \frac{\text{APR}}{n}\right)^n$$

- Annual percentage rates (APR) are annualized using simple rather than compound interest

$$\text{APR} = n \times [(1 + \text{EAR})^{1/n} - 1]$$

Risk and Risk Premiums: Holding Period Returns

- Sources of investment risk
  - Macroeconomic fluctuations
  - Changing fortunes of various industries
  - Firm-specific unexpected developments
- *Holding period return (HPR)*, or realized rate of return, is based on the price per share at year's end and any cash dividends collected

$$\text{HPR} = \frac{\text{Ending price of a share} - \text{Beginning price} + \text{Cash dividend}}{\text{Beginning price}}$$

Risk and Risk Premiums: Expected Return and Standard Deviation

- Expected returns

$$E(r) = \sum_{s} p(s) \times r(s)$$

  - $p(s)$ = probability of each scenario
  - $r(s)$ = HPR in each scenario
  - s = scenario

Risk and Risk Premiums: Expected Return and Standard Deviation

- Variance (VAR):

$$\sigma^2 = \sum_s p(s) \times \left[r(s) - E(r) \right]^2$$

- Standard Deviation (STD):

$$\mathrm{STD} = \sqrt{\sigma^2}$$

 Excess Returns and Risk Premiums

- **Risk premium** is the difference between the *expected* HPR and the **risk-free rate**
  - Provides compensation for the risk of an investment
- **Risk-free rate** is the rate of interest that can be earned with certainty
  - Commonly taken to be the rate on short-term T-bills
- Difference between actual rate of return and risk-free rate is called **excess return**
- **Risk aversion** dictates the degree to which investors are willing to commit funds to stocks

Learning from Historical Returns

Expected Returns and the Arithmetic Average

- When using historical data, each observation is treated as an equally likely "scenario"
- Expected return, *E(r)*, is estimated by arithmetic average of sample rates of return

$$E(r) = \sum_{s=1}^{n} p(s)r(s) = \frac{1}{n}\sum_{s=1}^{n} r(s)$$
$$= \text{Arithmetic average of historic rates of return}$$

Geometric (Time-Weighted) Average Return

- Geometric rate of return
  - Intuitive measure of performance over the sample period is the (fixed) annual HPR that would compound over the period to the same terminal value obtained from the sequence of actual returns in the time series

$$(1+g)^n = \text{Terminal value}$$

$$g = \text{Terminal value}^{1/n} - 1$$

Speculation

- Taking considerable risk for a commensurate gain
- Parties have heterogeneous expectations

Gambling

- Bet on an uncertain outcome for enjoyment
- Parties assign the same probabilities to the possible outcomes

Risk and Risk Aversion

- Utility Values
  - Investors are willing to consider:
    - Risk-free assets
    - Speculative positions with positive risk premiums
  - Portfolio attractiveness
    - Increases with expected return
    - Decreases with risk
    - What happens when return increases with risk?

Available Risky Portfolios

Portfolio	Risk Premium	Expected Return	Risk (SD)
L (low risk)	2%	7%	5%
M (medium risk)	4	9	10
H (high risk)	8	13	20

Table 6.1

Available risky portfolios (risk-free rate = 5%)

Each portfolio receives a utility score to assess the investor's risk/return trade off

Risk Aversion and Utility Values

- Utility Function
  - U = Utility
  - $E(r)$ = Expected return on the asset or portfolio
  - A = Coefficient of risk aversion
  - σ^2 = Variance of returns
  - ½ = A scaling factor

$$U = E(r) - \tfrac{1}{2} A\sigma^2$$

Utility Scores of Portfolios with Varying Degrees of Risk Aversion

Investor Risk Aversion (A)	Utility Score of Portfolio L [$E(r) = .07; \sigma = .05$]	Utility Score of Portfolio M [$E(r) = .09; \sigma = .10$]	Utility Score of Portfolio H [$E(r) = .13; \sigma = .20$]
2.0	$.07 - ½ \times 2 \times .05^2 = .0675$	$.09 - ½ \times 2 \times .1^2 = .0800$	**$.13 - ½ \times 2 \times .2^2 = .09$**
3.5	$.07 - ½ \times 3.5 \times .05^2 = .0656$	**$.09 - ½ \times 3.5 \times .1^2 = .0725$**	$.13 - ½ \times 3.5 \times .2^2 = .06$
5.0	$.07 - ½ \times 5 \times .05^2 = .0638$	**$.09 - ½ \times 5 \times .1^2 = .0650$**	$.13 - ½ \times 5 \times .2^2 = .03$

Investor Types

- Risk Averse Investors:

$$A > 0$$

- Risk-Neutral Investors:

$$A = 0$$

- Risk Lovers:

$$A < 0$$

Where A = Coefficient of risk aversion

Trade-Off Between Risk and Return

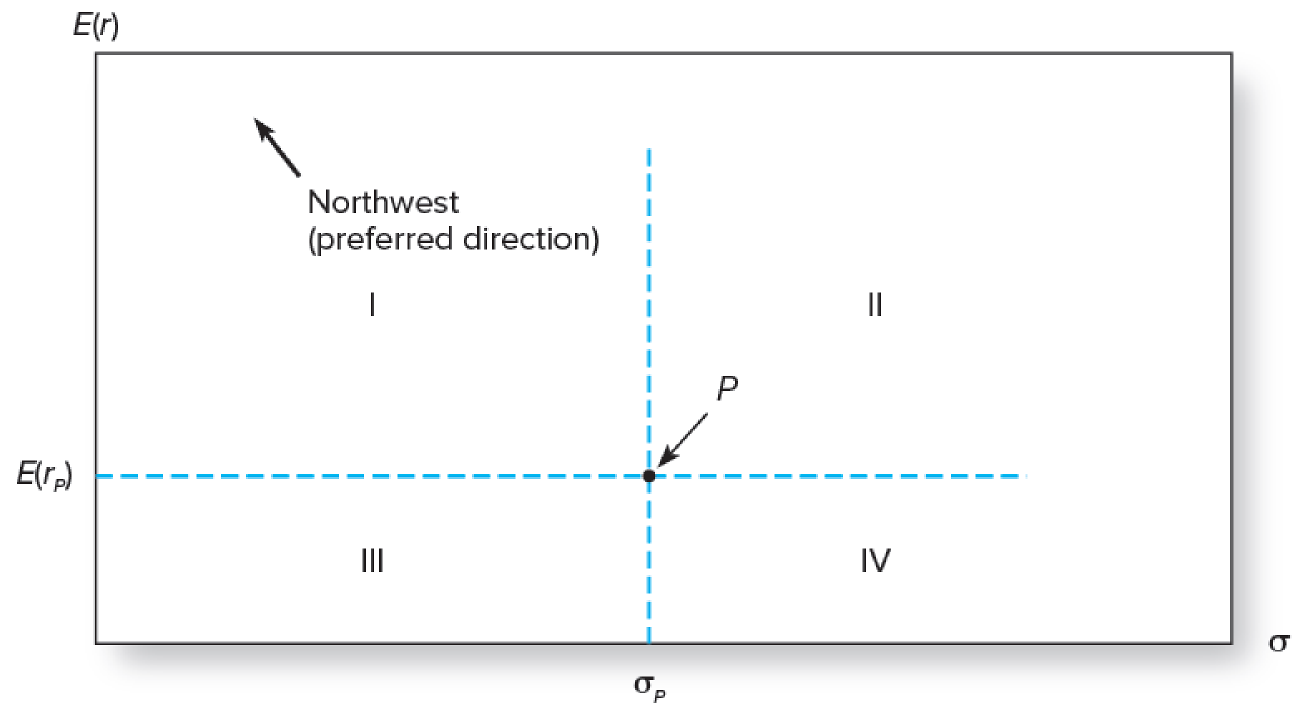


Figure 6.1 The trade-off between risk and return of a potential investment portfolio, *P*

Estimating Risk Aversion

- Use questionnaires
- Observe individuals' decisions when confronted with risk
- Observe how much people are willing to pay to avoid risk

- Mean-Variance (M-V) Criterion
  - Portfolio X dominates portfolio Y if:

$$E\left(r_X\right) \geq E\left(r_Y\right)$$

and

$$\sigma_X \leq \sigma_Y$$

and at least one inequality is strict

Indifference Curves

Equally preferred portfolios will lie in the mean–standard deviation plane on an **indifference curve,** which connects all portfolio points with the same utility value

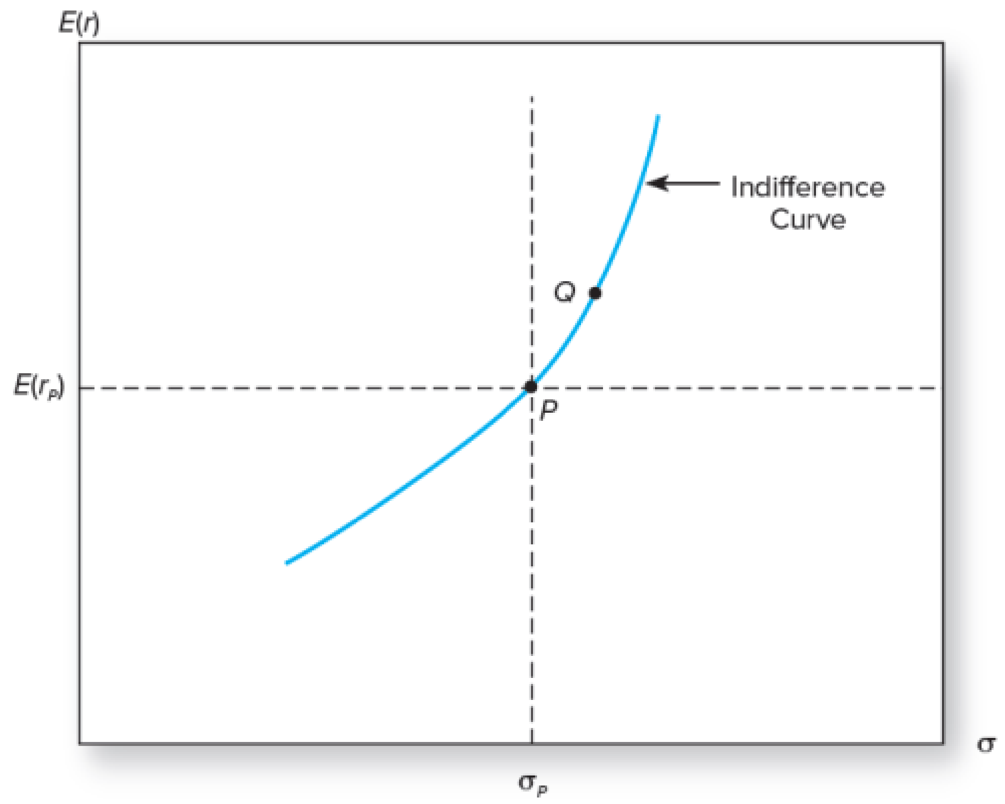


Figure 6.2 The indifference curve

Capital Allocation Across Risky and Risk-Free Portfolios

- Asset Allocation:

- Simplest way to control risk is to manipulate the ratio of risky assets to risk-free assets

Basic Asset Allocation Example

Total market value	\$300,000
Risk-free money market fund	\$90,000
Equities	\$113,400
Bonds (long-term)	\$96,600
Total risk assets	\$210,000

$$W_E = \frac{\$113{,}400}{\$210{,}000} = 0.54 \qquad W_B = \frac{\$96{,}600}{\$210{,}00} = 0.46$$

Basic Asset Allocation Example

Let

- y = Weight of the risky portfolio, P, in the complete portfolio
- $(1-y)$ = Weight of risk-free assets

$$y = \frac{\$210{,}000}{\$300{,}000} = 0.7 \qquad 1 - y = \frac{\$90{,}000}{\$300{,}000} = 0.3$$

$$E: \frac{\$113{,}400}{\$300{,}000} = .378 \qquad B: \frac{\$96{,}600}{\$300{,}000} = .322$$

The Risk-Free Asset

- Only the government can issue default-free securities
  - A security is risk-free in real terms only if
    - Its price is indexed
    - Maturity is equal to investor's holding period
- T-bills viewed as "the" risk-free asset
- Money market funds are also considered risk-free in practice

Portfolios: Risky Asset and Risk-Free Asset

- It's possible to create a complete portfolio by splitting investment funds between safe and risky assets

Let

  - y = Portion allocated to the risky portfolio, P
  - $(1 - y)$ = Portion to be invested in risk-free asset, F

One Risky Asset and a Risk-Free Asset: Example (1 of 2)

$r_f = 7\%$ $\qquad$ $\sigma_{rf} = 0\%$

$E(r_p) = 15\%$ $\qquad$ $\sigma_p = 22\%$

- The expected return on the complete portfolio: $E(r_c) = 7 + y \times (15 - 7)$
- The risk of the complete portfolio:

$$\sigma_C = y \times \sigma_P = 22 \times y$$

One Risky Asset and a Risk-Free Asset: Example (2 of 2)

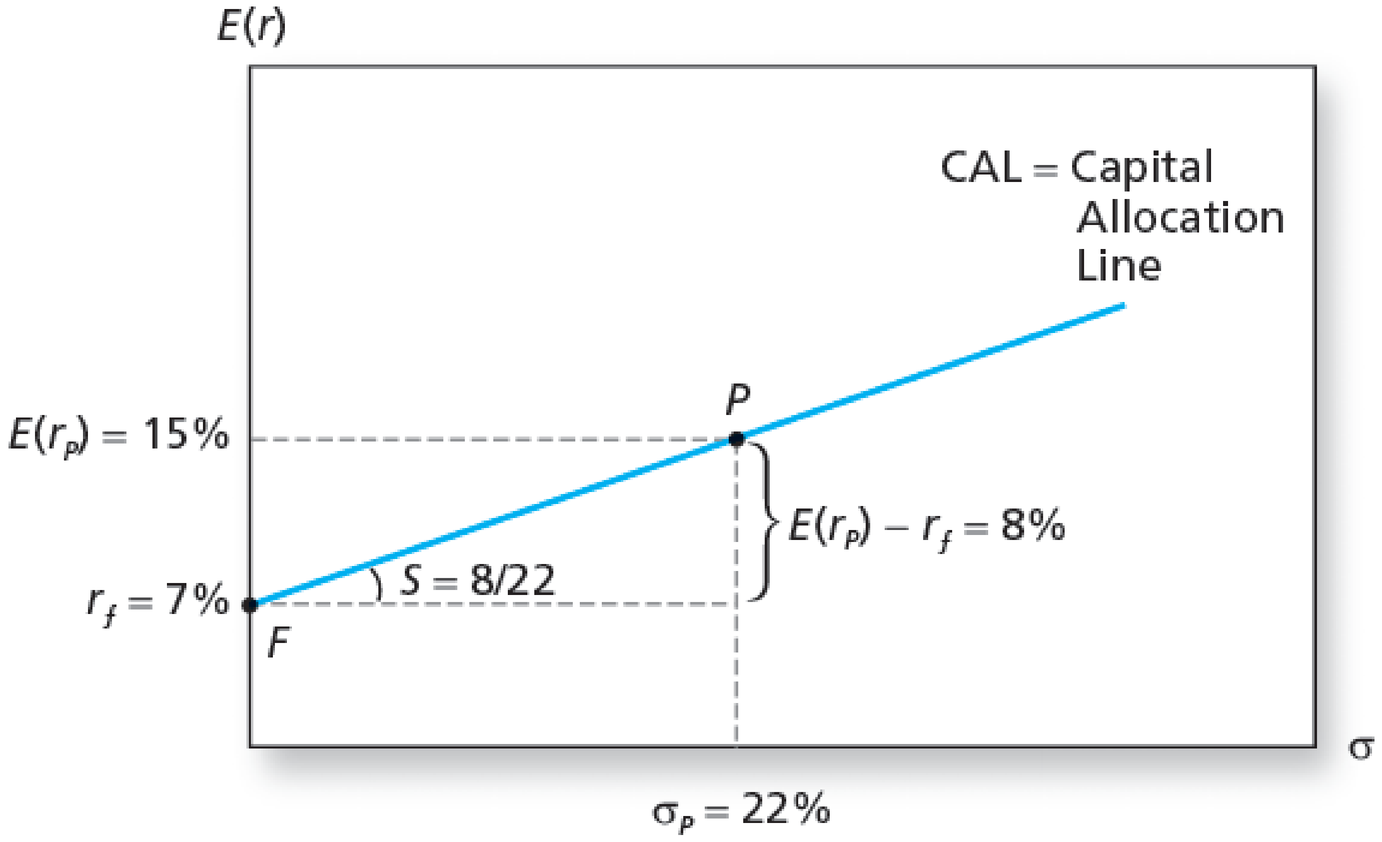
- Rearrange and substitute $y = \sigma_C/\sigma_P$:

$$E(r_C) = r_f + \frac{\sigma_C}{\sigma_P} \times \left[E(r_P) - r_f \right] = 7 + \frac{8}{22} \times \sigma_C$$

- Sharpe ratio: risk adjusted return

$$\text{Slope} = \frac{E(r_P) - r_f}{\sigma_P} = \frac{8}{22}$$

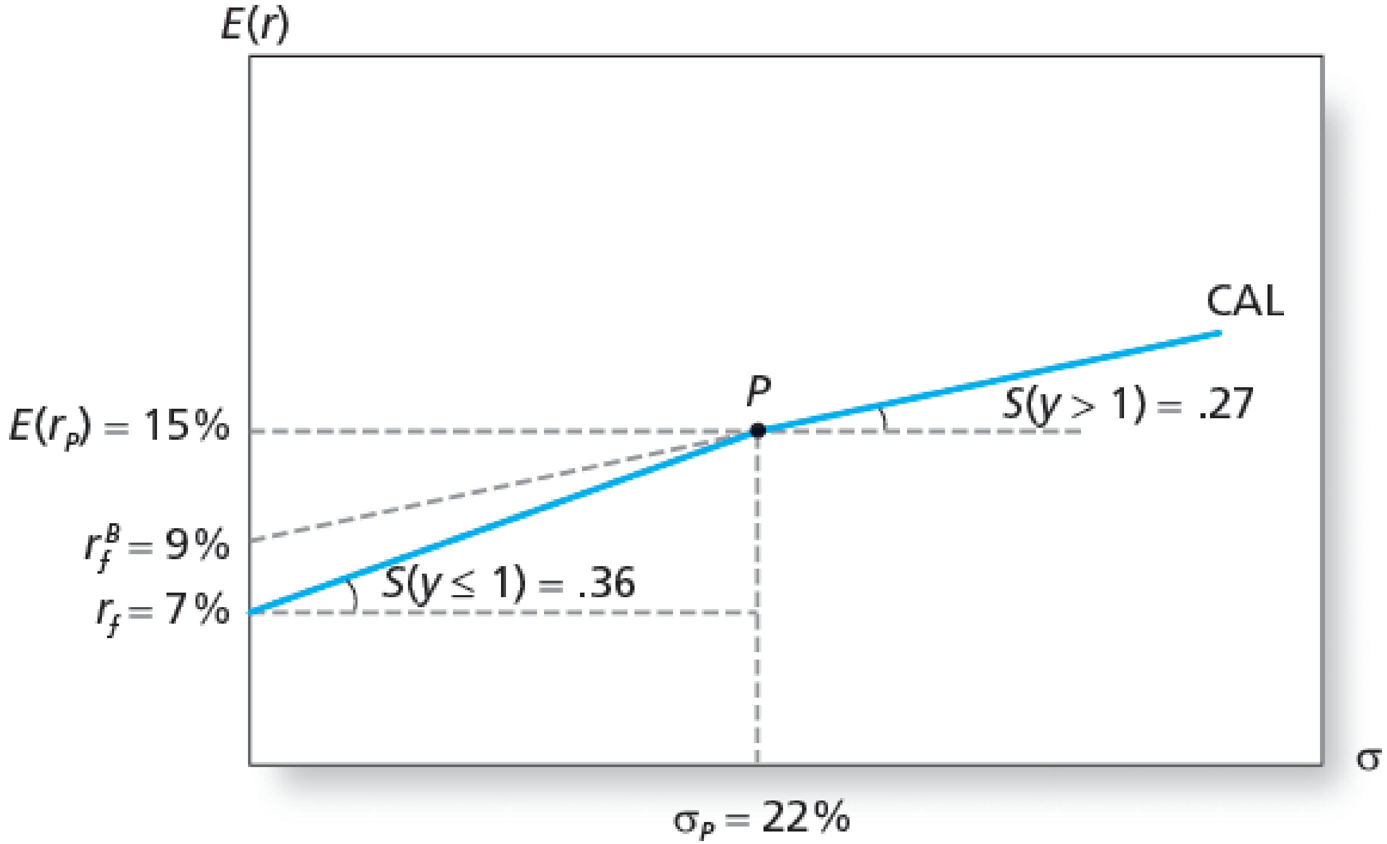
The Investment Opportunity Set



One Risky Asset and a Risk-Free Asset Portfolios

- Capital allocation line with leverage
  - Lend at $r_f = 7\%$ and borrow at $r_f = 9\%$
    - Lending range slope = 8/22 = 0.36
    - Borrowing range slope = 6/22 = 0.27
    - CAL kinks at *P*

The Opportunity Set with Different Borrowing and Lending Rates



Risk Tolerance and Asset Allocation

- The investor must choose one optimal portfolio, *C*, from the set of feasible choices
  - Expected return of the complete portfolio:

$$E(r_c) = r_f + y \times \left[E(r_p) - r_f \right]$$

  - Variance:

$$\sigma_c^2 = y^2 \times \sigma_p^2$$

Utility Levels for Various Positions in Risky Assets

(1) y	(2) $E(r_C)$	(3) σ_C	(4) $U = E(r) - ½A\sigma^2$
0	0.070	0	0.0700
0.1	0.078	0.022	0.0770
0.2	0.086	0.044	0.0821
0.3	0.094	0.066	0.0853
0.4	0.102	0.088	0.0865
0.5	0.110	0.110	0.0858
0.6	0.118	0.132	0.0832
0.7	0.126	0.154	0.0786
0.8	0.134	0.176	0.0720
0.9	0.142	0.198	0.0636
1.0	0.150	0.220	0.0532

Table 6.4

Utility levels for various positions in risky assets (y) for an investor with risk aversion $A = 4$

Utility as a Function of Allocation to the Risky Asset, *y*

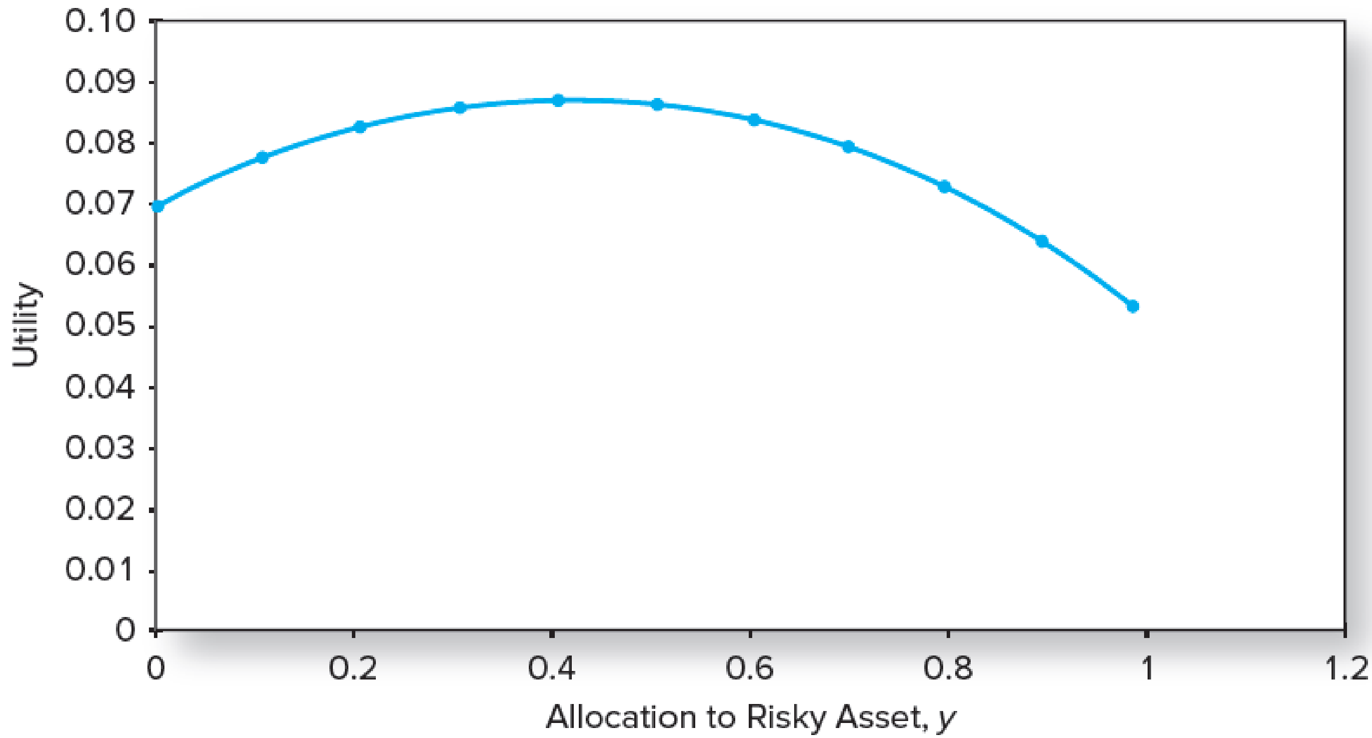


Figure 6.5 Utility as a function of allocation to the risky asset, *y*

Utility as a Function of Allocation to the Risky Asset, *y*

$$\underset{y}{Max}\, U = r_f + y[E(r_P) - r_f] - \tfrac{1}{2} A y^2 \sigma_P^2$$

To find max, take derivative w.r.t. y and set equal to 0

$$[E(r_P) - r_f] - A y \sigma_P^2 = 0$$

Solve for y

$$y^* = \frac{E(r_P) - r_f}{A \sigma_P^2}$$

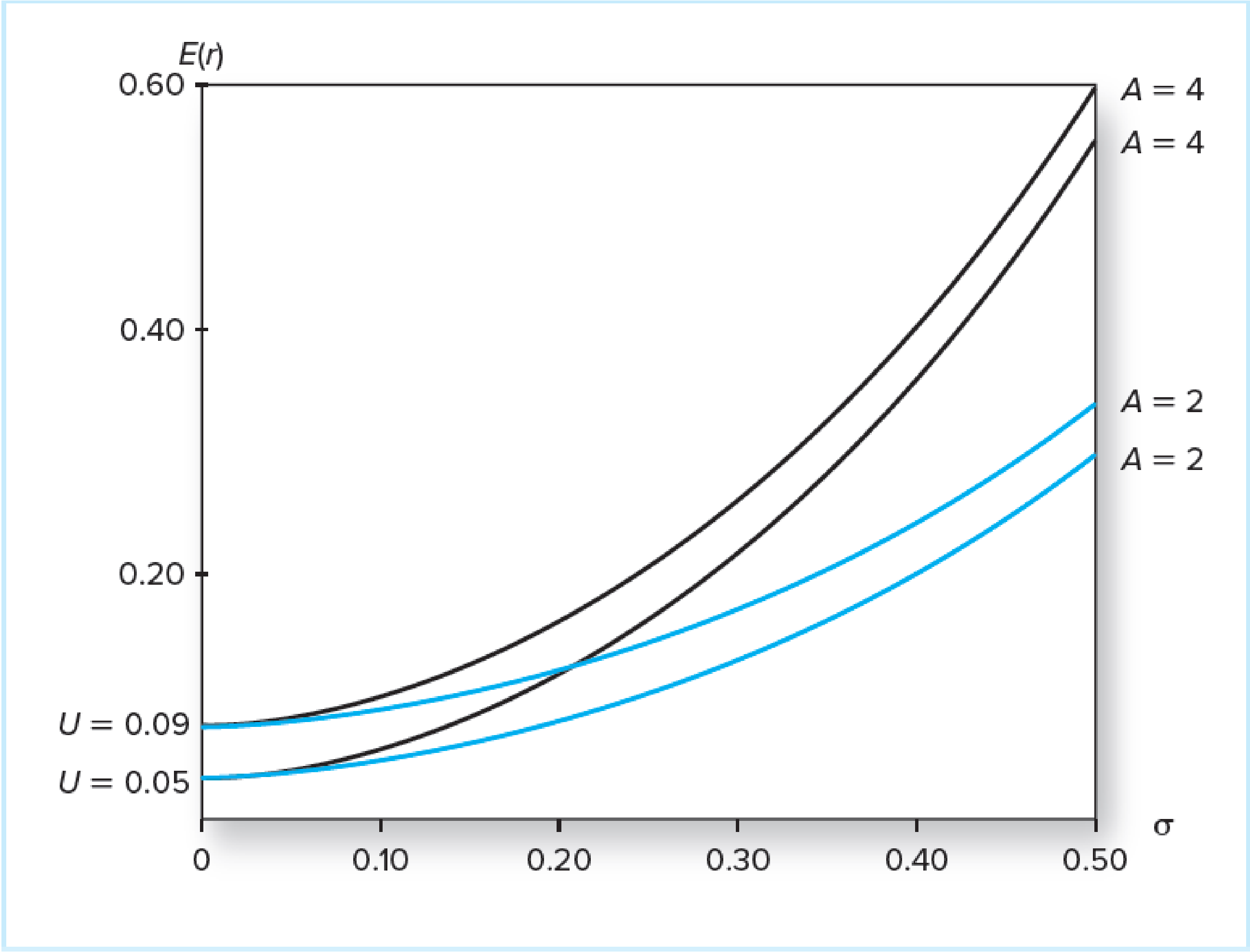
Calculations of Indifference Curves

σ	A = 2		A = 4	
	U = 0.05	U = 0.09	U = 0.05	U = 0.09
0	0.0500	0.0900	0.050	0.090
0.05	0.0525	0.0925	0.055	0.095
0.10	0.0600	0.1000	0.070	0.110
0.15	0.0725	0.1125	0.095	0.135
0.20	0.0900	0.1300	0.130	0.170
0.25	0.1125	0.1525	0.175	0.215
0.30	0.1400	0.1800	0.230	0.270
0.35	0.1725	0.2125	0.295	0.335
0.40	0.2100	0.2500	0.370	0.410
0.45	0.2525	0.2925	0.455	0.495
0.50	0.3000	0.3400	0.550	0.590

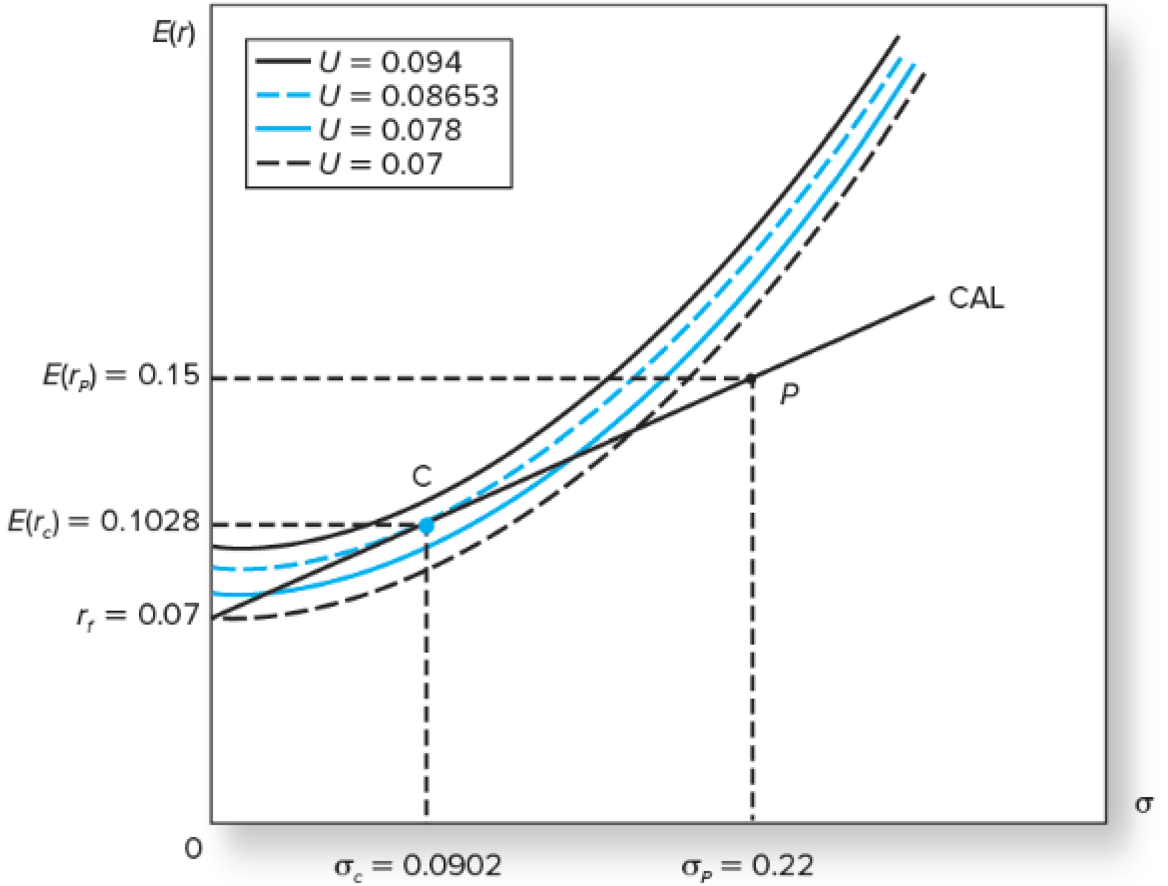
Table 6.5

Spreadsheet calculations of indifference curves (Entries in columns 2–4 are expected returns necessary to provide specified utility value.)

Indifference Curves for $U = .05$ and $U = .09$ with $A = 2$ and $A = 4$



Finding the Optimal Complete Portfolio



Expected Returns on Four Indifference Curves and the CAL

σ	$U = 0.07$	$U = 0.078$	$U = 0.08653$	$U = 0.094$	CAL
0	0.0700	0.0780	0.0865	0.0940	0.0700
0.02	0.0708	0.0788	0.0873	0.0948	0.0773
0.04	0.0732	0.0812	0.0897	0.0972	0.0845
0.06	0.0772	0.0852	0.0937	0.1012	0.0918
0.08	0.0828	0.0908	0.0993	0.1068	0.0991
0.0902	0.0863	0.0943	0.1028	0.1103	0.1028
0.10	0.0900	0.0980	0.1065	0.1140	0.1064
0.12	0.0988	0.1068	0.1153	0.1228	0.1136
0.14	0.1092	0.1172	0.1257	0.1332	0.1209
0.18	0.1348	0.1428	0.1513	0.1588	0.1355
0.22	0.1668	0.1748	0.1833	0.1908	0.1500
0.26	0.2052	0.2132	0.2217	0.2292	0.1645
0.30	0.2500	0.2580	0.2665	0.2740	0.1791

Table 6.6

Expected returns on four indifference curves and the CAL (Investor's risk aversion is $A = 4$.)

Non-Normal Returns

- Above analysis implicitly assumes normality
- VaR and ES* assess exposure to extreme losses
- “Black swan” events should concern investors

* Discussed in Chapter 5

Passive Strategies: The Capital Market Line

- The passive strategy avoids security analysis
- Supply/demand forces may make this strategy reasonable for many investors
- A natural candidate for a passively held risky asset would be the S&P 500

Passive Strategies: The Capital Market Line

- The Capital Market Line (CML)
  - Is a capital allocation line formed investment in two passive portfolios:
    1. Virtually risk-free short-term T-bills (or a money market fund)
    2. Fund of common stocks that mimics a broad market index

From 1926 to 2015, the passive risky portfolio offered an average risk premium of 8.3% with a standard deviation of 20.59%, resulting in a reward-to-volatility ratio of .40