Index model

Source: Bodie, Kane and Marcus, Investments, 12 ed., McGraw-Hill, 2021

Overview

- Drawbacks to Markowitz procedure
  - Requires a huge number of estimates to fill the covariance matrix
  - Model does not provide any guidelines for finding useful estimates of these covariances or the risk premiums
- Introduction of index models
  - Simplifies estimation of the covariance matrix
  - Enhances analysis of security risk premiums
- Optimal risky portfolios constructed using the index model

A Single-Factor Security Market

- Advantages of the single-index model
  - Number of estimates required is a small fraction of what would otherwise be needed
  - Specialization of effort in security analysis

$$r_i = E(r_i) + \beta_i m + e_i$$

- β_i = sensitivity coefficient for firm I
- m = market factor that measures unanticipated developments in the macroeconomy
- e_i = firm-specific random variable

Single-Index Model 1

- **Regression equation**

$$R_i(t) = \alpha_i + \beta_i R_M(t) + e_i(t)$$

- Expected return-beta relationship

$$E(R_i) = \alpha_i + \beta_i E(R_M)$$

Single-Index Model 2

- Total risk = Systematic risk + Firm-specific risk

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma^2(e_i)$$

- Covariance = Product of betas × Market-index risk

$$\mathrm{Cov}\left(r_i, r_j\right) = \beta_i \beta_j \sigma_M^2$$

- Correlation = Product of correlations with the market index

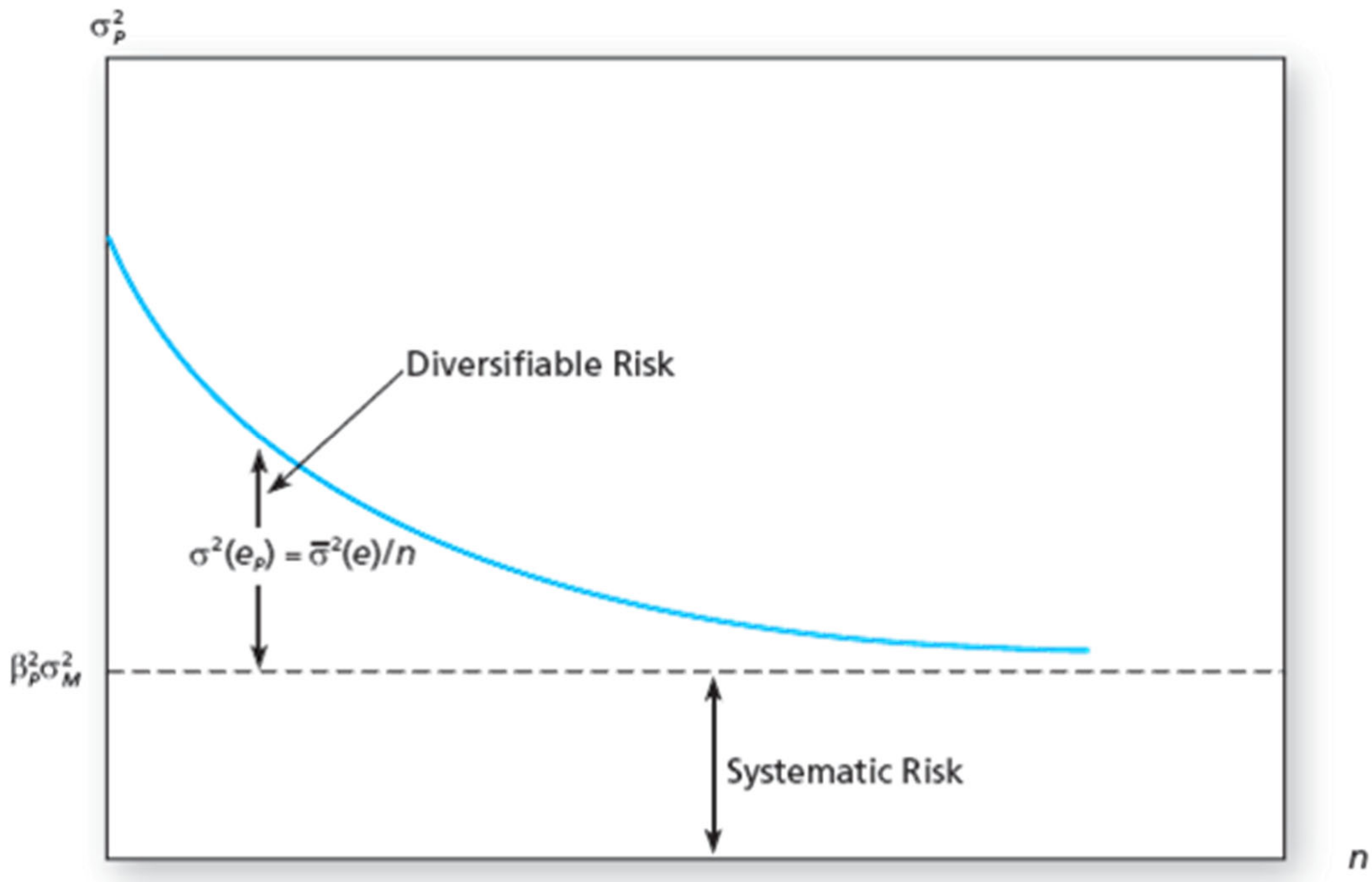
Index Model and Diversification

- Variance of the equally-weighted portfolio of firm-specific components:

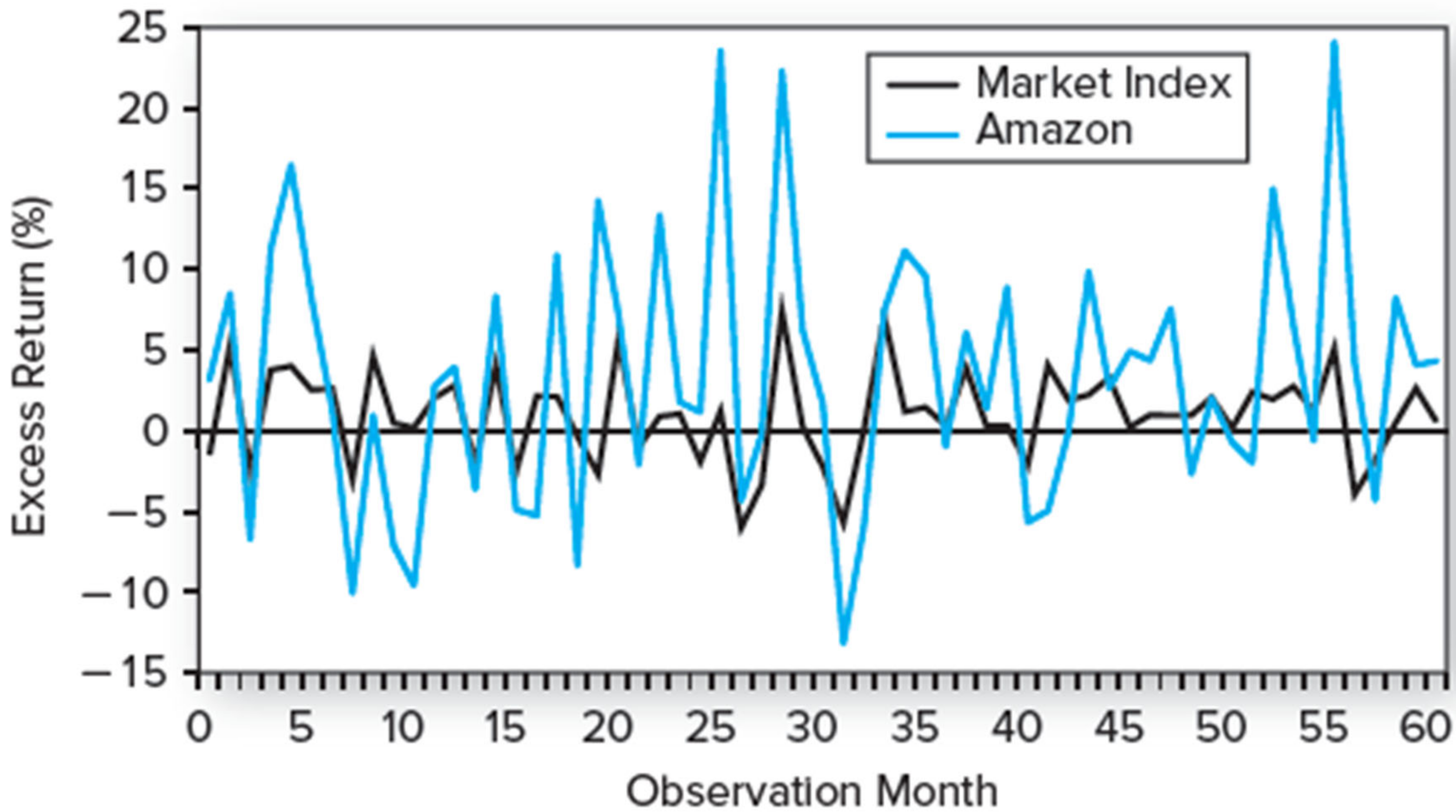
$$\sigma^2(e_P) = Var\left(\frac{1}{n}\sum_{i=1}^{n} e_i\right) = \sum_{i=1}^{n}\left(\frac{1}{n}\right)^2 \sigma^2(e_i) = \frac{1}{n}\sum_{i=1}^{n}\frac{\sigma^2(e_i)}{n} = \frac{1}{n}\overline{\sigma}^2(e)$$

- When n gets large, $\sigma^2(e_p)$ becomes negligible
- As diversification increases, the total variance of a portfolio approaches the systematic variance

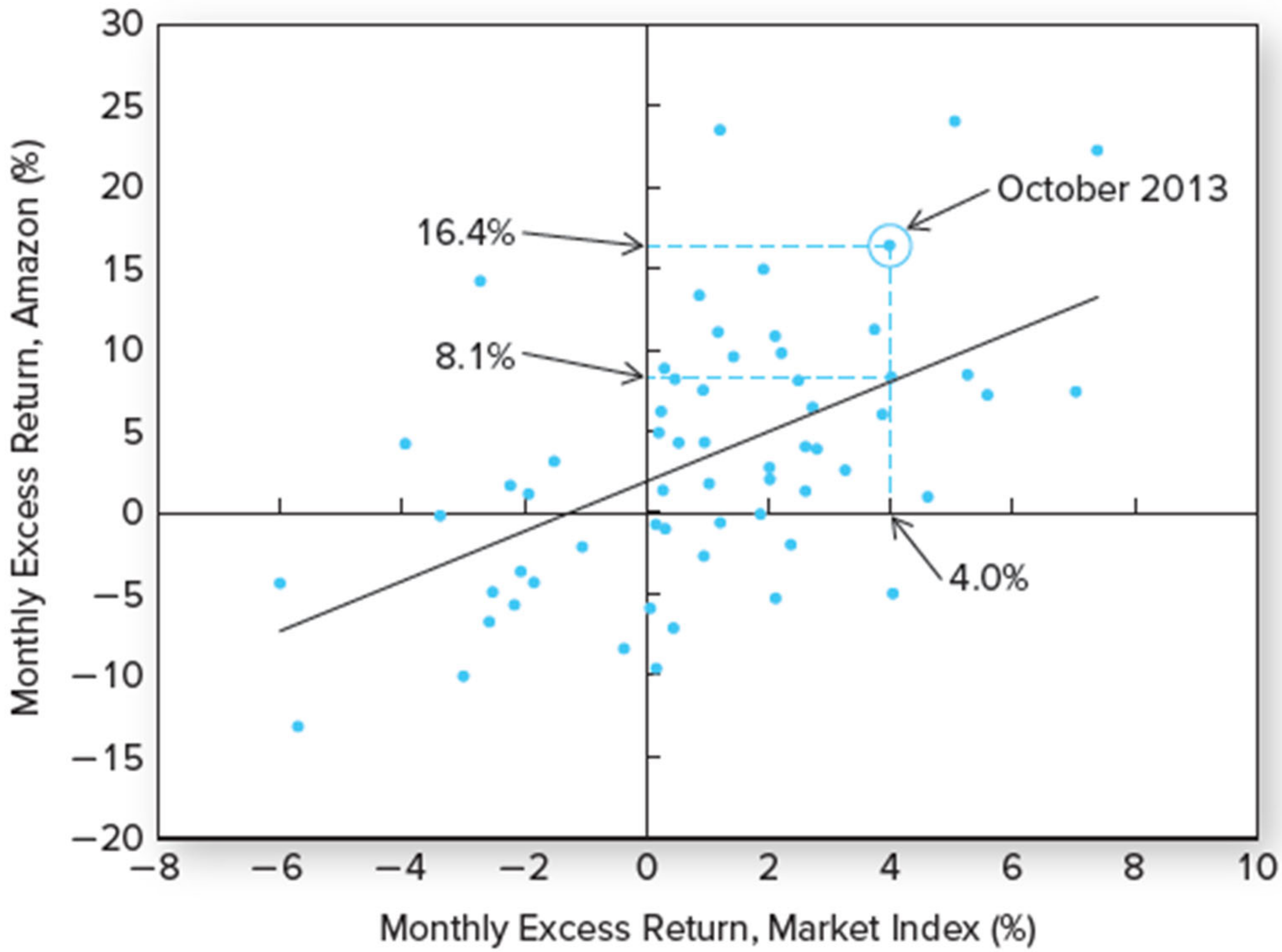
The Variance of an Equally Weighted Portfolio with Risk Coefficient, β_p



Excess Monthly Returns on Amazon and the Market Index



Scatter Diagram



- Access the text alternative for slide images.

Security Characteristic Line (SCL)

- Excess return of security *i*

$$R_i(t) = \alpha_i + \beta_i R_{S\&P500}(t) + e_i(t)$$

- Expected excess return when the market excess return is zero

- Zero-mean, firm-specific surprise in security *i*'s return in month *t*. (the *residual*)

- Sensitivity of security *i*'s return to changes in the return of the market

- Expected excess return of the market

Excel Output: Regression Statistics

• **Regression Statistic**				
Multiple *R*	0.5351			
R-Square	0.2863			
Adjusted *R*-Square	0.2742			
Standard Error	0.0686			
Observations	60			
	Coefficients	**Standard Error**	***t*-statistic**	***p*-value**
Intercept	0.0192	0.0093	2.0645	0.0434
Market index	1.5326	0.3150	4.8648	0.0000

The Industry Version of the Index Model

- Predicting betas
  - Betas tend to drift to 1 over time
- Rosenberg and Guy found the following variables to help predict betas:
  1. Variance of earnings
  2. Variance of cash flow
  3. Growth in earnings per share
  4. Market capitalization (firm size)
  5. Dividend yield
  6. Debt-to-asset ratio

Portfolio Construction and the Single-Index Model 1

- Alpha and security analysis

1. Macroeconomic analysis used to estimate the risk premium and risk of the market index
2. Statistical analysis used to estimate beta coefficients and residual variances, $\sigma_2(e_i)$, of all securities
3. Establish expected return of each security *absent any contribution from security analysis*
4. Security-specific expected return forecasts are derived from various security-valuation models

Portfolio Construction and the Single-Index Model

- Single-index model input list:

1. Risk premium on the S&P 500 portfolio
2. Standard deviation of the S&P 500 portfolio
3. *n* sets of estimates of:
   - Beta coefficients
   - Stock residual variances
   - Alpha values

- Optimal risky portfolio in the single-index model
  - Objective is to select portfolio weights to maximize the Sharpe ratio of the portfolio

$$E(R_P) = \alpha_P + E(R_M)\beta_P = \sum_{i=1}^{n+1} w_i\alpha_i + E(R_M)\sum_{i=1}^{n+1} w_i\beta_i$$

$$\sigma_P = [\beta_P^2\sigma_M^2 + \sigma^2(e_P)]^{1/2} = \left[\sigma_M^2\left(\sum_{i=1}^{n+1} w_i\beta_i\right)^2 + \sum_{i=1}^{n+1} w_i^2\sigma^2(e_i)\right]^{1/2}$$

$$S_P = \frac{E(R_P)}{\sigma_P}$$

Portfolio Construction: The Process

- Optimal risky portfolio in the single-index model is a combination of two component portfolios:
  - Active portfolio, denoted by *A*
  - Market-index portfolio (that is, passive portfolio), denoted by *M*

Summary of Optimization Procedure 1

1. Compute the initial position of each security:

$$w_i^0 = \frac{\alpha_i}{\sigma^2(e_i)}$$

2. Scale those initial positions:

$$w_i = \frac{w_i^0}{\sum_{i=1}^{n} w_i^0}$$

3. Compute the alpha of the active portfolio:

$$\alpha_A = \sum_{i=1}^{n} w_i \alpha_i$$

Summary of Optimization Procedure 2

4. Compute the residual variance of A:

$$\sigma^2(e_A) = \sum_{i=1}^{n} w_i^2 \sigma^2(e_i)$$

5. Compute the initial position in A:

$$w_A^0 = \frac{\alpha_A / \sigma^2(e_A)}{E(R_M) / \sigma_M^2}$$

6. Compute the beta of A:

$$\beta_A = \sum_{i=1}^{n} w_i \beta_i$$

Summary of Optimization Procedure 3

7. Adjust the initial position in A:

$$w_A^* = \frac{w_A^0}{1+(1-\beta_A)\times w_A^0}$$

$$Note\,When\,\beta_A = 1$$

$$\rightarrow w_A^* = w_A^0$$

Summary of Optimization Procedure 4

8. Optimal risky portfolio now has weights:

$$w_M^* = 1 - w_A^*$$
$$w_i^* = w_A^* \times w_i$$

9. Calculate the risk premium of P (Optimal risky portfolio):

$$E(R_P) = (w_M^* + w_A^* \beta_A) \times E(R_M) + w_A^* \alpha_A$$

10. Compute the variance of P:

$$\sigma_P^2 = (w_M^* + w_A^* \beta_A)^2 \sigma_M^2 + [w_A^* \sigma(e_A)]^2$$

Optimal Risky Portfolio: Information Ratio

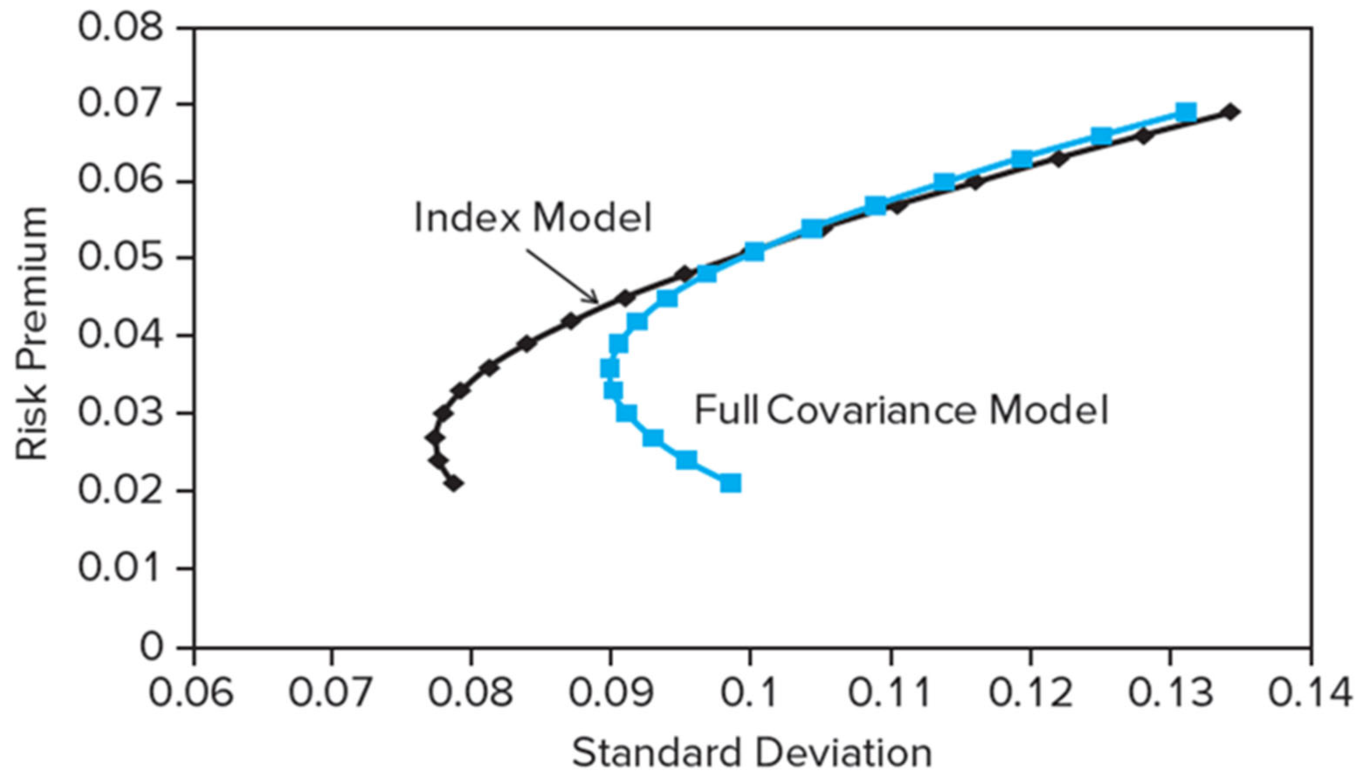
- **Information Ratio**
  - The contribution of the active portfolio depends on the ratio of its alpha to its residual standard deviation (Step 5)
  - Calculated as the ratio of alpha to the standard deviation of diversifiable risk
  - The information ratio measures the extra return we can obtain from security analysis

Optimal Risky Portfolio: Sharpe Ratio

- The Sharpe ratio of an optimally constructed risky portfolio will exceed that of the index portfolio (the passive strategy):

$$S_P^2 = S_M^2 + \left[\frac{\alpha_A}{\sigma(e_A)} \right]^2$$

Efficient Frontier: Index Model and Full-Covariance Matrix



- **Figure 8.4** Efficient frontier constructed from the index model and the full covariance matrix

Portfolios from the Index and Full-Covariance Models

	Index Model	Full-Covariance Model
A. Weights in Optimal Risky Portfolio		
Market index	0.82	0.90
WMT (Walmart)	0.13	0.17
TGT (Target)	−0.07	−0.14
VZ (Verizon)	−0.05	−0.18
T (AT&T)	0.10	0.19
F (Amazon)	0.07	0.08
GM (General Motors)	0.01	−0.03
B. Portfolio Characteristics		
Risk premium	0.0605	0.0639
Standard deviation	0.1172	0.1238
Sharpe ratio	0.5165	0.5163

Is the Index Model Inferior to the Full-Covariance Model?

- Full Markowitz model is better in principle, but
  - The full-covariance matrix invokes estimation risk of thousands of terms
  - Cumulative errors may result in a portfolio that is actually inferior
  - The single-index model is practical and decentralizes macro and security analysis