Efficient Diversification: Markowitz model

Source: Bodie, Kane and Marcus, Investments, 12 ed., McGraw-Hill, 2021

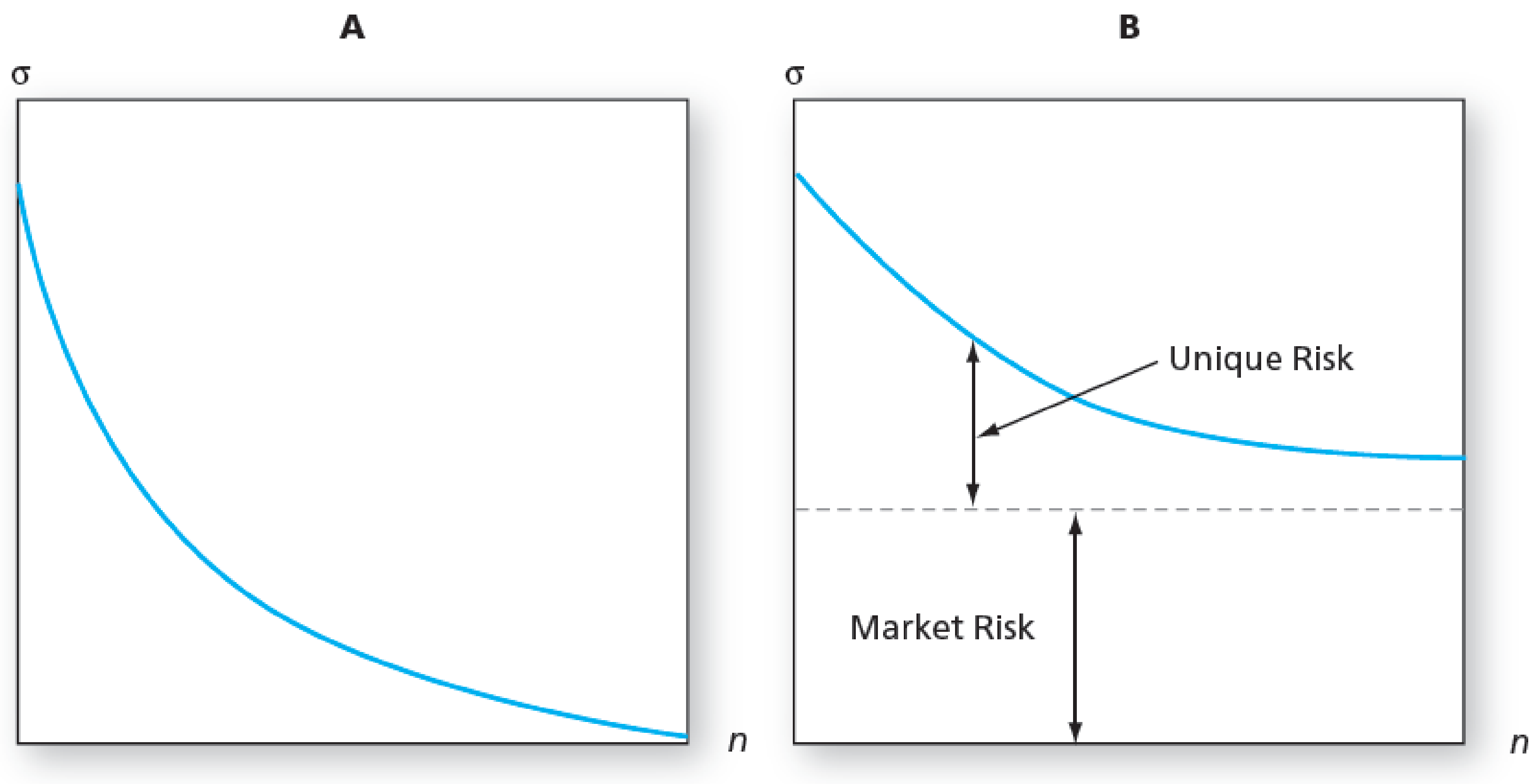
Investment Decision

- The investment decision is a top-down process
  1. Capital allocation (risky versus risk-free)
  2. Asset allocation
  3. Security selection
- Optimal risky portfolio construction
- Efficient diversification
- Long-term vs. short-term investment horizons

Diversification and Portfolio Risk

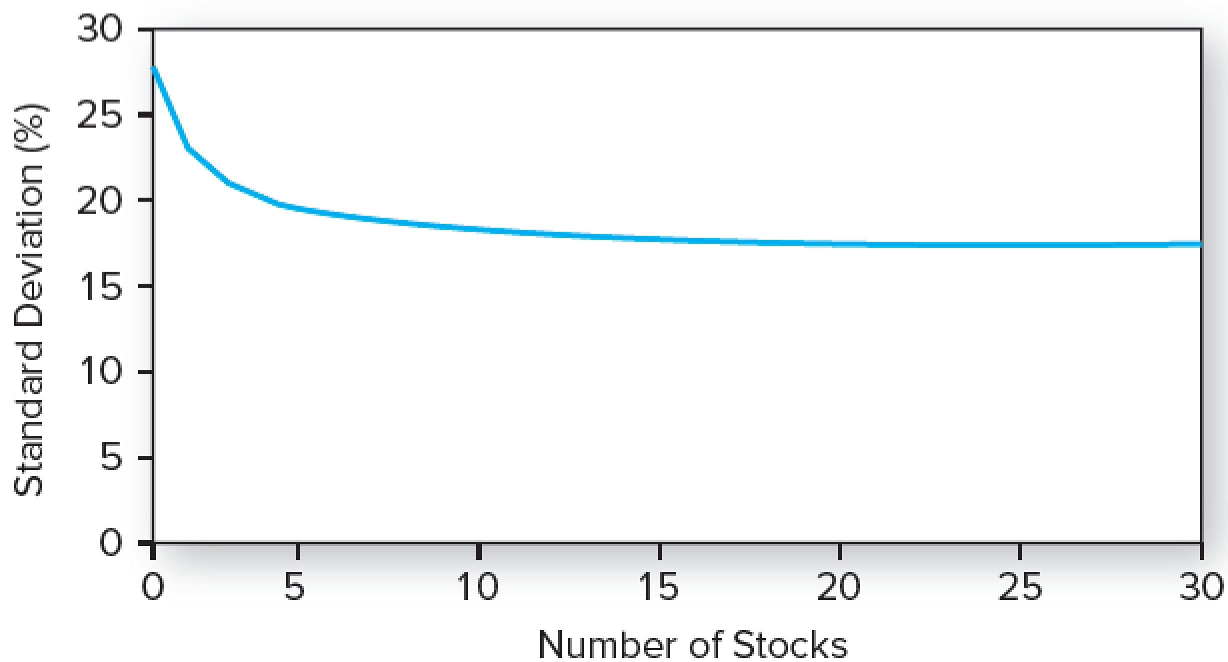
- **Market risk**
  - Attributable to marketwide risk sources
  - Remains even after diversification
  - Also called **systematic** or **nondiversifiable risk**
- **Firm-specific risk**
  - Risk that can be eliminated by diversification
  - Also called **diversifiable** or **nonsystematic risk**

Portfolio Risk as a Function of the Number of Stocks in the Portfolio



- **Panel A:** All risk is firm specific
- **Panel B:** Some risk is systematic

Portfolio Diversification



Portfolios of Two Risky Assets

- Expected return
  - Weighted average of expected returns on the component securities
- Portfolio risk
  - Variance of the portfolio is a weighted sum of covariances, and each weight is the product of the portfolio proportions of the pair of assets

Portfolios of Two Risky Assets: Expected Return

- Consider a portfolio made up of equity (stocks) and debt (bonds)…

$$r_p = w_D r_D + w_E r_E$$

- *where* r_P = *rate of return on portfolio*
- w_D = *proportion invested in the bond fund*
- w_E = *proportion invested in the stock fund*
- r_D = *rate of return on the debt fund*
- r_E = *rate of return on the equity fund*

$$E(r_p) = w_D E(r_D) + w_E E(r_E)$$

Portfolios of Two Risky Assets: Risk

- Variance of r_P

$$\sigma_p^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E \mathrm{Cov}\left(r_D, r_E\right)$$

- Bond variance

$$\sigma_D^2$$

- Equity variance

$$\sigma_E^2$$

- Covariance of returns for bond and equity

$$\mathrm{Cov}\left(r_D, r_E\right)$$

Portfolios of Two Risky Assets: Covariance

- Covariance of returns on bond and equity

$$Cov(r_D, rE) = \rho_{DE}\sigma_D\sigma_E$$

- ρ_{DE} = Correlation coefficient of returns
- σ_D = Standard deviation of bond returns
- σ_E = Standard deviation of equity returns

Portfolios of Two Risky Assets: Correlation Coefficients 1

- Range of values for correlation coefficient

$$-1.0 \leq \rho \leq 1.0$$

  - If $\rho = 1.0$ → perfectly positively correlated securities
  - If $\rho = 0$ → the securities are uncorrelated
  - If $\rho = -1.0$ → perfectly negatively correlated securities

Portfolios of Two Risky Assets: Correlation Coefficients 2

- When $\rho_{DE} = 1$, there is no diversification

$$\sigma_P = w_E \sigma_E + w_D \sigma_D$$

- When $\rho_{DE} = -1$, a perfect hedge is possible

$$w_E = \frac{\sigma_D}{\sigma_D + \sigma_E} = 1 - w_D$$

Portfolios of Two Risky Assets: Example — 50%/50% Split

- **Table 7.1** Descriptive statistics for two mutual funds

	Debt		Equity
Expected return, $E(r)$	8%		13%
Standard deviation, σ	12%		20%
Covariance, Cov(r_D, r_E)		72	
Correlation coefficient, ρ_{DE}		0.30	

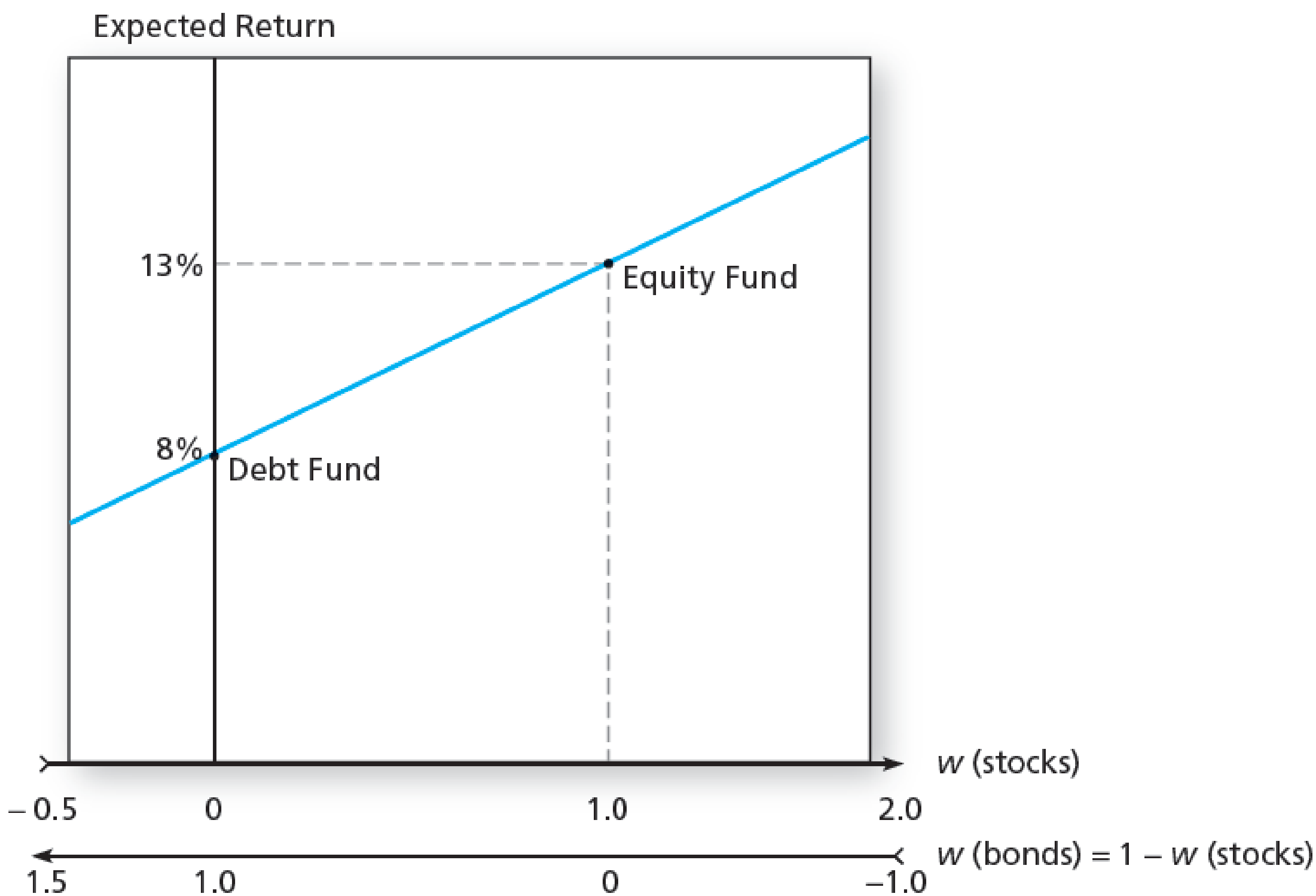
- **Expected Return:**

$$E(r_p) = w_D E(r_D) + w_E E(r_E)$$
$$= .50 \times 8\% + .50 \times 13\% = 10.5\%$$

- **Variance:**

$$\sigma_p^2 = w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 + 2 w_D w_E \mathrm{Cov}\left(r_D, r_E\right)$$
$$= .50^2 \times 12^2 + .50^2 \times 20^2 + 2 \times .5 \times .5 \times 72 = 172$$
$$\sigma_P = \sqrt{172} = 13.23\%$$

Portfolio Expected Return



Computation of Portfolio Variance from the Covariance Matrix

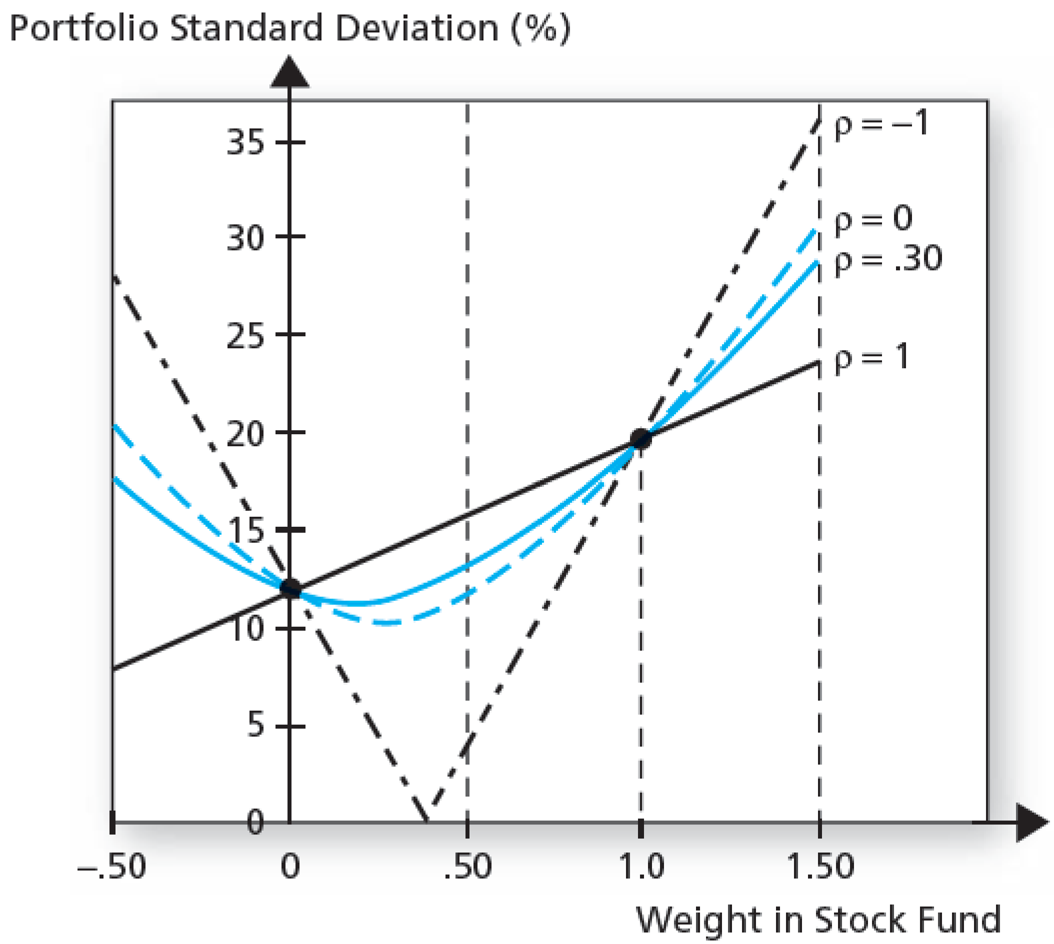
A. Bordered Covariance Matrix

Portfolio Weights	w_D	w_E
w_D	$\text{Cov}(r_D, r_D)$	$\text{Cov}(r_D, r_E)$
w_E	$\text{Cov}(r_E, r_D)$	$\text{Cov}(r_E, r_E)$

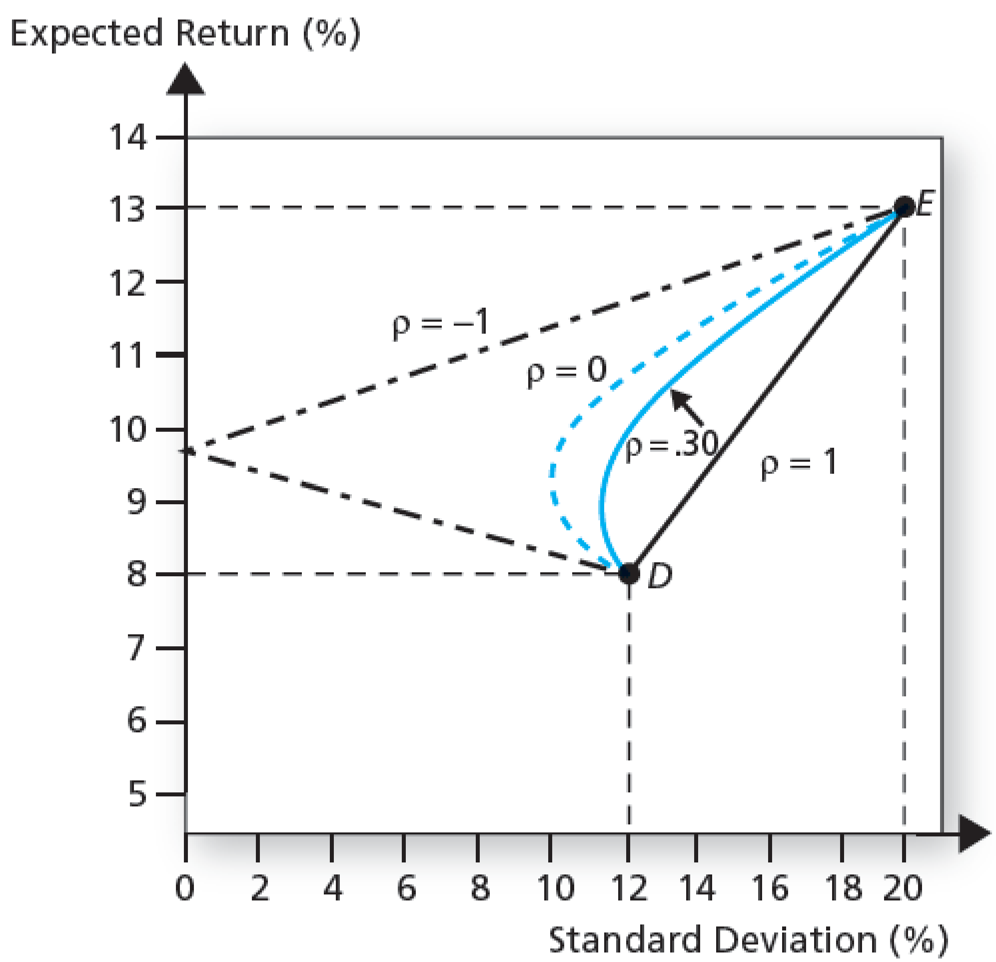
B. Border-Multiplied Covariance Matrix

Portfolio Weights	w_D	w_E
w_D	$w_D w_D \text{Cov}(r_D, r_D)$	$w_D w_E \text{Cov}(r_D, r_E)$
w_E	$w_E w_D \text{Cov}(r_E, r_D)$	$w_E w_E \text{Cov}(r_E, r_E)$
$w_D + w_E = 1$	$w_D w_D \text{Cov}(r_D, r_D) + w_E w_D \text{Cov}(r_E, r_D)$	$w_D w_E \text{Cov}(r_D, r_E) + w_E w_E \text{Cov}(r_E, r_E)$
Portfolio variance	$w_D w_D \text{Cov}(r_D, r_D) + w_E w_D \text{Cov}(r_E, r_D) + w_D w_E \text{Cov}(r_D, r_E) + w_E w_E \text{Cov}(r_E, r_E)$	

Portfolio Standard Deviation



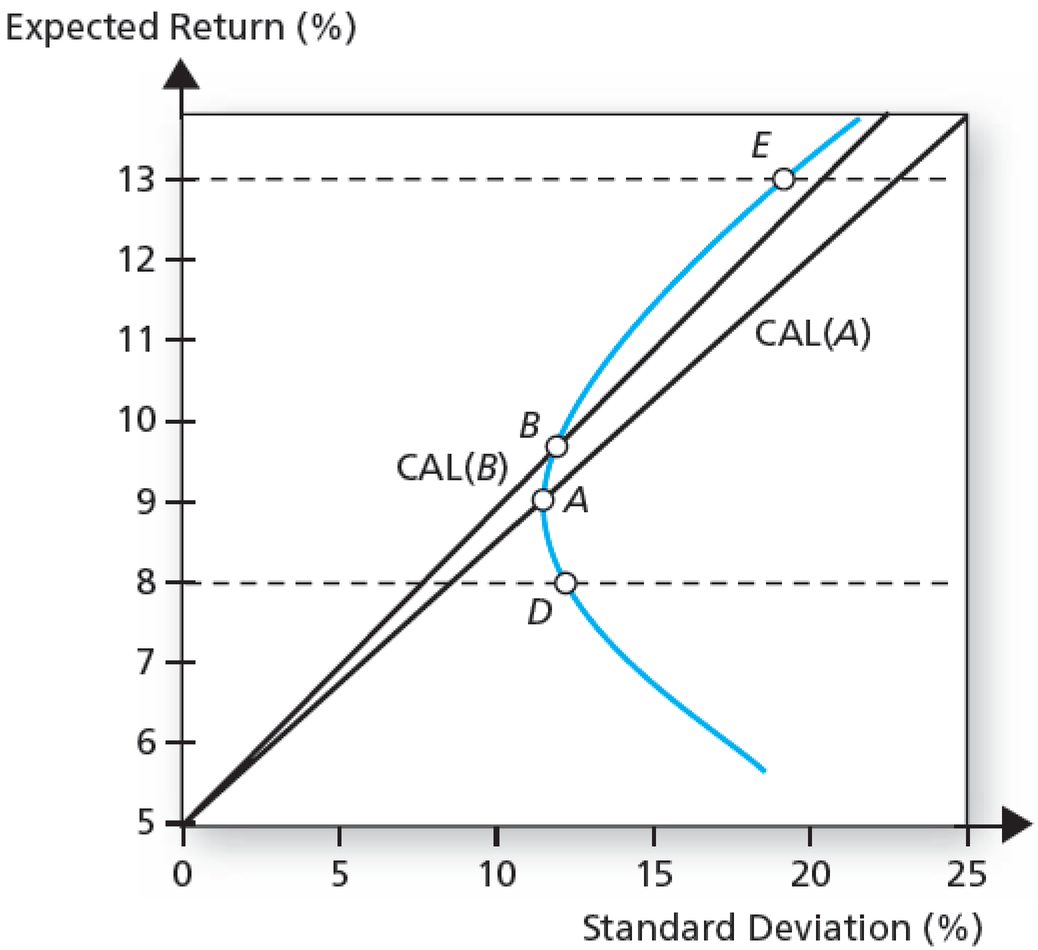
Portfolio Expected Return as a Function of Standard Deviation



The Minimum-Variance Portfolio

- The **minimum-variance portfolio** has a standard deviation *smaller than that of either of the individual component assets*
- Risk reduction depends on the correlation:
  - If $\rho = +1.0$, no risk reduction is possible
  - If $\rho = 0$, σ_P may be less than the standard deviation of either component asset
  - If $\rho = -1.0$, a riskless hedge is possible

The Opportunity Set of the Debt and Equity Funds and Two Feasible CALs



Portfolio A

$E(r_A) = 8.9\%$

$\sigma_A = 11.45\%$

Portfolio B

$E(r_B) = 9.5\%$

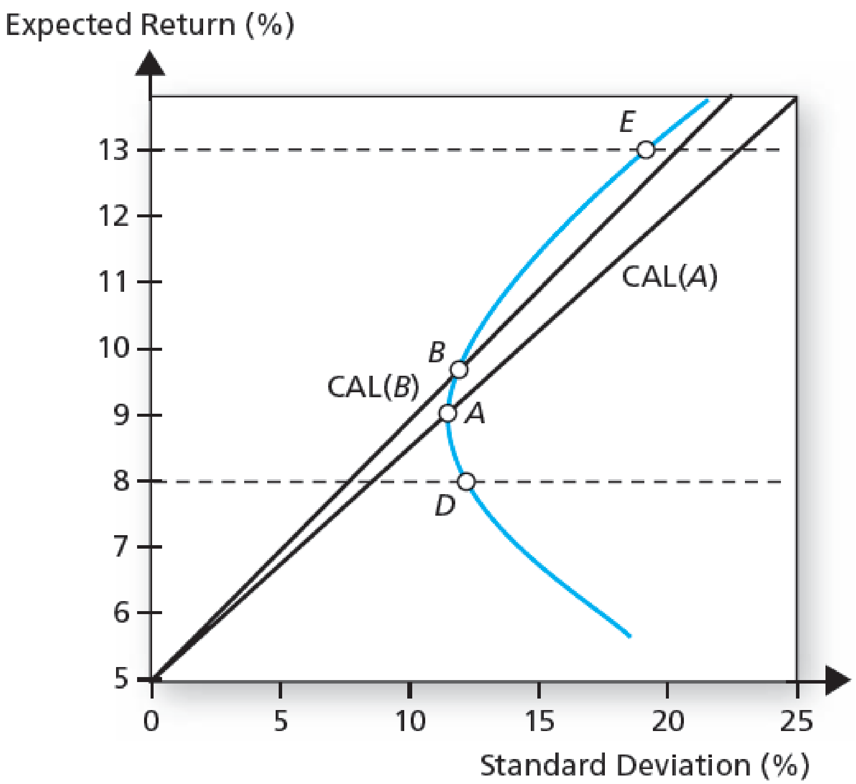
$\sigma_B = 11.70\%$

The Sharpe Ratio

- Objective is to find the weights w_D and w_E that result in the highest slope of the CAL
- Thus, our *objective function* is the Sharpe ratio:

$$S_p = \frac{E\left(r_p\right) - r_f}{\sigma_p}$$

The Sharpe Ratio: Example



Portfolio A

$$E(r_A) = 8.9\%$$

$$\sigma_A = 11.45\%$$

$$S_A = \frac{E(r_A) - r_f}{\sigma_A} = \frac{8.9\% - 5\%}{11.45\%} = .34$$

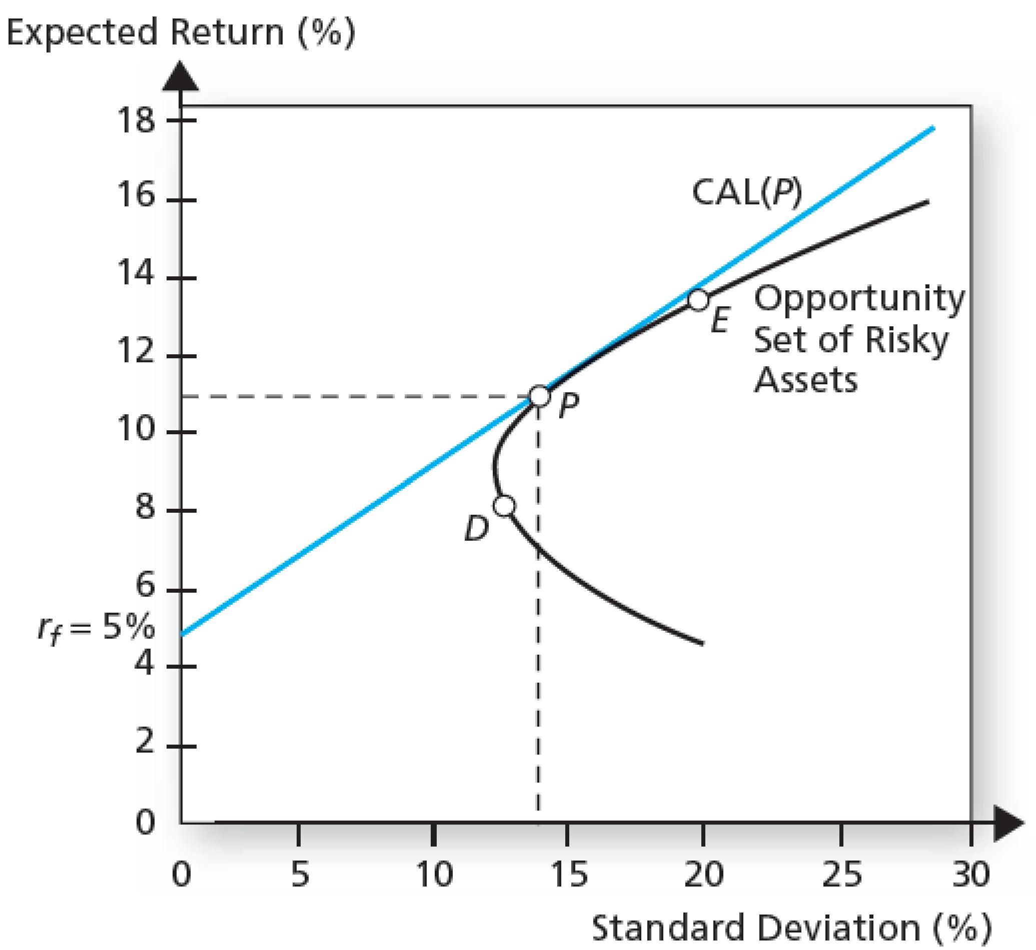
Portfolio B

$$E(r_B) = 9.5\%$$

$$\sigma_B = 11.70\%$$

$$S_B = \frac{E(r_B) - r_f}{\sigma_B} = \frac{9.5\% - 5\%}{11.70\%} = .38$$

Debt and Equity Funds with the Optimal Risky Portfolio



Optimal Risky Portfolio

$$E(r_P) = 11\%$$

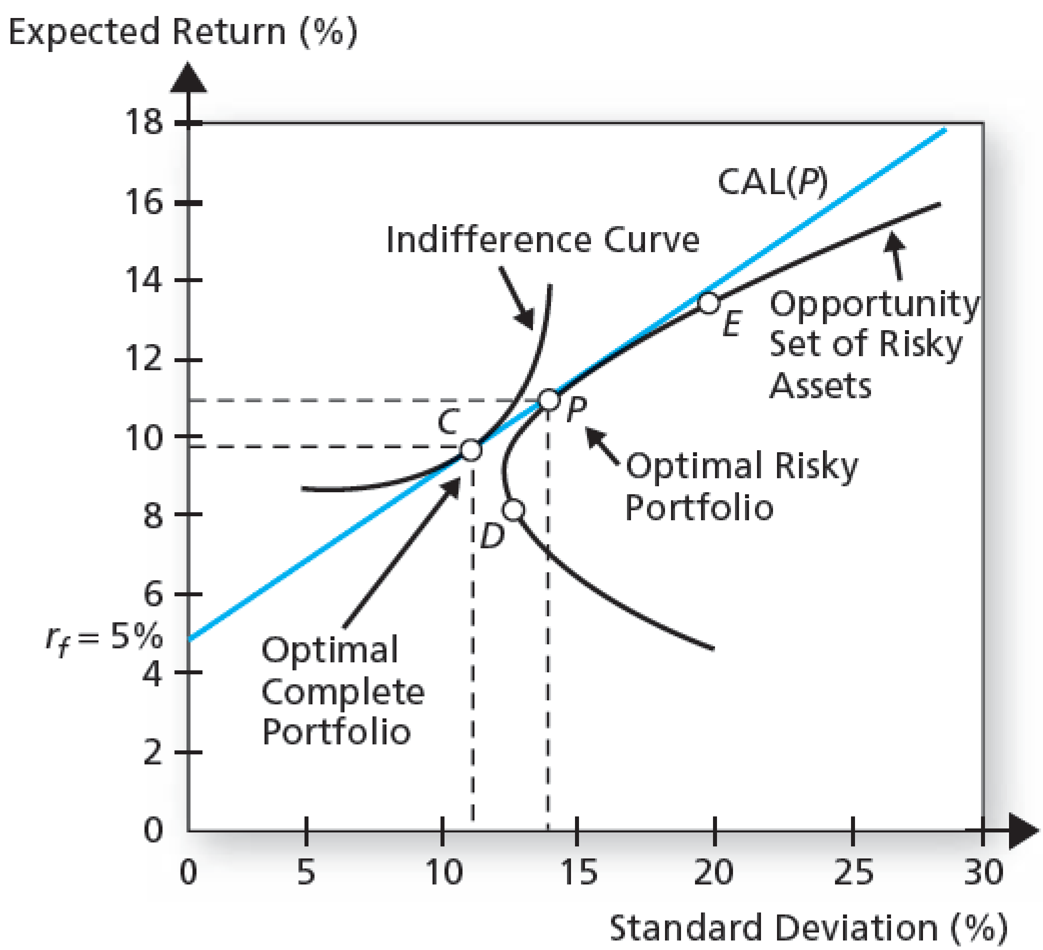
$$\sigma_P = 14.2\%$$

$$S_P = \frac{E(r_P) - r_f}{\sigma_P}$$

$$= \frac{11\% - 5\%}{14.2\%}$$

$$= .42$$

Determination of the Optimal Complete Portfolio



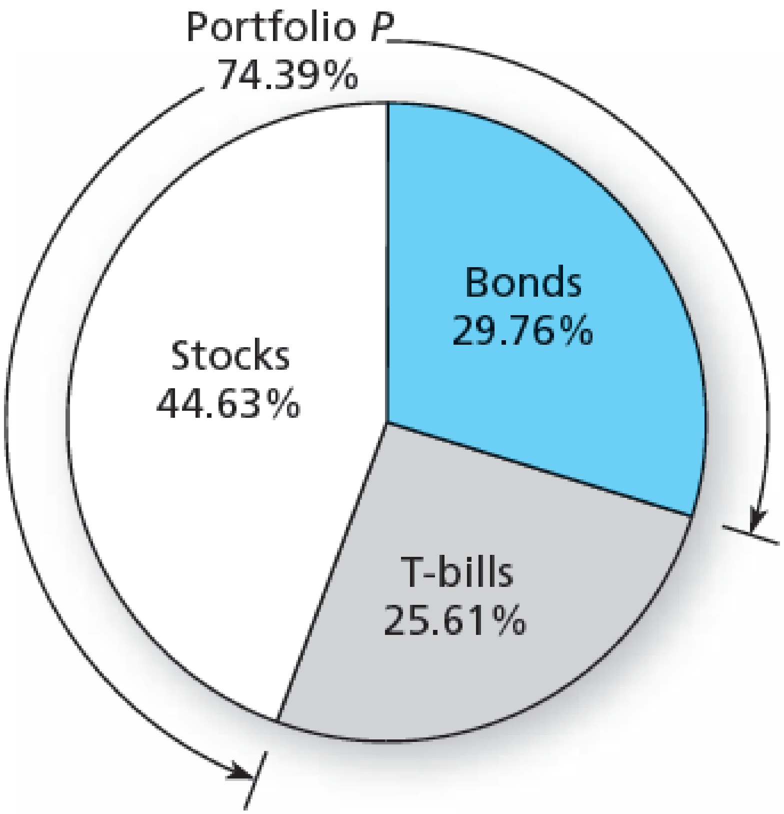
Optimal Allocation to P

$$A = 4$$

$$y = \frac{E(r_P) - r_f}{A\sigma_P^2}$$

$$= \frac{11\% - 5\%}{4 \times (14.2\%)^2} = .7439$$

The Proportions of the Optimal Complete Portfolio



Overall Portfolio

$E(r_P) = 11\% \quad y = .7439$

$\sigma_P = 14.2\% \quad r_f = 5\%$

$$E(r_{Overall}) = y \times E(r_p) + (1-y) \times r_f$$
$$= .7439 \times 11\% + .2561 \times 5\%$$
$$= 9.46\%$$

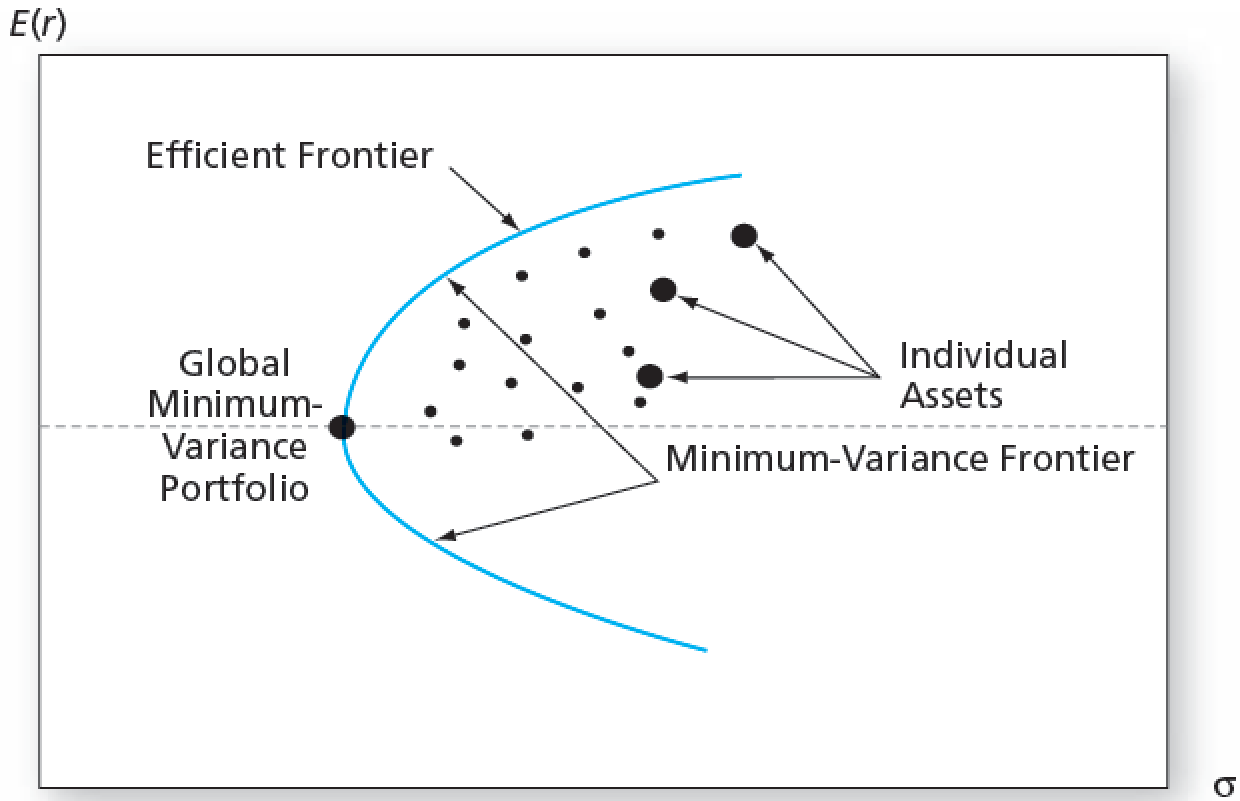
$$\sigma_{Overall} = .7439 \times 14.2\% = 10.56\%$$

$$S_{Overall} = \frac{9.46\% - 5\%}{10.56\%} = .42$$

Markowitz Portfolio Optimization Model 1

- Security selection
  - Determine the risk-return opportunities available
  - **Minimum-variance frontier** of risky assets
  - All portfolios that lie on the minimum-variance frontier from the global minimum-variance portfolio and upward provide the best risk-return combinations
  - Efficient frontier of risky assets is the portion of the frontier that lies above the global minimum-variance portfolio

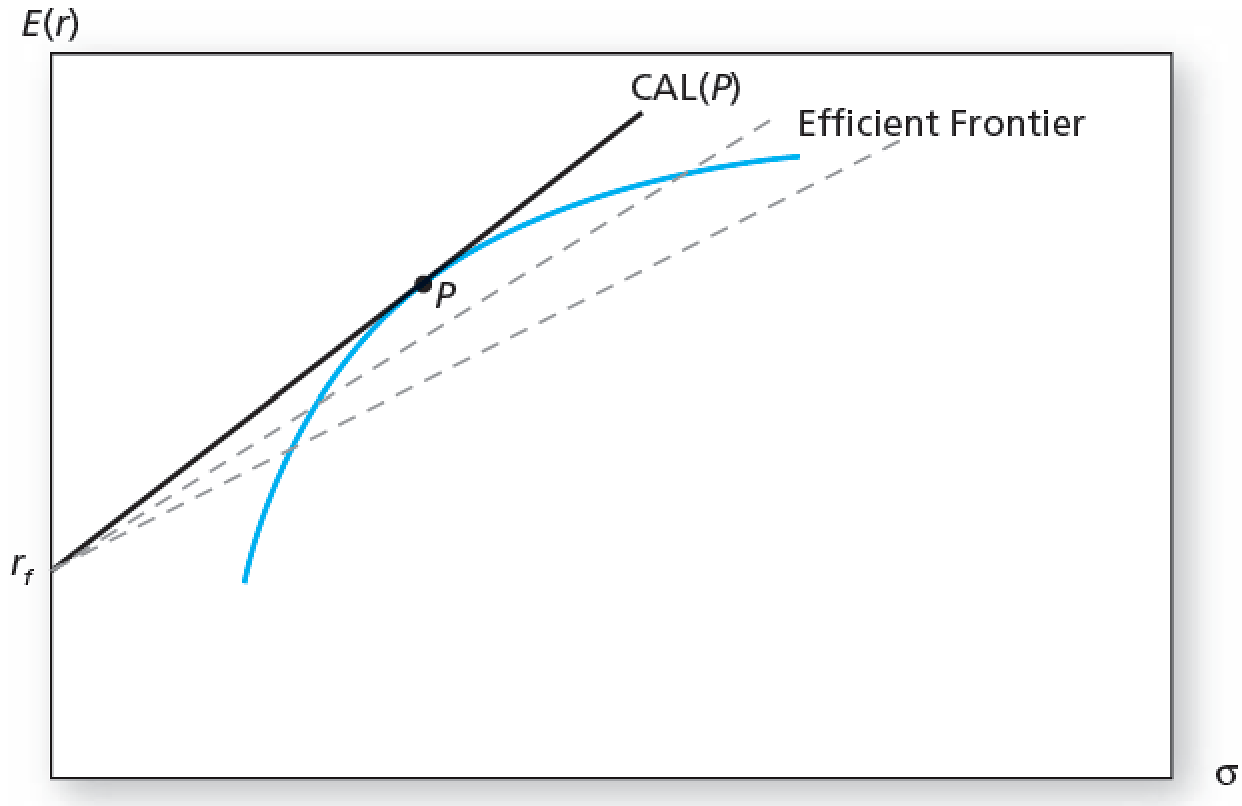
The Minimum-Variance Frontier of Risky Assets



Markowitz Portfolio Optimization Model 2

- Security selection (continued)
  - Search for the CAL with the highest Shape ratio (that is, the steepest slope)
  - Individual investor chooses the appropriate mix between the optimal risky portfolio *P* and T-bills
  - Everyone invests in *P*, regardless of their degree of risk aversion
    - – More risk averse investors put less in *P*
    - – Less risk averse investors put more in *P*

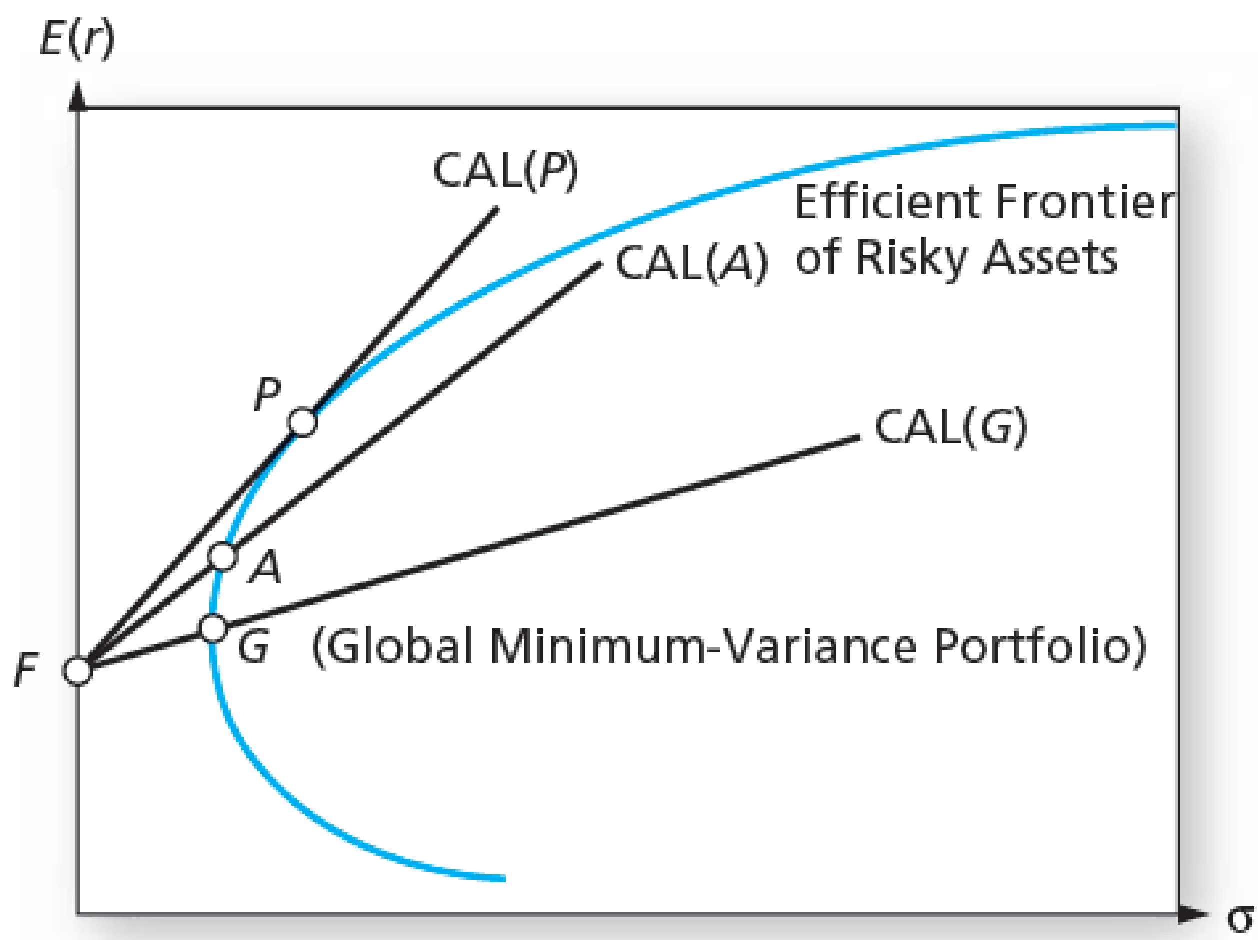
The Efficient Frontier of Risky Assets with the Optimal CAL



Markowitz Portfolio Optimization Model 3

- Capital allocation and the **separation property**
  - Portfolio choice problem may be separated into two independent tasks
    - Determination of the optimal risky portfolio is purely technical
    - Allocation of the complete portfolio to risk-free versus the risky portfolio depends on personal preference

Capital Allocation Lines with Various Portfolios from the Efficient Set



Markowitz Portfolio Optimization Model 4

- The power of diversification
  - Recall: $$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \mathrm{Cov}\left(r_i, r_j\right)$$
  - Assume we define the average variance and average covariance of the securities as:

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}\sigma_i^2$$

$$\mathrm{Cov} = \frac{1}{n\left(n-1\right)}\sum_{\substack{j=1 \\ j\neq i}}^{n}\sum_{i=1}^{n}\mathrm{Cov}\left(r_i, r_j\right)$$

Markowitz Portfolio Optimization Model 5

- The power of diversification (continued)
  - We can then express portfolio variance as

$$\sigma_p^2 = \frac{1}{n}\bar{\sigma}^2 + \frac{n-1}{n}\mathrm{Cov}$$

  - Portfolio variance can be driven to zero if the average covariance is zero
  - The risk of a highly diversified portfolio depends on the covariance of the returns of the component securities

Risk Reduction of Equally Weighted Portfolios

		$\rho = 0$		$\rho = 0.40$	
Universe Size *n*	**Portfolio Weights *w* = 1/*n* (%)**	**Standard Deviation (%)**	**Reduction in σ**	**Standard Deviation (%)**	**Reduction in σ**
1	100	50.00	14.64	50.00	8.17
2	50	35.36		41.83	
5	20	22.36	1.95	36.06	0.70
6	16.67	20.41		35.36	
10	10	15.81	0.73	33.91	0.20
11	9.09	15.08		33.71	
20	5	11.18	0.27	32.79	0.06
21	4.76	10.91		32.73	
100	1	5.00	0.02	31.86	0.00
101	0.99	4.98		31.86	

Markowitz Portfolio Optimization Model 6

- Optimal portfolios and non-normal returns
  - Fat-tailed distributions can result in extreme values of VaR and ES
  - Practice way to estimate values of VaR and ES in the presence of fat tails is called *bootstrapping*
  - If other portfolios provide sufficiently better VaR and ES values than the mean-variance efficient portfolio, we may prefer these when faced with fat-tailed distributions

Risk Pooling, Risk Sharing, and Time Diversification

- Risk pooling vs. risk sharing
  - Variance of *average* insurance policy payoff decreases with the number of policies

$$\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2} \times n\sigma^2 = \frac{\sigma^2}{n}$$

  - Variance of the *total* payoff becomes more uncertain

$$\mathrm{Var}\left(\sum_{i=1}^{n} x_i\right) = n\sigma^2$$

Time Diversification

- True diversification
  - Requires holding fixed the total funds put at risk, and spreading the exposure across multiple sources of uncertainty