Analysis of Algorithms, I

CSOR W4231

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Asymptotic notation, mergesort, recurrences

Outline

1. Asymptotic notation
2. The divide & conquer principle; application: mergesort
3. Solving recurrences and running time of `mergesort`

Review of the last lecture

- Introduced the problem of **sorting**.
- Analyzed `insertion-sort`.
  - Worst-case running time: $T(n) = \frac{3n^2}{2} + \frac{7n}{2} - 4$
  - Space: in-place algorithm
- Worst-case running time analysis: a reasonable measure of algorithmic efficiency.
- Defined polynomial-time algorithms as “efficient”.
- Argued that detailed characterizations of running times are not convenient for understanding scalability of algorithms.

Running time in terms of # primitive steps

We need a coarser classification of running times of algorithms; exact characterizations

- are **too detailed**;
- do not reveal similarities between running times in an immediate way as n grows large;
- are often **meaningless**: high-level language steps will **expand** by a constant factor that depends on the hardware.

Today

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2. The divide & conquer principle; application: mergesort
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Aymptotic analysis

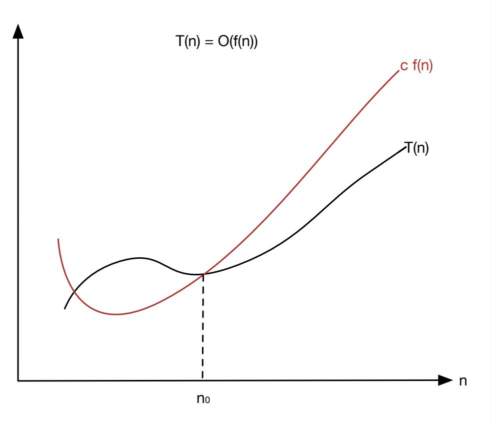
A framework that will allow us to compare the rate of growth of different running times as the input size n grows.

- We will express the running time as a function of the number of primitive steps; the latter is a function of the input size n.
- To compare functions expressing running times, we will ignore their low-order terms and focus solely on the highest-order term.

Asymptotic upper bounds: Big-O notation

Definition 1 (O).

We say that $T(n) = O(f(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ s.t. for all $n \geq n_0$, we have $T(n) \leq c \cdot f(n)$.



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Examples: Show that $T(n) = O(f(n))$ when

- $T(n) = an^2 + b$, $a, b > 0$ constants and $f(n) = n^2$.
- $T(n) = an^2 + b$ and $f(n) = n^3$.

Asymptotic lower bounds: Big-Ω notation

Definition 2 (Ω).

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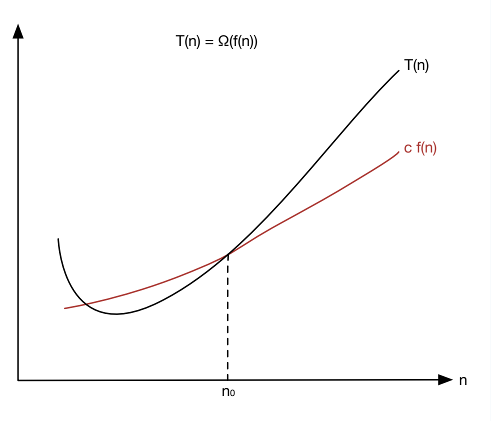

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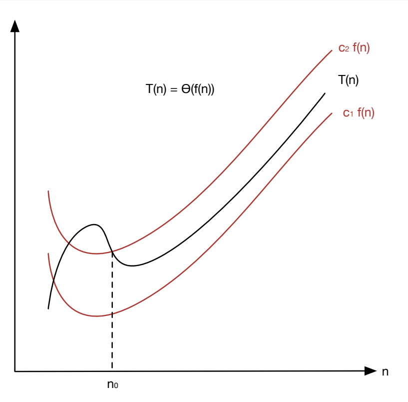
- $T(n) = an^2 + b$, $a, b > 0$ constants and $f(n) = n$.

Asymptotic tight bounds: Θ notation

Definition 3 (Θ).

We say that $T(n) = \Theta(f(n))$ if there exist constants $c_1, c_2 > 0$ and $n_0 \geq 0$ s.t. for all $n \geq n_0$, we have

$$c_1 \cdot f(n) \leq T(n) \leq c_2 \cdot f(n).$$



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Equivalent definition

$T(n) = \Theta(f(n)) \quad \text{if} \quad T(n) = O(f(n)) \; \textit{and} \; T(n) = \Omega(f(n))$

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Notational convention: $\log n$ stands for $\log_2 n$

Examples: Show that $T(n) = \Theta(f(n))$ when

- $T(n) = an^2 + b$, $a, b > 0$ constants and $f(n) = n^2$
- $T(n) = n \log n + n$ and $f(n) = n \log n$

Asymptotic upper bounds that are **not** tight: little-o

Definition 4 (o).

We say that $T(n) = o(f(n))$ if, for any constant $c > 0$, there exists a constant $n_0 \geq 0$ such that for all $n \geq n_0$, we have $T(n) < c \cdot f(n)$.

Asymptotic upper bounds that are **not** tight: little-o

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- Intuitively, $T(n)$ becomes insignificant relative to $f(n)$ as $n \to \infty$.
- Proof by showing that $\lim_{n \to \infty} \frac{T(n)}{f(n)} = 0$ (if the limit exists).

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Asymptotic lower bounds that are **not** tight: little-ω

Definition 5 (ω).

We say that $T(n) = \omega(f(n))$ if, for any constant $c > 0$, there exists a constant $n_0 \geq 0$ such that for all $n \geq n_0$, we have $T(n) > c \cdot f(n)$.

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- $T(n) = \omega(f(n))$ implies that $\lim_{n\to\infty} \frac{T(n)}{f(n)} = \infty$, if the limit exists. Then $f(n) = o(T(n))$.

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Examples: Show that $T(n) = \omega(f(n))$ when

- $T(n) = n^2$ and $f(n) = n \log n$.
- $T(n) = 2^n$ and $f(n) = n^5$.

Basic rules for omitting low order terms from functions

1. Ignore **multiplicative** factors: e.g., $10n^3$ becomes n^3
2. n^a dominates n^b if $a > b$: e.g., n^2 dominates n
3. Exponentials dominate polynomials: e.g., 2^n dominates n^4
4. Polynomials dominate logarithms: e.g., n dominates $\log^3 n$

$\Rightarrow$ For large enough n,

$$\log n < n < n \log n < n^2 < 2^n < 3^n < n^n$$

Properties of asymptotic growth rates

1. **Transitivity**
   1.1 If $f = O(g)$ and $g = O(h)$, then $f = O(h)$.
   1.2 If $f = \Omega(g)$ and $g = \Omega(h)$, then $f = \Omega(h)$.
   1.3 If $f = \Theta(g)$ and $g = \Theta(h)$, then $f = \Theta(h)$.
2. **Sums** of up to a constant number of functions
   2.1 If $f = O(h)$ and $g = O(h)$, then $f + g = O(h)$.
   2.2 Let k be a fixed constant, and let $f_1, f_2, \ldots, f_k, h$ be functions such that for all i, $f_i = O(h)$. Then $f_1 + f_2 + \ldots + f_k = O(h)$.
3. **Transpose symmetry**
   - $f = O(g)$ if and only if $g = \Omega(f)$.
   - $f = o(g)$ if and only if $g = \omega(f)$.

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The divide & conquer principle

- **Divide** the problem into a number of subproblems that are smaller instances of the same problem.
- **Conquer** the subproblems by solving them recursively.
- **Combine** the solutions to the subproblems to get the solution to the overall problem.

Divide & Conquer applied to sorting

- **Divide** the problem into a number of subproblems that are smaller instances of the same problem.
  Divide the input array into two lists of equal size.
- **Conquer** the subproblems by solving them recursively.
  Sort each list recursively. (Stop when lists have size 2.)
- **Combine** the solutions to the subproblems into the solution for the original problem.
  Merge the two sorted lists and output the sorted array.

mergesort: pseudocode

```
mergesort (A, left, right)
    if right == left then return
    end if
    mid = left + ⌊(right − left)/2⌋
    mergesort (A, left, mid)
    mergesort (A, mid + 1, right)
    merge (A, left, right, mid)
```

Remarks

- `mergesort` is a recursive procedure *(why?)*
- Initial call: `mergesort`$(A, 1, n)$
- Subroutine `merge` merges two sorted lists of sizes $\lfloor n/2 \rfloor$, $\lceil n/2 \rceil$ into one sorted list of size n. *How can we accomplish this?*

`merge`: intuition

Intuition: To merge two sorted lists of size $n/2$ repeatedly

- compare the two items in the front of the two lists;
- extract the smaller item and append it to the output;
- update the front of the list from which the item was extracted.

Example: $n = 8, L = \{1, 3, 5, 7\}, R = \{2, 6, 8, 10\}$

`merge`: pseudocode

`merge` $(A, left, right, mid)$

$L = A[left, mid]$
$R = A[mid + 1, right]$
Maintain two pointers p_L, p_R, initialized to point to the first elements of L, R, respectively
while both lists are nonempty **do**
  Let x, y be the elements pointed to by p_L, p_R
  Compare x, y and append the smaller to the output
  Advance the pointer in the list with the smaller of x, y
end while
Append the remainder of the non-empty list to the output.

Remark: the output is stored directly in $A[left, right]$, thus the subarray $A[left, right]$ is sorted after `merge`$(A, left, right, mid)$.

`merge`: optional exercises

Optional exercise 1: write detailed pseudocode or actual code for `merge`

Optional exercise 2: write a recursive `merge`

Analysis of `merge`

1. **Correctness**

2. **Running time**

3. **Space**

Analysis of `merge`: correctness

1. **Correctness:** by induction on the size of the two lists (recommended exercise)
2. **Running time**
3. **Space**

`merge`: pseudocode

`merge` $(A, left, right, mid)$

$L = A[left, mid]$ →**not** a primitive computational step!
$R = A[mid + 1, right]$ →**not** a primitive computational step!
Maintain two pointers p_L, p_R initialized to point to the first elements of L, R, respectively
while both lists are nonempty **do**
  Let x, y be the elements pointed to by p_L, p_R
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end while
Append the remainder of the non-empty list to the output.

Remark: the output is stored directly in $A[left, right]$, thus the subarray $A[left, right]$ is sorted after `merge`$(A, left, right, mid)$.

Analysis of `merge`: running time

1. **Correctness:** by induction on the size of the two lists (recommended exercise)
2. **Running time:**
   - Suppose L, R have $n/2$ elements each
   - *How many iterations before all elements from both lists have been appended to the output?*
   - *How much work within each iteration?*
3. **Space**

Analysis of `merge`: space

1. **Correctness:** by induction on the size of the two lists (recommended exercise)
2. **Running time:**
   - L, R have $n/2$ elements each
   - *How many iterations before all elements from both lists have been appended to the output?* At most $n-1$.
   - *How much work within each iteration?* Constant.

   $\Rightarrow$ `merge` takes $O(n)$ time to merge L, R (*why?*).
3. **Space:** extra $\Theta(n)$ space to store L, R (the output of `merge` is stored directly in A).

Refreshing your memory on recursive algorithms

Exercise (recommended): run `mergesort` on input $1, 7, 4, 3, 5, 8, 6, 2$.

Analysis of `mergesort`

1. **Correctness**

2. **Running time**

3. **Space**

`mergesort`: correctness

For simplicity, assume $n = 2^k$ for integer $k \geq 0$.
We will use induction on k.

- **Base case:** For $k = 0$, the input consists of 1 item; `mergesort` returns the item.
- **Induction Hypothesis:** For $k \geq 0$, assume that `mergesort` correctly sorts any list of size 2^k.
- **Induction Step:** We will show that `mergesort` correctly sorts any list A of size 2^{k+1}.

  From the pseudocode of `mergesort`, we have:

  - Line 3: mid takes the value 2^k
  - Line 4: `mergesort`$(A, 1, 2^k)$ correctly sorts the leftmost half of the input, by the induction hypothesis.
  - Line 5: `mergesort`$(A, 2^k + 1, 2^{k+1})$ correctly sorts the rightmost half of the input, by the induction hypothesis.
  - Line 6: `merge` correctly merges its two sorted input lists into one sorted output of size $2^k + 2^k$.

  $\Rightarrow$ `mergesort` correctly sorts any input of size 2^{k+1}.

Running time of `mergesort`

The running time of `mergesort` satisfies:

$$T(n) = 2T(n/2) + cn, \text{ for } n \geq 2, \text{ constant } c > 0$$
$$T(1) = c$$

This structure is typical of recurrence relations

- an **inequality** or **equation** bounds $T(n)$ in terms of an expression involving $T(m)$ for $m < n$
- a base case generally says that $T(n)$ is constant for small constant n

Remarks

- We ignore floor and ceiling notations.
- A recurrence does **not** provide an asymptotic bound for $T(n)$: to this end, we must **solve** the recurrence.

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Solving recurrences, method 1: recursion trees

The technique consists of three steps

1. Analyze the first few levels of the tree of recursive calls
2. Identify a pattern
3. Sum the work spent over all levels of recursion

Example: give an asymptotic bound for the recurrence describing the running time of `mergesort`

$$T(n) = 2T(n/2) + cn, \text{ for } n \geq 2, \text{ constant } c > 0$$
$$T(1) = c$$

A general recurrence and its solution

The running times of many recursive algorithms can be expressed by the following recurrence

$$T(n) = aT(n/b) + cn^k, \text{for } a, c > 0,\ b > 1, k \geq 0$$

What is the recursion tree for this recurrence?

- a is the branching factor
- b is the factor by which the size of each subproblem shrinks

$\Rightarrow$ at level i, there are a^i subproblems, each of size n/b^i

$\Rightarrow$ each subproblem at level i requires $c(n/b^i)^k$ work

- the height of the tree is $\log_b n$ levels

$\Rightarrow$ Total work: $\sum_{i=0}^{\log_b n} a^i c(n/b^i)^k = cn^k \displaystyle\sum_{i=0}^{\log_b n} \left(\frac{a}{b^k}\right)^i$

Solving recurrences, method 2: Master theorem

Theorem 6 (Master theorem).

If $T(n) = aT(\lceil n/b \rceil) + O(n^k)$ *for some constants* $a > 0$, $b > 1$, $k \geq 0$, *then*

$$T(n) = \begin{cases} O(n^{\log_b a}) & , \textit{ if } a > b^k \\ O(n^k \log n) & , \textit{ if } a = b^k \\ O(n^k) & , \textit{ if } a < b^k \end{cases}$$

Example: running time of `mergesort`

- $T(n) = 2T(n/2) + cn$:
  $a = 2, b = 2, k = 1, b^k = 2 = a \Rightarrow T(n) = O(n \log n)$

Solving recurrences, method 3: the substitution method

The technique consists of two steps

1. Guess a bound
2. Use (strong) induction to prove that the guess is correct

(See your textbook for more details on this technique.)

Remark 1 (simple vs strong induction).

1. **Simple induction:** *the induction step at n requires that the inductive hypothesis holds at step $n-1$.*
2. **Strong induction** *is just a variant of simple induction where the induction step at n requires that the inductive hypothesis holds at all previous steps* $1, 2, \ldots, n-1$.

How would you solve...

1. $T(n) = 2T(n-1) + 1, T(1) = 2$
2. $T(n) = 2T^2(n-1), T(1) = 4$
3. $T(n) = T(2n/3) + T(n/3) + cn$