Analysis of Algorithms, I
CSOR W4231

Eleni Drinea
Computer Science Department

Columbia University

The greedy principle: cache maintenance

Outline

1. Cache maintenance (the offline problem)

2. An optimal greedy algorithm for the offline problem: Farthest-into-Future (FF)

3. Proof of optimality of FF

4. The online problem

Today

1. Cache maintenance (the offline problem)
2. An optimal greedy algorithm for the offline problem: Farthest-into-Future (FF)
3. Proof of optimality of FF
4. The online problem

Caching

- Caching: the process of storing a small amount of data in a fast memory, so as to reduce the amount of time spent interacting with slow memory.

- **Goal:** when attempting to download a page, it should be in cache.

$\Rightarrow$ Need a cache maintenance algorithm to determine what to keep and what to **evict** from the cache.

The input

Input

- n, the number of pages in the main memory
- k, the size of the cache memory
- a sequence of m requests $r_1, r_2, \ldots, r_m$ for memory pages

Example:

- #main memory pages $n = 3$
- set of main memory pages $= \{a, b, c\}$
- cache size $k = 2$
- #requests $m = 7$
- sequence of requests: a, b, c, b, c, a, b

The model

- To service a request, the corresponding page must be in the cache.
  - $\Rightarrow$ After the first k requests for distinct pages the cache is full.
- A request is received and serviced within the same time step.
- Cache miss: a request for a page that is not in the cache.
  - $\Rightarrow$ If there is a cache miss, we must evict a page from the cache to bring in the requested page.

Remark 1.

The expensive operation is the eviction; every cache miss incurs an eviction.

Our objective

At each time step $1 \leq t \leq m$, we must decide which page (if any) to evict from the cache.

Definition 1 (Scheduling algorithm).

A **schedule** is a sequence of eviction decisions so that all m requests are serviced at time m. An algorithm that provides such a schedule is a **scheduling** algorithm.

Goal: find the schedule that minimizes the total number of cache misses.

Example 2.

- # pages in main memory: $n = 3$
- cache size: $k = 2$
- sequence of $m = 7$ requests: a, b, c, b, c, a, b

time t:	1	2	3	4	5	6	7
requests:	a,	b,	c,	b,	c,	a,	b

Example 2.

- # pages in main memory: $n = 3$
- cache size: $k = 2$
- sequence of $m = 7$ requests: a, b, c, b, c, a, b

time t:	1	2	3	4	5	6	7
requests:	$a,$	$b,$	$c,$	$b,$	$c,$	$a,$	b
eviction schedule S:	$-,$	$-,$	$a,$	$-,$	$-,$	$c,$	$-$
cache contents:	$\{a\}$	$\{a, b\}$	$\{b, c\}$	$\{b, c\}$	$\{b, c\}$	$\{b, a\}$	$\{b, a\}$

- $-$ stands for "no eviction"
- $S = \{-, -, a, -, -, c, -\}$ evicts a at time 3, c at time 6
- S incurs 2 cache misses *(can't do better)*

Offline and online problems

- Offline problem: the entire sequence of requests $\{r_1, r_2, \ldots, r_m\}$ is part of the **input** (known at time $t = 0$).
- Online problem (more natural): requests arrive one at a time; r_t must be serviced at time t, **before** future requests $r_{t+1}, \ldots, r_m$ are seen.
- A scheduling algorithm for the online problem can only base its eviction decision at time t on
  1. the requests it has seen so far,
  2. the eviction decisions it has made so far.
- The optimal offline algorithm provides a lower bound on the performance of **any** online algorithm.

Today

1. Cache maintenance (the offline problem)
2. An optimal greedy algorithm for the offline problem: Farthest-into-Future (FF)
3. Proof of optimality of FF
4. The online problem

The Farthest-into-Future (FF) rule

Definition 3 (Farthest-into-Future (FF) rule).

When the page requested at time i is not in the cache, evict from the cache the page that is needed the farthest into the future and bring in the requested page.

Notation: we will denote the schedule produced by this algorithm S_{FF}.

Example: the schedule S in Example 2 is the schedule produced by FF.

Reduced schedules

Definition 4 (Reduced schedule).

A reduced schedule brings a page in the cache at time t only if

1. the page is requested at time t; *and*
2. the page is not already in the cache.

Remark 2.

1. *In a sense, a reduced schedule performs the least amount of work at every time step.*
2. *FF is a reduced schedule.*

There is an optimal *reduced* schedule

Fact 5.

We can transform a non-reduced schedule into a reduced one that is at least as good, that is, incurs at most the same number of evictions.

Remark 3.

- *The expensive memory operation is the eviction: so we should be minimizing #evictions and not #cache misses.*
- *By Definition 4, #cache misses = #evictions in reduced schedules.*
- *By Fact 5, we can focus solely on reduced schedules to find the optimal schedule.*

$\Rightarrow$ *Thus our original goal of minimizing #cache misses (rather than #evictions) is justified.*

Proof of Fact 5

- Let S' be a schedule that is not reduced and solves an instance of cache maintenance.
- We will construct a reduced schedule S that incurs at most as many evictions as S'.
  - Suppose that at time i, there is a request $r_i \neq a$ but S' evicts a page from the cache to bring in page a, although a is not requested at time i.
  - Then S *pretends* to bring in a but in fact does nothing: at the first time step $j > i$ such that $r_j = a$, S brings in a.

  $\Rightarrow$ We can *charge* the cache miss of S at time j to the eviction of S' at the earlier time i.
- Thus S performs at most as many evictions as S'.

Example 6.

- # pages in main memory: $n = 4$
- cache size: $k = 3$
- sequence of $m = 9$ requests: $a, b, c, d, b, c, a, d, b$
- r_t = request at time t
- $C_S(t)$ = contents of the cache of schedule S at end of time t

Non-reduced schedule S

t	1	2	3	4	5	6	7	8	9
r_t	a	b	c	d	b	c	a	d	b
bring	a	b	c	d	c	d	b	-	-
evict	-	-	-	c	d	b	c	-	-
$C_S(t)$	a	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, d\}$	{a,b,c}	$\{a, c, d\}$	{a,b,d}	$\{a, b, d\}$	$\{a, b, d\}$

Example 6.

- \# pages in main memory: $n = 4$
- cache size: $k = 3$
- sequence of $m = 9$ requests: $a, b, c, d, b, c, a, d, b$
- r_t = request at time t
- $C_S(t)$ = contents of the cache of schedule S at end of time t

Non-reduced schedule S

t	1	2	3	4	5	6	7	8	9
r_t	a	b	c	d	b	c	a	d	b
bring	a	b	c	d	c	d	b	-	-
evict	-	-	-	c	d	b	c	-	-
$C_S(t)$	a	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, d\}$	{a,b,c}	$\{a, c, d\}$	{a,b,d}	$\{a, b, d\}$	$\{a, b, d\}$

Reduced schedule S'

t	1	2	3	4	5	6	7	8	9
r_t	a	b	c	d	b	c	a	d	b
bring	a	b	c	d	-	c	-	d	b
evict	-	-	-	c	-	d	-	b	c
$C_{S'}(t)$	a	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, d\}$	{a,b,d}	$\{a, b, c\}$	{a,b,c}	$\{a, c, d\}$	$\{a, b, d\}$

Blue entries denote cache misses.
Red entries denote evictions when no cache miss occurred.
S evicts pages d, b, c at times $5, 6, 7$, even though these pages are not requested at these times (no cache misses). S' performs the exact same evictions that S performed at times $5, 6, 7$ at the **later** times $6, 8, 9$ respectively, when these pages were actually requested (hence cache misses were incurred).

Today

1. Cache maintenance (the offline problem)
2. An optimal greedy algorithm for the offline problem: Farthest-into-Future (FF)
3. Proof of optimality of FF
4. The online problem

Optimality of Farthest-into-Future

Claim 1.

Let S_i be a reduced schedule that makes the same eviction decisions as S_{FF} up to time i, that is, up to request i. Then there is a reduced schedule S_{i+1} that

1. *makes the same eviction decisions as S_{FF} up to time $t = i + 1$, that is, up to request $i + 1$;*
2. *S_{i+1} incurs no more total cache misses than S_i.*

Proposition 1.

The schedule S_{FF} provided by the Farthest-into-Future algorithm is optimal.

Proof of Proposition 1: case $i = 0$

Notation

- $cm(S) =$ total #cache misses of schedule S
- S^* is an optimal reduced schedule
- *Schedule S follows schedule S' up to request i* if S makes the same eviction decisions as S' up to the i-th request

$i = 0$: trivially, S^* follows S_{FF} up to request $i = 0$. By Claim 1, we can construct a reduced schedule S_1 such that

1. S_1 follows S_{FF} up to request $i = 1$,
2. $cm(S_1) \leq cm(S^*)$.

Proof of Proposition 1: case $i > 0$

Notation: $cm(S) =$ total #cache misses of schedule S

- $i = 1$: now S_1 is a reduced schedule that follows S_{FF} up to request $i = 1$. By Claim 1, we can construct a reduced schedule S_2 such that
  1. S_2 follows S_{FF} up to request $i = 2$,
  2. $cm(S_2) \leq cm(S_1)$.
- $i = 2$: now S_2 is a reduced schedule that follows S_{FF} up to request $i = 2$. By Claim 1, we can construct a reduced schedule S_3 such that
  1. S_3 follows S_{FF} up to request $i = 3$,
  2. $cm(S_3) \leq cm(S_2)$.

Proof of Proposition 1: $S_m = S_{FF}$

Notation: $cm(S)$ = total #cache misses of schedule S

- Applying the claim for every $3 \leq i \leq m-1$, we obtain a reduced schedule S_m that
  1. follows S_{FF} up to time m,
  2. $cm(S_m) \leq cm(S_{m-1})$.

  Tracing back all the inequalities, we obtain $cm(S_m) \leq cm(S^*)$.

- Finally, since S_m follows S_{FF} up to time m, $S_{FF} = S_m$.

  Hence $cm(S_{FF}) = cm(S_m) \leq cm(S^*)$.

  Thus S_{FF} is optimal.

Optimality of Farthest-into-Future

Before we prove Claim 1, we slightly simplify notation to obtain the following claim.

Claim 2.

Let S be a reduced schedule that makes the same eviction decisions as S_{FF} up to time $t = i$, that is, up to request i. Then there is a reduced schedule S' such that

1. *S' makes the same eviction decisions as S_{FF} up to time $t = i + 1$, that is, up to request $i + 1$;*
2. *S' incurs no more total cache misses than S.*

Proof of Claim 2: a case-by-case analysis

Notation:

- $cm(S)$ = total #cache misses of schedule S
- $C_i(S)$ = contents of the cache of schedule S at the end of time step i

Since S and S_{FF} have made the same scheduling decisions up to time i, the following statements hold at the end of time step i.

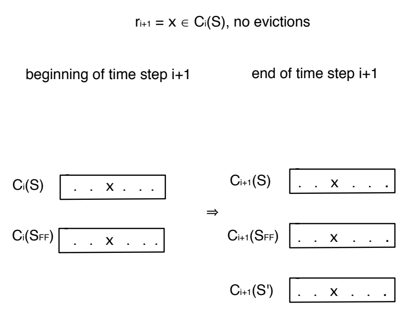
1. The contents of their caches are identical, that is,

$$C_i(S) = C_i(S_{FF}).$$

2. So far, S has the same number of cache misses as S_{FF}.

Case 1: request $r_{i+1} = x \in C_i(S)$

1. If page x requested at time $i + 1$ is in $C_i(S)$, then
   - $x \in C_i(S_{FF})$ (recall that $C_i(S) = C_i(S_{FF})$);
   - no cache miss for either schedule.

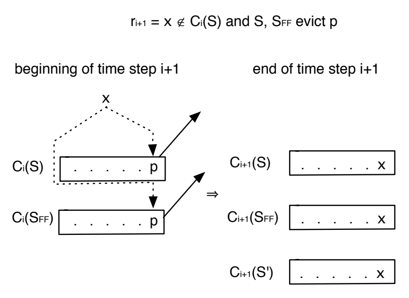


   - Set $S' = S$; then
     1. S' follows S_{FF} up to time $i + 1$ (S does!);
     2. $cm(S') \leq cm(S)$.

Case 2: $r_{i+1} = x \notin C_i(S)$ (cont'd)

2. If page x requested at time $i+1$ is not in $C_i(S)$, then
   - $x \notin C_i(S_{FF})$ (recall that $C_i(S) = C_i(S_{FF})$);
   - both schedules must bring x in, hence incur a cache miss.

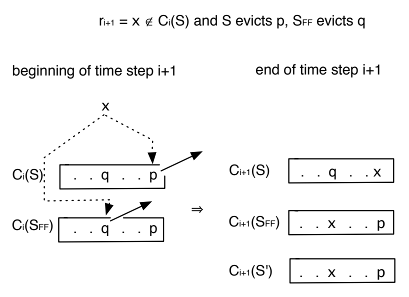
   2.1: If S and S_{FF} both evict the same page p, set $S' = S$:
   1. S' follows S_{FF} up to time $i+1$ (S does!),
   2. $cm(S') \leq cm(S)$.



Case 2.2: $r_{i+1} = x \notin C_i(S)$

2.2: If S evicts p but S_{FF} evicts q:

- By construction of S_{FF}, q must be requested later in the future than p (recall the *Farthest-into-Future* rule).
- At the end of time step $i+1$, the cache contents for the two schedules will differ in exactly one item.



Case 2.2: S evicts p, S_{FF} evicts q

At the end of time step $i+1$

- the cache of S contains q;
- the cache of S_{FF} contains p;
- the remaining $k-1$ items in both caches are the same;
- thus
$$C_{i+1}(S_{FF}) = C_{i+1}(S) - \{q\} + \{p\}.$$
- Since we want S' to *follow* S_{FF} up to time $i+1$, S' evicts q from its cache as well. Hence

$$C_{i+1}(S') = C_{i+1}(S_{FF}) = C_{i+1}(S) - \{q\} + \{p\}.$$

Roadmap for case 2.2: S evicts p, S_{FF} evicts q

Notation: $cm(S)$ = total #cache misses of schedule S

- At the end of time step $i + 1$,
  - the cache contents of S, S' differ in exactly one item;
  - S' follows S_{FF} up to time $i + 1$;
  - #cache misses of S = #cache misses of S'.

- **Goal:** Ensure that S' does not incur more misses than S for $i + 1 < t \leq m$, so that $cm(S') \leq cm(S)$.

- **Idea:** Set $S' = S$ *as soon as* the cache contents of S, S' are the same again.

  1. Make $C_t(S')$ equal $C_t(S)$ at the earliest $t > i + 1$ possible, while not incurring unnecessary misses.
  2. Once $C_t(S') = C_t(S)$, set $S' = S$.

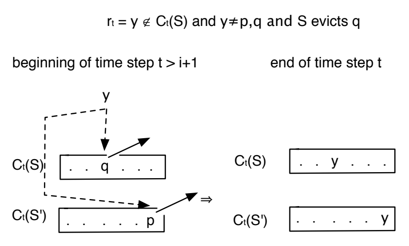
  $\Rightarrow$ If S' has not incurred more misses than S between steps $i + 2$ and t, then $cm(S') \leq cm(S)$ and the claim holds.

Case 2.2.1: $r_t = x \notin \{p, q\}$, $x \notin C_t(S)$, S evicts q

For all $t > i + 1$, S' follows S **until** one of the following happens for the first time:

2.2.1: $r_t = y \notin \{p, q\}$, **and** $y \notin C_t(S)$, **and** S evicts q.

Since $C_t(S)$ and $C_t(S')$ only differ in p, q, then $y \notin C_t(S')$. Set S' to evict p and bring in y. Then $C_t(S') = C_t(S)$!

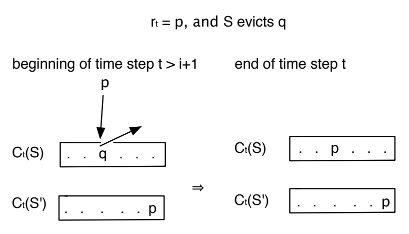


Set $S' = S$ henceforth: S' follows S_{FF} up to time $i + 1$ and $cm(S') \leq cm(S)$.

Case 2.2.2.1: $r_t = p$, S evicts q

2.2.2: $r_t = p$

2.2.2.1: If S evicts q, $C_t(S) = C_t(S')$!

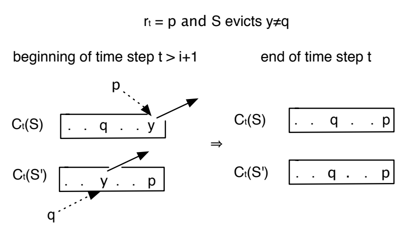


Set $S' = S$ henceforth: S' follows S_{FF} up to time $i+1$ and $cm(S') < cm(S)$.

Case 2.2.2.2: $r_t = p$, S evicts $y \neq q$

2.2.2: $r_t = p$

2.2.2.2: If S evicts $y \neq q$ from its cache, then S' evicts y as well and brings in q. Then $C_t(S') = C_t(S)$.



Set $S' = S$ henceforth: S' follows S_{FF} up to time $i + 1$ and $cm(S') \leq cm(S)$.

2.2.2.2: S' is no longer *reduced*

- S' is no longer reduced: q was brought in when there was no request for q at time t (recall that $r_t = p$).
- Fortunately, we can use Fact 1 to transform S' into a reduced schedule $\overline{S}$ that
  - incurs at most the same total #evictions as S';
  - still follows S_{FF} up to time $i+1$: all the real evictions of the reduced $\overline{S}$ will happen after time $i+1$.
- Hence we return $\overline{S}$ as the schedule that satisfies Claim 2.

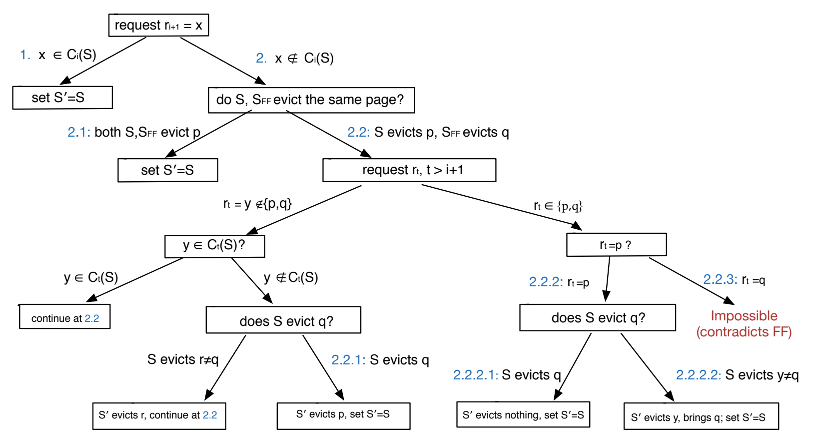
2.2.3: $r_t = q$

Can't happen!

S_{FF} evicted q and not p, hence q appears farther in the future than p.

Hence one of the cases 2.2.1, 2.2.2 will happen first.

Complete roadmap



Today

1. Cache maintenance (the offline problem)
2. An optimal greedy algorithm for the offline problem: Farthest-into-Future (FF)
3. Proof of optimality of FF
4. The online problem

The online problem

- Offline problem: the entire sequence of requests $\{r_1, r_2, \dots, r_m\}$ is part of the **input** (known at time $t = 0$).
- Online problem (more natural): requests arrive one at a time; r_t must be serviced at time t, **before** $r_{t+1}, \dots, r_m$ are seen.
- An online scheduling algorithm can only base its eviction decision at time t on
    1. the requests it has seen so far;
    2. the eviction decisions it has made so far.
- The optimal offline algorithm provides a lower bound on the performance of **any** online algorithm.

The Least Recently Used principle

- The Least Recently Used (LRU) principle: evict the page that was requested the longest ago.
- **Intuition**: a running program will generally keep accessing the things it's just been accessing (locality of reference).
- Essentially Farthest-into-Future (FF) reversed in time.
- LRU behaves well on average inputs.
- However an adversary can devise a specific sequence of online requests that will cause LRU to perform very badly compared to the optimal offline algorithm (*how?*).

Worst-case input to LRU

Example

- #pages in main memory: $n = 3$
- size of the cache: $k = 2$
- sequence of online requests

$$\underbrace{a, b, c}_{1}, \underbrace{a, b, c}_{2}, \ldots, \underbrace{a, b, c}_{M}$$

$\Rightarrow$ LRU: every request starting at time $t = 3$ is a miss, hence $3M - 2$ misses.

$\Rightarrow$ FF: no more than $3M/2$ misses.

Competitive ratio

- More generally, if we have a sequence of nM online requests as above, for some integer M, LRU will incur $nM - k$ misses, while FF will incur no more than $\lceil nM/k \rceil$ misses.
- Hence LRU may perform up to a factor of k times worse than FF.
- (Online analysis) competitive ratio: the worst-case ratio between the performance of the online algorithm and the performance of the optimal offline algorithm.