Analysis of Algorithms, I

CSOR W4231

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Insertion sort, efficient algorithms

Outline

1. Overview
2. A first algorithm: insertion sort
3. Analysis of algorithms
4. Efficiency of algorithms

Today

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Algorithms

- An algorithm is a **well-defined** computational procedure that transforms the **input** (a set of values) into the **output** (a new set of values).
- The desired input/output relationship is specified by the statement of the **computational problem** for which the algorithm is designed.
- An algorithm is correct if, *for every input*, it **halts** with the correct output.

Efficient Algorithms

- In this course we are interested in algorithms that are correct and efficient.
- Efficiency is related to the resources an algorithm uses: **time**, **space**
  - *How much time/space are used?*
  - *How do they scale as the input size grows?*

We will primarily focus on efficiency in **running time**.

Running time

Running time = number of primitive computational steps performed; typically these are

1. arithmetic operations: add, subtract, multiply, divide fixed-size integers
2. data movement operations: load, store, copy
3. control operations: branching, subroutine call and return

We will use pseudocode for our algorithm descriptions.

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Sorting

- **Input:** A list A of n integers $x_1, \ldots, x_n$.
- **Output:** A permutation $x'_1, x'_2, \ldots, x'_n$ of the n integers where they are sorted in non-decreasing order, i.e., $x'_1 \leq x'_2 \leq \ldots \leq x'_n$

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Example

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Example

- Input: $n = 6$, $A = \{9, 3, 2, 6, 8, 5\}$
- Output: $A = \{2, 3, 5, 6, 8, 9\}$

What data structure should we use to represent the list?

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Example

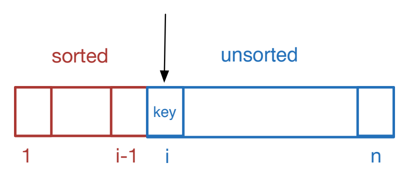
- Input: $n = 6$, $A = \{9, 3, 2, 6, 8, 5\}$
- Output: $A = \{2, 3, 5, 6, 8, 9\}$

What data structure should we use to represent the list?

Array: collection of items of the same data type

- allows for *random access*
- "zero" indexed in C++ and Java

Main idea of insertion sort



1. Start with a (trivially) sorted subarray of size 1 consisting of $A[1]$.

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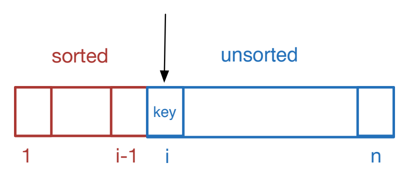

$\color{blue}{1.}$ Start with a (trivially) sorted subarray of size 1 consisting of $A[1]$.

$\color{blue}{2.}$ Increase the size of the sorted subarray by 1, by inserting the next element of A, call it `key`, in the **correct** position in the **sorted** subarray to its left. *How?*

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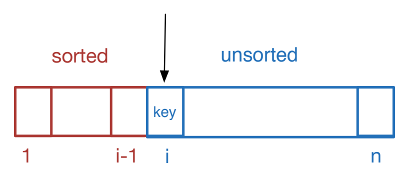

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- Compare `key` with every element x in the **sorted** subarray to the left of `key`, starting from the right.
  - If $x >$ `key`, move x one position to the right.
  - If $x \leq$ `key`, insert `key` after x.

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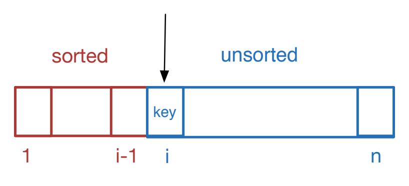

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   - Compare `key` with every element x in the **sorted** subarray to the left of `key`, starting from the right.
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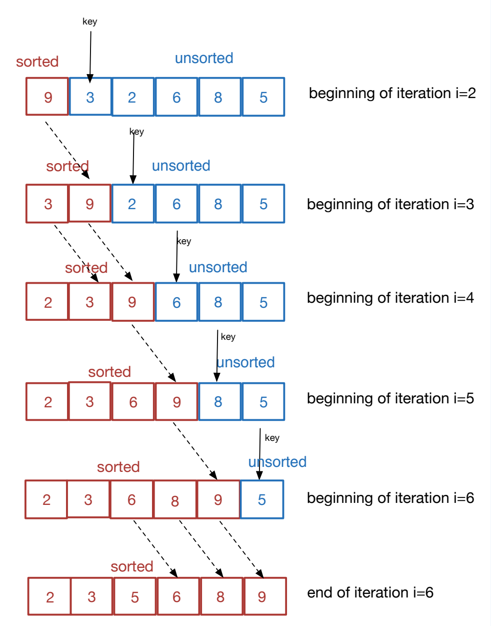
3. Repeat Step 2. until the sorted subarray has size n.

Example of insertion sort: $n = 6, A = \{9, 3, 2, 6, 8, 5\}$



Pseudocode

Let A be an array of n integers.

```
insertion-sort(A)
  for i = 2 to n do
     key = A[i]
     //Insert A[i] into the sorted subarray A[1, i − 1]
     j = i − 1
     while j > 0 and A[j] > key do
        A[j + 1] = A[j]
        j = j − 1
     end while
     A[j + 1] = key
  end for
```

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Analysis of algorithms

- **Correctness**
- **Running time**
- **Space**

Analysis of algorithms

- **Correctness:** formal proof often by induction
- **Running time:** number of primitive computational steps
  - Not the same as **time** it takes to execute the algorithm.
  - We want a measure that is independent of hardware.
  - We want to know how running time scales with the size of the input.
- **Space:** how much space is required by the algorithm

Analysis of `insertion sort`

Notation: $A[i, j]$ is the subarray of A that starts at position i and ends at position j.

- **Correctness:** follows from the key observation that after loop i, the subarray $A[1, i]$ is sorted
- **Running time:** number of primitive computational steps
- **Space:** in place algorithm (at most a constant number of elements of A are stored outside A at any time)

Example of induction

Fact 1.

For all $n \geq 1$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Example of induction

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For all $n \geq 1$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Proof.

- **Base case:** $n = 1$
- **Inductive hypothesis:** Assume that the statement is true for $n \geq 1$, that is, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
- **Inductive step:** We show that the statement is true for $n+1$. That is, $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$. (Show this!)
- **Conclusion:** It follows that the statement is true for all n since we can apply the inductive step for $n = 2, 3, \ldots$.

$\square$

Correctness of `insertion-sort`

Notation: $A[i, j]$ is the subarray of A that starts at position i and ends at position j.

Minor change in the pseudocode: in line 1, start from $i = 1$ rather than $i = 2$. *How does this change affect the algorithm?*

Claim 1.

Let $n \geq 1$ be a positive integer. For all $1 \leq i \leq n$, after the i-th loop, the subarray $A[1, i]$ is sorted.

Correctness of `insertion-sort` follows if we show Claim 1 (*why?*).

Proof of Claim 1

By induction on i.

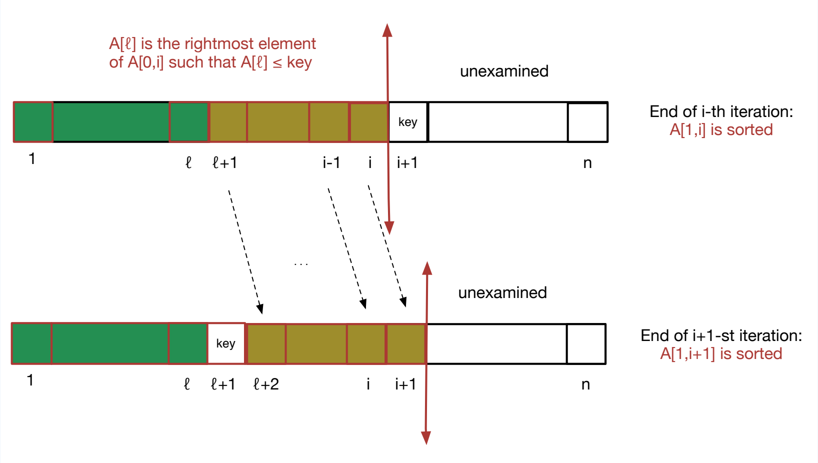
- **Base case:** $i = 1$, trivial.
- **Induction hypothesis:** assume that the statement is true for some $1 \leq i < n$.
- **Inductive step:** Show it true for $i + 1$.

  In loop $i + 1$, element `key` $= A[i + 1]$ is inserted into $A[1, i]$. By the induction hypothesis, $A[1, i]$ is sorted. Since

  1. `key` is inserted after the last element $A[\ell]$ such that $0 \leq \ell \leq i$ and $A[\ell] \leq$ `key`;
  2. all elements in $A[\ell + 1, j]$ are pushed one position to the right with their order preserved,

  the statement is true for $i + 1$.

Proof of the inductive step in a picture



Running time $T(n)$ of `insertion-sort`

for $i = 2$ to n **do**
    key $= A[i]$
    //Insert $A[i]$ into the sorted subarray $A[1, i-1]$
    $j = i - 1$
    while $j > 0$ and $A[j] >$ key **do**
        $A[j+1] = A[j]$
        $j = j - 1$
    end while
    $A[j+1] =$ key
end for

- *How many primitive computational steps are executed by the algorithm?*
- *Equivalently, what is the running time $T(n)$? Bounds on $T(n)$?*

Running time $T(n)$ of `insertion-sort`

```
for i = 2 to n do                                   line 1
    key = A[i]                                      line 2
    //Insert A[i] into the sorted subarray A[1, i − 1]
    j = i − 1                                       line 3
    while j > 0 and A[j] > key do                   line 4
        A[j + 1] = A[j]                             line 5
        j = j − 1                                   line 6
    end while
    A[j + 1] = key                                  line 7
end for
```

- For $2 \leq i \leq n$, let $t_i = \#$ times line 4 is executed.

Running time $T(n)$ of `insertion-sort`

```
for  i = 2 to n do                                  line 1
    key = A[i]                                      line 2
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      A[j + 1] = A[j]                               line 5
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end for
```

- For $2 \leq i \leq n$, let $t_i = \#$ times line 4 is executed. Then

$$T(n) = n + 3(n-1) + \sum_{i=2}^{n} t_i + 2\sum_{i=2}^{n}(t_i - 1) = 3\sum_{i=2}^{n} t_i + 2n - 1$$

- *Which input yields the smallest* (best-case) *running time?*
- *Which input yields the largest* (worst-case) *running time?*

Running time $T(n)$ of `insertion-sort`

```
for i = 2 to n do                                   line 1
    key = A[i]                                      line 2
    //Insert A[i] into the sorted subarray A[1, i − 1]
    j = i − 1                                       line 3
    while j > 0 and A[j] > key do                   line 4
        A[j + 1] = A[j]                             line 5
        j = j − 1                                   line 6
    end while
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end for
```

- For $2 \leq i \leq n$, let $t_i = \#$ times line 4 is executed. Then

$$T(n) = 3\sum_{i=2}^{n} t_i + 2n - 1$$

- Best-case running time: $5n - 4$
- Worst-case running time: $\frac{3n^2}{2} + \frac{7n}{2} - 4$

Worst-case analysis

Definition 2.

Worst-case running time: largest possible running time of the algorithm over all inputs of a given size n.

Why worst-case analysis?

- It gives well-defined computable bounds.
- Average-case analysis can be tricky: how do we generate a "random" instance?

The worst-case running time of `insertion-sort` is quadratic.
Is `insertion-sort` *efficient?*

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Efficiency of insertion-sort and the brute force solution

Compare to brute force solution:

- At each step, generate a new permutation of the n integers.
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Worst-case analysis: generate $n!$ permutations. *Is brute force solution efficient?*

Efficiency of insertion-sort and the brute force solution

Compare to brute force solution:

- At each step, generate a new permutation of the n integers.
- If sorted, stop and output the permutation.

Worst-case analysis: generate $n!$ permutations. *Is brute force solution efficient?*

- Efficiency relates to the performance of the algorithm as n grows.
- Stirling's approximation formula: $n! \approx \left(\frac{n}{e}\right)^n$.
  - For $n = 10$, generate $3.67^{10} \geq 2^{10}$ permutations.
  - For $n = 50$, generate $18.3^{50} \geq 2^{200}$ permutations.
  - For $n = 100$, generate $36.7^{100} \geq 2^{700}$ permutations!

$\Rightarrow$ Brute force solution is not efficient.

Efficient algorithms –Attempt 1

Definition 3 (Attempt 1).

An algorithm is efficient if it achieves better worst-case performance than brute-force search.

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Caveat: fails to discuss the scaling properties of the algorithm; *if the input size grows by a constant factor, we would like the running time $T(n)$ of the algorithm to increase by a constant factor as well.*

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Polynomial running times: on input of size n, $T(n)$ is at most $c \cdot n^d$ for $c, d > 0$ constants.

- Polynomial running times scale well!
- The **smaller** the exponent of the polynomial the better.

Efficient algorithms

Definition 4.

An algorithm is efficient if it has a polynomial running time.

Caveat

- What about huge constants in front of the leading term or large exponents?

However

- Small degree polynomial running times exist for most problems that can be solved in polynomial time.
- Conversely, problems for which no polynomial-time algorithm is known tend to be very hard in practice.
- So we can distinguish between easy and hard problems.

Remark 1.

Today's big data: even low degree polynomials might be too slow!

Are we done with sorting?

Insertion sort is efficient. *Are we done with sorting?*

Are we done with sorting?

Insertion sort is efficient. *Are we done with sorting?*

1. *Can we do better?*
2. *And what is better?*
   - *E.g., is* $T(n) = n^2$ *better than* $\frac{3n^2}{2} + \frac{7n}{2} - 4$*?*

Running time in terms of # primitive steps

To discuss this, we need a coarser classification of running times of algorithms; exact characterizations

- are **too detailed**;
- do not reveal similarities between running times in an immediate way as n grows large;
- are often **meaningless**: pseudocode steps will **expand** by a constant factor that depends on the hardware.

Asymptotic notation

A framework that will allow us to compare the rate of growth of different running times as the input size n grows.

- We will express the running time as a function of the number of primitive steps, which is a function of the size of the input n.
- To compare functions expressing running times, we will ignore their low-order terms and focus solely on the highest-order term.

A faster algorithm for sorting using the divide-and-conquer principle.