Analysis of Algorithms, I
CSOR W4231

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Depth-first search, topological sorting

Outline

Today

1. Recap
2. Applications of BFS
   - Testing bipartiteness
3. Depth-first search (DFS)
4. Applications of DFS
   - Cycle detection
   - Topological sorting

Review of the last lecture

- Graphs (directed, undirected, weighted, unweighted)
  - Notation: $G = (V, E)$, $|V| = n$, $|E| = m$
- Representing graphs
  1. Adjacency matrix
  2. Adjacency list
- Trees, bipartite graphs, the degree theorem
- Linear graph algorithms
- Breadth-first search (BFS)

Claim 1.

Let T be a BFS tree, let x and y be nodes in T belonging to layers L_i and L_j respectively, and let (x, y) be an edge in G. Then i and j differ by at most 1.

An algorithm for s-t connectivity: breadth-first search

Breadth-first search ($\texttt{BFS}(G, s)$): explore G starting from s **outward in all possible directions**, adding reachable nodes one **layer** at a time.

- First add all nodes that are joined by an edge to s: these nodes form the first layer.
  If G is unweighted, these are the nodes at distance 1 from s.
- Then add all nodes that are joined by an edge to a node in the first layer: these nodes form the second layer.
  If G is unweighted, these are the nodes at distance 2 from s.
- And so on and so forth.

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Testing bipartiteness & graph 2-colorability

Testing bipartiteness

- **Input:** a graph $G = (V, E)$
- **Output: yes** if G is bipartite, **no** otherwise

Equivalent problem (*why?*)

- **Input:** a graph $G = (V, E)$
- **Output: yes** if and only if we can color all the vertices in G using at most 2 colors –say red and white– so that no edge has two endpoints with the same color.

Why wouldn't we be able to 2-color a graph?

Fact: If a graph contains an odd-length cycle, then it is not 2-colorable.

So a **necessary** condition for a graph to be 2-colorable is that it does not contain odd-length cycles.

Is this condition also **sufficient***, that is, if a graph does not contain odd-length cycles, then is it* 2*-colorable?*

In other words, are odd cycles the only obstacle to bipartiteness?

Algorithm for 2-colorability

BFS provides a natural way to 2-color a graph $G = (V, E)$:

- Start BFS from any vertex; color it red.
- Color white all nodes in the first layer L_1 of the BFS tree. If there is an edge between two nodes in L_1, output **no** and stop.
- Otherwise, continue from layer L_1, coloring red the vertices in even layers and white in odd layers.
- If BFS terminates and all nodes in V have been explored (hence 2-colored), output **yes**.

Analyzing the algorithm

Upon termination of the algorithm

- either we successfully 2-colored all vertices and output **yes**, that is, declared the graph bipartite;
- or we stopped at some level because there was an edge between two vertices of that level and output **no**; in this case, we declared the graph non-bipartite.

This algorithm is **efficient**. *Is it a* **correct** *algorithm for 2-colorability?*

Showing correctness

To prove correctness, we must show the following statement.

If our algorithm outputs

1. **yes**, then the 2-coloring it returns is a valid 2-coloring of G;
2. **no**, then indeed G cannot be 2-colored by **any** algorithm (e.g., because it contains an odd-length cycle).

The next claim proves that this is indeed the case by examining the possible outputs of our algorithm. Note that the output depends solely on whether *there is an edge in G between two nodes in the same BFS layer*.

Correctness of algorithm for 2-colorability

Claim 2.

Let G be a connected graph, and let L_1, L_2, $\ldots$ be the layers produced by BFS starting at node s. Then exactly one of the following is true.

1. *There is no edge in G joining two nodes in the same BFS layer.* *Then G is bipartite and has no odd length cycles.*
2. *There is an edge in G joining two nodes in the same BFS layer.* *Then G contains an odd length cycle, hence is not bipartite.*

Corollary 1.

A graph is bipartite if and only if it contains no odd length cycle.

Proof of Claim 2, part 1

1. **Assume** that no edge in G joins two nodes of the same layer of the BFS tree.

By Claim 1, all edges in G not belonging to the BFS tree are

- either edges between nodes in the same layer;
- or edges between nodes in adjacent layers.

Our assumption implies that all edges of G not appearing in the BFS tree are between nodes in adjacent layers.

Since our coloring procedure gives such nodes different colors, the whole graph can be 2-colored, hence it is bipartite.

Proof of Claim 2, part 2

2. **Assume** that there is an edge $(u, v) \in E$ between two nodes u and v on the same layer.

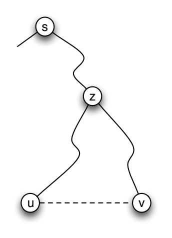
Obviously G is not 2-colorable by our algorithm: both endpoints of edge (u, v) are assigned the same color.

Our algorithm returns **no**, hence declares G non-bipartite.

Can we show existence of an odd-length cycle and prove that G indeed is not 2-colorable by **any** *algorithm?*

Proof of correctness, part 2

- Let u, v appear at layer L_j and edge $(u, v) \in E$.
- Let z be the common ancestor at max depth of u and v in the BFS tree (z might be s). Suppose z appears at layer L_i with $i < j$.
- Consider the following path in G: from z to u follow edges of the BFS tree, then to v via edge (u, v) and back to z following edges of the BFS tree. This is a cycle starting and ending at z, consisting of $(j - i) + 1 + (j - i) = 2(j - i) + 1$ edges, hence of odd length.



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Finding your way in a maze

Depth-first search (DFS): starting from a vertex s, explore the graph as deeply as possible, then backtrack

1. Try the first edge out of s, towards some node v.
2. Continue from v until you reach a dead end, that is a node whose neighbors have all been explored.
3. Backtrack to the first node with an unexplored neighbor and repeat 2.

Remark: DFS answers s-t connectivity

DFS vs BFS

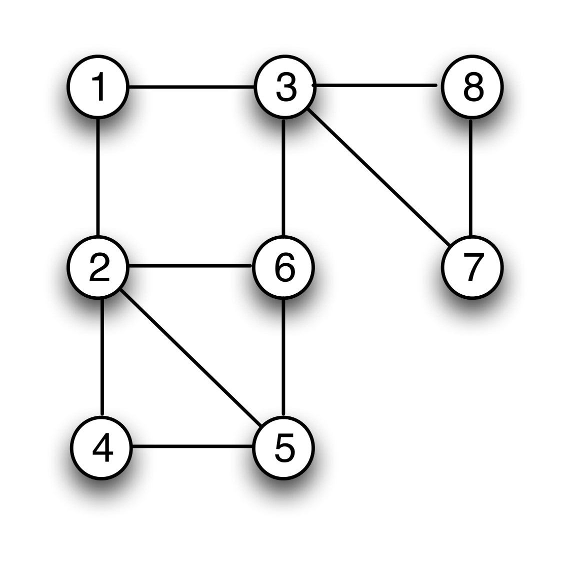
Similarities

- Linear-time algorithms that essentially can be used to perform the same tasks

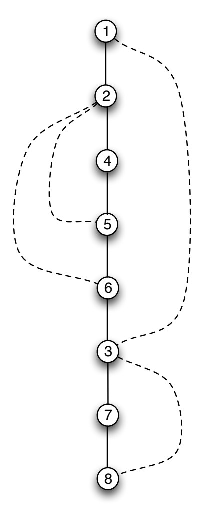
Differences

- DFS is more *impulsive*: when it *discovers* an *unexplored* node, it moves on to exploring it right away; BFS defers exploring until all nodes in the layer have been discovered.
- DFS is naturally recursive and implemented using a stack.
  - A stack is a LIFO (Last-In First-Out) data structure implemented as a doubly linked list: **insert** (push)/**extract** (pop) the top element requires $O(1)$ time.

An undirected graph G_1

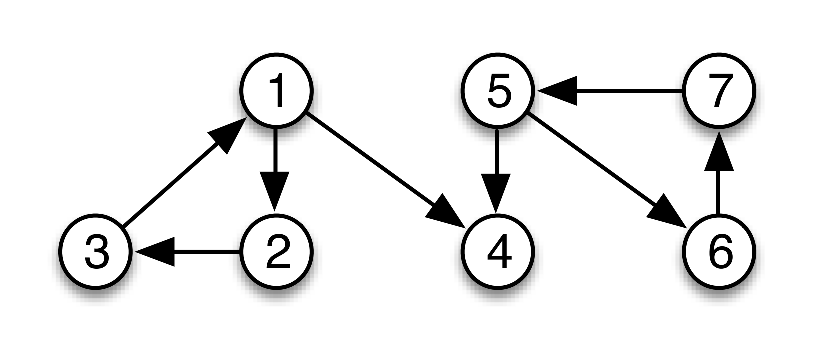


The DFS tree for the undirected graph G_1

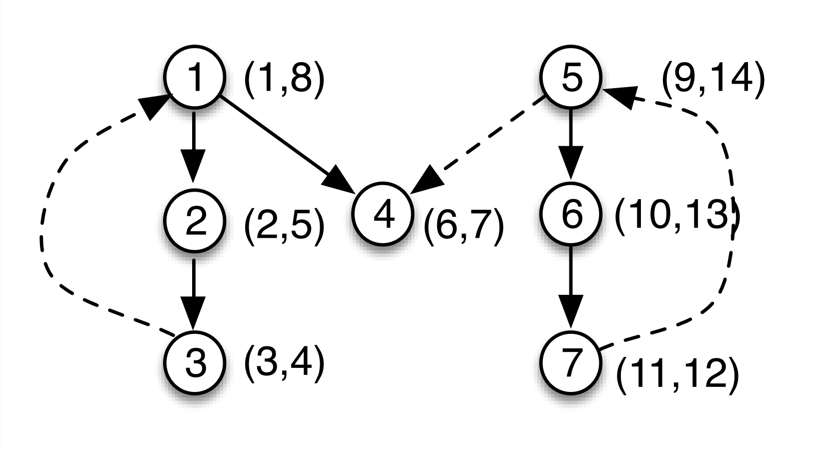


Dashed edges belong to the graph but not to the DFS tree. Ties are broken by considering nodes by increasing index.

A directed graph G



The DFS forest for the directed graph G



Dashed edges belong to G but not to the trees in the DFS forest.
$(start, finish)$ intervals appear to the right of every node.

Pseudocode for DFS exploration of the entire graph

`DFS`$(G = (V, E))$
  for $u \in V$ **do**
    $explored[u] = 0$
  end for
  for $u \in V$ **do**
    if $explored[u] == 0$ **then** `Search`(u)
    end if
  end for

`Search`(u)
  `previsit`(u)
  $explored[u] = 1$
  for $(u, v) \in E$ **do**
    if $explored[v] == 0$ **then** `Search`(v)
    end if
  end for
  `postvisit`(u)

Running time for `DFS` if `previsit`, `postvisit` take $O(1)$ time?

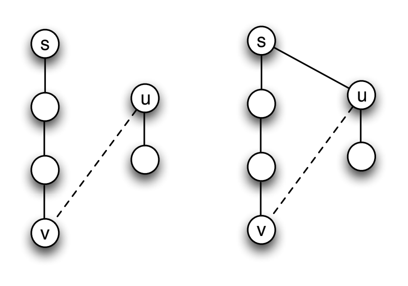
Directed graphs: classification of edges

Graph edges that do not belong to the DFS tree(s) may be

1. forward: from a vertex to a *descendant* (other than a *child*)
2. back: from a vertex to an *ancestor*
   Examples: edges $(3, 1), (7, 5)$ in G
3. cross: from right to left (no ancestral relation), that is
   - from tree to tree (example: edge $(5, 4)$ in G)
   - between nodes in the same tree but on different branches

Undirected graphs: classification of edges

Cross and *forward* edges do not exist in undirected graphs.



In undirected graphs, DFS only yields *back* and *tree* edges.

Time intervals for vertices

Subroutines `previsit`(u), `postvisit`(u) may be used to maintain a notion of time:

- In `DFS`(G), initialize a counter *time* to 0.
- Increment the counter by 1 every time `previsit`(u), `postvisit`(u) are accessed.
- Store the times $start(u)$ and $finish(u)$ corresponding to the first and last time u was visited during `DFS`(G).

`previsit`(u)

$$time = time + 1$$
$$start(u) = time$$

`postvisit`(u)

$$time = time + 1$$
$$finish(u) = time$$

On the time intervals of vertices u, v

If we use an explicit stack, then

- $start(u)$ is the time when u is pushed in the stack
- $finish(u)$ is the time when u is popped from the stack (that is, all of its neighbors have been explored).

1. *How do intervals* $[start(u), finish(u)]$, $[start(v), finish(v)]$ *relate?*
2. *What do the contents of the stack correspond to in the DFS tree (and the graph), if* s *was the first vertex pushed in the stack and* v *the last?*

Identifying back edges using time

1. Intervals $[start(u), finish(u)]$ and $[start(v), finish(v)]$
   - either contain each other (u is an ancestor of v or vice versa)
   - or they are disjoint.
2. If s was the first vertex pushed in the stack and v is the last, the vertices currently in the stack form an s-v path.

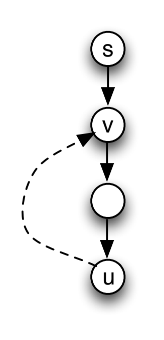
Claim 3 (Back edges).

Let $(u, v) \in E$. Edge (u, v) is a back edge in a DFS tree if and only if

$$start(v) < start(u) < finish(u) < finish(v).$$

Proof of Claim 3 (identifying back edges)

Proof.



If (u, v) is a back edge, the claim follows.

Otherwise, v was pushed in the stack before u and is still in the stack when u is pushed into it. Then there is a $v - u$ path in the DFS tree, so v is an ancestor of u and (u, v) is a back edge. $\square$

Identifying forward and cross edges

What conditions must the start and finish numbers satisfy if

1. $(u, v) \in E$ *is a* *forward* *edge in the DFS tree?*
2. $(u, v) \in E$ *is a* *cross* *edge in the DFS tree?*

Identifying forward and cross edges

What conditions must the start and finish numbers satisfy if

1. $(u, v) \in E$ *is a forward edge in the DFS tree?*
2. $(u, v) \in E$ *is a cross edge in the DFS tree?*

1. If edge $(u, v) \in E$ is a forward edge, then

$$start(u) < start(v) < finish(v) < finish(u).$$

2. If edge $(u, v) \in E$ is a cross edge, then

$$start(v) < finish(v) < start(u) < finish(u).$$

Today

Application I: Cycle detection

Claim 4.

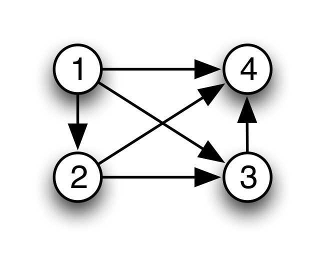
$G = (V, E)$ *has a cycle if and only if* $\texttt{DFS}(G)$ *yields a back edge.*

Proof.

If (u, v) is a back edge, together with the path on the DFS tree from v to u, it forms a cycle.
Conversely, suppose G has a cycle. Let v be the first vertex from the cycle discovered by $\texttt{DFS}(G)$. Let (u, v) be the preceding edge in the cycle. Since there is a path from v to *every* vertex in the cycle, all vertices in the cycle are now discovered **and** fully explored **before** v is popped from the stack. Hence the interval of u is contained in the interval of v.
By Claim 1, (u, v) is a back edge. □

Application II: Topological sorting in DAGs

- An undirected acyclic graph has an extremely simple structure: it is a tree, hence a sparse graph ($O(n)$ edges).
- A directed acyclic graph (DAG) may be dense ($\Omega(n^2)$ edges): e.g., $V = \{1, \ldots, n\}$, $E = \{(i, j) \text{ if } i < j \}$.



Topological sorting: motivation

Input:

- a set of tasks $\{1, 2, \ldots, n\}$ that need to be performed
- a set of dependencies, each of the form (i, j), indicating that task i must be performed before task j.

Output: a valid order in which the tasks may be performed, so that all dependencies are respected.

Example: tasks are courses and certain courses must be taken before others.

How can we model this problem using a graph? What kind of graph must arise and why?

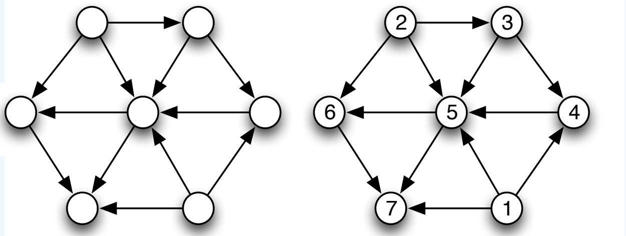
Topological ordering: definition

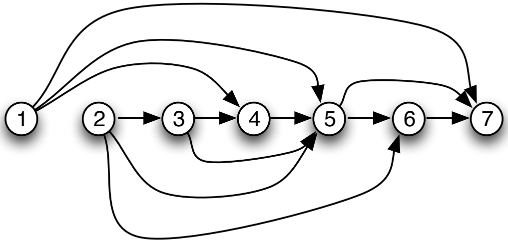
Definition 2.

A topological ordering of G is an ordering of its nodes as $1, 2, \ldots, n$ such that for every edge (i, j), we have $i < j$.

- All edges point **forward** in the topological ordering.
- It provides an order in which all tasks can be safely performed: when we try to perform task j, all tasks required to precede it have already been done.

Example of DAG and its topological sorting





A DAG (top left), its topological sort (top right) and a drawing emphasizing the topological sort (bottom).

Topological sorting in DAGs

Claim 5.

If G has a topological ordering, then G is a DAG.

Proof: By contradiction (*exercise*).

A visual proof is provided by the linearized graph of the previous slide: vertices appear in increasing order, edges go from left to right, hence no cycles.

Is the converse true: does every DAG have a topological ordering? And how can we find it?

Structural properties of DAGs

In a DAG, can **every** vertex have

- *an outgoing edge?*
- *an incoming edge?*

Definition 3 (source and sink).

A source is a node with no incoming edges.
A sink is a node with no outgoing edges.

Fact 4.

Every DAG has at least one source and at least one sink.

How can we use Fact 4 to find a topological order?

The node that we label *first* in the topological sorting must have no incoming edges. Fact 4 guarantees that such a node exists.

Fact 5.

Let G' be the graph after a source node and its adjacent edges have been removed. Then G' is a DAG.

Proof: removing edges from G cannot yield a cycle!

This gives rise to a recursive algorithm for finding the topological order of a DAG. Its correctness can be shown by induction (use Facts 4, 5 to show induction step).

Algorithm for topological sorting

`TopologicalOrder`(G)

1. Find a source vertex s and order it first.
2. Delete s and its adjacent edges from G; let G' be the new graph.
3. `TopologicalOrder`(G')
4. Append the order found after s.

Running time: $O(n^2)$. Can be improved to $O(n+m)$.

Topological sorting via DFS

Let $G = (V, E)$ be a DAG.

- Run `DFS`(G); compute $finish$ times.
- Process the tasks in decreasing order of $finish$ times.

Running time: $O(m + n)$

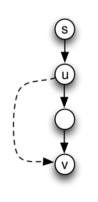
Intuition behind this algorithm

- The task v with the largest $finish$ has no incoming edges (if it had an incoming edge from some other task u, then u would have the largest $finish$). Hence v does not depend on any other task and it is safe to perform it first.
- The same reasoning shows that the task w with the second largest $finish$ has no incoming edges from any other task except (maybe) task v. Hence it is safe to perform w second.
- And so on and so forth.

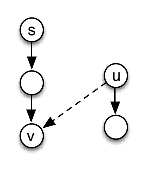
Formal proof of correctness

By Claim 4 there are no back edges in the DFS forest of a DAG. Thus every edge $(u, v) \in E$ is either

1. forward/tree: $start(u) < start(v) < finish(v) < finish(u)$



2. or cross edge: $finish(v) < start(u) < finish(u)$



Proof of correctness (cont'd)

Hence for every $(u, v) \in E$, $finish(v) < finish(u)$.

Consider a task v. All tasks u upon which v depends, that is, all tasks u such that there is an edge $(u, v) \in E$, satisfy $finish(v) < finish(u)$.

Since we are processing tasks in decreasing order of finish times, all tasks u upon which v depends have already been processed before we start processing v.