Analysis of Algorithms
CSOR W4231

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Data compression and huffman coding

Outline

Today

Motivation

Data compression: find compact representations of data

Data compression standards

- `jpeg` for image transmission
- `mp3` for audio content, `mpeg2` for video transmission
- utilities: `gzip`, `bzip2`

All of the above use the Huffman algorithm as a basic building block.

Data representation

- An organism's genome consists of *chromosomes* (giant linear DNA molecules)
- *Chromosome maps*: sequences of hundreds of millions of bases (symbols from $\{A, C, G, T\}$).
- **Goal:** store a chromosome map with 200 million bases.

How do we represent a chromosome map?

Data representation

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How do we represent a chromosome map?

- Encode every symbol that appears in the sequence separately by a fixed length binary string.
- Codeword $c(x)$ for symbol x: a binary string encoding x of length $\ell(x)$

Example code

- Alphabet $\mathcal{A} = \{A, C, G, T\}$ with 4 symbols
- Encode each symbol with 2 bits

alphabet symbol x	codeword $c(x)$
A	00
C	01
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Example

- Input sequence: $ACGTAA$

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Output of encoding: the concatenation of the codewords for every symbol in the input sequence

Example

- Input sequence: $ACGTAA$
- Output: $c(A)c(C)c(G)c(T)c(A)c(A) = 000110110000$
- Total length of encoding $= 6 \cdot 2 = 12$ bits.

Today

1. Data compression
2. Symbol codes and optimal lossless compression
3. Prefix codes
4. Prefix codes and trees
5. The Huffman algorithm

Symbol codes

Symbol code: a set of codewords where every input symbol is encoded **separately**.

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Examples of symbol codes

- $C_0 = \{00, 01, 10, 11\}$ is a symbol code for $\{A, C, G, T\}$.
- ASCII encoding system: every character and special symbol on the computer keyboard is encoded by a different 7-bit binary string.

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Remark 1.

C_0 and ASCII are fixed-length symbol codes: each codeword has the same length.

Unique decodability

Decoding C_0?

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- read 2 bits of the output;
- print the symbol corresponding to this codeword;
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- *This decoding algorithm works for ASCII (replace 2 by 7)*
- C_0*, ASCII: distinct input sequences have distinct encodings*

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- *This decoding algorithm works for ASCII (replace 2 by 7)*
- *C_0, ASCII: distinct input sequences have distinct encodings*

Definition 1.

A symbol code is uniquely decodable if, for any two distinct input sequences, their encodings are distinct.

Lossless compression

- Lossless compression: compress and decompress without errors.
- Uniquely decodable codes allow for lossless compression.
- A symbol code achieves optimal lossless compression when it produces an encoding of minimum length for its input (among all uniquely decodable symbol codes).
- Huffman algorithm: provides a symbol code that achieves optimal lossless compression.

Fixed-length vs variable-length codes

Chromosome map consists of 200 million bases as follows:

alphabet symbol x	frequency $freq(x)$
A	110 million
C	5 million
G	25 million
T	60 million

Fixed-length vs variable-length codes

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alphabet symbol x	frequency $freq(x)$
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- A appears much more often than the other symbols.

$\Rightarrow$ It might be best to encode A with fewer bits.

- Unlikely that the fixed-length encoding C_0 is optimal

Today

Prefix codes

Variable-length encodings

Code C_1

alphabet symbol x	codeword $c(x)$
A	0
C	00
G	10
T	1

Prefix codes

Variable-length encodings

Code C_1

alphabet symbol x	codeword $c(x)$
A	0
C	00
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- C_1 is not unique decodable! E.g., 101110: *how to decode it?*

Prefix codes

Variable-length encodings

Code C_2

alphabet symbol x	codeword $c(x)$
A	0
C	110
G	111
T	10

Prefix codes

Variable-length encodings

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- C_2 is uniquely decodable.
- C_2 is such that no codeword is a prefix of another.

Prefix codes

Variable-length encodings

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Definition 2 (prefix codes).

A symbol code is a prefix code if no codeword is a prefix of another.

Decoding prefix codes

1. Scan the binary string from left to right until you've seen enough bits to match a codeword;
2. Output the symbol corresponding to this codeword.
   - Since no other codeword is a prefix of this codeword or contains it as a prefix, this sequence of bits cannot be used to encode any other symbol.
3. Continue starting from the next bit of the bit string.

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Thus prefix codes allow for

- unique decoding;
- fast decoding (the end of a codeword is instantly recognizable).

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Examples of prefix codes: C_0, C_2

Prefix codes and optimal lossless compression

- Decoding a prefix code is very fast.

$\Rightarrow$ Would like to focus on prefix codes (rather than **all** uniquely decodable symbol codes) for achieving optimal lossless compression.

- Information theory guarantees this: for every uniquely decodable code, exists a prefix code with the **same codeword lengths**
- So we can solely focus on prefix codes for optimal compression.

Compression gains from variable-length prefix codes

Chromosome map: *do we gain anything by using C_2 instead of C_0 when compressing the map of* 200 *million bases?*

Input

symbol x	$freq(x)$
A	110 million
C	5 million
G	25 million
T	60 million

Code C_0

x	$c(x)$
A	00
C	01
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- C_0: 2 bits $\times 200$ million symbols $= 400$ million bits
- C_2: $1 \cdot 110 + 3 \cdot 5 + 3 \cdot 25 + 2 \cdot 60 = 320$ million bits
- Improvement of 20% in this example

The optimal prefix code problem

Input:

- Alphabet $\mathcal{A} = \{a_1, \ldots, a_n\}$
- Set $P = \{p_1, \ldots, p_n\}$ of empirical probabilities over $\mathcal{A}$ such that $p_i = \Pr[a_i]$

Output: a binary prefix code $C^* = \{c(a_1), c(a_2), \ldots, c(a_n)\}$ for $(\mathcal{A}, P)$, where codeword $c(a_i)$ has length ℓ_i and is such that its expected length

$$L(C^*) = \sum_{a_i \in \mathcal{A}} p_i \cdot \ell_i$$

is **minimum** among all binary prefix codes.

Example

Chromosome example

Input

symbol x	$\Pr(x)$
A	110/200
C	5/200
G	25/200
T	60/200

Code C_0

x	$c(x)$
A	00
C	01
G	10
T	11

Code C_2

x	$c(x)$
A	0
C	110
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- $L(C_0) = 2$
- $L(C_2) = 1.6$
- *Coming up:* C_2 is the output of the Huffman algorithm, hence an optimal encoding for $(\mathcal{A}, P)$.

Today

Prefix codes and trees

- A binary tree T is a rooted tree such that each node that is not a leaf has at most two children.
- Binary tree for a prefix code: a branch to the left represents a 0 in the encoding and a branch to the right a 1.

Code C_0

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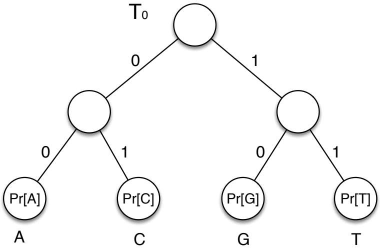

Binary tree for prefix code C_0

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Code C_2

x	$c(x)$
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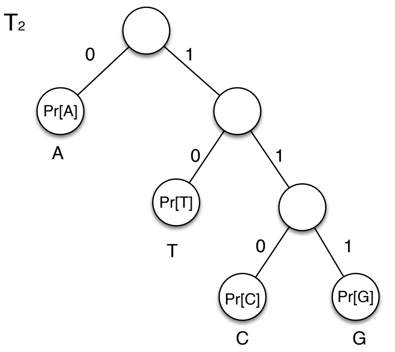
G	111
T	10



Binary tree for prefix code C_2

Properties of binary trees representing prefix codes

1. *Where do alphabet symbols appear in the tree?*
2. *What do codewords correspond to in the tree?*
3. *Consider the tree corresponding to the optimal prefix code. Can it have internal nodes with one child?*

Properties of binary trees representing prefix codes

1. Symbols must appear at the leaves of the tree T (*why?*)
   $\Rightarrow$ T has n leaves.

2. Codewords $c(a_i)$ are given by root-to-leaf paths.

   Recall that ℓ_i is the length of the codeword $c(a_i)$ for input symbol a_i. Therefore, on the tree T, ℓ_i corresponds to the depth of a_i (we assume that the root is at depth 0).

   $\Rightarrow$ Can rewrite the **expected length** of the prefix code as:

$$L(C) = \sum_{a_i \in \mathcal{A}} p_i \cdot \ell_i = \sum_{1 \leq i \leq n} p_i \cdot \mathrm{depth}_T(a_i) = L(T).$$

3. Optimal tree must be full: all internal nodes must have exactly two children (*why?*).

More on optimal tree

Claim 1.

There is an optimal prefix code, with corresponding tree T^*, in which the two lowest frequency characters are assigned to leaves that are siblings in T^* at maximum depth.

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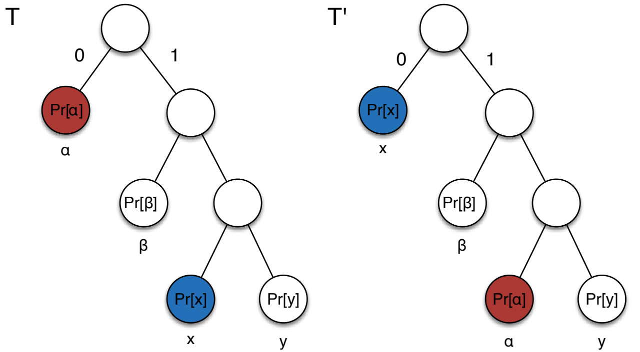
Proof.

By an exchange argument: start with a tree for an optimal prefix code and transform it into T^*. □

Proof of Claim 1

- Let T be the tree for the optimal prefix code.
- Let α, β be the two symbols with the smallest probabilities, that is, $\Pr[\alpha] \leq \Pr[\beta] \leq \Pr[s]$ for all $s \in \mathcal{A} - \{\alpha, \beta\}$.
- Let x and y be the two siblings at maximum depth in T.



We want L(T) ≥ L(T′)

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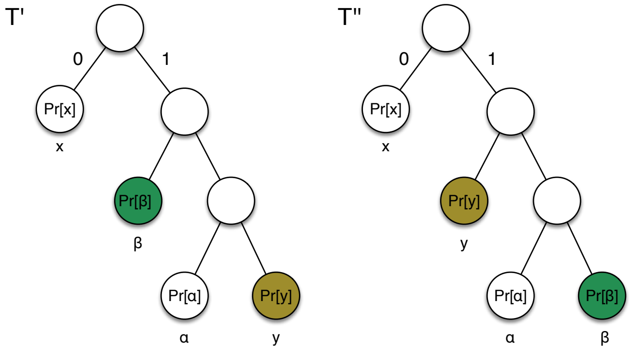


We want $L(T') \geq L(T'')$

How do the expected lengths of the two trees compare?

$$
\begin{aligned}
L(T) - L(T') &= \sum_{a_i \in \mathcal{A}} \Pr[a_i] \cdot \mathrm{depth}_T(a_i) - \sum_{a_i \in \mathcal{A}} \Pr[a_i] \cdot \mathrm{depth}_{T'}(a_i) \\
&= \Pr[\alpha] \cdot \mathrm{depth}_T(\alpha) + \Pr[x] \cdot \mathrm{depth}_T(x) \\
&\quad - \Pr[\alpha] \cdot \mathrm{depth}_{T'}(\alpha) - \Pr[x] \cdot \mathrm{depth}_{T'}(x) \\
&= \Pr[\alpha] \cdot \mathrm{depth}_T(\alpha) + \Pr[x] \cdot \mathrm{depth}_T(x) \\
&\quad - \Pr[\alpha] \cdot \mathrm{depth}_T(x) - \Pr[x] \cdot \mathrm{depth}_T(\alpha) \\
&= (\Pr[\alpha] - \Pr[x]) \cdot (\mathrm{depth}_T(\alpha) - \mathrm{depth}_T(x)) \geq 0
\end{aligned}
$$

- The third equality follows from the exchange.
- Similarly, exchanging β and y in T' yields $L(T') - L(T'') \geq 0$.
- Hence $L(T) - L(T'') \geq 0$.
- Since T is optimal, it must be $L(T) = L(T'')$.
- So T'' is also optimal.

The claim follows by setting T^* to be T''.

Building the optimal tree

Claim 1 tells us how to build the optimal tree greedily!

1. Find the two symbols with the lowest probabilities.
2. Remove them from the alphabet and replace them with a new meta-character with probability equal to the sum of their probabilities.
   - **Idea:** this meta-character will be the parent of the two deleted symbols in the tree.
3. Recursively construct the optimal tree using this process.

Greedy *algorithms: make a local (myopic) decision at every step that optimizes some criterion and eventually show that this is the optimal way for building the entire solution.*

Today

Huffman algorithm

`Huffman`$(\mathcal{A}, P)$

if $|\mathcal{A}| = 2$ **then**

Encode one symbol using 0 and the other using 1

end if

Let α and β be the two symbols with the lowest probabilities

Let ν be a new meta-character with probability $\Pr[\alpha] + \Pr[\beta]$

Let $\mathcal{A}_1 = \mathcal{A} - \{\alpha, \beta\} + \{\nu\}$

Let P_1 be the new set of probabilities over $\mathcal{A}_1$

$T_1 =$ `Huffman`$(\mathcal{A}_1, P_1)$

`return` T as follows: replace leaf node ν in T_1 by an internal node, and add two children labelled α and β below ν.

Remark 3.

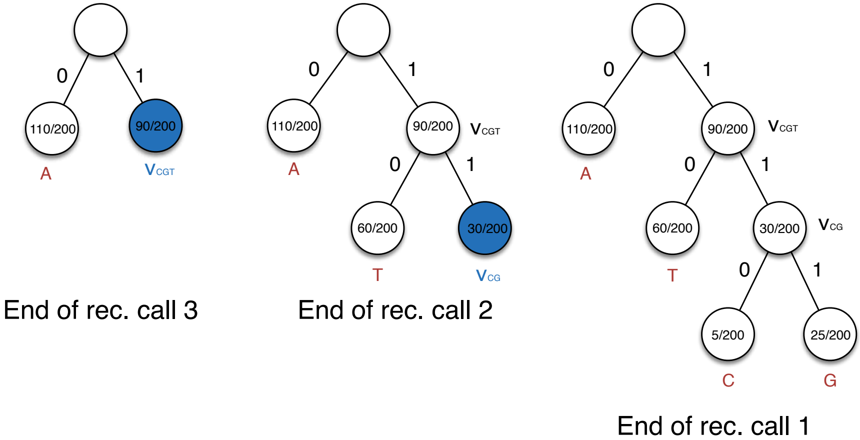
Output of `Huffman` *procedure is a binary tree* T*; the code for* $(\mathcal{A}, P)$ *is its corresponding prefix code.*

Example: recursive Huffman for chromosome map

Recursive call 1: $\texttt{Huffman}(\{A, C, G, T\}, \{\frac{110}{200}, \frac{5}{200}, \frac{25}{200}, \frac{60}{200}\})$

Recursive call 2: $\texttt{Huffman}(\{A, \nu_{CG}, T\}, \{\frac{110}{200}, \frac{30}{200}, \frac{60}{200}\})$

Recursive call 3: $\texttt{Huffman}(\{A, \nu_{CGT}\}, \{\frac{110}{200}, \frac{90}{200}\})$



Correctness

Proof: by induction on the size of the alphabet $n \geq 2$.

- **Base case.** For $n = 2$, Huffman is optimal.
- **Hypothesis.** Assume that Huffman returns the optimal prefix code for alphabets of n symbols.
- **Induction Step.** Let $\mathcal{A}$ be an alphabet of size $n + 1$, P the corresponding set of probabilities.

  Let T_1 be the optimal (by the hypothesis) tree returned by our algorithm for $(\mathcal{A}_1, P_1)$, where $\mathcal{A}_1, P_1, T_1$ as in the pseudocode. Let T be the final tree returned for $(\mathcal{A}, P)$ by our algorithm. We claim that T is optimal.

  We will prove the claim by contradiction. **Assume** T^* is the optimal tree for $(\mathcal{A}, P)$ such that

$$L(T^*) < L(T). \tag{1}$$

A useful fact

Fact 3.

Let T be a binary tree representing a prefix code. If we replace sibling leaves α, β in T by a meta-character ν where $\Pr[\nu] = \Pr[\alpha] + \Pr[\beta]$, we obtain a tree T_1 such that

$$L(T) = L(T_1) + (\Pr[\alpha] + \Pr[\beta]).$$

Proof.

Notation: $d_T(a_i) = \text{depth}_T(a_i)$

- α, β are sibling leaves in T, hence $d_T(\alpha) = d_T(\beta)$.
- T differs from T_1 only in that α, β are replaced by ν. Since $d_{T_1}(\nu) = d_T(\alpha) - 1$, we obtain

$$\begin{aligned} L(T) - L(T_1) &= \Pr[\alpha]d_T(\alpha) + \Pr[\beta]d_T(\beta) - (\Pr[\alpha] + \Pr[\beta])d_{T_1}(\nu) \\ &= \Pr[\alpha] + \Pr[\beta]. \end{aligned} \tag{2}$$

□

Correctness (cont'd)

- Claim 1 guarantees there is such an optimal tree for $(\mathcal{A}, P)$ where α, β appear as siblings at maximum depth.
- W.l.o.g. assume that T^* is such an optimal tree. By Fact 3, if we replace siblings α, β in T^* by ν' where $\Pr[\nu'] = \Pr[\alpha] + \Pr[\beta]$, the resulting tree T_1^* satisfies $L(T^*) = L(T_1^*) + (\Pr[\alpha] + \Pr[\beta])$.
- Similarly, the tree T returned by the Huffman algorithm satisfies $L(T) = L(T_1) + (\Pr[\alpha] + \Pr[\beta])$.
- By the induction hypothesis, we have $L(T_1^*) \geq L(T_1)$ since T_1 is optimal for alphabets of size n. Hence

  $$L(T^*) = L(T_1^*) + \Pr[\alpha] + \Pr[\beta] \geq L(T_1) + \Pr[\alpha] + \Pr[\beta] = L(T), \quad (3)$$

  where the inequality follows from the induction hypothesis.
- Equation (3) contradicts Assumption (1). Thus T must be optimal.

Implementation and running time

1. Straightforward implementation: $O(n^2)$ time
2. Store the alphabet symbols in a min priority queue implemented as a binary min-heap with keys their probabilities
   - **Operations:** `Initialize` ($O(n)$), `Extract-min` ($O(\log n)$), `Insert` ($O(\log n)$)

   Total time: $O(n \log n)$ time

For an iterative implementation of Huffman, see your textbook.

Example: iterative Huffman for chromosome map

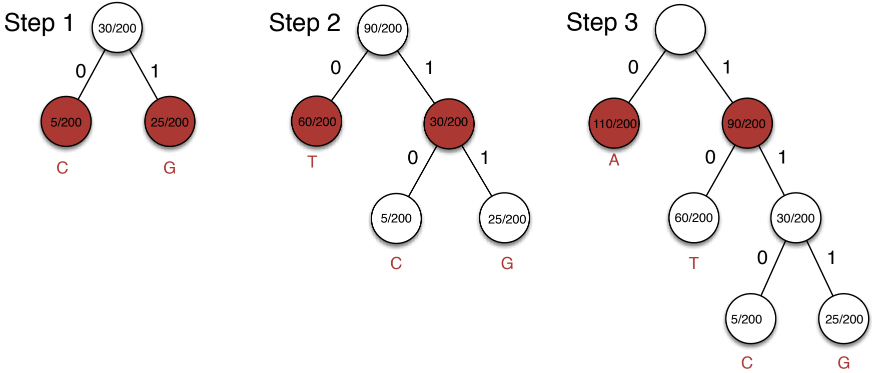
Input $(\mathcal{A}, P)$

symbol x	$\Pr(x)$
A	$110/200$
C	$5/200$
G	$25/200$
T	$60/200$

$\rightarrow$

Output code

symbol x	$c(x)$
A	0
C	110
G	111
T	10



Beyond Huffman coding

- Huffman algorithm provides an optimal symbol code.
- Codes that encode larger blocks of input symbols might achieve better compression.
- Storage on noisy media: *what if a bit of the output of the compressor is flipped?*
  - Decompression cannot carry through.
  - Need error correction on top of compression.