Analysis of Algorithms, I
CSOR W4231

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Representative NP-complete problems: `TSP`, `Set Cover`

Outline

Today

Complexity classes $\mathcal{P}$, $\mathcal{NP}$ and $\mathcal{NP}$-complete

Definition 1.

We define $\mathcal{P}$ to be the set of problems that can be solved by polynomial-time algorithms.

Definition 2.

We define $\mathcal{NP}$ to be the set of decision problems that have an efficient certifier.

Fact 3.

$\mathcal{P} \subseteq \mathcal{NP}$

Definition 4.

A problem $X(D)$ is $\mathcal{NP}$-complete if

1. $X(D) \in \mathcal{NP}$ and
2. for all $Y \in \mathcal{NP}$, $Y \leq_P X$.

How do we show that a problem is $\mathcal{NP}$-complete?

Suppose we had an $\mathcal{NP}$-complete problem X.

To show that another problem Y is $\mathcal{NP}$-complete, we use transitivity of reductions. So we “only” need show that

1. $Y \in \mathcal{NP}$
2. $X \leq_P Y$

The first $\mathcal{NP}$-complete problem

Theorem 5 (Cook-Levin).

Circuit SAT is $\mathcal{NP}$-complete.

Satisfiability of boolean functions

`SAT`: Given a formula ϕ in CNF with n variables and m clauses, is ϕ satisfiable?

`3SAT`: Given a formula ϕ in CNF with n variables and m clauses such that each clause has exactly 3 literals, is ϕ satisfiable?

`Circuit-SAT`: Given a boolean combinatorial circuit C, is there an assignment of truth values to its inputs that causes the output to evaluate to 1?

Lemma 6.

Circuit-SAT $\leq_P$ *SAT*, *SAT* $\leq_P$ *3SAT and 3SAT* $\leq_P$ $IS(D)$

Common pitfalls when showing $\mathcal{NP}$-completeness

1. Carry out the reduction in the wrong direction
2. Reduce from a problem not known to be $\mathcal{NP}$-complete
3. Exponential-time transformations
   - Subsets, permutations
4. Neglect to carefully prove both directions of equivalence of the original and the derived instances; that is, x is a **yes** instance of X *if and only if* $y = R(x)$ is a **yes** instance of Y
5. Neglect to show that the problem is in $\mathcal{NP}$

Suggestions

- You should think carefully which problem is most suitable to reduce from
- In absence of other ideas, reduce from 3SAT

Today

The Traveling Salesman Problem (TSP)

Tour: a *simple* cycle that visits *every* vertex exactly once.

Definition 7 (TSP(D)).

Given n cities $\{1, \ldots, n\}$, a set of non-negative distances d_{ij} between every pair of cities and a budget B, is there a tour of length $\leq B$?

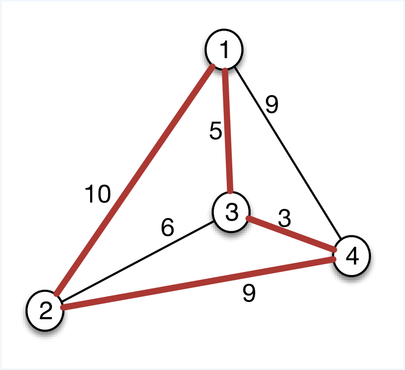
Equivalently, is there a permutation π such that

1. $\pi(1) = \pi(n+1) = 1$; that is, we start and end at city 1
2. the total distance travelled satisfies

$$\sum_{i=1}^{n} d_{\pi(i)\pi(i+1)} \leq B$$

Application: Google street view car

Example instance of TSP



Depending on the distances, TSP instances may be

- *Asymmetric*: $d_{ij} \neq d_{ji}$
- *Symmetric*: $d_{ij} = d_{ji}$
- *Metric*: satisfy the triangle inequality $d_{ij} \leq d_{ik} + d_{kj}$
- *Euclidean*: e.g., cities are in $\mathcal{R}^2$ hence city i corresponds to point (x_i, y_i); then $d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$

A related problem and hardness of `TSP(D)`

`Hamiltonian Cycle`: Given a graph $G = (V, E)$, is there a simple cycle that visits every vertex exactly once?

Claim 1.

`Hamiltonian Cycle` *is* $\mathcal{NP}$*-complete.*

Proof: Reduction from `3SAT` (e.g., see your textbook).

Claim 2.

`Symmetric TSP(D)` *is* $\mathcal{NP}$*-complete.*

Proof: reduction from undirected `Hamiltonian Cycle`.

Proof of Claim 2 (`Hamiltonian Cycle` $\leq_P$ `TSP(D)`)

1. Start from an arbitrary instance of undirected **`Hamiltonian Cycle`**, that is, an undirected graph $G = (V, E)$.
2. Construct the following instance $(G' = (V', E', w), B)$ of `TSP(D)`: G' is a *complete* weighted graph with $V' = V$ such that for every edge $e \in E'$,
$$w_e = \begin{cases} 1, & \text{if } e \in E \\ 2, & \text{otherwise} \end{cases}$$
3. Set the budget $B = n$.

This completes the reduction transformation.

Equivalence of the instances is straightforward:

- If G has a hamiltonian cycle, that cycle is a tour of length n in G'.
- If G' has a tour of length n, it must consist of edges of weight 1 *(why?)*; thus all these edges appear in G.

Concluding remarks on TSP

- Claim 1 also holds for directed Hamiltonian cycle. An exact analog of the proof of Claim 2 then shows that asymmetric TSP is $\mathcal{NP}$-complete.

- It is possible to reduce `Hamiltonian cycle` to `Euclidean TSP`, thus showing that even `Euclidean TSP` is $\mathcal{NP}$-complete.

- However, these problems are not similar in terms of how well they can be approximated: it is possible to provide very good approximate solutions to `Euclidean TSP`, which is not the case for `Symmetric TSP`.

Packing and partitioning problems

- `Set Packing`: given a set U of a elements, a collection $S_1, S_2, \ldots, S_b$ of subsets of U, and a number k, is there a collection of at least k subsets such that no two of them intersect?

- `3D-Matching`: Given disjoint sets B, G, H, each of size n, and a set of triples $T \subseteq B \times G \times H$, is there a set of n triples in T, no two of which have an element in common?

  Reduction from `3SAT`.

Numerical problems

- `Subset sum`: Given natural numbers $w_1, \ldots, w_n$ and a (large) target weight W, is there a subset of $w_1, \ldots, w_n$ that adds up exactly to W?

  Applications: cryptography, scheduling

- `Minimum-weight solution to linear equations`: Given a system of linear equations in n variables with integer constants, and an integer $B \leq n$, does it have a rational solution with at most B non-zero entries?

  Applications: coding theory, signal processing

Similar problems with very different complexities

$\mathcal{NP}$-complete	$\mathcal{P}$
max cut	min cut
longest path	shortest path
3D matching	matching
Hamiltonian cycle	Euler cycle
3-colorability	2-colorability
3-SAT	2-SAT
LCS of n sequences	LCS of 2 sequences

More on $\mathcal{NP}$-completeness:

- *Computers and Intractability: A guide to the theory of $\mathcal{NP}$-completeness*, by Garey and Johnson
- *Computational Complexity*, by C. Papadimitriou

Today

Integer Programming

Integer programming (IP(D)): Given a system of linear inequalities in n variables and m constraints with integer coefficients and a integer target value k, does it have an integer solution of value k?

- Applications: production planning, scheduling trains, etc.

Example:

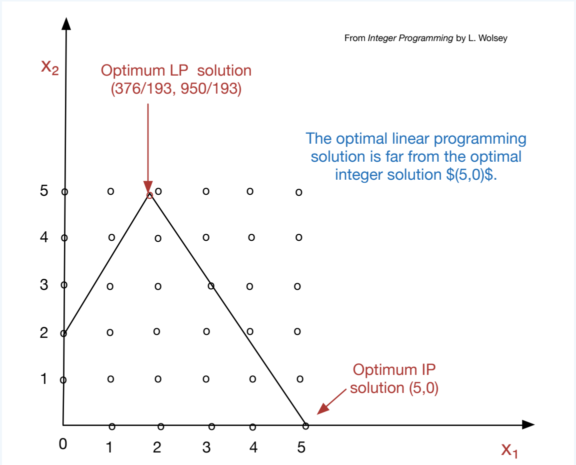
$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \in \mathbf{Z}^n \end{aligned}$$

Here A is an $m \times n$ matrix, $\mathbf{b} \in \mathbf{R}^m$, $\mathbf{c} \in \mathbf{R}^n$, $\mathbf{x}$ is an integer vector with n components.

What does the set of feasible solutions look like?

Rounding the LP is often insufficient

$$\begin{aligned} \max_{x_1 \geq 0, x_2 \geq 0} \quad & 1.00x_1 + 0.64x_2 \\ \text{subject to} \quad & 50x_1 + 31x_2 \leq 250 \\ & 3x_1 - 2x_2 \geq -4 \\ & x_1, x_2 \text{ integer} \end{aligned}$$



Is `IP(D)` *hard?*

- `IP(D)` is in $\mathcal{NP}$.
- We can quickly solve LPs with several thousands of variables and constraints but there exist integer programs with 10 variables and 10 constraints that are very hard to solve.

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- `IP(D)` is in $\mathcal{NP}$.
- We can quickly solve LPs with several thousands of variables and constraints but there exist integer programs with 10 variables and 10 constraints that are very hard to solve.

- This is not too surprising: integer programs restricted to solutions $\mathbf{x} \in \{0,1\}^n$ model **yes**/**no** decisions, which are generally hard.

- To formalize this intuition, we will reduce an $\mathcal{NP}$-complete problem to `IP(D)`.

Integer Programs for `Vertex Cover` and `IS`

First we formulate integer programs for two $\mathcal{NP}$-hard problems.

IP for `Independent Set`:

$$\begin{array}{rll} \max & \displaystyle\sum_{i=0}^{n} x_i & \\ \text{subject to} & x_i + x_j \leq 1, & \text{for every } (i,j) \in E \\ & x_i \in \{0,1\}, & \text{for every } i \in V \end{array}$$

IP for `Vertex Cover`:

$$\begin{array}{rll} \min & \displaystyle\sum_{i=0}^{n} x_i & \\ \text{subject to} & x_i + x_j \geq 1, & \text{for every } (i,j) \in E \\ & x_i \in \{0,1\}, & \text{for every } i \in V \end{array}$$

`IP(D)` is $\mathcal{NP}$-complete

Claim 3.

`VC(D)` $\leq_P$ `IP(D)`

Proof.

Reduction from arbitrary instance $(G = (V, E), k)$ of `VC(D)` to the following integer program with target value k:

$$\begin{array}{rll} \min & 0 & \\ \text{subject to} & x_i + x_j \geq 1, & \text{for every } (i,j) \in E \\ & \displaystyle\sum_{i=1}^{n} x_i \leq k & \\ & x_i \in \{0,1\}, & \text{for every } i \in V \end{array}$$

Equivalence of the instances is straightforward. □

Similar problems with very different complexities *(new)*

$\mathcal{NP}$-complete	$\mathcal{P}$
max cut	min cut
longest path	shortest path
3D matching	matching
Hamiltonian cycle	Euler cycle
3-colorability	2-colorability
3-SAT	2-SAT
LCS of n sequences	LCS of 2 sequences
integer programming	linear programming

The theory of integer and linear programming and duality
can guide the design of approximation algorithms,
and exact solutions, for hard problems.

Today

Minimum-weight Set Cover

Input

- a set $E = \{e_1, e_2, \ldots, e_n\}$ of n elements
- a collection of subsets of these elements $S_1, S_2, \ldots, S_m$, where each $S_j \subseteq E$
- a non-negative weight w_j for every subset S_j

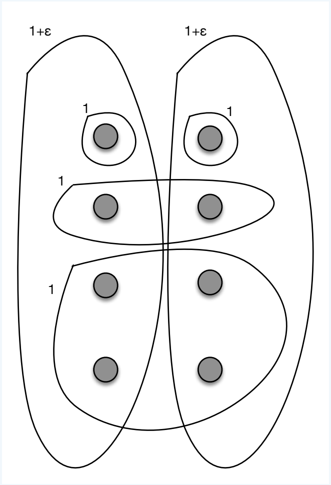
Output

A minimum-weight collection of subsets that cover all of E.

In symbols: find an $I \subseteq \{1, \ldots, m\}$ such that $\cup_{i \in I} S_i = E$ and $\sum\limits_{i \in I} w_i$ is minimum.

(`Unweighted Set Cover`: $w_j = 1$ for all j)

Example instance of `Set Cover`



$n = 8$ ground elements, $m = 6$ subsets with weights
$w_1 = w_2 = w_3 = w_4 = 1$, $w_5 = w_6 = 1 + \epsilon$.

Motivation: detect computer viruses

Motivation (IBM AntiVirus): detect features of boot sector viruses that do not occur in typical applications; then use them to *discover* more viruses

- **Ground elements:** known boot sector viruses ($n \approx 150$)
- **Sets:** labelled by some three-byte sequence occurring in these viruses but not occurring in typical computer applications ($m \approx 21000$); each set consisted of all the viruses that contained the three-byte sequence
- **Output**: a small number of such sequences—much smaller than 150—that *cover* all known viruses

$\Longrightarrow$ use the small set cover as features in a *neural classifier* to determine presence of a boot sector virus

$\Longrightarrow$ *detect* new viruses (many boot sector viruses are written by modifying existing ones)

Reduction via generalization

Claim 4.

`Set-Cover(D)` *is $\mathcal{NP}$-complete.*

Proof.

Reduction from `VC(D)`. Input instance: $(G = (V, E), k)$.

- Set $E = \{e_1, \ldots, e_m\}$ to be the set of ground elements we want to *cover*.
- For every vertex j, set S_j to be the set of edges (ground elements) that are incident to–hence *covered* by–vertex j.
- Set $w_j = 1$ for all $1 \leq j \leq n$.

Equivalence of instances: input graph has a vertex cover of size k if and only if E has a set cover of weight k. □

Forming the integer program for `Set Cover`

Variables: we introduce one variable per set S_j; intuitively,

$$x_j = \begin{cases} 1, & \text{if } S_j \text{ is included in the solution} \\ 0, & \text{otherwise} \end{cases}$$

Constraints: ensure that every element is *covered*:

for every element e_i, at least one of the sets S_j
containing e_i appears in the final solution

Objective function: minimize the sum of the weights of the sets included in the solution

An integer programming formulation of `Set Cover`

Integer program for `Set Cover`:

$$
\begin{aligned}
\min \quad & \sum_{i=0}^{m} w_j x_j \\
\text{subject to} \quad & \sum_{j : e_i \in S_j} x_j \geq 1, \quad \text{for every } 1 \leq i \leq n \\
& x_j \in \{0, 1\}, \quad \text{for every } 1 \leq j \leq m
\end{aligned}
$$

An integer programming formulation of `Set Cover`

Integer program for `Set Cover`:

$$\begin{aligned} \text{min} \quad & \sum_{i=0}^{m} w_j x_j \\ \text{subject to} \quad & \sum_{j: e_i \in S_j} x_j \geq 1, \quad \text{for every } 1 \leq i \leq n \\ & x_j \in \{0, 1\}, \quad \text{for every } 1 \leq j \leq m \end{aligned}$$

Let Z^*_{IP} be the optimum value of this integer program; OPT be the value of the optimum solution to `Set Cover`.

$$Z^*_{IP} = OPT.$$

△ We cannot solve this integer program efficiently *(why?)*.

LP relaxation: a bound for the value of the IP

LP relaxation for `Set Cover`:

$$\begin{aligned} \min_{\mathbf{x}\geq\mathbf{0}} \quad & \sum_{i=0}^{m} w_j x_j \\ \text{subject to} \quad & \sum_{j:e_i\in S_j} x_j \geq 1, \quad \text{for every } 1\leq i\leq n \end{aligned}$$

LP relaxation: a bound for the value of the IP

LP relaxation for `Set Cover`:

$$\begin{array}{rl} \min\limits_{\mathbf{x}\geq\mathbf{0}} & \displaystyle\sum_{i=0}^{m} w_j x_j \\ \text{subject to} & \displaystyle\sum_{j:e_i\in S_j} x_j \geq 1, \quad \text{for every } 1 \leq i \leq n \end{array}$$

- Every feasible solution to the original IP is a feasible solution to the LP relaxation.
- The value of any feasible solution to the original IP is the same in the LP (the objectives are the same).
- Let Z_{LP}^* be the optimum value of the LP relaxation.

$$Z_{LP}^* \leq Z_{IP}^* = OPT$$

Today

Rounding the solution to the LP

LP relaxation for `Set Cover`:

$$\begin{array}{ll} \min\limits_{\mathbf{x} \geq \mathbf{0}} & \displaystyle\sum_{i=0}^{n} w_j x_j \\ \text{subject to} & \displaystyle\sum_{j : e_i \in S_j} x_j \geq 1, \quad \text{for every } 1 \leq i \leq n \end{array}$$

- Let x^* be an optimal solution to the LP relaxation.
- Let $f_i = \#$ subsets S_j where element e_i appears.
- Let $f = \max\limits_{1 \leq i \leq n} f_i$.
- Set

$$\hat{x}_j = \begin{cases} 1, & \text{if } x_j^* \geq 1/f \\ 0, & \text{if } x_j^* < 1/f \end{cases}$$

Rounding yields a feasible solution to the original IP

The collection of sets S_j with $\hat{x}_j = 1$ cover all the elements.

- Consider the optimal solution x^* for the LP relaxation.
- Fix any element e_i; recall that e_i appears in f_i subsets.
- For simplicity, relabel these subsets as $S_1, S_2, \ldots, S_{f_i}$. Then the optimal solution satisfies the constraint

  $$x_1^* + x_2^* + \ldots + x_{f_i}^* \geq 1$$

  Let x_m^* be the maximum of $x_1^*, x_2^*, \ldots, x_{f_i}^*$. Then

  $$x_m^* \geq \frac{1}{f_i} \geq \frac{1}{f}$$

$\Rightarrow$ Our rounding procedure guarantees that, for every element e_i, at least one set S_j that *covers* e_i is chosen.

An f-approximation algorithm for `Set Cover`

How far is the solution obtained by the rounding procedure above from to the optimal solution to `Set Cover`*?*

- We do **not** know OPT!
- **But** we have a bound for it: the value Z_{LP}^* of the LP relaxation!

Recall that we set $\hat{x}_j = 1$ if and only if $x_j^* \geq 1/f$. Then

$$\begin{aligned} \sum_j w_j \hat{x}_j &\leq \sum_j w_j (f x_j^*) = f \sum_j w_j x_j^* \\ &= f \cdot Z_{LP}^* \leq f \cdot OPT \end{aligned}$$

Approximation algorithms

Definition 8.

An α-approximation algorithm for an optimization problem is a polynomial-time algorithm that, for all instances of the problem, produces a solution whose value is within a factor of α of the value of the optimal solution.

Remark 1.

- α *is the* approximation ratio *or* approximation factor
- *For minimization problems,* $\alpha > 1$.
- *For maximization problems,* $\alpha < 1$.

Examples

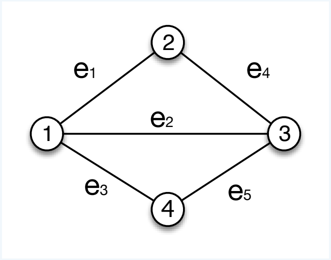
Example 1: the rounding procedure described on slide 53 gives an f-approximation algorithm for `Set Cover`:

- it can be completed in polynomial-time
- it always returns a solution whose value is at most f times the value of the optimal solution.

Remark: if an element appears in too many sets (e.g., $f = \Omega(n)$), this algorithm does not provide a good approximation guarantee.

Example 2: a 2-approximation algorithm for `VC` is a polynomial-time algorithm that always returns a solution whose value is at most twice the value of the optimal solution.

A 2-approximation algorithm for VC



- Let $E = \{e_1, \dots, e_m\}$ be the set of edges in the graph.
- Let S_j be the set of edges (ground elements) that are covered by vertex j.
- For every edge e_i, $f_i = 2$: e_i appears in exactly two subsets *(why?)*.
- Hence $f = \max\limits_{1 \le i \le m} f_i = 2$.