Lecture 2: Bandit Problem and MDP

Max(Chong) Li

Outline

- Bandit problem
- Markov Process
- Markov Reward Process
- Markov Decision Process

(Readings: Chapter 3 & 4.3 in RL-CPS book)

*Materials of MDP are modified from David Silver's RL lecture notes

n-armed bandit problem

- You need to repeatedly take a choice among n different options/actions
- You receive a numerical reward chosen from a stationary probability distribution that depends on the action you selected
- Your objective is to maximize the expected total reward over some time period.
- Invented in early 1950s by Robbins to model decision making under uncertainty when the environment is unknown

Multiple slot machines

- Each machine has a different distribution for rewards with unknown expectation
- Assume independence of successive plays and rewards across machines
- A policy is an algorithm that chooses the next machine to play based on the sequence of the past plays and obtained rewards

Questions

- If the expected reward from each machine is known, we are done: just pull the lever with the highest expected reward
- However, expected reward is unknown
- What should we do?
- Basic idea: estimate the expectation from the average of the reward received so far

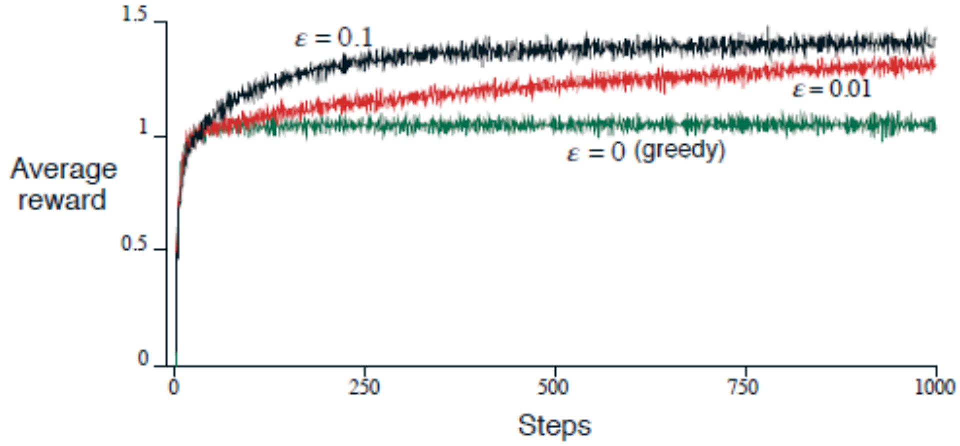
Action-Value Methods

- $Q_t(a)$: estimated value of action "*a*" at the "*t*-th" play
- K_a : times that action "*a*" has been chosen
- $R_1, R_2, \ldots, R_{K_a}$: reward received from each play
- Then, we define the action value as

$$Q_t(a) = \frac{R_1 + R_2 + \cdots + R_{K_a}}{K_a}$$

Policies

- Greedy policy
  - always choose the machine with current best expected reward $Q_t(a)$
- Ɛ-greedy policy (Total *N* machines)
  - Choose machine with current best expected reward with probability 1-Ɛ
  - Choose a random machine with probability Ɛ/*N*



Discussion

- The greedy method performs significantly worse in the long run because it often gets stuck performing suboptimal actions
- *Exploitation vs. exploration* dilemma:
  - Should you exploit the information you've learned or explore new options in the hope of greater payoff?
- Examples:
  - Restaurant selection: go to your favorite restaurant or try a new one?
  - Oil drilling: drill at the best known location or at a new location?

Softmax

- Another policy to balance exploration and exploitation
- Use Gibbs (or Boltzmann) distribution to choose action "*a*" at "*t*-th" play with probability

$$\frac{e^{Q_t(a)/\tau}}{\sum_{i=1}^{n} e^{Q_t(i)/\tau}}$$

- τ is a positive parameter called the *temperature*
  - *Large temp. -> all (nearly) equiprobable selection*
  - *Small temp. -> greedy action selection*
- *whether softmax or* Ɛ-greedy is better? depends on the task and on human factors

Algorithm Implementation

- In applications, do we need to save all rewards to compute Q_k ? *NO!*
- Incremental implementation to save memory: let Q_k be the average of its first *k-1* rewards,

$$
\begin{aligned}
Q_{k+1} &= \frac{1}{k}\sum_{i=1}^{k} R_i \\
&= \frac{1}{k}\left(R_k + \sum_{i=1}^{k-1} R_i\right) \\
&= \frac{1}{k}\Big(R_k + kQ_k - Q_k\Big) \\
&= Q_k + \frac{1}{k}\Big[R_k - Q_k\Big],
\end{aligned}
$$

- Require memory only for Q_k , k, and the new reward
- The general form

$$
NewEstimate \leftarrow OldEstimate \ + \ StepSize \Big[Target - OldEstimate\Big]
$$

- This expression will be frequently used in RL

Discussion on step size

- $\alpha_t(a)$: the step-size parameter of action "*a*" at time step "*t*"
- The following conditions (in stochastic approximation theory) are required to assure convergence of the incremental implementation with probability 1:

$$\sum_{k=1}^{\infty} \alpha_k(a) = \infty \qquad \text{and} \qquad \sum_{k=1}^{\infty} \alpha_k^2(a) < \infty.$$

- Choose constant step size for nonstationary problem
  - Weight recent reward more heavily than long-past ones

Markov Decision Process

- MDP formally describe an environment for reinforcement learning, where the environment is fully observable
- Almost all RL problems can be formalized as MDPs
  - Bandits are MDPs with one state
  - Optimal control primarily deals with continuous MDPs

Markov Property

- Definition: the future is independent of the past given the present

A state S_t is *Markov* if and only if

$$\mathbb{P}[S_{t+1} \mid S_t] = \mathbb{P}[S_{t+1} \mid S_1, ..., S_t]$$

The state captures all relevant information from the history

Once the state is known, the history may be thrown away

i.e. The state is a sufficient statistic of the future

State Transition Matrix

For a Markov state s and successor state s', the *state transition probability* is defined by

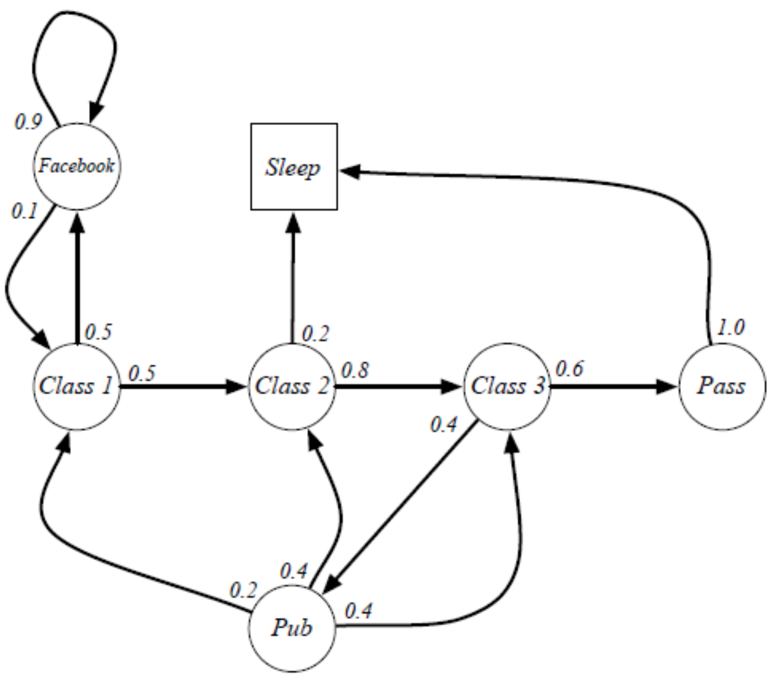
$$\mathcal{P}_{ss'} = \mathbb{P}\left[S_{t+1} = s' \mid S_t = s\right]$$

State transition matrix $\mathcal{P}$ defines transition probabilities from all states s to all successor states s',

$$\mathcal{P} \; = \begin{array}{cc} & \textit{to} \\ \textit{from} & \begin{bmatrix} \mathcal{P}_{11} & \ldots & \mathcal{P}_{1n} \\ \vdots & & \\ \mathcal{P}_{n1} & \ldots & \mathcal{P}_{nn} \end{bmatrix} \end{array}$$

where each row of the matrix sums to 1.

Example: Transition Matrix



$$\mathcal{P} = \begin{array}{cc} & \begin{array}{ccccccc} C1 & C2 & C3 & Pass & Pub & FB & Sleep \end{array} \\ \begin{array}{c} C1 \\ C2 \\ C3 \\ Pass \\ Pub \\ FB \\ Sleep \end{array} & \left[\begin{array}{ccccccc} & 0.5 & & & & 0.5 & \\ & & 0.8 & & & & 0.2 \\ & & & 0.6 & 0.4 & & \\ & & & & & & 1.0 \\ 0.2 & 0.4 & 0.4 & & & & \\ 0.1 & & & & & 0.9 & \\ & & & & & & 1 \end{array} \right] \end{array}$$

Markov Process

A Markov process is a memoryless random process, i.e. a sequence of random states $S_1, S_2, ...$ with the Markov property.

A *Markov Process* (or *Markov Chain*) is a tuple $\langle \mathcal{S}, \mathcal{P} \rangle$

- $\mathcal{S}$ is a (finite) set of states
- $\mathcal{P}$ is a state transition probability matrix,
  $\mathcal{P}_{ss'} = \mathbb{P}\left[S_{t+1} = s' \mid S_t = s\right]$

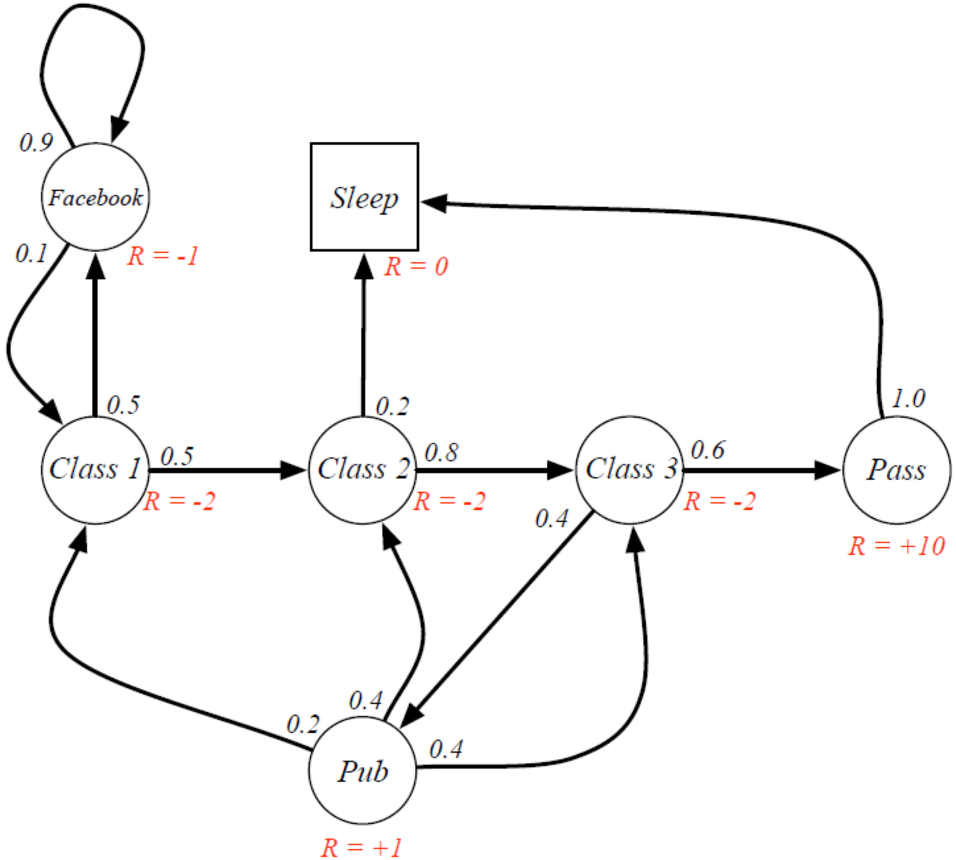
Markov Reward Process

- A Markov reward process is a Markov chain with reward values

A *Markov Reward Process* is a tuple $\langle \mathcal{S}, \mathcal{P}, \mathcal{R}, \gamma \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{P}$ is a state transition probability matrix, $\mathcal{P}_{ss'} = \mathbb{P}\left[S_{t+1} = s' \mid S_t = s\right]$
- $\mathcal{R}$ is a reward function, $\mathcal{R}_s = \mathbb{E}\left[R_{t+1} \mid S_t = s\right]$
- γ is a discount factor, $\gamma \in [0, 1]$

Example: MRP



0.9
Facebook
0.1
R = -1
Sleep
R = 0
0.5
0.2
1.0
Class 1
0.5
Class 2
0.8
Class 3
0.6
Pass
R = -2
R = -2
0.4
R = -2
R = +10
0.4
0.2
0.4
Pub
R = +1

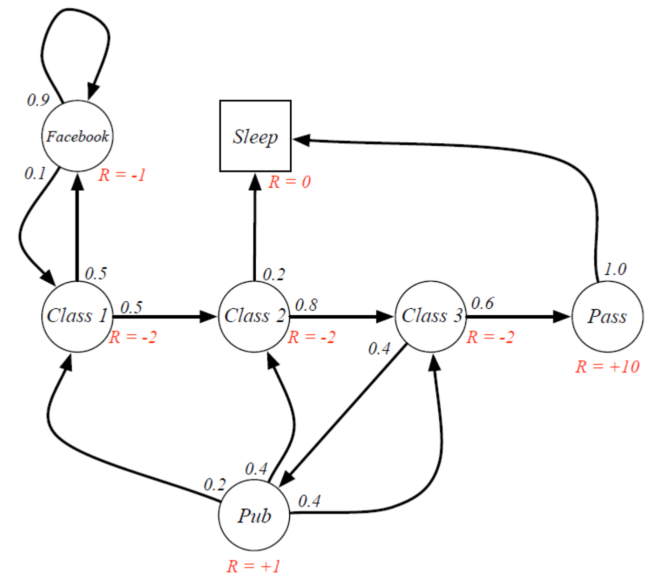
Return

The *return* G_t is the total discounted reward from time-step t.

$$G_t = R_{t+1} + \gamma R_{t+2} + ... = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

- The *discount* $\gamma \in [0, 1]$ is the present value of future rewards
- The value of receiving reward R after $k + 1$ time-steps is $\gamma^k R$.
- This values immediate reward above delayed reward.
  - γ close to 0 leads to "myopic" evaluation
  - γ close to 1 leads to "far-sighted" evaluation

State-Value Function



Sample returns for Student MRP:
Starting from $S_1 = \text{C1}$ with $\gamma = \frac{1}{2}$

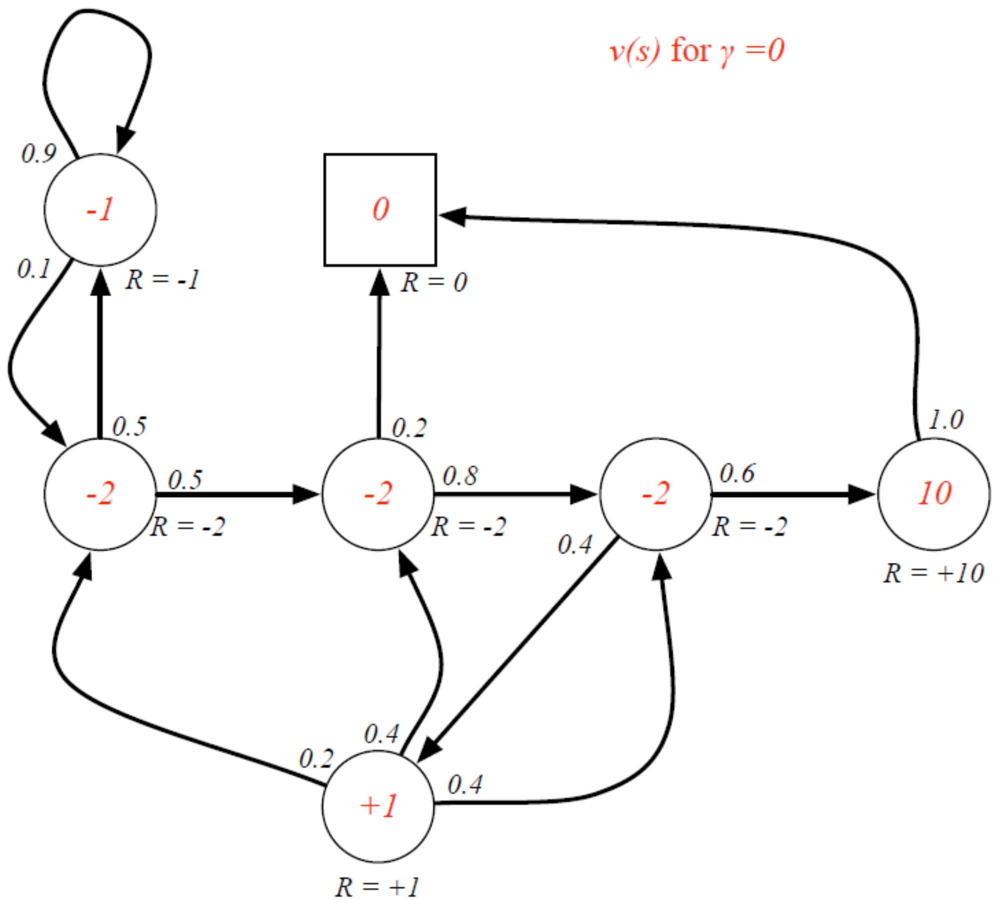
$$G_1 = R_2 + \gamma R_3 + ... + \gamma^{T-2} R_T$$

C1 C2 C3 Pass Sleep	$v_1 = -2 - 2 * \frac{1}{2} - 2 * \frac{1}{4} + 10 * \frac{1}{8}$	$=$	-2.25
C1 FB FB C1 C2 Sleep	$v_1 = -2 - 1 * \frac{1}{2} - 1 * \frac{1}{4} - 2 * \frac{1}{8} - 2 * \frac{1}{16}$	$=$	-3.125
C1 C2 C3 Pub C2 C3 Pass Sleep	$v_1 = -2 - 2 * \frac{1}{2} - 2 * \frac{1}{4} + 1 * \frac{1}{8} - 2 * \frac{1}{16} \dots$	$=$	-3.41
C1 FB FB C1 C2 C3 Pub C1 ...	$v_1 = -2 - 1 * \frac{1}{2} - 1 * \frac{1}{4} - 2 * \frac{1}{8} - 2 * \frac{1}{16} \dots$	$=$	-3.20
FB FB FB C1 C2 C3 Pub C2 Sleep			

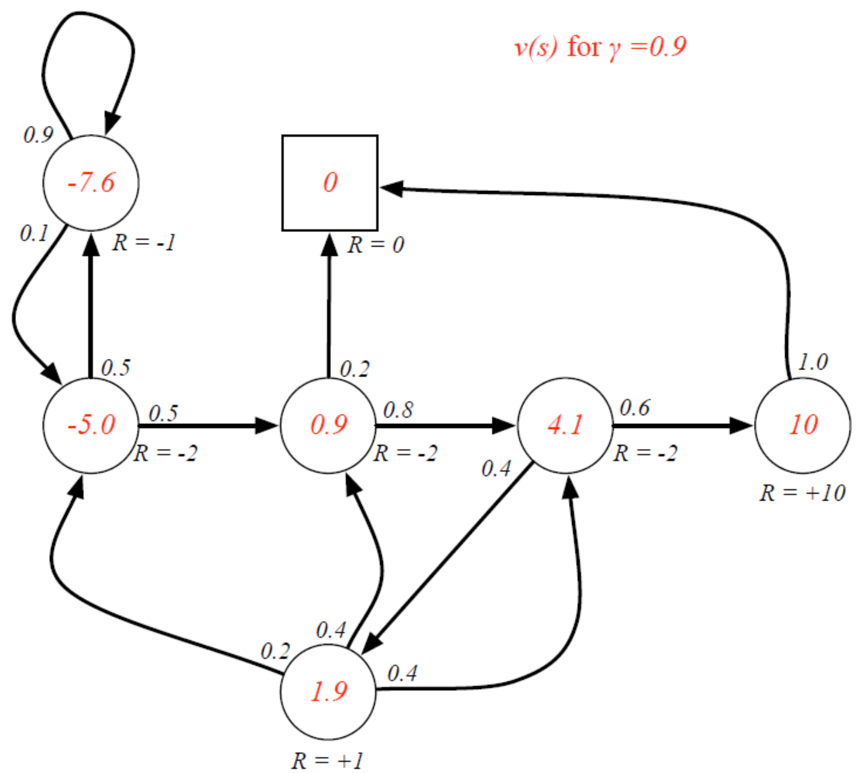
The *state value function* $v(s)$ of an MRP is the expected return starting from state s

$$v(s) = \mathbb{E}\left[G_t \mid S_t = s\right]$$

Example: state-value function



Example: state-value function



- A simple way to compute state-values ?

Bellman Equation for MRP

The value function can be decomposed into two parts:

- immediate reward R_{t+1}
- discounted value of successor state $\gamma v(S_{t+1})$

$$
\begin{aligned}
v(s) &= \mathbb{E}\left[G_t \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma \left(R_{t+2} + \gamma R_{t+3} + \dots\right) \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma G_{t+1} \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma v(S_{t+1}) \mid S_t = s\right]
\end{aligned}
$$

Bellman Equation: $v(s) = \mathcal{R}_s + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'} v(s')$

Bellman Equation in Matrix Form

The Bellman equation can be expressed concisely using matrices,

$$v = \mathcal{R} + \gamma \mathcal{P} v$$

where v is a column vector with one entry per state

$$\begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix} = \begin{bmatrix} \mathcal{R}_1 \\ \vdots \\ \mathcal{R}_n \end{bmatrix} + \gamma \begin{bmatrix} \mathcal{P}_{11} & \dots & \mathcal{P}_{1n} \\ \vdots & & \\ \mathcal{P}_{11} & \dots & \mathcal{P}_{nn} \end{bmatrix} \begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix}$$

Solving Bellman Equation

The Bellman equation is a linear equation

It can be solved directly:

$$
\begin{aligned}
v &= \mathcal{R} + \gamma \mathcal{P} v \\
(I - \gamma \mathcal{P})\, v &= \mathcal{R} \\
v &= (I - \gamma \mathcal{P})^{-1} \mathcal{R}
\end{aligned}
$$

Computational complexity is $O(n^3)$ for n states

Direct solution only possible for small MRPs

There are many iterative methods for large MRPs, e.g.

- Dynamic programming
- Monte-Carlo evaluation
- Temporal-Difference learning

Example:

- Transition Matrix =

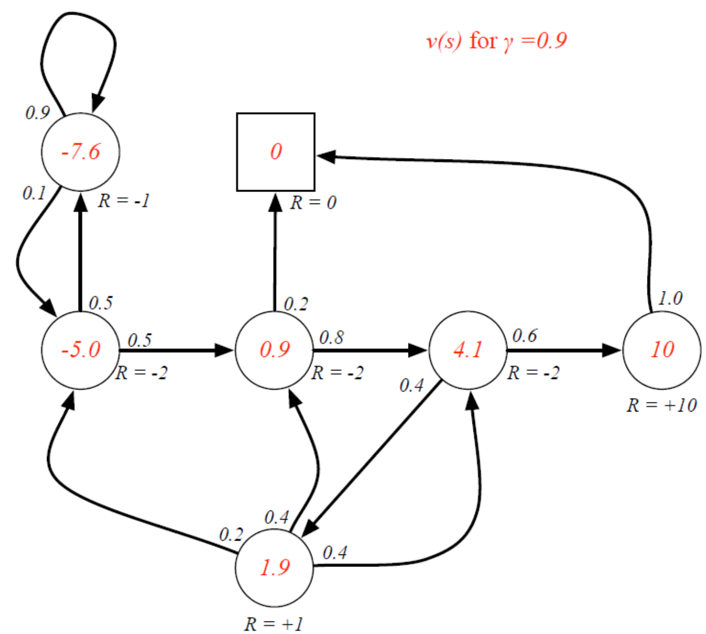
```
FB       [[ 0.9  0.1  0.   0.   0.   0.   0.  ]
Class 1   [ 0.5  0.   0.5  0.   0.   0.   0.  ]
Class2    [ 0.   0.   0.   0.8  0.   0.   0.2]
Class 3   [ 0.   0.   0.   0.   0.6  0.4  0.  ]
Pass      [ 0.   0.   0.   0.   0.   0.   1.  ]
Pub       [ 0.   0.2  0.4  0.4  0.   0.   0.  ]
sleep     [ 0.   0.   0.   0.   0.   0.   0.  ]]
```

- Reward Vector =

```
FB       [[-1]
Class 1   [-2]
Class2    [-2]
Class 3   [-2]
Pass      [10]
Pub       [ 1]
sleep     [ 0]]
```

- State Value =

```
FB       [[ -7.63760843]
Class 1   [ -5.01272891]
Class2    [  0.9426553 ]
Class 3   [  4.08702125]
Pass      [ 10.        ]
Pub       [  1.90839235]
sleep     [  0.        ]]
```



MDP

A Markov decision process (MDP) is a Markov reward process with decisions. It is an *environment* in which all states are Markov.

A *Markov Decision Process* is a tuple $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{A}$ is a finite set of actions
- $\mathcal{P}$ is a state transition probability matrix,
  $\mathcal{P}^a_{ss'} = \mathbb{P}\left[S_{t+1} = s' \mid S_t = s, A_t = a\right]$
- $\mathcal{R}$ is a reward function, $\mathcal{R}^a_s = \mathbb{E}\left[R_{t+1} \mid S_t = s, A_t = a\right]$
- γ is a discount factor $\gamma \in [0, 1]$.

Policies

A *policy* π is a distribution over actions given states,

$$\pi(a|s) = \mathbb{P}\left[A_t = a \mid S_t = s\right]$$

- A policy fully defines the behaviour of an agent
- MDP policies depend on the current state (not the history)

  i.e. Policies are *stationary* (time-independent),
  $A_t \sim \pi(\cdot|S_t), \forall t > 0$

Policies

- Given an MDP $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$ and a policy π
- The state sequence $S_1, S_2, ...$ is a Markov process $\langle \mathcal{S}, \mathcal{P}^\pi \rangle$
- The state and reward sequence $S_1, R_2, S_2, ...$ is a Markov reward process $\langle \mathcal{S}, \mathcal{P}^\pi, \mathcal{R}^\pi, \gamma \rangle$

  where

$$\mathcal{P}^\pi_{s,s'} = \sum_{a \in \mathcal{A}} \pi(a|s) \mathcal{P}^a_{ss'}$$

$$\mathcal{R}^\pi_s = \sum_{a \in \mathcal{A}} \pi(a|s) \mathcal{R}^a_s$$

Value Functions for MDP

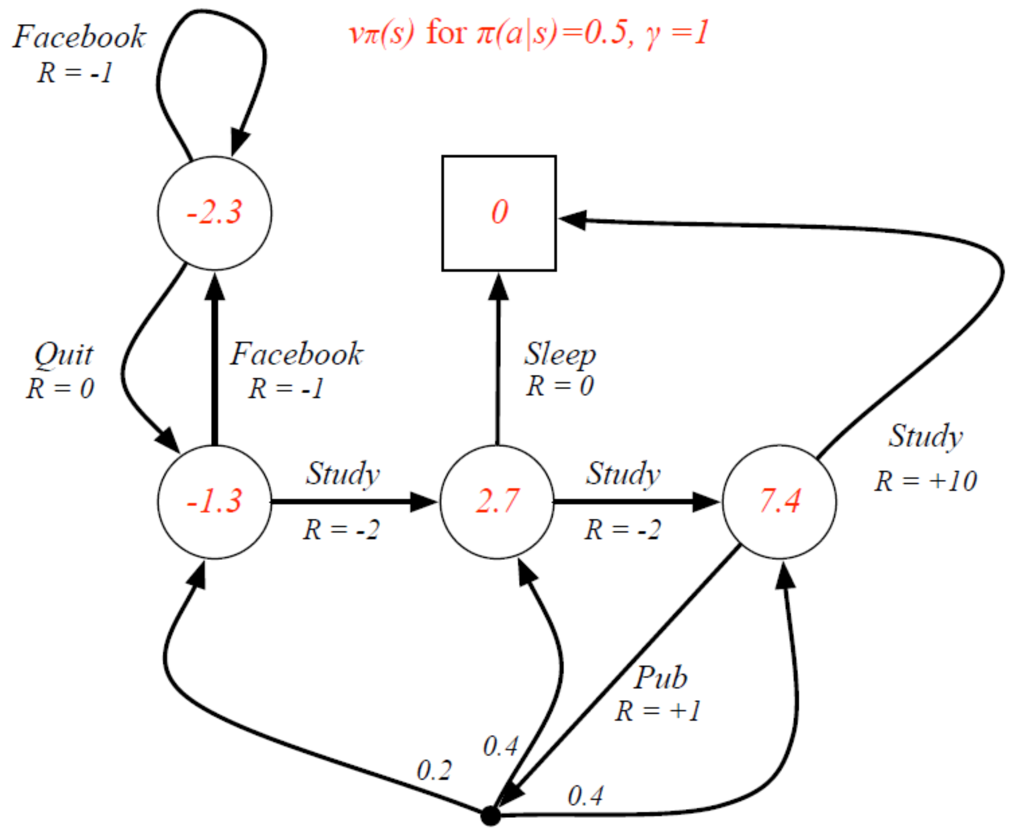
The *state-value function* $v_\pi(s)$ of an MDP is the expected return starting from state s, and then following policy π

$$v_\pi(s) = \mathbb{E}_\pi \left[G_t \mid S_t = s \right]$$

The *action-value function* $q_\pi(s, a)$ is the expected return starting from state s, taking action a, and then following policy π

$$q_\pi(s, a) = \mathbb{E}_\pi \left[G_t \mid S_t = s, A_t = a \right]$$

Example: State-Value Function for MDP



Bellman Expectation Equation

The state-value function can again be decomposed into immediate reward plus discounted value of successor state,

$$v_\pi(s) = \mathbb{E}_\pi \left[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s \right]$$

The action-value function can similarly be decomposed,

$$q_\pi(s, a) = \mathbb{E}_\pi \left[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1}) \mid S_t = s, A_t = a \right]$$

Bellman Expectation Equation

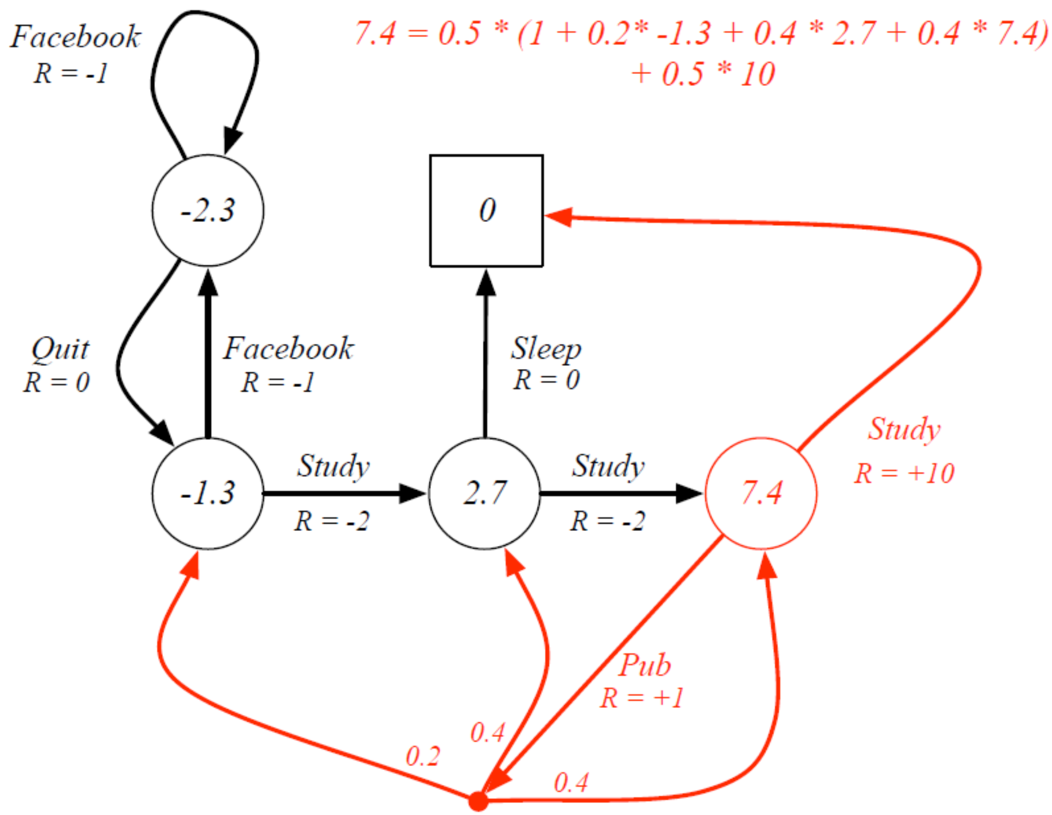
$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) q_\pi(s, a)$$

$$q_\pi(s, a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s')$$

$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s') \right)$$

$$q_\pi(s, a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a \sum_{a' \in \mathcal{A}} \pi(a'|s') q_\pi(s', a')$$

Example: Bellman Expectation Equation for MDP



Bellman Expectation Equation in Matrix Form

The Bellman expectation equation can be expressed concisely
using the induced MRP,

$$v_\pi = \mathcal{R}^\pi + \gamma \mathcal{P}^\pi v_\pi$$

with direct solution

$$v_\pi = \left(I - \gamma \mathcal{P}^\pi\right)^{-1} \mathcal{R}^\pi$$

Optimal Value Function

The *optimal state-value function* $v_*(s)$ is the maximum value function over all policies

$$v_*(s) = \max_\pi v_\pi(s)$$

The *optimal action-value function* $q_*(s, a)$ is the maximum action-value function over all policies

$$q_*(s, a) = \max_\pi q_\pi(s, a)$$

- The optimal value function specifies the best possible performance in the MDP.
- An MDP is "solved" when we know the optimal value

Optimal Policy

Define a partial ordering over policies

$$\pi \geq \pi' \text{ if } v_\pi(s) \geq v_{\pi'}(s), \forall s$$

For any Markov Decision Process

- *There exists an optimal policy* π_* *that is better than or equal to all other policies,* $\pi_* \geq \pi, \forall \pi$
- *All optimal policies achieve the optimal value function,* $v_{\pi_*}(s) = v_*(s)$
- *All optimal policies achieve the optimal action-value function,* $q_{\pi_*}(s, a) = q_*(s, a)$

Optimal Policy

An optimal policy can be found by maximising over $q_*(s, a)$,

$$\pi_*(a|s) = \begin{cases} 1 & \text{if } a = \underset{a \in \mathcal{A}}{\operatorname{argmax}}\, q_*(s, a) \\ 0 & \textit{otherwise} \end{cases}$$

- There is always a deterministic optimal policy for any MDP
- If we know $q_*(s, a)$, we immediately have the optimal policy

Bellman Optimality Equation

$$v_*(s) = \max_a q_*(s,a)$$

$$q_*(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_*(s')$$

$$v_*(s) = \max_a \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_*(s')$$

$$q_*(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a \max_{a'} q_*(s',a')$$

Solving Bellman Optimality Equation

Bellman Optimality Equation is non-linear

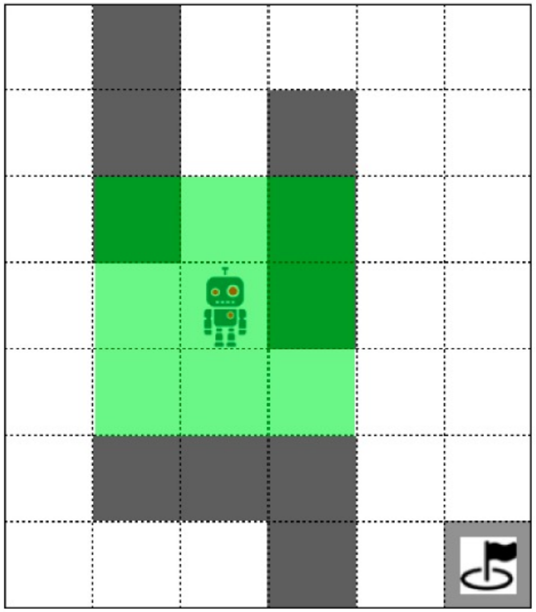
No closed form solution (in general)

Many iterative solution methods

- Value Iteration
- Policy Iteration
- Q-learning
- Sarsa

Extended MDPs

- Partially Observable MDP (POMDP)
  - Example: sailing at night w/o any navigation instruments
  - System dynamics are determined by an MDP, but the agent cannot directly observe the underlying state
  - Solved by introducing belief MDP (see wiki for more details)
- Continuous state MDP
  - Example: position, orientation and velocity of a car
  - Discretization on the continuous-state
  - Value function approximation



Given the green area as the only observable neighboring grids, the robot does not know the state, i.e., which grid on the map it is standing

Belief MDP

- Maintain a probability distribution over the states and then updates this probability distribution based on its real-time observations

A *history* H_t is a sequence of actions, observations and rewards,

$$H_t = A_0, O_1, R_1, \dots, A_{t-1}, O_t, R_t$$

A *belief state* $b(h)$ is a probability distribution over states, conditioned on the history h

$$b(h) = \left(\mathbb{P}\left[S_t = s^1 \mid H_t = h\right], \dots, \mathbb{P}\left[S_t = s^n \mid H_t = h\right]\right)$$

- Belief state satisfies Markov property, check!