Calculus

Introduction to Computing Foundations

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Calculus of infinitesimals (a.k.a.) calculus is interested in rates of change and areas under functions

Key observations:

- Rate of change of $f(x) = e^x$ at point x is also e^x
- Area under $f(x) = e^x$ up to point x is also e^x
- Rate of change of $f(x) = \ln x$ at point x is $1/x$

In general, given $f(x)$, how do we find the rate of change and area under f?

Differentiation

Differentiation

Given a function $f(x)$, the function describing its **rate of change** is derivative of f, written $f'(x)$

Process of finding derivative is differentiation

Rate of change of f at a can also be seen as the **gradient** or slope of f at $x = a$

How to find gradient of a function at a point? Estimate!

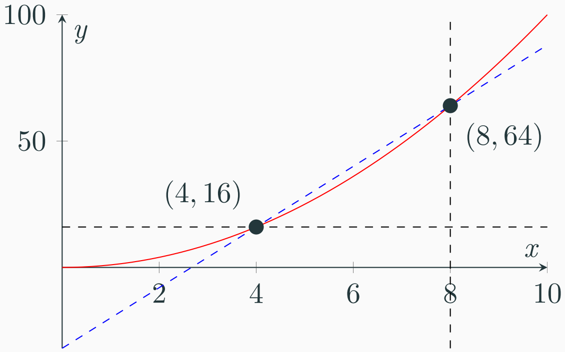

Figure 1: Graph of $y = x^2$ plotted showing intersections with line at $x = 4$ and $x = 8$.

Refine approximation by bring points closer together

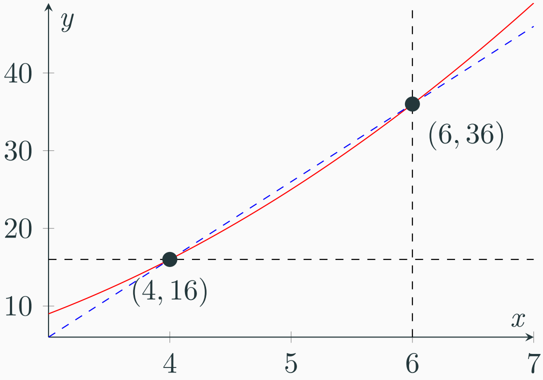

Figure 2: Graph of $y = x^2$ plotted showing intersections with line at $x = 4$ and $x = 6$.

Differentiation

For an infinitesimally small change in x (temporarily written dx), the rate of change of f at x is

$$\frac{f(x+dx)-f(x)}{dx}$$

Example, if $f(x)=x^2$, rate of change is

$$\begin{aligned}\frac{f(x+dx)-f(x)}{dx} &= \frac{(x+dx)^2-x^2}{dx}\\ &= \frac{x^2+2xdx+(dx)^2-x^2}{dx}\\ &= \frac{dx(2x+dx)}{dx}\\ &= 2x+dx\end{aligned}$$

Since dx is infinitesimally small, $2x+dx=2x$

Definition 2.1 (Derivative).

Assume that f is *differentiable* at x. Then, the derivative of f at x is given by

$$f'(x) = \frac{df}{dx}(x) = \lim_{n \to 0} \frac{f(x+n) - f(x)}{n}$$

If $y = f(x)$ we may also write

$$\frac{dy}{dx} = f'(x)$$

Analysis on limits too laborious for every function; find rules of differentiation to simplify analysis!

Goal: recap some popular rules of differentiation and show you how some are derived

Rules of Differentiation

Chain Rule

Proposition 2.1 (Derivative of composites).

$$(g \circ f)' = (g' \circ f) \times f'$$

Alternatively:

$$h(x) = g(f(x)) \Leftrightarrow h'(x) = g'(f(x)) \cdot f'(x)$$

Using $\frac{dy}{dx}$ notation, if we have $y = f(x)$ and $z = g(y)$ then

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

Exponentials and Chain Rule

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}\ln x = \frac{1}{x}$$

Goal: find derivative of any exponential function $f(x) = a^x$ for positive a.

Recall:

$$a = e^{\ln a}$$

Therefore:

$$a^x = e^{\ln a^x} = e^{x \ln a}$$

Let $u = x \ln a$. By chain rule:

$$\frac{d}{dx}a^x = \frac{d}{du}e^u \times \frac{d}{dx}u = e^{x\ln a} \times \frac{d}{dx}x \ln a$$

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Rate of change of linear function $y = mx + c$ is just gradient m, therefore

$$\frac{d}{dx}x \ln a = \ln a$$

Thus,

$$\frac{d}{dx}a^x = e^{x \ln a} \times \ln a = a^x \ln a$$

Exerise 2.1.

Find the derivative of $f(x) = 4^{(2x+3)}$.

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Solution 2.1.

Let $g(x) = 2x + 3$ and $f(u) = 4^u$.

By the chain rule:

$$(f \circ g)\,'(x) = 4^{g(x)} \ln 4 g\,'(x) = 2 \ln 4 \times 4^{(2x+3)}$$

Products

Goal: given $h(x) = f(x)g(x)$, find $h'(x)$.

$$\begin{aligned}
h'(x) &= \lim_{n\to 0} \frac{h(x+n) - h(x)}{n} \\
&= \lim_{n\to 0} \frac{f(x+n)g(x+n) - f(x)g(x)}{n} \\
&= \lim_{n\to 0} \frac{f(x+n)g(x+n) - f(x)g(x+n) + f(x)g(x+n) - f(x)g(x)}{n} \\
&= \lim_{n\to 0} \frac{(f(x+n) - f(x))g(x+n) + f(x)(g(x+n) - g(x))}{n} \\
&= \lim_{n\to 0} \left(\frac{f(x+n) - f(x)}{n} g(x+n) + f(x) \frac{g(x+n) - g(x)}{n} \right)
\end{aligned}$$

Products

$$
\begin{aligned}
&= \lim_{n\to 0}\left(\frac{f(x+n)-f(x)}{n}g(x+n)+f(x)\frac{g(x+n)-g(x)}{n}\right)\\
&= \lim_{n\to 0}\frac{f(x+n)-f(x)}{n}\cdot\lim_{n\to 0}g(x+n)+\lim_{n\to 0}f(x)\cdot\lim_{n\to 0}\frac{g(x+n)-g(x)}{n}\\
&= \lim_{n\to 0}\frac{f(x+n)-f(x)}{n}\cdot g(x)+f(x)\cdot\lim_{n\to 0}\frac{g(x+n)-g(x)}{n}\\
&= f'(x)g(x)+f(x)g'(x)
\end{aligned}
$$

Product rule: if $h(x)=f(x)g(x)$, $h'(x)=f'(x)g(x)+f(x)g'(x)$.

Quotients

Goal: given $h(x) = f(x) \div g(x)$, find $h'(x)$.

By definition, $f(x) = g(x)h(x)$, so by the product rule, $f'(x) = g'(x)h(x) + g(x)h'(x)$.

$$
\begin{aligned}
& & f'(x) &= g'(x)h(x) + g(x)h'(x) & \\
&\Leftrightarrow & g(x)h'(x) &= f'(x) - g'(x)h(x) & \triangleright \text{ rearranging terms} \\
&\Leftrightarrow & g(x)h'(x) &= f'(x) - g'(x)\frac{f(x)}{g(x)} & \triangleright \text{ by definition of } h(x) \\
&\Leftrightarrow & g(x)h'(x) &= \frac{f'(x)g(x) - g'(x)f(x)}{g(x)} & \\
&\Leftrightarrow & h'(x) &= \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} & \triangleright \textbf{ Quotient rule}
\end{aligned}
$$

Powers

Goal: find derivative of $f(x) = x^n$.

Recall:

$$a = e^{\ln a}$$

Therefore:

$$x^n = e^{\ln x^n} = e^{n \ln x}$$

Thus, Let $f(x) = e^{n \ln x}$. Then,

$$\begin{aligned} f'(x) &= e^{n \ln x} \cdot \frac{n}{x} && \triangleright \text{ by chain rule and product rule} \\ &= x^n \frac{n}{x} \\ &= n x^{n-1} && \triangleright \textbf{ Power rule} \end{aligned}$$

Sums

Goal: find derivative of $h(x) = f(x) + g(x)$.

$$
\begin{aligned}
h'(x) &= \lim_{n \to 0} \frac{h(x+n) - h(x)}{n} \\
&= \lim_{n \to 0} \frac{f(x+n) + g(x+n) - f(x) - g(x)}{n} \\
&= \lim_{n \to 0} \frac{f(x+n) - f(x) + g(x+n) - g(x)}{n} \\
&= \lim_{n \to 0} \left(\frac{f(x+n) - f(x)}{n} + \frac{g(x+n) - g(x)}{n} \right) \\
&= \lim_{n \to 0} \frac{f(x+n) - f(x)}{n} + \lim_{n \to 0} \frac{g(x+n) - g(x)}{n} \\
&= f'(x) + g'(x) && \triangleright \text{ Sum rule}
\end{aligned}
$$

Polynomials

Suppose we have $y = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$. Then,

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} c_n x^n + \frac{d}{dx} c_{n-1} x^{n-1} + \cdots + \frac{d}{dx} c_1 x + \frac{d}{dx} c_0 && \triangleright \text{ sum rule} \\
&= n c_n x^{n-1} + (n-1) c_{n-1} x^{n-2} + \cdots + c_1 && \triangleright \text{ power rule}
\end{aligned}
$$

Rules of Differentiation

Chain Rule	$h(x) = g(f(x)) \Rightarrow h'(x) = g'(f(x))f'(x)$
Exponential Rule	$f(x) = a^x \Rightarrow f'(x) = a^x \ln a$
Product Rule	$f(x) = g(x)h(x) \Rightarrow f'(x) = g'(x)h(x) + h'(x)g(x)$
Quotient Rule	$f(x) = g(x)/h(x) \Rightarrow f'(x) = (g'(x)h(x) - h'(x)g(x))/(h(x))^2$
Power Rule	$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$
Sum Rule	$f(x) = g(x) + h(x) \Rightarrow f'(x) = g'(x) + h'(x)$

Example 2.2.

Find the derivative of $f(x) = (x^2 + 2x + 4)/(x - 5)$.

By definition $f(x) = g(x)/h(x)$ where $g(x) = x^2 + 2x + 4$ and $h(x) = x - 5$.

First compute g' and h':

$$g'(x) = 2x + 2$$
$$h'(x) = 1$$

Using the quotient rule,

$$
\begin{aligned}
f'(x) &= \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2} \\
&= \frac{(2x+2)(x-5) - (x^2+2x+4)(1)}{(x-5)^2} \\
&= \frac{2x^2 - 10x + 2x - 10 - x^2 - 2x - 4}{x^2 - 10x + 25} \\
&= 1 - \frac{39}{x^2 - 10x + 25}
\end{aligned}
$$

Exercise 2.2.

Find the derivative of $f(x) = 4x^2 e^{2x}$.

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Find the derivative of $f(x) = 4x^2 e^{2x}$.

Solution 2.2.

By the product rule,

$$f'(x) = \frac{d}{dx}(4x^2) \times e^{2x} + 4x^2 \times \frac{d}{dx}(e^{2x}) = 8xe^{2x} + 8x^2 e^{2x} = (1+x)(8xe^{2x})$$

Use rules of differentiation to find other rules of differentiation and to find the derivatives of complicated functions easily

Higher Order Derivatives

Higher Order Derivatives

If we have function $f(x)$:

- Differentiating once gives $f'(x)$
- Differentiating $f'(x)$ gives $f''(x)$
- Differentiating f n times gives $f^{(n)}(x)$

Second derivatives can help us classify interesting points on a function

Stationary Points

Given $f(x) = x^2 - 4x + 10$, $x = 2$ is a minimum point, rate of change at $x = 2$ is 0

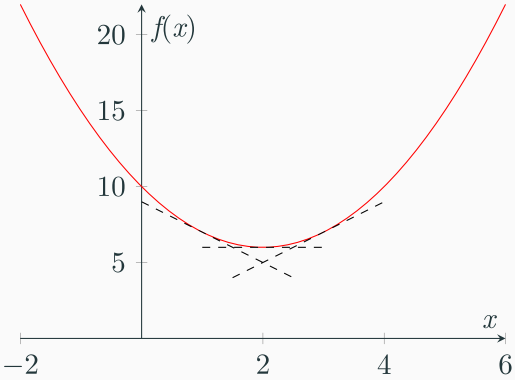

Figure 3: Graph of $f(x) = x^2 - 4x + 10$.

Given $g(x) = -2x^2 + 5x + 7$, $x = 5/4$ is a maximum point, rate of change at $x = 5/4$ is 0

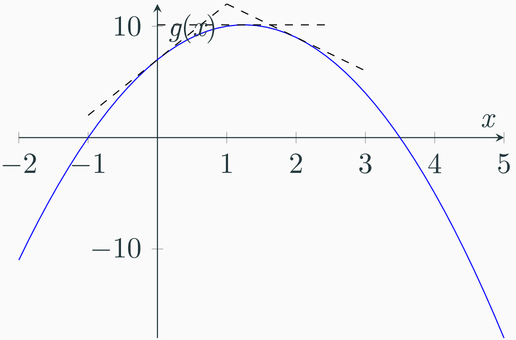

Figure 4: Graph of $g(x) = -2x^2 + 5x + 7$.

Stationary Points

How to find stationary points?

Find the root of the derivative!

$$f(x) = x^2 - 4x + 10 \Rightarrow f'(x) = 2x - 4 \Rightarrow f'(2) = 0$$

$$g(x) = -2x^2 + 5x + 7 \Rightarrow g'(x) = -4x + 5 \Rightarrow g'\left(\frac{5}{4}\right) = 0$$

How do we know if a stationary point is a minimum or maximum point?

1. Observe derivatives to the left and right of stationary point
2. Find the rate of change of gradient at stationary point, i.e. second derivative!

$$f(x) = x^2 - 4x + 10 \Rightarrow f'(x) = 2x - 4 \Rightarrow f''(x) = 2$$

$$g(x) = -2x^2 + 5x + 7 \Rightarrow g'(x) = -4x + 5 \Rightarrow g''(x) = -4$$

Since $f''(2) > 0$ and $g''(5/4) < 0$, $x = 2$ is a minimum point of f, and $x = 5/4$ is a maximum point of g!

Find the roots of derivatives to obtain stationary points

Use higher derivatives to determine if those stationary points are maxima or minima

More Calculus?

Other topics you can explore:

- Partial derivatives: differentiating an n-nary function by fixing other variables as constants
- Integration: the opposite of differentiation