Algebra

Introduction to Computing Foundations

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Equations and Inequalities

Equations

$$x = 5001$$

LHS (x, which is a *variable*) is equal to the RHS (the number 5001)

$$y = 500 + 1$$

y is equal to the quantity $500 + 1 = 501$

Numbers

$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$
Natural numbers $0, 1, 2, \ldots$	Integers $0, 1, -1, \ldots$	Rational numbers $0, 1, 1/2, -2/3, \ldots$	Real numbers $0, 1.15, \pi, \sqrt{2}, \ldots$

These are all sets or collections of numbers

Standard form: $1234.56 = 1.23456 \times 10^3$

$x \in A$ means x is a member of the collection A, $y \in \mathbb{N}$ means y is a natural number

Operations on Numbers

$a+b$	$a-b$	$a \times b$	$a \div b$	a^b
Addition	Subtraction	Multiplication	Division	Exponentiation

First four operators are left-associative:

$$a+b+c+d = ((a+b)+c)+d$$

Order of precedence follows PEMDAS

$$\begin{aligned} 1+2\times 3-4\div 5^2 &= 1+(2\times 3)-(4\div(5^2)) \\ &= 1+6-\frac{4}{25} \\ &= 6.84 \end{aligned}$$

Operations on Numbers

What is $6 \div 2(1+2)$?

ab (instead of $a \times b$) is *implicit multiplication by juxtaposition*, higher precedence than multiplication/division

Systems of Equations

Use variables as ‘placeholders’ for unknown quantities to solve for.

Construct systems of equations on these variables; once solved, quantities will be known

Example 1.1

Alice has 1.5 times as many oranges than Bob. Bob has 5 more oranges than Charlie. Finally, Alice has 15 more oranges than Charlie. How many oranges do Alice, Bob and Charlie each have?

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Alice has 1.5 times as many oranges than Bob. Bob has 5 more oranges than Charlie. Finally, Alice has 15 more oranges than Charlie. How many oranges do Alice, Bob and Charlie each have?

Let a, b and c be the number of oranges that Alice, Bob and Charlie have respectively.

$$a = 1.5b \tag{1}$$

$$b = c + 5 \tag{2}$$

$$a = c + 15 \tag{3}$$

Systems of Equations

Manipulate this system of equations via some rules:

1. If in one equation we have $X = Y$, we can substitute X for Y or Y for X in other equations
2. We can do the same operation to both sides of any equation

Systems of Equations

For example, by eq. (3), $a = c + 15$ and a occurs in eq. (1). Replace a with $c + 15$ in eq. (1):

$$c + 15 = 1.5b \tag{4}$$

Or, we can also multiply both sides of eq. (4) by 2:

$$
\begin{aligned}
& c + 15 = 1.5b && (4)\\
\Rightarrow \quad & 2(c + 15) = 2(1.5b) && \triangleright \text{ multiply both sides by 2}\\
\Rightarrow \quad & 2c + 30 = 3b && (5)
\end{aligned}
$$

Systems of Equations

Trick to solving equations: isolate one variable then substitute into another equation:

$$
\begin{aligned}
& & 2c + 30 &= 3b && && (5)\\
&\Rightarrow & (2c + 30) - 30 &= 3b - 30 && \triangleright \text{ subtract 30 from both sides} && (6)\\
&\Rightarrow & 2c &= 3b - 30 && && (7)\\
&\Rightarrow & \frac{2c}{2} &= \frac{3b - 30}{2} && \triangleright \text{ divide both sides by 2} && (8)\\
&\Rightarrow & c &= \frac{3b - 30}{2} && && (9)
\end{aligned}
$$

Systems of Equations

Now express b without any other variables by substituting eq. (9) into eq. (2):

$$
\begin{aligned}
& b = c + 5 && && (2) \\
& c = \frac{3b - 30}{2} && && (9) \\
\Rightarrow \quad & b = \frac{3b - 30}{2} + 5 && \triangleright \text{ by substitution of (9) onto (2)} && (10) \\
\Rightarrow \quad & 2b = 3b - 30 + 10 && \triangleright \text{ multiply both sides by 2} && (11) \\
\Rightarrow \quad & b = 20 && \triangleright \text{ add } 20 - 2b \text{ to both sides} && (12)
\end{aligned}
$$

Systems of Equations

Substitute $b = 20$ into original equations to solve other variables:

$$a = 1.5b \tag{1}$$

$$b = c + 5 \tag{2}$$

$$a = c + 15 \tag{3}$$

Exercise 1.1.

Solve the system of equations (1) $a = 2b$ and (2) $b = a + 3$.

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Solution 1.1.

Substitute a for $2b$ in (2):

$$b = 2b + 3$$
$$b = -3$$

Thus, $a = -6$.

What is wrong with this?

Let $a = b$. Then:

$$
\begin{aligned}
& & a &= b & & \\
&\Rightarrow & a^2 &= ab & &\triangleright \text{ multiply both sides by } a \\
&\Rightarrow & a^2 - b^2 &= ab - b^2 & &\triangleright \text{ subtract } b^2 \text{ from both sides} \\
&\Rightarrow & (a+b)(a-b) &= b(a-b) & &\triangleright \text{ factorize both sides} \\
&\Rightarrow & a+b &= b & &\triangleright \text{ divide both sides by } (a-b) \\
&\Rightarrow & b+b &= b & &\triangleright \text{ substitute } a = b \\
&\Rightarrow & 2b &= b & & \\
&\Rightarrow & 1 &= 2 & &\triangleright \text{ divide both sides by } b
\end{aligned}
$$

Inequalities

$$x \leq 1$$

This is an inequality stating that x is less than or equal to 1

Inequalities

Four kinds of inequalities:

1. $x < y$: x strictly less than y
2. $x \leq y$: x less than or equal to y
3. $x > y$: x strictly greater than y
4. $x \geq y$: x greater than or equal to y

Inequalities

Another way to denote inequalities: interval notation

$$x < y < z \Leftrightarrow x < y \text{ and } y < z$$

As intervals:

1. $x < y < z \Leftrightarrow y \in (x, z)$
2. $x < y \leq z \Leftrightarrow y \in (x, z]$
3. $x \leq y < z \Leftrightarrow y \in [x, z)$
4. $x \leq y \leq z \Leftrightarrow y \in [x, z]$

Proposition (Multiplication by -1 on inequalities).

For all real numbers x and y, if $x < y$ then $-x > -y$, and if $x \leq y$ then $-x \geq -y$.

Proof Sketch.

$$x < y \Rightarrow x + k < y + k$$

Substracting $x + y$ from both sides

$$x < y \Rightarrow x - x - y < y - x - y$$
$$\Rightarrow -y < -x$$

Argue similarly for $\leq$. $\square$

Systems of Inequalities

Systems of inequalities help solve optimization problems

Usually done with linear programming (which requires linear algebra; stay tuned!)

Use variables, mathematical operations and rules for manipulating systems of equations to solve for unknown quantities.

Functions

Functions

$$f(x) = 2x + 3$$

f is a function receiving input called x and will output $2x + 3$

Arity: number of parameters a function receives: f has arity 1

Function with arity 1 is *unary*, arity 2 is *binary*

Functions

$$f : A \to B$$

f has domain (collection of all possible inputs) A and codomain (collection of possible outputs) B

Example 1.2

$f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x + 3$ receives real numbers as inputs and produce real numbers as output.

Functions

$$g : \mathbb{R} \to \mathbb{N}$$
$$g(x) = 1$$

If $f : A \to B$ then image of A under f is the collection of all (and only) the possible outputs of f that are also members of B

Example 1.3

Let g be the following function:

$$g : \mathbb{R} \to \mathbb{N}$$
$$g(x) = 1$$

Domain of g is $\mathbb{N}$ but image of $\mathbb{R}$ under g is the collection only containing 1

Exercise 1.2.

Suppose we define a binary function $f(x, y) = x/y$. Would $\mathbb{N} \times \mathbb{N}^+ \to \mathbb{N}$ (where $\mathbb{N}^+$ represents *positive* natural numbers) be a correct description of f? If not, why?

Exercise 1.2.

Suppose we define a binary function $f(x, y) = x/y$. Would $\mathbb{N} \times \mathbb{N}^+ \to \mathbb{N}$ (where $\mathbb{N}^+$ represents *positive* natural numbers) be a correct description of f? If not, why?

Solution 1.2.

No because x/y could be a rational number and not a natural number, so the codomain of f is not $\mathbb{N}$.

How Functions Work

The function $f(x) = 2x + 3$ receives some input and produces output.

Given some number a, $f(a)$ is what you get by replacing x with a everywhere.

Example: $f(2) = 2(2) + 3 = 7$.

Plotting a Graph of a Function

Plot $f(x) = 2x + 3$ where horizontal axis is input x and vertical axis is output $f(x)$:

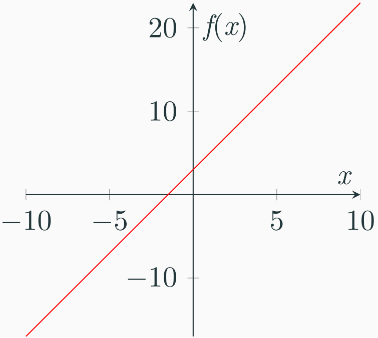

Figure 1: Graph of $f(x) = 2x + 3$.

Root of Function

Given $f(x) = 2x + 3$, what values of x is $f(x) = 0$?

$$
\begin{aligned}
& & 2x + 3 &= 0 & & \\
&\Rightarrow & 2x &= -3 & &\triangleright \text{ subtract } 3 \text{ from both sides} \qquad (13)\\
&\Rightarrow & x &= -\frac{3}{2} & &\triangleright \text{ divide both sides by } 2 \qquad (14)
\end{aligned}
$$

f intersects the x-axis when $x = -\frac{3}{2}$

This value of x is also known as the root of the equation/function

Roots are important because they are, frequently, solutions to problems formulated as equations

$f(x) = 2x + 3$ is a linear function, which is a polynomial function.

Definition 1.1 (Polynomial function).

A (uni-variate) *polynomial function* of degree n is a function in the form

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots c_1 x + c_0$$

where n is a natural number and c_n to c_0 are real numbers.

Quadratic Function

A quadratic equation is a polynomial with degree 2 in the form of $f(x) = c_2x^2 + c_1x + c_0$ with nonzero c_2.

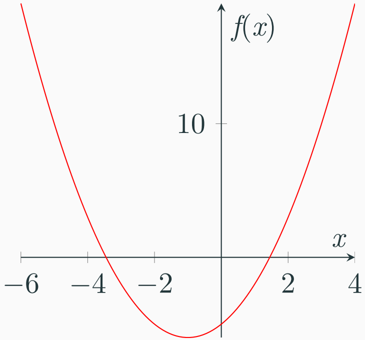

Figure 2: Graph of $f(x) = x^2 + 2x - 5$.

Root(s) of Quadratic Functions

Given quadratic function $f(x) = ax^2 + bx + c$, roots of the equation are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Why?

Root(s) of Quadratic Functions (Derivation)

$$
\begin{aligned}
& & ax^2 + bx + c &= 0 & & \\
&\Rightarrow & x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 & & \triangleright \text{ divide by } a \\
&\Rightarrow & x^2 + \frac{b}{a}x &= -\frac{c}{a} & & \triangleright \text{ subtract } \frac{c}{a} \\
&\Rightarrow & x^2 + 2\frac{b}{2a}x &= -\frac{c}{a} & & \triangleright\ \frac{x}{y} = 2\frac{x}{2y} \\
&\Rightarrow & x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 & & \triangleright \text{ add } \left(\frac{b}{2a}\right)^2 \\
&\Rightarrow & \left(x + \frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 & & \triangleright \text{ factorize LHS}
\end{aligned}
$$

Root(s) of Quadratic Functions (Derivation)

$$\Rightarrow \quad \left(x+\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \triangleright \text{ factorize LHS}$$

$$\Rightarrow \quad \left(x+\frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} \qquad \triangleright \left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2}$$

$$\Rightarrow \quad \left(x+\frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} \qquad \triangleright \frac{x}{y} = \frac{4ax}{4ay}$$

$$\Rightarrow \quad \left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2} \qquad \triangleright \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

$$\Rightarrow \quad x+\frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}} \qquad \triangleright \text{ take square roots}$$

Root(s) of Quadratic Functions (Derivation)

$$\Rightarrow \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \triangleright \text{ take square roots}$$

$$\Rightarrow \quad x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \qquad \triangleright \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

$$\Rightarrow \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \triangleright \text{ subtract } \frac{b}{2a}$$

$$\Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \triangleright \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

Root(s) of Quadratic Functions

For $f(x) = x^2 + 2x - 5$ we have $a = 1$, $b = 2$, $c = -5$, so we have two roots:

$$\frac{-2 + \sqrt{4 + 20}}{2} = \sqrt{6} - 1 \qquad \frac{-2 - \sqrt{4 + 20}}{2} = -1 - \sqrt{6}$$

N-nary Functions

A function that has arity n is also called n-nary

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

$$f(x, y) = 2x + 3y$$

$$g : \mathbb{R} \times \mathbb{N} \times \mathbb{R} \to \mathbb{R}$$

$$g(x, y, z) = x + y - z$$

Objects in the collection $\mathbb{R} \times \mathbb{R}$ are *pairs* of real numbers, like $(1.5, -3)$.

In general, things like $(1, 2)$ or $(1, 2, 3, 4)$ are known as tuples.

N-nary Functions

Functions can also produce outputs as tuples too

$$f : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$$

$$f(x) = (2x, 3x)$$

Binary Functions

Binary functions have arity 2

They can be written in different styles of notation:

- Prefix notation: $+\ x\ y$ or $+(x, y)$
- Infix notation $x + y$
- Postfix notation $x\ y\ +$

Syntax vs Semantics

Two different-looking statements can mean the same thing!

- If f is a binary function that $f(x, y)$ can be written $x\ f\ y$
- $f(x) = 2x$ is the same as $f(y) = 2y$
- If $g(x, y) = x + 2y$ and we let $* = g$ then $a * b = *(a, b) = g(a, b)$

Being able to separate **syntax** (how things look) and **semantics** (what things mean) is crucial to learning programming

Plotting Binary Functions

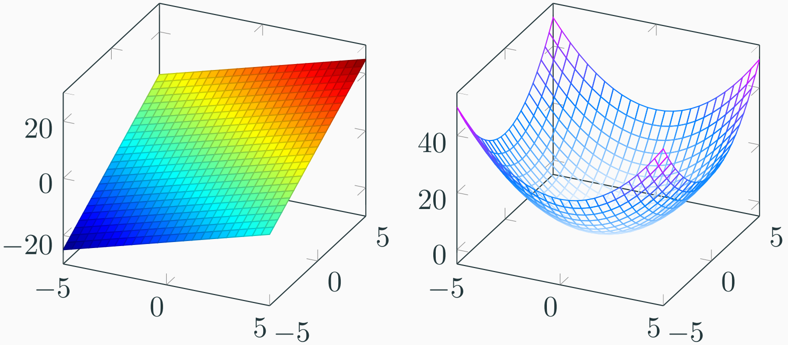

Figure 3: Example binary functions: $f(x, y) = 2x + 3y$ (left) and $g(x, y) = x^2 + y^2$ (right).

Function Composition

If machine A receives eggs and produce chicks and machine B receives chicks and produces chickens, we can make a new machine $B \circ A$ that receives eggs and produces chickens

This is known as function composition

Definition 1.2 (Function composition)

Suppose we have functions $f : A \to B$ and $g : B \to C$. The composite of f and g, denoted $g \circ f$ (read as g after f) is the function $(g \circ f) : A \to C$ where $(g \circ f)(x) = g(f(x))$.

Example 1.5

Suppose we have $h : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ and $k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ where $h(x) = (2x, 3x)$ and $k(x, y) = 2x + 3y$. Then, $k \circ h : \mathbb{R} \to \mathbb{R}$ is

$$(k \circ h)(x) = k(h(x)) = k(2x, 3x) = 4x + 9x = 13x$$

Functions are like machines that receive input and produce output;

We can easily find the roots of linear and quadratic functions;

Functions can be plotted on graphs and composed

Exponentials, Logarithms and Absolutes

If we started with \$1 in a bank that pays 10% interest per year compounded annually, how much money would we have after n years?

$$\$(1 + 1/10)^n$$

Exponential Functions

$$f(x) = 2^x$$

f is an exponential function

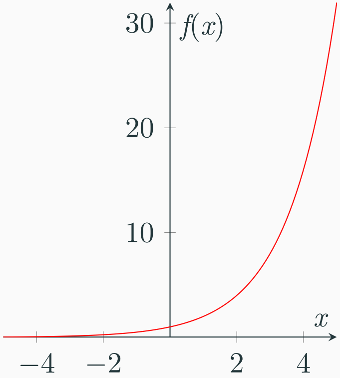

Figure 4: Graph of $f(x) = 2^x$.

Exponential Functions

Suppose we have \$1 in a bank giving 100% interest annually.

If in a year, the bank compounds our interest n times, we would have $\$(1 + 1/n)^n$ after one year.

What if bank compounds interest **continuously**?

We would have $\$2.71828\ldots$, which is Euler's number e

$$\exp x = e^x$$

Inverses

Definition (Inverse function).

For a function $f(x)$, its inverse f^{-1} is a function where $(f^{-1} \circ f)(x) = x$

Example (Inverse).

The inverse of $f(x) = 2x$ is $f^{-1}(x) = x/2$.

Logarithms

The inverse of $f(x) = a^x$ is $f^{-1}(b) = \log_a b$

$$a^x = b \Leftrightarrow \log_a b = x$$

$$e^x = a \Leftrightarrow \log_e a = \ln a = x$$

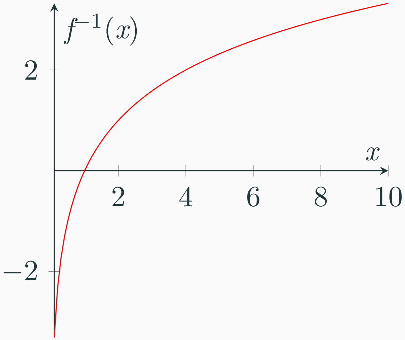

Figure 5: Graph of $f^{-1}(x) = \log_2 x$.

If we started with one cell and every cell divides into 2 every day: how many cells would we have after n days? How many days required to get m cells?

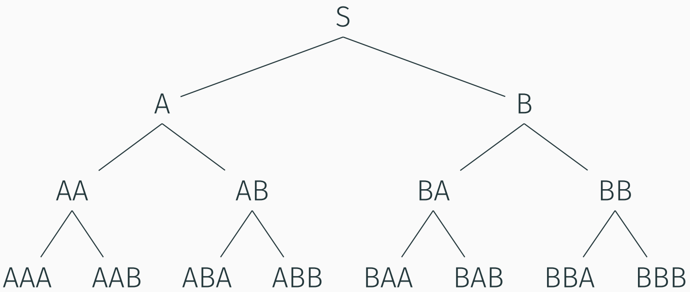

Figure 6: Tree showing cell division after 3 days.

Exponentials and Logarithms

- If we had a piece of paper that only had space for n digits, what is the largest number we can write on it?

$$10^n - 1$$

- Given a number m, what is least amount of space needed to write it?

$$\lceil \log_{10} m \rceil$$

$\lceil x \rceil$: ceiling of x, rounding x **up** to nearest integer

$\lfloor x \rfloor$: floor of x, rounding x **down** to nearest integer

Laws of exponentials and logarithms

$$a^b a^c = a^{b+c} \qquad \left(a^b\right)^c = a^{bc} \qquad a^c b^c = (ab)^c$$

$$\frac{a^b}{a^c} = a^{b-c} \qquad \frac{a^c}{b^c} = \left(\frac{a}{b}\right)^c \qquad a^0 = 1$$

$$\log_a b + \log_a c = \log_a (bc) \qquad \log_a b - \log_a c = \log_a \frac{b}{c} \qquad \log_a b = \frac{\log_c b}{\log_c a}$$

$$\log_a (b^x) = x \log_a b \qquad a^{\log_a x} = x \qquad \log_a a = 1$$

$$x = y \Leftrightarrow a^x = a^y \Leftrightarrow \log_a x = \log_a y$$

Absolutes

"Size" of a number

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise} \end{cases}$$

$$|2| = |-2| = 2$$

$$|x| \geq 0 \qquad |xy| = |x||y| \qquad |f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{otherwise} \end{cases}$$

$$|x^n| = |x|^n \qquad |-x| = x \qquad |x| = y \Leftrightarrow x = y \text{ or } -x = y$$

Absolutes

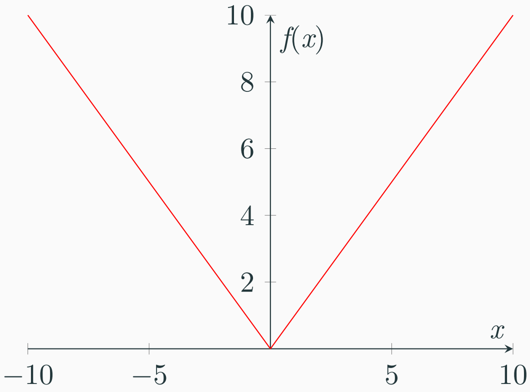

Figure 7: Graph of $f(x) = |x|$.

Exponential functions model exponential growth; its inverse is the logarithm.

Absolute values are the ‘size’ of the value

Series

Complete the sequences:

$$3, 5, 7, 9, \ldots$$

$$1, 2, 4, 8, \ldots$$

Series

We can construct a sequence of numbers $u_0, u_1, u_2, \ldots$

Summation:

$$\sum_{x=0}^{n} u_x = u_0 + u_1 + \cdots + u_{n-1} + u_n$$

Product:

$$\prod_{x=0}^{n} u_x = u_0 \times u_1 \times \cdots \times u_{n-1} \times u_n$$

Series

Factorial

$$n! = n \times (n-1) \times \cdots \times 2 \times 1 = \prod_{i=1}^{n} i$$

For example, $4! = 1 \times 2 \times 3 \times 4 = 24$

e

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \frac{1}{1 \times 2 \times 3 \times 4} + \ldots$$

Arithmetic Progression

$$3, 5, 7, 9, \ldots$$

This is an arithmetic progression because difference between adjacent terms is always constant (2)

$$u_i = u_{i-1} + c$$

The i^{th} term is $f(i)$ where i starts from 0:

$$f(x) = cx + u_0$$

Arithmetic Progression

Let the i^{th} term be $f(i)$ where $f(x) = 2x + 3$

$$3, 5, 7, 9, 11, 13, 15, 17, 19, 21, \ldots$$

What is the sum of the first 10 terms?

$$(3 + 21) + (5 + 19) + \cdots + (11 + 13) = 5 \times 24 = 120$$

For any AP with each term given by $f(x) = ax + b$, sum of first n terms is

$$\sum_{x=0}^{n-1} (ax + b) = \frac{n}{2} (a(n-1) + 2b) = \frac{an^2 - an + 2bn}{2}$$

What is the sum of all terms of AP?

As n gets larger, $|(an^2 - an + 2bn)/2|$ also gets larger (unless $a = b = 0$)

Sum of all terms of AP is infinite, series is divergent

Geometric Progression

$$1, 2, 4, 8, \ldots$$

This is a geometric progression because ratio of adjacent terms is always constant (2)

$$u_i = c \times u_{i-1}$$

The i^{th} term is $f(i)$:

$$f(x) = u_0 \times c^x$$

Suppose we have GP where each term is $f(x) = ab^x$

Let $S(n)$ be sum of first n terms of GP.

$$S(n) = a + ab + ab^2 + \cdots + ab^{n-1}$$

Multiply both sides by b:

$$b \times S(n) = ab + ab^2 + \cdots + ab^n$$

Geometric Progression

$$S(n) = a + ab + ab^2 + \cdots + ab^{n-1}$$
$$b \times S(n) = ab + ab^2 + \cdots + ab^n$$

Take top minus bottom:

$$S(n) - b \times S(n) = a - ab^n$$
$$S(n) - b \times S(n) = a(1 - b^n)$$
$$(1 - b)(S(n)) = a(1 - b^n)$$
$$S(n) = \frac{a(1 - b^n)}{1 - b}$$

Geometric Progression

$$\frac{a(1-b^n)}{1-b}$$

What is the sum of all terms of a GP?

- If $|b| > 1$, as n gets infinitely large, $|b^n|$ also gets infinitely large, so series is divergent
- If $|b| < 1$, as n gets larger, b^n gets closer to 0!

$$\lim_{n \to \infty} b^n = 0 \qquad |b| < 1$$

Thus,

$$\sum_{x=0}^{\infty} ab^x = \frac{a}{1-b} \qquad |b| < 1$$

When $|b| < 1$, GP is convergent

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$$

Recurrence Relations

In sequences, terms can depend on previous terms

AP: $u_i = u_{i-1} + c$

GP: $u_i = u_{i-1} \times c$

In general, when a term depends on previous terms, the sequence of terms is a recurrence relation

$$n! = \prod_{i=1}^{n}$$

Alternative formulation: We have

$$\begin{aligned}(n+1)! &= (n+1) \times n \times (n-1) \times \cdots \times 1 \\ &= (n+1) \times n!\end{aligned}$$

Therefore,

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \times (n-1)! & \text{otherwise} \end{cases}$$

Recurrence Relations

Fibonacci Sequence: $1, 1, 2, 3, 5, 8, 13, \ldots$

$$\mathrm{fib}_n = \begin{cases} 1 & \text{if } n = 0 \text{ or } 1 \\ \mathrm{fib}_{n-1} + \mathrm{fib}_{n-2} & \text{otherwise} \end{cases}$$