Geometry

Introduction to Computing Foundations

Foo Yong Qi

11 January 2024 | Day 1 AM Session 3

Department of Computer Science
School of Computing
National University of Singapore

Table of contents

Geometry

Geometry is all about lines, curves and shapes!

Trigonometry

Right-Angled Triangles

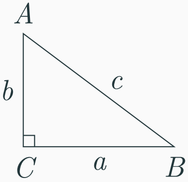


Figure 1: A right-angled triangle.

Pythagoras' Theorem

Let a, b and c be the lengths of the three sides of a right-angled triangle, where c is the length of its hypotenuse. Then,

$$a^2 + b^2 = c^2$$

Example 3.1.

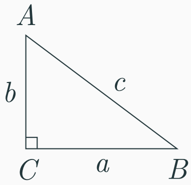
If $a = 4$ and $b = 3$, what is the length of c?

By Pythagoras' Theorem,

$$c^2 = 4^2 + 3^2 = 25$$

$$c = 5$$

Notations



$$|AB| = c$$

$$\pi \text{ rad} = 180^\circ$$

$$\angle ACB = \frac{\pi}{2} \text{ rad}$$

Similarity

Two shapes are similar if one shape can be obtained from the other by uniform scaling (shrinking or enlarging), rotation, translation or reflection.

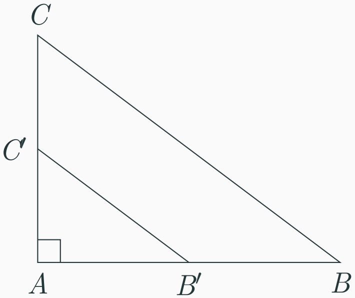


Figure 2: Two similar right-angled triangles.

Similarity

Two triangles are similar if each corresponding angle is equal, i.e. $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$ and $\angle CAB = \angle FDE$.

In fact, two triangles are similar if two pairs of corresponding angles are equal.

Why?

Angles of a Polygon

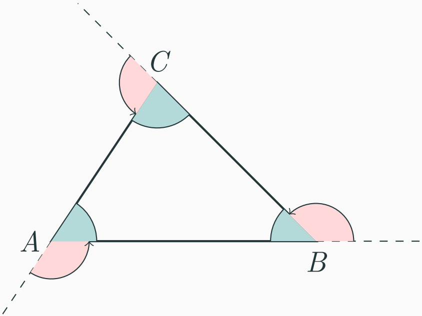


Figure 3: Internal and external angles of a triangle.

Suppose polygon has n sides

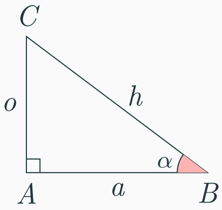
Sum of pink angles is 2π rad

Sum of all coloured angles is $\pi \times n$ rad

Sum of teal angles is $\pi \times n - 2\pi = (n-2)\pi$ rad

Sum of all internal angles of a triangle is always π rad!

sin, cos, tan



$$\sin\alpha = \frac{o}{h} \qquad \cos\alpha = \frac{a}{h} \qquad \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{o}{a}$$

$$\arcsin\frac{o}{h} = \arccos\frac{a}{h} = \arctan\frac{o}{a} = \alpha$$

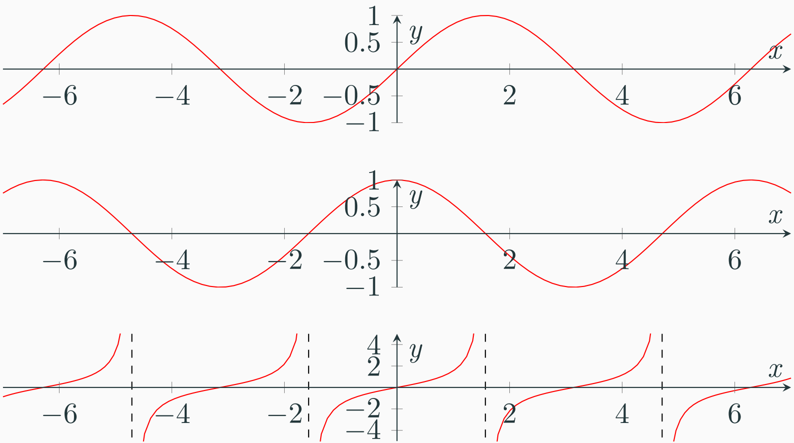
$$\sin\beta = \sin\left(\frac{\pi}{2} - \alpha\right) = \frac{a}{h} = \cos\alpha \qquad \cos\beta = \cos\left(\frac{\pi}{2} - \alpha\right) = \frac{o}{h} = \sin\alpha$$

sin, cos, tan

θ	0° (0 rad)	30° $\left(\frac{\pi}{6}\text{ rad}\right)$	45° $\left(\frac{\pi}{4}\text{ rad}\right)$	60° $\left(\frac{\pi}{3}\text{ rad}\right)$	90° $\left(\frac{\pi}{2}\text{ rad}\right)$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Table 1: Special values for sin, cos and tan.

sin, cos, tan



Analytic Geometry

Points, lines, curves and shapes can be represented as equations on a graph

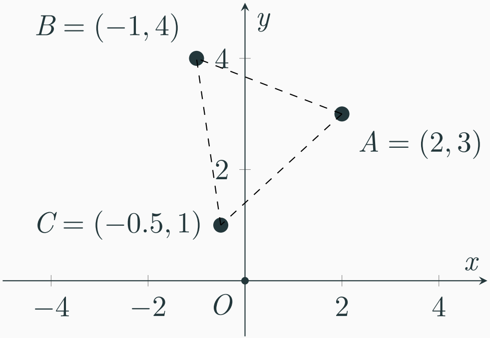


Figure 4: A triangle ABC.

A line is formed by linear equation $y = mx + c$, where m is the *gradient* and c is the value of y when $x = 0$

Given two points (a, b) and (a', b'), how do we find the equation of the line segment intersecting them?

Construct a system of linear equations!

Line Segments

Two points (a, b) and (a', b') must satisfiy line equation:

$$b = am + c$$
$$b' = a'm + c$$

From the first equation we have $c = b - am$, so:

$$\begin{aligned} && b' &= a'm + b - am \\ \Leftrightarrow && b' - b &= a'm - am \\ \Leftrightarrow && b' - b &= m(a' - a) \\ \Leftrightarrow && m &= \frac{b' - b}{a' - a} \end{aligned}$$

As expected, gradient is difference in y-values over difference in x-values

Exercise 3.2.1

Find the equation of the line intersecting $(-1, 1)$ and $(2, -4)$.

Exercise 3.2.1.

Find the equation of the line intersecting $(-1, 1)$ and $(2, -4)$.

Solution 3.2.1.

Gradient is $(-4 - 1)/(2 - -1) = -5/3$.

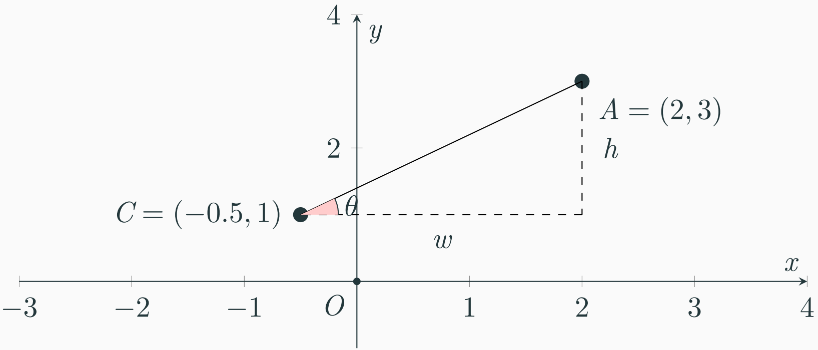
Solving for the y-intercept, $c = -4 - (-5/3)(2) = -2/3$.

Therefore, equation of line is

$$y = -\frac{5}{3}x - \frac{2}{3}$$

Line Segments

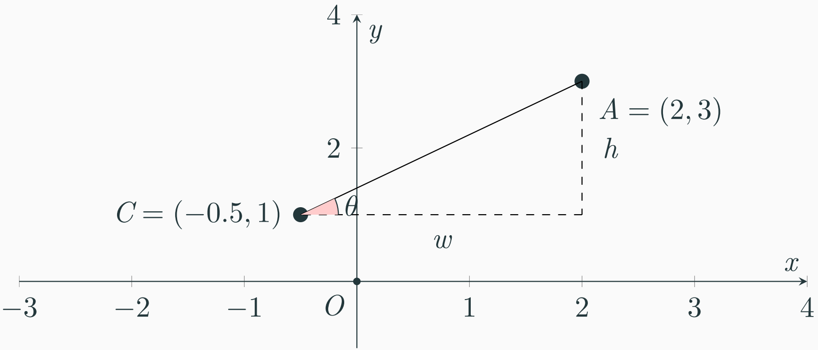
How do we find the length of a line segment between two points?



$$|AC|^2 = w^2 + h^2$$

Line Segments

What about the angle made by a line segment?



$$\tan\theta = \frac{h}{w}$$

Line Segments

The line segment intersecting two points (a, b) and (a', b') has:

- length $\sqrt{(a' - a)^2 + (b' - b)^2}$
- angle $\arctan m$ where m is gradient of line segment

Angles are at the leftmost point, with respect to x-axis. Lines going to the top right have positive angles, lines going to the bottom right have negative angles.

Exercise 3.2.2

Find the length and angle of the line intersecting $(-1, 1)$ and $(2, -4)$.

Exercise 3.2.2.

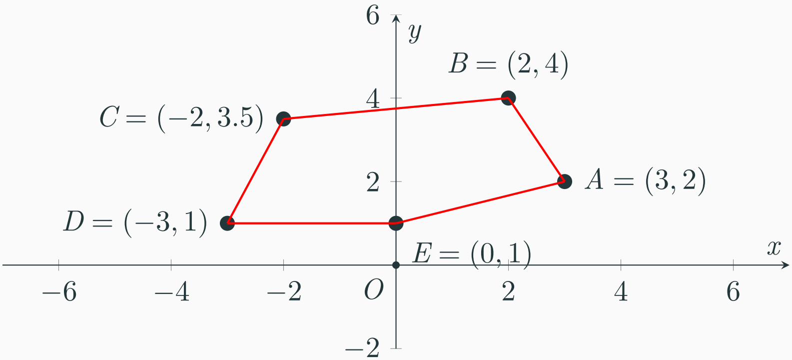
Find the length and angle of the line intersecting $(-1, 1)$ and $(2, -4)$.

Solution 3.2.2.

By Pythagoras' Theorem the length of the line segment = $\sqrt{(2 - -1)^2 + (-4 - 1)^2} \approx 5.83$.

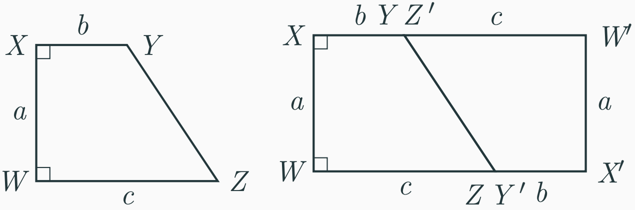
From earlier, gradient is $-5/3$, so angle made by the line segment is $\arctan -5/3 \approx -1.03$ rad.

How do we find the area of a polygon?



Area of Polygon

How do we find the area of a **trapezium** with two adjacent right angles?



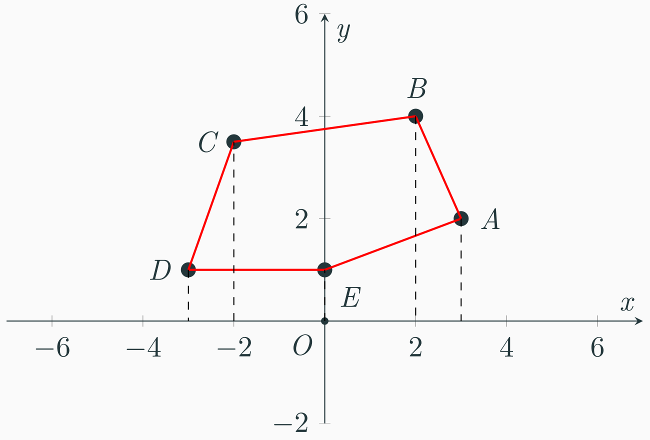
Put two of them together to get a rectangle!

Area of $WXYZ$:

$$\frac{1}{2}(a)(b+c)$$

Area of Polygon

Draw vertical lines from each point to get trapeziums; **signed** sum of area of trapeziums is area of polygon!



Area of Polygon

Area of trapezium under two points from (a, b) to (a', b') is

$$\frac{1}{2}(b' + b)(a' - a)$$

Suppose we had n points $(x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$ going **clockwise**, then the area of the polygon is

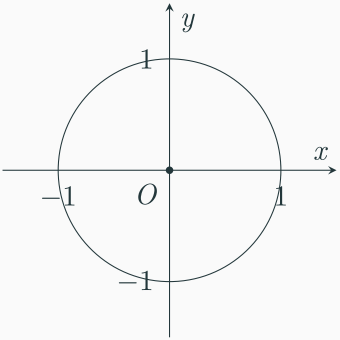
$$\frac{1}{2}(y_1 + y_0)(x_1 - x_0) + \cdots + \frac{1}{2}(y_n + y_{n-1})(x_n - x_{n-1}) = \frac{1}{2}\sum_{i=0}^{n-1}(y_{i+1} + y_i)(x_{i+1} - x_i)$$

Known as **shoelace formula**

Circles

To draw circle of radius R, each (x, y) point of circle has distance R from $(0, 0)$, i.e $x^2 + y^2 = R^2$

This is the equation of a circle!



How do we draw a circle centered somewhere else? In general, how do we move or translate a graph?

Circles

Suppose we have $y = f(x)$, and we want to translate the graph upwards by b.

Add b to the RHS!

$$y = f(x) + b \Leftrightarrow y - b = f(x)$$

Replacing y with $y - b$, our graph moves upwards by b.

Circles

Suppose we have $x = f(y)$, and we want to translate the graph rightwards by a.

Add a to the RHS!

$$x = f(y) + a \Leftrightarrow x - a = f(y)$$

Replacing x with $x - a$, our graph moves rightwards by a.

Circles

To draw a circle centered at (a, b) is to move our normal circle rightwards by a and upwards by b, giving us the general circle equation

$$(x - a)^2 + (y - b)^2 = R^2$$