Sets and Numbers

Introduction to Computing Foundations

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12 January 2024 | Day 2 AM Session 1

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Set Theory

Set Theory is all about sets, which are collections of things

Set theory forms the foundation of all of mathematics

Many important and relevant mathematical disciplines are built on top of set theory

Sets

Sets

A set is an *unordered* collection of *unique* objects, known as elements.

Each element can only occur once in a set, and the order of the elements in the set is irrelevant.

A multiset is a set that can contain duplicate elements.

Set Roster Notation

$$\{1, 2, 3\}$$

Set containing 1, 2 and 3

$$\{4, \{5\}\}$$

Set containing 4 and a set that contains 5

$$\{6, \texttt{'abc'}\}$$

Set containing 6 and *'abc'*

Sets

$$\{1, 2, 3\} = \{3, 2, 1\} = \{1, 1, 2, 3\}$$

In sets, order and duplicates are irrelevant

For multisets:

$$\{1, 2, 3\} = \{3, 2, 1\} \neq \{1, 1, 2, 3\}$$

Order is irrelevant but duplicates are

Cardinality of Sets

The cardinality of a set is its number of elements

$$|\{1,2,3\}| = |\{1,2,3,1\}| = 3$$

For multisets,

$$|\{1,2,3\}| \neq |\{1,2,3,1\}| = 4$$

Empty set

$$\emptyset = \{\}$$

Empty set has no elements, is unique

$$|\emptyset| = 0$$

Set Membership

$$x \in A$$

Proposition that x is an element of A

$$x \notin A$$

Proposition that x is **not** an element of A

- $1 \in \{1, 2, 3\}$ is true
- $4 \in \{1, 2, 3\}$ is false
- $4 \notin \{1, 2, 3\}$ is true

Set Equality

Two sets are equal if and only if they have the same elements.

$$\{1, 2, 3\} = \{3, 2, 1\} \neq \{4, 5\}$$

$$\emptyset \neq \{\emptyset\}$$

Subsets

A is a subset of B if and only if every element of A is also an element of B

If A is a subset of B but is not equal to B, then A is a proper subset of B

The opposite of subset is superset

$$\{1, 2\} \subseteq \{1, 2, 3\} \text{ and } \{6, 5, 4\} \supseteq \{4, 5, 6\}$$

$$\{1, 2\} \subset \{1, 2, 3\}, \text{ but } \{6, 5, 4\} \not\supset \{4, 5, 6\}$$

Subsets

Is $\mathbb{N} \subset \mathbb{Z}$?

Yes because every natural number is an integer, but negative integers are not natural numbers

Set Builder

We can construct a subset of another set by keeping only the elements satisfying a condition

$$\{x \in A \mid \phi(x)\}$$

A new set containing all elements x in A such that $\phi(x)$ is a true statement

Example 4.2

Let $A = \{1, 2, 3, 4\}$. We can construct a subset B of A containing only the even numbers

$$B = \{x \in A \mid x \text{ is even}\} = \{2, 4\}$$

Set Builder

We can even obtain the image of such a subset under a function

$$\{f(x) \mid x \in A, \phi(x)\}$$

A new set containing $f(x)$ for all elements x in A such that $\phi(x)$ is a true statement

Example 4.3

Let $A = \{1, 2, 3, 4\}$. We can construct a subset B of A containing only the even numbers, then increase each number by 1. Let $f(x) = x + 1$, then

$$B = \{f(x) \mid x \in A, x \text{ is even}\} = \{x + 1 \mid x \in A, x \text{ is even}\} = \{3, 5\}$$

Exercise 4.1.

Let $A = \{0, 1, 2, 3\}$. What is the set $\{x + 2 \mid x \in A, x > 1\}$?

Set Builder

Exercise 4.1.

Let $A = \{0, 1, 2, 3\}$. What is the set $\{x + 2 \mid x \in A, x > 1\}$?

Solution 4.1.

$$\{x \in A \mid x > 1\} = \{2, 3\}$$

$$\{x + 2 \mid x \in A, x > 1\} = \{4, 5\}$$

Power Sets

The power set of a set A is the set of all subsets of A

$$\mathcal{P}(A) = \{x \mid x \subseteq A\}$$

Example 4.4

Let $A = \{1, 2, 3\}$. Then, $\mathcal{P}(A)$ contains all the subsets of A, i.e.

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Power Sets

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Let $|A| = n$, What is $|\mathcal{P}(A)|$?

To construct a subset, two choices for each element: (1) include, (2) exclude

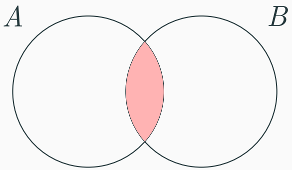
There are n elements, so there are 2^n ways to create a subset

$$|\mathcal{P}(A)| = 2^{|A|}$$

Operations on Sets

The intersection of two sets A and B is the set containing all the elements that are both in A and in B

$$A \cap B = \{x \in A \mid x \in B\}$$



$$\{1, 2, 3, 4\} \cap \{3, 4, 5, 6\} = \{3, 4\}$$

Exercise 4.3.

Let $A = \{\emptyset, \{\emptyset\}\}$ and $B = \{\emptyset\}$. What is $A \cap B$?

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Let $A = \{\emptyset, \{\emptyset\}\}$ and $B = \{\emptyset\}$. What is $A \cap B$?

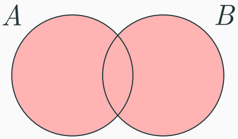
Solution 4.3.

Common element of A and B is $\emptyset$, so

$$A \cap B = \{\emptyset\}$$

The union of two sets A and B is the set containing all the elements that are either in A and/or in B

$$A \cap B = \{x \mid x \in A \text{ and/or } x \in B\}$$



$$\{1, 2, 3, 4\} \cup \{3, 4, 5, 6\} = \{1, 2, 3, 4, 5, 6\}$$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Exercise 4.4.

Let $A = \emptyset$ and $B = \{1, 2\}$. What is $A \cup B$?

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Let $A = \emptyset$ and $B = \{1, 2\}$. What is $A \cup B$?

Solution 4.3.

$$\emptyset \cup B = B = \{1, 2\}$$

Operations on Sets

Given a set of sets $\mathcal{F}$, we can take intersection/union of all sets in $\mathcal{F}$

Let $\mathcal{F} = \{S_0, S_1, \ldots, S_n\}$ where S_i are all sets

$$\bigcap \mathcal{F} = \bigcap_{i=0}^{n} S_i = S_1 \cap S_2 \cap \cdots \cap S_n$$

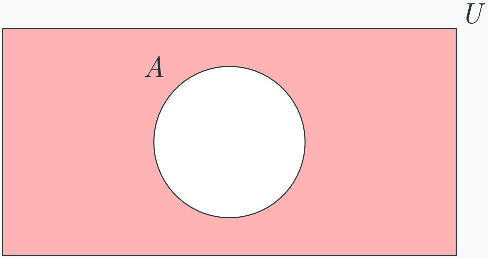
$$\bigcup \mathcal{F} = \bigcup_{i=0}^{n} S_i = S_1 \cup S_2 \cup \cdots \cup S_n$$

$$\bigcup \{\{1,2,3,4\},\{3,4,5,6\},\{7,8\}\} = \{1,2,3,4,5,6,7,8\}$$

Operations on Sets

The complement of a set A with respect to a set U such that $U \supseteq A$, is given by

$$A^{\complement} = \{x \in U \mid x \notin A\}$$



With respect to $\mathbb{N}$, $\{1, 2, 3\}^{\complement} = \{0, 4, 5, 6, \ldots\}$

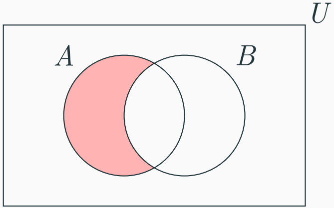
$$|A^{\complement}| = |U| - |A|$$

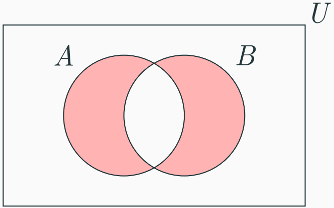
The difference of two sets A and B is the set of all elements in A that are not in B:

$$A \backslash B = \{x \in A \mid x \notin B\} = A \cap B^{\complement}$$

The symmetric difference of two sets A and B, is the set of all elements either in A, or in B, but **not both**:

$$A \,\Delta\, B = (A \cup B) \backslash (A \cap B) = (A \backslash B) \cup (B \backslash A)$$





$$|A \backslash B| = |A| - |A \cap B|$$

$$|A \,\Delta\, B| = |A \cup B| - |A \cap B|$$

Operations on Sets

The cartesian product of two sets A and B are all pairs (a, b) where $a \in A$ and $b \in B$

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

$$\{1, 2\}\times\{3, 4\} = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$$

$$|A \times B| = |A| \times |B|$$

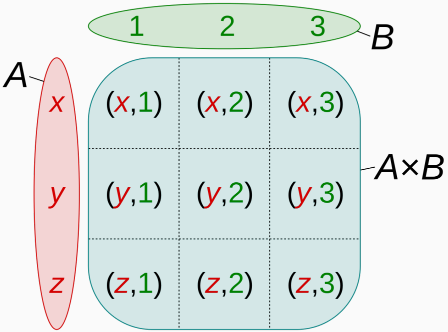


Figure 1: Visualization of the cartesian product of two sets. Source: *https://en.wikipedia.org/wiki/Cartesian_product*

Exercise.

We know that given two numbers a and b, $a \times b = b \times a$.

However, given two sets A and B is $A \times B = B \times A$?

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However, given two sets A and B is $A \times B = B \times A$?

Solution.

No. For example, $A = \{1, 2\}$ and $B = \{3\}$.

$$A \times B = \{(1, 3), (2, 3)\} \neq \{(3, 1), (3, 2)\} = B \times A$$

Given a set of sets $\mathcal{F}$, we can take product of all sets in $\mathcal{F}$

Let $\mathcal{F} = \{S_0, S_1, \ldots, S_n\}$ where S_i are all sets

$$\prod \mathcal{F} = \prod_{i=0}^{n} S_i = S_1 \times S_2 \times \cdots \times S_n$$

Sets are collections of things

Functions (Again)

Function

A function f from a set A to another set B is a set of pairs (a, b) such that $a \in A$ and $b \in B$ (i.e. $f \subseteq A \times B$).

Given a pair $(a, b) \in f$, b is the image of a under f, and $b = f(a)$.

A is the domain, B is the codomain, and $\{f(a) \mid a \in A\}$ is the image of A under f.

Restriction on functions

- If $f(a) = b$ and $f(a) = b'$ then $b = b'$; Every element can only have one image
- Every $a \in A$ must have an image; function is total

Function

Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$.
Define a function $f\colon A \to B$ given by

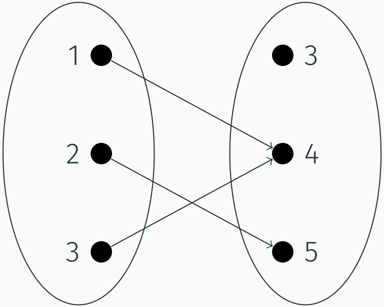
$$f = \{(1,4), (2,5), (3,4)\}$$

$$f(1) = 4,\ f(2) = 5 \text{ and } f(3) = 4$$

Domain A

Codomain B

Image $\{4, 5\}$



Exercise.

Let $f : \mathbb{R}^+ \to \mathbb{R}$ be $f(x) = \sqrt{x}$, i.e. $f(4) = 2$ and $f(9) = -3$.

Is f a function?

Exercise.

Let $f : \mathbb{R}^+ \to \mathbb{R}$ be $f(x) = \sqrt{x}$, i.e. $f(4) = 2$ and $f(9) = -3$.

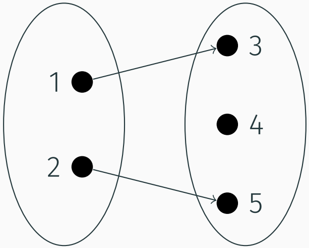
Is f a function?

Solution.

No. $f(4) = 2$ and $f(4) = -2$, but functions can only have one output for each input

Properties of Functions

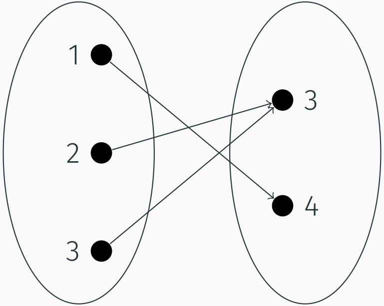
A function $f\colon A \to B$ is an injection if every $b \in B$ has at most one $a \in A$ such that $f(a) = b$



Every injection f has a left inverse g such that $g(f(x)) = x$ (reverse all the arrows!)

Properties of Functions

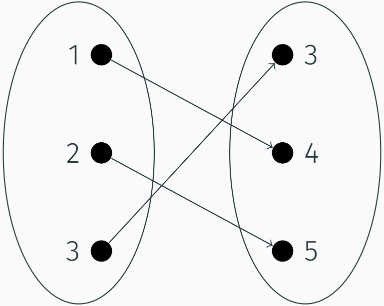
A function $f : A \to B$ is a surjection if every $b \in B$ has at least one $a \in A$ such that $f(a) = b$



Every surjection f has a right inverse g such that $f(g(x)) = x$ (reverse all the arrows!)

Properties of Functions

A function $f: A \to B$ is a bijection if every $b \in B$ has exactly one $a \in A$ such that $f(a) = b$



Every bijection f has an inverse f^{-1} such that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$ (reverse all the arrows!)

Cardinalities

Functions let us compare cardinalities of sets.

- If there exists injection $f\colon A \to B$ then $|A| \leq |B|$
- If there exists surjection $f\colon A \to B$ then $|A| \geq |B|$
- If there exists bijection $f\colon A \to B$ then $|A| = |B|$

Cardinalities

Let $2\mathbb{N}$ be the set of all even natural numbers $\{0, 2, 4, 6, \dots\}$

Which is true?

- $|\mathbb{N}| \geq |2\mathbb{N}|$
- $|\mathbb{N}| \leq |2\mathbb{N}|$
- $|\mathbb{N}| = |2\mathbb{N}|$

There exists a bijection $f : \mathbb{N} \to 2\mathbb{N}$ given by

$$f(x) = 2x$$

and inverse $f^{-1} : 2\mathbb{N} \to \mathbb{N}$

$$f^{-1}(x) = x \div 2$$

Therefore, $|\mathbb{N}| = |2\mathbb{N}|$

Cardinalities

Which is true?

- $|\mathbb{N}| \geq |\mathbb{Z}|$
- $|\mathbb{N}| \leq |\mathbb{Z}|$
- $|\mathbb{N}| = |\mathbb{Z}|$

Construct bijection $f : \mathbb{N} \to \mathbb{Z}$ that yo-yos around the positive and negative integers

$$f = \{(0,0),(1,1),(2,-1),(3,2),(4,-2),\dots\}$$

$$f(x) = \begin{cases} \frac{x+1}{2} & \text{if } x \text{ is odd} \\ -\frac{x}{2} & \text{otherwise} \end{cases}$$

$$f^{-1}(x) = \begin{cases} 2x-1 & \text{if } x > 0 \\ -2x & \text{otherwise} \end{cases}$$

Therefore, $|\mathbb{N}| = |\mathbb{Z}|$

Functions are sets of input/output pairs

Numbers

Natural Numbers

The natural numbers 0, 1, 2, … can be defined in the following way:

- 0 is a natural number
- Every natural number n has a successor $S(n)$ that is also a natural number

We can define common arithmetic operations as functions from this formulation:

$$+(a, b) = \begin{cases} a & \text{if } b = 0 \\ +(S(a), z) & \text{if } b = S(z) \end{cases}$$

$$S(S(0)) + S(0) = S(S(S(0))) + 0 = S(S(S(0)))$$

Prime Numbers

Prime number: natural number $n \geq 2$ that is only divisible by 1 and itself

$$2, 3, 5, 7, 11, 13, \ldots$$

Fundamental theorem of arithmetic: every natural number greater than 1 can be expressed as a unique product of prime factors

$$60 = 2^2 \times 3 \times 5$$

Numbers that are not prime are composite

Exercise.

Express 45 as a product of prime factors

Exercise.

Express 45 as a product of prime factors

Solution.

$$45 = 3^2 \times 5$$

gcd and lcm

The greatest common divisor of two natural numbers a and b is largest number that divides both a and b

$$\gcd(18, 24) = 6$$

The lowest common multple of a and b is smallest number that is divisible by both a and b

$$\text{lcm}(4, 6) = 12$$

How to find gcd and lcm?

gcd and lcm

Naive solution: find all divisors (multiples) of a and b and take the largest (smallest). Alternatively:

- Find prime factors of a and b as multisets
- Take intersection (union) of multisets
- Product of elements in multiset is gcd (lcm)

Intersection of multiset: for each element a in either of the two sets, intersection contains minimum occurrence between the two sets

$$\{1, 1, 1\} \cap \{1, 1\} = \{1, 1\}$$

Union of multiset: for each element a in either of the two sets, union is maximum occurrence between the two sets

$$\{1, 1, 1\} \cup \{1, 1\} = \{1, 1, 1\}$$

gcd and lcm

Let $a = 20$ and $b = 15$.

Prime factors of a as multiset: $2 \times 2 \times 5 \to \{2, 2, 5\}$

Prime factors of b as multiset: $3 \times 5 \to \{3, 5\}$

Intersection of prime factors: $\{5\}$ so

$$\gcd(a, b) = 5$$

Union of prime factors: $\{2, 2, 3, 5\}$ so

$$\mathrm{lcm}(a, b) = 2 \times 2 \times 3 \times 5 = 60$$

Positional Number Systems

Given any natural number $x > 1$ called the radix or base, every number n can be expressed as a unique polynomial

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

such that for all i, $0 \leq c_i < x$ and $c_i \in \mathbb{N}$

Let our radix x be 10. Then:

$$12345 = 1 \cdot 10^4 + 2 \cdot 10^3 + 3 \cdot 10^2 + 4 \cdot 10^1 + 5$$

Coefficients are the digits of the number! Can we have different bases?

Positional Number Systems

Let n_m be the number represented by the sequence of digits n with base m.

Binary: base 2 (two digits, 0 and 1)

Hexadecimal: base 16 (16 digits, 0 to 9 and A to F)

$$\begin{aligned} 127_{10} &= 1 \cdot 10^2 + 2 \cdot 10^1 + 7 \\ &= 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \\ &= \mathit{7} \cdot 16^1 + \mathit{F} \end{aligned}$$

$$127_{10} = 1111111_2 = \mathit{7F}_{16}$$

Positional Number Systems

How do we re-express a number in different bases?

Recall: if $a \div b = q\frac{r}{b}$, then q is quotient and r is remainder

$$9 \div 4 = 2\frac{1}{4} = 2 \text{ remainder } 1$$

1. Let $n \div b$ give quotient q and remainder r.
2. Write (from right to left) r.
3. Update the new value of n to be q.
4. Go back to step 1 if $n > 0$, otherwise, what you have written down is the new number.

Let us express 11_{10} in binary.

Division	Quotient	Remainder	Digits
$11 \div 2$	5	1	1
$5 \div 2$	2	1	11
$2 \div 2$	1	0	011
$1 \div 2$	0	1	1011
0			1011

$$11_{10} = 1011_2$$

Natural numbers are unique products of primes and unique combinations of digits in a base