NUS | Computing
National University of Singapore

IT5001 Software Development Fundamentals

5. Recursion Vs Iterations and Nested Functions

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Recursion vs Iteration

Reversing a String

- How about reversing a string? Of course, we can just use string slicing

```
>>> s = 'abcde12345'
>>> s[::-1]
'54321edcba'
>>>
```

- How about we write a function for it?

```
>>> reverseStringI(s)
'54321edcba'
>>>
```

Reverse String (Iterative Version 1)

```
def reverseStringI(s):
    output = ''
    l = len(s)
    for i in range(l):
        output += s[l-i-1]
    return output
```

```
>>> reverseStringI('abcde')
'edcba'
```

i
0
1
2
3
4

Reverse String (Iterative Version 1)

```
def reverseStringI(s):
    output = ''
    l = len(s)
    for i in range(l):
        output += s[l-i-1]
    return output
```

```
>>> reverseStringI('abcde')
'edcba'
```

i	l-i-1
0	4
1	3
2	2
3	1
4	0

Reverse String (Iterative Version 1)

```
def reverseStringI(s):
    output = ''
    l = len(s)
    for i in range(l):
        output += s[l-i-1]
    return output
```

```
>>> reverseStringI('abcde')
'edcba'
```

i	l-i-1	s[l-i-1]
0	4	e
1	3	d
2	2	c
3	1	b
4	0	a

Reverse String (Iterative Version 1)

```
def reverseStringI(s):
    output = ''
    l = len(s)
    for i in range(l):
        output += s[l-i-1]
    return output
```

```
>>> reverseStringI('abcde')
'edcba'
```

i	l-i-1	s[l-i-1]	output
0	4	e	e
1	3	d	ed
2	2	c	edc
3	1	b	edcb
4	0	a	edcba

Reverse String (Iterative Version 2)

```
def reverseStringI(s):
    output = ''
    for c in s:
        output = c + output
    return output
```

```
>>> reverseStringI('abcde')
'edcba'
```

c	output
a	a
b	ba
c	cba
d	dcba
e	edcba

Reversing String (Recursive Version)

```
def reverseStringR(s):
    if not s:
        return ''
    return reverseStringR(s[1:])+s[0]
```

- reverseStringR(‘abcde’)
- reverseStringR(‘bcde’)+’a’
- reverseStringR(‘cde’)+’b’+’a’
- reverseStringR(‘de’)+’c’+’b’+’a’
- reverseStringR(‘e’)+’d’+’c’+’b’+’a’
- reverseStringR(‘’)+’e’+’d’+’c’+’b’+’a’
- ‘’+’e’+’d’+’c’+’b’+’a’
- ‘edcba’

Taylor Series

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad \text{for all } x$$

$$\tan x = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n (1-4^n)}{(2n)!} x^{2n-1} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \quad \text{for } |x| < \frac{\pi}{2}$$

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots \quad \text{for } |x| < \frac{\pi}{2}$$

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots \quad \text{for } |x| \le 1$$

$$\arccos x = \frac{\pi}{2} - \arcsin x$$

$$= \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} = \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} - \cdots \quad \text{for } |x| \le 1$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \quad \text{for } |x| \le 1,\ x \neq \pm i$$

Taylor Series

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad = \underbrace{x}_{n = 0} - \underbrace{\frac{x^3}{3!}}_{n = 1} + \underbrace{\frac{x^5}{5!}}_{n = 2} - \cdots \qquad \text{for all } x$$

- We do not need the infinite precision
- We may just sum up to k terms

$$\sin x = \sum_{n=0}^{\cancel{\infty}\ k} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \text{for all } x$$

Computing sine by Iteration

$$\sin x = \sum_{n=0}^{\cancel{\infty}\ k} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \text{for all } x$$

- Using iteration

```
def sinI(x,k):
    result = 0
    for n in range(0,k):
        result += ((-1)**n / fact(2*n+1)) * x**(2*n+1)
    return result
```

```
>>> print(sinI(PI/6,10))
0.5000000000592083
>>> from math import sin
>>> sin(PI/6)
0.5000000000592083
```

Python Library version of "sin()"

Computing sine by Recursion

$$\sin x = \sum_{n=0}^{\cancel{\infty}\ k} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = \overbrace{\underbrace{x - \frac{x^3}{3!}}_{\text{Sum up to n = 1}} + \frac{x^5}{5!}}^{\text{Sum up to n = 2}} - \cdots \qquad \text{for all } x$$

- In general, if we want to sum up to the k terms

$$\sin x = \sum_{n=0}^{\cancel{\infty}\ k} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = \overbrace{\underbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}_{\text{Sum up to n = k - 1}}\ \underbrace{\boxed{\text{k}^{\text{th}}}}_{\text{n = k}}}^{\text{Sum up to n = k}} \ \text{for all } x$$

Computing sine by Recursion

- Assuming that if the function `sinR(x,k)` sums until n = k, then

`sinR(x,k)` = `sinR(x,k-1)` + the kth term

- In general, if we want to sum up to the k terms

$$\sin x = \sum_{n=0}^{\cancel{\infty}\ k} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$= \overbrace{\underbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}_{\text{Sum up to n = k - 1}} \boxed{k^{th}}}^{\text{Sum up to n = k}} \text{ for all } x$$

n = k

Computing sine by Recursion

- Assuming that if the function `sinR(x,k)` sums until n = k, then

`sinR(x,k)` = `sinR(x,k-1)` + the kth term

```
def sinR(x,k):
    if k < 0:
        return 0
    return sinR(x,k-1) + ((-1)**k / fact(2*k+1)) * x**(2*k+1)
```

```
>>> sinR(PI/6,6)
0.5000000000592083
>>> from math import sin
>>> sin(PI/6)
0.5000000000592083
```

More Taylor Series

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad \text{for all } x$$

$$\tan x = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n (1-4^n)}{(2n)!} x^{2n-1} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \quad \text{for } |x| < \frac{\pi}{2}$$

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots \quad \text{for } |x| < \frac{\pi}{2}$$

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots \quad \text{for } |x| \le 1$$

$$\begin{aligned} \arccos x &= \frac{\pi}{2} - \arcsin x \\ &= \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} = \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} - \cdots \quad \text{for } |x| \le 1 \end{aligned}$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \quad \text{for } |x| \le 1,\ x \ne \pm i$$

Recursion Common Patterns

```
def reverseStringR(s):
    if not s:
        return ''
    return reverseStringR(s[1:])+s[0]

def sinR(x,k):
    if k < 0:
        return 0
    return sinR(x,k-1) + ((-1)**k / fact(2*k+1)) * x**(2*k+1)
```

Base cases

Recursion step to reduce the problem one-by-one

Iteration Common Patterns

```
def reverseStringI(s):
    output = ''
    l = len(s)
    for i in range(l):
        output += s[l-i-1]
    return output

def sinI(x,k):
    result = 0
    for n in range(0,k):
        result += ((-1)**n / fact(2*n+1)) * x**(2*n+1)
    return result
```

Accumulate element one-by-one

Initial the final answer to “nothing” at the beginning.
Accumulate and return the final answer

Iteration/Recursion Conversion

```
def sinR(x,k):
    if k < 0:
        return 0
    return sinR(x,k-1) + ((-1)**k / fact(2*k+1)) * x**(2*k+1)
```

Base case

The answer for previous k – 1 terms

The kth term

```
def sinI(x,k):
    result = 0
    for n in range(0,k):
        result += ((-1)**n / fact(2*n+1)) * x**(2*n+1)
    return result
```

Iteration/Recursion Conversion

```
def reverseStringR(s):
    if not s:
        return ''
    return reverseStringR(s[1:])+s[0]
```

Base case

The answer for previous k – 1 terms

The kth term

```
def reverseStringI(s):
    output = ''
    l = len(s)
    for i in range(l):
        output += s[l-i-1]
    return output
```

“Homework”

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} \qquad = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots \qquad \text{for } |x| \le 1$$

- The answer for all k-1 terms?
- Base case?
- Kth term?

Another Example

Recursion vs Iteration

- SumDigits
  - Given a positive number n, the sum of all digits is obtained by adding the digit one-by-one
    - For example, the sum of 52634 = 5 + 2 + 6 + 3 + 4 = 20
  - Write a function `sum(n)` to compute the sum of all the digits in n
- Factorial
  - Factorial is defined (recursively) as $n! = n * (n-1)!$ such that $0! = 1$
  - Write a function `fact(n)` to compute the value of `n!`

- Can you do it in both recursion and iteration?

SumDigits

Iteration

```
def sum(n):
  res = 0
  while n > 0:
    res = res + n%10
    n = n//10
  return res
```

base/initial value

computation

continuation/next value

Recursion

```
def sum(n):
  if n == 0:
    return 0
  else:
    return n%10 + sum(n//10)
```

stop/base case (*they are related, how?*)

temporary result variables
not needed in recursion (*why?*)

Factorial

Iteration

```
def fact(n):
   res = 1
   while n > 0:
      res = res * n
      n = n-1
   return res
```

Recursion

```
def fact(n):
   if n == 0:
      return 1
   else:
      return n * fact(n-1)
```

base/initial value

computation

continuation/next value

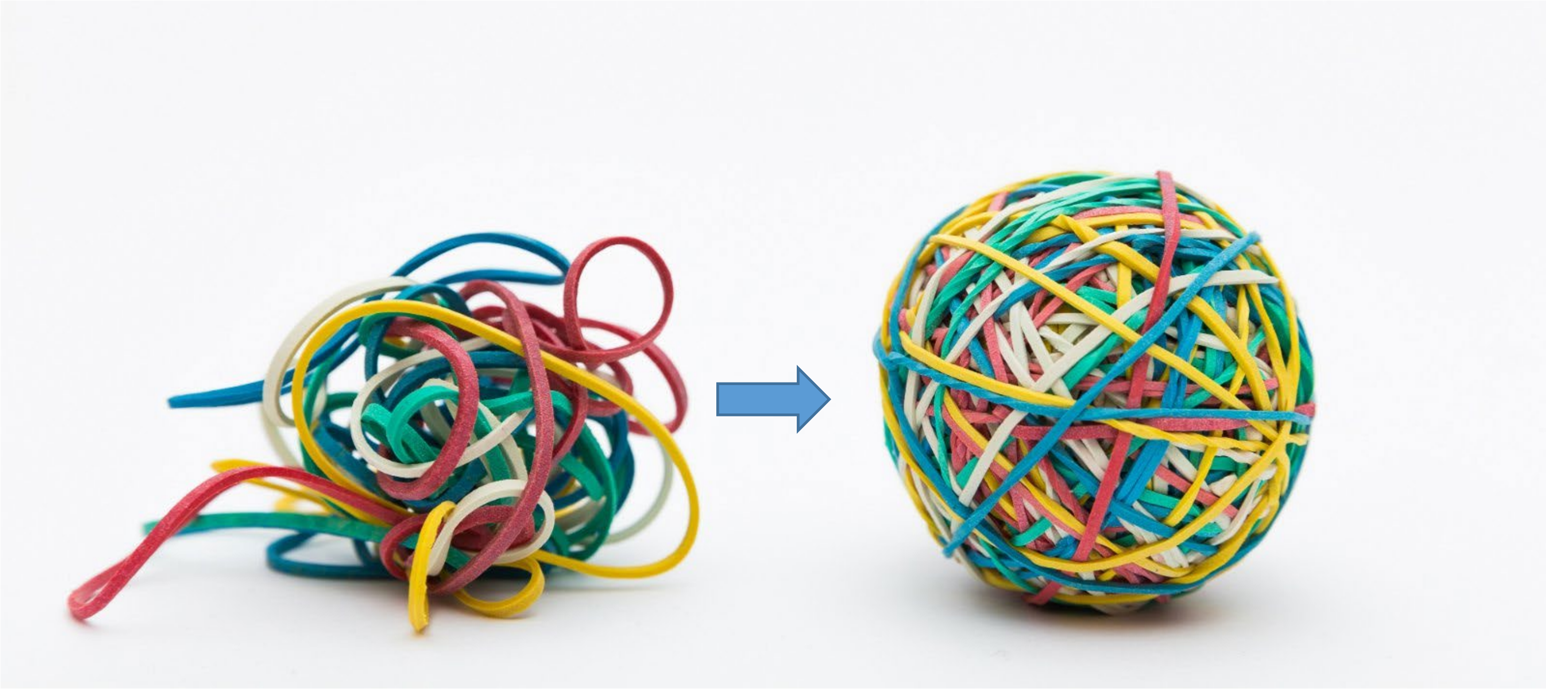
stop/base case (*they are related, how?*)

temporary result variables
not needed in recursion (*why?*)

“Homework”

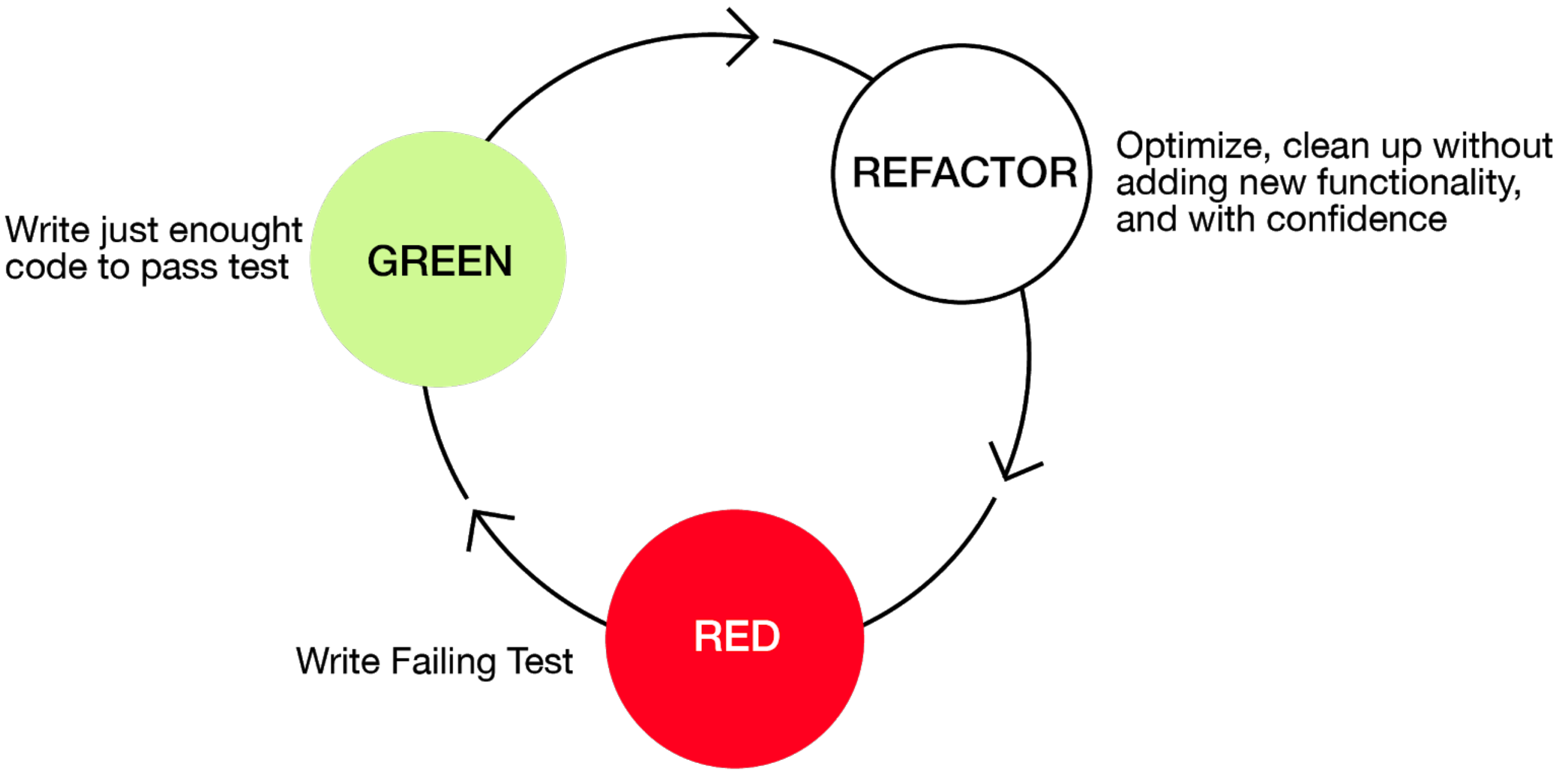
- How to re-write your code with both iterative/recursion version mentioned in this course before?
  - `burgerPrice()`
  - `checkAllAlpha()`
  - Etc.
- The answer for all k-1 terms?
- Base case?
- Kth term?

Code Refactoring



Code Refactoring

- Refactoring is a disciplined technique for restructuring an existing body of code, altering its internal structure without changing its external behavior.



Nested Functions

Nested Functions

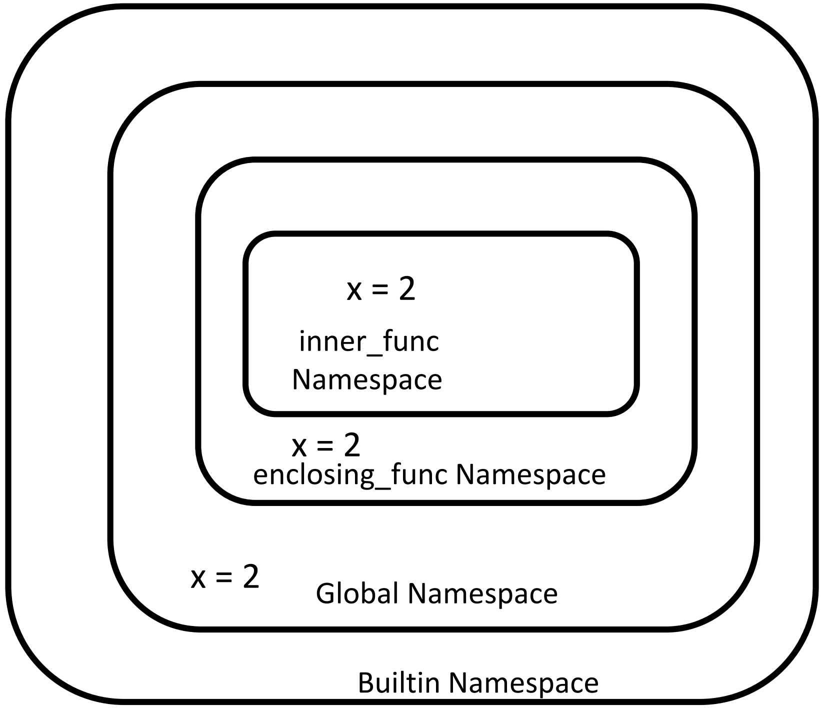
- Functions defined inside other functions

```python
x = 2
def enclosing_func(x):
    def inner_func(x):
        return x**2
    output = inner_func(x)
    return output

print(enclosing_func(x))
```

Output:

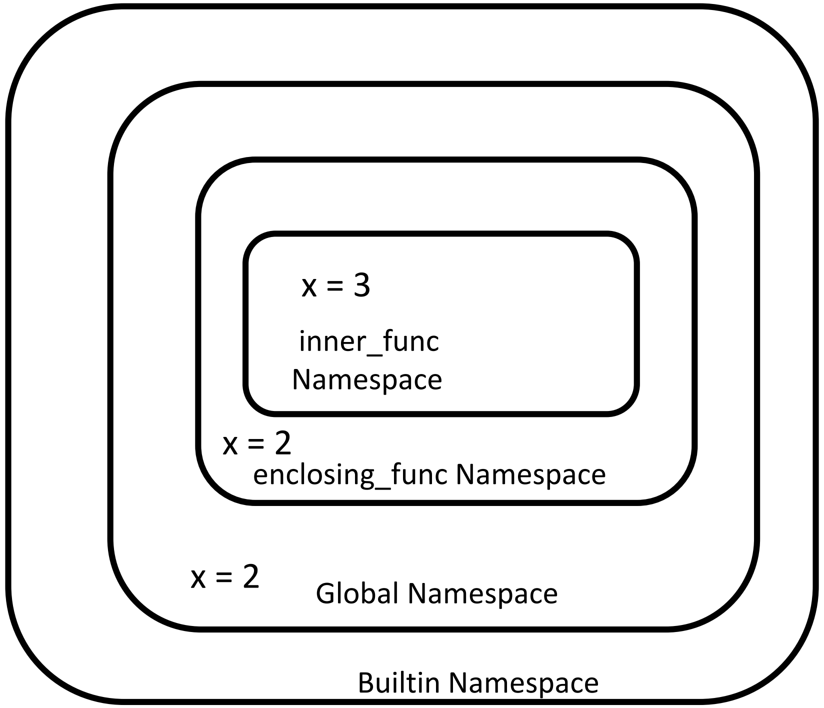
```
4
```



Nested Functions

Namespace of inner function is different from enclosing function and global namespace

```
def enclosing_func(x):
    def inner_func(x):
        x += 1
        return x**2
    print(x)
    output = inner_func(x)
    print(x)
    return output
x = 2
print(x)
print(enclosing_func(x))
print(x)
```



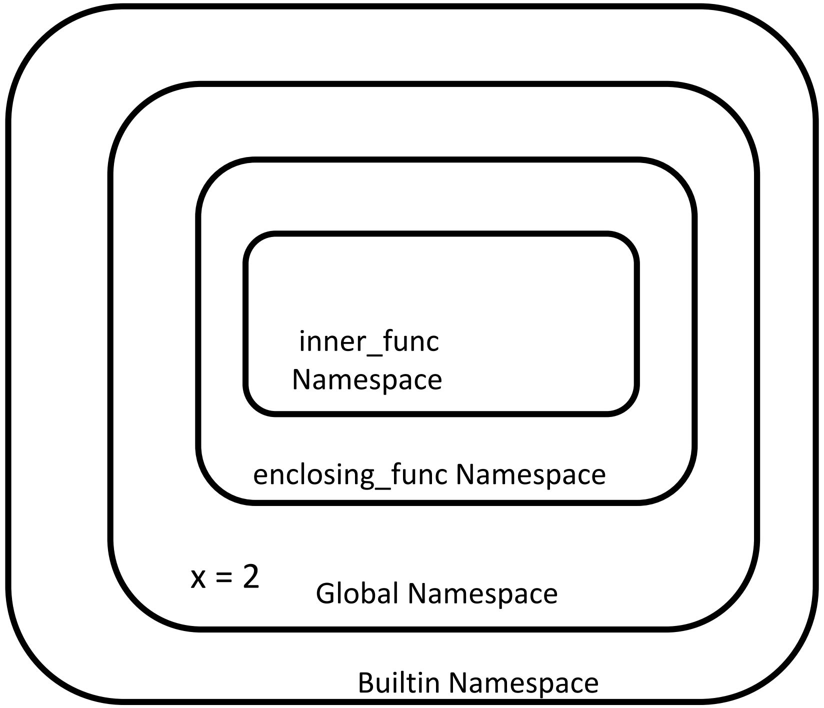
Output:

```
2
2
2
9
2
```

Nested Functions

Inner functions can access global variables

```
def enclosing_func():
    def inner_func():
        return x**2
    output = inner_func()
    return output
x = 2
print(enclosing_func())
```



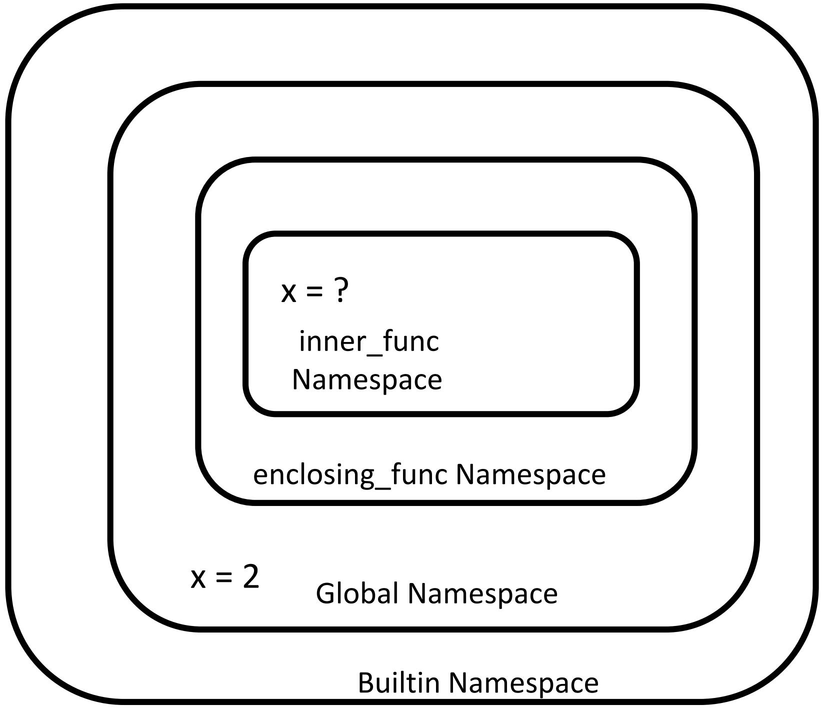
Output:

```
4
```

Nested Functions

Inner functions cannot modify global variables

```
def enclosing_func():
    def inner_func():
        x =  x+2
        return x**2
    output = inner_func()
    return output
x = 2
print(enclosing_func())
```




```
UnboundLocalError: local variable 'x' referenced before assignment
```

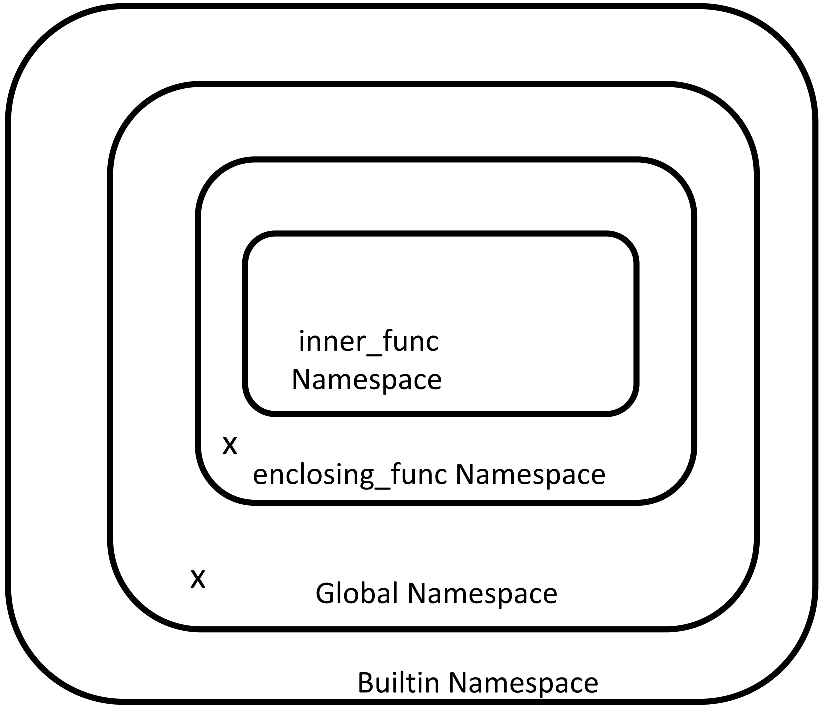
Nested Functions: *global* keyword

Modifying global variable from inner function

```
def enclosing_func():
    x = 3
    def inner_func():
        global x
        print(x)
        x = 1
        print(x)
    print(x)
    inner_func()
    print(x)
x = 5
print(x)
print(enclosing_func())
print(x)
```

nonlocal/enclosing namespace for inner function

global keyword binds this variable to global variable x



Output:

```
5
3
5
1
3
None
1
```

Nested Functions

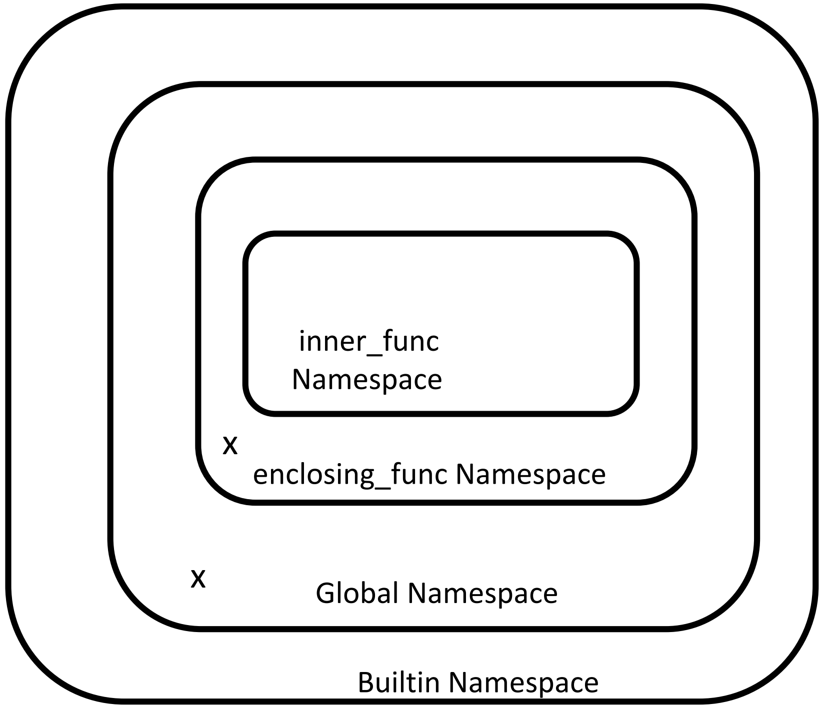
Inner functions can access nonlocal variables

```
def enclosing_func():
    x = 3
    def inner_func():
        print(x)

    print(x)
    inner_func()
    print(x)
x = 2
print(x)
print(enclosing_func())
print(x)
```

Output:

```
2
3
3
3
None
2
```

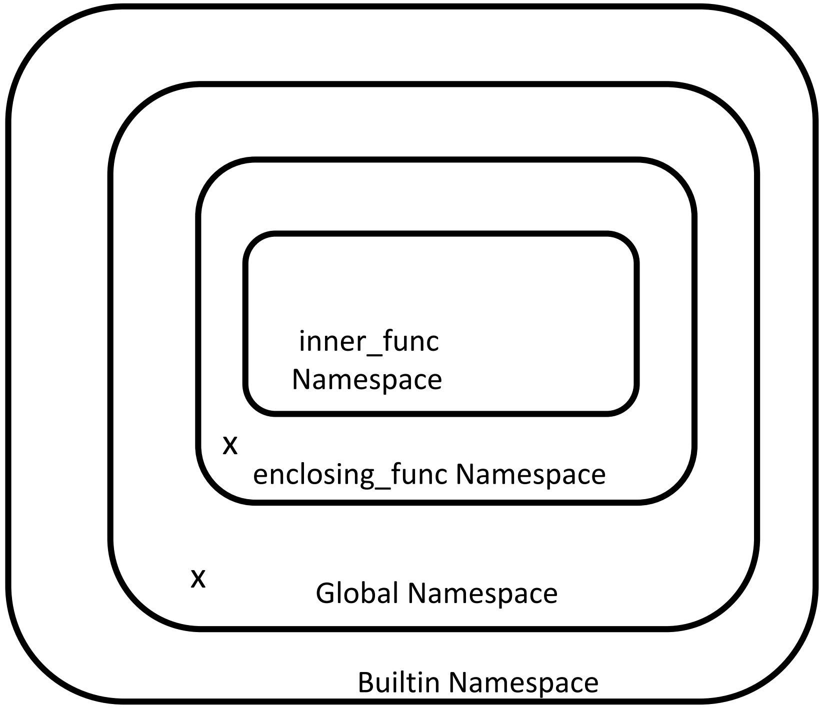


Nested Functions

Inner functions cannot modify nonlocal variables

```
def enclosing_func():
    x = 3
    def inner_func():
        x = x+1
        print(x)

    print(x)
    inner_func()
    print(x)
x = 2
print(x)
print(enclosing_func())
print(x)
```



Output:

```
UnboundLocalError: local variable 'x' referenced before assignment
```

Nested Functions: *nonlocal* keyword

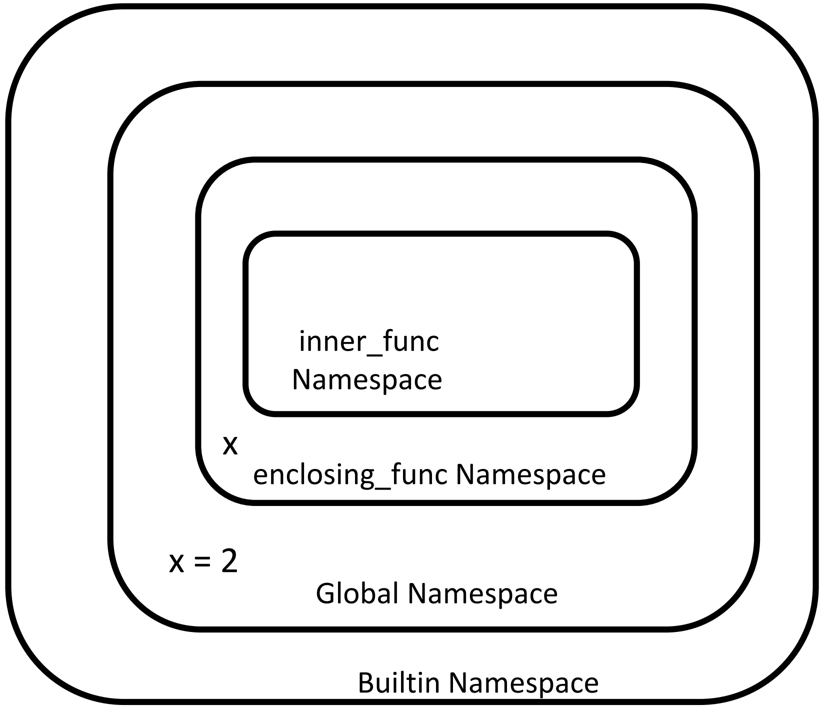
Names in enclosing_func namespace
are nonlocal variables for inner_func

Binds this name to variable
in nearest enclosing namespace

```
def enclosing_func():
    x = 3
    def inner_func():
        nonlocal x
        print(x)
        x = x+1
        print(x)
    print(x)
    inner_func()
    print(x)
x = 2
print(x)
print(enclosing_func())
print(x)
```

Output:

```
2
3
3
4
4
None
2
```



Nested Functions

```
def p1(x):
    y = 2
    print('Entering p1')
    def p2(x):
        print('Entering p2')
        def p3(x):
            print('Entering p3')
            output = x**2
            print('Leaving p3')
            return output
        output = p3(x)
        print('Leaving p2')
        return output
    output = p2(x)
    print('Leaving p1')
    return output
```

enclosing function
for p1() and p3()

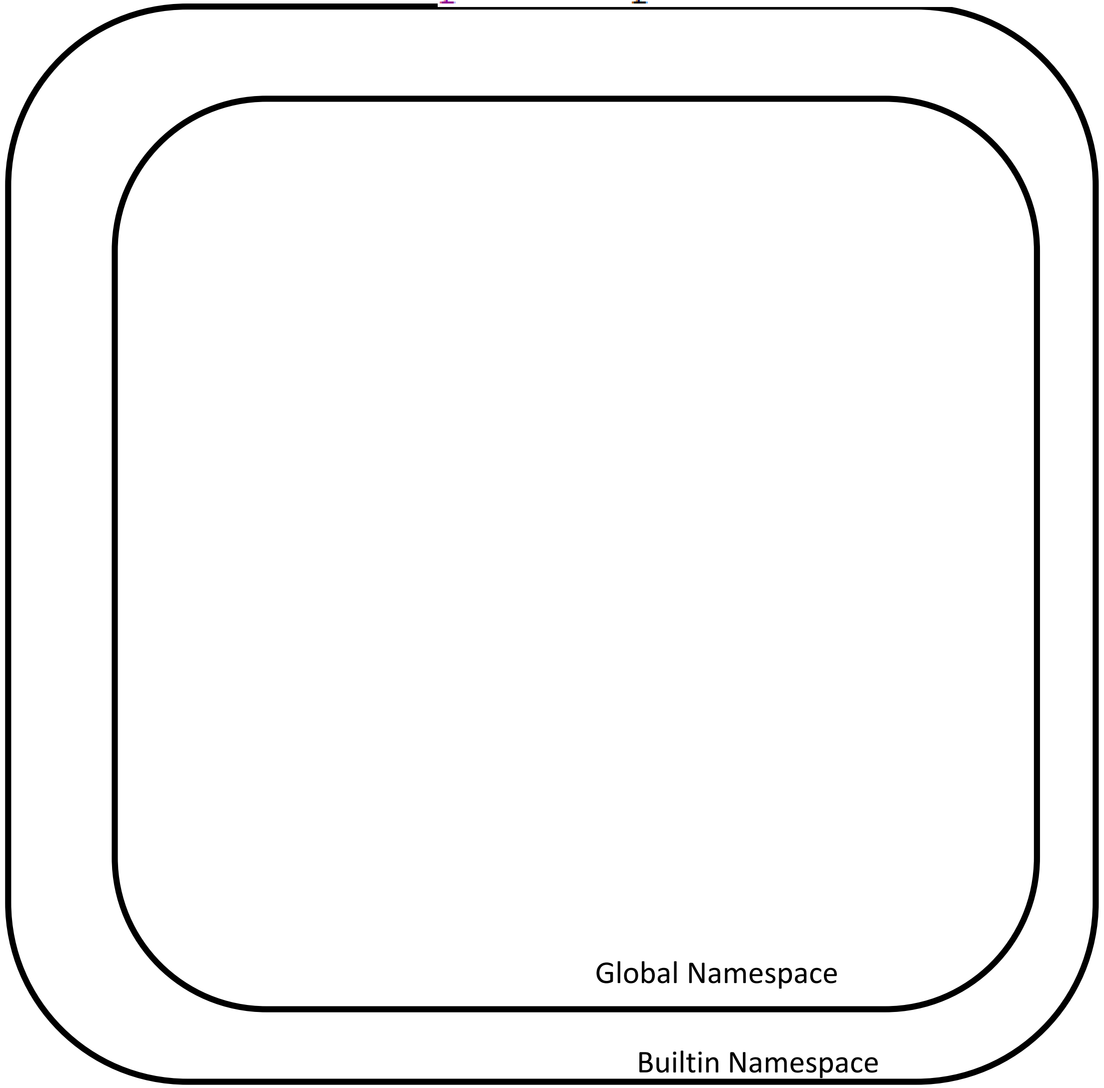
Inner function for
p1() and
enclosing function for p3()

Inner function for
p1() and p2()

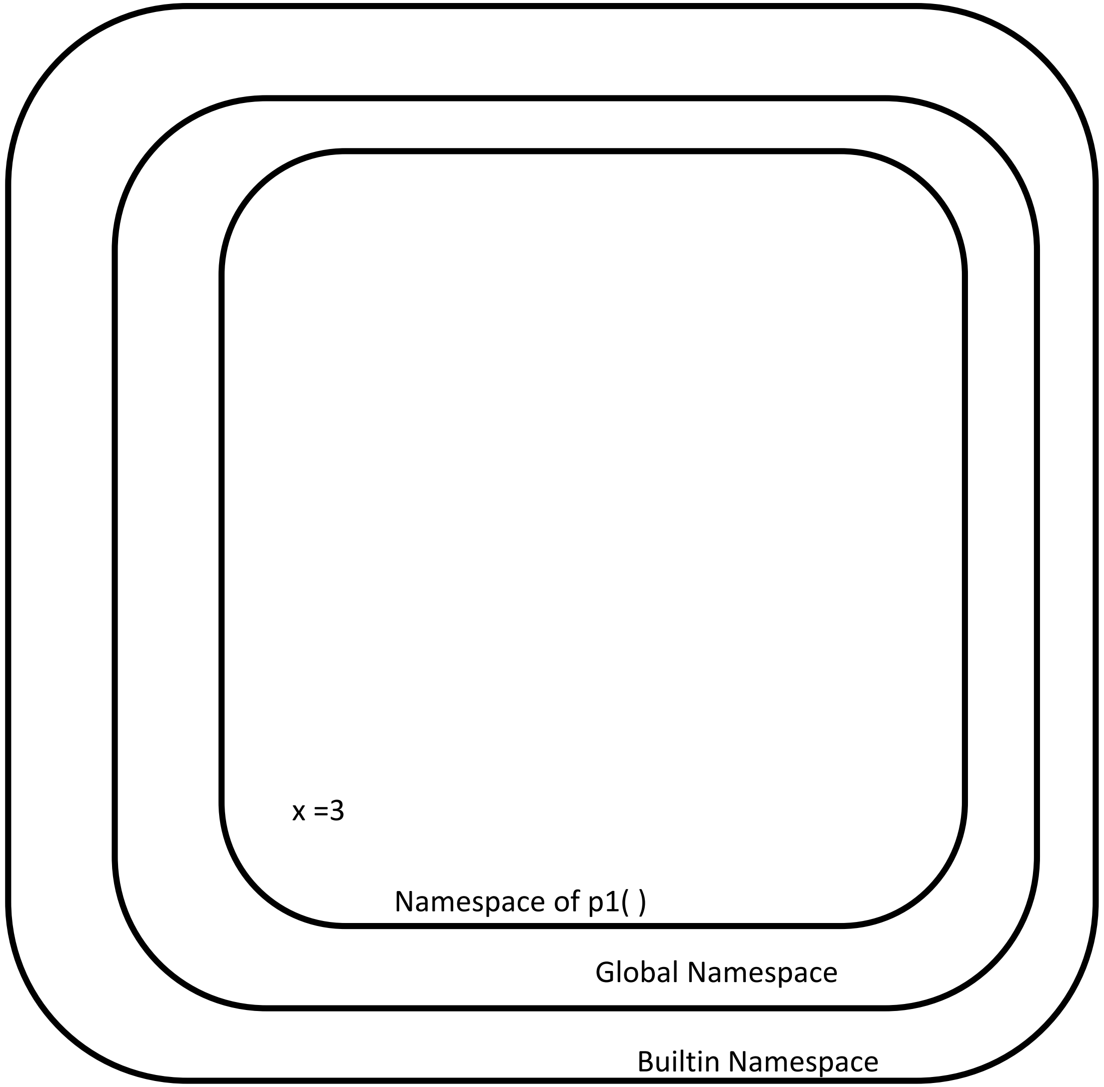
```
print(p1(3))
```

Namespaces: Nested Functions

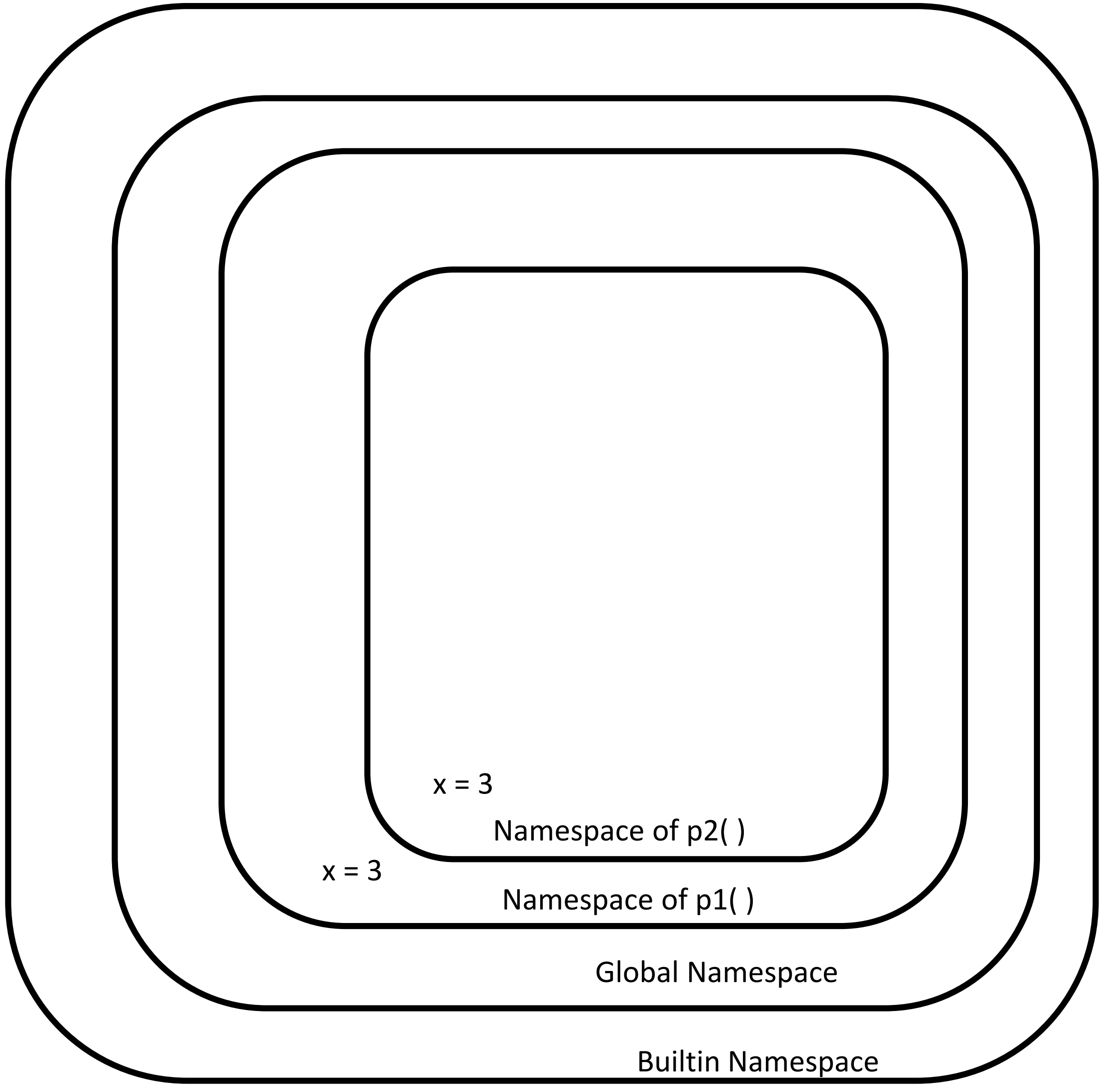
```
print(p1(3))
```



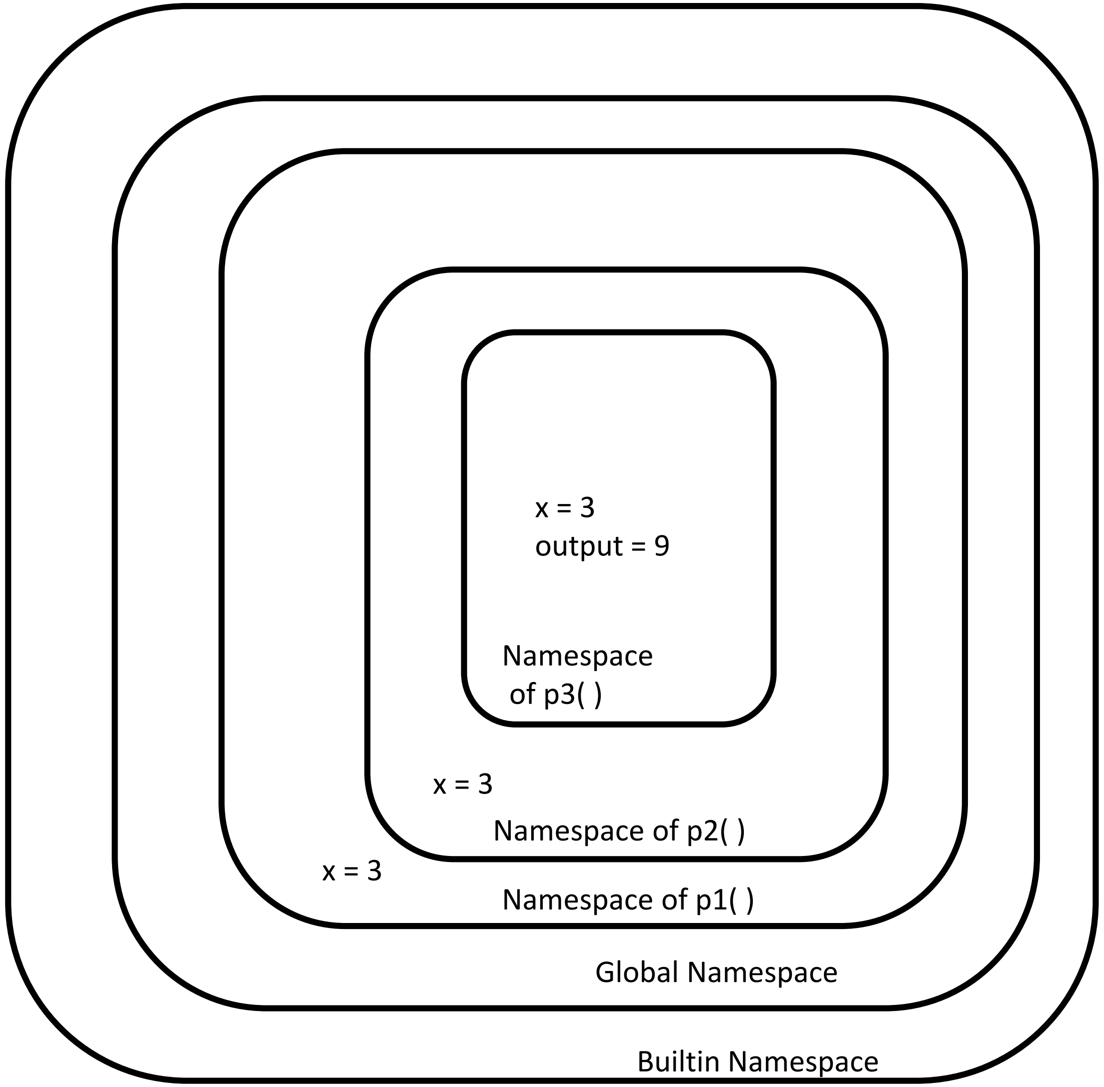
Namespaces: Nested Functions



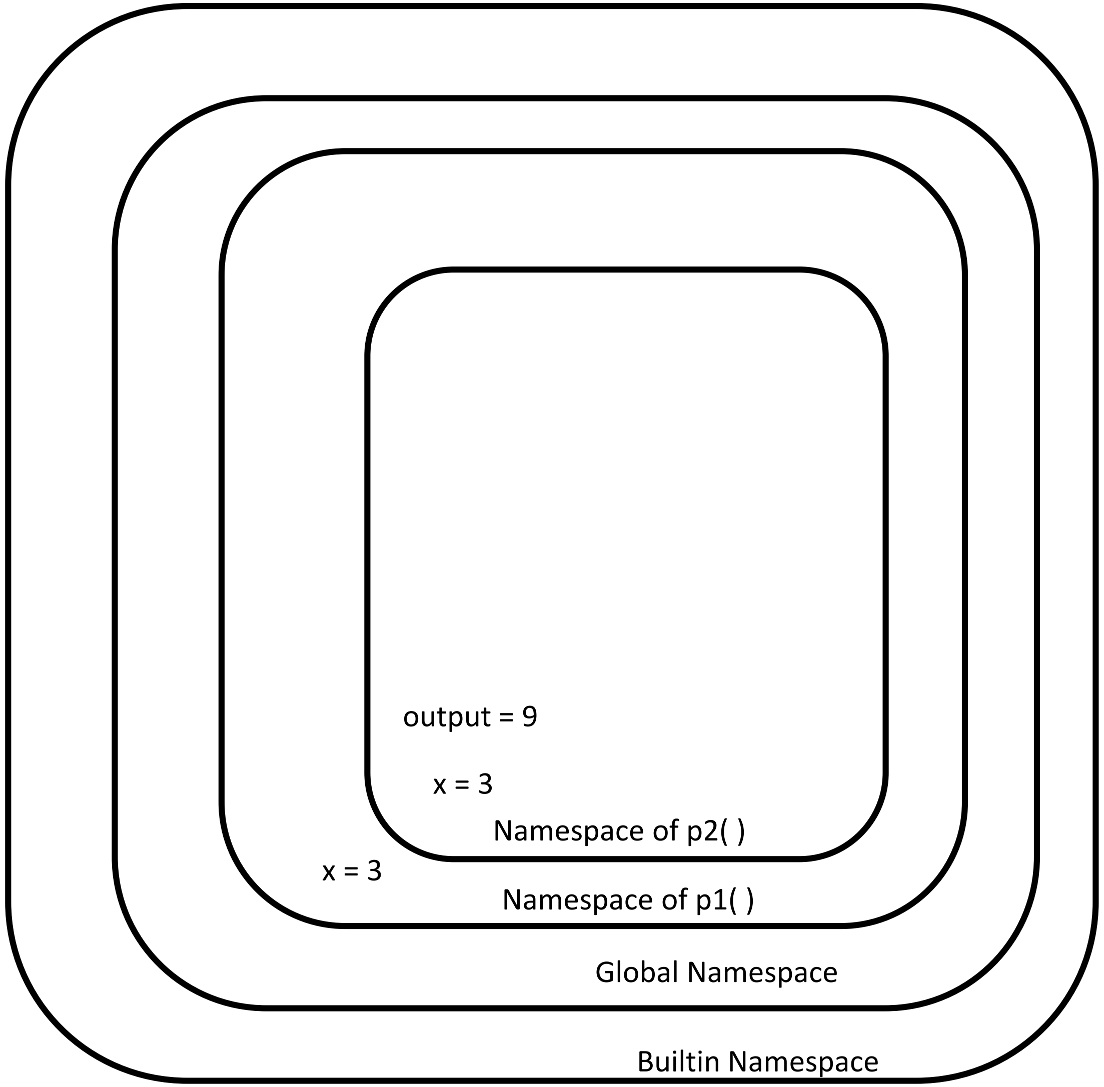
Namespaces: Nested Functions



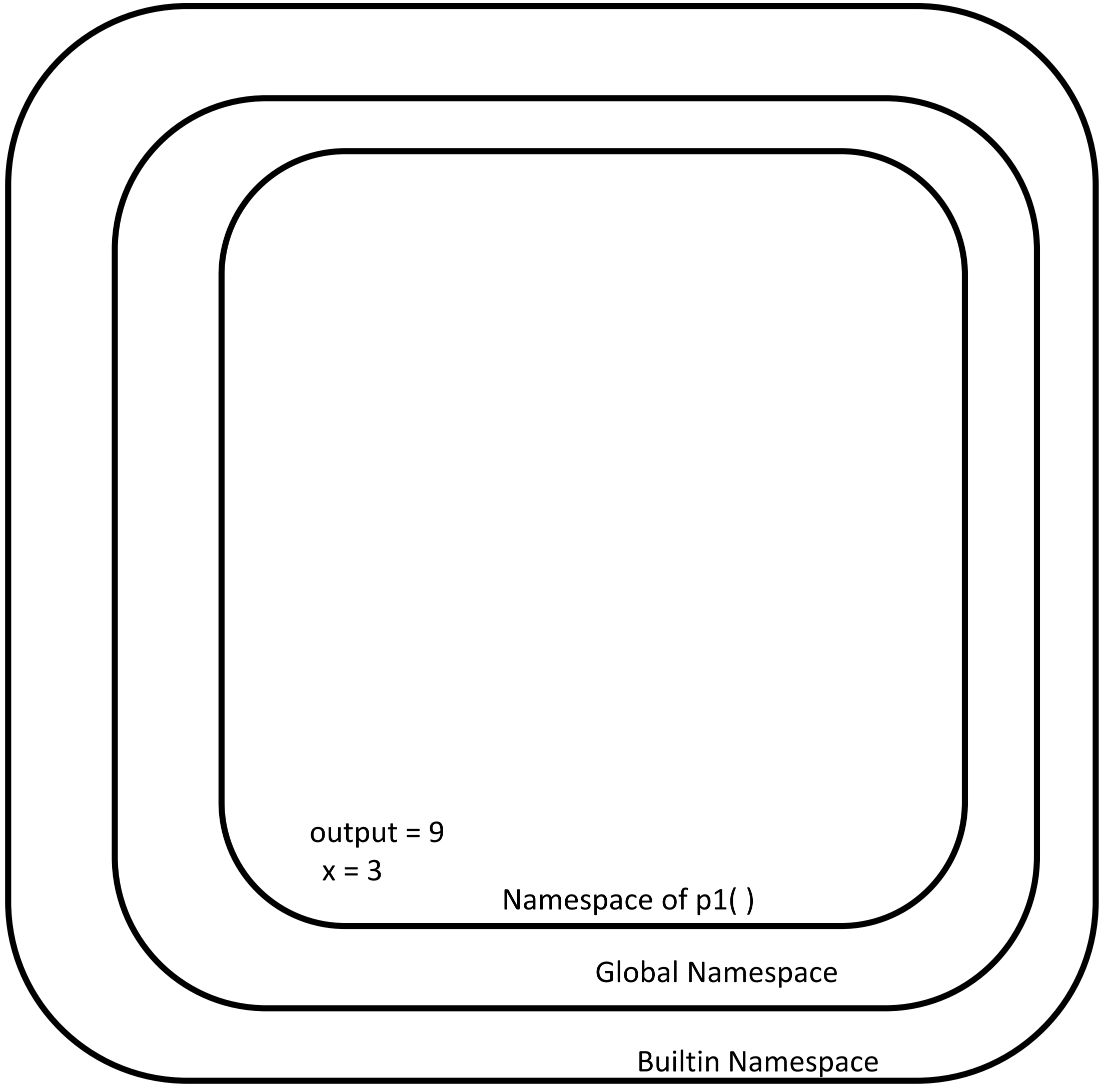
Namespaces: Nested Functions



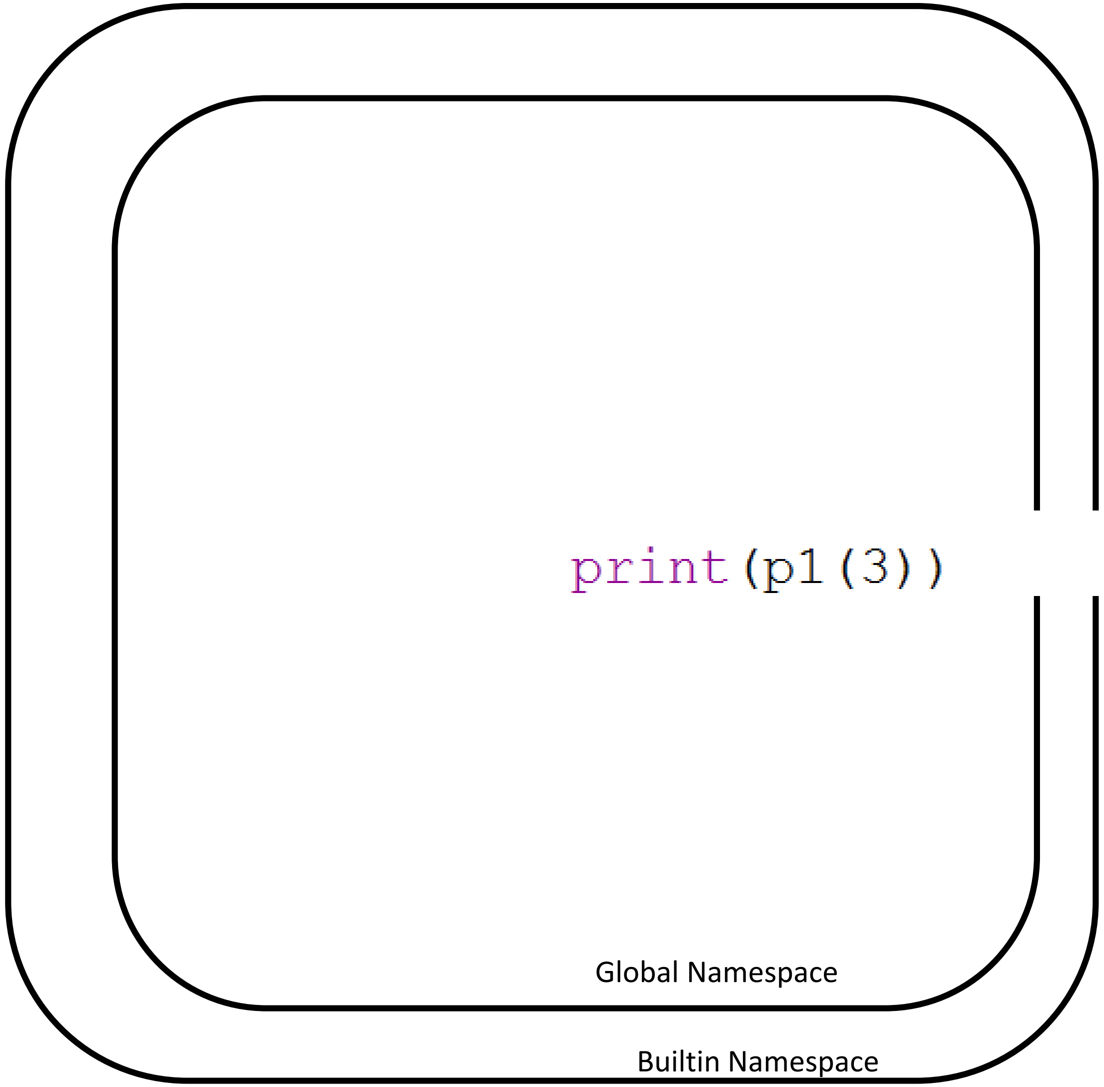
Namespaces: Nested Functions



Namespaces: Nested Functions



Namespaces: Nested Functions



What is the output?

```
def p1(x):
    y = 2
    print('Entering p1')
    def p2(x):
        print('Entering p2')
        z = 4
        def p3(x):
            print('Entering p2')
            output = x**2 +y**2+z**2
            print('Leaving p3')
            return output
        output = p3(x)
        print('Leaving p2')
        return output
    output = p2(x)
    print('Leaving p1')
    return output


print(p1(3))
```

What is the output?

```
def p1(x):
    y = 2
    print('Entering p1')
    def p2(x):
        print('Entering p2')
        z = 4
        def p3(x):
            print('Entering p2')
            output = x**2
            print('Leaving p3')
            return output
        output = p3(x)
        print('Leaving p2')
        return output
    output = p2(x)+z**2
    print('Leaving p1')
    return output

print(p1(3))
```

Where do we use inner functions?

- Higher-Order Functions (Week 5)