IT5005 Artificial Intelligence
Tutorial 3

1. Represent the following sentences in first order logic, using a consistent vocabulary defined as follows:

    ***Took*(*x,y,z*)** : is true if student *x* took subject *y* in semester *z*
    ***Score*(*x,y,z*)** : is true if student *x* obtains score *z* in subject *y*
    ***Passed*(*x,y*)** : is true if student *x* passed subject *y*
    ***Buys*(*x,p*)** : is true if person *x* buys policy *p*
    ***IsSmart*(*x*)** : is true if person *x* is smart
    ***IsExpensive*(*x*)** : is true if *x* is expensive
    ***Sells*(*x,y,p*)**: is true if person *x* sells policy *p* to person *y*
    ***IsInsured*(*x*)**: is true if person *x* is insured
    ***IsBarber*(*x*)**: is true if *x* is a barber
    ***Shaves*(*x,y*)**: is true if person *x* shaves person *y*

    (a) Some students took French in Spring 2001.

    $\exists x \; Took(x, French, Spring2001)$

    (b) Every student who takes French passes it.

    $\forall x,y \; (Took(x, French, y) \rightarrow Passed(x, French))$

    (c) Only one student took Greek in Spring 2001.

    $\exists! \, x: Took(x, Greek, Spring2001)$

    Alternatively:

    $\exists x \; (Took(x, Greek, Spring2001) \land \forall y : (Took(y, Greek, Spring2001) \rightarrow y = x))$

    (d) The best score in Greek is always higher than the best score in French.

    There are many ways of defining this, but here is one:

    $\forall x,s \; (Score(x, French, s) \rightarrow \exists y,t : Score(y, Greek, t) \land t > s)$.

    This roughly translates to 'for every student *x* who took French and got a score *s* there is some student *y* who took Greek, got *t* and *t* > *s*. One can first define the property *Best*(*t*) which is the best score of any student who took topic *t*, and then state that *Best*(*Greek*) > *Best*(*French*). One can also insist that there is some student who took Greek and some student who took French (add existential quantifiers appropriately).

(e) Everyone who buys a policy is smart.

$\forall x,p\ (Buys(x,p) \rightarrow IsSmart(x)\ )$

Alternatively:

$\forall x, p(\sim IsSmart(x) \rightarrow \sim Buys(x, p))$

(f) No person buys an expensive policy.

$\neg\exists x,p(Buys(x,p) \land IsExpensive(p))$

Alternatively:

$\forall x,p(Buys(x,p) \rightarrow \neg IsExpensive(p))$

(g) There is an agent who sells policies only to those people who are not insured.

$\exists x\forall y,p(Sells(x,y,p) \rightarrow \neg IsInsured(y))$

(h) There is a barber who shaves all men in town who do not shave themselves.

$\exists x(IsBarber(x) \land \forall y(\neg Shaves(y,y) \rightarrow Shaves(x,y)))$

(i) There is a barber who shaves all men in town who does not shave himself.

$$\exists x(IsBarber(x) \land \forall y(shaves(x, y) \rightarrow x \neq y))$$

2. Given the following logical statements, use model checking to show that $KB \models \alpha$. In other words, write down all possible true/false assignments to the variables, the ones for which KB is true and the one for which α is true, and see whether one is a subset of the other

.

(a)

$$KB = (x_1 \vee x_2) \wedge (x_1 \rightarrow x_3) \wedge \neg x_2$$

$$\alpha = x_3 \vee x_2$$

(b)

$$KB = (x_1 \vee x_3) \wedge (x_1 \rightarrow \neg x_2)$$

$$\alpha = \neg x_2$$

Solution:For the first KB and α pair we have (0 = False, 1 = True)

x_1	x_2	x_3	$x_1 \vee x_2$	$x_1 \rightarrow x_3$	$\sim x_2$	KB	$\alpha = x_3 \vee x_2$
0	0	0	0	1	1	0	0
0	0	1	0	1	1	0	1
0	1	0	1	1	0	0	1
0	1	1	1	1	0	0	1
1	0	0	1	0	1	0	0
1	0	1	1	1	1	1	1
1	1	0	1	0	0	0	1
1	1	1	1	1	0	0	1

When $KB = True$ (yellow ow), α is true as well. Thus $KB \models \alpha$.

For the next KB and α pair, we have (0 is False, 1 is True)

x_1	x_2	x_3	$x_1 \vee x_3$	$\sim x_2$	$x_1 \rightarrow \sim x_2$	KB	$\alpha = \sim x_2$
0	0	0	0	1	1	0	1
0	0	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	1	1	1	0	1	1	0
1	0	0	1	1	1	1	1
1	0	1	1	1	1	1	1
1	1	0	1	0	0	0	0
1	1	1	1	0	0	0	0

In order to show that $KB \models \alpha$ we need to show that whenever KB is true, so is α. However this is not the case: note that in the yellow line $KB = True$ but $\alpha = False$. Thus KB does not entail α.

3. Here are two sentences in the language of first-order logic:

   (a) $\forall x \exists y \; (x \geq y)$
   (b) $\exists y : \forall x : (x \geq y)$

   (i) Assume that the variables range over all the natural numbers $0,1,2,\ldots$ and that the "$\geq$" predicate means "is greater than or equal to". Under this interpretation, translate (a) and (b) into English.

   (a) is translated as "For every natural number x, there is a natural number y that is less than or equal to x". (b) is translated as "There is a natural number y that is less than or equal to all natural numbers".

   (ii) Is (a) true under this interpretation? Is (b) true under this interpretation?

   Yes, (a) is true under this interpretation. You can always pick the number x itself to be the number y.

   Yes, (b) is true under this interpretation. You can pick 0 to be the number y.

   (iii) Does (a) logically entail (b)? Does (b) logically entail (a)? Justify your answers.

   (a) entails (b) under the domain of natural numbers (shown in (ii))

   However (a) does not entail (b) under the domain of integers. (a) states that for every integer there is an integer smaller than or equal to it, which is true since the set of integers is infinitely negative.

   However (b) is false since the set of integers, being infinitely negative, mean that for any integer y there will be some integer x s.t. $\sim(x \geq y)$, i.e. x<y.

   (b) logically entails (a). We will prove by resolution.

   First we Skolemize (b) to convert to CNF:

$$\exists y \forall x (x \geq y)$$
$$\forall x (x \geq D)$$

   Now let's assume ~(a). Convert to CNF:

$$\sim\left(\forall x \exists y(x \geq y)\right)$$
$$\equiv \exists x \forall y \sim(x \geq y)$$

Skolemize (a) to obtain the CNF:

$$\equiv \forall y \sim(E \geq y)$$

Now we have $\forall x(x \geq D) \wedge \forall y \sim(E \geq y)$

We unify both clauses with $\{x \backslash E, y \backslash D\}$ giving us:

$$(E \geq D) \wedge \sim(E \geq D)$$

This yields the empty clause. Hence ~(a) is False and (a) must be True. So (b) entails (a)

Next, convert ¬(a) to CNF:

$$\neg(\forall x : \exists y : (x \geq y))$$
$$\exists x : \forall y : \neg(x \geq y)$$
$$\forall y : \neg(D \geq y) \qquad \text{(Skolemization)}$$
$$\neg(D \geq y)$$

The two clauses unify with $\{x \backslash D, y \backslash C\}$ to derive the empty clause.

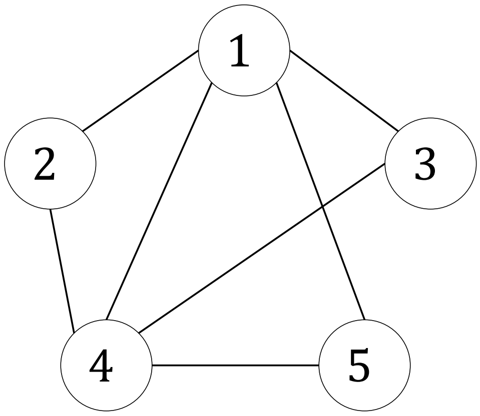
This can be written in intuitive natural language as well: suppose by contradiction that (b) does not entail (a). For simplicity we write $\neg(A \geq B)$ as $A < B$. By (b), there is some constant C such that for every x, $x \geq C$. Now, suppose for contradiction that there exists some D such that for every y, $D < y$ (both steps thus far were Skolemization steps). This, in particular, holds for C, thus $D < C$; moreover, $D \geq C$ by (b) (this is the unification step!). Putting them together we obtain $D \geq C > D$, a contradiction (this is the empty clause resolution step). Intuitively, any resolution can be written as a proof by contradiction.

4. Given a graph G = <V, E> we say that a subset of vertices $X \subseteq V$ is an *independent set* if no two vertices in X share an edge. Here we assume *pairwise* independent sets. I.e. each independent set has at most two vertices.

   a. Given a set of vertices $X \subseteq V$, write the constraint "no two vertices in X share an edge" in propositional logic. You may only use Boolean variables of the form $x_v \in \{\text{False, True}\}$ which indicate that x_v is part of the independent set. You may not refer to the set X in your solution, only the set V, and the edges in E. You can use basic arithmetic operators and basic logical operators.

$$\forall(u, v) \in E\big(\sim(x_u \wedge x_v)\big)$$

b. Write down the independent set constraints for the following graph in propositional logic. We assume that all independent sets have at most two vertices.



We have the following edge set $E = \{(1,2), (1,3), (1,4), (1,5), (2,4), (3,4), (5,6)\}$ giving us:

We go through every element of E. For the first pair we have (1, 2). From our rule:

$$\forall(u, v) \in E\big(\sim(x_u \wedge x_v)\big)$$

We unify such that {u\1, v\2} giving us $\sim(x_1 \wedge x_2)$. Repeat for all other tuples (i.e. coordinates) in E. We get this set of propositional sentences:

$$\sim(x_1 \wedge x_2)$$
$$\sim(x_1 \wedge x_3)$$
$$\sim(x_1 \wedge x_4)$$
$$\sim(x_1 \wedge x_5)$$
$$\sim(x_2 \wedge x_4)$$
$$\sim(x_3 \wedge x_4)$$
$$\sim(x_4 \wedge x_5)$$

c. Using resolution show that vertex 1 is not in any independent set. We assume that the independent set size is 2, and this is equivalent to showing that if x_1 is True, then x_2, x_3, x_4 are all False.

Convert to CNF:

$$\sim x_1 \vee \sim x_2$$
$$\sim x_1 \vee \sim x_3$$
$$\sim x_1 \vee \sim x_4$$
$$\sim x_1 \vee \sim x_5$$

We prove by contradiction by introducing the clause x_1, which means that vertex 1 is in the independent set. Thus we have:

$$x_1 \wedge \sim x_1 \vee \sim x_2$$
$$\equiv False \ \vee \sim x2$$
$$\equiv \sim x2$$

Similary too for the other clauses giving us:

$$\sim x_2$$
$$\sim x_3$$
$$\sim x_4$$
$$\sim x_5$$

Here x_2 to x_5 must all be False. We can prove this by resolution again. Let's assume x_2 is True, then:

$$x_2 \wedge \sim x_2 = False$$

Thus x_2 cannot be True. Same argument follows for x_3 to x_5