ELEC 5660 **Name:** ____________________
Midterm
March 20, 2024
Time Limit: 120 Minutes **Student ID:** ____________________

Please write down your answer on blank sheets clearly.
Please write down the steps to get the answer. A single final result gets no marks.
You are not allowed to take away the exam questions.

Question	Points	Score
1	25	
2	40	
3	40	
Total:	105	

1. Consider a robot moving in a bounded grid-based discretized map shown in Figure.1. White grids are traversable spaces, and gray grids represent obstacles and walls. The green and red grids represent the start and goal positions. The robot can only move in horizontal and vertical directions (from one grid to its four neighboring grids). The cost of one horizontal or vertical movement is 1. Each grid is denoted by a coordinate (x, y) as shown in the figure. Here we want to apply some path search algorithms to find a path between the start and the goal positions. Each free grid is associated with a graph node, and undirected edges are connected to its four neighbors if the neighbor is free and within the bound.

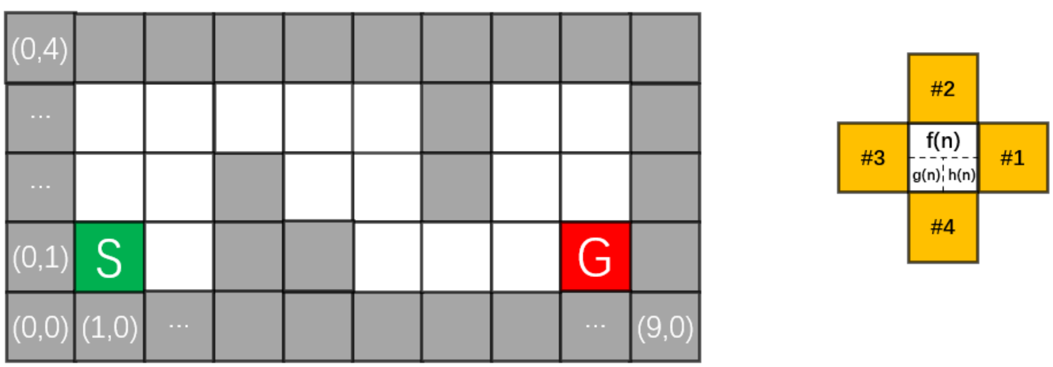

Figure 1: An illustration of the grid map.

The priority of picking a node in the open list is defined as:

1. Pick the node with lowest $f(n)$ in the list;
2. If some nodes have same $f(n)$, choose the one with smallest $h(n)$;
3. If some nodes have the same $h(n)$, choose the one with the parent node on the left. If none have a left parent, choose the one with the parent node at the bottom. If none

have a bottom parent, choose the one with the parent node on the right or at the top. **The figure on the right of Figure.1 shows the priority after expansion of the center node**;

4. If some nodes have the same parent node direction, choose the one with the largest $x+y$;
5. If some nodes have same $x+y$, choose the one with largest x;

(a) (10 points) Using the A* algorithm, perform a path search from the start (The grid named "S") to the goal position (the grid named "G"). Write down the coordinates of those expanded nodes in sequence and the final path. Suppose that a Manhattan heuristic is used, i.e., estimated cost at a point (x_i, y_j) is the Manhattan distance to the goal position (x_g, y_g) (the grid named "G"): $\|x_i - x_g\| + \|y_i - y_g\|$.

(b) (4 points) Does this heuristic give an optimal result in (b)? If yes, what property that the heuristic has, and why this property guarantees optimality? If no, give a heuristic producing the optimal path and write down the optimal path.

(c) (6 points) In practice, we often slightly modify the Manhattan heuristic by multiplying a factor 1.001. What advantage and problem will this modification bring?

(d) (5 points) Briefly describe the difference and advantage of A* algorithm over Dijkstra's algorithm.

Background for Question 2 & 3:

A quadrotor with a cable-suspended payload system is a typical robotic system with dual motion modes, depending on whether the cable is slack. As shown in Figure.2 right, if the cable is slack, i.e., the robotic system is in the slack motion mode, the movements of the quadrotor and payload do not interfere with each other (the motion of the quadrotor follows the classical dynamics model). If the cable is taut, i.e., the robotic system is in the taut motion mode, the dynamics model of the robotic system can be slightly more complex in this case.

We defined the states of this robot system as

$$\mathbf{X} = [x_{\mathcal{L}}, y_{\mathcal{L}}, z_{\mathcal{L}}, x_{\mathcal{Q}}, y_{\mathcal{Q}}, z_{\mathcal{Q}}, \phi_{\mathcal{Q}}, \theta_{\mathcal{Q}}, \psi_{\mathcal{Q}}, \dot{x}_{\mathcal{Q}}, \dot{y}_{\mathcal{Q}}, \dot{z}_{\mathcal{Q}}, \omega_{x_{\mathcal{Q}}}, \omega_{y_{\mathcal{Q}}}, \omega_{z_{\mathcal{Q}}}],$$

where $\mathbf{X}[0:3]$ is the position of the payload in the world frame, $\mathbf{X}[3:15]$ is the state of the quadrotor as the same in lecture notes.

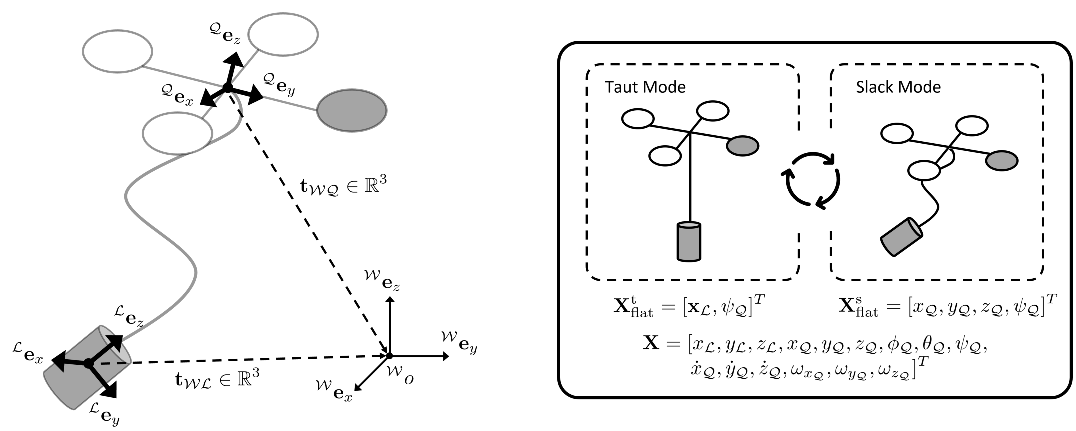

Figure 2: An illustration of the payload system.

2. We assume that the robotic system, as shown in Figure.2, is in slack mode. That means we only consider the movement of the quadrotor.

   Given elementary rotation matrices:

$$\mathbf{R_x}(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}, \quad \mathbf{R_y}(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \quad \mathbf{R_z}(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   (a) (4 points) Write down the rotation matrix from the body frame of the quadrotor $\mathcal{Q}$ to the world frame $\mathcal{W}$ in the $Z-Y-X$ Euler angle representation ($\mathbf{R}_{\mathcal{WQ}}(\phi, \theta, \psi) = \mathbf{R_z}(\psi) \cdot \mathbf{R_y}(\theta) \cdot \mathbf{R_x}(\phi)$).

   (b) (6 points) What conditions can cause the singularity of the Euler angle representation? Explain by providing the rotation matrices under the conditions.

   (c) (6 points) Write down the instantaneous body angular velocity $[\omega_x, \omega_y, \omega_z]^T$ given the derivative of Euler angle $[\dot{\phi}, \dot{\theta}, \dot{\psi}]^T$.

   (d) (4 points) It often uses a homogeneous representation of $SE(3)$ to represent a rigid body transformation. Given a transformation from body frame to the world frame $g_{\mathbf{t},\mathbf{R}}(\mathbf{p}) \in SE(3)$, write down its homogeneous representation $\mathbf{T}$. What do you think about the advantage of using such a homogeneous representation?

(e) (4 points) Please prove that a rigid body transformation g preserves **distance between points**, i.e., $\|\mathbf{p}_a - \mathbf{p}_b\| = \|g(\mathbf{p}_a) - g(\mathbf{p}_b)\|$ for all points $\mathbf{p}_a, \mathbf{p}_b \in \mathbb{R}^3$.

(f) (6 points) Please prove that a rigid body transformation g preserves **orientation between vectors**, i.e., $g_*(\mathbf{v}_a \times \mathbf{v}_b) = g_*(\mathbf{v}_a) \times g_*(\mathbf{v}_b)$ for all vectors $\mathbf{v}_a, \mathbf{v}_b \in \mathbb{R}^3$.

(g) (10 points) Given the rotation matrix $\mathbf{R}_{\mathcal{WQ}}$ and the translation vector $\mathbf{t}_{\mathcal{WQ}}$, as well as their first order derivative $\dot{\mathbf{R}}_{\mathcal{WQ}}$ and $\dot{\mathbf{t}}_{\mathcal{WQ}}$, please calculate the linear velocity of the quadrotor in the world frame and the body frame.

3. We consider the dynamic models of the suspended payload system now. We assume that the mass of the quadrotor $m_{\mathcal{Q}}$, the mass of the payload $m_{\mathcal{L}}$, the cable length l_0, and the gravity g are constant and known.

   (a) (10 points) First, we assume the robot system is in slack mode, i.e., we only consider the system dynamics of the quadrotor. A quadrotor's states and inputs can be written as algebraic functions of four carefully selected flat outputs and their derivatives. In our case, the flat outputs are $\mathbf{X}^{\mathrm{s}}_{\mathrm{flat}} = [x_{\mathcal{Q}}, y_{\mathcal{Q}}, z_{\mathcal{Q}}, \psi_{\mathcal{Q}}]$. Please write down the full states of the quadrotor, i.e., $\mathbf{X}_{\mathcal{Q}} = \mathbf{X}[3:15]$, given the flat outputs.

   (b) (15 points) When the robot system is in the taut mode, the tension in the cable is not zero and the distance between the quadrotor and the payload is equal to the cable length. In this case, the robot system has different flat outputs $\mathbf{X}^{\mathrm{t}}_{\mathrm{flat}} = [x_{\mathcal{L}}, y_{\mathcal{L}}, z_{\mathcal{L}}, \psi_{\mathcal{Q}}]$. Please write down the full states of the robot system, i.e., $\mathbf{X}$, given the flat outputs.

   (c) (5 points) What are the advantages of the system's differential flatness? Please list at least three (3) advantages.

   (d) (10 points) (Bonus) As shown above, this robotic system has two different subsystem dynamics. For each subsystem, the number of flat outputs is different. Now we want the system state to be calculated using the same set of flat outputs regardless of the robot's motion mode. What should be a suitable set of flat outputs? Given this set of flat outputs, how can we calculate the full state of the system?