Parameter Estimation of Linear FM Signals Embedded in White Gaussian Noise

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Abstract

The subject of parameter estimation in linear FM signals embedded in White Gaussian Noise has been extensively studied. This paper will present an accurate means of estimating the unknown initial frequency f_0 and frequency sweep rate m for a sinusoidal signal. Experimental results from an arbitrary signal will also be presented.

1 Introduction

Estimating parameters of unknown sinusoidal signals has many important applications in signal processing and communication theory [1]. Using the observations of a sinusoid embedded in White Gaussian Noise (WGN)

$$x[n] = \cos(2\pi(f_0 n + (1/2)mn^2)) + w[n] \tag{1}$$

where w[n] is WGN with variance σ^2, f_0 is the initial frequency, m is the sweep rate, and n = 0, 1, ..., N-1, it is possible to accurately estimate the unknown parameters f_0 and m from a linear FM signal. For this paper, the maximum likelihood estimator (MLE) will be used to estimate the unknown parameters of an arbitrary signal (fig. 1) with the form of (1). An estimation scheme based on the positive instantaneous frequency (PIF) from the analytic signal will also be examined [2].

2 Maximum Likelihood Estimation

The maximum likelihood estimation (MLE) method is based on the idea that the best estimate of an unknown parameter (or parameters) is the one that maximizes the likelihood function. The likelihood function can be viewed as a probability density function (PDF) that, with x fixed, is dependent on some unknown parameter θ such that

$$p(\mathbf{x};\boldsymbol{\theta}) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-\theta)^2\right]. \tag{2}$$

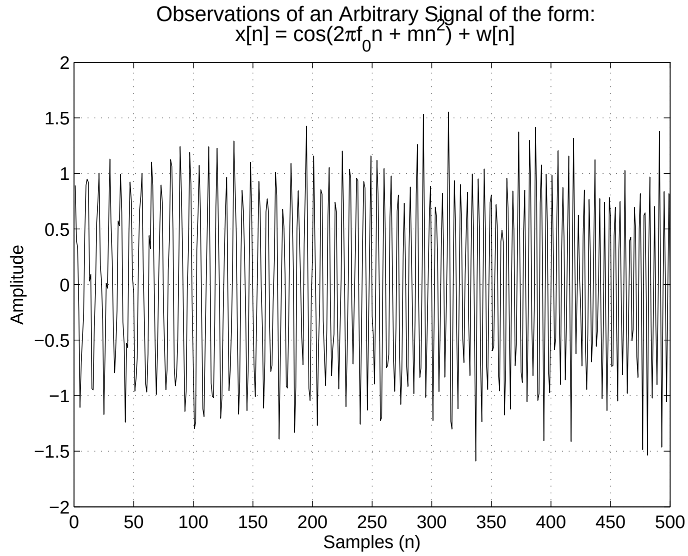

Figure 1: Observations of arbitrary signal.

Clearly, the quantity θ is the parameter to affect the likelihood since all other parameters are fixed. Of course taking the logarithm of this function isolates the argument. The result is termed the log-likelihood function. Taking the partial derivative with respect to the unknown parameter θ and setting the result equal to zero minimizes the log-likelihood function, which, of course, maximizes the likelihood function. For estimating the intial frequency and sweep rate of (1), where there are N independent and identically distributed (iid) observations, the likelihood function is maximized by minimizing J for a signal $s[n]$ parameterized by θ

$$J(\theta) = \sum_{n=0}^{N-1} (x[n] - s[n;\theta]))^2. \tag{3}$$

For a signal $s[n]$ parameterized by $\theta = [f_0 \; m]^T$, (3) can be written as

$$J(f_0, m) = \sum_{n=0}^{N-1} (x[n] - \cos(2\pi(f_0 n + (1/2)mn^2)))^2. \tag{4}$$

By expanding the square, the function becomes

$$J(f_0, m) = \sum_{n=0}^{N-1} (x^2[n] - 2\cos(2\pi(f_0 n + (1/2)mn^2)) + \cos^2(2\pi(f_0 n + (1/2)mn^2))). \tag{5}$$

Under the assumption that f_0 is not near 0 or 0.5, it is clear that by maximizing the cross term $2\cos(2\pi f_0 n + (1/2)mn^2)$, the function will be minimized since $x[n]$ is fixed and $\cos^2(2\pi(f_0 n + (1/2)mn^2))$, after applying some trigonometric properties, converges to $\frac{N}{2}$.

While the MLE is not optimal in general, for large enough data records or as $N \to \infty$, the MLE is asymptotically the minimum variance unbiased (MVU) estimator [1]. In fact, if the MVU exists, the MLE will produce it [1].

2.1 Sinusoidal Parameter Estimation

The MLE of the initial frequency of a sinusoid is well known and is obtained by maximizing the periodogram

$$I(f) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]\exp(-j2\pi f_0 n)\right|^2 \quad (6)$$

provided that the intitial frequency f_0 over the range $0 < f_0 < 0.5$, is not near 0 or 0.5 [1]. From fig. 2, it can be seen that the sinusoid embedded in WGN has an initial frequency of $f_0 \approx 0.1$ that sweeps to approximately 0.25. This occurs over N number of samples where N = 500. This gives an approximate sweep rate of 0.0003, which is in the known range of m, since $0 \leq m \leq 0.001$. Of course, this type of subjective estimation is unacceptable and we must resort to a more robust estimation scheme. Additionally, the problem at hand is to find an MLE for *both* f_0 and m, which renders (6) of limited use for this purpose.

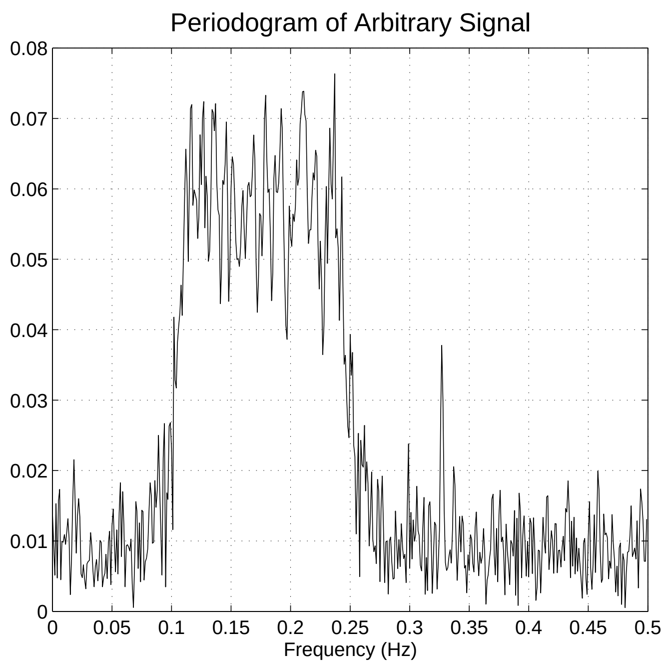

Figure 2: Power spectrum of arbitrary signal (Frequency is normalized).

As stated in (4), Estimation of the two parameters f_0 and m can be computed by extending θ to $\theta = [f_0\ m]^T$ and maximizing the crossterm from (5)

$$J(f_0, m) = \sum_{n=0}^{N-1} 2\cos(2\pi(f_0 n + (1/2)mn^2)). \quad (7)$$

This can be visualized as a *real* Fourier Transform of a polynomial in n. Recall from (5) that this is the function which needs to be maximized in order to minimize the the argument of the likelihood function.

A coarse grid search was used over the known ranges of f_0 and m. The estimator yields $f_0 = 0.1$ and $m = 0.0003$. It is worth pointing out that maximizing (7) produces a sharper maximum than using the Fourier transform, which tends to smear the region around the true parameter values (fig. 3). Recall from fig. 2 that the periodogram provides approximately the same values by inspection, but the determination of the initial frequency $f_0 = 0.1$ and the final frequency $f_{final} = 0.25$ was subjective, at best.

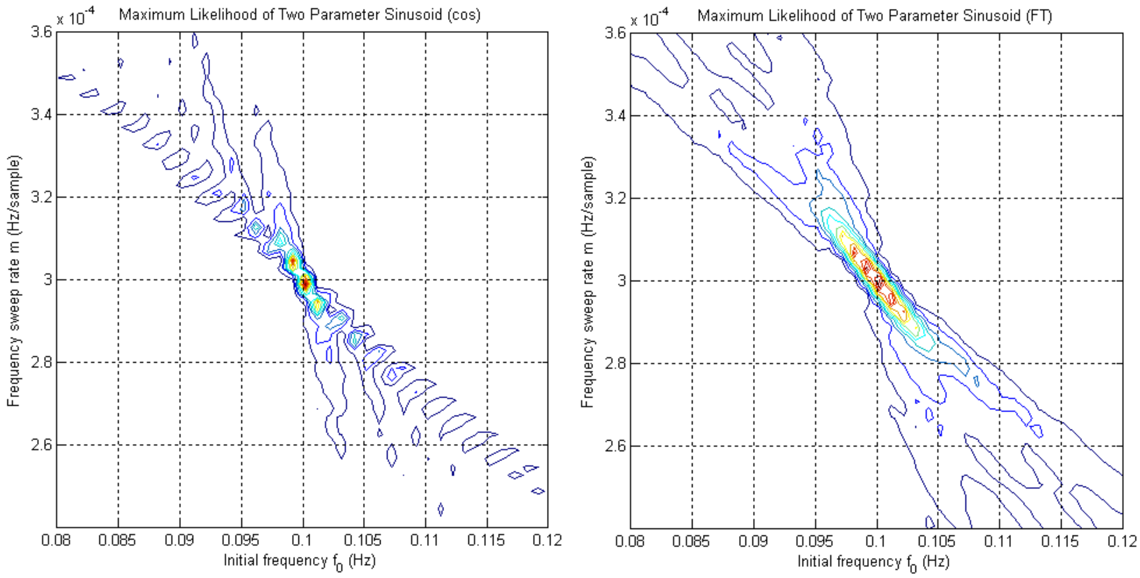

Figure 3: Estimation of f_0 and m through maximizing the likelihood function (Frequency is normalized).

Using the estimated parameters, the model of the arbitrary signal (1) is constructed without noise. The purpose of this is to isolate the range of the parameters over the grid which are most likely to produce the maximum, since noise will artificially produce false maxima. The result allows the grid parameters to be reduced to a finer scale and improves the estimate. (See figs. 4 and 5)

3 Noise

Often, it is desirable, if not necessary, to have an estimate of the noise variance in the unknown signal. To be sure, there are many techniques suitable for this task. Two very simple ways in which to determine an estimate involve the pdf of the noise in a region of the frequency transform known not to contain any signal. Recall from fig. 2, that the frequency content of the signal was determined to be in the approximate range of $0.1 \leq f \leq 0.25$. By isolating a portion of the transform outside this range, say $0.3 \leq f \leq 0.5$, the noise is isolated and the pdf is due to the noise variance. In effect, this is filtering out the signal from the noise. Alternatively, if the amplitude of the signal were known, reconstructing the model of the signal without noise and comparing to the noise corrupted signal would give an estimate of the power in the noise. Since the estimation of the initial frequency and sweep rate have been determined, those parameters can be fixed and the problem becomes one of estimating the amplitude

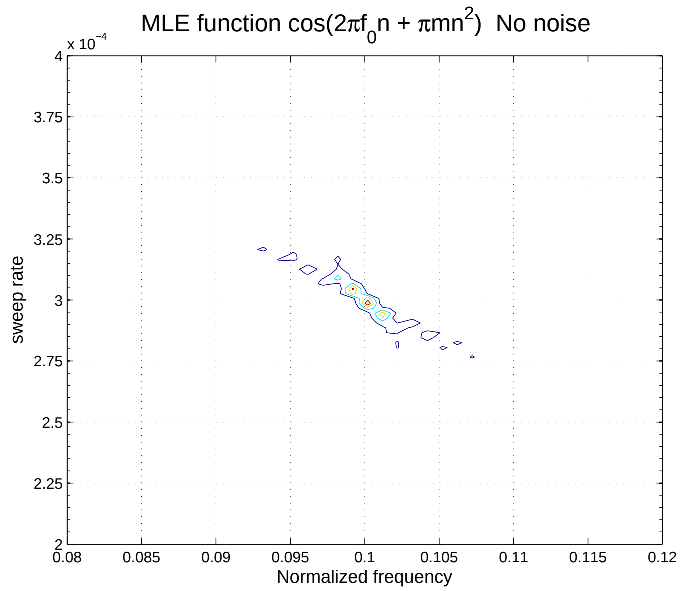

Figure 4: Maximizing function that isolates the likely values of the model with the estimated parameters in the absence of noise.

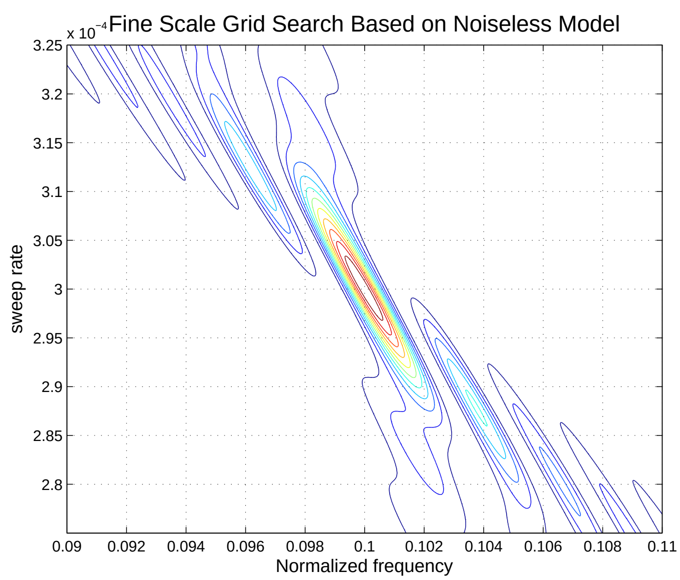

Figure 5: The MLE for the intial frequency f_0 and sweep rate m of the arbitrary (observed) signal, recalculated using the the finer grid search based on the noiseless model. The estimation of f_0 and m from the test data shows greater precision with one strong maximum over the parameters.

of a sinusoid. This is a straightforward calculation since it is known that the amplitude A of a sinusoid is given by

$$\hat{A} = \sqrt{\hat{\alpha}_1^2 + \hat{\alpha}_2^2} \tag{8}$$

where $\hat{\alpha}_1 = \frac{2}{N}\sum_{n=0}^{N-1} x[n]\cos 2\pi \hat{f}_0 n$ and $\hat{\alpha}_2 = \frac{2}{N}\sum_{n=0}^{N-1} x[n]\sin 2\pi \hat{f}_0 n$. (The sweep parameter m has no effect on the amplitude of the sinusoid and so it has been left out. Of course it could easily be included for completeness.) The amplitude of the signal is estimated as $\hat{A} = 1.073 \approx 1$. Estimating the noise from the power spectrum in this case is quite simple since the difference in power levels is due strictly to the noise (fig. 6). Using a least squares fit over the region not associated with the linear FM signal, the power difference between the signal and the noise is approximately 4dB. This gives an approximate variance of 0.04. Incorporating this variance into the model shows excellent agreement between the arbitrary data record and the model (figs 7 and 8).

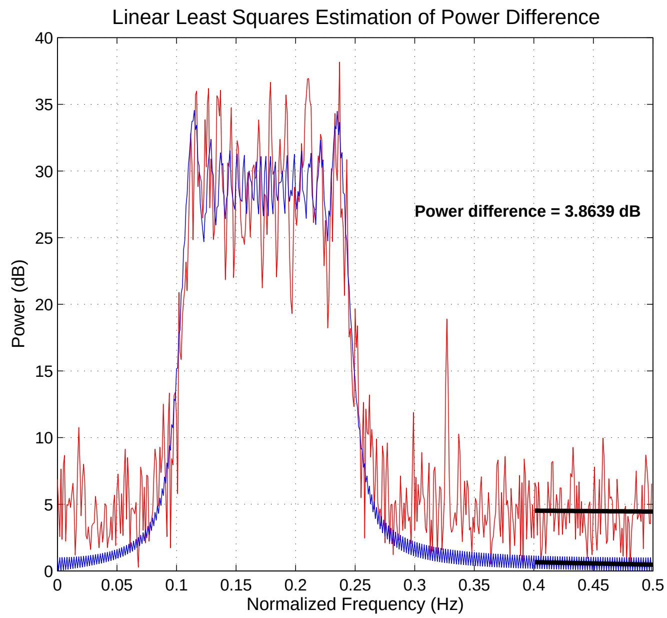

Figure 6: Power spectrum of signal (noise) versus model (no noise). The difference is due to the noise power, approximately 4dB.

4 Estimation Using Phase Differences

In order to use the estimator listed in [2], it is necessary to compute the analytic signal. Since the model given in (1) is real by construction, the analytic signal is defined as

$$x_{analytic}[n] = x[n] + j\hat{x}[n] \tag{9}$$

where $\hat{x}[n]$ is the Hilbert transform of $x[n]$ [4]. Upon taking the Hilbert transform of the model (1), the result can be written as

$$x_{analytic}[n] = (\cos(2\pi(f_0 n + (1/2)mn^2)) + j\sin(2\pi(f_0 n + (1/2)mn^2))) + w[n] = e^{j(2\pi(f_0 n+(1/2)mn^2)+w[n])} \tag{10}$$

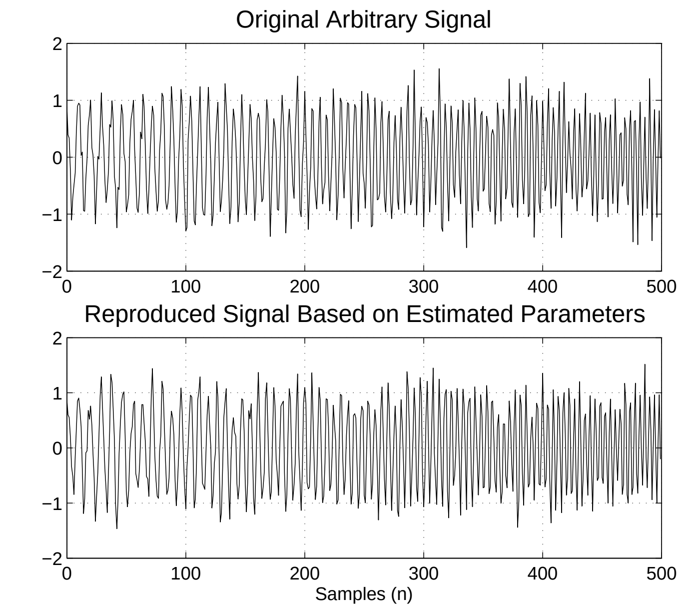

Figure 7: Signal comparison - original arbitrary (above) and reconstructed figure using estimated parameters.

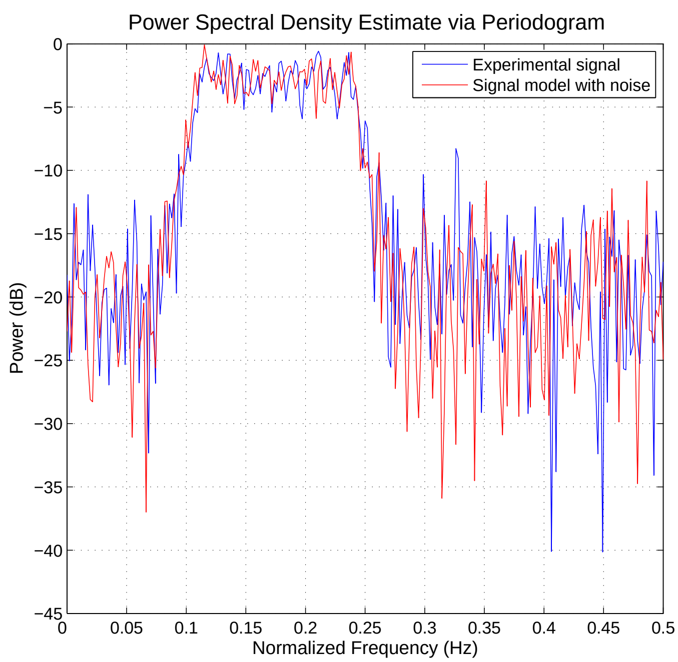

Figure 8: Power spectral density comparison - original test signal (blue) model signal with estimated noise parameter (red).

In this form, the conjugate of (6) is easily obtained by negating the argument of the exponent. The phase difference can now be obtained by multiplying the analytic signal by a one-sample delay of the conjugate of the analytic signal, such that

$$\Delta phase = \angle x_n * \angle x^*_{n+1} \qquad n = 0, 1, ..., N-1. \tag{11}$$

This is the so-called instantaneous frequency of the signal and provides an estimate of frequency with repect to time [2]. What is immediately clear from fig. 9 is that the estimate is quite noisy. To compensate for this somewhat, a linear least squares was fit to the estimate. Now, we see that the initial frequency $f_0 \approx 0.1$ and linearly increases to 0.25, which is an increase of 0.15 over 500 samples giving a sweep rate of $m = \frac{0.15Hz}{500samples} = 0.0003$, in very good agreement to the estimate provided by the MLE. The major advantage to this approach is that it is very computationally efficient [2].

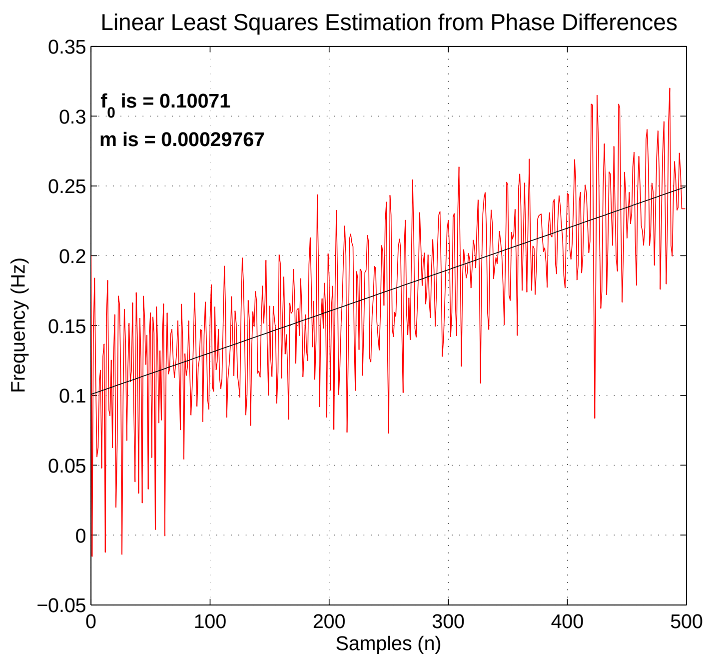

Figure 9: Instantaneous frequency analysis of arbitrary signal

References

[1] S. Kay, *Fundamentals of statistical signal processing: estimation theory.* Upper Saddle River, NJ: Prentice Hall, Inc., 1993.

[2] S. Kay, "A fast and accurate single frequency estimator," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 37, pp. 1987-1989, Dec. 1989.

[3] P. Djuric and S. Kay, "Parameter estimation of chirp signals," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 38, pp. 2118-2126, Dec. 1990.

[4] A. Rao and R. Kumaresan, "A Parametric Modeling Approach to Hilbert Transformation," *IEEE Signal Processing Letters*,vol. 5, pp. 15-17, Jan. 1998.

[5] L. B. Jackson, *Digital Filters and Signal Processing, 3rd ed.*, Boston, MA: Kluwer, 1996.

A Appendix - MATLAB™code

```
clear all
close all

load ELE661data2.mat

%%%%%%%%%%%%%%%%%%%%Positive Instantaneous Frequency Estimation
fs = 1;
N = length(x);
n = 0:N-1;
f = (0:N-1)/(2*N);
hil_x = hilbert(x);
sig_conj = conj(hil_x);
pha_dif = angle(sig_conj(1:(N-1),:).*hil_x(2:N,:));
pha_dif(N,:) = angle(hil_x(N,:));
pha_dif = pha_dif.*(fs/(2*pi));
pha_dif(N-2:N) = pha_dif(N-3);
%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%%%%Least Squares Estimation for Noisy PIF
H = [ones(1,500)' n'];
lse = inv(H'*H)*H'*pha_dif;
llse = lse(1) + n*lse(2);
plot(n,pha_dif,'r',n,llse,'k')
text(4,0.32,['f_0 is = ',num2str(lse(1))])
text(4,0.3,['m is = ',num2str(lse(2))])
title('Linear Least Squares Estimation from Phase Differences')
xlabel('Samples (n)'),ylabel('Frequency (Hz)'),grid
%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%%%%%Coarse Search
load ELE661data2.mat
%x = cos(2*pi*(0.1*n + (1/2)*0.0003*n.^2))'; noise free model
mm = linspace(0,0.001,N);
freq = linspace(0.0,0.5,N);
for kk = 1:500
    for k = 1:500
```

```
        hol2(kk,k) = sum(2*x'.*cos(2*pi*freq(k)*n + pi*mm(kk)*(n.^2)));
    end
end
%%%%%%%%%%%%%%%%%%%%%

% %%%%%%%%%%%%%%%%%%%%%Coarse Search FT
load ELE661data2.mat
%x = cos(2*pi*(0.1*n + (1/2)*0.0003*n.^2))'; noise free model
mm = linspace(0,0.001,500);
freq = linspace(0.0,0.5,500);
for kk = 1:500
    for k = 1:500
        hol3(kk,k) = sum(2*x'.*exp(-j*(2*pi*n*freq(k) + pi*mm(kk)*n.^2)));
    end
end
% %%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%%%%Fine Search Based on Noise Free Model
load ELE661data2.mat
mm = linspace(2.75e-4,3.25e-4,N);
freq = linspace(0.09,0.11,N);
for kk = 1:500
    for k = 1:500
        hol2(kk,k) = sum(2*x'.*cos(2*pi*freq(k)*n + pi*mm(kk)*(n.^2)));
    end
end
figure
subplot(121)
contour(freq,mm,hol2.^2),grid,axis([0.08 0.12 0.24e-3 0.36e-3])
title('Maximum Likelihood of Two Parameter Sinusoid (cos)')
xlabel('Initial frequency f_0'),ylabel('Frequency sweep rate m')
subplot(122),contour(freq,mm,abs(hol3)),grid,axis([0.08 0.12 0.24e-3 0.36e-3])
title('Maximum Likelihood of Two Parameter Sinusoid (FT)')
xlabel('Initial frequency f_0'),ylabel('Frequency sweep rate m')
```