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11.14

(a) $y_i = z_i'\hat{\Gamma}\beta + e_i$

$$\Rightarrow \hat{\beta} = \left(\sum_{i=1}^{n} \hat{\Gamma}' z_i z_i' \hat{\Gamma}\right)^{-1} \left(\sum_{i=1}^{n} \hat{\Gamma}' z_i y_i\right)$$

$$\hat{\beta} = \left(\sum_{i=1}^{n} \hat{\Gamma}' z_i z_i' \hat{\Gamma}\right)^{-1} \left(\sum_{i=1}^{n} \hat{\Gamma} z_i \cdot z_i \Gamma\right) \cdot \beta + \left(\sum_{i=1}^{n} \hat{\Gamma}' z_i z_i' \hat{\Gamma}\right)^{-1} \left(\sum_{i=1}^{n} \hat{\Gamma}' z_i e_i\right)$$

Since

$$\hat{\Gamma} \xrightarrow[p]{} \Gamma \quad \text{and} \quad E[e_i | z_i] = 0 \Rightarrow E[z_i e_i] = 0$$

We have

$$\hat{\beta} \xrightarrow[p]{} \beta$$

(b) When $\beta = 0$

$$\sqrt{n}(\hat{\beta} - \beta) = \left(\frac{1}{n}\sum_{i=1}^{n} \hat{\Gamma}' z_i z_i' \hat{\Gamma}\right)^{-1} \left(\sqrt{n} \cdot \frac{1}{n}\sum_{i=1}^{n} \hat{\Gamma}' z_i e_i\right)$$

$$\xrightarrow[d]{CLT} \cancel{Q_{\Gamma z_i}} \; N\left(0, Q_{\Gamma z_i}^{-1} \cdot E[\hat{\Gamma}' z_i z_i \hat{\Gamma}' e_i^2] \cdot Q_{\Gamma z_i}^{-1}\right)$$

where $Q_{\Gamma z_i} = E[\hat{\Gamma}' z_i z_i' \hat{\Gamma}]$

(c) Setting $\beta = 0$ is useful, since we do not know the true value of Γ, which shows up in $\hat{\beta}$.

(d) Reject H_0, if $T > c$ for a given significance level α, where $T = \frac{\hat{\beta}}{\cancel{\Sigma}} \div Q_{\Gamma z_i}^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} \hat{\Gamma}' z_i z_i \hat{\Gamma} \hat{e}_i^2\right) Q_{\Gamma z_i}$ and c satisfying $\alpha = 2(1 - \Phi(c))$.

12.2

GLS:

$$\tilde{\beta}_{GLS} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y, \text{ where } \Sigma^{-1} = \begin{bmatrix} \sigma_1^{-2} & & & \\ & \sigma_2^{-2} & & \\ & & \ddots & \\ & & & \sigma_n^{-2} \end{bmatrix}$$

~~OLS:~~

IV:

$$\hat{\beta}_{IV} = (Z'X)^{-1}Z'y$$

Let $Z = \Sigma^{-1}X$ or $z_i = \sigma_i^{-2} \cdot x_i$

then $\tilde{\beta}_{GLS} = \hat{\beta}_{IV}$

12.3

$$\hat{e}_{OLS} = y - X\hat{\beta}$$

$$\hat{e}_{OLS} = y - X(X'X)^{-1}X'y$$

$$\hat{e}_{OLS}^2 = y'(I - X(X'X)^{-1}X')y \qquad \text{Idempodent matrix } P_X$$

$$\tilde{e}_{IV} = y - X(X'P_ZX)^{-1}X'P_Zy \qquad P_Z \triangleq Z(Z'Z)^{-1}Z'$$

$$\tilde{e}_{IV}^2 = y'(I - X(X'P_ZX)^{-1}X'P_Z)y$$

There is no way to determine which one is larger.

Another way to see this is

$\hat{y}_{OLS}$ is project y on span of X.

$\hat{y}_{IV}$ is $X \cdot (X'P_ZX)^{-1}X'P_Z \cdot y$, equals to $P_Z^{-1} \cdot P_ZX(X'P_ZX)^{-1}X'P_Zy$

equals to $P_Z^{-1}\hat{X}(\hat{X}'\hat{X})^{-1}\hat{X}y$,

which is projecting y to z then times P_Z^{-1}, which is undetermined.