Reproducing Kernel Hilbert Spaces

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Let $\mathcal{X}$ be an arbitrary set and $\mathcal{H}$ be a Hilbert space of real-valued functions on $\mathcal{X}$, endowed by the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Let L_x be the evaluation functional over $\mathcal{H}$ that evaluates each function at a point $x \in$: a

$$L_x : f \to f(x), \quad \forall f \in \mathcal{H}.$$

A Hilbert space $\mathcal{H}$ is called a *reproducing kernel Hilbert space* if, for all $x \in \mathcal{X}$, the map L_x is continuous at any $f \in \mathcal{H}$, namely, there exists some $C > 0$ such that

$$|L_x(f)| = |f(x)| \leq C\|f\|_{\mathcal{H}}, \quad \forall f \in \mathcal{H}.$$

By the Riesz representation theorem, for all $x \in \mathcal{X}$, there exists a unique element $K_x \in \mathcal{H}$ with the reproducing property

$$f(x) = L_x(f) = \langle f, K_x \rangle_{\mathcal{H}}, \quad \forall f \in \mathcal{H}.$$

Also

$$K_x \in \mathcal{H} \implies \forall x' \in \mathcal{X}, \exists K_{x'} \in \mathcal{H}, \ni L_{x'}(K_x) \equiv K_x(x') = \langle K_x, K_{x'} \rangle_{\mathcal{H}}.$$

This defines the **reproducing kernel** $K(x, x') = \langle K_x, K_{x'} \rangle_{\mathcal{H}}$.

Define the class of functions

$$\mathcal{H}_0 := \left\{ f : f(x) = \sum_{i=1}^{m} \alpha_i K(x, x_i), m \in \mathbb{N}, \alpha_i \in \mathbb{R}, x_i \in \mathcal{X} \right\}.$$

For some $f, g \in \mathcal{H}_0$ with $f(\cdot) = \sum_{i=1}^{m} \alpha_i K(\cdot, x_i)$ and $g(\cdot) = \sum_{j=1}^{n} \beta_j K(\cdot, z_j)$. Define an inner product on $\mathcal{H}_0$ by

$$\langle f, g \rangle = \left\langle \sum_{i=1}^{m} \alpha_i K(\cdot, x_i), \sum_{j=1}^{n} \beta_j K(\cdot, z_j) \right\rangle := \sum_{i,j} \alpha_i \beta_j K(x_i, z_j).$$

Let $\mathcal{H}$ be the completion of $\mathcal{H}_0$, i.e.

$$\mathcal{H} = \overline{\mathcal{H}}_0.$$

Properties

1. $\forall f \in \mathcal{H}, x \in \mathcal{X}$,
$$\langle f, K(\cdot, x)\rangle = f(x)$$

Proof. 1. For $f \in \mathcal{H}$, by definition there exists a Cauchy sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H}_0$ such that $f_n \to f$, namely, $f(x) = \lim_{n\to\infty} f_n(x), \forall x \in \mathcal{X}$. For each f_n, it can be expressed as

$$f_n(\cdot) = \sum_{i=1}^{m} \alpha_i^n K(\cdot, x_i)$$

So

$$\langle f_n, K(\cdot, x)\rangle = \sum_{i=1}^{m} \alpha_i^n K(x, x_i)$$

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