AS.180.633: Econometrics **Spring 2020**

Homework 2: Suggested Solutions

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$\widehat{\boldsymbol{\beta}}$

3.3

Using matrix algebra, show $\boldsymbol{X}'\widehat{\boldsymbol{e}} = \mathbf{0}$

Proof. By $\widehat{\boldsymbol{e}} = \boldsymbol{M}\boldsymbol{y}$,

$$\begin{aligned}\boldsymbol{X}'\widehat{\boldsymbol{e}} &= \boldsymbol{X}'\boldsymbol{M}\boldsymbol{y}\\ &= (\underbrace{\boldsymbol{M}\boldsymbol{X}}_{=\mathbf{0}})'\boldsymbol{y}\\ &= \mathbf{0},\end{aligned}$$

where $\boldsymbol{M}$ is the annihilator matrix associated with $\boldsymbol{X}$. □

3.5

Let $\widehat{\boldsymbol{e}}$ be the OLS residual from a regression of $\boldsymbol{y}$ on $\boldsymbol{X}$. Find the OLS coefficient from a regression of $\widehat{\boldsymbol{e}}$ on $\boldsymbol{X}$.

Proof. Suppose the population regression function is $\widehat{\boldsymbol{e}} = \boldsymbol{X}\boldsymbol{\gamma} + \boldsymbol{\eta}$. The OLS coefficient from a regression of $\widehat{\boldsymbol{e}}$ on $\boldsymbol{X}$ is

$$\widehat{\boldsymbol{\gamma}}_{\mathrm{OLS}} = \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\underbrace{\boldsymbol{X}'\widehat{\boldsymbol{e}}}_{=\mathbf{0}} = \mathbf{0}.$$

The intuition is that $\widehat{\boldsymbol{e}}$ is the "leftover" effect on $\boldsymbol{y}$ after $\boldsymbol{X}$ "absorbs" relevant information in $\boldsymbol{y}$. So, it is not surprising that $\boldsymbol{X}$ has no (linear) effect on $\widehat{\boldsymbol{e}}$.

Or you can argue that the orthogonality between $\boldsymbol{X}$ and $\widehat{\boldsymbol{e}}$ indicated in Problem 3.3 implies the projection of $\widehat{\boldsymbol{e}}$ onto the column space of $\boldsymbol{X}$ is a zero vector. □

3.7

Show that if $\boldsymbol{X} = [\boldsymbol{X}_1 \ \boldsymbol{X}_2]$ then $\boldsymbol{P}\boldsymbol{X}_1 = \boldsymbol{X}_1$ and $\boldsymbol{M}\boldsymbol{X}_1 = \boldsymbol{0}$.

Proof. It is apparent that

$$\begin{aligned}[\boldsymbol{X}_1 \ \boldsymbol{X}_2] &= \boldsymbol{X} \\ &= \boldsymbol{P}\boldsymbol{X} \\ &= \boldsymbol{P}[\boldsymbol{X}_1 \ \boldsymbol{X}_2] \\ &= [\boldsymbol{P}\boldsymbol{X}_1 \ \boldsymbol{P}\boldsymbol{X}_2].\end{aligned}$$

Since $\boldsymbol{X}_1$ and $\boldsymbol{P}\boldsymbol{X}_1$ have the same dimensions, it must be the case that $\boldsymbol{P}\boldsymbol{X}_1 = \boldsymbol{X}_1$ and $\boldsymbol{P}\boldsymbol{X}_2 = \boldsymbol{X}_2$. Also, $\boldsymbol{0} = \boldsymbol{M}\boldsymbol{X} = [\boldsymbol{M}\boldsymbol{X}_1 \ \boldsymbol{M}\boldsymbol{X}_2]$ implies $\boldsymbol{M}\boldsymbol{X}_1 = \boldsymbol{0}$ and $\boldsymbol{M}\boldsymbol{X}_2 = \boldsymbol{0}$.

The intuition is that $\boldsymbol{P}\boldsymbol{X}_1$ is the projection of $\boldsymbol{X}_1$ onto the column space of $\boldsymbol{X}$. Since $\boldsymbol{X}_1$ already lives in the space, the projection matrix $\boldsymbol{P}$ does not act on it, implying $\boldsymbol{P}\boldsymbol{X}_1 = \boldsymbol{X}_1$.

A similar argument applies to $\boldsymbol{M}$. □

Remark. See Fig. 1 below for an illustration.

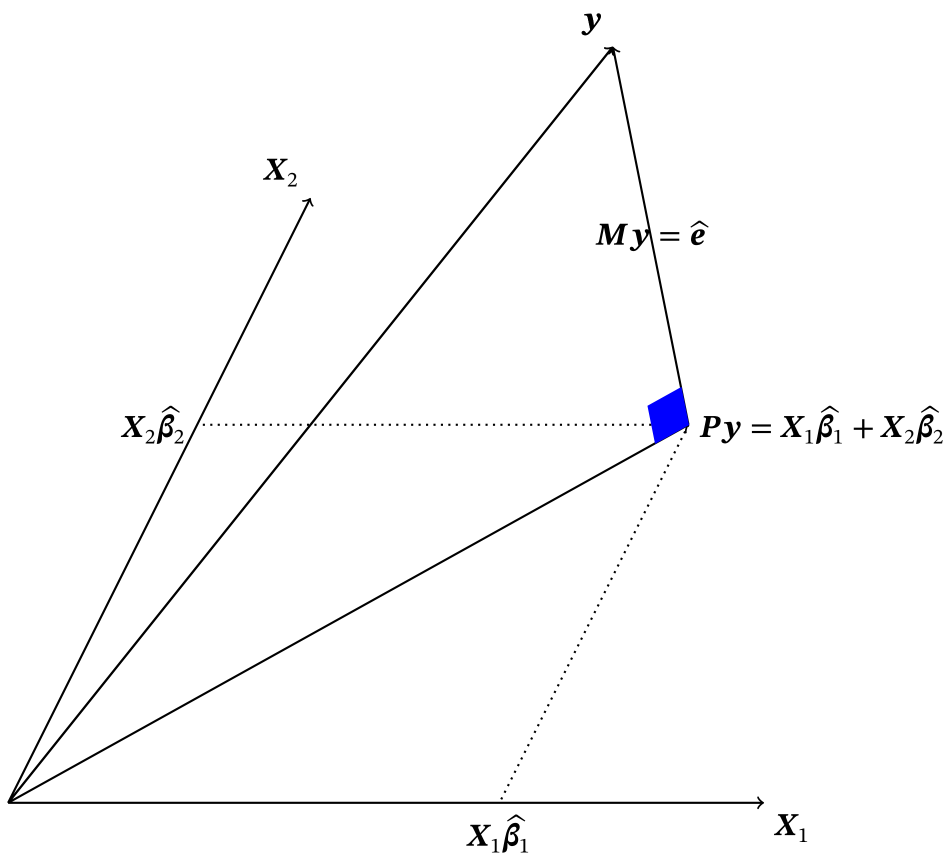

Figure 1: The Orthogonal Projection of $\boldsymbol{y}$ onto span$(\boldsymbol{X}_1, \boldsymbol{X}_2)$.

Frisch-Waugh-Lovell Theorem