Uniform Integrability

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1 Motivation

The original motivation of uniform integrability is to extend the classical dominated convergence theorem (DCT). Roughly speaking, the DCT asserts that if a sequence of R.V.-s $X_n \xrightarrow{\text{a.s.}} X$, and $|X_n| \le Y \in \mathcal{L}^1(\Omega)$ for all n, then the moments convergence holds: $\mathbb{E}[X_n] \to \mathbb{E}[X]$ and $X_n \xrightarrow{L_1} X$. The extension comes with two steps. In step one, the almost surely convergence can be weakended to convergence in probability, i.e., $X_n \xrightarrow{\mathsf{P}} X$. In step two, the relaxation is not trivial, we shall show that such condition in place is called *uniform integrability*. This extension works because we obtain such an equivalence relation between convergence in probability and convergence in moments:

$$X_n \xrightarrow{\mathsf{P}} X \ + \ \{X_n\}_n \ \text{ is U.I.} \iff X_n \xrightarrow{L_1} X$$

2 definition of uniform integrability

definition:

A collection of R.V.-s $\{X_\alpha, \alpha \in \mathcal{I}\}$ is called *uniform integrability* (U.I.) if

$$\lim_{M\to\infty} \sup_{\alpha \in \mathcal{I}} \mathbb{E}[|X_\alpha| \mathbb{1}\{|X_\alpha| > M\}] = 0.$$

Lemma 1. *Let Y be integrable and suppose that $|X_\alpha| \le Y$ for all α, then $\{X_\alpha\}$ is U.I.. In particular, any finite collection of integrable R.V.-s is U.I.. Further, if X_α is U.I. then* $\sup_\alpha \mathbb{E}[|X_\alpha|] < \infty$.

Proof.

□

However, $\sup_\alpha \mathbb{E}[|X_\alpha|]$ does not ensure $\{X_\alpha, \alpha \in \mathcal{I}\}$ is U.I. This can be seen by the following example:

Example 1: Let X_n be binary R.V.-s with $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n}$ and $\mathbb{P}(X_n = n) = \frac{1}{n}$. Then $\mathbb{E}[X_n] = 1$ for each $n \in \mathbb{N}$. For any constant $M \in \mathbb{R}$, $\mathbb{E}[|X_n| \mathbb{1}\{|X_n| > M\}] = \mathbb{E}[|X_n|], \forall n > \lceil M \rceil$. It follows that

$$\sup_{n\in\mathbb{N}} \mathbb{E}[|X_n| \mathbb{1}\{|X_n| > M\}] = 1$$

and therefore

$$\lim_{M\to\infty} \sup_{n\in\mathbb{N}} \mathbb{E}[|X_n \mathbb{1} \{|X_n| > M\}|] = 1 \neq 0.$$

♣