AS.180.633: Econometrics **Spring 2020**

Homework 8: Suggested Solutions

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11.14

Take the model

$$
\begin{aligned}
y_i &= \boldsymbol{\pi}_i' \boldsymbol{\beta} + e_i \\
\boldsymbol{\pi}_i &= \mathbb{E}[\boldsymbol{x}_i | \boldsymbol{z}_i] = \boldsymbol{\Gamma}' \boldsymbol{z}_i \\
\mathbb{E}[e_i | \boldsymbol{z}_i] &= 0
\end{aligned}
$$

where y_i is scalar, $\boldsymbol{x}_i$ is a k vector and $\boldsymbol{z}_i$ is an ℓ vector. $\boldsymbol{\beta}$ and $\boldsymbol{\pi}_i$ are $k \times 1$ and $\boldsymbol{\Gamma}$ is $\ell \times k$. The sample is $(y_i, \boldsymbol{x}_i, \boldsymbol{z}_i : i = 1, \cdots, n)$ with $\boldsymbol{\pi}_i$ unobserved.

Consider the estimator $\widehat{\boldsymbol{\beta}}$ for $\boldsymbol{\beta}$ by OLS of y_i on $\widehat{\boldsymbol{\pi}}_i = \widehat{\boldsymbol{\Gamma}}' \boldsymbol{z}_i$, where $\widehat{\boldsymbol{\Gamma}}$ is the OLS coefficient from the multivariate regression of $\boldsymbol{x}_i$ on $\boldsymbol{z}_i$.

(a) Show that $\widehat{\boldsymbol{\beta}}$ is consistent for $\boldsymbol{\beta}$.

(b) Find the asymptotic distribution $\sqrt{n}\left(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right)$ as $n \to \infty$, assuming that $\boldsymbol{\beta} = \mathbf{0}$

(c) Why is the assumption $\boldsymbol{\beta} = \mathbf{0}$ an important simplifying condition in part (b)?

(d) Using the result in (c), construct an appropriate asymptotic test for the hypothesis $\mathbb{H}_0 : \boldsymbol{\beta} = \mathbf{0}$.

Sketch of Proof:

$\square$

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Proof. **(a)**

Since $\boldsymbol{y} = \boldsymbol{Z}\boldsymbol{\Gamma}\boldsymbol{\beta} + \boldsymbol{e}$,

$$\begin{aligned}
\widehat{\boldsymbol{\beta}} &= \left(\widehat{\boldsymbol{\Gamma}}'\boldsymbol{Z}'\boldsymbol{Z}\widehat{\boldsymbol{\Gamma}}\right)^{-1}\widehat{\boldsymbol{\Gamma}}'\boldsymbol{Z}'\boldsymbol{y} \\
&= \left(\widehat{\boldsymbol{\Gamma}}'\boldsymbol{Z}'\boldsymbol{Z}\widehat{\boldsymbol{\Gamma}}\right)^{-1}\widehat{\boldsymbol{\Gamma}}'\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{\Gamma}\boldsymbol{\beta} + \left(\widehat{\boldsymbol{\Gamma}}'\boldsymbol{Z}'\boldsymbol{Z}\widehat{\boldsymbol{\Gamma}}\right)^{-1}\widehat{\boldsymbol{\Gamma}}'\boldsymbol{Z}'\boldsymbol{e} \\
&= \left(\widehat{\boldsymbol{\Gamma}}'\left(\frac{1}{n}\boldsymbol{Z}'\boldsymbol{Z}\right)\widehat{\boldsymbol{\Gamma}}\right)^{-1}\widehat{\boldsymbol{\Gamma}}'\left(\frac{1}{n}\boldsymbol{Z}'\boldsymbol{Z}\right)\boldsymbol{\Gamma}\boldsymbol{\beta} + \left(\widehat{\boldsymbol{\Gamma}}'\left(\frac{1}{n}\boldsymbol{Z}'\boldsymbol{Z}\right)\widehat{\boldsymbol{\Gamma}}\right)^{-1}\widehat{\boldsymbol{\Gamma}}'(\frac{1}{n}\boldsymbol{Z}'\boldsymbol{e}).
\end{aligned}$$

By $\mathbb{E}[\boldsymbol{x}_i|\boldsymbol{z}_i] = \boldsymbol{\Gamma}'\boldsymbol{z}_i$, $\widehat{\boldsymbol{\Gamma}} \xrightarrow{\mathrm{P}} \boldsymbol{\Gamma}$. Also, $\frac{1}{n}\boldsymbol{Z}'\boldsymbol{Z} \xrightarrow{\mathrm{P}} \boldsymbol{Q}_{zz} \equiv \mathbb{E}[\boldsymbol{z}_i\boldsymbol{z}_i']$ and $\frac{1}{n}\boldsymbol{Z}'\boldsymbol{e} \xrightarrow{\mathrm{P}} \boldsymbol{0}$ by $\mathbb{E}[e_i|\boldsymbol{z}_i] = 0$. Therefore, we find

$$\widehat{\boldsymbol{\beta}} \xrightarrow{\mathrm{P}} \left(\boldsymbol{\Gamma}'\boldsymbol{Q}_{zz}\boldsymbol{\Gamma}\right)^{-1}\left(\boldsymbol{\Gamma}'\boldsymbol{Q}_{zz}\boldsymbol{\Gamma}\right)\boldsymbol{\beta} \equiv \boldsymbol{\beta}$$

(b)

Following (a), we have

$$\sqrt{n}\left(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) = \left(\widehat{\boldsymbol{\Gamma}}'\left(\frac{1}{n}\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1}\widehat{\boldsymbol{\Gamma}}\right)^{-1}\widehat{\boldsymbol{\Gamma}}'\left(\frac{1}{n}\boldsymbol{Z}'\boldsymbol{Z}\right)\boldsymbol{\Gamma}\boldsymbol{\beta} - \boldsymbol{\beta} \tag{1}$$

$$+ \left(\widehat{\boldsymbol{\Gamma}}'\left(\frac{1}{n}\boldsymbol{Z}'\boldsymbol{Z}\right)\widehat{\Gamma}\right)^{-1}\widehat{\boldsymbol{\Gamma}}'\left(\frac{1}{\sqrt{n}}\boldsymbol{Z}'\boldsymbol{e}\right) \tag{2}$$

Under the assumption $\boldsymbol{\beta} = \boldsymbol{0}$, it reduces to

$$\sqrt{n}\left(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) = \left(\widehat{\boldsymbol{\Gamma}}'\left(\frac{1}{n}\boldsymbol{Z}'\boldsymbol{Z}\right)\widehat{\Gamma}\right)^{-1}\widehat{\boldsymbol{\Gamma}}'\left(\frac{1}{\sqrt{n}}\boldsymbol{Z}'\boldsymbol{e}\right)$$

By the CLT, $\frac{1}{\sqrt{n}}\boldsymbol{Z}'\boldsymbol{e} \xrightarrow{\mathrm{d}} \mathcal{N}\left(\boldsymbol{0}, \Omega\right)$, where $\Omega = \mathbb{E}[\boldsymbol{z}_i\boldsymbol{z}_i'e_i^2]$. Hence,

$$\sqrt{n}\left(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) \xrightarrow{\mathrm{d}} \mathcal{N}\left(\boldsymbol{0}, \left(\boldsymbol{\Gamma}'\boldsymbol{Q}_{zz}\boldsymbol{\Gamma}\right)^{-1}\left(\boldsymbol{\Gamma}'\boldsymbol{\Omega}\boldsymbol{\Gamma}\right)\left(\boldsymbol{\Gamma}'\boldsymbol{Q}_{zz}\boldsymbol{\Gamma}\right)^{-1}\right)$$

(c)

Without $\boldsymbol{\beta} = \boldsymbol{0}$, the term Eq. (1) is $O_p(1)$ and therefore contaminates the asymptotic distribution in (b)

(d)

Under $\boldsymbol{\beta} = \boldsymbol{0}$, apply the result in (b), a Wald statistic can be used

$$\mathcal{W}_n = n\widehat{\boldsymbol{\beta}}'\widehat{\boldsymbol{V}}_{\boldsymbol{\beta}}^{-1}\widehat{\boldsymbol{\beta}}$$

where $\widehat{\boldsymbol{V}}_\beta = \left(\widehat{\boldsymbol{\Gamma}}'\widehat{\boldsymbol{Q}}_{zz}\widehat{\boldsymbol{\Gamma}}\right)^{-1}\left(\widehat{\boldsymbol{\Gamma}}'\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{\Gamma}}\right)\left(\widehat{\boldsymbol{\Gamma}}'\widehat{\boldsymbol{Q}}_{zz}\widehat{\boldsymbol{\Gamma}}\right)^{-1}$. □

12.2

In the linear model

$$\begin{aligned} y_i &= \boldsymbol{x}_i'\boldsymbol{\beta} + e_i \\ \mathbb{E}[e_i|\boldsymbol{x}_i] &= 0 \end{aligned}$$

suppose $\sigma_i^2 = \mathbb{E}[e_i^2|x_i]$ is known. Show that the GLS estimator of $\boldsymbol{\beta}$ can be written as an IV estimator using some instrument $\boldsymbol{z}_i$. (Find an expression for $\boldsymbol{z}_i$.)

Proof. The GLS estimator can be written as :

$$\begin{aligned} \widehat{\boldsymbol{\beta}}_{GLS} &= \left(\boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{X}\right)^{-1}\left(\boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{y}\right) \\ &= \left(\boldsymbol{W}'\boldsymbol{X}\right)^{-1}\boldsymbol{W}'\boldsymbol{y} \end{aligned}$$

where $\boldsymbol{\Omega} = \operatorname{diag}\left(\sigma_1^2, \ldots, \sigma_n^2\right)$ and $\boldsymbol{W} = \boldsymbol{\Omega}^{-1}\boldsymbol{X}$. □

12.3

Take the linear model

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{e}$$

Let the OLS estimator for $\boldsymbol{\beta}$ be $\widehat{\boldsymbol{\beta}}$ and the OLS residual be $\widehat{e} = y - \boldsymbol{X}\widehat{\boldsymbol{\beta}}$.

Let the IV estimator for $\boldsymbol{\beta}$ using some instrument $\boldsymbol{Z}$ be $\widetilde{\boldsymbol{\beta}}$ and the IV residual be $\widetilde{\boldsymbol{e}} = \boldsymbol{y} - \boldsymbol{X}\widetilde{\boldsymbol{\beta}}$. If $\boldsymbol{X}$ is indeed endogenous, will IV "fit" better than OLS, in the sense that $\widetilde{\boldsymbol{e}}'\widetilde{\boldsymbol{e}} < \widehat{\boldsymbol{e}}'\widehat{\boldsymbol{e}}$?

Proof. Define $\widetilde{\boldsymbol{M}} = \boldsymbol{I} - \boldsymbol{X}\left(\boldsymbol{X}'\boldsymbol{P_Z}\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\boldsymbol{P_Z}\boldsymbol{y}$, where $\boldsymbol{P_Z} = \boldsymbol{Z}\left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1}\boldsymbol{Z}'$.

Then $\widetilde{\boldsymbol{e}} = \widetilde{\boldsymbol{M}}\boldsymbol{e}$, and the proof is similar to that in Problem 6 of Midterm. Observing $\widetilde{\boldsymbol{M}}'\boldsymbol{M}\widetilde{\boldsymbol{M}} = \boldsymbol{M}$, it is easy to show

$$\widetilde{\boldsymbol{e}}'\widetilde{\boldsymbol{e}} \geqslant \widehat{\boldsymbol{e}}'\widehat{\boldsymbol{e}}$$

□