Lower semi-continuity and upper semi-continuity

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Definition 1 (lower semi-continuity). *Suppose $(\mathbb{S}, \rho)$ is a metric space. A function $g : \mathbb{S} \to [-\infty, \infty]$ is called lower semi-continuous (l.s.c.) if $\liminf_{\rho(y,x)\downarrow 0} g(y) \geqslant g(x)$, for all $x \in \mathbb{S}$. A function g is said to be upper semi-continuous (u.s.c.) if $-g$ is l.s.c.*

Exercise 1 (1.2.20.).

1. *Show that if g is l.s.c. then $\{x : g(x) \leqslant b\}$ is closed for each $b \in \mathbb{R}$.*
2. *Conclude that semi-continuous functions are Borel measurable.*
3. *Conclude that continuous functions are Borel measurable.*
4. *Show that g is l.s.c. at x if and only if*

$$g(x) = \lim_{\delta \searrow 0} \left[\inf_{y \in \mathbb{S}, \rho(x,y) < \delta} g(y) \right].$$

5. *Show that g is l.s.c. at x if and only if for any given $\varepsilon > 0$ there is $\delta > 0$ such that $g(y) > g(x) - \varepsilon$, if $\rho(y, x) < \delta$.*

Proof.

1. - **Sequence method:** suppose a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \{g \leqslant b\}$ with $x_n \to x$ for $n \to \infty$. It suffices to show $x \in \{g \leqslant b\}$, i.e., $g(x) \leqslant b$.
   By definition, the l.s.c of g implies
   $$\liminf_{n\to\infty} g(x_n) \geqslant g(x).$$
   Also, since $g(x_n) \leqslant b$, for all $b \in \mathbb{R}$, the following inequality immediately follows
   $$g(x) \leqslant \liminf_{n\to\infty} g(x_n) \leqslant b, \forall b \in \mathbb{R}.$$
   Thus, $x \in \{g \leqslant b\}$ holds.
2. Consider the set $\{g \leqslant b\}$ for any $b \in \mathbb{R}$. If g is l.s.c., the set $\{g \leqslant b\}$ is closed and hence a Borel set, i.e., $\{g \leqslant b\} \in \mathcal{B}(\mathbb{S})$. It is then concluded that g is Borel measurable.
3. A continuous function g is also lower semi-continuous, so it must be also Borel measurable.
4. - Show RHS $\implies$ LHS. If $g(x) = \lim_{\delta\searrow 0} \left[\inf_{\rho(x,y)<\delta} g(y)\right]$ holds, define $\delta_k = \frac{1}{k}$. Then the condition $\delta \to 0$ is equivalent to $k \to \infty$. Now fixing $k \in \mathbb{N}$, for any y_k with $\rho(x, y_k) < \delta_k$, it is obvious that
   $$\inf_{\rho(x,y)<\delta_k} g(y) \leqslant g(y_k).$$

Taking lim inf on both sides yields

$$g(x) = \lim_{k\to\infty} \inf_{\rho(x,y)<\delta_k} g(y) \leqslant \liminf_{k\to\infty} g(y_k).$$

Thus, we proved g is l.s.c.

- Show LHS $\Longrightarrow$ RHS. If g is l.s.c., i.e., $\liminf_{k\to\infty} g(y_k) \geqslant g(x)$, where $\rho(x, y_k) < \delta_k = 1/k$.

  Since $\rho(x,x) = 0 < \delta_k$, then the following holds

  $$g(x) \geqslant \inf_{\rho(x,y)<\delta_k} g(y), \quad \text{for any } k \in \mathbb{N}$$

  and let $k \to \infty$, then

  $$g(x) \geqslant \lim_{k\to\infty} \inf_{\rho(x,y)<\delta_k} g(y).$$

  It then suffices to show

  $$g(x) \leqslant \lim_{k\to\infty} \inf_{\rho(x,y)<\delta_k} g(y).$$

  Define $\alpha_k := \inf_{\rho(x,y)<\delta_k} g(y)$. Then for any $\varepsilon > 0$, $\exists y_k$ with $\rho(x, y_k) < \delta_k$ such that

  $$\alpha_k + \varepsilon > g(y_k).$$

  Since this holds for any $k \in \mathbb{N}$, $y_k \to x$ as $k \to \infty$. Bring the sequence $(y_k)_{k\in\mathbb{N}}$ to the definition of g is l.s.c.:

  $$\lim_{k\to\infty} \inf_{\rho(x,y)<\delta_k} g(y) + \varepsilon = \lim_k \alpha_k + \varepsilon \geqslant \liminf_k g(y_k) \geqslant g(x).$$

  Letting $\varepsilon \searrow 0$, then we have the desired inequality:

  $$\lim_{k\to\infty} \inf_{\rho(x,y)<\delta_k} g(y) \geqslant g(x).$$

Combine those inequalities for the two directions, then it is concluded that

$$\lim_{k\to\infty} \inf_{\rho(x,y)<\delta_k} g(y) = g(x).$$

5. Simply use 4. Since $\inf_{\rho(x,y)<\delta} g(y) \nearrow g(x)$ for $\delta \to 0$, given $\varepsilon > 0$, there exists $\delta > 0$ such that for any y with $\rho(x, y) < \delta$,

$$g(y) \geqslant \inf_{\rho(x,y)<\delta} g(y) > g(x) - \varepsilon.$$

For the other direction, it suffices to show

$$g(x) \leqslant \lim_{\delta\searrow 0} \left[\inf_{\rho(x,y)<\delta} g(y) \right].$$

By the assumption that $g(y) + \varepsilon > g(x)$, for any y with $\rho(y, x) < \delta$, then

$$\inf_{\rho(x,y)<\delta} g(y) + \varepsilon \geqslant g(x).$$

Letting $\delta \searrow 0$, then we have

$$\lim_{\delta \searrow 0} \inf_{\rho(x,y)<\delta} g(y) + \varepsilon \geqslant g(x).$$

Finally, letting $\varepsilon \searrow 0$, we have the desired inequality

$$\lim_{\delta \searrow 0} \inf_{\rho(x,y)<\delta} g(y) \geqslant g(x).$$

□

1. APPLICATIONS

EXAMPLE 1.1. *If $F \in \mathcal{B}(\mathbb{R})$ is a closed set, the indicator function $\mathbb{1}_F(x)$ is upper semi-continuous bounded by one.*

PROOF.
It suffices to show that for any $x_n \to x$ for $n \to \infty$, then $\mathbb{1}_F(x) \geqslant \limsup_{n\to\infty} \mathbb{1}_F(x_n)$.
If $\{x_n\}_{n\in\mathbb{N}} \subseteq F$, then $x \in F$. □

REMARK 1.
It can be used to prove one result in the Portmanteau Theorem : $\nu_n \overset{w}{\Rightarrow} \nu_\infty \implies$ For every closed set F, one has $\limsup_{n\to\infty} \nu_n(F) \leqslant \nu_\infty(F)$.
Proof: By Fatou's lemma

$$\limsup_{n\to\infty} \nu_n(F) = \limsup_{n\to\infty} \mathbb{E}[\mathbb{1}_F(Y_n)] \leqslant \mathbb{E}[\limsup_{n\to\infty} \mathbb{1}_F(Y_n)] \leqslant \mathbb{E}[\mathbb{1}_F(Y_\infty)] = \nu_\infty(F).$$