# A short notes on conditional independence testing

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October 9, 2020

This notes is based on the following papers:

- [NBW 2020] *Minimax Optimal Conditional Independence Testing*, by Matey Neykov, Sivaraman Balakrishnan, Larry Wasserman, *Preprint*, January 2020. click here
- [SP2018] *The Hardness of Conditional Independence Testing and the Generalized Covariance Measure*, by Rajen D. Shah, Jonas Peters, **AoS**, 2018.

The following notations are from **SP2018**:

1. $\mathbf{E}_P[\cdot]$: expectations of random variables whose distribution is determined by $P$.
2. $\mathcal{P}$: a potentially composite null hypothesis consisting of a collection of distributions for $(X, Y, Z)$.
3. $\psi_n : \mathbb{R}^{(d_X+d_Y+d_Z)\cdot n} \times [0,1] \to \{0,1\}$ : **a test function (or critical function)**, which is measurable.
4. Given $\alpha \in (0,1)$ and null hypothesis $\mathcal{P}$, a sequence of tests $\{\psi_n\}_{n=1}^{\infty}$ has

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Notions

(a) *valid level at sample size $n$* if

$$\sup_{P\in\mathcal{P}} \mathbf{P}_P\left(\psi_n = 1\right) \leqslant \alpha, \tag{1}$$

(b) *uniformly asymptotic level* if

$$\limsup_{n\to\infty} \sup_{P\in\mathcal{P}} \mathbf{P}_P(\psi_n = 1) \leqslant \alpha, \tag{2}$$

(c) *pointwise asymptotic level* if

$$\sup_{P\in\mathcal{P}} \limsup_{n\to\infty} \mathbf{P}_P(\psi_n = 1) \leqslant \alpha \tag{3}$$

Obviously, Equation (1) $\Longrightarrow$ Equation (2) $\Longrightarrow$ Equation (3) . The proof (2) $\Longrightarrow$ (3) is sketched as follows: since $\forall P \in \mathcal{P}, \mathbf{P}_P\left(\psi_n = 1\right) \leqslant \sup_{P\in\mathcal{P}} \mathbf{P}_P\left(\psi_n = 1\right)$. Taking lim sup on both sides yield:

$$\limsup_{n\to\infty} \mathbf{P}_P\left(\psi_n = 1\right) \leqslant \limsup_{n\to\infty} \sup_{P\in\mathcal{P}} \mathbf{P}_P\left(\psi_n = 1\right) \leqslant \alpha,$$

This relation holds for all $P \in \mathcal{P}$, taking sup on the LHS gives the desired implication.

Now let us dig deeper into Equation (3), more specifically about why it is called "pointwise" and its relation to the "uniform" sense. First observe that

$$\sup_{P\in\mathcal{P}} \limsup_{n\to\infty} \mathbf{P}_P\left(\psi_n = 1\right) \leqslant \alpha \Longleftrightarrow \limsup_{n\to\infty} \mathbf{P}_P\left(\psi_n = 1\right) \leqslant \alpha, \forall P \in \mathcal{P}.$$

The RHS makes it a clear interpretation of why it is called a "pointwise" asymptotic level, because this inequality holds conditional on a given $P \in \mathcal{P}$. The interpretation of *limsup*[1] is as usual what we learnt in any analysis courses. Hence, another alternative statement about Equation (3) could be: <u>given $P \in \mathcal{P}$ and given $\varepsilon > 0$, there exists $N = N(P, \varepsilon) \in \mathbb{N}$ such that $\mathbf{P}_P\left(\psi_n = 1\right) < \alpha + \varepsilon, \forall n > N$</u>.

It has two implications. First, there is still possibility that $\mathbf{P}_P\left(\psi_n = 1\right) \geqslant \alpha + \varepsilon$ for

[1]The statement $\limsup_{n\to\infty} a_n = b \in \mathbb{R}$ holds if and only if 1. $\forall \varepsilon > 0, \exists N \in \mathbb{N}$, s.t. $a_m < b + \varepsilon$, whenever $m \geqslant N$ and 2. For any $\varepsilon > 0$ and any $N \in \mathbb{N}$, $\exists m > N$ s.t. $a_m > b - \varepsilon$.

some $n = 1, 2, \cdots, N$. Second, the above chosen $N = N(P, \alpha)$ depends on $P \in \mathcal{P}$ and the nominal level $\alpha$. Hence, "pointwise" asymptotic level and "uniform" asymptotic level can be associated by this following observation:

> **INTERPRETATION OF POINTWISE ASYMPTOTIC LEVEL**
>
> Suppose $\sup_{P \in \mathcal{P}} \limsup_{n \to \infty} \mathbf{P}_P \left(\psi_n = 1\right) = a \in \mathbb{R}$, where $a \leqslant \alpha$ and the test $\psi_n$ satisfies *pointwise aymptotic level* but **NOT** *uniformly asymptotic level*. Then there may exist $\varepsilon > 0$, such that $a + \varepsilon > \alpha$, but $N(\varepsilon) \coloneqq \sup_{P \in \mathcal{P}} N(\varepsilon, P) = \infty$. It implies for any $n_0 \in \mathbb{N}$, there exists some $P' \in \mathbb{N}$, such that $N^* \coloneqq N(\varepsilon, P') > n_0$ and $\mathbf{P}_P \left(\psi_{N^*} = 1\right) \geqslant a + \varepsilon$.
>
> In summary, we showed that if $\left\{\psi_n\right\}_{n \in \mathbb{N}}$ only satisfies *pointwise asymptotic level* condition, then for any $n_0 \in \mathbb{N}$, there exists some "bad" $P \in \mathcal{P}$ such that $N = N(\varepsilon, P) > n_0$ with $\mathbf{P}_P \left(\psi_N = 1\right) \geqslant a + \varepsilon > \alpha$.

As the aforementioned discussions, a sequence of tests $\psi_n$ having *pointwise asymptotic level* may not serve a good purpose, because even for large $n$ there may exist some test for this their size exceeds the nominal level $\alpha$ by some fixed amount $\delta > 0$[2]. That is one reason why we require that a test should have at least *uniform asymptotic level*.

[2]The proof is relegated to the appendix

# Appendix

### Proof of the claim