MIDTERM SPRING 2015

1. In a linear model,

$$\boldsymbol{y} = \boldsymbol{X}\beta + \boldsymbol{e}, E(\boldsymbol{e}|\boldsymbol{X}) = 0, \text{var}(\boldsymbol{e}|\boldsymbol{X}) = \sigma^2\boldsymbol{\Omega}$$

with $\boldsymbol{\Omega}$ a known function of $\boldsymbol{X}$. The GLS estimator is:

$$\tilde{\beta} = (\boldsymbol{X'}\boldsymbol{\Omega}^{-1}\boldsymbol{X})^{-1}(\boldsymbol{X}\boldsymbol{\Omega}^{-1}\boldsymbol{y}),$$

the residual vector is $\hat{\boldsymbol{e}} = \boldsymbol{y} - \boldsymbol{X}\tilde{\beta}$, and an estimate of σ^2 is

$$s^2 = \frac{1}{n-k}\hat{\boldsymbol{e}}'\boldsymbol{\Omega^{-1}}\hat{\boldsymbol{e}}$$

(a) Find $E(\tilde{\boldsymbol{\beta}}|\boldsymbol{X})$.

(b) Find $\text{var}(\tilde{\boldsymbol{\beta}}|\boldsymbol{X})$.

(c) Prove that $\hat{\boldsymbol{e}} = \boldsymbol{M_1 e}$, where $\boldsymbol{M_1} = \boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X'}\boldsymbol{\Omega^{-1}}\boldsymbol{X})^{-1}\boldsymbol{X'}\boldsymbol{\Omega^{-1}}$.

(d) Prove that $\boldsymbol{M_1'}\boldsymbol{\Omega^{-1}}\boldsymbol{M_1} = \boldsymbol{\Omega^{-1}} - \boldsymbol{\Omega^{-1}}\boldsymbol{X}(\boldsymbol{X'}\boldsymbol{\Omega^{-1}}\boldsymbol{X})^{-1}\boldsymbol{X'}\boldsymbol{\Omega^{-1}}$.

(e) Find $E(s^2|\boldsymbol{X})$.

(f) Is s^2 a reasonable estimator for σ^2?

2. Let $(y_i, \boldsymbol{x_i})$ be a random sample with $E(y|\boldsymbol{X}) = \boldsymbol{X\beta}$. Consider the **Weighted Least Squares** (WLS) estimator of $\boldsymbol{\beta}$

$$\tilde{\beta} = (\boldsymbol{X'WX})^{-1}(\boldsymbol{X'Wy})$$

where $\boldsymbol{W} = diag(w_1, w_2, \cdots, w_n)$ and $w_i = x_{ji}^{-2}$, where x_{ji} is one of the $\boldsymbol{x_i}$.

(a) In which contexts would $\tilde{\boldsymbol{\beta}}$ be a good estimator?

(b) Using your intuition, in which situations would you expect $\tilde{\boldsymbol{\beta}}$ would perform better than the OLS estimator?

3. Let $\boldsymbol{y}$ be $n \times 1$, $\boldsymbol{X}$ be $n \times k$ (rank k). $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{e}$ with $E(\boldsymbol{x_i}e_i) = 0$. Define the *ridge regression estimator*

$$\hat{\beta} = (\sum_{i=1}^{n} \boldsymbol{x_i x_i'} + \lambda \boldsymbol{I_k})^{-1}(\sum_{i=1}^{n} \boldsymbol{x_i} y_i)$$

where $\lambda > 0$ is a fixed constant. Find the probability limit of $\hat{\boldsymbol{\beta}}$ as $n \to \infty$. Is $\hat{\boldsymbol{\beta}}$ consistent for $\boldsymbol{\beta}$?

4. For the ridge regression estimator in 3, set $\lambda = cn$ where $c > 0$ is fixed as $n \to \infty$. Find the probability limit of $\hat{\boldsymbol{\beta}}$ as $n \to \infty$.

5. The model is

$$
\begin{gathered}
y_i = \boldsymbol{x_i'}\boldsymbol{\beta} + e_i \\
E(\boldsymbol{x_i} e_i) = 0 \\
\boldsymbol{\Omega} = E(\boldsymbol{x_i}\boldsymbol{x_i'} e_i^2)
\end{gathered}
$$

(a) Find the method of moments estimators $(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\Omega}})$ for $(\boldsymbol{\beta}, \boldsymbol{\Omega})$.

(b) In this model, are $(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\Omega}})$ efficient estimators of $(\boldsymbol{\beta}, \boldsymbol{\Omega})$?

(c) If so, in what sense are they efficient?