AS.180.633: Econometrics **Spring 2020**

Homework 5: Suggested Solutions

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5.4

Show that $\widehat{\theta} = \arg\max_{\theta \in \Theta} \ell_n(\theta) = \arg\max_{\theta \in \Theta} \mathscr{L}_n(\theta)$.

Proof. Since $\mathscr{L}_n(\theta)$ is merely a monotonic transformation of $\ell_n(\theta)$ by definition, maximizers will agree for this two objective functions □

5.5

For the regression in-sample predicted values $\widehat{y}_i$ show that $\widehat{y}_i|\boldsymbol{X} \sim \mathcal{N}\left(\boldsymbol{x}_i'\boldsymbol{\beta}, \sigma^2 h_{ii}\right)$ where h_{ii} are leverage values (3.41)

Proof. From **Theorem 5.4**,

$$\widehat{\boldsymbol{\beta}} \sim \mathcal{N}(\boldsymbol{\beta}, \sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1}).$$

Then conditional on $\boldsymbol{X}$, $\boldsymbol{x}_i'$ becomes constant for each i. By **Theorem 5.2** of Hansen's notes, it follows that $\widehat{y}_i = \boldsymbol{x}_i'\widehat{\boldsymbol{\beta}}$ follows a normal distribution with mean $\boldsymbol{x}_i'\boldsymbol{\beta}$ and variance $\sigma^2\boldsymbol{x}_i'(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{x}_i$, i.e.

$$\widehat{y}_i|\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{x}_i'\boldsymbol{\beta}, \sigma^2 h_{ii})$$

where $h_{ii} = \boldsymbol{x}_i'(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{x}_i$ by (3.41). □

5.6

In the normal regression model, show that the leave-one out prediction errors $\widetilde{e}_i$ and the standardized residuals $\bar{e}_i$ are independent of $\widehat{\boldsymbol{\beta}}$, conditional on $\boldsymbol{X}$. (Hint: Use (3.46) and (4.24)).

Proof. By **Theorem 5.6**, $\widehat{\boldsymbol{e}}$ is independent of $\widehat{\beta}$. From (3.46) and (4.24), since $\boldsymbol{M}^*$ and $\boldsymbol{M}$ become constant conditional on $\boldsymbol{X}$, $\widetilde{\boldsymbol{e}}$ and $\bar{\boldsymbol{e}}$ are also independent of $\widehat{\beta}$. □

5.7

In the normal regression model show that the robust covariance matrix $\widehat{V}^{HC0}_{\widehat{\boldsymbol{\beta}}}$ $\widehat{V}^{HC1}_{\widehat{\boldsymbol{\beta}}}$, $\widehat{V}^{HC2}_{\widehat{\boldsymbol{\beta}}}$ and $\widehat{V}^{HC3}_{\widehat{\boldsymbol{\beta}}}$ are independent of the OLS estimator $\widehat{\boldsymbol{\beta}}$, conditional on $\boldsymbol{X}$.

Proof. Since the randomness of those four variance estimators is completely sourced from $\hat{e}_i^2$ for each i when $\boldsymbol{X}$ is controlled, they must be also independent of $\widehat{\boldsymbol{\beta}}$ (because $\widehat{\boldsymbol{e}}$ is independent of $\widehat{\boldsymbol{\beta}}$.) $\square$