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The Borel-Cantelli Lemmas

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The Borel-Cantelli lemmas play a vital role in proving strong law of large numbers. Here are some definitions and results that are useful for later analysis.

DEFINITION 1 (limits superior / limits inferior). *For a sequence of subsets $A_n \subseteq \Omega$, define*

$$\begin{aligned} A^\infty := \limsup A_n &= \bigcap_{m=1}^{\infty} \bigcup_{\ell=m}^{\infty} A_\ell \\ &= \{\omega : \omega \in A_n \text{ for infinitely many } n\text{'s } \} \\ &= \{\omega : \omega \in A_n \text{ infinitely often } \} \\ &= \{A_n \text{ i.o.}\} \end{aligned}$$

Similarly,

$$\begin{aligned} \liminf A_n &= \bigcup_{m=1}^{\infty} \bigcap_{\ell=m}^{\infty} A_\ell \\ &= \{\omega : \omega \in A_n \text{ for all but finitely many } n\text{'s } \} \\ &= \{\omega : \omega \in A_n \text{ eventually } \} \\ &= \{A_n \text{ ev.}\} \end{aligned}$$

Borel-Cantelli Lemma I and II

THEOREM 1 (Borel-Cantelli Lemma I). *Suppose $A_n \in \mathcal{F}$ and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$. Then, $\mathbb{P}(A_n \text{ i.o.}) = 0$.*

THEOREM 2 (Borel-Cantelli Lemma II). *Suppose $A_n \in \mathcal{F}$ are mutually independent and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$. Then, necessarily $\mathbb{P}(A_n \text{ i.o.}) = 1$.*

To dig more into the difference between the two concepts, let's see the following exercise.

EXERCISE 1 (Amir Exercise 2.2.5). *Suppose $A_n \in \mathcal{F}$ are such that $\sum_{n=1}^{\infty} \mathbb{P}(A_n \bigcap A_{n+1}^c) < \infty$ and $\mathbb{P}(A_n) \to 0$. Show that then $\mathbb{P}(A_n \text{ i.o.}) = 0$.*

PROOF. Let $B_k = A_k \bigcap A_{k+1}^c$. The key is to observe that

$$\{A_n \text{ i.o.}\} = \{B_n \text{ i.o.}\} \bigcup \{A_n \text{ ev.}\}.$$

Then we have

$$\mathbb{P}(A_n \text{ i.o.}) = \mathbb{P}(B_n \text{ i.o.}) + \mathbb{P}(A_n \text{ ev.}).$$

By the assumptions $A_n \in \mathcal{F}$ and $\sum_{n=1}^{\infty} B_n < \infty$, we have $\mathbb{P}(B_n \text{ i.o.}) = 0$.

It then suffices to show $\mathbb{P}(A_n \text{ ev.}) = 0$. Since $\{A_n \text{ ev.}\} = \bigcup_{m=1}^{\infty} \bigcap_{k \geqslant m} A_k$, by the property of continuity from above of a measure, we have **1**

$$\mathbb{P}(A_n \text{ ev.}) = \lim_{m\to\infty} \mathbb{P}(\cap_{k\geqslant m} A_k) \leqslant \lim_{m\to\infty} \mathbb{P}(A_m) = 0.$$

The inequality is simply justified by the monotonicity relation $\bigcap_{k\geqslant m} A_k \subseteq A_m, \forall m \in \mathbb{N}$. Thus, we must have

$$\mathbb{P}(A_n \text{ ev.}) = 0.$$

- $Y_n \geqslant 0$ and $\mathbb{E}[Y_n^2] < \infty$.
- $0 \leqslant a_n < \mathbb{E}[Y_n]$.

Use the Exercise 1.3.21, we have

$$\mathbb{P}(Y_n > a_n) \geqslant (1-\lambda)^2 \frac{(\mathbb{E}[Y_n])^2}{\mathbb{E}[Y_n^2]}.$$

Next we shall show the two events satisfy $\bigcup_{k>n} A_k \supseteq \{Y_n > \lambda \mathbb{E}[Y_n]\}$. Or equivalent, we prove

Suppose $\omega \notin \{A_n \text{ i.o.}\}$, which means $\omega \in \left\{A_n^c \text{ ev.}\right\}$, i.e., $\omega \notin A_n$ for large n. Then for such ω, $Y_n(\omega) = \sum_{k=1}^n I_{A_k}(\omega) < \infty$. But by assumption $\sum_k \mathbb{P}(A_k) = \infty$, $\omega \notin \left\{Y_n > \lambda \sum_{k=1}^n \mathbb{P}(A_k)\right\}$ for large n. Hence, we have shown that

$$\omega \notin \{A_n \text{ i.o.}\} \implies \omega \notin \left\{Y_n > \lambda \sum_{k=1}^n \mathbb{P}(A_k)\right\} \quad \text{for large } n.$$

It is equivalent that

$$\left\{Y_n > \lambda \sum_{k=1}^n \mathbb{P}(A_k)\right\} \subseteq \{A_n \text{ i.o.}\} \quad \text{for large } n.$$

It follows that

$$\mathbb{P}(A_n \text{ i.o.}) \geqslant \mathbb{P}(Y_n > a_n) \geqslant (1-\lambda^2) \frac{(\mathbb{E}[Y_n])^2}{\mathbb{E}[Y_n^2]}.$$

By assumption we know

$$\limsup_{n\to\infty} \frac{(\mathbb{E}[Y_n])^2}{\mathbb{E}[Y_n^2]} = \alpha.$$

Then

$$\mathbb{P}(A_n \text{ i.o.}) \geqslant (1-\lambda)^2 \alpha, \forall \lambda \in (0,1).$$

Let $\lambda \downarrow 0$, then we have the desired

$$\mathbb{P}(A_n \text{ i.o.}) \geqslant \alpha.$$

$\square$

Convergence a.s. is invariant under continuous mapping

EXERCISE 3 (Amir Exercise 2.2.12). *Let $g : \mathbb{R} \to \mathbb{R}$ be a Borel function and denote by $\mathbb{D}_g$ its set of discontinuities. Show that if $X_n \xrightarrow{a.s.} X_\infty$ finite valued, and $\mathbb{P}(X_\infty \in \mathbb{D}_g) = 0$, then $g(X_n) \xrightarrow{a.s.} g(X_\infty)$ as well. This applies for a continuous function g in which case $\mathbb{D}_g = \emptyset$.*

Proof. From $X_n \xrightarrow{\text{a.s.}} X_\infty$, there exists a $\mathbb{P}$-null set N_1, i.e., $\mathbb{P}(N_1) = 0$, such that for any $\omega \in \Omega \setminus N_1$, we have $X_n(\omega) \to X_\infty(\omega)$. Define $N_2 = \{X_\infty \in \mathbb{D}_g\}$. Then N_2 is also a $\mathbb{P}$-null set. Define $N = N_1 \bigcup N_2$, it is clear that N is also a $\mathbb{P}$-null set.

Pick any $\omega \notin N$, then

- $\omega \notin N_1 \implies X_n(\omega) \to X_\infty(\omega)$.
- $\omega \notin N_2 \implies g(X_n(\omega)) \to g(X_\infty(\omega))$ (g is continuous at $X_\infty(\omega)$).

Thus, except a $\mathbb{P}$-null set N, for any $\omega \in \Omega \setminus N$, we have $g(X_n(\omega)) \to g(X_\infty(\omega))$. Or equivalently, $g(X_n) \xrightarrow{\text{a.s.}} g(X_\infty)$. □

zero-one law implied by B-C lemmas

Corollary 1. *If $A_n \in \mathcal{F}$ are $\mathbb{P}$-mutually independent then $\mathbb{P}(A_n$ i.o.$)$ is either 0 or 1. In other words, for any given sequence of mutually independent events, either almost all outcomes are in infinitely many of these events, or almost all outcomes are in finitely many of them.*

1. APPENDIX

Exercise 4 (Amir 1.3.21). *Let $Y \geqslant 0$ with $v = \mathbb{E}[Y^2] < \infty$.*

1. *Show that for any $0 \leqslant a < \mathbb{E}[Y]$,*

   $\mathbb{P}(Y > a) \geqslant \frac{(\mathbb{E}[Y]-a)^2}{\mathbb{E}[Y^2]}$.