Solutions for Homework 2

Tong Zhou

Problems: 2.22, 3.2, 3.4, 3.5.

PROBLEM 2.22

Proof. To check conditional heteroskedasticity, it suffices to see if $\mathsf{Var}(X_2'\beta_2 + e \mid X_1)$ is a function of X_1 or not.

First project X_2 on X_1, we have

$$X_2 = \mathbb{E}[X_2 \mid X_1] + e_2 = \Gamma X_1 + e_2 \tag{1}$$

where $e_2 := X_2 - \mathbb{E}[X_2 \mid X_1]$ is the projection error.

Observe that

$$\mathbb{E}[e_2 \mid X_1, X_2] \overset{(1)}{=} e_2$$

Equality (1) holds because e_2 is a function of X_2 and X_1. Use Eq. (1),

$$\begin{aligned}
\mathsf{Var}(X_2'\beta_2 + e \mid X_1) &= \mathsf{Var}(X_1'\Gamma'\beta_2 + e_2'\beta_2 + e \mid X_1) \\
&\overset{(2)}{=} \mathsf{Var}(e_2'\beta_2 + e \mid X_1) \\
&\overset{(3)}{=} \mathbb{E}[(e_2'\beta_2 + e)^2 \mid X_1] \\
&= \mathbb{E}[\beta_2'e_2e_2'\beta_2 + e^2 + 2e_2'\beta_2 e \mid X_1] \\
&\overset{(4)}{=} \mathbb{E}[\beta_2'e_2e_2'\beta_2 \mid X_1] + \sigma^2
\end{aligned}$$

Equality (2) holds because $X_1'\Gamma'\beta_2$ is a constant given X_1; Equality (3) is true because $\mathbb{E}[e_2'\beta_2 + e \mid X_1] = 0$; Equality (4) is by equality (1)

$$\mathbb{E}[e_2'\beta_2 e \mid X_1] = \mathbb{E}[e_2'\beta_2 \mathbb{E}[e \mid X_1, X_2] \mid X_1] = 0.$$

Thus, whether conditional heteroskedasticity is an issue depends on the term

$$\beta_2'\mathbb{E}[e_2e_2' \mid X_1]\beta_2$$

is constant or not. If, for example, suppose X_1, X_2 have the same dimension and

$$e_2 \mid X_1 \sim \mathcal{N}(0, X_1X_1'),$$

then $\mathbb{E}[e_2e_2' \mid X_1] = X_1X_1'$. As a result,

$$\mathsf{Var}(X_2'\beta_2 + e \mid X_1) = \beta_2'X_1X_1'\beta_2 + \sigma^2.$$

It implies conditional heteroskedasticity does exist.

So without further information, we cannot conclude conditional homoskedasticity. □

PROBLEM 3.2

Proof. The estimates from regressing $\boldsymbol{y}$ on $\boldsymbol{X}$ is

$$\widehat{\beta} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}.$$

The estimates from regressing $\boldsymbol{y}$ on $\boldsymbol{Z}$ is

$$\begin{aligned}
\widehat{\beta}_z &= (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{y} \\
&= (\boldsymbol{C}'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{C})^{-1}\,\boldsymbol{C}'\boldsymbol{X}'\boldsymbol{y} \\
&= \boldsymbol{C}^{-1}(\boldsymbol{X}'\boldsymbol{X})^{-1}(\boldsymbol{C}')^{-1}\boldsymbol{C}'\boldsymbol{X}'\boldsymbol{y} \\
&= \boldsymbol{C}^{-1}\widehat{\beta}.
\end{aligned}$$

It is not hard to verify that their residuals are algebraically identical. □

PROBLEM 3.4

Proof. Since

$$\begin{aligned}
\boldsymbol{X}'\hat{\boldsymbol{e}} &= \begin{pmatrix} \boldsymbol{X}_1' \\ \boldsymbol{X}_2' \end{pmatrix} \hat{\boldsymbol{e}} \\
&= \begin{pmatrix} \boldsymbol{X}_1'\hat{\boldsymbol{e}} \\ \boldsymbol{X}_2'\hat{\boldsymbol{e}} \end{pmatrix} \\
&= \boldsymbol{0},
\end{aligned}$$

then

$$\boldsymbol{X}_2'\hat{\boldsymbol{e}} = 0.$$

□

PROBLEM 3.5

Proof. Suppose the regression function is $\widehat{\boldsymbol{e}} = \boldsymbol{X}\boldsymbol{\lambda} + \boldsymbol{\eta}$. The OLS coefficient from a regression of $\widehat{\boldsymbol{e}}$ on $\boldsymbol{X}$ is

$$\widehat{\lambda}_{\text{OLS}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\underbrace{\boldsymbol{X}'\widehat{\boldsymbol{e}}}_{=\boldsymbol{0}} = \boldsymbol{0}.$$

The intuition is that $\widehat{\boldsymbol{e}}$ is the "leftover" effect on $\boldsymbol{y}$ after $\boldsymbol{X}$ "absorbs" relevant information in $\boldsymbol{y}$. So, it is not surprising that $\boldsymbol{X}$ has no (linear) effect on $\widehat{\boldsymbol{e}}$. □