The Arzelà-Ascoli Theorem

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1. MOTIVATION

The original motivation aims to look for conditions under which the pointwise convergence of a sequence of functions is equivalent to its uniform convergence. In other words, the strengthened conditions should be neither too strong nor too weak. We shall show that such condition in place is called *equi-continuity*.

A further more fruitful motivation is to characterize the form of compact sets in the infinite-dimensional space $(\mathscr{C}(K;V), \|\cdot\|_\infty)$. We know that a compact set in any finite-dimensional normed vector space $(X, \|\cdot\|)$ (i.e. $\dim(X) < \infty$) must be closed and bounded:

LEMMA 1. *Let $(X, \|\cdot\|)$ be a finite-dimensional normed vector space,* i.e. $\dim(X) < \infty$. *Suppose $A \subseteq X$ is closed and bounded, then A is compact.*

PROOF. We shall show that A is totally bounded and complete.

- **complete:** Suppose a sequence $\{x_n\}_n \subseteq A$ is Cauchy, i.e. for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon$, whenever $m, n > N$.

□

A further more fruitful result is to take a broader perspective: consider a sequence of functions $\{f_k\}_{k=1}^\infty$ in the space $\mathscr{C}(K;V)$. Equipped with the sup-norm $\|\cdot\|_\infty$, $\{f_k\}_{k=1}^\infty$ that converges uniformly to a function f is equivalent to f_k converging to f in $\mathscr{C}(K;V)$, in which $f_k, f \in \mathscr{C}(K;V)$ can be regarded as *elements* in the space $\mathscr{C}(K;V)$.

DEFINITION 1 (equi-continuity). *Let (M, d) be a metric space, $(V, \|\cdot\|)$ be a normed vector space, and $A \subseteq M$ be a subset. A subset $B \subseteq \mathscr{C}_b(A;V)$ is said to be **equi-continuous** if*

$$\forall \varepsilon > 0, \exists \delta > 0, \ni \|f(x_1) - f(x_2)\| < \varepsilon \quad \textit{whenever } d(x_1, x_2) < \delta, x_1, x_2 \in A, \ \textit{and } f \in B.$$

DEFINITION 2 (modulus of continuity of f). *The modulus of continuity of $f : A \to V$ is defined by*

$$w_f(\delta) := \sup_{x,y \in A} \{\|f(x) - f(y)\| : d(x,y) < \delta\}.$$

Note that a function f is uniformly continuous if and only if $\lim_{\delta \to 0} w_f(\delta) = 0$.

Implications of equi-continuity

- The definition is for *a class of functions.* Here is the space of functions B.
- Where does the uniform come from?
  Now suppose each $f \in B$ is uniformly continuous, *i.e.* $\forall \varepsilon > 0, \exists \delta_f \ni \|f(x) - f(y)\| < \varepsilon$ whenever $d(x, y) < \delta_f$. Take

$$\delta := \inf_{f \in B} \delta_f$$

  If $\delta > 0$, we say B is *equi-continuous.* However, if $\delta = 0$, then B is NOT equi-continuous.
- By the concept of *modulus of continuity of f*, it characterizes that the collection of functions B is uniformly continuous with the **same** "degree" of uniform continuity for each $f \in B$.
- If B is finite, it must be equi-continuous.

Let $f_k : [0, 1] \to \mathbb{R}$ be a sequence of functions given by

$$f_k(x) = \begin{cases} kx, & \text{if } 0 \leqslant x \leqslant \frac{1}{k} \\ 2 - kx, & \text{if } \frac{1}{k} \leqslant x \leqslant \frac{2}{k} \\ 0, & \text{if } x \geqslant \frac{2}{k} \end{cases}$$

However, the sequence $\{f_k\}_{k=1}^{\infty}$ is NOT equi-continuous.

Let us now check this by the *modulus of continuity.* First fix a small δ and $k \in \mathbb{N}$, then $w_k(\delta) := k\delta$. Hence given $k \in \mathbb{N}$, $\lim_{\delta \to 0} w_k(\delta) = \lim_{k \to 0} k\delta = 0$, implying that each f_k is uniformly continuous. However, when considered as a whole class of of functions, $\{f_k\}_{k=1}^{\infty}$ is NOT equi-continuous, which can be justified by

$$\sup_{k \in \mathbb{N}} w_k(\delta) = \infty,$$

and as a result,

$$\lim_{\delta \to 0} \sup_{k \in \mathbb{N}} w_k(\delta) = \infty.$$

That is why $\{f_k\}_{k=1}^{\infty}$ is not equi-continuous.

Here is the plot for the case when $k = 2$, *i.e.* the plot of $f_2(x)$. By staring at the following plot, it is not hard to see that for any $\varepsilon \in (0, 1)$, $\delta_{\varepsilon,k} = \frac{\varepsilon}{k}$. It indicates that δ also depends on k. Then it follows that

$$\delta := \inf_k \delta_{\varepsilon,k} = \lim_{k \to \infty} \frac{\varepsilon}{k} = 0, \forall \varepsilon \in (0, 1).$$

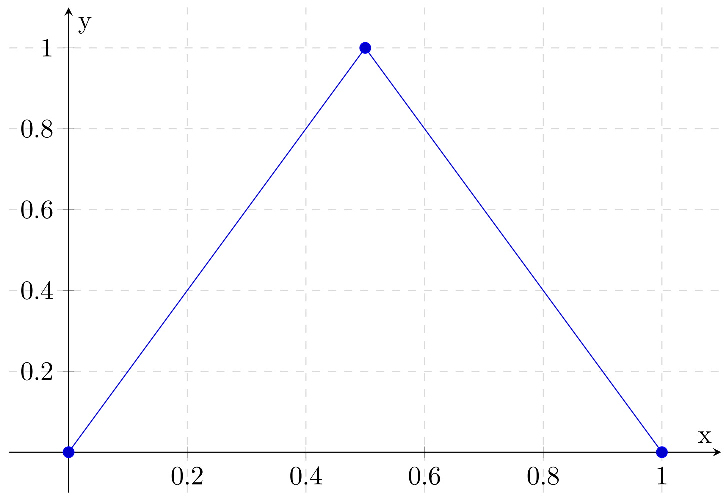

FIGURE 1. *The plot of f_k when $k = 2$*

2. PROPERTIES

Note that (M, d) denotes the universal metric space and $(V, \|\cdot\|)$ denotes the normed vector space.

A sufficient condition: Pre-compact $\implies$ equi-continuity

Suppose $K \subseteq M$ is compact. If $B \subseteq \mathscr{C}(K; V)$ is pre-compact, then B is equi-continuous.

COROLLARY 1. *Under the same conditions, if $\{f_k\}_{k=1}^{\infty}$ converges uniformly on K, then $\{f_k\}_{k=1}^{\infty}$ is equi-continuous.*

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What the corollary says?

$\{f_k\}_{k=1}^{\infty}$ converges uniformly on K
$\iff$ $\{f_k\}_{k=1}^{\infty}$ converges in $(\mathscr{C}(K; V), \|\cdot\|_{\infty})$
$\implies$ The space $\{f_k\}_{k=1}^{\infty}$ is pre-compact in $(\mathscr{C}(K; V), \|\cdot\|_{\infty})$.
$\implies$ $\{f_k\}_{k=1}^{\infty}$ is equi-continuous.

Next, we shall investigate under what conditions the inverse is true, *i.e.* equi-continuity of a sequence of functions, defined on a compact set K, converges uniformly.

equi-continuity + pointwise convergence $\implies$ uniform convergence

THEOREM 1. *If $(V, \|\cdot\|)$ is a Banach space, $K \subseteq M$ is compact and $\{f_k\}_{k=1}^{\infty} \subseteq \mathscr{C}(K; V)$ is equi-continuous. Further, if $\{f_k\}_{k=1}^{\infty}$ converges pointwise on a dense subset E of K, then $\{f_k\}_{k=1}^{\infty}$ converges uniformly on K.*

Proof. **Goal:** $\forall \varepsilon > 0, \exists N \in \mathbb{N}$, s.t. $\|f_k(x) - f_\ell(x)\| < \varepsilon, \forall x \in K$, whenever $k, \ell > N$.

(i) By equi-continuity, for such ε, there exists δ, such that

$$\|f_k(x) - f_k(y)\| < \frac{\varepsilon}{3}, \forall x, y \in K, \text{ with } d(x, y) < \delta, \forall k \in \mathbb{N}.$$

(ii) Also, since K is compact, then K is *totally bounded*, which means there exists $y_1, \cdots, y_m \in K$ such that

$$K \subseteq \bigcup_{i=1}^{m} D\left(y_i, \frac{\delta}{2}\right).$$

Since E is dense in K, pick some $z_i \in E \bigcap D(y_i, \frac{\delta}{2})$ for $i = 1, \cdots, m$. Then we end up having $z_1, \cdots, z_m$ with the property that $K \subseteq \bigcup_{i=1}^{m} D(y_i, \frac{\delta}{2}) \subseteq \bigcup_{i=1}^{m} D(z_i, \delta)$. By the pointwise convergence $\{f_k\}_{k=1}^{\infty}$ on E, for each z_j, there exists $N_j \in \mathbb{N}$, such that

$$\|f_k(z_j) - f_\ell(z_j)\| < \frac{\varepsilon}{3}, \forall k, \ell > N_j$$

Let

$$N := \max_{j=1,\cdots,m} N_j.$$

Then it implies $\|f_k(z_t) - f_\ell(z_t)\| < \frac{\varepsilon}{3}$ for any $z_t \in \{z_1, \cdots, z_m\}$ whenever $k, \ell > N$

(iii) So for any $x \in K$, x must be in some $D(y_J, \frac{\delta}{2})$ and $z_J \in E \bigcap D\left(y_J, \frac{\delta}{2}\right)$. So we have

$$d(x, z_J) \leqslant d(x, y_J) + d(y_J, z_J) < \delta.$$

It implies

(a) $\|f_k(x) - f_k(z_J)\| < \frac{\varepsilon}{3}$, for any $k \in \mathbb{N}$.

(b) And therefore whenever $k, \ell > N$, we have

$$\begin{aligned}\|f_k(x) - f_\ell(x)\| &\leqslant \|f_k(x) - f_k(z_J)\| + \|f_k(z_J) - f_\ell(z_J)\| + \|f_\ell(z_J) - f_\ell(x)\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon\end{aligned}$$

Hence, $\{f_k\}_{k=1}^{\infty}$ is uniformly Cauchy. Along with the condition that V is a Banach space, it is concluded that $\{f_k\}_{k=1}^{\infty}$ converges uniformly on K. □

Remark 1. *E is dense in K means that*

$$E \subseteq K \subseteq \overline{E}$$

Learn from this proof:

- Use the *Cauchy criterion* to prove uniform convergence.
- Use equivalent conditions of compactness of K: *totally bounded*. In this way, problems can be simplified to *finiteness*.

THE FIRST EQUIVALENCE

By Corollary 1 and Theorem 1, we have the first important equivalent theorem:

THEOREM 2. *A sequence $\{f_k\}_{k=1}^{\infty} \subseteq \mathscr{C}(K;V)$ converges uniformly on $K \iff \{f_k\}_{k=1}^{\infty}$ is equi-continuous* AND *pointwise convergent (on a dense subset of K).*

REMARK 2. *Left $\implies$ Right does not need V to be a Banach space.*

REMARK 3. *Right $\implies$ Left dictates that pointwise convergence only holds on a dense subset of K.*

REMARK 4. *It is easier to check pointwise convergence and equi-continuity than uniform convergence,e.g. $\{f_k\}_{k=1}^{\infty} \subseteq \mathscr{C}([0,1];\mathbb{R})$ and pointwise convergence. Suppose that $\left|f_k'(x)\right| \leqslant 1, \forall k \in \mathbb{N}, x \in [0,1]$. It is easy to check $\{f_k\}_{k=1}^{\infty}$ is equi-continuous on $[0,1]$. Thus, $\{f_k\}_{k=1}^{\infty}$ is uniformly convergent. The advantage is that we avoid checking the thornier condition for uniform convergence.*

Theorem 2 states that if we have a sequence of pointwise convergent functions, then its uniform convergence is equivalent to its equi-continuity.

EXAMPLE 2.1 (Revisit Example 1). *Let's see whether $\{f_k\}_{k=1}^{\infty}$ is uniformly convergent. Check the condition for uniform convergence:*

$$\lim_{k\to\infty} \sup_{x\in[0,1]} |f_k(x) - 0| = 1.$$

Therefore, $\{f_k\}_{k=1}^{\infty}$ is not uniformly convergent on $[0,1]$. Thus, $\{f_k\}_{k=1}^{\infty}$ is not equi-continuous either.

3. THREE NOTIONS OF COMPACTNESS IN $\mathscr{C}(K;V)$

'kjdd: We have investigated the relationship between uniform convergence and pointwise convergence for a sequence of functions $\{f_k\}_{k=1}^{\infty}$ in $\mathscr{C}(K;V)$ (See Theorem 2). As is discussed in the motivation, we will proceed to generalize this result to classes of functions and see what a compact set will look like in $\mathscr{C}(K;V)$.

To better fulfil this goal, we first study three notions that are closely relevant to our subsequent discussions: *pointwise compact / pointwise pre-compact / pointwise bounded*.

The general settings are the same as before: suppose (M,d) is a metric space and $(V,\|\cdot\|)$ is a normed vector space.

Definition 3 (pointwise bounded). *A subset $B \subseteq \mathscr{C}_b(A; V)$ is said to be pointwise bounded is the set $B_x := \{f(x) : f \in B\}$ is bounded in $(V, \|\cdot\|)$ for all $x \in A$.*

Definition 4 (pointwise pre-compact). *A subset $B \subseteq \mathscr{C}_b(A; V)$ is said to be pointwise pre-compact is the set $B_x := \{f(x) : f \in B\}$ is pre-compact in $(V, \|\cdot\|)$ for all $x \in A$.*

Definition 5 (pointwise compact). *A subset $B \subseteq \mathscr{C}_b(A; V)$ is said to be pointwise compact is the set $B_x := \{f(x) : f \in B\}$ is compact in $(V, \|\cdot\|)$ for all $x \in A$.*

Remark 5. *Note that $B_x \subseteq V$ (but $B \subseteq \mathscr{C}(K; V)$) for any $x \in A$. So it is called "pointwise" bounded/ pre-compact/compact.*

Remark 6. *Those notions are useful because they are related to pointwise convergence of a sequence of functions, which is the first step to further discuss stronger uniform convergence.*

3.1 Diagonal Process

An important tool in analysis is *diagonal process*.

pointwise pre-compact $\Longrightarrow$ pointwise convergence of a subsequence

Lemma 2. *Let E be a countable set, $(V, \|\cdot\|)$ be a Banach space, and $f_k : E \to V$ be a sequence of functions. Suppose that for each $x \in E$, $\{f_k(x)\}_{k=1}^{\infty}$ is pre-compact in V. Then there exists a subsequence of $\{f_k\}_{k=1}^{\infty}$ that converges pointwise on E.*

Proof. 1. Since E is countable, define $E = \{x_i\}_{i=1}^{\infty}$.
2. Plugging $x_1, x_2, \cdots$ into $\{f_k\}_{k=1}^{\infty}$ sequentially, which allows us to have the desired subsequence.

why it is called "diagonal" process?

- $\{f_k(x_1)\}_k \subseteq V$ is pre-compact $\Longrightarrow \exists f_{k_j}(x_1) \to f, j \to \infty$ $\Longrightarrow S_1 := \left\{f_{k_1}, f_{k_2}, \cdots\right\}$: converges at x_1.
- $f_{k_j}(x_2) \subseteq V$ is pre-compact $\Longrightarrow \exists f_{k_{j_\ell}}(x_2) \to f, \ell \to \infty \Longrightarrow S_2 := \left\{f_{k_{j_1}}, f_{k_{j_2}}, \cdots\right\}$: converges at x_1 and x_2.
- $\cdots$
- $\cdots$, S_k : converges at $x_1, x_2, \cdots, x_k$.
- $\cdots$.

Continue this process, we obtain a further subsequence of its last step's subsequence that has the following properties:

- $S_k \supseteq S_{k+1}$ for each k.
- S_k converges at $x_1, x_2, \cdots, x_k$.

3. Creating the new sequence $\{g_k\}_{k=1}^{\infty}$ by picking up each its element by: g_j is the j-th element of S_k. By construction, after discarding the first k elements, the sequence becomes a subsequence S_k, implying that $\{g_i\}_{i=k}^{\infty} \subseteq S_k$, for each $k \in \mathbb{N}$. Thus, the sequence $g_1, g_2, \cdots$ converges at $\{x_i\}_{i=1}^{\infty}$ eventually.

$\square$

How to understand pre-compact?

Observe that
A is pre-compact $\iff \bar{A}$ is compact $\iff$ Given $\{x_k\}_{k=1}^{\infty} \subseteq A$, $\exists \left\{x_{k_j}\right\}_j$ s.t. $x_{k_j} \to x \in A$, as $j \to \infty$.
We often use the sub-sequence argument to characterize the pre-compactness of a set.

3.2 Why compactness matters?

Compact $\implies$ Separable

Lemma 3. *A compact set K in a metric space (M, d) is separable; that is, there exists a countable subset E of K such that $\overline{E} = K$.*

equi-continuity + pointwise pre-compact $\implies$ pre-compact

Theorem 3. *Suppose $(V, \|\cdot\|)$ is a Banach space, $K \subseteq M$ is compact and $B \subseteq \mathscr{C}(K; V)$ is equi-continuous and pointwise pre-compact. Then B is pre-compact in $(\mathscr{C}(K; V), \|\cdot\|_{\infty})$.*

Proof. By Lemma 3, Lemma 2 and Theorem 1, the desired result is obtained. $\square$

Then the "general" version of Arzelà-Ascoli theorem immediately follows

The second equivalence (The "general" version of Arzelà-Ascoli Theorem)

Theorem 4 (The "general" version of Arzelà-Ascoli Theorem). *A set $B \subseteq \mathscr{C}(K; V)$ is pre-compact $\iff B$ is equi-continuous* AND *pointwise pre-compact.*

Notice that

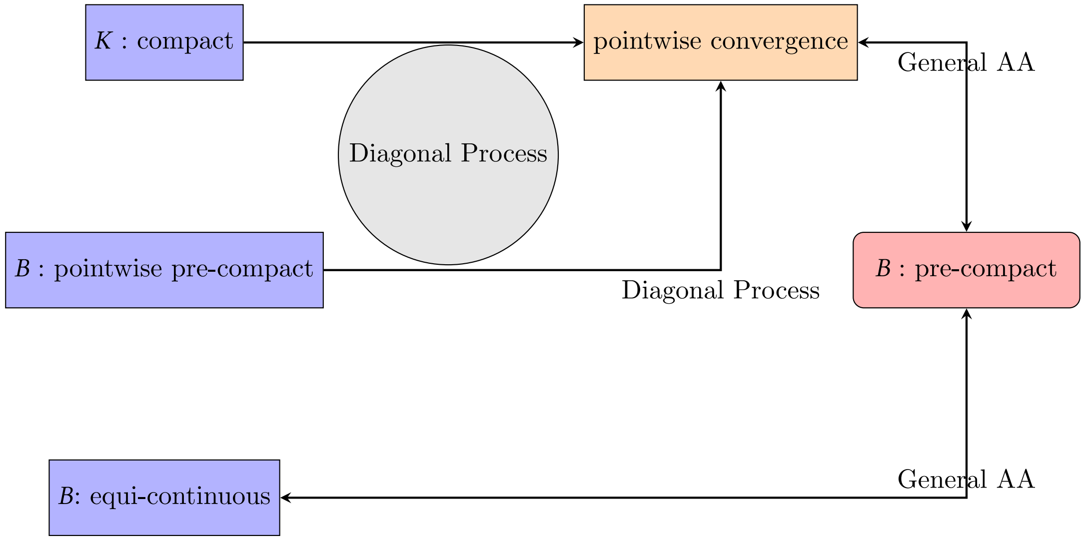

FIGURE 2. *The roadmap of the "general" AA theorem*

When $\{f_k\}_{k=1}^{\infty}$ is a sequence of real-valued functions, *i.e.* $\{f_k\}_{k=1}^{\infty} \subseteq \mathscr{C}(K;\mathbb{R})$ (sometimes $\mathscr{C}(K;\mathbb{R})$ can be abbreviated as $\mathscr{C}(K)$), by the Bolzano-Weierstrass theorem, pointwise pre-compact condition is equivalent to pointwise bounded condition. Hence, we have the following corollary:

The "general" Arzelà-Ascoli Theorem in $\mathbb{R}$

COROLLARY 2 (The "general" Arzelà-Ascoli theorem in $\mathbb{R}$). *Let $K \subseteq M$ be compact and $B \subseteq \mathscr{C}(K;\mathbb{R})$ be equi-continuous and pointwise bounded on K. Then every sequence in B has a uniformly convergent subsequence.*

The third equivalence

Theorem 5 (The Arzelà-Ascoli Theorem). *Let (M, d) be a metric space, $(V, \|\cdot\|)$ be a Banach space, $K \subseteq M$ be a compact set, and $B \subseteq \mathscr{C}(K; V)$. Then B is compact in $(\mathscr{C}(K; V), \|\cdot\|_\infty) \Longleftrightarrow B$ is closed, equi-continuous, and pointwise compact.*

4. STOCHASTIC EQUI-CONTINUITY

Recall that given a sequence of deterministic functions $f_n(\theta)$, the (uniform) equi-continuity condition enables equivalence of the uniform convergence and the pointwise convergence of this sequence.

4.1 asymptotic equi-continuity

However, the apparent shortcoming is that $f_n(\theta)$ must be continuous and defined on a compact set, which precludes its practicality. For example,

$$f_n(\theta) = \begin{cases} 0, & \text{if } \theta \in [0, \frac{1}{2}] \\ \frac{1}{n}, & \text{if } \theta \in (\frac{1}{2}, 1] \end{cases}$$

Even though f_n is not continuous for any n, $f_n \longrightarrow 0$ uniformly. In this example, it is not appropriate to use the *modulus of continuity* to check uniform convergence of f_n, because given n and δ, $w_n(\delta) = \sup_{\theta,\theta' \in [0,1]} \{|f_n(\theta) - f_n(\theta')| : d(\theta, \theta') < \delta\} = \frac{1}{n}$, and thus

$$\lim_{\delta \to 0} \sup_n w_n(\delta) = 1 \neq 0.$$

A modified condition is called *asymptotic (uniform)* equi-continuity then is employed to deal with such scenario:

$$\lim_{\delta \to 0} \limsup_{n \to \infty} w_n(\delta) = 0$$

It is not hard to see that for the above example, the asymptotic equi-continuity condition holds. The following theorem also follows:

Theorem 6. *Suppose Θ is compact, then the following two statements are equivalent*

1. $\sup_{\theta \in \Theta} |f_n(\theta)| \longrightarrow 0$.
2. $f_n(\theta) \longrightarrow 0$ *for each* θ, *and* f_n *is asymptotically (uniformly) equi-continuous.*

4.2 stochastic equi-continuity

In this section, the class of *deterministic* functions is relaxed to *random* functions. This relaxation matters because showing consistency of proposed estimators is of paramount importance in both econometric and statistics. In particular, when the estimators are obtained through some implicit minimization/maximization process, some well-behaved properties on the objective functions need to be imposed, one of which is *stochastic (uniform) equi-continuity*.

Definition 6 (stochastic (uniform) equi-continuity). *A sequence of random functions $Q_n(\theta)$ is* *stochastic uniform equi-continuity* *if* $\forall \varepsilon > 0$,

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}\left(\sup_{|\theta - \theta'| < \delta} |Q_n(\theta) - Q_n(\theta')| > \varepsilon \right) = 0.$$

Alternative definition: $\forall \varepsilon > 0$, $\exists \delta > 0$ *such that*

$$\limsup_{n \to \infty} \mathbb{P}\left(\sup_{|\theta - \theta'| < \delta} |Q_n(\theta) - Q_n(\theta')| > \varepsilon \right) < \varepsilon.$$

Yet another equivalent definition: $\forall \varepsilon > 0, \eta > 0$, $\exists \delta > 0$ *such that*

$$\limsup_{n \to \infty} \mathbb{P}\left(\sup_{|\theta - \theta'| < \delta} |Q_n(\theta) - Q_n(\theta')| > \varepsilon \right) < \eta.$$

Similar to the Arzelà-Ascoli theorem which asserts that a pointwise convergent sequence of deterministic functions is also uniformly convergent under the equi-continuity condition, the stochastic analogue is that pointwise consistency also implies uniform consistency under the stochastic equi-continuity condition.

Theorem 7. *If $Q_n(\theta) \xrightarrow{P} 0$ for each $\theta \in \Theta$ and $Q_n(\theta)$ is stochastic equi-continuous on Θ which is compact, then $\sup_{\theta \in \Theta} |Q_n(\theta)| \xrightarrow{P} 0$.*

Definition 7. *Let $X_n \in L_\infty(T)$ be random variables. We say X_n are* *asymptotically equi-continuous* *if for all $\eta, \varepsilon > 0$, there is a finite partition $T_1, \cdots, T_k$ of T such that*

$$\limsup_{n \to \infty} \mathbb{P}\left(\max_i \sup_{s,t \in T_i} \left| X_{n,s} - X_{n,t} \right| \geqslant \varepsilon \right) \leqslant \eta.$$

Recall that we have a sequence Ω_n of sample spaces, with $X_n : \Omega_n \to L_\infty(T)$, and $X_{n,t}(\omega)$ is the value of $X_n(\omega)$ at t.

5. EXERCISE

Exercise 1. *Show that $B \subseteq \mathscr{C}_b(K; V)$ is equi-continuous if*

$$\lim_{\delta \to 0} \sup_{f \in B} w_f(\delta) = 0.$$

Equivalently, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\sup_{d(x,y) < \delta} \|f(x) - f(y)\| \leqslant \varepsilon, \ \forall f \in B.$$

Exercise 2. *A consequence of Definition 2 is that if a function class B is bounded at some $x_0 \in A$, then it is bounded for all $x \in A$.*

Exercise 3. *Define $\mathscr{C}^0(K)$ to be the family of real-valued continuous functions on K, i.e.*

$$\mathscr{C}^0(K) := \{f : K \to \mathbb{R} | f \text{ is continuous}\}$$

and the metric associated with the space is:

$$d(f,g) := \sup_{x\in K} |f(x) - g(x)| .$$

Prove

(i) $(\mathscr{C}^0(K), d)$ *is a complete metric space.*
(ii) $f_n \to f \iff f_n \to f$ *uniformly on* K.
(iii) $\{f_n\}$ *is bounded in* $\mathscr{C}^0(K) \iff \{f_n\}$ *is uniformly bounded on* K.
(iv) A subset $\mathscr{F} \subseteq \mathscr{C}^0(K)$ *is compact with respect to the metric* d *if and only if* $\mathscr{F}$ *is closed, bounded and equi-continuous*

Exercise 4. *Suppose that X and Y are two compact metric spaces, V is a metric space and $f : X \times Y \to V$ is a continuous function. For every $x \in X$, let f_x be the continuous function $Y \ni y \mapsto f(x,y) \in V$. We furnish $\mathscr{C}(Y;V)$ with the supremum metric:*

$$d^\infty(g,h) = \max_{y\in Y} d_V(g(y),h(y)), \quad \forall g,h \in \mathscr{C}(Y;V).$$

Show that the function $x \mapsto f_x$ is continuous from X into $\mathscr{C}(Y;V)$ and $\{f_x\}_{x\in X}$ is an equi-continuous family in $\mathscr{C}(Y;V)$.

Exercise 5. *Let X and Y be two Banach spaces and let $\mathcal{Y} \subseteq \mathcal{L}(X;Y)$. Show that $\mathcal{Y}$ is equi-continuous if and only if it is equi-bounded (i.e. , there exists $M > 0$ sucht that $\|A\|_{\mathcal{L}} \leqslant M$ for all $A \in \mathcal{Y}$).*

Exercise 6. *Show that the three definitions of stochastic equi-continuity in Definition 6 are equivalent.*

6. SOLUTIONS

Exercise 4

Goal

Define $g : X \to \mathscr{C}(Y;V)$ such that $g(x) = f_x$. The goal is to show for any $\{x_n\}_{n=1}^\infty \subseteq X$ with $d_X(x_n,x) \to 0$, then $d^\infty(g(x_n),g(x)) \to 0$.

Proof. Let $z = (x,y) \in (X\times Y)$, the continuity of $f(x,y)$ means that whenever $d_X(x_n,x) \to 0$ and $d_Y(y_n,y) \to 0$, we have $d_V(f(z_n),f(z)) \to 0$.

For any $y \in Y$ with $d_X(x_n,x) \to 0$ and $d_Y(y,y) \equiv 0$, by the continuity of $f(x,y)$, then it follows that

$$d_V(f_{x_n},f_x) = d_V(f(x_n,y),f(x,y)) \to 0, \text{ as } n \to \infty.$$

By definition

$$\begin{aligned} d^{\infty}(g(x_n), g(x)) &= \max_{y \in Y} d_V(f_{x_n}, f_x) \\ &= \max_{y \in Y} d_V\left(f(x_n, y), f(x, y)\right) \end{aligned}$$

□