The Strong Law of Large Numbers

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The first strong law of large numbers is motivated by the Borel-Cantelli lemmas:

Proposition 1 (Proposition 2.2.16). *Suppose that $\mathbb{E}[Z_n^2] \leqslant C$ for some $C < \infty$ and all n. Then, $\frac{Z_n}{n} \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$.*

Proof. Fix $\delta > 0$, consider the set

$$A_k = \{\omega \in \Omega : |Z_k(\omega)| > k\delta\} \quad \text{for each } k \in \mathbb{N}.$$

By the Chebyshev's inequality,

$$\mathbb{P}(A_k) \leqslant \frac{\mathbb{E}[Z_k^2]}{k^2\delta^2} \leqslant \frac{C}{\delta^2 k^2}.$$

By $\sum_k \mathbb{P}(A_k) \leqslant (C/\delta^2) \sum_k \frac{1}{k^2} < \infty$, apply the Borel-Cantelli lemma I,

$$\mathbb{P}(A_k \text{ i.o.}) = \mathbb{P}(\cap_{m \geqslant 1} \cup_{k \geqslant m} A_k) = 0.$$

Define $N = \bigcap_{m \geqslant 1} \bigcup_{k \geqslant m} A_k$. Pick any $\omega \in N^c$, i.e., $\exists m \in \mathbb{N}$, s.t., $\forall k \geqslant m$, we have

$$\left| \frac{Z_k(\omega)}{k} \right| \leqslant \delta, \forall k \geqslant m.$$

It is true for any k and $\delta > 0$. So let $\delta \downarrow 0$, we have

$$\limsup_k \left| \frac{Z_k(\omega)}{k} \right| = 0.$$

and therefore,

$$\lim_n \frac{Z_n(\omega)}{n} = 0,$$

which concludes that $n^{-1} Z_n \xrightarrow{\text{a.s.}} 0$. □

independent but non-identical R.V.s

Exercise 1. *Suppose X_i are mutually independent random variables such that $\mathbb{P}(X_n = n^2 - 1) = 1 - \mathbb{P}(X_n = -1) = n^{-2}$ for $n = 1, 2, \cdots$. Show that $\mathbb{E}[X_n] = 0$, for all n, while $n^{-1} \sum_{i=1}^n X_i \xrightarrow{\text{a.s.}} -1$ for $n \to \infty$.*

Additional properties of convergence a.s.

EXERCISE 2 (2.2.19). *Show that for any R.V.* X_n

1. $X_n \xrightarrow{a.s.} 0 \iff \mathbb{P}(|X_n| > \varepsilon \text{ i.o.}) = 0$ *for each* $\varepsilon > 0$.
2. *There exist non-random constants* $b_n \uparrow \infty$ *such that* $X_n/b_n \xrightarrow{a.s.} 0$.

PROOF.

1. For the sufficiency, suppose $X_n \xrightarrow{\text{a.s.}} 0$. It means for any $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that $|X_n| < \varepsilon$ with probability 1 for any $n \geqslant m$. Or equivalently, given $\varepsilon > 0$,

$$\mathbb{P}\left(\bigcup_{m\in\mathbb{N}}\bigcap_{n\geqslant m}\{|X_n| < \varepsilon\}\right) = 1$$

Taking the complement of the insider set:

$$\left(\bigcup_{m\in\mathbb{N}}\bigcap_{n\geqslant m}\{|X_n| < \varepsilon\}\right)^c = \{|X_n| > \varepsilon \text{ i.o.}\}.$$

So we have

$$\mathbb{P}(|X_n| > \varepsilon \text{ i.o.}) = 0.$$

For the necessary part, replace $\varepsilon = 1/k, k \in \mathbb{N}$. The condition can be rephrased as

$$\mathbb{P}\left(|X_n| > \frac{1}{k} \text{ i.o.}\right) = 0 \quad \text{for any } k \in \mathbb{N}.$$

Define the set

$$N := \bigcup_{k=1}^{\infty}\bigcap_{m=1}^{\infty}\bigcup_{\ell>m}\left\{|X_n| > \frac{1}{k}\right\}$$

So N is a $\mathbb{P}$-null set, i.e., $\mathbb{P}(N) = 0$.
Pick any $\omega \in N^c$. Then for any $k \in \mathbb{N}$, there exists $m \in \mathbb{N}$, such that $|X_n(\omega)| \leqslant \frac{1}{k}$, $\forall \ell > k$. This indicates that $X_n \xrightarrow{\text{a.s.}} 0$.

2.

$\square$