Solutions for Homework 7

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PROBLEMS: 12.12, 12.15, 12.16, 12.23.

PROBLEM 12.12

PROOF. (a) By the OLS of Y_i on $\widehat{X}_i^2$, we have

$$\begin{aligned}\widehat{\beta} &= \frac{\sum_i Y_i \widehat{X}_i^2}{\sum_i \widehat{X}_i^4} \\ &= \frac{1}{\widehat{\gamma}^2} \frac{\sum_i Y_i Z_i^2}{\sum_i Z_i^4} \\ &= \frac{(\sum_i Z_i^2)^2 \sum_i Y_i Z_i^2}{(\sum_i X_i Z_i)^2 \sum_i Z_i^4}\end{aligned}$$

(b) By weak law of large numbers,

$$\begin{aligned}\widehat{\beta} &\xrightarrow{\mathsf{P}} \frac{\mathbb{E}[Y_i Z_i^2](\mathbb{E}[Z_i^2])^2}{(\mathbb{E}[Z_i^4])(\mathbb{E}[X_i Z_i])^2} \\ &= \frac{\beta\left(\gamma^2 \mathbb{E}[Z_i^4] + \mathbb{E}[Z_i^2 u_i^2] + 2\gamma \mathbb{E}[Z_i^3 u_i]\right) + \mathbb{E}[Z_i^2 e_i]}{\mathbb{E}[Z_i^4]\gamma^2}.\end{aligned}$$

(c) In general, $\widehat{\beta}$ is not consistent for β, unless Z^2 satisfies $\mathbb{E}[Z^2 e] = \mathbb{E}[Z^2 u] = 0$.

□

PROBLEM 12.15

PROOF. (a) No. Perfect collinearity would be an issue.

(b) Yes, if certain conditions are satisfied.

(c) Exclusion condition is $\mathbb{E}[Z^2 e] = 0$.

(d) Regress Y_2 on Z and Z^2, both projection coefficients cannot be zero.

(e) In a generic application, folks are not comfortable about it. It seems that there is free lunch to remedy endogeneity.

□

PROBLEM 12.16

PROOF. (a) It is straightforward that

$$Y = X\beta + e = \gamma\beta Z + u_2\beta + e.$$

So

$$\lambda = \gamma\beta$$
$$u_1 = e + u_2\beta$$

(b) It is not hard to obtain

$$\sqrt{n}(\widehat{\theta} - \theta) = \sqrt{n}\frac{\sum_i Z_i u_i}{\sum_i Z_i^2}. \tag{1}$$

where $u_i = (u_{1i}, u_{2i})'$.

(c) The assumptions $\mathbb{E}[Ze] = \mathbb{E}[Zu_2] = 0$ imply $\mathbb{E}[Zu] = 0$.

(d) Apply central limit theorem on the RHS of Eq. (1)

$$\sqrt{n}(\widehat{\theta} - \theta) \xrightarrow{\mathsf{d}} \mathcal{N}\left(\mathbf{0}, \frac{1}{(\mathbb{E}[Z_i^2])^2}\Omega_u\right)$$

(e) The matrix

$$R = \begin{pmatrix} \frac{1}{\gamma} \\ -\frac{\lambda}{\gamma^2} \end{pmatrix}.$$

Thus we have

$$\sqrt{n}(\widehat{\beta} - \beta) \xrightarrow{\mathsf{d}} \mathcal{N}\left(\mathbf{0}, \frac{1}{(\mathbb{E}[Z_i^2])^2}R'\Omega_u R\right) \tag{2}$$

(f) By theorem 12.2 in Hansen's text, the asymptotic variance of 2SLS estimator is

$$\frac{1}{\gamma^2(\mathbb{E}[Z_i^2])^2}\mathbb{E}[Z_i^2 e_i^2]$$

Next, we shall show the asymptotic variance in Eq. (2) is the same. First observe that

$$R = \frac{1}{\gamma}\begin{pmatrix} 1 \\ -\beta \end{pmatrix}.$$

Then it can be shown that

$$\frac{1}{(\mathbb{E}[Z_i^2])^2}R'\Omega R = \frac{1}{\gamma^2(\mathbb{E}[Z_i])^2}\mathbb{E}[Z_i^2 e_i^2].$$

Thus they are numerically identical.

□