AS.180.633: Econometrics **Spring 2020**

Homework 6: Suggested Solutions

Instructor: Yingyao Hu *By: Tong Zhou*

7.2

Let $\boldsymbol{y}$ be $n\times 1$, $\boldsymbol{X}$ be $n\times k$ (rank k). $\boldsymbol{y}=\boldsymbol{X\beta}+\boldsymbol{e}$ with $\mathbb{E}[x_i e_i]=0$. Define the *ridge regression* estimator

$$\widehat{\boldsymbol{\beta}}=\left(\sum_{i=1}^{n}\boldsymbol{x}_i\boldsymbol{x}_i'+\lambda\boldsymbol{I}_k\right)^{-1}\left(\sum_{i=1}^{n}\boldsymbol{x}_i y_i\right)$$

here $\lambda>0$ is a fixed constant. Find the probability limit of $\widehat{\boldsymbol{\beta}}$ as $n\to\infty$. Is $\widehat{\boldsymbol{\beta}}$ consistent for $\boldsymbol{\beta}$?

Proof. By WLLN,

$$\frac{1}{n}\lambda\boldsymbol{I}_k\to\boldsymbol{0},\quad \text{as}\quad n\to\infty$$

therefore,

$$\widehat{\boldsymbol{\beta}}=\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{x}_i\boldsymbol{x}_i'+\frac{1}{n}\lambda\boldsymbol{I}_k\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{x}_i y_i\right)\xrightarrow{P}\left(\mathbb{E}[\boldsymbol{x}_i\boldsymbol{x}_i']\right)^{-1}\mathbb{E}[x_i y_i]\equiv\boldsymbol{\beta}$$

Thus, $\widehat{\boldsymbol{\beta}}$ is consistent for $\boldsymbol{\beta}$. □

7.7

Of the variables $(y_i^*, y_i, \boldsymbol{x}_i)$ only the pair $(y_i,\boldsymbol{x}_i)$ are observed. In this case we say that y_i^* is a *latent* variable. Suppose

$$\begin{aligned} y_i^* &= \boldsymbol{x}_i'\boldsymbol{\beta}+e_i\\ \mathbb{E}[\boldsymbol{x}_i e_i] &= \boldsymbol{0}\\ y_i &= y_i^*+u_i \end{aligned}$$

where u_i is a measurement error satisfying

$$\begin{aligned} \mathbb{E}[\boldsymbol{x}_i u_i] &= \boldsymbol{0}\\ \mathbb{E}[y_i^* u_i] &= 0. \end{aligned}$$

Let $\widehat{\boldsymbol{\beta}}$ denote the OLS coefficient from the regression of y_i on $\boldsymbol{x}_i$.

(a) Is $\boldsymbol{\beta}$ the coefficient from the linear projection of y_i on $\boldsymbol{x}_i$?

(b) Is $\widehat{\boldsymbol{\beta}}$ consistent for $\boldsymbol{\beta}$ as $n \to \infty$?

(c) Find the asymptotic distribution of $\sqrt{n}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ as $n \to \infty$.

Proof. **(a)**

Define

$$v_i = e_i + u_i$$

for each i, then

$$y_i = \boldsymbol{x}_i'\boldsymbol{\beta} + v_i.$$

Hence, $\boldsymbol{\beta}$ is the coefficient from the linear projection of y_i on $\boldsymbol{x}_i$.

(b)

we have

$$\mathbb{E}[\boldsymbol{x}_i v_i] = \mathbf{0}.$$

So,

$$\begin{aligned}
\widehat{\boldsymbol{\beta}} &= \left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i\boldsymbol{x}_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i y_i\right) \\
&= \boldsymbol{\beta} + \left(\frac{1}{n}\sum_{i}^{n} \boldsymbol{x}_i\boldsymbol{x}_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i v_i\right) \\
&\xrightarrow{P} \boldsymbol{\beta} + \left(\mathbb{E}[\boldsymbol{x}_i\boldsymbol{x}_i']\right)^{-1} \underbrace{\mathbb{E}[\boldsymbol{x}_i v_i]}_{=\mathbf{0}} \\
&= \boldsymbol{\beta}.
\end{aligned}$$

That is, $\widehat{\boldsymbol{\beta}} \xrightarrow{P} \boldsymbol{\beta}$.

(c)

From (b),

$$\sqrt{n}\left(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) = \left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i\boldsymbol{x}_i'\right)^{-1} \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \boldsymbol{x}_i v_i \tag{1}$$

By the weak law of large numbers,

$$\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{x}_i\boldsymbol{x}_i' \xrightarrow{P} \mathbb{E}[\boldsymbol{x}_i\boldsymbol{x}_i'] \equiv \boldsymbol{Q}_{\boldsymbol{xx}}. \tag{2}$$

By the central limit theorem,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\boldsymbol{x}_i\nu_i \xrightarrow{d} \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Omega}_\nu\right) \tag{3}$$

where $\boldsymbol{\Omega}_\nu = \mathbb{E}[\boldsymbol{x}_i\boldsymbol{x}_i'\nu_i^2]$.

Plugging Eq. (2) and Eq. (3) into Eq. (1), by the Slustky Theorem

$$\sqrt{n}\left(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \boldsymbol{Q}_{\boldsymbol{xx}}^{-1}\boldsymbol{\Omega}_\nu\boldsymbol{Q}_{\boldsymbol{xx}}^{-1}).$$

□

7.8

Find the asymptotic distribution of $\sqrt{n}(\hat{\sigma}^2 - \sigma^2)$ as $n \to \infty$.

Proof. By Equation 7.18 of Hansen's book, we have

$$\begin{aligned}
\sqrt{n}(\hat{\sigma}^2 - \sigma^2) = {} & \sqrt{n}\left(\frac{1}{n}\sum_i e_i^2 - \sigma^2\right) \\
& - \underbrace{\sqrt{n}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})'}_{=O_p(1)} \overbrace{\frac{2}{n}\sum_i(\boldsymbol{x}_i e_i)}^{=o_p(1)} \\
& + \underbrace{\sqrt{n}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})'}_{=O_p(1)} \overbrace{\frac{1}{n}\sum_i(\boldsymbol{x}_i\boldsymbol{x}_i')}^{=O_p(1)} \underbrace{(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})}_{=o_p(1)} \\
& = \sqrt{n}\left(\frac{1}{n}\sum_i e_i^2 - \sigma^2\right) + o_p(1) \\
& \xrightarrow{\mathrm{d}} \mathcal{N}\left(0, \mathsf{Var}(e_i^2)\right).
\end{aligned}$$

Thus, $\sqrt{n}(\hat{\sigma}^2 - \sigma^2) \xrightarrow{d} \mathcal{N}(0, \mathsf{Var}(e_i^2))$.

□

7.14

Take the model

$$\begin{aligned} y_i &= x_{1i}\beta_1 + x_{2i}\beta_2 + e_i \\ \mathbb{E}[x_i e_i] &= 0 \end{aligned}$$

with both $\beta_1 \in \mathbb{R}$ and $\beta_2 \in \mathbb{R}$, and define the parameter

$$\theta = \beta_1\beta_2.$$

(a) What is the appropriate estimator $\widehat{\theta}$ for θ?

(b) Find the asymptotic distribution of $\widehat{\beta}$ under standard regularity conditions.

(c) Show how to calculate an asymptotic 95% confidence interval for θ.

Proof. **(a)**

A natural estimator $\widehat{\theta}$ is that

$$\widehat{\theta} = \widehat{\beta}_1\widehat{\beta}_2.$$

(b)

By **Theorem 7.9**,

$$\sqrt{n}\left(\widehat{\theta} - \theta\right) = \sqrt{n}\left(\widehat{\beta}_1\widehat{\beta}_2 - \beta_1\beta_2\right) \xrightarrow{d} \mathcal{N}(0, V_\theta)$$

where $V_\theta = \boldsymbol{R}'\boldsymbol{V}_\beta\boldsymbol{R} = \begin{pmatrix}\beta_2 & \beta_1\end{pmatrix}\begin{pmatrix}V_{11} & V_{12} \\ V_{21} & V_{22}\end{pmatrix}\begin{pmatrix}\beta_2 \\ \beta_1\end{pmatrix} = \beta_2^2 V_{11} + 2\beta_1\beta_2 V_{12} + \beta_1^2 V_{22}.$

(c)

By **Theorem 7.14**, the 95% confidence interval is:

$$\left[\widehat{\theta} - 1.96 \times s(\widehat{\theta}), \widehat{\theta} + 1.96 \times s(\widehat{\theta})\right]$$

where $s(\widehat{\theta}) = \sqrt{\widehat{\boldsymbol{R}}'\widehat{\boldsymbol{V}}_{\widehat{\beta}}\widehat{\boldsymbol{R}}} = \sqrt{\frac{\widehat{\boldsymbol{R}}'\widehat{\boldsymbol{V}}_{\beta}\widehat{\boldsymbol{R}}}{n}}.$

□

7.28

As in Exercise 3.26, use the CPS dataset and the subsample of white male Hispanics. Estimate the regression

$$\widehat{\log(\text{wage})} = \beta_1 \text{education} + \beta_2 \text{experience} + \beta_3 \text{experience}^2/100 + \beta_4.$$

(a) Report the coefficient estimates and robust standard errors.

(b) let θ be the ratio of the return to one year of education to the return to one year of experience. Write θ as a function of the regression coefficients and variables. Compute $\widehat{\theta}$ from the estimated model.

(c) Write out the formula for the asymptotic standard error for $\widehat{\theta}$ as a function of the covariance matrix for $\widehat{\boldsymbol{\beta}}$. Compute $s\left(\widehat{\theta}\right)$ from the estimated model.

(d) Construct a 90% asymptotic confidence interval for θ from the estimated model.

(e) Compute the regression function at education = 12 and experience = 20. Compute 95% confidence interval for the regression function at the point.

(f) Consider an out-of-sample individual with 16 years of education and 5 years experience. Construct an 80% forecast interval for their log wage and wage. (To obtain the forecast interval for the wage, apply the exponential function to both endpoints.)

Proof. **(a)** The codes is similar to Exercise 3.26 of Homework 3.

Using *Stata*, it can be obtained that

$$\widehat{\boldsymbol{\beta}} = \begin{pmatrix} 0.090 \\ 0.035 \\ -0.047 \\ 1.185 \end{pmatrix}$$

and

$$\widehat{\boldsymbol{V}}_{\widehat{\boldsymbol{\beta}}} = \begin{pmatrix} 0.085 & 0.006 & 0.003 & -1.097 \\ 0.006 & 0.067 & 0.131 & -0.697 \\ 0.003 & 0.131 & 0.282 & 1.056 \\ -1.097 & -0.691 & 1.056 & 21.252 \end{pmatrix} \times 10^{-4}$$

(b)

$$\theta = \frac{\beta_1}{\beta_2 + \beta_3 \exp/50}$$

So

$$\widehat{\theta} = \frac{\widehat{\beta}_1}{\widehat{\beta}_2 + \widehat{\beta}_3 \exp/50}$$

(c)

Let $\theta = r(\boldsymbol{\beta})$, then

$$R = \frac{\partial r(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = \begin{pmatrix} \partial\theta/\partial\beta_1 \\ \partial\theta/\partial\beta_2 \\ \partial\theta/\partial\beta_3 \\ \partial\theta/\partial\beta_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta_2+\beta_3\exp/50} \\ -\frac{\beta_1}{(\beta_2+\beta_3\exp/50)^2} \\ -\frac{\beta_1\exp}{(\beta_2+\beta_3\exp/50)^2 50} \\ 0 \end{pmatrix}$$

It follows that

$$V_\theta = R' V_{\widehat{\boldsymbol{\beta}}} R$$

and

$$s\left(\widehat{\theta}\right) = \sqrt{\widehat{R}' \widehat{V}_{\widehat{\boldsymbol{\beta}}} \widehat{R}}$$

(d)

The statistic $\frac{\widehat{\theta}-\theta}{s(\widehat{\theta})}$ follows asymptotically normal distribution. The 90% confidence interval is

$$\widehat{C} = \left[\widehat{\theta} - 1.645 \times s(\widehat{\theta})\ ,\ \widehat{\theta} + 1.645 \times s(\widehat{\theta})\right].$$

(e)

Given edu = 12 and exp = 20, $\boldsymbol{x}'\widehat{\boldsymbol{\beta}} = \begin{pmatrix} 12 & 20 & 4 & 1 \end{pmatrix} \begin{pmatrix} 0.090 \\ 0.035 \\ -0.047 \\ 1.185 \end{pmatrix} = 2.777$

The 95% confidence interval for the regression function is

$$\left[\boldsymbol{x}'\widehat{\boldsymbol{\beta}} - 1.96 \times \sqrt{\boldsymbol{x}'\widehat{V}_{\widehat{\boldsymbol{\beta}}}\boldsymbol{x}}\ ,\ \boldsymbol{x}'\widehat{\boldsymbol{\beta}} + 1.96 \times \sqrt{\boldsymbol{x}'\widehat{V}_{\widehat{\boldsymbol{\beta}}}\boldsymbol{x}}\right]$$

The variance matrix $\widehat{V}_{\widehat{\boldsymbol{\beta}}}$ can be obtained from (a), and it can be calculated that

$$\boldsymbol{x}'\widehat{V}_{\widehat{\boldsymbol{\beta}}}\boldsymbol{x} = 1.363 \times 10^{-4}$$

and the confidence interval is

$$\left[2.777 \pm 1.96\sqrt{1.363 \times 10^{-4}}\right] = [2.754\ ,\ 2.8]$$

(f)

Using Equation 7.36, assume homoskedasticity, the 80% forecast interval is

$$\left[\boldsymbol{x}'\widehat{\boldsymbol{\beta}} \pm 1.28 \times \widehat{s}(\boldsymbol{x})\right]$$

where $\widehat{s}(\boldsymbol{x}) = \sqrt{\widehat{\sigma} + \boldsymbol{x}'\widehat{V}_{\widehat{\boldsymbol{\beta}}}\boldsymbol{x}}$ and $\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\widehat{e}_i^2$

It can be obtained that $\boldsymbol{x}'\widehat{\boldsymbol{\beta}} = 2.798$, $\widehat{\sigma}^2 = 0.329$, $\boldsymbol{x}'\widehat{V}_{\widehat{\boldsymbol{\beta}}}\boldsymbol{x} = 0.0004$, and therefore $\widehat{s}(\boldsymbol{x}) = 0.574$.

So the forecast interval for the log-wage is

$$[2.063, 3.533]$$

Apply the exponential function, the forecast interval for wage is

$$[7.872, 34.214]$$

□