Weak Convergence

TONG ZHOU

JHU

1. CONVERGENCE IN DISTRIBUTION

Any random variable $X : \Omega \to \mathbb{R}$ can induce a measure $\mathbb{P}_X$ on $\mathbb{R}$. Since $\mathbb{P}_X$ is defined on every Borel set in $\mathbb{R}$, when considering a specific Borel set $(-\infty, \alpha]$, $F_X(\alpha) := \mathbb{P}_X((-\infty, \alpha])$ is defined to be the distribution function of X. The distribution function of any random variable is instrumental to exploring many important properties of a random variable, such that convergence in distribution. However, when we study more general random variables, for example a random vector, the concept of distribution function is not able to be defined. This limitation motivates a new notion called weak convergence.

DEFINITION 1. *We say that R.V.-s X_n converge in distribution to a R.V. X_∞, denoted by $X_n \xrightarrow{d} X_\infty$, if $F_{X_n}(\alpha) \to F_{X_\infty}(\alpha)$ as $n \to \infty$ for each fixed α which is a continuity point of F_{X_∞}.*

Similarly, we say that distribution functions F_n converge weakly to F_∞, denoted by $F_n \xrightarrow{w} F_\infty$, if $F_n(\alpha) \to F_\infty(\alpha)$ as $n \to \infty$ for each fixed α which is a continuity point of F_∞.

Conv. in distribution $\Longleftrightarrow$ Point-wise Conv. of corresponding CFs

LEMMA 1. *If the limit R.V. X_∞ has a probability density function, or more general whenever F_{X_∞} is a continuous function, then*

$$X_n \xrightarrow{d} X_\infty \Longleftrightarrow F_n(\alpha) \to F_\infty(\alpha), \forall \alpha \in \mathbb{R}.$$

EXERCISE 1 (1.2.50). *The support of a distribution function F is the set*

$$S_F := \{x \in \mathbb{R} \text{ such that } F(x+\varepsilon) - F(x-\varepsilon) > 0 \text{ for all } \varepsilon > 0\}.$$

(a) Show that all points of discontinuity of $F(\cdot)$ belong to S_F, and that any isolated point of S_F (i.e., $x \in S_F$ such that $(x-\delta, x+\delta) \cap S_F = \{x\}$ for some $\delta > 0$) must be a point of discontinuity of $F(\cdot)$.

(b) Show that the support of the law $\mathbb{P}_X$ of a random variable X, as defined in Exercise 1.2.48, is the same as the support of its distribution function F_X.

PROOF.

(a) Since F is monotone increasing and right-continuous, if b is a discontinuity point of F, it must be the case that $F(b+\varepsilon) \geqslant F(b) > F(b-\varepsilon)$ for all $\varepsilon > 0$. It implies that $b \in S_F$.

If a is any isolated point of S_F, then $(a-\delta, a+\delta) \cap S_F = \{a\}$ for some $\delta > 0$. To show a is a discontinuity point of $F(\cdot)$, we need to show $F(a) > F(a-)$.
Pick a small $\varepsilon < \delta$, then $(a-\varepsilon, a+\varepsilon) \cap S_F = \{a\}$. We first shall show that $F(a) = F(x)$ for any $x \in (a, a+\varepsilon)$.
Suppose on the contrary that $\exists b \in (a, a+\varepsilon)$ such that $F(a) < F(b)$. Note that since $F(\cdot)$ is monotone increasing and right-continuous, F must be continuous on (a, b). Otherwise its discontinuity point must be in S_F. Hence by the *Intermediate Value Theorem*, $\exists c \in (a, b)$ such that $F(c) \in (F(a), F(b))$. Define

$$\ell := \sup_d \{d : F(c-d) = F(c) = F(c+d)\}.$$

Then we have the following observations:

- If $\ell = 0$, it implies $c \in S_F$. contradiction !
- If $\ell > 0$, then $\ell < \min\{c-a, b-c\}$. Otherwise $F(c) = F(a)$ or $F(c) = F(b)$. Contradiction with $F(c) \in (F(a), F(b))$.
- Fix $\ell < \min\{c-a, b-c\}$. Then $a < c-\ell < c < c+\ell < b$ and $F(c-\ell) = F(c) = F(c+\ell) \in (F(a), F(b))$. Moreover, $c-\ell, c+\ell \in S_F$. This is true because for any $x \in (a, c-\ell)$ and $y \in (c+\ell, b)$, there must be the case that

$$F(x) < F(c-\ell) \quad \text{and} \quad F(c+\ell) < F(y),$$

  which implies that $c-\ell, c+\ell \in S_F$. Contradiction with $(a-\varepsilon, a+\varepsilon) \cap S_F = \{a\}$.

Therefore, all the contradictions indicate that the assumption $F(a) < F(b)$ is untrue. Hence, there must be that $F(a) = F(b)$ for any $b \in (a, a+\varepsilon)$.

The analogous argument can be applied to the interval $(a-\varepsilon, a)$ if F is left-continuous and therefore $F(\cdot)$ must be constant on $(a-\varepsilon, a+\varepsilon)$. However, this would imply $a \notin S_F$. This contradiction is owing to the fact that we mistakenly assume $F(\cdot)$ is left-continuous.

□

EXERCISE 2. *Show that if* $F_n \xrightarrow{w} F_\infty$ *and* F_∞ *is a continuous function then also* $\sup_x |F_n(x) - F_\infty(x)| \longrightarrow 0$.

PROOF.
Suppose $\varepsilon > 0$ is small and $k = \frac{2}{\varepsilon}$ is an integer. Then the interval $[0,1]$ is equal-divided into k sub-intervals $[0, \frac{\varepsilon}{2}), \cdots, (1-\frac{\varepsilon}{2}, 1]$. For each knot except for the leftmost and rightmost ends 0 and 1: $\frac{\varepsilon}{2}, \cdots, 1-\frac{\varepsilon}{2}$, in total there are $\frac{2}{\varepsilon} - 1 (= k-1)$ knots, denoted by $t_1, \cdots, t_{k-1}$ with $t_1 < t_2 < \cdots < t_{k-1}$ and the length of any adjacent knots is $\left|t_j - t_{j-1}\right| = \varepsilon/2$.
Since $F_\infty(\cdot)$ is increasing and continuous on $\mathbb{R}$, for each knot t_j, $\exists x_j \in \mathbb{R}$ such that $F_\infty(x_j) = t_j$, for $j = 1, \cdots, k-1$ with the monotonicity relation of F_∞ implying that $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{k-1}$.

More specifically, the equal-division $\left(\left|t_j - t_{j-1}\right| = \varepsilon/2\right)$ implies $F_\infty(x_{j+1}) = F_\infty(x_j) + \varepsilon/2$ for each j. Fixing each x_j, by the point-wise convergence of $F_n(x_j) \to F_\infty(x_j)$ for $n \to \infty$, there exists $N_j \in \mathbb{N}$, such that $\left|F_n(x_j) - F_\infty(x_j)\right| < \varepsilon/2$ whenever $n > N_j$, and let

$$N := \max_{1 \leqslant j \leqslant k-1} N_j < \infty.$$

By construction, we have $x_1, \cdots, x_{k-1}$ partitioning $\mathbb{R} = (-\infty, x_1] \cup \bigcup_{j=2}^{k-2} (x_j, x_{j+1}] \cup (x_{k-1}, \infty)$. To ease subsequent derivation, let $x_0 = -\infty$ and $x_k = \infty$, then $\mathbb{R} = \bigcup_{j=0}^{k-1} (x_j, x_{j+1})$. Also, $F_\infty(x_0) = t_0 = 0$ and $F_\infty(x_k) = t_k = 1$.

Pick any $x \in \mathbb{R}$, there must be case that $x \in (x_j, x_{j+1}]$ for some $j = 0, \cdots, k-1$ and hence $F_\infty(x) \in (t_j, t_{j+1}]$ by monotonicity. For any $n > N$, it follows that

$$\begin{aligned} F_n(x) &\leqslant F_n(x_{j+1}) \\ &< F_\infty(x_{j+1}) + \frac{\varepsilon}{2} \\ &= F_\infty(x_j) + \varepsilon \\ &< F_\infty(x) + \varepsilon \end{aligned}$$

So we obtain an one-side inequality:

$$F_n(x) < F_\infty(x) + \varepsilon, \quad \forall n > N \tag{1}$$

Similarly, we have the other side

$$\begin{aligned} F_n(x) &\geqslant F_n(x_j) \\ &> F_\infty(x_j) - \frac{\varepsilon}{2} \\ &= F_\infty(x_j) - \frac{\varepsilon}{2} + F_\infty(x) - F_\infty(x) \quad \left[F_\infty(x_j) - F_\infty(x) > -\frac{\varepsilon}{2}\right]. \\ &> F_\infty(x) - \varepsilon \end{aligned}$$

So we have another one-side inequality :

$$F_n(x) > F_\infty(x) - \varepsilon, \quad \forall n > N. \tag{2}$$

Combine *Equations* (1)*and* (2), it is concluded that for any such $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that

$$|F_n(x) - F_\infty(x)| < \varepsilon, \forall n > N \quad \text{for any } x \in \mathbb{R}$$

Thus, $F_n \to F_\infty$ uniformly on $\mathbb{R}$. Or equivalently,

$$\sup_{x \in \mathbb{R}} |F_n(x) - F_\infty(x)| \to 0, \text{ as } n \to \infty.$$

Note that we only consider the case where $2/\varepsilon$ is an integer. If it is not, the argument is similar, but we should be careful about dividing the interval $[0, 1]$. ◻

conv. in probability $\Longrightarrow$ conv. in distribution

LEMMA 2. *If $X_n \xrightarrow{P} X_\infty$, then $X_n \xrightarrow{d} X_\infty$. Conversely, if $X_n \xrightarrow{d} X_\infty$ and X_∞ is a.s. a non-random constant, then $X_n \xrightarrow{P} X_\infty$.*

PROOF.
Suppose R.V.-s X_n and its limit X_∞ are all in the same probability space. Given $\varepsilon > 0$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned}\mathbb{P}(X_n \leqslant \alpha) &= \mathbb{P}(X_n \leqslant \alpha, |X_n - X_\infty| \leqslant \varepsilon) + \mathbb{P}(X_n \leqslant \alpha, |X_n - X_\infty| > \varepsilon) \\ &\leqslant \mathbb{P}(X_\infty < \alpha + \varepsilon) + \mathbb{P}(|X_n - X_\infty| > \varepsilon).\end{aligned}$$

Thus we got the first ingredient:

$$\mathbb{P}(X_n \leqslant \alpha) \leqslant \mathbb{P}(X_\infty < \alpha + \varepsilon) + P(|X_n - X_\infty| > \varepsilon).$$

Moreover, taking limits on both sides yields:

$$\limsup_{n\to\infty} \mathbb{P}(X_n \leqslant \alpha) \leqslant \mathbb{P}(X_\infty \leqslant \alpha + \varepsilon) \tag{3}$$

Similar argument applies for the set $\{X < \alpha - \varepsilon\}$:

$$\begin{aligned}\mathbb{P}(X_\infty < \alpha - \varepsilon) &= \mathbb{P}(X_\infty < \alpha - \varepsilon, |X_\infty - X_n| \leqslant \varepsilon) + \mathbb{P}(X_\infty < \alpha - \varepsilon, |X_\infty - X_n| \geqslant \varepsilon) \\ &\leqslant \mathbb{P}(X_n \leqslant \alpha) + \mathbb{P}(|X_n - X_\infty| > \varepsilon).\end{aligned}$$

So we got the second ingredient:

$$\mathbb{P}(X_n \leqslant \alpha) \geqslant \mathbb{P}(X_\infty < \alpha - \varepsilon) - \mathbb{P}(|X_n - X_\infty| > \varepsilon).$$

Taking limits gives:

$$\liminf_{n\to\infty} \mathbb{P}(X_n \leqslant \alpha) \geqslant \mathbb{P}(X_\infty \leqslant \alpha - \varepsilon). \tag{4}$$

Combining Equations (3) and (4), and letting $\varepsilon \searrow 0$, we have

$$F_{X_\infty}(\alpha) \leqslant \liminf_{n\to\infty} \mathbb{P}(X_n \leqslant \alpha) \leqslant \limsup_{n\to\infty} \mathbb{P}(X_n \leqslant \alpha) \leqslant F_{X_\infty}(\alpha),$$

which implies that

$$\lim_{n\to\infty} F_{X_n}(\alpha) = F_{X_\infty}(\alpha), \forall \alpha \text{ that is the continuity point of } F_{X_\infty}.$$

Now let's see when X_∞ is a constant and $X_n \xrightarrow{\mathsf{d}} X_\infty$. Given $\varepsilon > 0$, w.l.o.g, suppose $X_\infty \leqslant \varepsilon$ a.s., i.e. $\mathbb{P}(X_\infty \leqslant \varepsilon) = 1$.

First note that $X_n \xrightarrow{\mathsf{d}} X_\infty$ implies $\mathbb{P}(X_n \leqslant \varepsilon) \to 1$. To prove $X_n \xrightarrow{\mathsf{P}} X_\infty$, we shall show $\mathbb{P}(|X_n - X_\infty| \leqslant \varepsilon) \to 1$.

Observe that

$$\mathbb{P}(|X_n - X_\infty| \leqslant \varepsilon) = F_{X_n}(X_\infty + \varepsilon) - F_{X_n}(X_\infty - \varepsilon),$$

and

$$F_{X_n}(X_\infty + \varepsilon) \longrightarrow 1$$
$$F_{X_n}(X_\infty - \varepsilon) \longrightarrow 0.$$

So we have

$$\mathbb{P}(|X_n - X_\infty| \leqslant \varepsilon) \longrightarrow 1.$$

Thus $X_n \xrightarrow{\mathsf{P}} X_\infty$. □

Slustky's Lemma (Coverging Together Lemma)

EXERCISE 3. *Suppose that $X_n \xrightarrow{d} X_\infty$ and $Y_n \xrightarrow{d} Y_\infty$, where Y_∞ is non-random and for each n the variables X_n and Y_n are defined on the same probability space.*

(a) Show that then $X_n + Y_n \xrightarrow{d} X_\infty + Y_\infty$.
(b) Deduce that if $Z_n - X_n \xrightarrow{d} 0$ then $X_n \xrightarrow{d} X$ if and only if $Z_n \xrightarrow{d} X$.
(c) Show that $Y_n X_n \xrightarrow{d} X_\infty Y_\infty$.

PROOF.

(a) We need to show that for all α that is the continuity point of $F_{X_\infty}(\cdot)$, $\mathbb{P}(X_n + Y_n \leqslant \alpha) \to \mathbb{P}(X_\infty + Y_\infty \leqslant \alpha) = F_{X_\infty}(\alpha - Y_\infty)$.
Note that $\alpha + Y_\infty$ is not necessarily a continuity point of F_{X_∞}, so choose $\varepsilon > 0$ such that $\alpha - Y_\infty + \varepsilon$ and $\alpha - Y_\infty - \varepsilon$ are continuity points of F_{X_∞}. Then we have

$$\begin{aligned}\mathbb{P}(X_n + Y_n \leqslant \alpha) &= \mathbb{P}(X_n + Y_n \leqslant \alpha, |Y_n - Y_\infty| \leqslant \varepsilon) + \mathbb{P}(X_n + Y_n \leqslant \alpha, |Y_n - Y_\infty| > \varepsilon) \\ &\leqslant \mathbb{P}(X_n \leqslant \alpha - Y_\infty + \varepsilon) + \mathbb{P}(|Y_n - Y_\infty| > \varepsilon).\end{aligned}$$

By $X_n \xrightarrow{\mathsf{d}} X_\infty$, first term $\mathbb{P}(X_n \leqslant \alpha - Y_\infty + \varepsilon) \to \mathbb{P}(X_\infty \leqslant \alpha - Y_\infty + \varepsilon)$. By $Y_n \xrightarrow{\mathsf{P}} Y_\infty$, the second term $\mathbb{P}(|Y_n - Y_\infty| > \varepsilon) \to 0$. So we obtain an one-side inequality

$$\limsup_{n\to\infty} \mathbb{P}(X_n + Y_n \leqslant \alpha) \leqslant F_{X_\infty}(\alpha - Y_\infty + \varepsilon). \tag{5}$$

Using an analogous argument, we bound the other side by

$$\begin{aligned}\mathbb{P}(X_n \leqslant \alpha - Y_\infty - \varepsilon) &= \mathbb{P}(X_n \leqslant \alpha - Y_\infty - \varepsilon, |Y_n - Y_\infty| \leqslant \varepsilon) + \mathbb{P}(X_n \leqslant \alpha - Y_\infty - \varepsilon, |Y_n - Y_\infty| > \varepsilon) \\ &\leqslant \mathbb{P}(X_n + Y_n \leqslant \alpha) + \mathbb{P}(|Y_n - Y_\infty| > \varepsilon).\end{aligned}$$

Taking limits then yields

$$\liminf_{n\to\infty} \mathbb{P}(X_n + Y_n \leqslant \alpha) \geqslant F_{X_\infty}(\alpha - Y_\infty - \varepsilon). \tag{6}$$

Combining Equations (5) and (6):

$$F_{X_\infty}(\alpha - Y_\infty - \varepsilon) \leqslant \liminf_{n\to\infty} \mathbb{P}(X_n + Y_n \leqslant \alpha) \leqslant \limsup_{n\to\infty} \mathbb{P}(X_n + Y_n \leqslant \alpha) \leqslant F_{X_\infty}(\alpha - Y_\infty + \varepsilon).$$

Let $\varepsilon \searrow 0$ with $\alpha - Y_\infty - \varepsilon$ and $\alpha - Y_\infty + \varepsilon$ being continuity points of F_{X_∞}, then

$$\mathbb{P}(X_n + Y_n \leqslant \alpha) \longrightarrow F_{X_\infty}(\alpha - Y_\infty).$$

(b) Just use results in (a).
(c) Mimic the strategy in (a).

□

renewal theory (application of the Slutsky' theorem)

EXERCISE 4.

(a) Suppose $\{N_m\}$ are non-negative integer-valued random variables and $b_m \to \infty$ are non-random integers such that $N_m/b_m \xrightarrow{P} 1$. Show that if $S_n = \sum_{k=1}^n X_k$ for i.i.d. random variables $\{X_k\}$ with $v = \mathsf{Var}(X_1) \in (0,\infty)$ and $\mathbb{E}[X_1] = 0$, then $S_{N_m}/\sqrt{vb_m} \xrightarrow{d} G$ as $m \to \infty$. (Hint: Use Kolmogorov's inequality to show that $S_{N_m}/\sqrt{vb_m} - S_{b_m}/\sqrt{vb_m} \xrightarrow{P} 0$.)

PROOF.

(a) The CLT implies that

$$\frac{S_{b_m}}{\sqrt{vb_m}} \xrightarrow{\mathsf{d}} G.$$

If we know

$$\frac{S_{N_m}}{\sqrt{vb_m}} - \frac{S_{b_m}}{\sqrt{vb_m}} \xrightarrow{\mathsf{P}} 0, \tag{7}$$

then by the Slutsky Theorem, then $S_{N_m}/\sqrt{vb_m} \xrightarrow{\mathsf{d}} G$, as $m \to \infty$. So what we need to prove is Equation (7).

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