Optimal Control and Trajectory Estimation of a Nonlinear Quadrotor

Antoni Rosinol[1]

***Abstract*—We present an optimal control problem to estimate the rotor speed inputs and state sequences to drive a nonlinear model of a quadcopter through a race track in a minimum time. We show how to solve this problem by using GPOPS-II, a matlab-based general purpose optimal control software.**

***Index Terms*—Optimal Control, Drone Racing**

SUPPLEMENTARY MATERIAL

Videos of the experiments: **TODO:** Add video url
Open-source code: **TODO:** Add code url

I. INTRODUCTION

THE advent of drone racing competitions such as **TODO:** cite alpha pilot requires the estimation of the fastest possible trajectory through a given race track. Finding this optimal trajectory can be useful for human pilots, and their tacticians, to evaluate their performance and improve upon it. For example, the teams in the Red Bull Air Race **TODO:** cite redbull and SH make extensive use of such information to train and perfect their manouevers. In this same way, drones can autonomously follow a dynamically feasible trajectory with high accuracy. Therefore being able to infer the optimal feasible trajectory could potentially make fully-autonomous drones beat human pilots. We are still not there, but great progress in this direction is being made.

Contributions.

- Formulation of the minimum-time problem for the specific problem of drone-racing through multiple gates.
- Presentation of a means to solve the problem using readily available software.

Paper Structure. Section III presents the mathematical formulation of our approach, and discusses the implementation of Section IV reports and discusses the experimental results and comparison against related work. Section V concludes the paper.

II. RELATED WORK

A. TODO

B. Optimal Control

III. APPROACH

The minimum time optimal control problem can be generically formulated as follows:

Cost:

$$J = \int_{t_0}^{t_f} dt$$

Nonlinear Dynamics:

$$\dot{x}(t) = f(x(t), u(t), t)$$

Constraints:

- Input constraints:

$$M_i^- \leq u_i(t) \leq M_i^+$$

- Path constraints: $\forall t, t_0 \leq t \leq t_f$

$$c(x(t), u(t), t) \leq 0,$$

Boundary Conditions: Given

- Initial: $n(x(t_0), t_0) = 0$
- Final: $m(x(t_f), t_f) = 0$

A. Mathematical Model of a Quadcopter

Absolute position: x, y, z.
Attitude: pitch θ, roll ϕ, yaw ψ.

$$\boldsymbol{\xi} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \boldsymbol{\eta} = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

Linear velocities (body frame): $\boldsymbol{V}_B$
Angular velocities (body frame): $\boldsymbol{\nu}$

$$\boldsymbol{V}_B = \begin{bmatrix} v_{x,B} \\ v_{y,B} \\ v_{z,B} \end{bmatrix}, \quad \boldsymbol{\nu} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

The rotation matrix from the body frame (B) to the inertial frame (G) is given by: $\mathbf{x}^B = \mathbf{R}_G^B \mathbf{X}^G = \mathbf{R}(\phi)\mathbf{R}(\theta)\mathbf{R}(\psi)\mathbf{X}^G$

$$\boldsymbol{R}_B^G = \boldsymbol{R} = \begin{bmatrix} C_\psi C_\theta & C_\psi S_\theta S_\phi - S_\psi C_\phi & C_\psi S_\theta C_\phi + S_\psi S_\phi \\ S_\psi C_\theta & S_\psi S_\theta S_\phi + C_\psi C_\phi & S_\psi S_\theta C_\phi - C_\psi S_\phi \\ -S_\theta & C_\theta S_\phi & C_\theta C_\phi \end{bmatrix}$$

where $S_x = \sin(x)$ and $C_x = \cos(x)$.

Time derivative of this rotation matrix provides us with the angular velocities. These are not simply the time derivatives of the independent Euler angles, instead:

$$\dot{\boldsymbol{\eta}} = \boldsymbol{W}_\eta^{-1}\boldsymbol{\nu}, \quad \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & S_\phi T_\theta & C_\phi T_\theta \\ 0 & C_\phi & -S_\phi \\ 0 & S_\phi / C_\theta & C_\phi / C_\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Conversely,

$$\boldsymbol{\nu} = \boldsymbol{W}_\eta \dot{\boldsymbol{\eta}}, \quad \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -S_\theta \\ 0 & C_\phi & C_\theta S_\phi \\ 0 & -S_\phi & C_\theta C_\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

[1]A. Rosinol is with the Laboratory for Information & Decision Systems (LIDS), Massachusetts Institute of Technology, Cambridge, MA, USA, arosinol@mit.edu

Symbol	Definition
$\boldsymbol{\xi}$	Absolute position (inertial frame)
$\boldsymbol{\eta}$	Attitude (inertial frame)
$\boldsymbol{V}_B$	Linear velocities (body frame)
$\boldsymbol{\nu}$	Angular velocities (body frame)
$\mathbf{R}$	Rotation matrix from body to inertial frame
$\mathbf{W}_{\boldsymbol{\eta}}$	Transformation matrix for angular velocities from inertial to body frame
$\boldsymbol{I}$	Inertia matrix
$\boldsymbol{G}$	Gravity

TABLE I: Definitions and Notations

where $T_x = \tan(x)$. The matrix $\boldsymbol{W}_\eta$ is invertible if $\theta \neq (2k-1)\phi/2, (k \in \mathbb{Z})$

Assumptions:

- The quadrotor structure is rigid and symmetrical with a center of mass aligned with the center of the body frame of the vehicle. The four arms of the quadcopter are aligned with the body x- and y-axes. Therefore, the inertia matrix $\mathbf{I}$ is diagonal:

$$\boldsymbol{I} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

  We further have that due to symmetry inertial components are equal: $I_{xx} = I_{yy}$.
- The thrust and drag of each motor is proportional to the square of the motor velocity.
- The propellers are considered to be rigid and therefore blade flapping is negligible (deformation of propeller blades due to high velocities and flexible material.
- The Earth is flat and non-rotating (difference of gravity by altitude or the spin of the earth is negligible)
- Surrounding fluid velocities (wind) are negligible.
- Ground effect is negligible.

These assumptions and basic dynamics lead to the model used in this work. The following derivations are a summary of the equations in [].

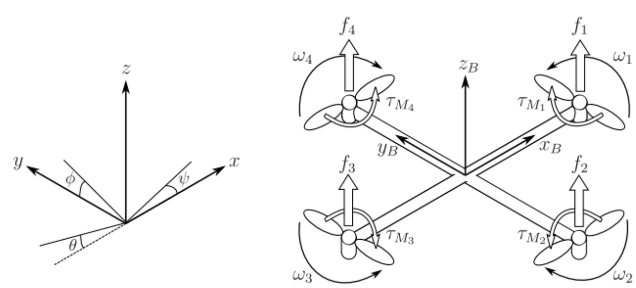

Fig. 1: The inertial and body frames of a quadcopter (ENU coordinates). Figure from [**?**].

B. Quadcopter Dynamics

- The angular velocity of rotor i, denoted with ω_i, creates force f_i in the direction of the rotor axis:

$$f_i = k\omega_i^2$$

  The combined forces of rotors create thrust T in the direction of the body z-axis.

$$T = \sum_{i=1}^{4} f_i = k \sum_{i=1}^{4} \omega_i^2, \qquad \boldsymbol{T}_B = \begin{bmatrix} 0 \\ 0 \\ T \end{bmatrix}$$

- The angular velocity and acceleration of the rotor also creates torque τ_{M_i} around the rotor axis:

$$I_M \dot{\omega}_i = \tau_{M_i} - b\omega_i^2$$

  in which the lift constant is k, the drag constant is b and the inertia moment of the rotor is I_M.
  Usually the acceleration of the rotor ($\dot{\omega}_i$) is considered small and thus it is omitted. This holds when we assume that the quadrotor is operating in stable flight and that the propellers are maintaining a constant thrust and not accelerating. This assumption results in the torque about the global z axis being equal to the torque due to drag.
  Torque $\boldsymbol{\tau_B}$ consists of the torques on the three axis $\tau_\phi, \tau_\theta, \tau_\psi$:

$$\boldsymbol{\tau}_B = \begin{bmatrix} \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} = \begin{bmatrix} lk\left(-\omega_2^2 + \omega_4^2\right) \\ lk\left(-\omega_1^2 + \omega_3^2\right) \\ \sum_{i=1}^{4}(-1)^{i+1}\tau_{M_i} \end{bmatrix}$$

$$= \begin{bmatrix} lk\left(-\omega_2^2 + \omega_4^2\right) \\ lk\left(-\omega_1^2 + \omega_3^2\right) \\ b(\omega_1^2 - \omega_2^2 + \omega_3^2 - \omega_4^2) \end{bmatrix}$$

  where l is the distance between the rotor and the center of mass of the quadcopter.

1) Translational dynamics: Newton's second law:

$$m\ddot{\boldsymbol{\xi}} = \boldsymbol{G} + \boldsymbol{R}\boldsymbol{T}_B$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = -g \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{T}{m} \begin{bmatrix} C_\psi S_\theta C_\phi + S_\psi S_\phi \\ S_\psi S_\theta C_\phi - C_\psi S_\phi \\ C_\theta C_\phi \end{bmatrix}$$

2) Rotational dynamics: The rotational equations of motion are defined in the body frame so that the rotations can be computed about the quadrotor's center and not the center of the global coordinate frame.

Applying Euler's second law:

$$\boldsymbol{I}\dot{\nu} + \boldsymbol{\nu} \times (\boldsymbol{I}\boldsymbol{\nu}) + \boldsymbol{\Gamma} = \boldsymbol{\tau}$$

where:

- Angular acceleration of the inertia: $\boldsymbol{I}\dot{\boldsymbol{v}}$
- Centripetal forces: $\boldsymbol{\nu} \times (\boldsymbol{I}\boldsymbol{\nu})$
- Gyroscopic forces: $\boldsymbol{\Gamma}$
- External torque: $\boldsymbol{\tau}_B$

Replacing terms by their definitions, multiplying both sides by $\boldsymbol{I}^{-1}$, and rearranging:

$$\dot{\boldsymbol{\nu}} = \boldsymbol{I}^{-1}\left(- \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} I_{xx}p \\ I_{yy}q \\ I_{zz}r \end{bmatrix} - I_r \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega_\Gamma + \boldsymbol{\tau}_B \right)$$

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} (I_{yy} - I_{zz})\, qr/I_{xx} \\ (I_{zz} - I_{xx})\, pr/I_{yy} \\ (I_{xx} - I_{yy}) pq/I_{zz} \end{bmatrix} - I_r \begin{bmatrix} q/I_{xx} \\ -p/I_{yy} \\ 0 \end{bmatrix} \omega_\Gamma + \begin{bmatrix} \tau_\phi/I_{xx} \\ \tau_\theta/I_{yy} \\ \tau_\psi/I_{zz} \end{bmatrix}$$

where $\omega_\Gamma = \omega_1 - \omega_2 + \omega_3 - \omega_4$, and I_r is the rotor moment of inertia.

The angular accelerations in world frame $\ddot{\boldsymbol{\eta}}$ are given by the time derivatives of the angular velocities $\left(\dot{\boldsymbol{\eta}} = \boldsymbol{W}_\eta^{-1}\boldsymbol{\nu}\right)$,

$$\ddot{\boldsymbol{\eta}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{W}_\eta^{-1}\boldsymbol{\nu}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{W}_\eta^{-1}\right)\boldsymbol{\nu} + \boldsymbol{W}_\eta^{-1}\dot{\boldsymbol{\nu}} =$$

$$\begin{bmatrix} 0 & \dot{\phi}C_\phi T_\theta + \dot{\theta}S_\phi/C_\theta^2 & -\dot{\phi}S_\phi C_\theta + \dot{\theta}C_\phi/C_\theta^2 \\ 0 & -\dot{\phi}S_\phi & -\dot{\phi}C_\phi \\ 0 & \dot{\phi}C_\phi/C_\theta + \dot{\phi}S_\phi T_\theta/C_\theta & -\dot{\phi}S_\phi/C_\theta + \dot{\theta}C_\phi T_\theta/C_\theta \end{bmatrix}\boldsymbol{\nu} + \boldsymbol{W_\eta^{-1}}\dot{\boldsymbol{\nu}}.$$

At this point, it is common to do a simplification by setting $[\dot{\phi} \quad \dot{\theta} \quad \dot{\psi}]^T = \begin{bmatrix} p & q & r \end{bmatrix}^T$, which holds true for small angles of movement [?].

$$\ddot{x} = \frac{(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi)u_1 - K_{fx}\dot{x}}{m}$$
$$\ddot{y} = \frac{(\cos\phi\sin\theta\cos\psi - \sin\phi\cos\psi)u_1 - K_{fy}\dot{x}}{m}$$
$$\ddot{z} = \frac{(\cos\phi\cos\theta)u_1 - K_{fz}\dot{z}}{m} - g$$
$$\ddot{\phi} = \frac{l\left(u_2 - K_l\dot{\phi}\right)}{I_x}$$
$$\ddot{\theta} = \frac{l\left(u_3 - K_l\dot{\theta}\right)}{I_y}$$
$$\ddot{\psi} = \frac{\left(u_4 - K_d\dot{\psi}\right)}{I_z}$$

C. Aerodynamic Drag

- We include the drag force generated by the air resistance. For this, we add a diagonal coefficient matrix that associates the linear velocities to the force slowing the movement

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = -g\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{T}{m}\begin{bmatrix} C_\psi S_\theta C_\phi + S_\psi S_\phi \\ S_\psi S_\theta C_\phi - C_\psi S_\phi \\ C_\theta C_\phi \end{bmatrix} - \frac{1}{m}\begin{bmatrix} A_x & 0 & 0 \\ 0 & A_y & 0 \\ 0 & 0 & A_z \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

  in which A_x, A_y and A_z are the drag force coefficients for velocities in the corresponding directions of the inertial frame.
  Several other aerodynamical effects could be included in the model. For example, dependence of thrust on angle of attack, blade flapping and airflow disruptions.
- We ignore rotational drag forces since we assume rotational velocities to be relatively low. Alternatively, we could add the components $\boldsymbol{\tau_w} = \begin{bmatrix} \tau_{wx} & \tau_{wy} & \tau_{wz} \end{bmatrix}^T$ to the overall torque.

D. State-space Model

We can write our nonlinear dynamics using the following state vector:
$\boldsymbol{X}^T = \begin{bmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} & \phi & \theta & \psi & p & q & r \end{bmatrix}^T$
Moreover, we define our inputs as follow:

- U_1: the resulting thrust of the four rotors.
- U_2: the difference of thrust between the motors on the x axis which results in roll angle changes and subsequent movement in the lateral x direction.
- U_3: the difference of thrust between the motors on the y axis which results in pitch angle changes and subsequent movement in the lateraly direction.
- U_4: the difference of torque between the clockwise and counterclockwise rotors which results in a moment that rotates the quadrotor around the vertical z axis.

$$U = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} T \\ \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} = \begin{bmatrix} k\sum_{i=1}^4 \omega_i^2 \\ lk\left(-\omega_2^2 + \omega_4^2\right) \\ lk\left(-\omega_1^2 + \omega_3^2\right) \\ b(\omega_1^2 - \omega_2^2 + \omega_3^2 - \omega_4^2) \end{bmatrix}$$

Our nonlinear dynamics can be written as a nonlinear differential equation of the states $\boldsymbol{X}$ and control inputs $\boldsymbol{U}$: $\dot{\boldsymbol{X}} = f(\boldsymbol{X}, \boldsymbol{U})$.

Or, more precisely:

$$\dot{\boldsymbol{X}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ -\frac{T}{m}[S_\phi S_\psi + C_\phi C_\psi S_\theta] \\ -\frac{T}{m}[C_\phi S_\psi S_\theta - C_\psi S_\phi] \\ g - \frac{T}{m}[C_\phi C_\theta] \\ p + r[C_\phi T_\theta] + q[S_\phi T_\theta] \\ q[C_\phi] - r[S_\phi] \\ r\frac{C_\phi}{C_\theta} + q\frac{S_\phi}{C_\theta} \\ \frac{I_y - I_z}{I_x}rq - \frac{I_r}{I_x}qw_\Gamma + \frac{\tau_x}{I_x} \\ \frac{I_z - I_x}{I_y}pr + \frac{I_r}{I_y}pw_\Gamma + \frac{\tau_y}{I_y} \\ \frac{I_x - I_y}{I_z}pq + \frac{\tau_z}{I_z} \end{bmatrix}$$

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E. Simplifications

The primary assumption made to simplify the model is that the quadrotor will be operated around a stable hover with small attitude angles and minimal rotational and translational velocities and accelerations. Since there are no aerodynamic lifting surfaces, we will assume that the aerodynamic forces and moments are negligible. The effects that are considered negligible are treated as disturbances in the control system and can be compensated for with appropriate control system design. These assumptions are described mathematically as well as the resulting simplified equations of motion. The equations below continue to reduce in complexity as more assumptions are made.

$$\begin{cases} f_t = b\left(\Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2\right) \\ \tau_x = bl\left(\Omega_3^2 - \Omega_1^2\right) \\ \tau_y = bl\left(\Omega_4^2 - \Omega_2^2\right) \\ \tau_z = d\left(\Omega_2^2 + \Omega_4^2 - \Omega_1^2 - \Omega_3^2\right) \end{cases}$$

Define control inputs as:

$$U = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} T_1 + T_2 + T_3 + T_4 \\ T_4 - T_2 \\ T_3 - T_1 \\ (T_1 + T_3) - (T_2 + T_4) \end{bmatrix}$$

where $T_i = b\omega_i^2$ is the thrust force of propeller i.

Using the following constants for simplifying the equations:

$$a_1 = \frac{I_{yy} - I_{zz}}{I_{xx}}, \quad a_2 = \frac{I_{zz} - I_{xx}}{I_{yy}}, \quad a_3 = \frac{I_{xx} - I_{yy}}{I_{zz}}$$

$$b_1 = \frac{l}{I_{xx}}, \quad b_2 = \frac{l}{I_{yy}}, \quad b_3 = \frac{l}{I_{zz}}$$

We get:

$$\begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} \frac{I_{yy}-I_{zz}}{I_{xx}}\dot{\theta}\dot{\psi} + \frac{\tau_x}{I_{xx}} \\ \frac{I_{zz}-I_{xx}}{I_{yy}}\dot{\phi}\dot{\psi} + \frac{\tau_y}{I_{yy}} \\ \frac{I_{xx}-I_{yy}}{I_{zz}}\dot{\phi}\dot{\theta} + \frac{\tau_z}{I_{zz}} \end{bmatrix} = \begin{bmatrix} a_1\dot{\theta}\dot{\psi} + b_1 u_2 \\ a_2\dot{\phi}\dot{\psi} + b_2 u_3 \\ a_3\dot{\phi}\dot{\theta} + b_3 u_4 \end{bmatrix}$$

Using these together with the absolute acceleration above:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = -g\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{T}{m}\begin{bmatrix} C_\psi S_\theta C_\phi + S_\psi S_\phi \\ S_\psi S_\theta C_\phi - C_\psi S_\phi \\ C_\theta C_\phi \end{bmatrix}$$

1) Calculus of Variations:

2) 3D Mesh Propagation:

IV. Experimental Results

V. Conclusion

VI. Supplementary Material

Euler angle rotation matrices:

$$\mathbf{R}(\psi) = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{bmatrix}$$