Построение сплайна через моменты

Введём обозначения:

$$S''(x_i) = M_i, i = \overline{0, n}$$

$$S(x) = y_i + c_i(x - x_i) + a_i\frac{(x - x_i)^2}{2} + b_i\frac{(x - x_i)^3}{6}, i = \overline{0, n-1}$$

$$S'(x) = c_i + a_i(x - x_i) + b_i\frac{(x - x_i)^2}{2}, i = \overline{0, n-1}$$

$$S''(x) = a_i + b_i(x - x_i), i = \overline{0, n-1}$$

$$S''(x_i) = a_i = M_i,\ S''(x_{i+1}) = a_i + b_ih_{i+1} = M_{i+1}$$

$$\boxed{b_i = \frac{M_{i+1} - M_i}{h_{i+1}}}$$

$$S(x_{i+1}) = y_i + c_ih_{i+1} + a_i\frac{h_{i+1}^2}{2} + b_i\frac{h_{i+1}^3}{6} = y_{i+1}$$

$$c_i = \frac{y_{i+1} - y_i}{h_{i+1}} - M_i\frac{h_{i+1}^2}{2h_{i+1}} - \frac{(M_{i+1} - M_i)h_{i+1}^3}{h_{i+1}6h_{i+1}}$$

$$\boxed{c_i = \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{h_{i+1}}{6}(M_{i+1} + 2M_i)}$$

$$S'(x) = c_i + a_i(x - x_i) + b_i\frac{(x - x_i)^2}{2}, x \in [x_i; x_{i+1}]$$

$$S'(x) = c_{i-1} + a_{i-1}(x - x_{i-1}) + b_{i-1}\frac{(x - x_{i-1})^2}{2}, x \in [x_{i-1}; x_i]$$

$$S'(x_i - 0) = S'(x_i + 0)$$

$$c_{i-1} + a_{i-1}h_i + b_{i-1}\frac{h_i^2}{2} = c_i$$

$$\frac{y_i - y_{i-1}}{h_i} - \frac{h_i}{6}(M_i + 2M_{i-1}) + M_{i-1}h_i + \frac{M_i - M_{i-1}}{h_i}\frac{h_i^2}{2} = \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{h_{i+1}}{6}(M_{i+1} + 2M_i)$$

$$\frac{h_{i+1}}{6}(M_{i+1} + 2M_i) - \frac{h_i}{6}(M_i + 2M_{i-1}) + M_{i-1}h_i + \frac{h_i}{2}(M_i - M_{i-1}) = \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}\ \Bigg|\ \times 6$$

$$h_i M_{i-1}+2M_i(h_{i+1}+h_i)+h_{i+1}M_{i+1}=6\left(\frac{y_{i+1}-y_i}{h_{i+1}}-\frac{y_i-y_{i-1}}{h_i}\right)\Bigg|\times\frac{1}{h_i+h_{i+1}}$$

$$\mu_i=\frac{h_{i+1}}{h_i+h_{i+1}};\lambda_i=\frac{h_i}{h_i+h_{i+1}};\lambda_i+\mu_i=1$$

$$\boxed{\lambda_i M_{i-1}+2M_i+\mu_i M_{i+1}=\underbrace{6\left(\mu_i\frac{y_{i+1}-y_i}{h_{i+1}^2}-\lambda_i\frac{y_i-y_{i-1}}{h_i^2}\right)}_{g_i}}$$

$i=\overline{1,n-1}; M_0,M_1,\ldots,M_{n-1},M_n$

Краевые условия:

$1)S'(a)=f'(a),S'(b)=f'(b);$

$$S'(a)=c_0+a_0(a-x_0)+b_0\frac{(a-x_0)^2}{2}=c_0=\frac{y_1-y_0}{h_1}-\frac{h_1}{6}(M_1+2M_0)=f'(a)$$

$$2M_0+M_1=\underbrace{6\frac{y_1-y_0}{h_1}-\frac{6}{h_1}f'(a)}_{g_0}$$

$$S'(b)=c_{n-1}+a_{n-1}(b-x_{n-1})+b_{n-1}\frac{(b-x_{n-1})^2}{2}=c_{n-1}+a_{n-1}h_n+b_{n-1}\frac{h_n^2}{2}=$$

$$\frac{y_n-y_{n-1}}{h_n}-\frac{h_n}{6}(M_n+2M_{n-1})+h_nM_{n-1}+\frac{M_n-M_{n-1}}{h_n}\frac{h_n^2}{2}=f'(b)$$

$$M_{n-1}+2M_n=\underbrace{\frac{6}{h_n}f'(b)-6\frac{y_n-y_{n-1}}{h_n^2}}_{g_n}$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \lambda_1 & 2 & \mu_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda_2 & 2 & \mu_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_{n-1} & 2 & \mu_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{pmatrix}=\begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_{n-1} \\ g_n \end{pmatrix}$$

$3)S''(a)=f''(a),S''(b)=f''(b);$

$M_0=f''(a),M_n=f''(b)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \lambda_1 & 2 & \mu_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda_2 & 2 & \mu_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_{n-1} & 2 & \mu_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{pmatrix} = \begin{pmatrix} f''(a) \\ g_1 \\ g_2 \\ \vdots \\ g_{n-1} \\ f''(b) \end{pmatrix}$$

$5) f(a) = f(b), f(x_{n+1}) = f(x_1), \ldots$

$M_0 = M_n, M_{n+1} = M_1, \ldots$

$h_{n+1} = h_1, h_{n+2} = h_2, \ldots$

$\mu_{n+1} = \mu_1, \ldots$

$i = 1 :$

$\lambda_1 M_0 + 2M_1 + M_2 = g_1$

$2M_1 + \mu_1 M_2 + \lambda_1 M_n = g_1$

$i = n :$

$\lambda_n M_{n-1} + 2M_n + \mu_n M_{n+1} = g_n$

$\mu_n M_1 + \lambda_n M_{n-1} + 2M_n = g_n$

$$\begin{pmatrix} 2 & \mu_1 & 0 & 0 & \cdots & 0 & 0 & \lambda_1 \\ \lambda_2 & 2 & \mu_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda_3 & 2 & \mu_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_{n-1} & 2 & \mu_{n-1} \\ \mu_n & 0 & 0 & 0 & \cdots & 0 & \lambda_n & 2 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{n-1} \\ M_n \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_{n-1} \\ g_n \end{pmatrix}$$

$6) S'''(x_1 + 0) = S'''(x_1 - 0), S'''(x_{n-1} + 0) = S'''(x_{n-1} - 0);$

$S'''(x) = b_i, x \in [x_i; x_{i+1}]$

$b_0 = b_1$

$$\frac{M_1 - M_0}{h_1} = \frac{M_2 - M_1}{h_2} \,\Bigg|\, \times h_1$$

$$\gamma_1 = \frac{h_1}{h_2}$$

$$M_0 - M_1(1+\gamma_1) + \gamma_1 M_2 = 0$$

$$i = 1:\ \lambda_1 M_0 + 2M_1 + \mu_1 M_2 = 6\left(\mu_1 \frac{y_2 - y_1}{h_2^2} - \lambda_1 \frac{y_1 - y_0}{h_1^2}\right)$$

$$M_0 - M_1(1+\gamma_1) + \gamma_1 M_2 = 0 \,|\, \times \lambda_1$$

$$M_2(\mu_1\gamma_1^2 - \mu_1) - M_1(\lambda_1 + \lambda_1\gamma_1 + 2) = -6\left(\mu_1 \frac{y_2 - y_1}{h_2^2} - \lambda_1 \frac{y_1 - y_0}{h_1^2}\right)$$

$$-\lambda_1 - \lambda_1\gamma_1 - 2 = -\lambda_1(1+\gamma_1) - 2 = -\frac{h_1}{h_2 + h_1}\frac{h_2 + h_1}{h_2} - 2 = -\gamma_1 - 2$$

$$M_1(-\gamma_1 - 2) + M_2(\mu_1\gamma_1^2 - \mu_1) = \underbrace{-6\left(\mu_1 \frac{y_2 - y_1}{h_2^2} - \lambda_1 \frac{y_1 - y_0}{h_1^2}\right)}_{\widehat{g}_1}$$

$b_{n-2} = b_{n-1}$

$$\frac{M_{n-1} - M_{n-2}}{h_{n-1}} = \frac{M_n - M_{n-1}}{h_n} \,\Bigg|\, \times h_n$$

$$\gamma_n = \frac{h_n}{h_{n-1}}$$

$$\gamma_n M_{n-2} - M_{n-1}(1+\gamma_n) + M_n = 0$$

$$i = n-1:\ \lambda_{n-1} M_{n-2} + 2M_{n-1} + \mu_{n-1} M_n = 6\left(\mu_{n-1} \frac{y_n - y_{n-1}}{h_n^2} - \lambda_{n-1} \frac{y_{n-1} - y_{n-2}}{h_{n-1}^2}\right)$$

$$\gamma_n M_{n-2} - M_{n-1}(1+\gamma_n) + M_n = 0 \,|\, \times \mu_{n-1}$$

$$M_{n-2}(\lambda_{n-1}\gamma_n^2 - \lambda_{n-1}) - M_{n-1}(\mu_{n-1}\gamma_n + \mu_{n-1} + 2) = -6\left(\mu_{n-1} \frac{y_n - y_{n-1}}{h_n^2} - \lambda_{n-1} \frac{y_{n-1} - y_{n-2}}{h_{n-1}^2}\right)$$

$$-\mu_{n-1} - \mu_{n-1}\gamma_n - 2 = \mu_{n-1}(1+\gamma_n) - 2 = -\frac{h_n}{h_n + h_{n-1}}\frac{h_n + h_{n-1}}{h_{n-1}} - 2 = -\gamma_n - 2$$

$$M_{n-2}(\lambda_{n-1}\gamma_n^2 - \lambda_{n-1}) + M_1(-\gamma_n - 2) = \underbrace{-6\left(\mu_{n-1} \frac{y_n - y_{n-1}}{h_n^2} - \lambda_{n-1} \frac{y_{n-1} - y_{n-2}}{h_{n-1}^2}\right)}_{\widehat{g}_{n-1}}$$

$$
\begin{pmatrix}
-\gamma_1 - 2 & \mu_1\gamma_1^2 - \mu_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\lambda_2 & 2 & \mu_2 & 0 & \cdots & 0 & 0 & 0 \\
0 & \lambda_3 & 2 & \mu_3 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \lambda_{n-2} & 2 & \mu_{n-2} \\
0 & 0 & 0 & 0 & \cdots & 0 & \lambda_{n-1}\gamma_n^2 - \lambda_{n-1} & -\gamma_n - 2
\end{pmatrix}
\begin{pmatrix}
M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{n-2} \\ M_{n-1}
\end{pmatrix}
=
\begin{pmatrix}
\widehat{g}_1 \\ g_2 \\ g_3 \\ \vdots \\ g_{n-2} \\ \widehat{g}_{n-1}
\end{pmatrix}
$$