Bauman Moscow State University
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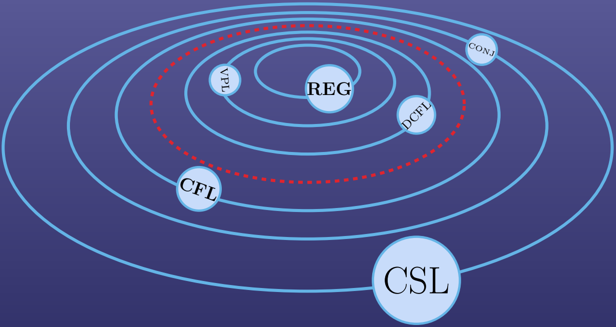
Equivalences of Finite Automata



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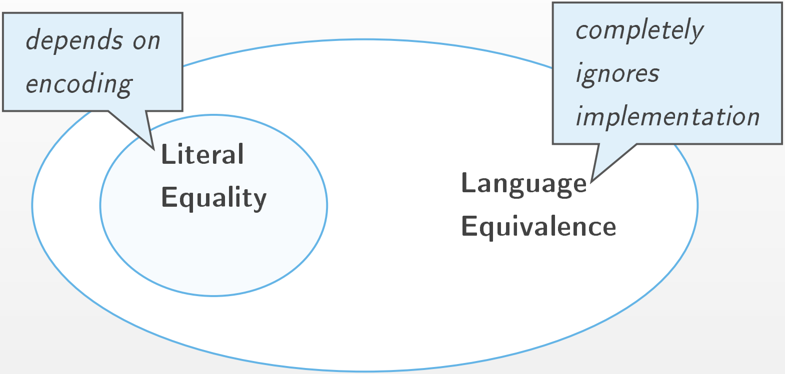
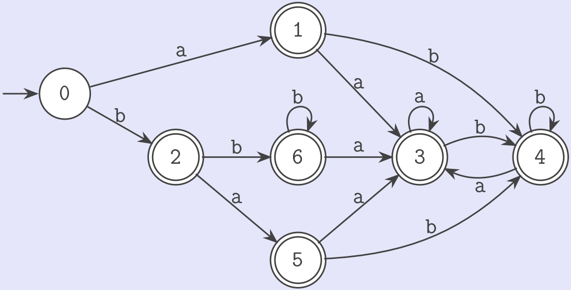
Machines Comparison



Problem: *how to find an equivalence that is sustainable to irrelevant implementation details (such as node naming) but tracks parsing-relevant properties?*

Equality of DFA and NFA

Given a DFA, its states can be *canonically named*: i.e. state number is determined by the string marking the shortest path from the starting state to the state considered. Then the numeration depends only on the chosen linear order on strings.



Above is the state numeration with respect to the military (length- lexicographic) order, given $\mathtt{a} \prec \mathtt{b}$*:*
$\varepsilon \prec \mathtt{a} \prec \mathtt{b} \prec \mathtt{aa} \prec \mathtt{ab} \prec \mathtt{ba} \prec \mathtt{bb}$.

Equality of DFA and NFA

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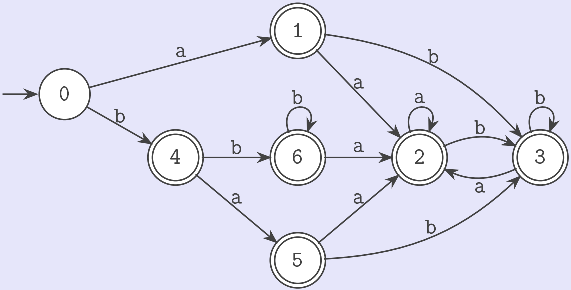


Another numeration is induced by the "vanilla" lexicographic order, given $\mathtt{a} \prec \mathtt{b}$*: now* $\varepsilon \prec \mathtt{a} \prec \mathtt{aa} \prec \mathtt{ab} \prec \mathtt{b} \prec \mathtt{ba} \prec \mathtt{bb}$.

Equality of DFA and NFA

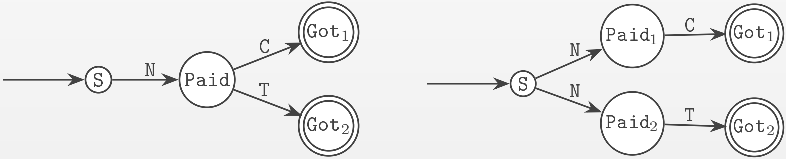
Given a DFA, its states can be *canonically named*: i.e. state number is determined by the string marking the shortest path from the starting state to the state considered. Then the numeration depends only on the chosen linear order on strings.
However, the canonical numeration does not work for NFA, provided that the string sets marking paths to the states can coincide.

Hence, DFA equality up to the state renaming can be thought as the literal coincidence of the canonically ordered DFA; but the canonical order does not work for recognising NFA equality.

Behavioral Equivalence and Language Equivalence

Language equivalence tracks only admissible actions, but not a way the actions are performed. The following example is well-known.

- `N` is «put a note into the machine»;
- `C` is «order a coffee»;
- `T` is «order a tea».



The given two machines have the same language, but are distinct from the point of view of a user:

- the first requires a note, and then asks what drink is ordered;
- the second asks for a choice when taking money, then requires to press the button that prepares it.

Bisimulation of Labelled Transition Systems

Bisimulation is a relation $\sim$ between states of the systems $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfying the following property:

- *If $q_1 \sim q_2$ ($q_1 \in \mathcal{T}_1$, $q_2 \in \mathcal{T}_2$), then for every transition $q_1 \xrightarrow{\gamma} q_1'$ in $\mathcal{T}_1$ there is a transition $q_2 \xrightarrow{\gamma} q_2'$ in $\mathcal{T}_2$ such that $q_1' \sim q_2'$, and vice versa.*

Starting and final (if any) states must be bisimilar.

Every state machine $\mathscr{A}$ can be represented as a labelled transition system.

- $\mathscr{A}_1$ and $\mathscr{A}_2$ are bisimilar $\Leftrightarrow$ their LTS $\mathcal{T}_1$ and $\mathcal{T}_2$ are bisimilar.

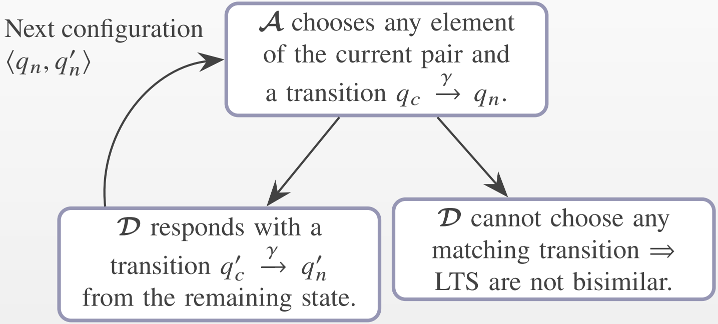
Labelled Transition Systems *versus* NFA

Labelled transition systems are not necessarily finite; moreover, LTS contain no final states.

Existence of final states can be modelled via introducing *endmarkers* (usually denoted $). Then every final state of an NFA has a transition by the endmarker to the unique «bottom» state.

Bisimulation Game

$\mathcal{T}_1$ and $\mathcal{T}_2$ bisimilarity checking technique can be formulated as a two-player game with an initial configuration $\langle q_S, q'_S \rangle$:



- Attacker's winning strategy always leads to the fact that any possible play is finite.
- In presence of final states, $\mathcal{A}$ can additionally declare «game-over» in any final state (and $\mathcal{D}$ must respond with the «game-over» as well).

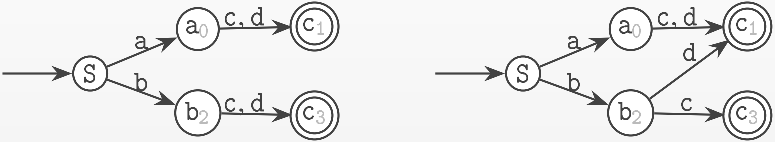
Equivalent Trim DFA are Bisimilar

Given two non-bisimilar trim DFA $\mathscr{A}_1$ and $\mathscr{A}_2$, we assume by the contradiction that the player $\boldsymbol{\mathcal{A}}$ has a winning strategy. The strategy is completely determined by a finite input string ω.

- If after reading the string ω one DFA ends up in a final state, while the other ends up in a non-final state, then ω witnesses that their languages do not coincide.
- If after reading the string ω one DFA (say $\mathscr{A}_1$) ends up in a state with an outgoing transition by some $\gamma \in \Sigma$, while the other does not has such a transition, then no string in $\mathscr{L}(\mathscr{A}_1)$ with the prefix $\omega\gamma$ can belong to $\mathscr{L}(\mathscr{A}_2)$. The DFA are trim $\Rightarrow$ at least one string prefixed by $\omega\gamma$ is in $\mathscr{L}(\mathscr{A}_1)$.

Bisimulation and Equality

Bisimilar DFA are not necessarily equal, even if cardinalities of their state sets are also equal.



In this example, the distinction between the states $\mathtt{c}_1$ and $\mathtt{c}_3$ is redundant: they are indistinguishable with respect to the languages that can be recognised from them, namely, $\mathscr{L}(\mathtt{c}_1) = \mathscr{L}(\mathtt{c}_3) = \{\varepsilon\}$. Hence, $\mathtt{c}_1 \sim \mathtt{c}_3$.

We could *merge* the bisimilar states with no impact to the recognised DFA language; conversely, if we know that the DFA states coincide wrt their languages, then we know they are behaviorally equivalent.

k-Bisimulation: Playing Backwards

Given an NFA $\mathscr{A}$, how do we know that its states q_i, q_j are bisimilar?

- if q_i is final, while q_j is non-final, then $\boldsymbol{\mathcal{A}}$ can choose q_i and declare game-over, winning the game. Hence, $\boldsymbol{\mathcal{A}}$ can win doing no move at all. If q_i, q_j are both final or both non-final, at least one move is required for $\boldsymbol{\mathcal{A}}$ to win, and we say that $q_i \sim_0 q_j$.

- if exists γ and q_i' s.t. $q_i \xrightarrow{\gamma} q_i'$, and for all q_j' s.t. $q_j \xrightarrow{\gamma} q_j'$ $q_i' \not\sim_k q_j'$, then $\boldsymbol{\mathcal{A}}$ wins in $k+1$ moves starting from the position $\langle q_i, q_j \rangle$. Otherwise, we say that $q_i \sim_{k+1} q_j$.

A closer look on the $\sim_{k+1}$-condition:

$$q_i \sim_{k+1} q_j \Leftrightarrow \forall q_i', \gamma\big(q_i \xrightarrow{\gamma} q_i' \Rightarrow \exists q_j'(q_j \xrightarrow{\gamma} q_j')\big)$$
$$\& \ \forall q_j', \gamma\big(q_j \xrightarrow{\gamma} q_j' \Rightarrow \exists q_i'(q_i \xrightarrow{\gamma} q_i')\big)$$

When we reach the fixpoint of $\sim_k$ (i.e. $\sim_k = \sim_{k+1}$), then we know that $\boldsymbol{\mathcal{A}}$ never can win in position $\langle q_i, q_j \rangle$ given $q_i \sim_k q_j$, hence $q_i \sim q_j$.

Fixpoints

Definition

Given a function $f : \mathcal{D} \to \mathcal{D}$, its fixpoint is a value $\tau \in \mathcal{D}$ such that $f(\tau) = \tau$.

If $f : \mathbb{R} \to \mathbb{R}$, then its fixpoints are located at the intersection of f-graph with the diagonal.

We have met the fixpoint construction before in the Arden lemma. Namely, the expression $\Phi^*\Psi$ is the fixpoint of the function $f(X) = \Phi X \mid \Psi$.

Moreover, the «Kleene star» construction is a fixpoint itself: Φ^* is the fixpoint language for the function $s(X) = \Phi X \mid \varepsilon$.

The fixpoint construction is particularly useful in computer science when some set is saturated wrt a well-founded relation.

Bisimilarity in DFA and Minimization

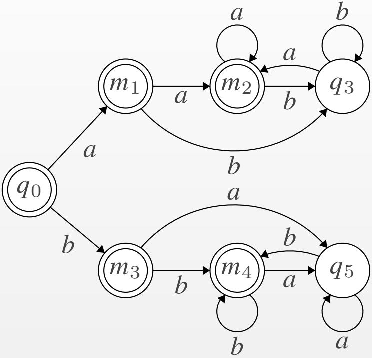
Given a DFA with state set Q, and set of final states F, we know that $q_i \sim q_j$ if and only if the sets of words recognised starting from q_i and starting from q_j are equal.
Hence, we can minimize the DFA by merging bisimilar states.

DFA Minimization Algorithm

1. Given $q_i, q_j \in Q$, mark the pairs $\{q_i, q_j\}$ s.t. $q_i \not\sim_0 q_j$ (i.e. $q_i \in F$, while $q_j \notin F$, or vice versa).
2. Mark all the pairs $\{q_i, q_j\}$ s.t. $\exists \gamma (q_i \xrightarrow{\gamma} q_i' \;\&\; q_j \xrightarrow{\gamma} q_j' \;\&\; \{q_i', q_j'\}$ is marked).
3. Repeat Step 2 until no new marked pair appear (i.e. fixpoint of $\sim_k$ is reached).
4. Merge the states in the all unmarked pairs: they are bisimilar.

Minimisation Example

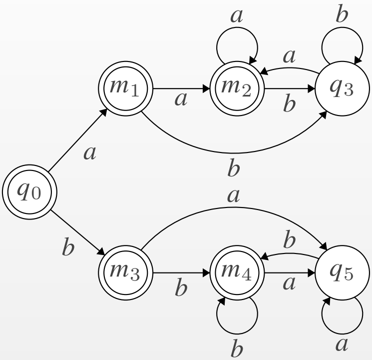
The initial DFA is:



	q_0	m_1	m_2	q_3	m_3	m_4
m_1						
m_2						
q_3						
m_3						
m_4						
q_5						

First, we construct the table in order to track marked pairs. We are required to track the pair only once, hence we can consider only the table part below its diagonal.

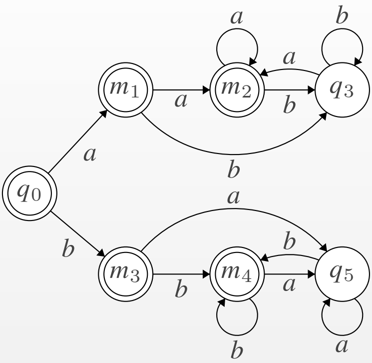
Minimisation Example



m_1						
m_2						
q_3	✓	✓	✓			
m_3				✓		
m_4				✓		
q_5	✓	✓	✓		✓	✓
	q_0	m_1	m_2	q_3	m_3	m_4

We have marked the states being not 0-bisimilar so far.

Minimisation Example



	q_0	m_1	m_2	q_3	m_3	m_4
m_1						
m_2						
q_3	✓	✓	✓			
m_3				✓		
m_4				✓		
q_5	✓	✓	✓		✓	✓

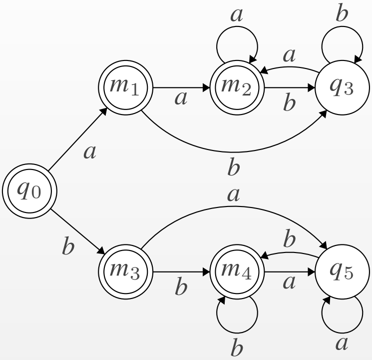
Now we are checking 1-bisimilarity:

$q_0 \xrightarrow{a} m_1, m_1 \xrightarrow{a} m_2 \qquad q_0 \xrightarrow{b} m_3, m_1 \xrightarrow{b} q_3 \qquad \{m_1, m_2\} \xrightarrow{a} m_2 \qquad \{m_1, m_2\} \xrightarrow{b} q_3$

$q_0 \xrightarrow{a} m_1, m_2 \xrightarrow{a} m_2 \qquad q_0 \xrightarrow{b} m_3, m_2 \xrightarrow{b} q_3$

The states q_0 and m_1 are distinguishable by b, as well as q_0 and m_2. The states m_1 and m_2 behave equally, i.e. we can say in advance that $m_1 \sim m_2$.

Minimisation Example



	q_0	m_1	m_2	q_3	m_3	m_4
m_1	✓					
m_2	✓					
q_3	✓	✓	✓			
m_3				✓		
m_4				✓		
q_5	✓	✓	✓		✓	✓

We continue checking 1-bisimilarity:

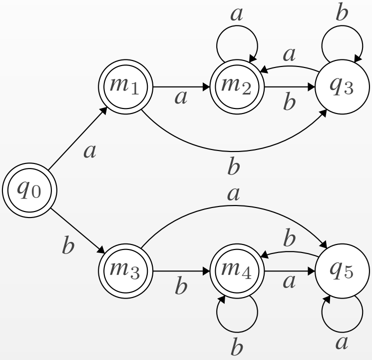
$q_0 \xrightarrow{a} m_1, m_3 \xrightarrow{a} q_5$ $\quad q_0 \xrightarrow{a} m_1, m_4 \xrightarrow{a} q_5$ $\quad m_1 \xrightarrow{a} m_2, m_3 \xrightarrow{a} q_5$

$m_2 \xrightarrow{a} m_2, m_3 \xrightarrow{a} q_5$ $\quad m_1 \xrightarrow{a} m_2, m_4 \xrightarrow{a} q_5$ $\quad m_2 \xrightarrow{a} m_2, m_4 \xrightarrow{a} q_5$

$q_3 \xrightarrow{a} m_2, q_5 \xrightarrow{a} q_5$

We find out that a-transitions distinguish almost all remaining pairs.

Minimisation Example



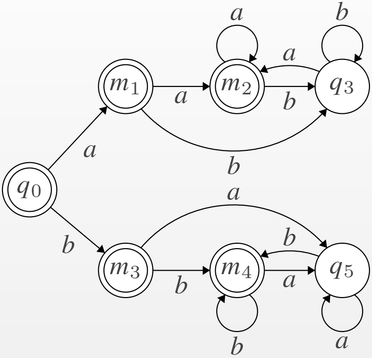
m_1	✓					
m_2	✓					
q_3	✓	✓	✓			
m_3	✓	✓	✓	✓		
m_4	✓	✓	✓	✓		
q_5	✓	✓	✓	✓	✓	✓
	q_0	m_1	m_2	q_3	m_3	m_4

The one more exception are the states m_3, m_4, behaving equally on both terminal letters.

$\{m_3, m_4\} \xrightarrow{a} q_5 \quad \{m_3, m_4\} \xrightarrow{b} m_4$

We cannot distinguish m_3 and m_4, as well as m_1 and m_2, by 2-bisimilarity relation, hence, the relation fixpoint is reached.

Minimisation Example

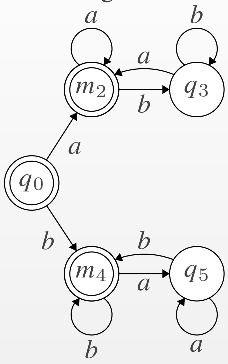


	q_0	m_1	m_2	q_3	m_3	m_4
m_1	✓					
m_2	✓					
q_3	✓	✓	✓			
m_3	✓	✓	✓	✓		
m_4	✓	✓	✓	✓		
q_5	✓	✓	✓	✓	✓	✓

Now we can merge the pairs m_1, m_2 and m_3, m_4, constructing a DFA with less number of states.

Minimisation Example

The resulting DFA is:



m_1	✓					
m_2	✓					
q_3	✓	✓	✓			
m_3	✓	✓	✓	✓		
m_4	✓	✓	✓	✓		
q_5	✓	✓	✓	✓	✓	✓
	q_0	m_1	m_2	q_3	m_3	m_4

Nice, we have reduced the DFA. But how we can guarantee that the resulted DFA is *the least one*, i.e. there is no DFA with less number of states recognising the same language?
We can guarantee so far only the fact that the states in the given DFA cannot be merged any more.

Suffix Congruence

Every state q_i of an NFA $\mathscr{A}$ determines a language of words $\mathscr{L}_p(q_i) = \{\omega \mid q_0 \xrightarrow{\omega} q_i\}$. If $\mathscr{A}$ a DFA, given $q_i \neq q_j$, $\mathscr{L}_p(q_i) \cap \mathscr{L}_p(q_j) = \varnothing$.

If every two states in the DFA $\mathscr{A}$ are not bisimilar, then

$$
\begin{aligned}
&\forall q_i, q_j \big(i \neq j \Rightarrow \\
&\qquad \exists \omega_{ij} \Big(\big(\mathscr{L}_p(q_i)\{\omega_{ij}\} \subseteq \mathscr{L}(\mathscr{A}) \;\&\; \mathscr{L}_p(q_j)\{\omega_{ij}\} \nsubseteq \mathscr{L}(\mathscr{A})\big) \\
&\qquad\qquad \vee \big(\mathscr{L}_p(q_j)\{\omega_{ij}\} \subseteq \mathscr{L}(\mathscr{A}) \;\&\; \mathscr{L}_p(q_i)\{\omega_{ij}\} \nsubseteq \mathscr{L}(\mathscr{A})\big) \Big) \big)
\end{aligned}
$$

Namely, suffix ω_{ij} distinguishes states q_i and q_j.

Meanwhile, all the words belonging to the same $\mathscr{L}_p(q_i)$ are congruent (undistinguishable) with respect to suffixes.

Forbidden Factors and Thin Languages

Definition

A word ω is said to be a forbidden factor in a language $\mathscr{L}$ is no word in $\mathscr{L}$ contains ω as a subword.

Languages with forbidden factors are said to be thin.

E.g. the word ba is forbidden in a^*b^*, hence the expression a^*b^* defines a thin language. On the other hand, language given by the expression $a(a|b)^*a$ has no forbidden factors.

All the words with forbidden factors in $\mathscr{L}$ are undistinguishable with respect to suffixes, i.e. lead to the trap state in any machine recognising it. However, even if $\mathscr{L}$ is not thin, its recognising machine can still contain trap states, e.g. the language of balanced parentheses has no forbidden factors, but requires the parentheses balance to be non-negative in any prefix of its words.

Language IS THE Minimal DFA

👀 **Theorem**

- *Every two equivalent DFA containing no bisimilar states are equal (up to the canonical numeration).*
- *Hence, the minimisation procedure is confluent for DFA, and the minimal DFA is unique for any regular language.*

Proof: Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be DFA both recognising the same language $\mathscr{L}$, s.t. neither $\mathscr{A}_1$ nor $\mathscr{A}_2$ contains bisimilar states. Equivalent trim DFA are bisimilar, thus non-trap parts of $\mathscr{A}_1$ and $\mathscr{A}_2$ are bisimilar, as well as the traps (if any).

For any q_i in $\mathscr{A}_1$, there is a non-empty set of states $q_{i,j}$ in $\mathscr{A}_2$ that are bisimilar to q_i (and vice versa). If this set is not a singleton, then all the states $q_{i,j}$ are mutually bisimilar, which contradicts the choice of $\mathscr{A}_1$ and $\mathscr{A}_2$. Therefore, $\mathscr{A}_1$ and $\mathscr{A}_2$ have equal state set cardinality.

Assume that the states are canonically numbered, and there is at least one prefix υ s.t. $q_0(\mathscr{A}_1) \xrightarrow{\upsilon} q_i$, $q_0(\mathscr{A}_2) \xrightarrow{\upsilon} q_j$, and $i \neq j$, i.e. $q_i \nsim q_j$. Hence, there exists a suffix ω_{ij} distinguishing q_i and q_j (i.e. the tests $\upsilon\omega_{ij} \in \mathscr{L}(\mathscr{A}_1)$ and $\upsilon\omega_{ij} \in \mathscr{L}(\mathscr{A}_2)$ have distinct values), which contradicts the choice of $\mathscr{A}_1$ and $\mathscr{A}_2$. $\square$

Suffix Congruence and Minimal DFA

Now we know that partition to $\mathscr{L}_p(q_i)$-sets wrt the minimal DFA can be considered as a relation wrt its language $\mathscr{L}$. Namely, we say that $\upsilon_1 \equiv_{\mathscr{L}} \upsilon_2$, iff $\forall\omega(\upsilon_1\omega \in \mathscr{L} \Leftrightarrow \upsilon_2\omega \in \mathscr{L})$, i.e. no suffix can distinguish υ_1 and υ_2.

Myhill–Nerode Theorem

Language $\mathscr{L}$ in an alphabet Σ is regular $\Leftrightarrow$ the quotient set $\Sigma^*/\equiv_{\mathscr{L}}$ is of a finite cardinality.

Proof *(⇐):* Let $\Sigma^*/\equiv_{\mathscr{L}}$ be of a finite cardinality. We are going to construct a DFA $\mathscr{A}$ recognising $\mathscr{L}$.

- For every equivalence class S_i wrt $\equiv_{\mathscr{L}}$, we introduce a state of $\mathscr{A}$. The initial state is S_0 s.t. $\varepsilon \in S_0$. Final states are S_k s.t. $\forall\omega \in S_k(\omega \in \mathscr{L})$.
- Given state S_i and $\gamma \in \Sigma$, add transition $S_i \xrightarrow{\gamma} S_j$ s.t. $\forall\omega \in S_i(\omega\gamma \in S_j)$.

(⇒): Was proven already. Elements of $\Sigma^*/\equiv_{\mathscr{L}}$ are $\mathscr{L}_p(q_i)$-sets of the minimal DFA. □

The suffix congruence $\equiv_{\mathscr{L}}$ is also called the Myhill–Nerode equivalence.

Matrix Method: Non-Regularity

Language Regularity Criterion, Negated

Language $\mathscr{L}$ in an alphabet Σ is non-regular $\Leftrightarrow$ the quotient set $\Sigma^*/\equiv_{\mathscr{L}}$ is infinite.

In order to prove that $\Sigma^*/\equiv_{\mathscr{L}}$ is infinite:

- construct an infinite series $\upsilon_1, \ldots, \upsilon_k, \ldots$ of prefixes of words in $\mathscr{L}$;
- construct an infinite series of distinguishing suffixes $\omega_1, \ldots, \omega_k$ such that $\forall i, j \exists k \left(\upsilon_i \omega_k \in \mathscr{L} \ \& \ \upsilon_j \omega_k \notin \mathscr{L}\right)$.

If $\mathscr{L}$ is "scarse", it is convenient to choose suffixes s.t. $\forall i \left(\upsilon_i \omega_i \in \mathscr{L}\right)$, and for most $k \neq i$, $\upsilon_k \omega_i \notin \mathscr{L}$.

Then the classes can be visualised as an infinite matrix with distinct rows containing 1's on its leading diagonal.

NB: if $\mathscr{L}$ is dense, then it is better to choose an opposite strategy, constructing a matrix containing 0's on its leading diagonal.

Matrix Method: Non-Regularity

If $\mathscr{L}$ is "scarse", it is convenient to choose suffixes s.t. $\forall i(\upsilon_i \omega_i \in \mathscr{L})$, and for most $k \neq i$, $\upsilon_k \omega_i \notin \mathscr{L}$.

Let us show non-regularity of the language

$$\left\{ w_1 w_2 \mid w_1 \neq \varepsilon \;\&\; |w_1| = 2 \cdot k \;\&\; w_1 = w_1^R \;\&\; w_i \in \{a, b\}^* \right\}$$

Given $\omega_i = \upsilon_i = ab^{2 \cdot i - 1}a$,

$$\upsilon_i \omega_j = \begin{cases} ab^{2 \cdot i - 1}aab^{2 \cdot i - 1}a, & \text{palindrome, if } i = j; \\ ab^{2 \cdot i - 1}aab^{2 \cdot j - 1}a, & \text{never starts with an even palindrome, if } i \neq j. \end{cases}$$

The classes matrix:

	aba	ab^3a	$\ldots$	$ab^{2 \cdot k - 1}a$	$\ldots$
aba	+	−		−	
ab^3a	−	+		−	
$\ldots$		$\ldots$			
$ab^{2 \cdot k - 1}a$	−	−		+	
$\ldots$					

There υ_i name rows, ω_i name columns, **+** at position (i, j) denotes the fact that $\upsilon_i \omega_j \in \mathscr{L}$; and **−** at position (i, j) denotes the fact that $\upsilon_i \omega_j \notin \mathscr{L}$.

Definition

Language $\mathscr{L}$ is said to be bounded if $\mathscr{L} \subseteq \omega_1^* \ldots \omega_n^*$, where $\omega_1, \ldots, \omega_n$ are fixed words.

The bounded languages are of extreme usefulness further in the course (when investigating about the context-freeness property), but they are still of interest even in the regular case.

When discussing “scarse” and “dense” languages, we assume that the notions of scarcity and density are given in the context of the enclosing bounded language.

E.g. the language $\mathscr{L}_{\neq} = \left\{a^m b^n \mid n \neq m\right\}$ is thin in $\{a, b\}^*$ (since its word contain no subword ba); but in the bounded context of $a^* b^*$ it is dense (for almost all pairs n, m, $a^n b^m \in \mathscr{L}_{\neq}$).

Indeed, the trap state equivalence class corresponding to the words with forbidden factors is useless in constructing infinite classes matrix, so it is natural to omit it.

Pumping Lemma vs. Classes Matrix Method

Pumping Lemma:

- requires a unique parameterized word.
- usually requires preliminary intersection with a bounded language, in order to narrow a set of pumped subwords.

Matrix Method:

- requires constructing series of words, not necessarily parameterized.
- normally does not require any intersection tricks, since we are free to restrict combinations of subwords by the given series.

The matrix method is far more universal and flexible. However, in a few cases the pumping lemma is more convenient.

Let us prove that $\mathscr{L} = \left\{\omega \mid \omega \textit{ does not contain a factor of power } 4\right\}$ *in* $\{a, b\}^*$ *is non-regular.*

- *We know that* $\mathscr{L}$ *is infinite, since Thue words (given by iterative application of Thue morphism* $h(a) = ab$*,* $h(b) = ba$*) belong to* $\mathscr{L}$*.*
- *Then, if* $\mathscr{L}$ *is regular and its pumping length is* N*, there is some Thue word* ω *s.t.* $|\omega| > N$*, and* $\omega = \omega_1\omega_2\omega_3$*,* $\forall i(\omega_1\omega_2^i\omega_3 \in \mathscr{L})$*. But that can never happen for* $i \geq 4$*, by definition of* $\mathscr{L}$*.*

Matrix Method: Minimality

DFA Minimality Criterion

A DFA $\mathscr{A}$ is minimal for given language $\mathscr{L}$ in alphabet Σ iff the number of states in $\mathscr{A}$ is equal to $\mathtt{card}(\Sigma^*/\equiv_{\mathscr{L}})$ (i.e. the number of Myhill–Nerode equivalence classes of $\mathscr{L}$).

An easy way to prove that a given DFA $\mathscr{A}$ with the state set $\{q_0, ..., q_n\}$ is minimal:

- construct a set of prefixes ε, υ_1, ..., υ_n s.t. $q_0 \xrightarrow{\upsilon_i} q_i$;
- construct a finite set of distinguishing suffixes ω_1,..., ω_k s.t. $\forall i, j \exists k \big(\upsilon_i \omega_k \in \mathscr{L} \;\&\; \upsilon_j \omega_k \notin \mathscr{L}\big)$.

Usually, it is enough to find $|O(\log_2 n)|$ distinguishing suffixes, i.e. the resulting matrix is not necessarily quadratic.

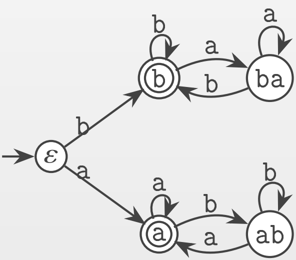
Matrix Method: Minimality

- construct a set of prefixes ε, υ_1, ..., υ_n s.t. $q_0 \xrightarrow{\upsilon_i} q_i$;
- construct a finite set of distinguishing suffixes ω_1,..., ω_k s.t. $\forall i, j \exists k \big(\upsilon_i \omega_k \in \mathscr{L}\ \&\ \upsilon_j \omega_k \notin \mathscr{L}\big)$.

Consider the DFA mentioned above for the language

$$\{\omega \mid \omega \text{ starts and ends with a same letter}\}$$

The given matrix verifies its minimality with respect to the language.



	ε	a	b
ε	0	1	1
a	1	1	0
b	1	0	1
ab	0	1	0
ba	0	0	1