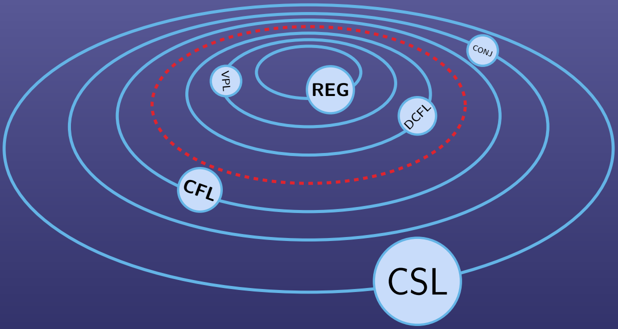
Bauman Moscow State University
Th. Computer Science Dept.

Introduction to Realms of Formal Languages



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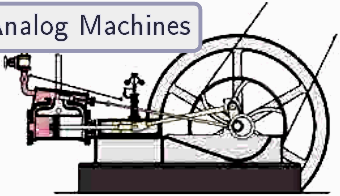
Lecture Outline

1. **Formal Languages by Examples**
   - Automata Around Us
   - Tons of Formalizations
2. **Course Details**
3. **Basics on Term Rewriting**
   - Rewriting and Reduction
   - Ordering and Induction
   - String Rewriting

Physical Concept of Automaton

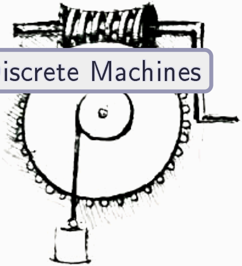
Analog Machines



Continuous control

Differential equations

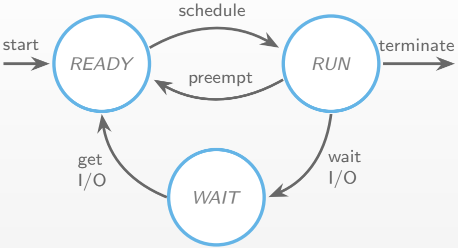
Discrete Machines



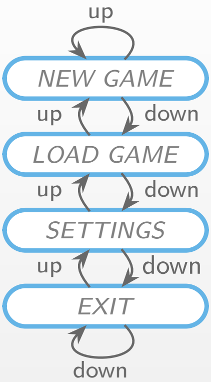
Discrete control

Algebra & Logics

Automata Specifications



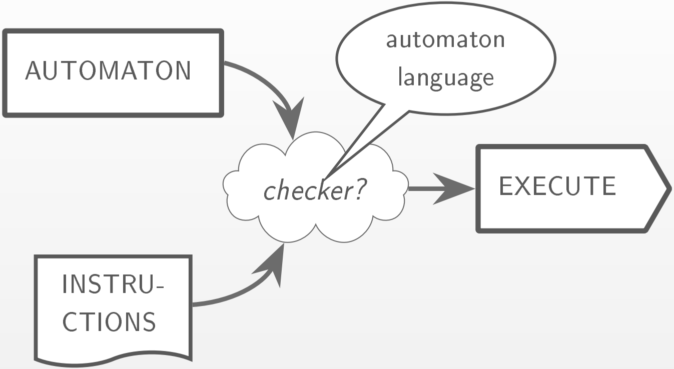
- CPU Scheduler scheme



- Simple game menu

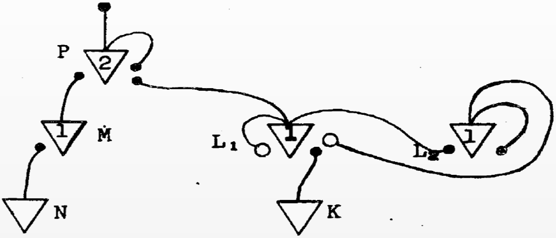
- Automata-based schemes are used widely:
  - communication protocols;
  - user interfaces;
  - control units;
  - optimisation cycles.

Programming: Automata Languages



- One-by-one input steps ⇒ automaton model is enough.
- Sequence of multiple steps given simultaneously ⇒ executor is required to verify the sequence wrt the automata model.
- Admissible step sequences — *a formal language* of the machine.

McCulloch& Pitts Neural Networks



- • — excitatory signal;
- ∘ — inhibitory signal;

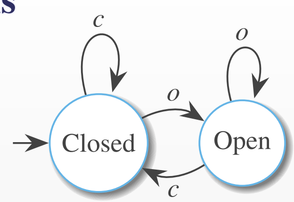
▽ — an input neuron;

▽k — an inner neuron firing whenever none of the inhibitory signals and at least k of excitatory signals fire.

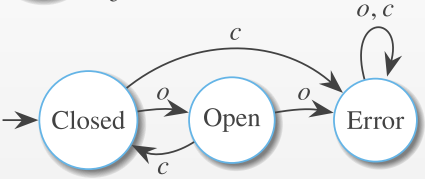
Kleene (1951) introduced regular languages, describing the events in McCulloch–Pitts NN in terms of three operations: $\{\cdot, \cup, ^*\}$.

Automata Models

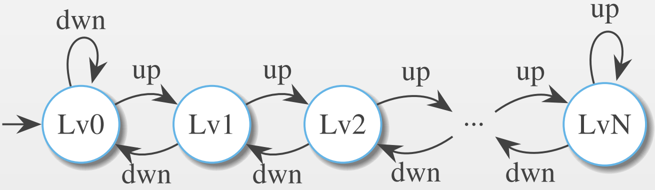
Door automaton:



Channel automaton:



Elevator:



Formal Languages Formally

Definition

Let us consider an algebra $\mathcal{A} = \langle \underbrace{\mathcal{M}}_{\textit{carrier}}, \overbrace{\mathcal{F}}^{\textit{signature}} \rangle$.

A formal language is a set $\mathcal{M}$ *of terms in algebra* $\mathcal{A}$.

- Classical case: if $\mathcal{A} = \langle \Sigma, \cdot \rangle$, where $\cdot$ is the concatenation operation, then $\mathcal{M} \subseteq \Sigma^*$, where * is iteration (Kleene star, Kleene closure).
- Wider scope: tree automata languages, syntax trees, process graphs...

Formal Languages Examples

syntax

- $\left\{\underbrace{aa\ldots a}_{n \text{ times}}\ \underbrace{bb\ldots b}_{n\cdot 3 \text{ times}} \mid n \in \mathbb{N}\right\}$ (henceforth $\left\{a^n b^{3n} \mid n \in \mathbb{N}\right\}$);
- words containing a square number of letters a

  $$\left\{w \mid \exists k(\underbrace{|w|_a}_{\text{number of letters } a \text{ in word } w} = k^2)\right\};$$
- Russian palindromes of the even length

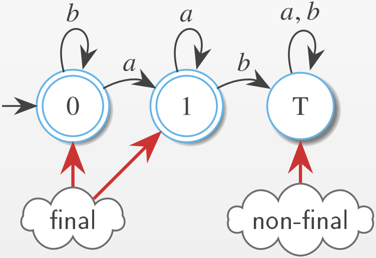
  («*Я слипся, я слился, я спился*»);
- sequences of balanced parentheses;
- well-formed arithmetic expressions over $\mathbb{N}$ and $\{\cdot, +\}$.

semantics

- all tautologies in the classical logic;
- all consistently typed programs in Python;
- formal languages with linear-time-decidable membership.

Varieties of FL Representations

- Set comprehension: $\{w \mid w \text{ does not contain subword } ab\}$.
- Algebraic expressions: $b^* a^*$.
- Term rewriting systems: $\begin{cases} a \to aa \\ b \to bb \end{cases}$, base set $\{\overbrace{\ \varepsilon\ }^{\text{empty word}}, a, b, ba\}$.
- Recognising machines:



- First-order or second-order logical formulas:
$\forall x, y\big(\underbrace{Q_a(x)}_{x \text{ is equal to } a} \ \& \ \underbrace{\mathcal{S}(x,\ y)}_{y \text{ succeeds } x} \ \Rightarrow \ \underbrace{\neg\, Q_b(y)}_{y \text{ is not equal to } b}\big)$.

Varieties of FL Representations

- Set comprehension: $\{w \mid w$ does not contain subword $ab\}$.
- Algebraic expressions: $b^* a^*$.
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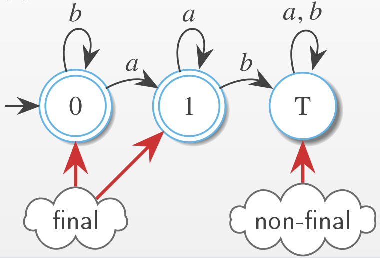
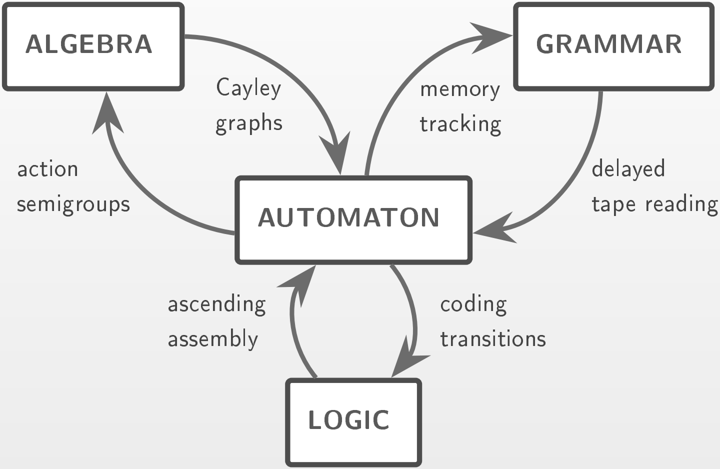


Automata-based representation easily reduces to Turing machines $\Rightarrow$ useful when estimating computational complexity of the language recognition.

Transforming & Analysing Representations

- *Efficient conversions between representation classes*
- *Constructing an optimal form of a representation inside a class*



Spherical Cows in RoFL Course

What is a reason to study artificial languages like «*words not containing subwords ba*»?

(common knowledge):
A cow is homeomorphic to a sphere with a couple of handles.

- Transducer images are shorter and their intermediate interpretations are more sheer;
- The transducer images reveal the essence of abstract formal properties rather than features of the ad-hoc model;
- In fact, morphisms and FSM transformations can be considered as lexers into the theoretical scope.

Course Structure

- Introduction and TRS

I part: automata
- Finite Automata;
- Visibly Pushdown Automata;
- Generic Pushdown Automata;
- Deterministic Pushdown Automata;
- Alternating Automata, Memory Automata;
- Cellular Automata, Tree Automata + *Midterm I*.

II part: languages
- Semirings;
- Syntactic Monoids;
- Ehrenfeucht–Fraïssé games and Pumping;
- Hardest Languages and Language Representations;
- Language Inference;
- Combinatorial & Computational Language Properties.

- *Midterm II*
- Concluding Lecture
- *«RoFL Farm»*

Course Scores

👀 Score Arrangement

- ***Midterms*** *× 2 × 15+.*
- ***Assignments*** *× 5 × 8+.*
  - *Java, Python, Go, JS — no score bonus*
  - *C/C++, Kotlin, Dart, TypeScript, Lua — +1 point*
  - *Rust, Lisp dialects, Scala, Julia — +2 points*
  - *Haskell, Erlang — +3 points*
  - *Рефал — +4 points for first time +3 afterwards*
- **Deadlines for assignments**:
  - 0-14 days — no penalty
  - 15-21 days — 1 score penalty
  - 22-28 days — 2 score penalty
  - 29-∞ — 3 score penalty

Reputation Count and Queuing

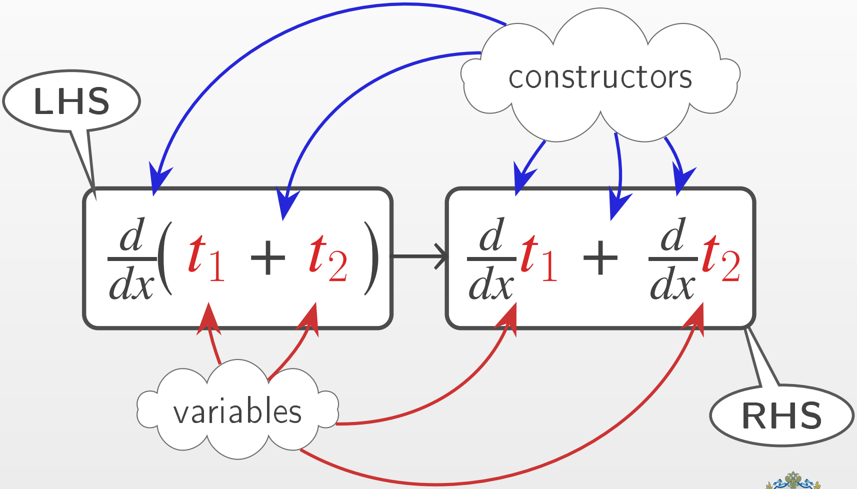
Initial rep = 100 points for everyone.

- Submit a work at deadline daytime: rep -2.
- Submit a work at deadline night: rep -5 .
- Submit a work on the eve of Reset Event: rep-10.
- Artefacts of someone's else code: rep-25.
- Unable to explain self code: rep-50.
- Submit a work on the eve of exam (daytime): rep-35.
- Submit a work on the eve of exam (night): rep-50.
- Submit a work not assigned to self: rep-75.
- Other (violating RoFL Farm rules, etc) — ad hoc.

> *The assignments are considered in* ***descending order*** *of rep!*
> rep $\leq 0 \Rightarrow$ *personal task arrangement.*

Term Rewriting and All That

Any symbolic computation is a term rewriting process, controlled by an appropriate set of *rewrite rules* — *term rewriting system*.

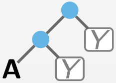


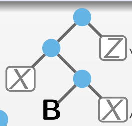
Unification and Pattern Matching

A rewriting process *unifies* term t and LHS s_1 of a rule $s_1 \to s_2$ by constructing a substitution.

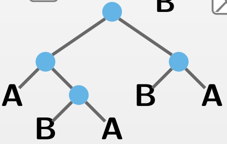
Unification Formally

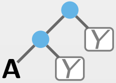
*Given two terms t_1, t_2, **a unifier** is a pair of variable substitutions $\langle \theta_1, \theta_2 \rangle$ s.t. $t_1\theta_1 = t_2\theta_2$.*

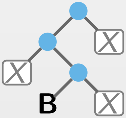
If $\mathrm{eq}(x, x) \to$True, when eq(

, 

) is rewritten?

Unification is possible when $x =$ 

Term eq(

, 

) is never unified with $\mathrm{eq}(x, x)$.

Notation Clash: Substitutions

In algebraic courses, given a substitution σ, its application to a term Φ is denoted with $\sigma(\Phi)$.
In mathematical logic & computer science, a ***postfix notation*** $\Phi\sigma$ is more usual.

Origins of the Notation

In classical mathematical logic textbooks (Tarski, Curry) the substitutions were denoted by $[x/A]$. Hence, the postfix notation was natural: formula $F(x,x)[x/A]$ is pretty more readable than $[x/A](F(x,x))$.

In modern computer science, the notations $[x := A]$ are $[x \mapsto A]$ are both widely used.

Reduction, Redexes, Normal Forms

Given a TRS $\left\{\Phi_i \to \Psi_i\right\}_{i=1}^{n}$ and a term T,

👀 Basic Notions of Rewriting

- ***A redex*** *is a subterm T_0 that can be unified with some Φ_i by unifier $\langle\theta_1, \theta_2\rangle$.*
- ***Reduction*** *replaces $T_0\theta_1$ in T by $\Psi_i\theta_2$.*
- *Term T is in **a normal form** if T contains no redex.*
- *Normalisation — reduction to the normal form: $T \twoheadrightarrow T'$.*

Given TRS $\begin{cases} 0 + x \to x \\ 0 \cdot x \to 0 \end{cases}$, the term $(0+1)\cdot 0$ contains a redex $0+1$, and the reduced term $1 \cdot 0$ is in the normal form.

Adding commutativity rules results in unrestricted reductions.

α-conversions and α-equivalence

- If x, z are variables, the rewriting rules $\{\mathrm{eq}(x, x) \to \mathrm{True}\}$ and $\{\mathrm{eq}(z, z) \to \mathrm{True}\}$ are *equivalent*; while the rule $\{\mathrm{eq}(x, z) \to \mathrm{True}\}$ is not equivalent to both.

- α-conversion is a semantic-preserving variable renaming. Non-trivial in case of bound variables.

λ-calculus

- Constructors: $\mathrm{Apply}(M, N)$ and $\lambda x.M$, bounding x in M.
- Rewriting rule: $\mathrm{Apply}(\lambda x.M), N) \to M[x := N]$.

Reduction + capture-avoiding substitution = universal computation model.

Proving Properties of FL and Term Ordering

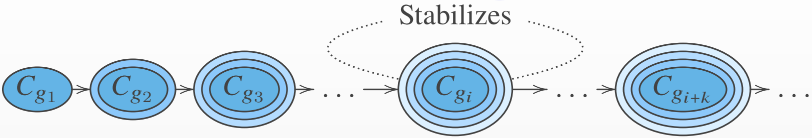
Check that the language $\mathscr{L} = \{w \mid |w|_a \text{ is even}\}$ is the set of normal forms generated from term srt $\{S\}$ using the following TRS T: $\quad S \to a\,S\,a \quad S \to b\,S \quad S \to S\,b \quad S \to \varepsilon$

- Check that all words from $\mathscr{L}$ can be generated by T.
- Check that all the normal forms η s.t $S \twoheadrightarrow \eta$ are in $\mathscr{L}$.

Assume that there exists w s.t. $S \twoheadrightarrow w$ and $|w|_a$ is odd. The reduction step before the application of $\{S \to \varepsilon\}$ can always be replaced by $\{S \to \varepsilon\} \Rightarrow$ making the word w shorter. We can repeat this process to ∞, contradicting that $|w|$ is finite.

Key moment: the length ordering admits no infinite descending chains.

Noetherian Orders and Well-Quasi-Orders



Ordering as an Induction Base

- *A preorder $\preceq$ is* ***well-founded*** *on set $\mathcal{A}$ if it admits no infinite descending chains, i.e. all chains $t_0 \succeq t_1 \succeq \dots$ are finite.*
- *A preorder is* ***Noetherian*** *on set $\mathcal{A}$ if it stabilizes upwards (no infinite strictly ascending chains, abbreviated ACC).*
- *A preorder $\preceq$ is* ***a well-quasi-order*** *on set $\mathcal{A}$ if every infinite term sequence from $\mathcal{A}$ contains t_i, t_j s.t. $\left(i < j \;\&\; t_i \preceq t_j\right)$.*

Well-Founded Monotone Algebras

A preorder $\preceq$ is monotone on $\mathcal{A}$, if
$\forall f, t_1, .., t_n, s, s' \in \mathcal{A}\Big(s \preceq s' \Rightarrow f(t_1, .., s, .., t_n) \preceq f(t_1, .., s', .., t_n)\Big)$,
and is strictly monotone, if converse always fails.

Well-Founded Monotone Algebras

WFMA over the signature F on a well-founded set $\langle \mathcal{A}, > \rangle$ is an algebra s.t. for any $f \in F$ there exists a function $f_{\mathcal{A}} : \mathcal{A}^n \rightarrow \mathcal{A}$ being strictly monotone wrt all its arguments.

- $\langle \mathbb{N}, > \rangle$ can be a WFMA over a signature $(f_1, 2), (f_2, 1)$, given e.g. interpretations $f_1(x, y) = x + y + 1$, $f_2(x) = 2 \cdot x$;
- $\langle \mathbb{Q}^+, > \rangle$ and $\langle \mathbb{Z}, > \rangle$ are not WFMAs.

TRS Termination

Given a variable set $\mathcal{V}$, extension of $\sigma : \mathcal{V} \to \mathcal{A}$ is defined as:

- $[x, \sigma] = x\sigma$;
- $[f(t_1, \dots, t_n), \sigma] = f_{\mathcal{A}}([t_1, \sigma], \dots, [t_n, \sigma])$.

*A TRS $\{l_i \to r_i\}$ is **compatible** with WFMA $\mathcal{A} \Leftrightarrow$ for all i, σ condition $[l_i, \sigma] > [r_i, \sigma]$ holds.*

Informally: the image of l_i is greater than the image of r_i for the extension of any substitution $\sigma : \mathcal{V} \to \mathcal{A}$.

Consider TRS $\{f(f(x)) \to f(x)\}$ and WFMA carrier $\mathcal{N} = \langle \mathbb{N}, > \rangle$.

- *Let $f_{\mathcal{N}}(x) = 2 \cdot x$ and $\sigma = [x := 0]$: $f_{\mathcal{N}}(f_{\mathcal{N}}(x))\sigma = f_{\mathcal{N}}(x)\sigma = 0$, $\Rightarrow$ WFMA is not compatible with the TRS.*
- *Let $f_{\mathcal{N}}(x) = x + 1$: $\forall x \big(f_{\mathcal{N}}(f_{\mathcal{N}}(x)) - f_{\mathcal{N}}(x) = 1\big) \Rightarrow$ the TRS and WFMA are compatible.*

WFMA and TRS

Main Theorem on TRS Termination

A TRS $\{l_i \to r_i\}$ terminates $\Leftrightarrow$ there exists a WFMA compatible with $\{l_i \to r_i\}$.

The TRS $\left\{\frac{d}{dx}(t_1 + t_2) \to \frac{d}{dx}t_1 + \frac{d}{dx}t_2\right\}$ *is terminating, which is verified by WFMA* $\mathcal{N}$ *over* $\langle \mathbb{N}, > \rangle$*:*

- $+_{\mathcal{N}}(u, v) = u + v + 1;$
- $\frac{d}{dx}_{\mathcal{N}}(u) = 2 \cdot u.$

Indeed,
$\frac{d}{dx}_{\mathcal{N}}(t_1 +_{\mathcal{N}} t_2) = 2 \cdot (t_1 + t_2 + 1);$
$\frac{d}{dx}_{\mathcal{N}} t_1 +_{\mathcal{N}} \frac{d}{dx}_{\mathcal{N}} t_2 = 2 \cdot t_1 + 2 \cdot t_2 + 1.$

Free Monoid (aka String DataType) as a FL Carrier

Usual case: the string data type is a carrier of a TRS; the only constructor is the string concatenation.
Then the term rewriting system becomes *a string rewriting system* (SRS): $\{l_i \to r_i\}$, where l_i, r_i are strings.

👀 **Terminal & Nonterminal Symbols**

Given disjoint alphabets N, Σ, we say that elements of Σ are terminal symbols, and elements of N are non-terminals in a given SRS $\{l_i \to r_i\}$, if words $w \in N \cup \Sigma$ containing letters from N are considered as partially computed (even if they are in the normal form).

Formal Grammars

👀 **Definition**

*A **grammar** is a tuple $G = \langle N, \Sigma, P, S \rangle$, where:*

- *N — non-terminal alphabet;*
- *Σ — terminal alphabet;*
- *P — string rewriting system $\{\alpha_i \to \beta_i\}$, where α_i is non-empty;*
- *$S \in N$ is an initial nonterminal.*

*A **language** $\mathscr{L}(G)$ of a grammar G is the set $\{u \mid u \in \Sigma^* \ \&\ S \to^* u\}$, where $\to^*$ is a composition of reductions.*