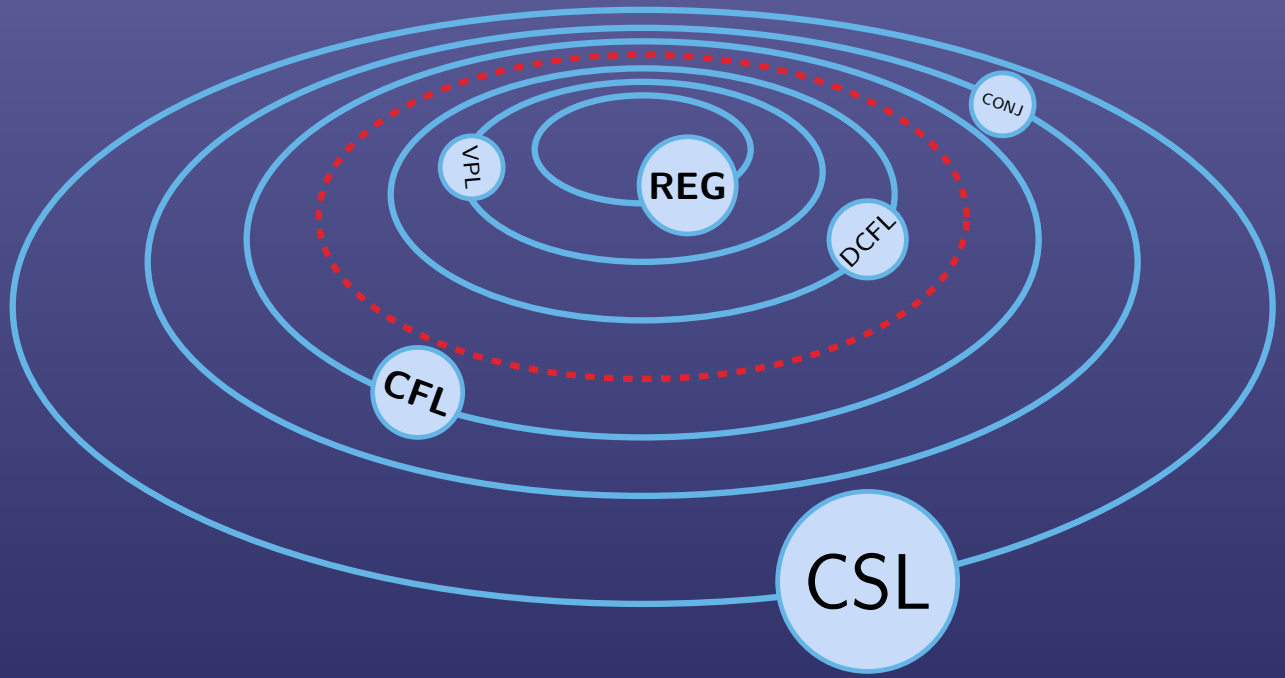
Bauman Moscow State University
Th. Computer Science Dept.

Introduction to Realms of Formal Languages



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Lecture Outline

1. **Formal Languages by Examples**
   - Automata Around Us
   - Tons of Formalizations
2. **Course Details**
3. **Basics on Term Rewriting**
   - Rewriting and Reduction
   - Ordering and Induction
   - String Rewriting

Standard English Abbreviations

All academia:

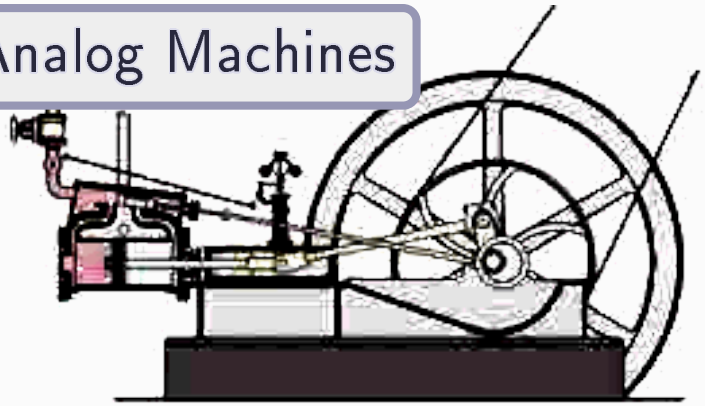
- *e.g.* (exempli gratia – *Latin*) — shortcut for “for example” (например – *рус.*).
- *i.e.* (id est – *Latin*) — shortcut for “that is” (то есть – *рус.*).
- *etc* (et cetera – *Latin*) — shortcut for “and so on” (и так далее – *рус.*).

math papers:

- *wrt* — shortcut for “with respect to” (относительно, по отношению к – *рус.*).
- *s.t.* — shortcut for “such that” (такой, что – *рус.*).
- *iff* — shortcut for “if and only if” (тогда и только тогда – *рус.*).

Physical Concept of Automaton

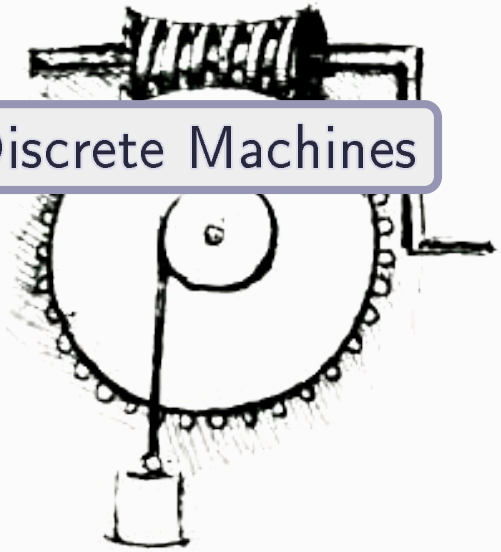
Analog Machines



Continuous control

Differential equations

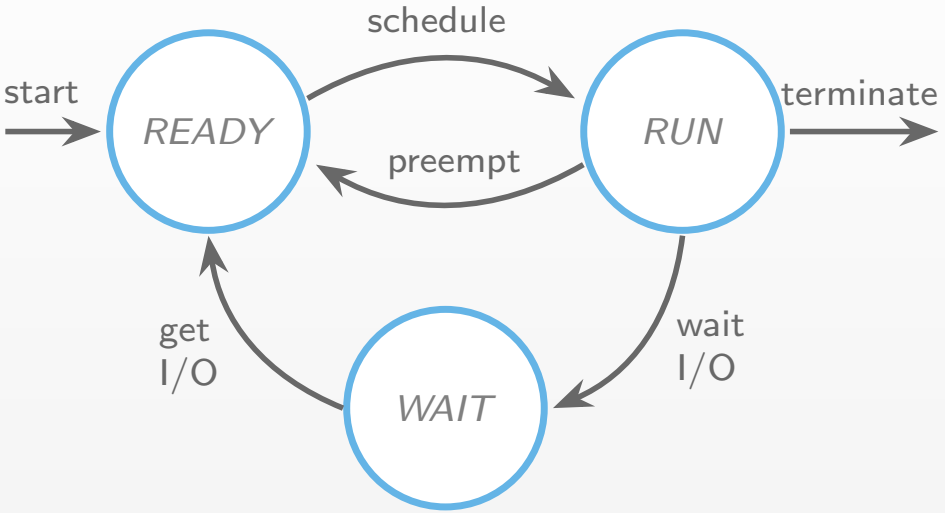
Discrete Machines



Discrete control

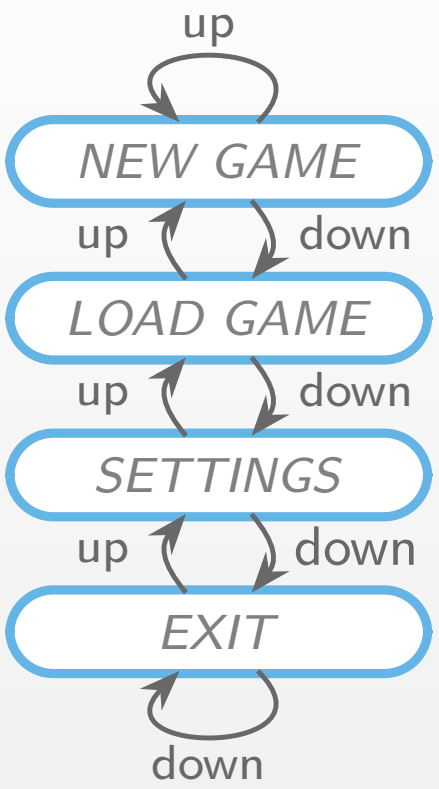
Algebra & Logics

Automata Specifications



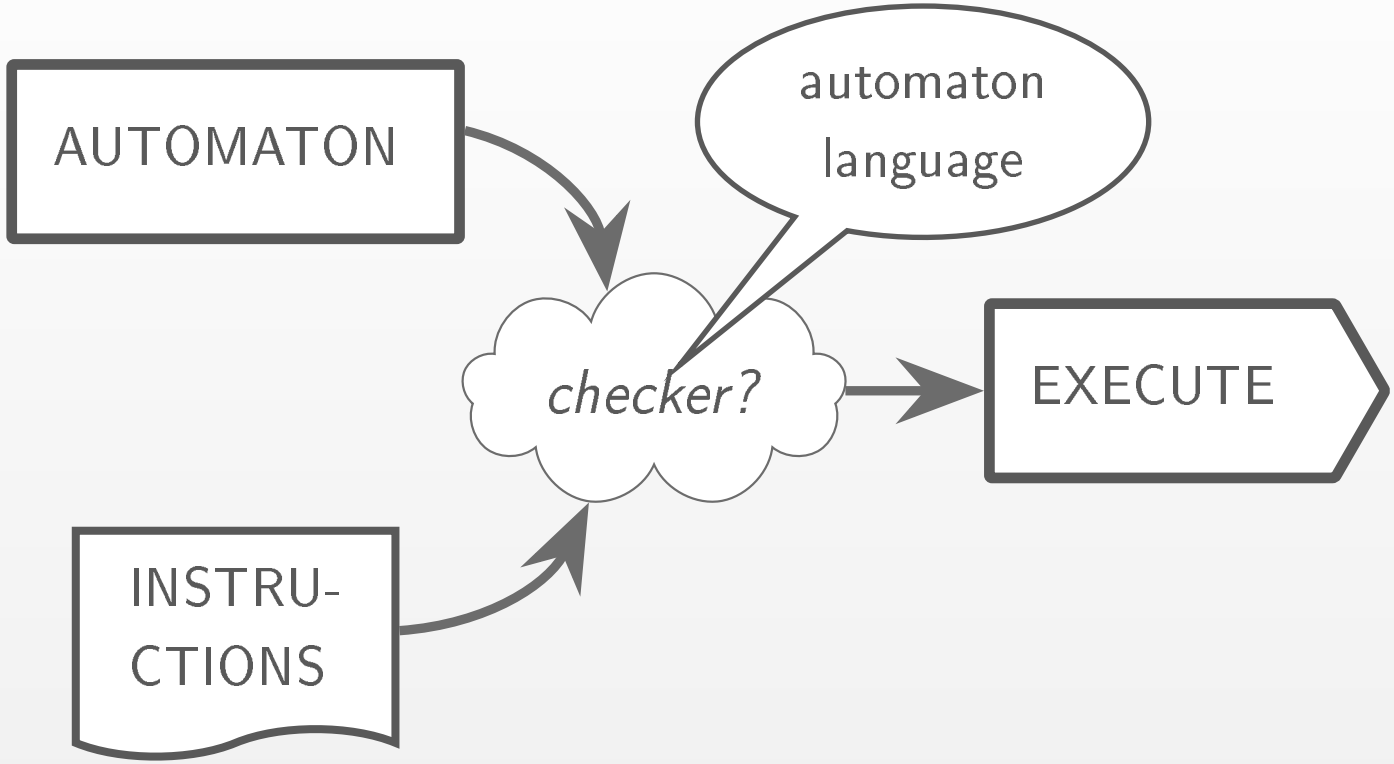
- CPU Scheduler scheme

- Automata-based schemes are used widely:
  - communication protocols;
  - user interfaces;
  - control units;
  - optimisation cycles.



- Simple game menu

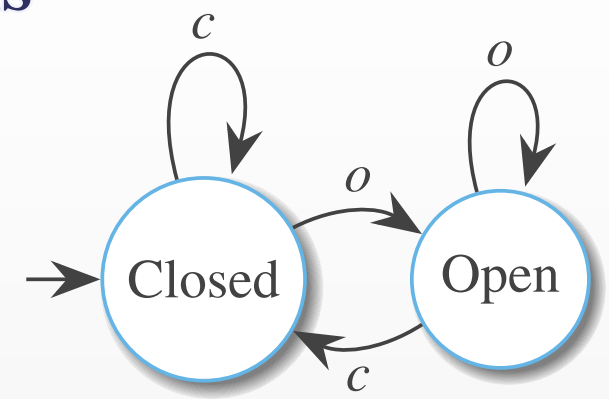
Programming: Automata Languages



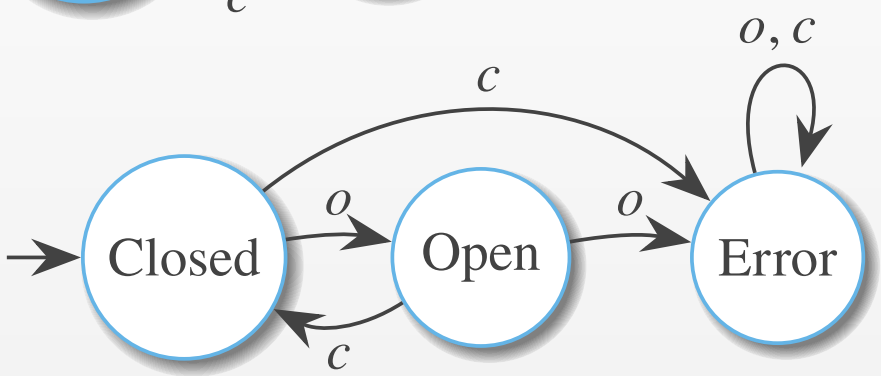
- One-by-one input steps ⇒ automaton model is enough.
- Sequence of multiple steps given simultaneously ⇒ executor is required to verify the sequence wrt the automata model.
- Admissible step sequences — *a formal language* of the machine.

Automata Models

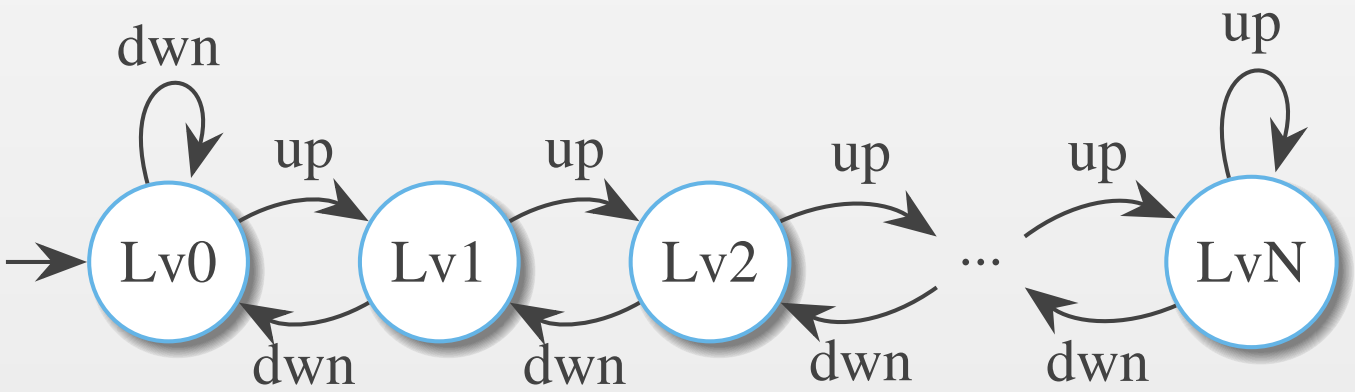
Door automaton:



Channel automaton:



Elevator:



Prehistoric Era of Formalization

Quando chel cubo con le cose appresso
Se aqquaglia a qualche numero discreto
Trouan duo altri di erenti in esso
Dapoi terrai questo per consueto
Che llor productto sempre sia equale
Alterzo cubo delle cose neto,
El residuo poi suo generale
Delli lor lati cubi ben sottrati
Varra la tua cosa principale.
...

Tartaglia, 1539

$3x^2 + 1 - (10x^3 + 2x) = 4$ in Diophantus's notation:

$$\Delta^{\upsilon}\gamma\mathring{M}\alpha \Uparrow K^{\upsilon}\iota\varsigma\beta\iota^{\sigma}\mathring{M}\delta$$

Antique Era of Formal Languages

Von Dyck's Theorem, 1882

Let X be a set and let R be a set of reduced words on X. Assume that a group G has the presentation $\langle X \mid R \rangle$. If H is any group generated by X and H satisfies the relations of G, i.e., $\omega = 1$ in H for all $\omega \in R$, then there is a surjective group homomorphism from G to H.

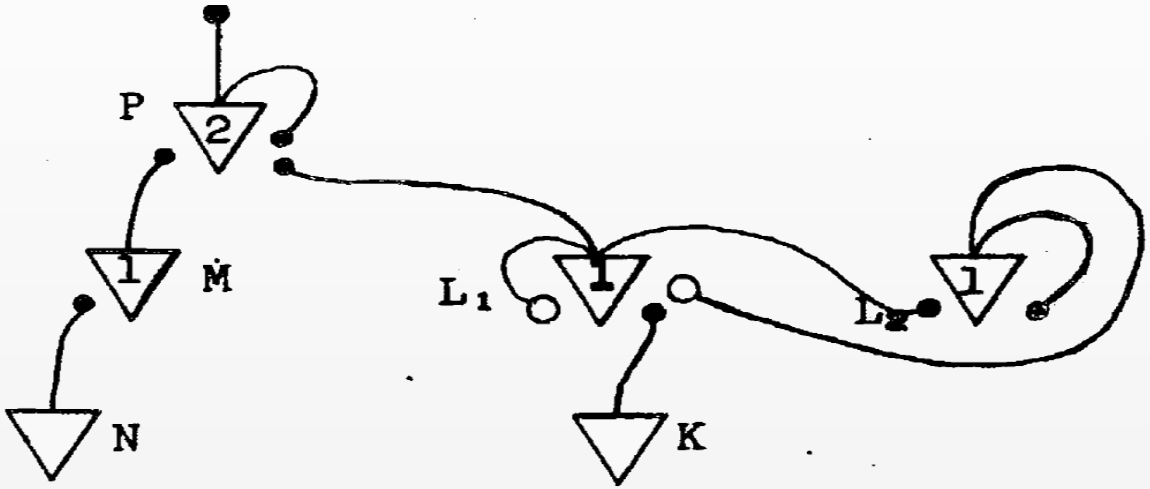
Axel Thue's Theorems on Unavoidable Patterns, 1906

In the alphabet $\{a, b, c, d\}$, there exists an infinite square-free word (i.e. not containing adjacent equal factors, or not matching a pattern $xwwy$ with w non-empty).

Later, in 1912, A. Thue improved his result for the alphabet $\{a, b, c\}$.

Renaissance of Formal Languages

McCulloch& Pitts Neural Networks:

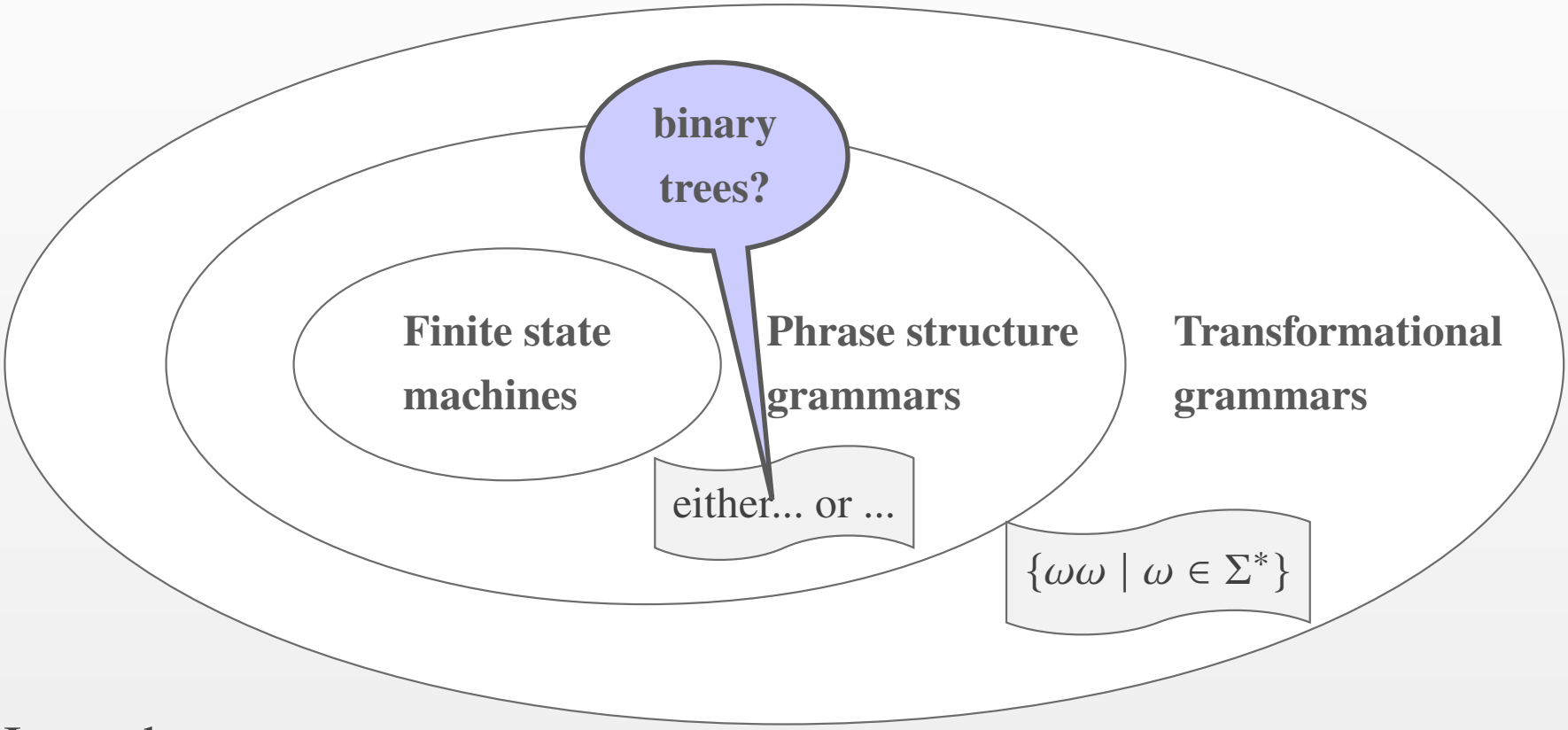


- ● — excitatory signal;
- ○ — inhibitory signal;
- ▽ — an input neuron;
- $\overset{k}{\triangledown}$ — an inner neuron firing whenever none of the inhibitory signals and at least k of excitatory signals fire.

Kleene (1951) introduced regular languages, describing the events in McCulloch–Pitts NN in terms of three operations: $\{\cdot, \cup, ^*\}$.

Noam Chomsky and Language Hierarchy

Three Models For The Description of Language, 1956:



In modern terms:

- phrase-structure grammars (1956) $\Rightarrow$ context-free grammars
- transformational grammars (1956) $\Rightarrow$ recursively-enumerable grammars

Binary Trees and Von Dyck's Language

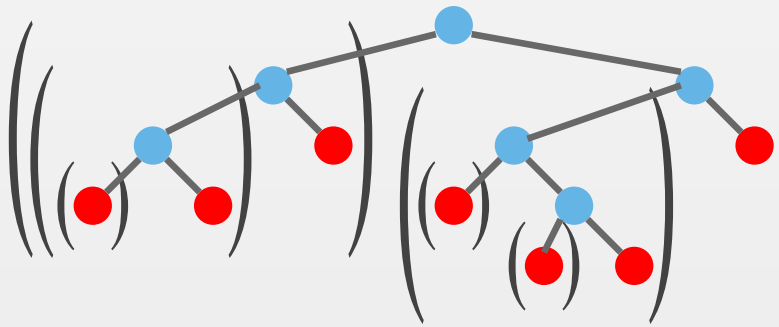
👀 Dyck's Language

One-sided (von) Dyck's language is the language of balanced parentheses.

One is able to code tree structures with strings using the parentheses language.

- One-to-one correspondence between words of length $2 \cdot n$ and binary trees with n inner vertices (using $V_P \longrightarrow (V_R)V_L$).

$$\Big(\big(\big(\,\big)\big)\Big)\big(\,\big)\big(\,\big) \longleftrightarrow$$



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Two-sided Dyck's languages are the languages of words in a free group that are reduced to 1.

Text Analysis and Forbidden Patterns

$\overbrace{xx\,yy\,zz\,ww}^{x,\,y,\,z,\,w \text{ non-empty}}$

- Russian and English texts naturally avoid the pattern $x^2y^2z^2w^2$.
- Still, pattern $x^2y^2z^2w^2$ occurs in everyday Chinese language, e.g.:

明明白白清清楚楚

- Any well-composed natural text and programming language source code written by hand should avoid the pattern x^2 (tandem repeats) with $\underbrace{|x|}_{\text{length of } x}$ large enough.
- Still, generated and auto-optimised code can contain this pattern naturally (e.g. in loop unrolling).
- Within DNA, tandem repeats (and even more, multiple repeats x^n with n large enough) are a crucial part of the gene code.

Capturing Invariants with Formal Languages

```
while (x > 0) {
 // some code not involving x and y
 x = x - y;
 }
```

- Minimal formal language validating loop termination: $\{= 0, \neq 0\}$.
- Better: $\{-, 0, +\}$.
- Possibly more precise, also for overflows: intervals.

```
while (x.contains('<script>')) {
 // x = x.replace('<', '< '); // invalid
 // x = x.replaceAll('<', '< '); // valid
 // x = x.replace('<', '&lt;'); // valid
  x = x.replace('<s', '< s'); } // valid
```

occurrences of substring <s

- Minimal language validating loop termination: tracking $\overbrace{|x|_{\texttt{<s}}}$.

Formal Languages Formally

👀 Definition

Let us consider an algebra $\mathcal{A} = \langle \underbrace{\mathcal{M}}_{\text{carrier}}, \overbrace{\mathcal{F}}^{\text{signature}} \rangle$.

A formal language is a set $\mathcal{M}$ *of terms in algebra* $\mathcal{A}$.

- Classical case: if $\mathcal{A} = \langle \Sigma, \cdot \rangle$, where $\cdot$ is the concatenation operation, then $\mathcal{M} \subseteq \Sigma^*$, where * is iteration (Kleene star, Kleene closure).
- Wider scope: tree automata languages, syntax trees, process graphs...

Formal Languages Examples

syntax

- $\{\underbrace{aa\ldots a}_{n \text{ times}}\ \underbrace{bb\ldots b}_{n\cdot 3 \text{ times}} \mid n \in \mathbb{N}\}$ (henceforth $\{a^n b^{3n} \mid n \in \mathbb{N}\}$);
- words containing a square number of letters a

  $$\left\{w \mid \exists k(\underbrace{|w|_a}_{\text{number of letters } a \text{ in word } w} = k^2)\right\};$$

- Russian palindromes of the even length

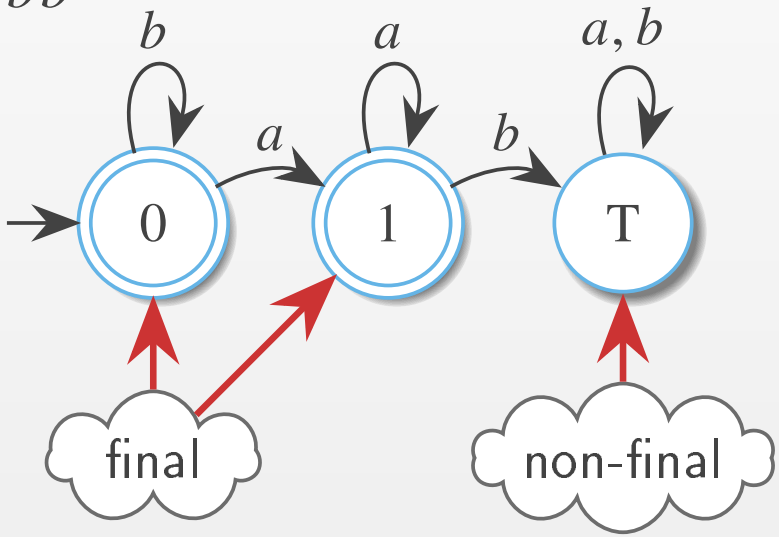
  («*Я слипся, я слился, я спился*»);
- sequences of balanced parentheses;
- well-formed arithmetic expressions over $\mathbb{N}$ and $\{\cdot, +\}$.

semantics

- all tautologies in the classical logic;
- all consistently typed programs in Python;
- formal languages with linear-time-decidable membership.

Varieties of FL Representations

- Set comprehension: $\{w \in \{a, b\}^* \mid w \text{ does not include } ab\}$, or $\{w \mid |w|_{ab} = 0 \;\&\; w \in \{a, b\}^*\}$.
- Algebraic expressions: $b^* a^*$.
- Term rewriting systems: $\begin{cases} a \to aa \\ b \to bb \end{cases}$, base set $\{\overbrace{\varepsilon}^{\text{empty word}}, a, b, ba\}$.
- Recognising machines:



- First-order or second-order logical formulas:
$\forall x, y \big(\underbrace{Q_a(x)}_{x \text{ is equal to } a} \;\&\; \underbrace{S(x, y)}_{y \text{ succeeds } x} \;\Rightarrow\; \underbrace{\neg Q_b(y)}_{y \text{ is not equal to } b}\big)$.

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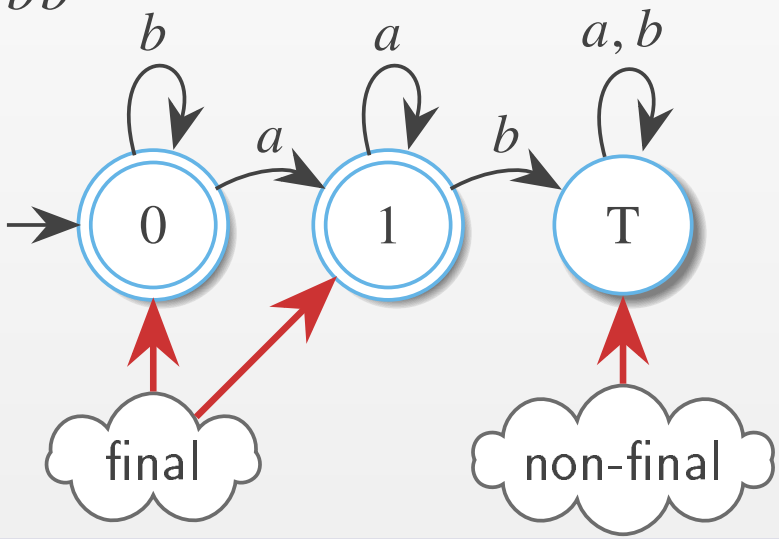


Automata-based representation easily reduces to Turing machines $\Rightarrow$ useful when estimating computational complexity of the language recognition.

The Empty Word Notation

- In modern computer science (2000s and later) — epsilon ε.
- Before 2000s — lambda λ.

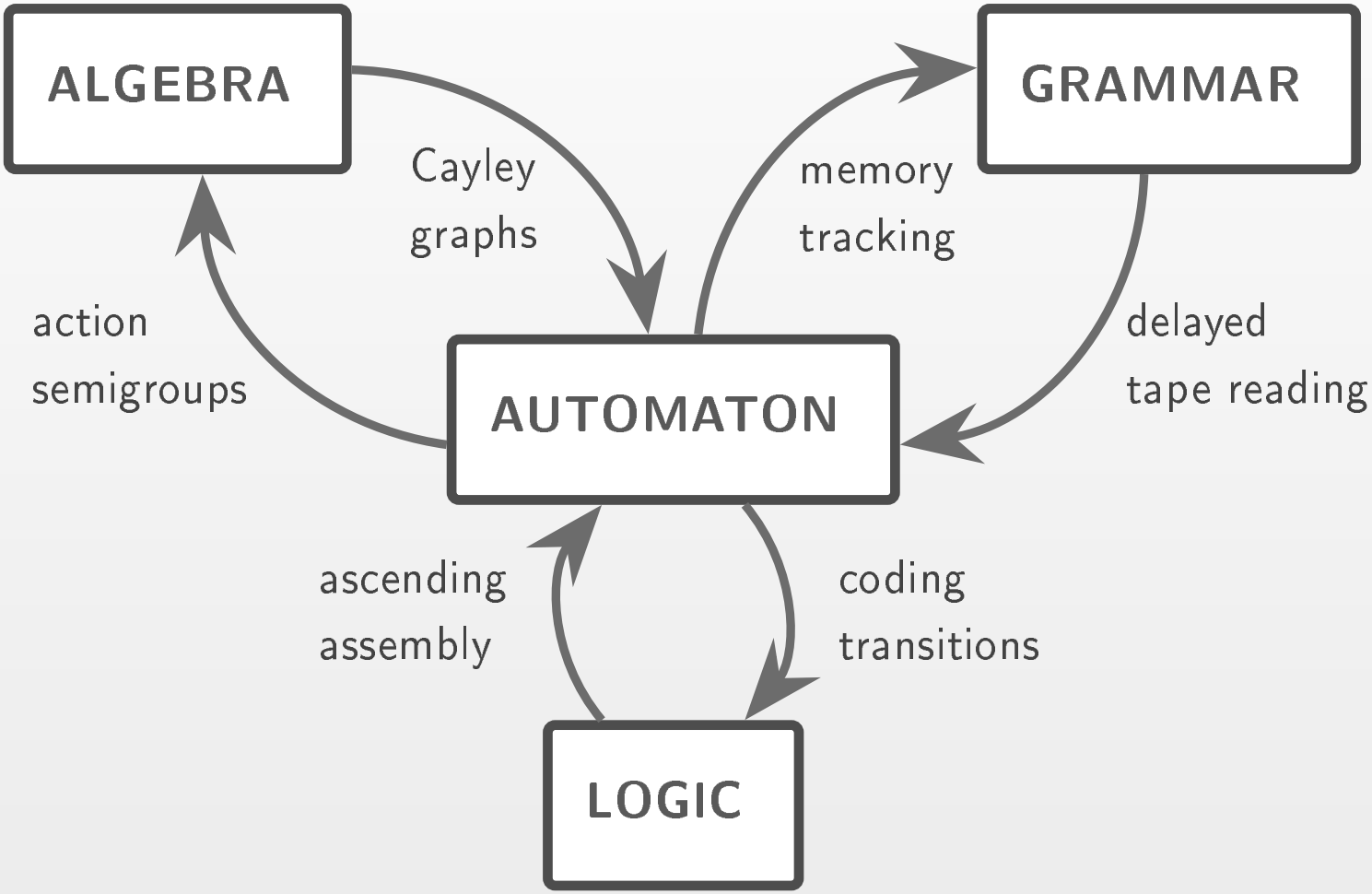
The reason: λ-notation for anonymous functions. $\overbrace{\lambda\text{Arg.Body}}^{\text{in theory}}$ or $\overbrace{\lambda\text{Arg} \to \text{Body}}^{\text{in applied research}}$, although known from 1930-s, became widespread and met formal language theory.

Compare: $\lambda\langle q_i, \lambda, q_j\rangle.\lambda$ and $\lambda\langle q_i, \varepsilon, q_j\rangle.\varepsilon$.

Or: $\lambda\left(q_i \xrightarrow{\lambda} q_j\right) \to \lambda$ and $\lambda\left(q_i \xrightarrow{\varepsilon} q_j\right) \to \varepsilon$.

Transforming & Analysing Representations

- *Efficient conversions between representation classes*
- *Constructing an optimal form of a representation inside a class*



Spherical Cows in RoFL Course

What is a reason to study artificial languages like «*words not containing subwords ba*»?

(common knowledge):
A cow is homeomorphic to a sphere with a couple of handles.

- Transducer images are shorter and their intermediate interpretations are more sheer;
- The transducer images reveal the essence of abstract formal properties rather than features of the ad-hoc model;
- In fact, morphisms and FSM transformations can be considered as lexers into the theoretical scope.

Course Structure

- Introduction and TRS
- I part: automata
  - Rewriting and Syntactic Monoids;
  - Finite Automata;
  - Visibly Pushdown Automata;
  - Generic Pushdown Automata;
  - Deterministic Pushdown Automata;
  - Alternating Automata, Memory Automata;
  - Cellular Automata, Tree Automata + *Midterm I*.
- II part: languages
  - Semirings;
  - Ehrenfeucht–Fraïssé games and Pumping;
  - Hardest Languages and Language Representations;
  - Language Inference;
  - Combinatorial & Computational Language Properties.
- *Midterm II*
- Concluding Lecture
- *«RoFL Farm»*

Course Scores

👀 Score Arrangement

- ***Midterms*** $\times$ *2* $\times$ *15+.*
- ***Assignments*** $\times$ *5* $\times$ *8+.*
  - *Java, Python, Go, JS — no score bonus*
  - *C/C++, Kotlin, Dart, TypeScript, Lua, Julia — +1 point*
  - *Rust, Lisp dialects, Scala — +2 points*
  - *Haskell, Erlang — +3 points*
  - *Рефал — +4 points for first time +3 afterwards*

- **Deadlines for assignments**:
  - 0-14 days — no penalty
  - 15-21 days — 1 score penalty
  - 22-28 days — 2 score penalty
  - 29-∞ — 3 score penalty

Reputation Count and Queuing

Initial rep = 100 points for everyone.

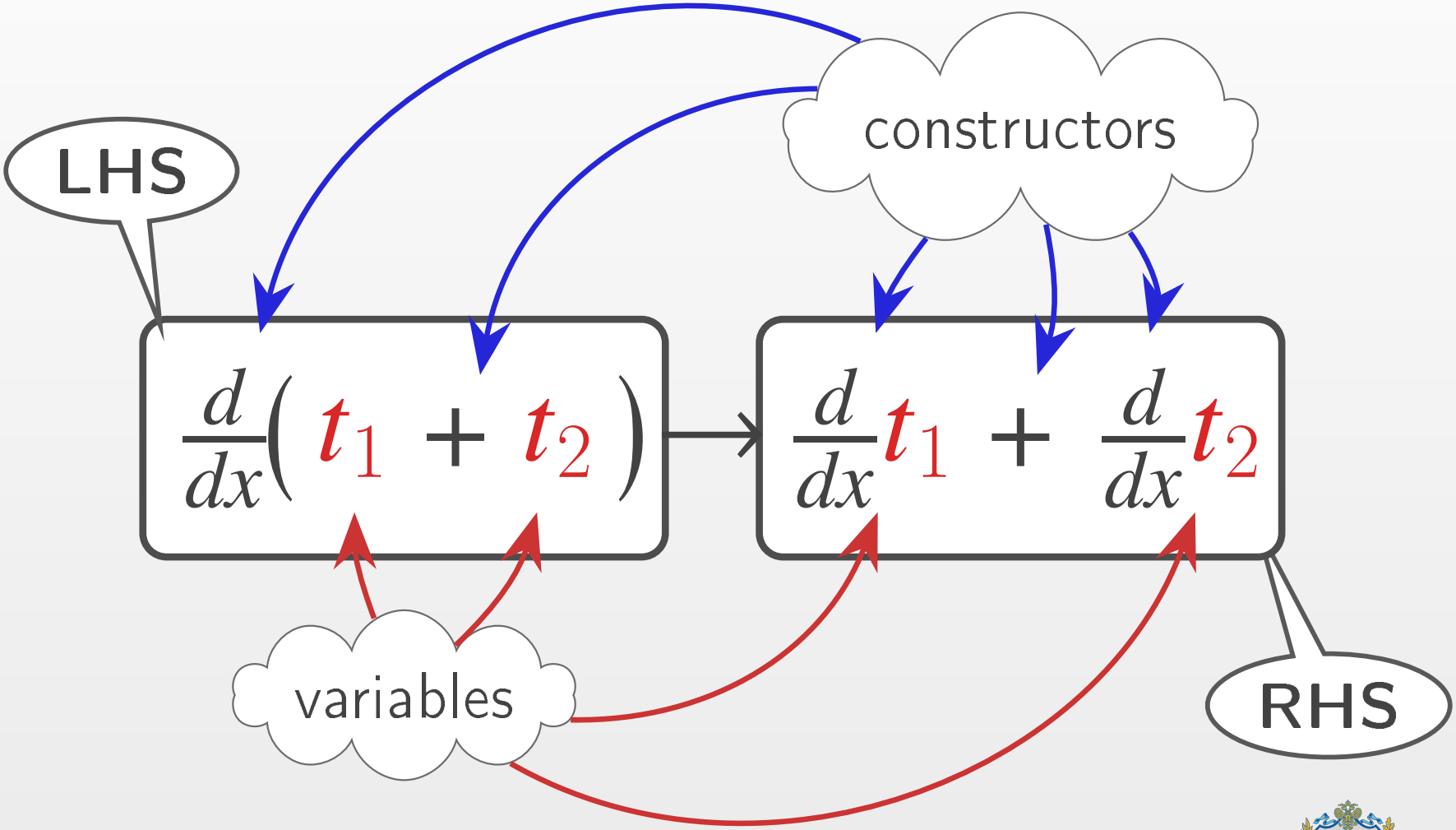
- Submit a work at deadline daytime: rep -2.
- Submit a work at deadline night: rep -5 .
- Submit a work on the eve of Reset Event: rep-10.
- Artefacts of someone's else code: rep-25.
- Unable to explain self code: rep-50.
- Submit a work on the eve of exam (daytime): rep-35.
- Submit a work on the eve of exam (night): rep-50.
- Submit a work not assigned to self: rep-75.
- Other (violating RoFL Farm rules, etc) — ad hoc.

The assignments are considered in ***descending order*** *of rep!*

$\text{rep} \leq 0 \Rightarrow$ *personal task arrangement.*

Term Rewriting and All That

Any symbolic computation is a term rewriting process, controlled by an appropriate set of *rewrite rules* — *term rewriting system*.

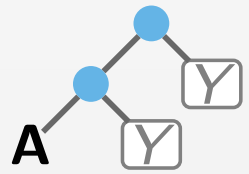


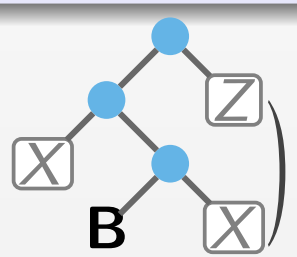
Unification and Pattern Matching

A rewriting process *unifies* term t and LHS s_1 of a rule $s_1 \longrightarrow s_2$ by constructing a substitution.

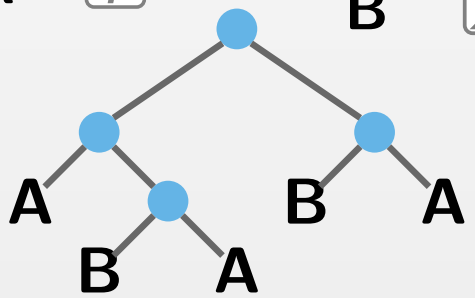
👀 Unification Formally

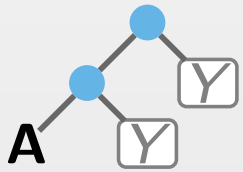
*Given two terms t_1, t_2, **a unifier** is a pair of variable substitutions $\langle \theta_1, \theta_2 \rangle$ s.t. $t_1\theta_1 = t_2\theta_2$.*

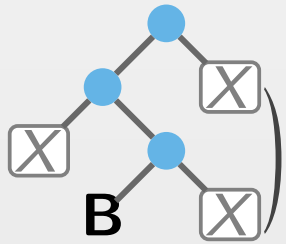
If Eq(x, x) $\longrightarrow$True, when Eq$\Big($ 

, 

$\Big)$ is rewritten?

Unification is possible when $x =$ 

Term Eq$\Big($ 

, 

$\Big)$ is never unified with Eq(x, x).

Notation Clash: Substitutions

In algebraic courses, given a substitution σ, its application to a term Φ is denoted with $\sigma(\Phi)$.
In mathematical logic & computer science, a ***postfix notation*** $\Phi\sigma$ is more usual.

Origins of the Notation

In classical mathematical logic textbooks (Tarski, Curry) the substitutions were denoted by $[x/A]$. Hence, the postfix notation was natural: formula $F(x,x)[x/A]$ is pretty more readable than $[x/A](F(x,x))$.

In modern computer science, the notations $[x := A]$ are $[x \mapsto A]$ are both widely used.

Reduction, Redexes, Normal Forms

Given a TRS $\left\{\Phi_i \to \Psi_i\right\}_{i=1}^{n}$ and a term T,

👀 Basic Notions of Rewriting

- *A **redex** is a subterm T_0 that can be unified with some Φ_i by unifier $\langle\theta_1, \theta_2\rangle$.*
- ***Reduction** replaces $T_0\theta_1$ in T by $\Psi_i\theta_2$.*
- *Term T is in **a normal form** if T contains no redex.*
- *Normalisation — reduction to the normal form: $T \twoheadrightarrow T'$.*

Given TRS $\begin{cases} 0 + x \to x \\ 0 \cdot x \to 0 \end{cases}$, the term $(0 + 1) \cdot 0$ contains a redex $0 + 1$, and the reduced term $1 \cdot 0$ is in the normal form.

Adding commutativity rules results in unrestricted reductions.

α-conversions and α-equivalence

- If x, z are variables, the rewriting rules $\{\mathrm{Eq}(x, x) \to \mathrm{True}\}$ and $\{\mathrm{Eq}(z, z) \to \mathrm{True}\}$ are *equivalent*; while the rule $\{\mathrm{Eq}(x, z) \to \mathrm{True}\}$ is not equivalent to both.

- α-conversion is a semantic-preserving variable renaming. Non-trivial in case of bound variables.

λ-calculus

- Constructors: $\mathrm{Apply}(M, N)$ and $\lambda x.M$, binding x in M.
- Rewriting rule: $\mathrm{Apply}\big((\lambda x.M),\ N\big) \to M[x := N]$

Reduction + capture-avoiding substitution = universal computation model.

Proving Properties of FL and Term Ordering

Check that the language $\mathscr{L} = \{w \mid |w|_a \text{ is even}\}$ *is the set of normal forms generated from term srt* $\{S\}$ *using the following TRS* $\mathcal{T}$*:* $S \to a\,S\,a \quad S \to b\,S \quad S \to S\,b \quad S \to \varepsilon$

- Check that all words from $\mathscr{L}$ can be generated by $\mathcal{T}$.
- Check that all the normal forms η s.t $S \to^* \eta$ are in $\mathscr{L}$.

arbitrary number of steps

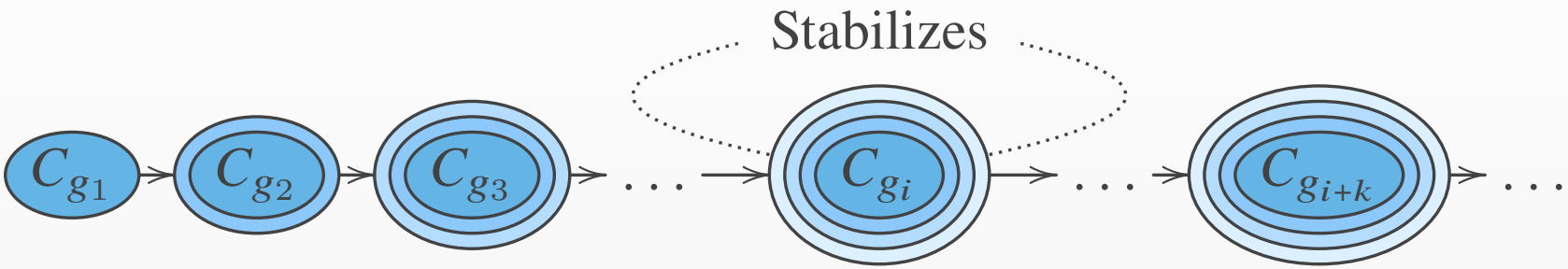
Assume that there exists w s.t. $S \to^* w$ and $|w|_a$ is odd. The reduction step before the application of $\{S \to \varepsilon\}$ can always be replaced by $\{S \to \varepsilon\} \Rightarrow$ making the word w shorter. We can repeat this process to ∞, contradicting that $|w|$ is finite.

Key moment: the length ordering admits no infinite descending chains.

Many Arrows, Many Notions

- **definitive arrows** (define functions, types, etc): plain $\to$.
  E.g. $\begin{cases} f : \Sigma^* \to \mathbb{N} & \text{(a function mapping strings to naturals)} \\ Sa \to aS & \text{(a rule replacing a subterm } Sa \text{ with } aS) \end{cases}$
- **closures wrt rewriting** (define closure functions): $\to^*$.
  E.g. $\begin{cases} S \to^* S & (S \text{ is a result of rewriting } S \text{ zero or more times}) \\ \omega \to^* \varepsilon & (\omega \text{ can be collapsed to the empty string eventually}) \end{cases}$
- **substitutions** (denote applications of concrete replacing rules): $\mapsto$.
  Depending on context, either exhaustive or one-time replacement.
  E.g. $\begin{cases} S \mapsto a & (S \text{ should be replaced with } a) \\ aaaa \xrightarrow{aa \mapsto a}{}^* aa & (aa \text{ results from a substitution composition}) \end{cases}$
- **normalizing arrows** (denote exhaustive rewriting): $\twoheadrightarrow$. E.g.
  $\begin{cases} S \twoheadrightarrow a & (S \text{ is rewritten to } a, \text{ and } a \text{ cannot be rewritten further}) \\ aaaa \overset{aa \mapsto a}{\not\twoheadrightarrow} aa & (aa \text{ is not a final result of the iterative replacements}) \end{cases}$

Noetherian Orders and Well-Founded Orders



👀 Ordering as an Induction Base

- *A preorder $\preceq$ is **well-founded** on set $\mathcal{A}$ if it admits no infinite descending chains, i.e. all chains $t_0 \succeq t_1 \succeq \ldots$ are finite.*
- *A preorder is **Noetherian** on set $\mathcal{A}$ if it stabilizes upwards (no infinite strictly ascending chains, abbreviated ACC).*

Well-Founded Monotone Algebras

A preorder $\preceq$ is monotone on $\mathcal{A}$, if

$$\forall f, t_1, .., t_n, s, s' \in \mathcal{A}\Big(s \preceq s' \Rightarrow f(t_1, .., s, .., t_n) \preceq f(t_1, .., s', .., t_n)\Big),$$

and is strictly monotone, if converse always fails.

Well-Founded Monotone Algebras

WFMA over the signature F on a well-founded set $\langle \mathcal{A}, > \rangle$ is an algebra s.t. for any $\langle f, n \rangle \in F$ there exists a function $f_{\mathcal{A}} : \mathcal{A}^n \to \mathcal{A}$, called **an interpretation** of f, being strictly monotone wrt all its arguments.

- $\langle \mathbb{N}, > \rangle$ can be a WFMA over a signature $(f_1, 2), (f_2, 1)$, given e.g. interpretations $f_1(x, y) = x + y + 1$, $f_2(x) = 2 \cdot x$;
- $\langle \mathbb{Q}^+, > \rangle$ and $\langle \mathbb{Z}, > \rangle$ are not WFMAs.

TRS Termination

Given a variable set $\mathcal{V}$, extension of $\sigma : \mathcal{V} \to \mathcal{A}$ is defined as:

- $[x, \sigma] = x\sigma$;
- $[f(t_1, \ldots, t_n), \sigma] = f_{\mathcal{A}}([t_1, \sigma], \ldots, [t_n, \sigma])$.

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*A TRS $\{l_i \to r_i\}$ is **compatible** with WFMA $\mathcal{A} \Leftrightarrow$ for all i, σ condition $[l_i, \sigma] > [r_i, \sigma]$ holds.*

Informally: the image of l_i is greater than the image of r_i for the extension of any substitution $\sigma : \mathcal{V} \to \mathcal{A}$.

Consider TRS $\{f(f(x)) \to f(x)\}$ and WFMA carrier $\mathcal{N} = \langle \mathbb{N}, > \rangle$.

- *Let $f_{\mathcal{N}}(x) = 2 \cdot x$ and $\sigma = [x := 0]$: $f_{\mathcal{N}}(f_{\mathcal{N}}(x))\sigma = f_{\mathcal{N}}(x)\sigma = 0$, $\Rightarrow$ WFMA is not compatible with the TRS.*
- *Let $f_{\mathcal{N}}(x) = x + 1$: $\forall x \big(f_{\mathcal{N}}(f_{\mathcal{N}}(x)) - f_{\mathcal{N}}(x) = 1\big) \Rightarrow$ the TRS and WFMA are compatible.*

WFMA and TRS

Main Theorem on TRS Termination

A TRS $\{l_i \to r_i\}$ terminates $\Leftrightarrow$ there exists a WFMA compatible with $\{l_i \to r_i\}$.

The TRS $\left\{\frac{d}{dx}(t_1 + t_2) \to \frac{d}{dx}t_1 + \frac{d}{dx}t_2\right\}$ *is terminating, which is verified by WFMA* $\mathcal{N}$ *over* $\langle \mathbb{N}, > \rangle$, $\mathbb{N}$ *not including* 0*:*

- $+_{\mathcal{N}}(u, v) = u + v + 1$;
- $\frac{d}{dx}_{\mathcal{N}}(u) = 2 \cdot u$.

Indeed, $\frac{d}{dx}_{\mathcal{N}}(t_1 +_{\mathcal{N}} t_2) = 2 \cdot (t_1 + t_2 + 1) = 2 \cdot t_1 + 2 \cdot t_2 + 2$, which is strictly greater than
$\frac{d}{dx}_{\mathcal{N}} t_1 +_{\mathcal{N}} \frac{d}{dx}_{\mathcal{N}} t_2 = 2 \cdot t_1 + 2 \cdot t_2 + 1$.

Choosing a Monotone Interpretation

If $\mathcal{A}$ is well-founded and $f_{\mathcal{A}}$ is strictly monotone, then $\forall a\big(f_{\mathcal{A}}(a) \succeq a\big)$.

Indeed, let $a \succ f_{\mathcal{A}}(a)$. Then the chain

$$a, f_{\mathcal{A}}(a), f_{\mathcal{A}}(f_{\mathcal{A}}(a)), \ldots, f^n_{\mathcal{A}}(a), \ldots$$

is infinitely descending.

Good choices of functions over $\mathcal{A}$:

- given $\mathcal{A} \subseteq \mathbb{N}$, extended polynomials strictly increasing on $\mathcal{A}$, e.g. $f(x) = x \cdot \lfloor \log_2 x \rfloor$ on $[4; +\infty)$;
- given $\mathcal{A} \subseteq \mathbb{N}^k$, functions strictly increasing lexicographically on $\mathcal{A}$, e.g. $f(\langle x, y \rangle) = \langle x + 1, 0 \rangle$ on $\mathbb{N}^2$.

Free Monoid (aka String DataType) as a FL Carrier

Usual case: the string data type is a carrier of a TRS; the only constructor is the string concatenation.
Then the term rewriting system becomes *a string rewriting system* (SRS): $\{l_i \to r_i\}$, where l_i, r_i are strings.

👀 Terminal & Nonterminal Symbols

Given disjoint alphabets N, Σ, we say that elements of Σ are terminal symbols, and elements of N are non-terminals in a given SRS $\{l_i \to r_i\}$, if words $w \in N \cup \Sigma$ containing letters from N are considered as partially computed (even if they are in the normal form).

Formal Grammars

👀 Definition

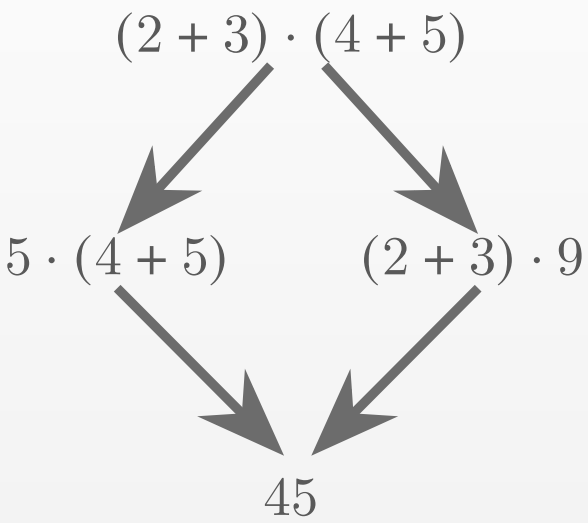
A grammar *is a tuple* $G = \langle N, \Sigma, P, S \rangle$*, where:*

- N *— non-terminal alphabet;*
- Σ *— terminal alphabet;*
- P *— string rewriting system* $\{\alpha_i \to \beta_i\}$*, where* α_i *is non-empty;*
- $S \in N$ *is an initial nonterminal.*

A language $\mathscr{L}(G)$ *of a grammar* G *is the set* $\{u \mid u \in \Sigma^* \ \& \ S \to^* u\}$*, where* $\to^*$ *is a composition of reductions.*

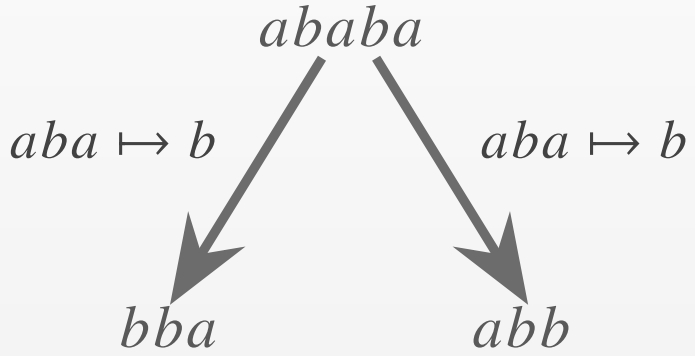
Diamond Property, or Confluence

Evaluation of an arithmetic expression:



A unique normal form exists.

Non-deterministic replacements (rewriting) in strings:



No unique normal form.

*Given a TRS $\mathcal{T}$, it is **confluent** iff*

$$\forall t_0, t_1, t_2 \exists t_3 \big(t_0 \xrightarrow{\mathcal{T}}{}^{*} t_1 \;\&\; t_0 \xrightarrow{\mathcal{T}}{}^{*} t_2 \Rightarrow t_1 \xrightarrow{\mathcal{T}}{}^{*} t_3 \;\&\; t_2 \xrightarrow{\mathcal{T}}{}^{*} t_3\big).$$

Global and Local Confluence

- (Global) confluence:

$$\forall t_0, t_1, t_2 \exists t_3 \big(\underbrace{t_0 \xrightarrow{\mathcal{T}}{}^* t_1}_{\text{arbitrary number of steps}} \;\&\; t_0 \overbrace{\xrightarrow{\mathcal{T}}{}^* t_2}^{\text{arbitrary number of steps}} \Rightarrow t_1 \xrightarrow{\mathcal{T}}{}^* t_3 \;\&\; t_2 \xrightarrow{\mathcal{T}}{}^* t_3 \big).$$

- Local confluence:

$$\forall t_0, t_1, t_2 \exists t_3 \big(\underbrace{t_0 \xrightarrow{\mathcal{T}} t_1}_{\text{one-step rewriting}} \;\&\; t_0 \overbrace{\xrightarrow{\mathcal{T}} t_2}^{\text{one-step rewriting}} \Rightarrow t_1 \xrightarrow{\mathcal{T}}{}^* t_3 \;\&\; t_2 \xrightarrow{\mathcal{T}}{}^* t_3 \big).$$

The following TRS is locally confluent, but not confluent:

$$\infty \to \infty + 1 \quad \mathrm{Eq}(x,\ x) \to \mathrm{True} \quad \mathrm{Eq}(x,\ x+1) \to \mathrm{False}$$

Global and Local Confluence

- (Global) confluence:

$$\forall t_0, t_1, t_2 \exists t_3 \big(\underbrace{t_0 \xrightarrow{\mathcal{T}}{}^* t_1}_{\text{arbitrary number of steps}} \;\&\; t_0 \overbrace{\xrightarrow{\mathcal{T}}{}^* t_2}^{\text{arbitrary number of steps}} \Rightarrow t_1 \xrightarrow{\mathcal{T}}{}^* t_3 \;\&\; t_2 \xrightarrow{\mathcal{T}}{}^* t_3 \big).$$

- Local confluence:

$$\forall t_0, t_1, t_2 \exists t_3 \big(\underbrace{t_0 \xrightarrow{\mathcal{T}} t_1}_{\text{one-step rewriting}} \;\&\; t_0 \overbrace{\xrightarrow{\mathcal{T}} t_2}^{\text{one-step rewriting}} \Rightarrow t_1 \xrightarrow{\mathcal{T}}{}^* t_3 \;\&\; t_2 \xrightarrow{\mathcal{T}}{}^* t_3 \big).$$

👀 **Newman's Lemma**

If TRS is terminating, then local and global confluence properties coincide.

Critical Pairs

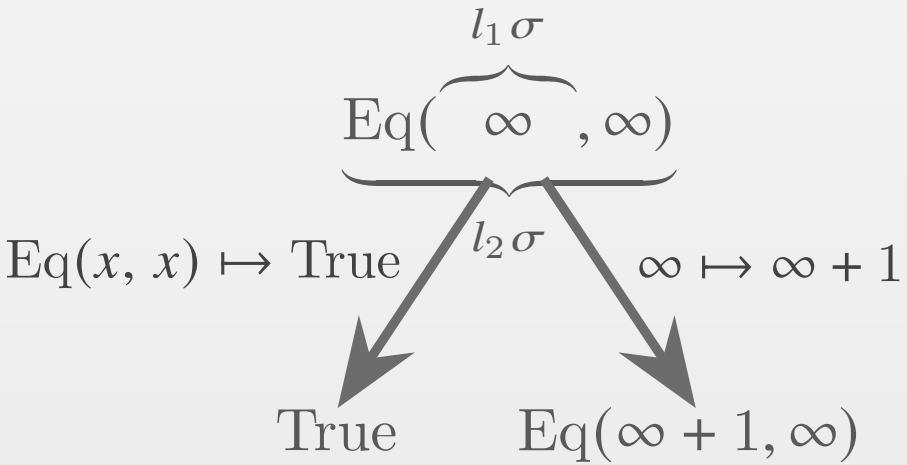
👀 Definition

Given TRS $\mathcal{T}$, its rules $l_1 \to r_1$ and $l_2 \to r_2$ are said to form ***a critical pair****, iff there exists a term t and substitution σ, s.t. $l_1\sigma$ and $l_2\sigma$ are overlapping subterms of t.*
The given critical pair is $\langle (t)[l_1\sigma \mapsto r_1\sigma],\ (t)[l_2\sigma \mapsto r_2\sigma] \rangle$.

Consider:

$$\infty \to \infty + 1 \quad \mathrm{Eq}(x,\ x) \to \mathrm{True} \quad \mathrm{Eq}(x,\ x+1) \to \mathrm{False}$$

Term $\mathrm{Eq}(\infty, \infty)$ has a critical pair. Indeed, given $\sigma = (x \mapsto \infty)$,



Local Confluence Criterion

The terms t_1, t_2 are said to be **convergent** iff they can be reduced to a single normal form.

$\mathrm{Eq}(\infty + 1, \infty)$ is convergent with True, given the reduction chain

$$\mathrm{Eq}(\infty + 1, \infty) \to \mathrm{Eq}(\infty + 1, \infty + 1) \to \mathrm{True}$$

👀 **Critical Pair Lemma**

A TRS is locally confluent ⇔ all its critical pairs are convergent.

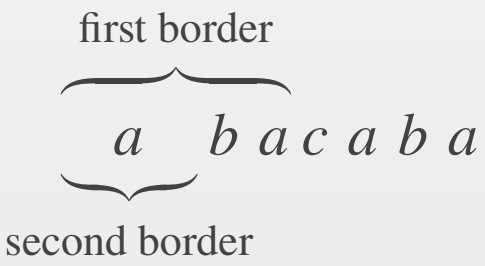
Critical Pairs in SRS

Since in an SRS substitutions are trivial, SRS rules $l_1 \to r_1$ and $l_2 \to r_2$ generate a critical pair iff a shortest common superstring t of l_1, l_2 satisfies the condition $|t| < |l_1| + |l_2|$.

NB: a rule $l \to r$ forms a critical pair with itself iff l has a non-empty **border** — i.e. a string both starting and ending l.

l has several borders $\Rightarrow$ $l \to r$ forms several critical pairs.

Given $abacaba \to c$, the terms $abacabacaba$ and $abacababacaba$ generate non-convergent critical pairs $\langle ccaba, abacc \rangle$, $\langle cbacaba, abacabc \rangle$, because the lhs $ababa$ has two distinct borders:



Knuth–Bendix Completion

Let $\succ$ be a well-founded order, $\mathcal{T} = \{l_i \to r_i\}$ be a TRS s.t. $\forall i \big(l_i \succ r_i\big)$. A sketch of the completion procedure is as follows.

$\mathcal{R} \leftarrow \mathcal{T}$ ▷ Resulting rewrite system

$P \leftarrow \bigcup_{l_i \to r_i \in \mathcal{T}} \big(\text{GetCriticalPairs}(l_i \to r_i, \mathcal{R})\big)$ ▷ Get all critical pairs in $\mathcal{T}$

while $P \neq \varnothing$ **do**

  $\langle t_1, t_2 \rangle = P.\text{pop}$ ▷ Get an unprocessed critical pair

  $\langle t_1', t_2' \rangle \leftarrow \langle \text{NormalForm}(t_1, \mathcal{R}), \text{NormalForm}(t2, \mathcal{R}) \rangle$ ▷ Reduce wrt current $\mathcal{R}$

  if $t_1' \neq t_2'$ **then**

    if $\big(t_1' \not\succ t_2' \;\&\; t_2' \not\succ t_1'\big)$ **then**

      return failure

    end if

    $l'' = \max_\succ(t_1', t_2'), r'' = \min_\succ(t_1', t_2')$ ▷ $\min_\succ$ and $\max_\succ$ are now valid

    $\mathcal{R} \leftarrow$Interreduce$(\mathcal{R} \cup \{l'' \to r''\})$ ▷ Reduce older rules wrt $l'' \to r''$

    $P \leftarrow P \cup \text{GetCriticalPairs}(l'' \to r'', \mathcal{R})$

  end if ▷ If $t_1' = t_2'$, the pair $\langle t_1, t_2 \rangle$ is convergent, so do nothing

end while

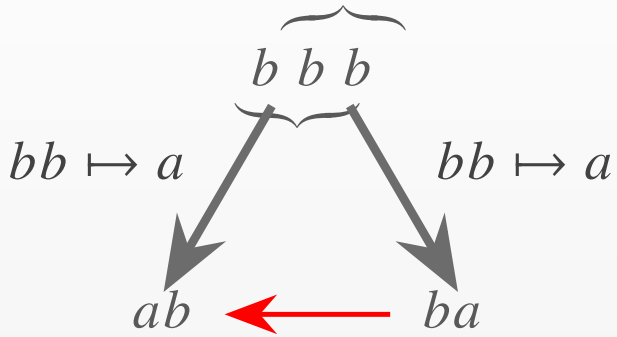
return $\mathcal{R}$

Details: pair ordering in P, efficient interreduction.

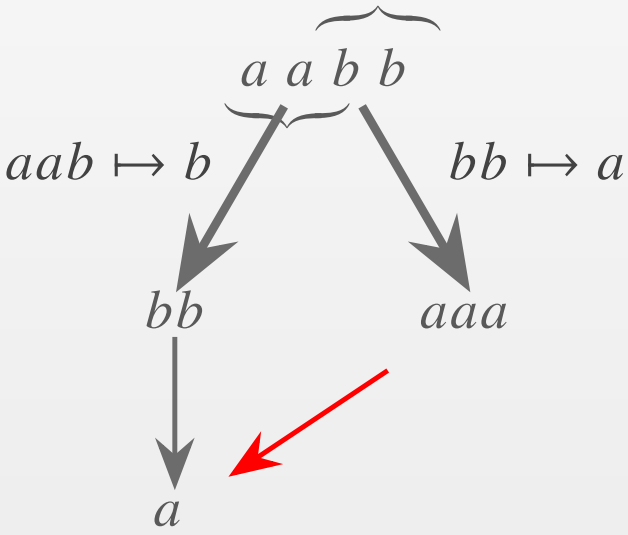
Completion Example

Take length-lexicographic order as $\succ$.

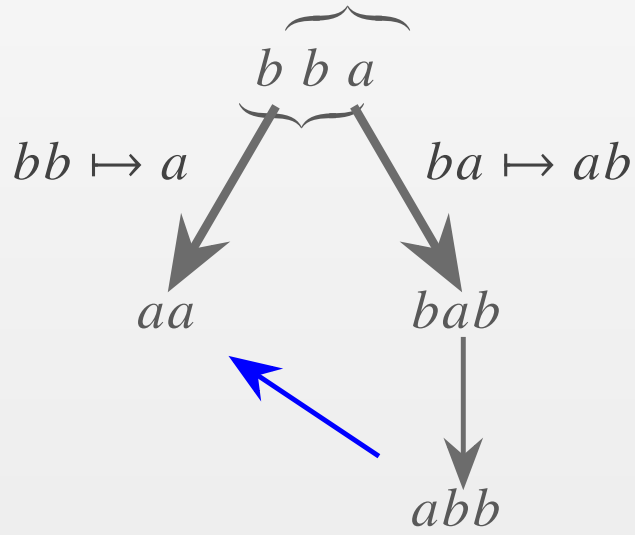
(I) $bb \to a$
(II) $aba \to b$



(I) $bb \to a$
(II) $aba \to aab \to b$
(III) $ba \to ab$



(I) $bb \to a$
(II) $aab \to b$
(III) $ba \to ab$
(IV) $aaa \to a$



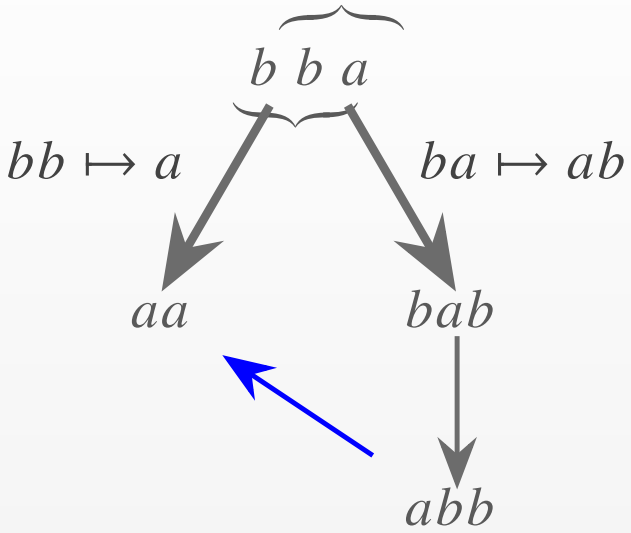
Completion Example

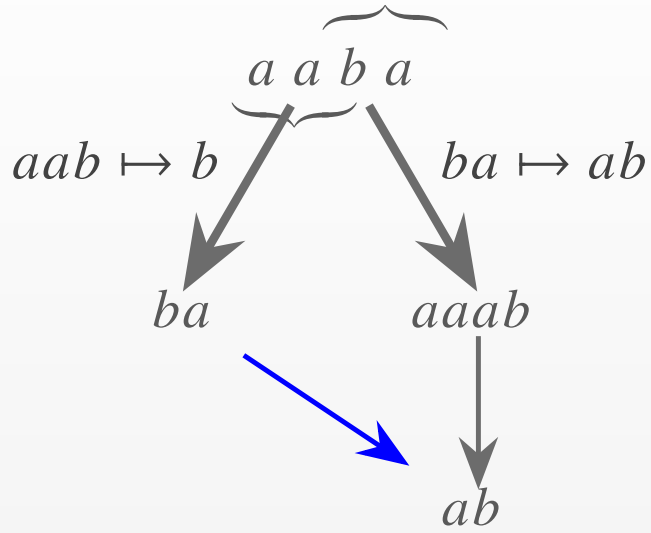
(I) $bb \to a$

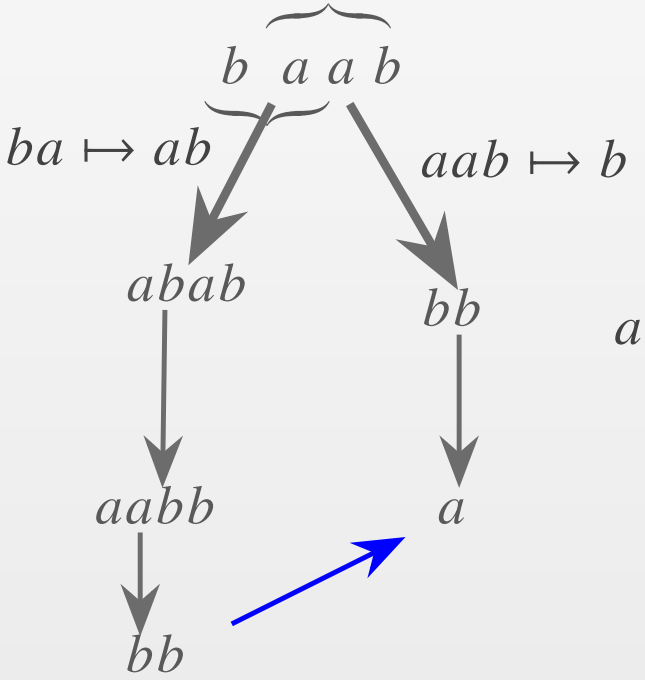
(II) $aab \to b$

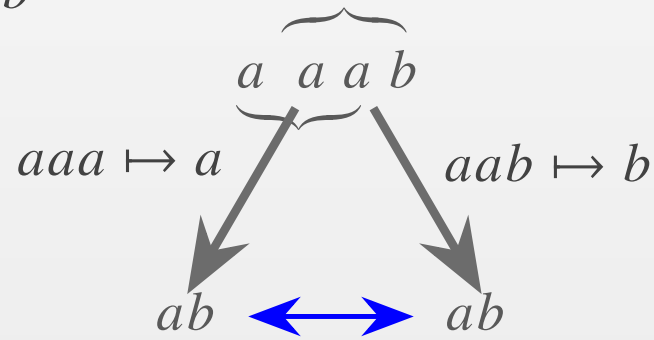
(III) $ba \to ab$

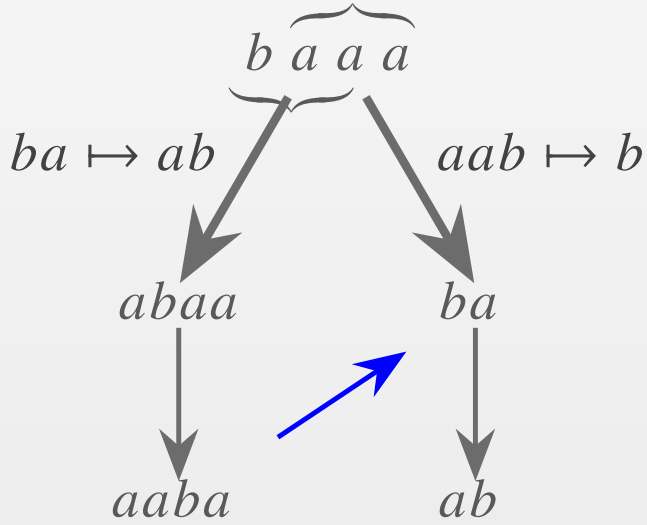
(IV) $aaa \to a$











Extracting Normal Forms

Each of normal forms in fact is a representative of a quotient set wrt the equivalence $\equiv_{\mathcal{T}} = \{\langle \omega_1, \omega_2 \rangle \mid \exists \upsilon_1, \upsilon_2 (\upsilon_1 \to^* \omega_1 \;\&\; \upsilon_1 \to^* \omega_2 \;\&\; \omega_1 \twoheadrightarrow \upsilon_2 \;\&\; \omega_2 \twoheadrightarrow \upsilon_2)\}$.

Brute-force approach: perform completion and then simply list all the irreducible strings.

(I) $bb \to a$

(II) $aab \to b$

(III) $ba \to ab$

(IV) $aaa \to a$

Irreducible words: $\{\varepsilon, a, b, aa, ab\}$.

Extracting Normal Forms

Each of normal forms in fact is a representative of a quotient set wrt the equivalence $\equiv_{\mathcal{T}} = \{\langle \omega_1, \omega_2 \rangle \mid \exists \upsilon_1, \upsilon_2 (\upsilon_1 \to^* \omega_1 \;\&\; \upsilon_1 \to^* \omega_2 \;\&\; \omega_1 \twoheadrightarrow \upsilon_2 \;\&\; \omega_2 \twoheadrightarrow \upsilon_2)\}$.

Heuristic approach: find invariants preserved by the rules and the corresponding minimal set of classes, and then prove there are no other classes.

(I) $bb \to a$

(II) $aba \to b$

Invariants:

- string non-emptiness (distinguishes ε)
- parity of $|\omega|_b$
- parity of $|\omega|_a + \left\lfloor \frac{|\omega|_b}{2} \right\rfloor$

} distinguishes a, b, aa, ab

While the heuristic approach seems involved, the skill to determine invariants is *extremely* useful. E.g. all the formal verification domain relies on finding invariant and monotone (wrt wfos) properties.

Final Twist: from NFs to Transition Systems

The SRS considered:

(I) $bb \to a$

(II) $aab \to b$

(III) $ba \to ab$

(IV) $aaa \to a$

Equivalence classes: $\overline{\varepsilon}, \overline{a}, \overline{b}, \overline{aa}, \overline{ab}$.

Finite number of classes $\Rightarrow$ finite set of transitions between them.

$\overline{\varepsilon}a \to \overline{a}$ $\quad$ $\overline{\varepsilon}b \to \overline{b}$

$\overline{a}a \to \overline{aa}$ $\quad$ $\overline{a}b \to \overline{ab}$

$\overline{b}a \to \overline{ab}$ $\quad$ $\overline{b}b \to \overline{a}$

$\overline{aa}a \to \overline{a}$ $\quad$ $\overline{aa}b \to \overline{b}$

$\overline{ab}a \to \overline{b}$ $\quad$ $\overline{ab}b \to \overline{aa}$