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Word Equations as Abstract Domain for String Manipulating Programs

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Abstract Program Properties

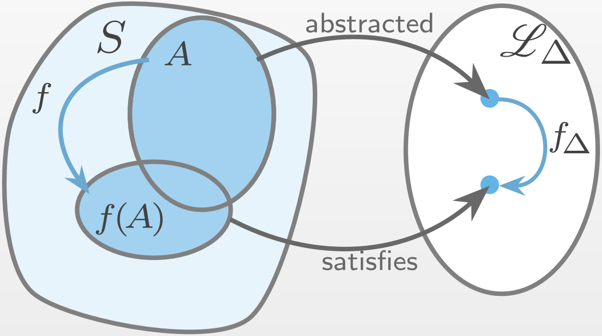
- We are interested in satisfiability of program properties w.r.t. some predicate set Δ over program data. The set Δ is to be expressed in a chosen language $\mathscr{L}_\Delta$.
- Then any program trace can be considered in terms of abstracted structures wrt preserving properties in Δ, i.e. abstracted to $\mathscr{L}_\Delta$.

Example

- Checking division by $0 \Rightarrow$ $\mathscr{L}_\Delta = \{\mathtt{IsZero}, \mathtt{IsNonZero}, \mathtt{Unknown}, \mathtt{Error}\}$.
- Checking overflows in arithmetics $\Rightarrow \mathscr{L}_\Delta$ can be an interval predicate set $\{(x \in (a, b)) \mid a, b \in \mathbb{R} \cup \{-\infty, \infty\}\} \cup \{\mathtt{Error}\}$.

Function Lifting

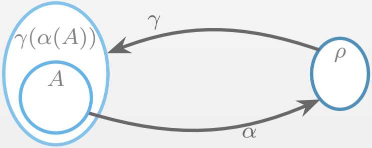
- Every function $f : S \to S$, where S is the program data domain, is to be lifted to the abstract values domain.
- The lifted image of f (denoted f_Δ) must respect the original function f properties.

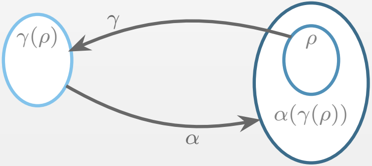


Galois Connection in Abstract Interpretation

Let S and $\mathscr{L}_\Delta$ be sets of concrete and abstract data values. We say that $a \propto \rho$ iff $a \in S$ satisfies the predicate $\rho \in \mathscr{L}_\Delta$.
The two functions are crucial to analyse the abstract properties:

- Abstraction $\alpha : 2^S \to \mathscr{L}_\Delta$;
  $\alpha(A) =$ most specific predicate $\rho \in \mathscr{L}_\Delta$ s. t. $\forall a \, (a \in A \Rightarrow a \propto \rho)$;
- Concretisation $\gamma : \mathscr{L}_\Delta \to 2^S$;
  $\gamma(\rho) = \{a \in S \mid a \propto \rho\}$.





$A \subseteq \gamma(\alpha(A))$

$\rho \overset{???}{\preceq} \alpha(\gamma(\rho))$

Such a relation between any two sets in called a Galois connection.

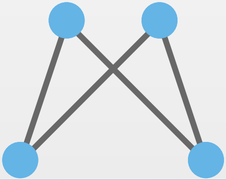
Partial Order on $\mathscr{L}_\Delta$ and Lattice Axioms

$\mathscr{L}_\Delta$ is equipped by partial order $\preceq$ and a pair of functions $\alpha : 2^S \to \mathscr{L}_\Delta$, $\gamma : \mathscr{L}_\Delta \to 2^S$ over-approximate elements of 2^S w.r.t. $\preceq$ and $\subseteq$.

- $A \subseteq \gamma(\alpha(A))$;
- $\rho \preceq \alpha(\gamma(\rho))$.

Collecting semantics

Given $\rho_1, \rho_2 \in \mathscr{L}_\Delta$, a joined value ρ' s.t. $\rho_1 \preceq \rho'$ and $\rho_2 \preceq \rho'$ is to be unique $\Rightarrow$ ρ' is a unique least upper bound of ρ_1, ρ_2.



An ordering with multiple least upper bounds cannot produce an unambiguous semantics of the joined values.

Partial Order on $\mathscr{L}_\Delta$ and Lattice Axioms

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- $A \subseteq \gamma(\alpha(A))$;
- $\rho \preceq \alpha(\gamma(\rho))$.

In a lattice, every two elements of $\mathscr{L}_\Delta$ have supremums ($\vee$) and infinums ($\wedge$) w.r.t. the partial order $\preceq$.

Lattice Axioms

- $\big(x \vee (x \wedge y) = x\big)$ & $\big(x \wedge (x \vee y) = x\big)$;
- $\big(x \vee y = y \vee x\big)$ & $\big(x \wedge y = y \wedge x\big)$;
- $\big(x \vee (y \vee z) = (x \vee y) \vee z\big)$ & $\big(x \wedge (y \wedge z) = (x \wedge y) \wedge z\big)$.

Existing Lattices for Strings

Let Σ be a letter alphabet, $S \subseteq \Sigma^*$.

- The simplest lattice over predicates $\{X \doteq \xi \mid \xi \in \Sigma^*\}$ (denoted $\mathcal{L}att_{\mathrm{Eq}}$). Finite height, imprecise.

mapping to commutative algebra

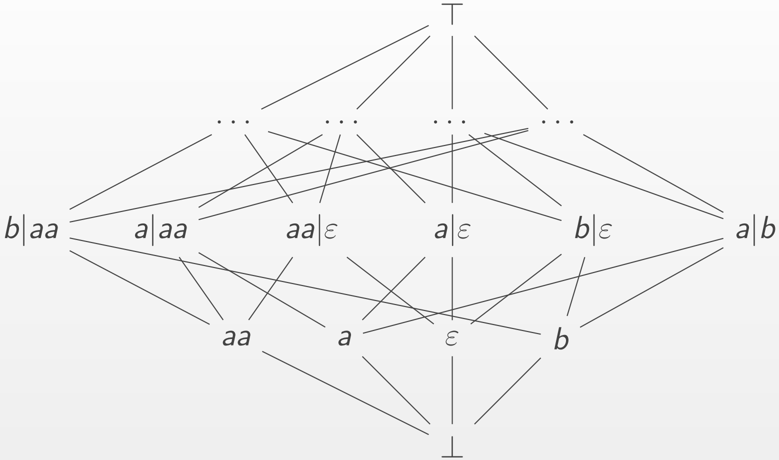
- Lattice tracking string lengths: $\{|X| = n \mid n \in \mathbb{N}\}$.
- Lattice tracking letter occurrences in strings: $\{(|X|_q > 0) \mid q \in \Sigma\}$.

incomplete

- Lattice tracking prefixes and suffixes of strings: $\{\exists \tau_1, \tau_2 (X = \xi_1 \tau_1) \;\&\; (X = \tau_2 \xi_2) \mid \xi_i \in \Sigma^*\}$.
- Lattice checking membership in regular languages: $\{X \in \tau \mid \tau$ is a regular expression$\}$.

- Lattice tracking program-specific predicates (JSAI, Kashyap et al, 2014).
  Desirable to be parameterizable with predicates.

Regular Languages Lattice



requires widening $\left\{ \begin{array}{l} \text{Both infinite ascending chains (e.g. } a,\ a|a^2, \ldots) \\ \text{and infinite descending chains (e.g. } a^*,\ (a^2)^*, \ldots). \end{array} \right.$

The Research Questions Arise

- Find an expressible language describing string predicates set, whose elements form a lattice of string abstract values:
  - being complete (and, moreover, finite-height);
  - extending the trivial lattice $\mathcal{L}att_{\mathrm{Eq}}$ considered above;
  - capturing non-commutative string properties.
- Improve flexibility of the given language set using lattice operations.

We suggest a word equation language as a candidate for such a set of abstract values.

Word Equations by Example

Given two alphabets:

- Σ is a set of constant (e.g. lowercase Latin) characters;
- $\mathcal{V}$ is a set of variables (e.g. capitalized Latin characters).

Consider two constant words:

$$aababaa \qquad aababaa$$

Word Equations by Example

Given two alphabets:

- Σ is a set of constant (e.g. lowercase Latin) characters;
- $\mathcal{V}$ is a set of variables (e.g. capitalized Latin characters).

Replace some of occurrences of their subwords with variables:

$$aababaaba \qquad aabababaa$$
$$a^2 \mapsto X, \quad a^2ba \mapsto Y, \quad b \mapsto Z$$
$$XZabY \qquad YbaZX$$

Word Equations by Example

Given two alphabets:

- Σ is a set of constant (e.g. lowercase Latin) characters;
- $\mathcal{V}$ is a set of variables (e.g. capitalized Latin characters).

Insert the equality sign between the resulted words:

$$\mathsf{aababaaba} \qquad \mathsf{aabababaa}$$

$$a^2 \mapsto X, \quad a^2ba \mapsto Y, \quad b \mapsto Z$$

$$XZabY \qquad YbaZX$$

$$XZabY \doteq YbaZX$$

Word Equations by Example

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$$aababaaba \qquad aabababaa$$

$$a^2 \mapsto X, \quad a^2ba \mapsto Y, \quad b \mapsto Z$$

$$XZabY \doteq YbaZX$$

We have obtained a word equation.

A solution of the equation is the following substitution:

$$X \mapsto a^2, \quad Y \mapsto a^2ba, \quad Z \mapsto b$$

Word Equations by Example

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$$aababaaba \qquad aababbabaa$$

$$a^2 \mapsto X, \quad a^2ba \mapsto Y, \quad b \mapsto Z$$

$$XZabY \quad \doteq \quad YbaZX$$

We have obtained a word equation.

A solution of the equation is the following substitution:

$$X \mapsto a^2, \quad Y \mapsto a^2ba, \quad Z \mapsto b$$

Another solution of the equation :

$X \mapsto \varepsilon, Y \mapsto a, Z \mapsto \varepsilon$, where ε is the empty word.

Word Equations

Given a finite alphabet Σ of constant characters and an alphabet $\mathcal{V}$ of variables; ε stands for the empty word. Small Greek letters stand for words in Σ^*.

Definition

A word equation is an equation of the form $\Psi \doteq \Phi$, where $\Psi, \Phi \in \{\Sigma \cup \mathcal{V}\}^*$. One may see the equation as a pair $\langle \Phi, \Psi \rangle$.

Given a word equation $\Psi \doteq \Phi$, where $\Psi, \Phi \in \{\Sigma \cup \mathcal{V}\}^*$. A solution of the equation is a morphism $\sigma : \{\Sigma \cup \mathcal{V}\}^* \rightarrow \Sigma^*$ s.t. $\Psi\sigma = \Phi\sigma$, where the morphism σ respects the concatenation operation, and $\forall a \in \Sigma \, (a\sigma = a)$.

Word Equations in Program Analysis

How do word equations naturally arise in the context of program analysis?

The variables X, X_1, Y, U, Z range over strings. Their values may be completely unknown or partially known in compile (analyse) time.

$$X \doteq UbaYabZ$$

```
if ( includes(X, 'ba' + Y + 'ab') )
  then { ...... }
  else { ...... }
```

$$aX_1 \doteq X_1 a$$

```
if ( startsWith(X,'a') && endsWith(X,'a') )
  then  { X1 := substring(X, 1, length(X))
          X2 := substring(X, 0, length(X)-1)
          if (X1 === X2) return true;  }
```

Word Equations in Program Analysis

Example source: P.A. Abdulla, M.F. Atig, Y. Chen, B.P. Diep, L. Holìk, A. Rezine, Ph. Rümmer: Flatten and conquer: a framework for efficient analysis of string constraints, PLDI, 2017, ACM, pp. 602–617.

Initial program

```
main(X,Y,Z)   =
       eq(g(Z)++X, 'a'++g(Z)++Y);
g('i'++X)     =   'a'++g(X)++'b';
g('s'++X)     =   g(X)++'b';
g(ε)          =   ε;
eq(X, X)      =   true;
eq(X, Y)      =   false;
```

Simplified program

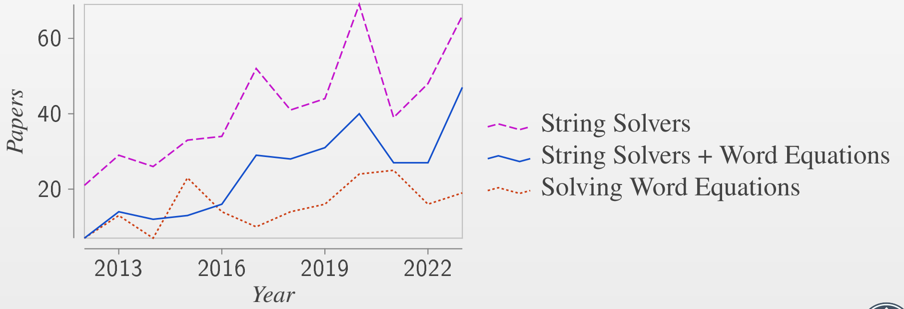
```
main'(X,Y,'i'++Z) = false;
main'(X,Y,'s'++Z) = false;
main'('a'++Y,Y,ε) = true;
main'(X,Y,ε)      = false;
```

- Quadratic equation $WX \doteq aWY$ ($W :=$ `g(Z)`) appears as a constraint generated by rule `eq(X, X) = true`, and its solution set includes $W \in a^*$, which is incompatible with any trace of g-computation except the trivial one.

String Solvers

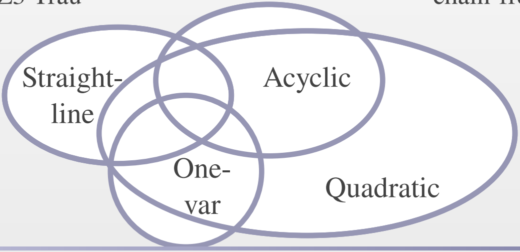
In recent years:

- Many studies are conducted on developing automated reasoning tools capable of proving or disproving statements involving words.
    - Their task is to automatically determine the satisfiability of that formula.



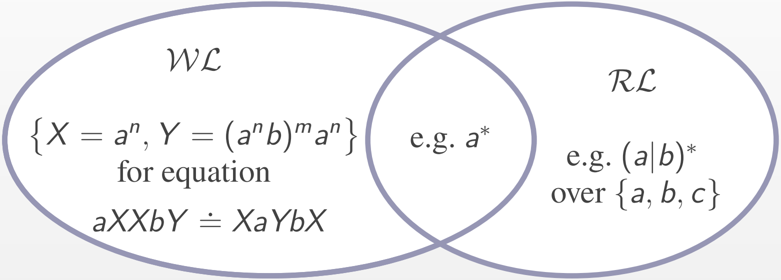
String Solvers

- Some string solving tools using word equations as constraint sets:
  - HAMPI — bounded
  - Norn — acyclic
  - OSTRICH — straight-line
  - Sloth — straight-line & acyclic
  - Woorpje — bounded
  - CertiStr — acyclic
  - MSCP-A — regularly-cyclic
  - CVC5, Z3Str3 — acyclic & bounded
  - Z3-Trau — chain-free



$\mathcal{WL}$ vs. $\mathcal{RL}$

The word equation language $\mathcal{WL}$ vs. the regular expression language $\mathcal{RL}$: intersect but are not subsets each of other:



- There is no word equation with solution-set represented by $(a|b)^*$ over the alphabet $\{a, b, c\}$.
  - J. Karhumäki, F. Mignosi, W. Plandowski. The expressibility of languages and relations by word equations. Journal of the ACM (JACM), 47(3), pp: 483-505, (2000).

Unary Quantifier-Free Predicates

Given a predicate P expressed by a word equation E, let us assume that E is a one-variable word equation.

- What string properties can be expressed by such equations?
- How can be such equations normalized, in order to represent language properties consistently?

Non-trivial question: *Can the solution-set be constructively described simpler as compared to the description provided by the original equation itself?*

Equations $aX \doteq Xa$, $aaX \doteq Xaa$, and $aXX \doteq XXa$ all have the same solution set a^*.

One-Variable Equations*

Theorem (Nowotka-Saarela, 2018).

Every one-variable word equation has either infinitely many or at most three solutions.

Definition

A word $\eta \in \Sigma^+$ is said to be primitive, if for any $\xi \in \Sigma^*$, $n \in \mathbb{N}$ s.t. $\eta = \xi^n$ the equality $n = 1$ holds.

If an X-variable word equation has infinitely many solutions, then its whole solution-set is described with the fractional powers

$$\left\{ (\underbrace{\xi\zeta}_{\text{primitive word}})^n \xi \mid n \in \mathbb{N} \right\}$$

Hence, such an equation can be normalised to $\overbrace{\xi\zeta}^{\text{primitive}} X \doteq X\zeta\xi$.

Join and Meet for One-Variable Equations

Given the language

$$\mathsf{WL}_0 = \left\{ \bot, \top, \{X \doteq \zeta\}, \{\xi_1\xi_2 X \doteq X\xi_2\xi_1\} \mid \zeta, \xi_i \in \Sigma^* \ \&\ \xi_1\xi_2 \text{ is primitive} \right\}$$

and the order $\preceq$ induced by the solution set inclusion, we are required to check that the following situation is impossible:

$$\begin{tikzcd}
\xi_5\xi_6 X \doteq X\xi_6\xi_5 \arrow[d, no head] \arrow[drr, no head] & & \xi_3\xi_4 X \doteq X\xi_4\xi_3 \arrow[dll, no head] \arrow[d, no head] \\
X \doteq \xi_2 & & X \doteq \xi_1
\end{tikzcd}$$

If supremums and infinums are uniquely determined, then WL_0 is a lattice w.r.t. $\preceq$.

Base Lattice WL_0 and its Sublattices

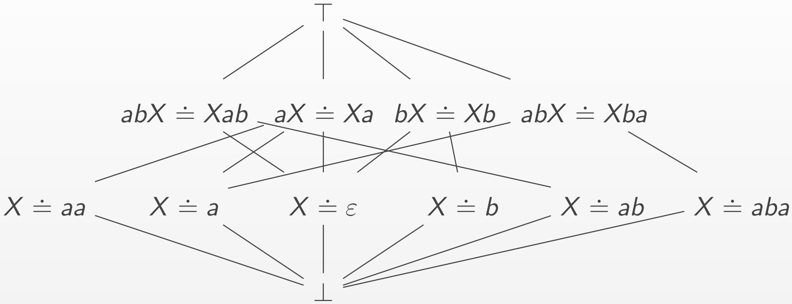
Lemma

Normalised one-variable equations with infinite solution sets and trivial equations of the form $X \doteq \eta$, together with $\top$, $\bot$ form a complete lattice, that is:

- Given any constant words ξ_1, ξ_2, at most one infinite-solution one-variable word equation in the normal form is satisfied by both.
- Given any two infinite-solution word equations $\xi_1\xi_2X \doteq X\xi_2\xi_1$, $\xi_3\xi_4X \doteq X\xi_4\xi_3$, at most one word in Σ^* satisfies both.

Base lattice WL_0 includes all elements $\{X \doteq \xi\}$, $\{\xi_1\xi_2X \doteq X\xi_2\xi_1\}$; its finite sublattices $\mathsf{WL}_0(\mathcal{P})$ depend on programs $\mathcal{P}$, i.e. on the finite set of strings explicitly occurring in $\mathcal{P}$.

A Lattice Based on $\{\varepsilon, a, a^2, b, ab, aba\}$



- A non-trivial one-variable word equation with an infinite solution set can be reduced to an equivalent one

$$\underbrace{\eta_1\eta_2 X \doteq X\eta_2\eta_1 \; (\eta_1\eta_2 \text{ is primitive})}_{\text{normal-form condition}}.$$

- $\left((\eta_1\eta_2)^n X \doteq X(\eta_2\eta_1)^n\right) \Leftrightarrow \left(\eta_1\eta_2 X \doteq X\eta_2\eta_1\right)$ $(n \geq 1)$, hence the widening problem is resolved.

Lifting JS String Operations to Abstract Domain

Simple example — string concatenation in JavaScript. We assume that $x \neq \bot$, $y \neq \bot$, otherwise concatenation results in an error.

$$x + y = \begin{cases} \{X \doteq \xi_1\xi_2\}, \text{ if } x = \{X \doteq \xi_1\}, y = \{X \doteq \xi_2\}; \\ \{\xi_4\xi_5 X \doteq X\xi_5\xi_4\}, \text{ if } x = \{X \doteq \xi_1\} \ \&\ y = \{\xi_2\xi_3 X \doteq X\xi_3\xi_2\} \\ \qquad \text{and } \xi_4, \xi_5 \text{ are s.t. } \xi_5\xi_4 = \xi_3\xi_2 \text{ and } \exists n\big(\xi_1\xi_2 = (\xi_4\xi_5)^n\xi_4\big); \\ \{\xi_4\xi_5 X \doteq X\xi_5\xi_4\}, \text{ if } x = \{\xi_1\xi_2 X \doteq X\xi_2\xi_1\} \ \&\ y = \{X \doteq \xi_3\} \\ \qquad \text{and } \xi_4, \xi_5 \text{ are s.t. } \xi_4\xi_5 = \xi_1\xi_2 \text{ and } \exists n\big(\xi_1 xi_3 = (\xi_4\xi_5)^n\xi_4\big); \\ \{\xi_5\xi_6 X \doteq X\xi_6\xi_5\}, \text{ if } x = \{\xi_1\xi_2 X \doteq X\xi_2\xi_1\} \ \&\ y = \{\xi_3\xi_4 X \doteq X\xi_4\xi_3\} \\ \qquad \text{and } \xi_2\xi_1 = \xi_3\xi_4 \\ \qquad \text{and } \xi_5, \xi_6 \text{ are s.t. } \xi_5\xi_6 = \xi_1\xi_2 \text{ and } \exists n\big(\xi_1\xi_3 = (\xi_1\xi_2)^n\xi_5\big); \\ \top, \text{ otherwise.} \end{cases}$$

Below we consider in details the only non-trivial case, when x and y are infinite-solution equations.

Lifting JS String Operations to Abstract Domain

$$\boxed{\mathrm{x} + \mathrm{y}} =$$

$$\begin{cases} \dots \\ \{\xi_5\xi_6 X \doteq X\xi_6\xi_5\}, \text{ if } x = \{\xi_1\xi_2 X \doteq X\xi_2\xi_1\} \;\&\; y = \{\xi_3\xi_4 X \doteq X\xi_4\xi_3\} \\ \hfill \text{and } \xi_2\xi_1 = \xi_3\xi_4 \\ \hfill \text{and } \xi_5, \xi_6 \text{ are s.t. } \xi_5\xi_6 = \xi_1\xi_2 \text{ and } \exists n\big(\xi_1\xi_3 = (\xi_1\xi_2)^n\xi_5\big); \\ \top, \text{ otherwise.} \end{cases}$$

Where do ξ_5, ξ_6 come from?

$$\exists \xi_5, \xi_6 \forall m, n \,\exists k \Big(\underbrace{(\xi_1\xi_2)^m\xi_1}_{x \text{ values}}\,\underbrace{(\xi_3\xi_4)^n\xi_3}_{y \text{ values}} = \underbrace{(\xi_5\xi_6)^k\xi_5}_{\boxed{\mathrm{x+y}} \text{ values}}\Big)$$

$$\Downarrow$$

$$\big(\xi_1\xi_2 = \xi_5\xi_6\big) \;\&\; \forall n \,\exists k \Big(\xi_1\,\underbrace{(\xi_3\xi_4)^n\xi_3}_{y \text{ values}} = \underbrace{\xi_1\big(\xi_2\xi_1\big)^k\xi_2}_{(\xi_5\xi_6)^{k+1}}\,\xi_5\Big)$$

String Morphisms and Fixpoints

Classical Lemma

If σ is a solution to equation $\Psi \doteq \Phi$, and h is a morphism, then $h \circ \sigma$ is a solution to $h(\Psi) \doteq h(\Phi)$.

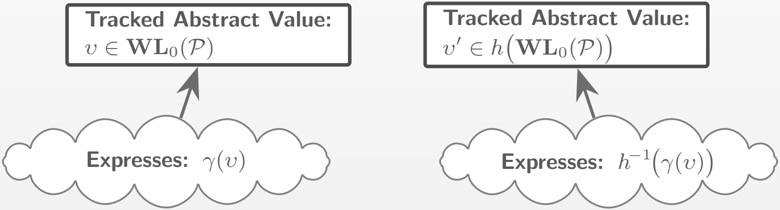
Corollary: string morphisms $h : \Sigma \to \Sigma^*$, extended to WL_0 elements by normalising the morphic images of the equation sides, are monotone lattice mappings. Namely,

$$\left(E_1 \preceq E_2\right) \Rightarrow \left(h(E_1) \preceq h(E_2)\right).$$

By Knaster–Tarski theorem, morphisms fixpoints form a complete sublattice of WL_0.

Expressibility via Transformations

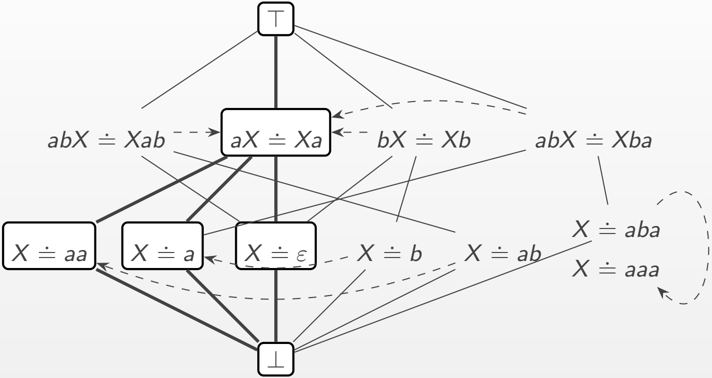
Given a monotone mapping h, we can track properties expressed by inverse morphisms of concretisations of the abstract values, refining expressibility of the basic abstract domain.



There $h^{-1} : 2^S \to 2^S$ is defined as usual:

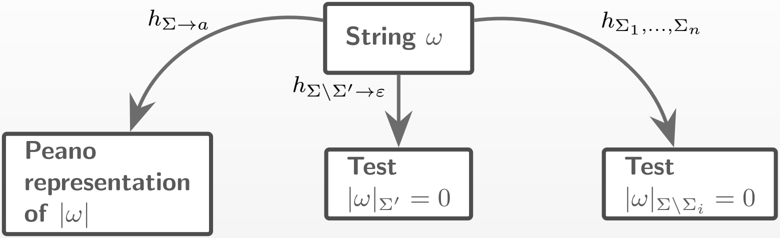
- $\forall s \in S \Big(h^{-1}\big(\{s\}\big) = \big\{ s' \mid h(s') = s \big\} \Big)$;
- $h^{-1}(\mathcal{M}_1 \cup \mathcal{M}_2) = h^{-1}(\mathcal{M}_1) \cup h^{-1}(\mathcal{M}_2)$.

Fixpoint Example



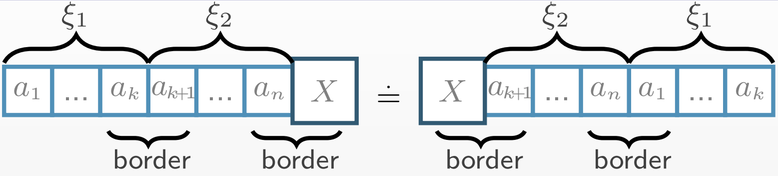
The sublattice generated by the morphism $h(x) = a$ tracks string lengths in a given program.

Expressibility of Fixpoint Lattices



- $h_{\Sigma\to a} \triangleq$ a morphism that maps a to itself and other letters to ε;
- $h_{\Sigma\setminus\Sigma'\to\varepsilon} \triangleq$ a morphism that maps all the letters from Σ' to a and all the other letters to ε.
- Let $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \dots \Sigma_n$, where $\forall i, j (i \neq j \Rightarrow \Sigma_i \cap \Sigma_j = \varnothing)$.
  $h_{\Sigma_1,\dots,\Sigma_k} \triangleq$ a morphism that maps all the elements of the disjoint sets to a single letter.

Inverse Images of String Morphisms



An inverse mapping of a morphism h is border-preserving wrt $\mathcal{L}$, if for any element $\{\xi_1\xi_2X \doteq X\xi_2\xi_1\}$ of $\mathcal{L}$ and for any $a \in \Sigma$, $h(a)$ either contains no border of $\xi_1\xi_2X \doteq X\xi_2\xi_1$, or is equal to ξ_2, and $\xi_1 = \varepsilon$.

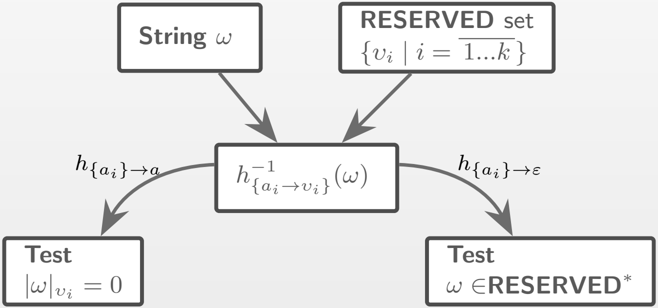
Lemma

Given lattice $\mathsf{WL}_0(\mathcal{P})$ and a string morphism h satisfying the conditions above, for any order induced on Σ, h^{-1}_{min} is a lattice morphism.

Expressibility of Inverse Mappings

Let RESERVED be a set of reserved words v_i (keywords, constants, etc), s.t. morphism $h_{\{a_i \to v_i\}}$ mapping fresh letters from Σ' to v_i is border-preserving.

Then an inverse image lattice $h^{-1}_{\{a_i \to v_i\}}$ can be used to track occurrences of the keywords inside strings.



Word Equations in Abstract Interpretation (Program Analysis)

Simple Example: abstract interpretation over $\mathsf{WL}_0(\mathcal{P})$-lattice shows the sanitization given in line 5 is sound (line 7 is unreachable).

```
1  z = ξ;
2  x = '<script' + z;
3  if (x != z + z) {
4     x = x + x };
5  replaceAll(x,'ip','ipv4');
6  if (contains(x,'script')) {
7     return 'Possible code injection'; }
```

Word Equations in Abstract Interpretation (Program Analysis)

The sanitization given in line 5 is sound.

1 `z = ξ;` $z \mapsto \{X \doteq \xi\}$

2 `x = '<script' + z;` $x \mapsto \{X \doteq \eta\}$

3 `if (x != z + z) {` (does not change)

4 `x = x + x };` $\{X \doteq \eta\} \vee \{X \doteq \eta + \eta\}$

$= \{\mathit{prm}(\eta)X \doteq X\,\mathit{prm}(\eta)\};\ x \mapsto \{\mathit{prm}(\eta)X \doteq X\,\mathit{prm}(\eta)\}$

5 `replaceAll(x, 'ip', 'ipv4');` $x \mapsto^{*} \{\eta' X \doteq X \eta'\}$

6 `if (contains(x, 'script')) {` (never holds for the abstract value of x)

7 `return 'Possible code injection'; }` (is pruned)

- $\mathit{prm}(\eta)$ is a primitive* root of η;
- $\eta =$ `'<script'` $+\,\xi$; $\eta' =$ `replaceAll(`$\mathit{prm}(\eta)$`, 'ip', 'ipv4')`.
- predicate `contains(x, 'script')` fails for any value of X satisfying $\eta' X \doteq X \eta'$.

Conclusion

- An infinite-solution subset of one-variable word equations together with predicates $\mathcal{L}att_{\mathrm{Eq}} = \{X \doteq \xi\}$ form a finite-height lattice WL_0. Sublattices of the lattice WL_0 can be used for abstract interpretation of string-manipulating programs.

- Monotone lattice mappings by means of string morphisms and special cases of inverses of string morphisms generate fixpoint lattices upon WL_0, which can track additional program properties, non-expressible in the word equations language directly.

Work-In-Progress

- A word-equation-lattice-based framework for analysing real-world string-manipulating programs.
- Lattice modifications capturing relations between values.

  A simple instance:

  $$\left\{ \mathsf{WL}_0(\mathcal{P})[X] \times \mathsf{WL}_0(\mathcal{P})[Y] \times \left\langle \top, \bot, \{XY \doteq YX\} \right\rangle \right\}.$$

Thanks for Attention

- Any questions?

Possible Extensions

$\mathcal{WL}$ — existential string theory

(concatenation + equality = word equations).

Theory	Replace All (Const Args)	letter count	length count	$\mathcal{RL}$	Complexity
$\mathcal{WL} + \mathcal{RL}$	✗	✗	✗	✓	PSPACE
$\mathcal{WL}$+len	✗	✗	✓	✗	???
$\mathcal{WL}$+count	✗	✓	✗	✗	Undec.
$\mathcal{WL}$+repl	✓	✗	✗	✗	Undec.

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