Combinatorics - Lecture 2
Distribution

Dr. Chintan Kr Mandal

The problem of distibution in combinatorics is equivalent to arrangement or selection problem with repeatations.

1 Modelling Distribution Problems

- Distribution of *distinct* objects are equivalent to **arrangements**
- Distribution of *identical* objects are equivalent to **selections**

2 Basic Models

2.1 Distinct Objects

Proposal 2.1 *The process of distributing* $\boldsymbol{r}$ *distinct objects into* $\boldsymbol{n}$ *different boxes is equivalent to putting the distinct objects in a row and stamping one of the n-different box names on each object.*

- The resulting sequence of box names is an arrangement of length $\boldsymbol{r}$ formed from n-items (box names) with repetations.

  $\therefore$, there are $n \times n \times \ldots \boldsymbol{r} - n = n^r$ distributions of the r-distinct objects

- If r_i objects must go in box i, $1 \leq i \leq n$, then there are $P(r : r_1, r_2, \ldots, r_n)$ distributions.

2.2 Identical Objects

Proposal 2.2 *The process of distributing* $\boldsymbol{r}$ *identical objects into* $\boldsymbol{n}$ *different boxes is equivalent to choosing an (un-ordered) subset of* $\boldsymbol{r}$ *box names with repetation from the n-choices of boxes*

The total number of distributions of the r-identical objects are

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}$$

Example 2.1 Distribution of a combination of identical and distinct objects : *How many ways are there to distribute 4 identical oranges and 6 distinct apples (each of different variety) into 5 distinct boxes.*

Answer 2.1 *The total number of cases are (n=5, r=4)*

1. *Distibute 4 identical oranges* **into** *5 distinct boxes :* $C(5+4-1, 4) = 70$

2. *Put the 6 distinct apples in 5 distinct boxes :* $5^6 = 15,625$

As both the processes are distinct, *the total number of ways to distribute the 4 identical oranges and 6 distinct apples are :* 70×15625

Example 2.2 *How many ways are there to distribute 4 identical oranges and 6 distinct apples (each of different variety) into 5 distinct boxes (maximum capacity is 2 items) such that*

(A) *2 identical oranges in each of 2 boxes and 6 distinct apples in 3 other boxes*

(B) *2 identical oranges in 1 box; 1 identical orange in each of 2 other boxes; 2 distinct apples each in box containing 1 orange and 2 distinct apples in the 2 empty boxes*

(C) *1 identical orange in each 4 of the 5 boxes and the distinct apples distributed.*

Answer 2.2

Case A

- *2 identical oranges in each of 2 boxes and 0 oranges in other 3 boxes :* $C(5, 2) = 10$
- *6 distinct apples to be distributed into 3 other boxes :* $P(6 : 2, 2, 2) = 90$

$\therefore$, *the total number of possible distibutions* $= 10 \times 90 = 900$

Case B

- *2 identical oranges in 1 box :* $C(5, 1)$
- *1 orange in each of 2 other boxes :* $C(4, 2)$

$\therefore$ *the total number of ways for distibution of the oranges is :* $C(5, 1) \times C(4, 2) = 30$

Note: The above can be interpreted as arranging the numbers 1,2,2,0,0 among 5 boxes : $P(5 : 1, 2, 2) = 30$

- *2 distinct apples will go into 2 boxes and 2 boxes with 1 orange will have 1 apple each :* $P(6 : 2, 2, 1, 1) = 180$

$\therefore$, *the total number of possible distibutions* $= 30 \times 180 = 5400$

Case C

- *1 identical orange in 4 of the 5 boxes :* $C(5, 4)$
- *6 distinct apples to be distributed among the boxes :* $P(6 : 2, 1, 1, 1, 1) = 360$

$\therefore$, *the total number of possible distibutions* $= 5 \times 360 = 1800$

$\therefore$, *the total number of possible distibutions for all the above cases is* $= 900 + 5400 + 1800 = 8100$

Example 2.3 Integer Solutions : *How many integer solutions are there to the equation :* $x_1 + x_2 + x_3 + x_4 = 12$ *with cases* **(A)** $x_i \geq 0$ **(B)** $x_i \geq 1$ **(C)** $x_1 \geq 2,\ x_2 \geq 2,\ x_3 \geq 4,\ x_4 \geq 0$

Answer 2.3 *An integer solution to an equation means : Order a set of integer values for* x_i*'s summing to 12 i.e.* $< x_1, x_2, x_3, x_4 >=< 2, 3, 3, 4 >$.

Notes : *One can model the system as* $(*)$ *distribution of identical objects* $(*)$ *selection with repeatation*

Case A. *Let* x_i *be the number of identical objects in box* i *or number of objects of type* i *chosen. The total number of integer solutions are then*

$$\binom{12 + 4 - 1}{12} = 455$$

Case B. *Solution with* $x_i \geq 1$ *: Can be interpreted as putting at least 1 object in each box or at least 1 object of 1 type.*

$$\binom{12-1}{4-1} = 165$$

Case C. *Can be interpreted as at least 2 objects in box 1, 2 objects in box 2, 4 objects in box 3 :*

$$\binom{(12-2-2-4)+4-1}{4-1} = \binom{7}{3} = 35$$

3 Diophantine Equations

Definition 3.1 *A Diophantine equationis a poly-nomial equation with integer coefficients, possibly in several variables, for whichwe require integer solutions.*

For instance, $x^2 + y^2 = 3$ is a Diophantine equation with no solutions. On the other hand, $x = 1, y = 2$ is a solution of the Diophantine equation $x^2 + y^2 = 5$. The most basic Diophantine problem that one can ask is the following: *given a Diophantine equation, does it have integer solutions?*

The linear Diophantine equaltion is given by $\sum_{i=1}^{n} x_i = r$ and it is required to find the non-negative integers satisfying the equation.

This problem has equivalent forms for *Selection with Repeatations*

1. The number of ways to select $\boldsymbol{r}$ objects with repeatations from $\boldsymbol{n}$-different types of objects

2. The number of ways to distribute $\boldsymbol{r}$ identical objects into $\boldsymbol{n}$-distinct boxes

Example 3.1 *What fraction of binary sequences of length 10 consists of a positive number of 1's, followed by 0's, followed by 1's, followed by a number of 0's.*

e.g. 1110111000

Answer 3.1 *There are* $2^{10} = 1024$ *binary sequences a length of 10 bits.*

Modelling the system, we can do it by :

$\boxed{\textit{Box 1 : 1s}}$ $\boxed{\textit{Box 2 : 0s}}$ $\boxed{\textit{Box 3 : 1s}}$ $\boxed{\textit{Box 4 : 0s}}$

1. *There are 10 identical markers (x-s)*

2. *Each box must have at least 1 marker, since subsequence of 0s and 1s must be non-empty*

   This is similar to putting 1 ball in each box with a false bottom to conceal the ball in each box **and then** *count the ways to distribute without restriction the remaining* $(r-n)$ *balls into the* $\boldsymbol{n}$*-boxes i.e*

$$\binom{(r-n)+n-1}{r-n} = \frac{[(r-n)+n-1]!}{(r-n)!(n-1)!} = \binom{r-1}{n-1}$$

3. *For each 4* $(\boldsymbol{n})$ *box, distibution of 10 x's is given as*

$$\binom{10-1}{4-1} = 84$$

$\therefore$ *there are 84 such binary sequences i.e* $\frac{84}{1024} \approx 0.08$ *of all the 10 bit binary sequences having the above properties.*

4 Synopsis of Distributions

	Arrangement (ordered outcome) OR Distribution of Distinct Objects	**Combination (Un-ordered Sequence) OR Distribution of Identical Objects**
No Repetition	$P(n, r)$	$C(n, r)$
Unlimited Repetition	n^r	$C(n + r - 1, r)$
Restricted Repetition	$P(n : r_1, r_2, \ldots, r_m)$	-