Generating Function

(1)

$\to$ Recurring function

$f(x_n) = a_0 + a_1 x^1 + a_2 x^2 + \cdots$ $\to$ generating function

$f(n) = f(A_0, A_1, \cdots, A_n)$

$= f(x_{n-1}, x_{n-2}, \cdots, a_0, a_1)$

$\boxed{\text{Solving Linear Recurrence Relation}}$

Solving linear homogeneous recurrence relation, of degree k,

Def: A linear homogeneous recurrence relation of with constant co-efficients is a recurrence relation of the form —

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \quad (1)$$

where $c_1, c_2, \cdots c_k \in \mathbb{R}$ & $c_k \neq 0$

(*) Linear refers to the fact that $a_{n-1}, a_{n-2}, \cdots a_{n-k}$ appear in separate terms & to the first powers.

(*) homogeneous refers to the fact that the total degree of each term is the same & that there is no constant term.

$\to$ if there is a constant term (which is allowed to be a fn. of n), (we refer it to be a linear non-homogeneous recurrence relation.)

$\to$ Constant co-efficients refers to the fact that $c_1, c_2 \cdots c_n$ are fixed real numbers that don't depent on n.

Degree k refers to the fact that the expression for a_n contain the previous terms up to a_{n-k}.

$\boxed{\text{Theorem 1}}$

If a_n is a solution of eq^n (1), then so is ca_n for some constant c

If a_n & b_n are two solutions of eq^n (1) then $a_n + b_n$ is another solution.

Characteristic Eq^n for a linear homogeneous R. R

let, $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots c_k a_{n-k}$

where $c_1, c_2, \cdots c_n \in \mathbb{R}$ & $c_k \neq 0$

$a_n = \lambda^n$ $(\lambda \neq 0)$ is a solution of eq^n (1) iff.

$\lambda^n = c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \cdots c_k \lambda^{n-k}$

$\Rightarrow \lambda^k = c_1 \lambda^{k-1} + c_2 \lambda^{k-2} + \cdots + c_k$ (dividing both side by λ^{n-k}) (2)

eq^n (2) is the characteristic eq^n of the eq^n (1)

Thus $a_n = \lambda^n$ is a sol^n of eq^n (1) iff λ is a sol^n of eq^n (2)

(*) from roots of this eq^n, one can find the solution of the eq^n (1)

where (a) roots are non-repeated.

(b) roots are repeated.

Characteristic root has non repeated roots

Theorem 2: Let $c_1, c_2 \cdots c_n$ be real numbers, suppose that the characteristic eq^n —

$$\lambda^k = c_1 \lambda^{k-1} + c_2 \lambda^{k-2} + \cdots + c_k$$

has k distinct roots, $\lambda_1, \lambda_2 \cdots \lambda_k$.

Then, the ~~sequence~~ sequence $\{a_n\}$ is a solution of the recurrence relation eq^n (1)

iff, $a_n = d_1 \lambda_1^n + d_2 \lambda_2^n + \cdots + d_k \lambda_k^n$

for all $n \geq 0$ & $d_1, d_2 \cdots d_k$ are constants.

Proof: $\to$ Let us proof the first 'if' part,

Some $\lambda_1, \lambda_2 \cdots \lambda_k$ are the roots of the eq^n (2), so for each $1 \leq i \leq k$, it is known that $\{\lambda_i^n\}_{n \geq 0}$ is a solution of the recurrence relation. Hence, from theorem (1), it implies that $a_n = c_1 \lambda_1^n + \cdots + c_k \lambda_k^n$ is a solution of the recurrence relation.

Let us proof the only if part

Suppose that $(a_n)_{n\geq 0}$ is a solution. One needs to find constants $d_1, d_2 \dots d_k$ such that $a_n = d_1\lambda_1^n + d_2\lambda_2^n + \dots + d_k\lambda_k^n$ holds for every $n \geq 0$

For any fixed $d_1, d_2 \dots d_k$, let
$b_n = d_1\lambda_1^n + d_2\lambda_2^n + \dots + d_k\lambda_k^n$

We shall find the constants such that b_n matches with a_n in the initial values. i.e, we need to find the constants such that

$$\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_k^2 \\ \vdots & & & \vdots \\ \lambda_1^{k-1} & \dots & \dots & \lambda_k^{k-1} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_k \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix}$$

→ Transpose of the Vandermont matrix.

The determination of the matrix is given by $\prod_{1\leq i<j\leq k}(d_j - d_i) \neq 0$

Since, $\lambda_1, \lambda_2 \dots \lambda_n$ are ~~constant~~ distinct

Thus, the matrix is invertible & therefore the above system of eq^ns has always a solution.

So, for any values of $a_0, \dots a_{k-1}$, there exists constants $d_1, d_2 \dots d_k$ such that $d_1\lambda_1^n + d_2\lambda_2^n + \dots + d_k\lambda_k^n = b_n$ matches with a_n at the initial k values,

Now, since $\lambda_1 \dots \lambda_k$ are the roots of the characteristics eq^n of a_n, so b_n satisfies same recurrence relation as a_n with same initial condition.

So, Principle of strong induction implies that b_n must equal as a_n at every n

In other words, we get $a_n = d_1\lambda_1^n + \dots + d_k\lambda_k^n$

eg: Let us find the gen^l sol^n of the Fibonacci series $a_n = a_{n-1} + a_{n-2}$, $n \geq 2$

⇒ If $a_n = n^n$, the characteristic eq^n of the above is given as —

$$n^n = n^{n-1} + n^{n-2}$$
$$\Rightarrow n^2 = n + 1$$

which has 2 distinct roots
$\lambda_1 = \left(\frac{1+\sqrt5}{2}\right)$ and $\lambda_2 = \left(\frac{1-\sqrt5}{2}\right)$

- From Theorem ②,

$$a_n = d_1\left(\frac{1+\sqrt5}{2}\right)^n + d_2\left(\frac{1-\sqrt5}{2}\right)^n \quad \forall\, n \geq 0$$

⟶ General solution.

- To find the particular solution for the sequence, $F_0 = 0$ & $F_1 = 1$ are the initial conditions.

So, d_1, d_2 must satisfy $d_1 + d_2 = 0$ & $d_1\lambda_1 + d_2\lambda_2 = 1$
which gives $d_1 = -d_2 = 1/\sqrt5$.

$$a_n = F_n = \frac{1}{\sqrt5}\left[\left(\frac{1+\sqrt5}{2}\right)^n - \left(\frac{1-\sqrt5}{2}\right)^n\right], \text{ for } \forall\, n \geq 0$$

Characteristic Solution has repeated roots

Suppose λ is a root of eq^n ① with multiplicity $m \geq 1$ i.e. $a_n = \lambda^n, n\lambda^n, \dots, n^{m-1}\lambda^n$, all satisfy the recurrence relation corresponding to that characteristic eq^n. Any general sol^n of the recurrence relation must be a linear combination of such solution.

$\boxed{\text{Theorem 3}}$ Let $c_1, c_2 \dots c_k$ be real numbers, Suppose, the characteristic eq^n $\lambda^k = c_1\lambda^{k-1} + c_2\lambda^{k-2} + \dots c_k$ has $t < k$ distinct roots, $\lambda_1, \lambda_2 \dots \lambda_t$ with ~~multiplication~~ multiplicities $m_1, m_2, \dots m_t$ respectively.

such that $m_i \geq 1$, for $i = 1, 2, \ldots t$ & $m_1 + m_2 + \cdots + m_t = k$

Then a sequence $\{a_n\}_{n \geq 0}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}, \quad n \geq k$$

iff, $$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1}) \lambda_1^n + \cdots$$
$$+ (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-2}) \lambda_t^n \cdots$$

for every $n \geq 0$; where $\alpha_{i,j}$ are constants for

$1 \leq i \leq t$ & $0 \leq j \leq m_i - 1$.

Thus for $n \geq k$

$$h_n = y_n - x_n = \left(\sum_{i=1}^{k} c_i y_{n-i} + g(n) \right) - \left(\sum_{i=1}^{k} c_i x_{n-i} + g(n) \right)$$

$$= \sum_{i=1}^{k} c_i (y_{n-i} - x_{n-i}) = \sum_{i=1}^{k} c_i h_{n-i}$$

which shows that h_n satisfies the homogeneous recurrence relation corresponding to (1)

Let us state that 2^n is a solution of the

$$a_n = 3a_{n-1} - 2^{n-1}$$

as $a_n = 3\,2^{n-1} - 2^{n-1} = 2^n$

Solution for the homogeneous part

$$a_n = 3a_{n-1}$$

which has characteristic eq^n of $x = 3$

So, the general solution to this homogeneous eq. is given by

$h_n = c\,3^n$ where c is a constant

∴ The total solution is given by-

$$a_n = c\,3^n + 2^n$$

Linear non-homogenous eq. with constant coefficient

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + g(n), \quad n \geq k$$

(1)

eg. $a_n = 3a_{n-1} - 2^{n-1}, \; n \geq 1$

Theorem 4: Suppose x_n is one solution of the eq (1). Then y_n is another solution if and only if $y_n = x_n + h_n \;\forall\; n \geq 0$, where h_n is a solution of the linear homogenous recurrence relation.

Proof

if part

Let $y_n = x_n + h_n \;\forall n \geq 0$ where x_n & h_n are as above.

Then, $y_n = x_n + h_n = \sum_{i=1}^{k} c_i x_{n-i} + g(n) + \sum_{i=1}^{k} c_i h_{n-i}$

$= \sum_{i=1}^{k} c_i (x_{n-i} + h_{n-i}) + g(n) = \sum_{i=1}^{k} c_i y_{n-i} + g(n)$

which shows that y_n satisfies the recurrence relation (1)

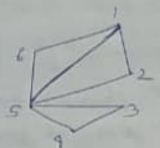
only if part

Let x_n & y_n both satisfies eq. (1).

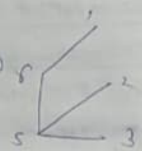
Define $h_n = y_n - x_n$

We show that h_n satisfies the homogenous eq. corresponding to (1)

$G(V,E)$



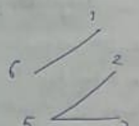
$H(V,F)$



<u>Induced Subgraph</u> If a subgraph has every possible edge, it is an induced subgraph.

vertex induced subgraph

$\langle U \rangle = \langle 1, 2, 5, 6 \rangle$



Edge induced subgraph of G

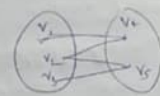
$\langle F \rangle = \{ \{1, 6\}, \{2, 5\} \{3, 5\} \}$

. when $F(H)$ consists of all edges of G ~~forming~~ joining pairs of $U(H)$; H is <u>vertex induced</u> subgraph of G.

. Given a subset $F \subseteq E$, let $U(H)$ be the set of end-vertices of the edge of $F(H)$, the ~~F(H)~~ H is the ~~x~~ edge induced subgraph of G.

Partite Graph

Def (r-partite) A graph $G(V,E)$ is said to be r-partite $(r > 0)$, if its vertex set can be partitioned as $V = V_1 \cup V_2 \cup \dots \cup V_p$, such that $uv \in E(G)$, then $u \in V_i$ & $v \in V_j$ i.e. everyone of the induced subgraph $\langle V_i \rangle$ is an empty graph.

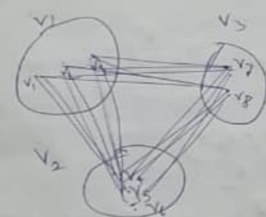


$\langle V_1 \rangle = \{1, 2, 3\}$

$\langle V_2 \rangle = \{4, 5\}$

$E(V_1) = E(V_2) = \phi$

If an r-partite graph has all possible edge, i.e. $uv \in E$ for all pairs $u \in V_i$, $v \in V_j$ then it is called a complete partite graph is denoted as $K_{n_1, n_2 \dots n_p}$



$r = 2$, bipartite graph

$K_{3,3,2}$

$K_{1,n} \rightarrow$ Stargraph

Hypergraph (Berge)

Let $X = \{x_1, x_2 \dots, x_n\}$ be a finite set

A hypergraph on X is a ~~form~~ family

$H = \{E_1, E_2 \dots, E_n\}$ of subsets of X such that

① $\cap E_i \neq \phi$, $i = 1, 2, \dots n$

② $\bigcup_{i=1}^{\infty} E_i = X$

A simple hypergraph (Sperner family) is a

Binomial distribution with parameters n & p

$M(t) = \sum_k e^{tk} \cdot p(k)$ $\qquad p(k) = {}^nC_k p^k q^{n-k}$

$= \sum_k e^{tk} \, {}^nC_k p^k \cdot q^{n-k}$

$= \sum_k \binom{n}{k} (pe^t)^k \cdot (1-p)^{n-k}$

$M(t) = (pe^t + 1 - p)^n$

$M'(t) = n(pe^t + 1 - p)^{n-1} \cdot pe^t$

$\boxed{M'(0) = np = E(x)}$

$M''(t) = np[(n-1)(pe^t+1-p)^{n-2} pe^t e^t + e^t (pe^t+1-p)^{n-1}]$

$M''(0) = np \cdot [(n-1)p + 1] = np(np - p + 1)$

$Var(x) = M''(0) - \{M'(0)\}^2$

$= \cancel{n^2p - np^2 = n}$

$= n^2p^2 - np^2 + np - n^2p^2$

$\boxed{Var(x) = np(1-p)}$

Poisson distr. with parameter λ

$p(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$

$M(t) = \sum_k e^{tk} e^{-\lambda} \frac{\lambda^k}{k!}$

$= e^{-\lambda} \sum_k \frac{(\lambda e^t)^k}{k!}$

$= e^{-\lambda} \cdot e^{\lambda e^t} = e^{\lambda(e^t - 1)}$

$M'(t) = e^{-\lambda} \cdot e^{\lambda e^t} \frac{d}{dt}(\lambda e^t) = e^{-\lambda} \cdot e^{\lambda e^t} \lambda e^t$

$M'(0) = e^{-\lambda} \cdot e^{\lambda} \cdot \lambda = \lambda$

$M''(t) = e^{-\lambda}\lambda[e^t e^{\lambda e^t} + e^t \cdot e^{\lambda e^t} \cdot \lambda e^t]$

$M''(0) = e^{-\lambda} \lambda [e^{\lambda} + e^{\lambda} \cdot \lambda]$

$= \lambda + \lambda^2$

$Var(x) = M''(0) - \{M'(0)\}^2$

$= \lambda + \lambda^2 - (\lambda)^2 = \lambda$

For two independent r.v.s X & Y, what is $M_{X+Y}(t)$,

$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

$M_{X+Y}(t) = E(e^{tX + tY})$

$= E(e^{tX} \cdot e^{tY})$

$= E(e^{tX}) \cdot E(e^{tY})$

$= M_X(t) \cdot M_Y(t)$

Q. Suppose, the M.G.F of a r.v. is

$M(t) = e^{3(e^t - 1)}$

Then, what is $P(X = 0)$?

poisson distr. by comparision $M(t) = e^{\lambda(e^t - 1)}$

$\lambda = 3$, $P(X=0) = e^{-3} \cdot \frac{3^0}{0!} = e^{-3}$

③ Removal of a vertex

Let $G(V,E)$ be a graph & $v \in V(G)$. Let $E_v(G)$ be the set of all edges incident with v. Then the graph $H = (V-\{v\}, E-E_v)$ is said to be obtained from G by the removal of the vertex v & is denoted by $G-v$ / G/v

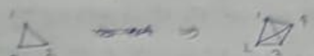
If $U \subset V$, the graph obtained by removing the vertex of G in U in any order is denoted by $G-U$

④ Addition of an edge

$G+e$, $H = (V, E \cup \{e\})$

⑤ Addition of a vertex:

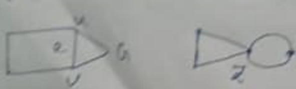
Let $G(V,E)$ be a graph & $v \notin V$. Then the graph H with $V \cup \{v\}$ & edge set $E \cup \{uv \mid u \in V\}$ is said to be obtained from G by adding a vertex v, & is denoted by $G+v$



$G(V,E)$ $\qquad$ $G+4$

⑥ Contraction of Graph

Let $G(V,E)$ be a graph & G contract $e(v,w)$ is the graph G/e obtained by ~~denoting~~ deleting e from $E(G)$ & then adding a new vertex z which is incident to all edges deleting, deleting v, w from $V(G)$ in $E(G)/e$ which were incident to v & w.



G $\qquad$ G/e

- If $e \in E(G)$ is a loop then by removal $G - e = E(G)/e$

- A contraction of $G(V,E)$ is any graph which can be obtained by recursively contracting edges in $G(V,E)$

A minor of a graph $G(V,E)$ is any graph which can be obtained by recursively deleting or contracting edges & deleting isolated vertices from $G(V,E)$

Joining / Union of a graph

Moment Generating Function:

A moment generating function M_t or $M(t)$ of a r.v. X is defined by $\boxed{M(t) = E(e^{tX})}$ where $t \in \mathbb{R}$

For discrete r.v. X $M(t) = \sum_x e^{tx} p(x) \rightarrow$ pmf

cont. r.v. X $M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$ $\rightarrow$ p.d.f

$M(t)$ or M_t

All moments of r.v. X can be generated by successive differentiation of $M(t)$.

$$M(t) = E(e^{tX})$$

$$\Rightarrow M'(t) = \frac{d}{dt}(E(e^{tX}))$$

$$= E\left(\frac{d}{dt}(e^{tX})\right)$$

$$= E(X.e^{tX})$$

Thus $M'(0) = E(X)$

Now, $M''(t) = \frac{d}{dt}(M'(t))$

$$= \frac{d}{dt}(E(xe^{xt}))$$

$$= E(x^2 e^{xt})$$

Thus $M''(0) = E(X^2)$

$\therefore$ $var(X) = E(X^2) - (E(X))^2 = M''(0) - (M'(0))^2$

Why recurrence relations:

Combinatorial problem depending on $n \in \mathbb{Z}^+$, where n can be -

① denote size of some set / multiset in problem

② Size of subsets

③ number of positions in a permutation

e.g. ① Let h_n ~~the~~ denote the permutation of $\{1, 2, \dots, n\}$. $\Rightarrow h_n = n!$

$\therefore h_0, h_1, h_2, \dots, h_n \equiv$ number of permutation for sets $S_0 = \{\}$, $S_1 = \{1\}$, $S_2 = \{1, 2\}$, ...

e.g. ② Let g_n be the number of non-negative integral solutions of the eqn

$$x_1 + x_2 + x_3 + x_4 = n$$

This is equivalent to finding subsets where summation is n, i.e. selecting n items from set of 4 elements so that

① x_1 has type 1

② x_2 —— 2

③ x_3 —— 3

④ x_4 —— 4

$\therefore$ the no. of solutions ~~[illegible]~~ is eqv. to the no. of n combination with repetition allowed from a set of 4 elements which is also the g.t.

$$g_n = \binom{4+n-1}{n} = \binom{n+3}{n}$$

Suppose $X_1, X_2, \ldots$ be a sequence of independent & identically distributed r.v.s.

Let N be a non-negative integer valued r.v. which is independent of X, then compute M.G.F. of

$$\left(Y = \sum_{i=1}^{N} X_i\right)$$

$$\Rightarrow \; E\left[e^{\left[t \cdot \sum_{i=1}^{N} X_i\right]} \mid N = n\right]$$

$$= E\left[e^{\left[t \sum_{i=1}^{n} X_i\right]}\right]$$

$$= (M_X(t))^n$$

$$M_X(t) = E(e^{tX_i})$$

Hence, $E(e^{tY} \mid N) = (M_X(t))^N$

$$E(E(X \mid Y = y)) = E(X)$$

$$\Rightarrow E(E(e^{tY} \mid N)) = E[(M_X(t))^N]$$

$$\Rightarrow E(e^{tY}) = M_Y(t) = E[(M_X(t))^N]$$

$$M_Y'(t) = E[N \cdot \{M_X(t)\}^{N-1} \cdot M_X'(t)]$$

$$M_X(0) = E(e^{0X}) = 1$$

$$M_Y'(0) = E[N \cdot \{M_X(0)\}^{N-1} \cdot M_X'(0)]$$

$$= E[N \, E(X)]$$

$$= E(N) \cdot E(E(X))$$

$$= E(N) \cdot E(X) = E(Y)$$

$$M_Y''(t) = E\left[N \cdot \left[M_X'(t) \cdot (N-1)\{M_X(t)\}^{N-2} \cdot M_X'(t) + \{M_X(t)\}^{N-1} \cdot M_X''(t)\right]\right]$$

$$M_Y''(0) = E[N \cdot [E(X) \cdot (N-1) \cdot E(X) + E(X^2)]]$$

$$E(Y^2) = E[N \cdot [(E(X))^2 (N-1) + E(X^2)]]$$

$$= E(N) \cdot [E((E(X))^2) \cdot E(N-1) + E(E(X^2))]$$

$$= E(N) \cdot [(E(X))^2 \cdot E(N-1) + E(X^2)]$$

$$\mathrm{var}(Y) = E(Y^2) - (E(Y))^2$$

$$= E(N) E(N-1) (E(X))^2 + E(N) \cdot E(X^2) - (E(N))^2 (E(X))^2$$

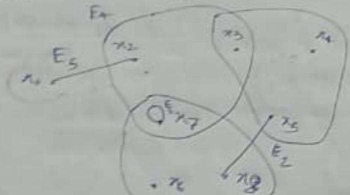
$$=$$

hypergraph $H = \{E_1, E_2, \ldots, E_m\}$ such that $\exists\ E_i \subset E_j\ (i \neq j)$

- The elements $x_1, x_2, \ldots, x_n$ of X are called vertex, and the sets $E_1, E_2, \ldots, E_m$ are the edge of the hypergraph.

A simple graph is a simple hypergraph each of whose edges has cardinality ~~=~~ 2

- A multigraph (with loops & multiple edges) is a hypergraph in which edge has cardinality ≤ 2.
- We ~~should~~ shall not consider isolated points of a graph to be vertex.

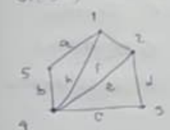


Incidence MATRIX ←

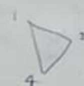
$x \downarrow$	E_1	E_2	E_3	E_4	E_5	E_6
1	0	0	0	0	1	0
2	0	0	0	1	1	0
3	[illegible]	0	0	1	0	0
4	1	0	0	0	0	0
5	1	1				
6			1			
7			1	1	0	1
8		1	[illegible]			

Operations on graph

$G(V,F)$



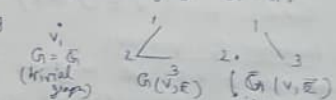
$H(V,F)$



- $\overline{\text{complete graph}}$ = empty graph

① Complement of a graph

The complement $\overline{G}(V, \overline{E})$ of a graph $G(V,E)$ has the same vertex set & its edge set is the complement $\overline{E}(G)$ of $E(G)$ in $V^{(2)}$ i.e. uv is an edge in $\overline{G}$ iff uv is not an edge of G. A graph is said to be self-complementary if $G = \overline{G}$.

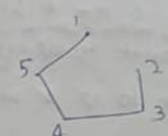


② Removal of an edge:

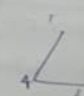
~~Def~~ Def: Let $G(V,E)$ be a graph & $F \subset E(G)$. Then the graph $H(V, E-F)$ is said to be obtained from G by removing the edges in the set F. It is denoted by $G-F$. If F consists of a single edge $e \in G$, the graph obtained by removing e is denoted by $G-e$

$F = \{\{1,2\}, \{2,4\}, \{1,4\}\}$

$G-F$

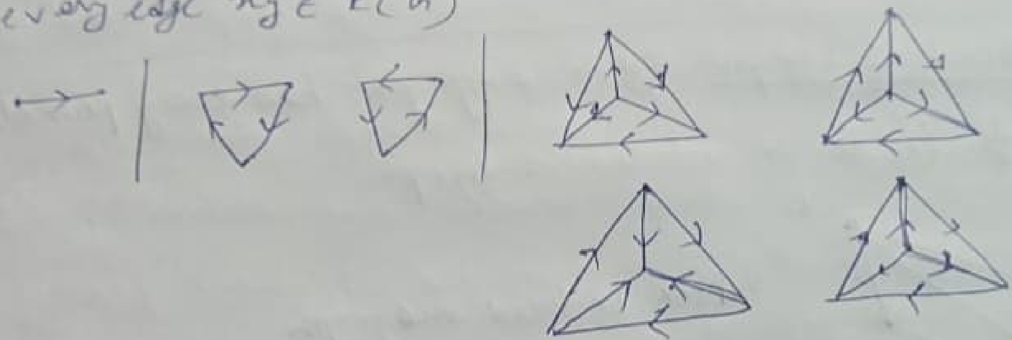


$H - \{1,2\} =$



Orientation & Tournaments

Orientation of a graph: An orientation of a graph $G(V,E)$ is a digraph $D(V,E)$ obtained from $G(V,E)$ by choosing an orientation $x \to y$ or $y \to x$ for every edge $xy \in E(G)$



Tournment / Tournament graphs

Tournament graph is an orientation of a complete graph. They are so called because an n-node tournament graph corresponds to a tournament in which each member of a graph of n-players plays all other $(n-1)$ players.

Subgraph

A subgraph of a graph $G(V,E)$ is a graph $H(U,F)$ with $U(H) \subseteq V(G)$ and $F(H) \subseteq E(H)$.

- When $U(H) = V(G)$, H is a spanning graph of G. If H is a subgraph of $G \Rightarrow G$ is a subgraph of H.

integers m_i is at least r, i.e. one of the boxes contains at least r-objects. $r-1 < av < r$

Note: ①. If the average of n non-negative integers $m_1, m_2, m_3, \ldots, m_n$ is less than $(r+1)$, i.e.

$$\frac{\sum m_i}{n} < r+1$$

then at least; one of the integers is less than $r+1$

② If the avg. of n non-negative integers

$m_1, m_2, \ldots m_n$ is at least equal to r

then at least one of the integers $m_1, m_2, \ldots m_n$ satisfies $m_i \geq r$

Recurrence Relation & Generating Functions

A recurrence relation for a sequence $\{a_n\}_{n\geq 0}$ is an equation that expresses a_n in terms of one or more of the previous terms $a_0, a_1, \ldots, a_{n-1}$

eg: $a_n = 2a_{n-1} - a_{n-2} \ \forall \ n \geq 2$
is a recurrence relation.

Def: A recurrent relation in an eqn of the form

$$a_n = f(a_{n-1}, a_{n-2}, \ldots, a_{n-k}\} \ \forall \ n \geq k$$

with ~~[illegible]~~ k initial conditions, which should completely determine the sequence.