Joint distribution functions

For any two random variables X & Y, the joint cummulative probability distribution func. of X & Y by,

$$F(a,b) = P\{X \leq a, Y \leq b\} \qquad -\infty < a, b < \infty.$$

The distribution of X can be obtained from the joint distr. of X & Y as follows:

$$F_X(a) = P(X \leq a)$$

$$= P(X \leq a, Y < \infty)$$

$$= P\left(\lim_{b\to\infty} (X \leq a, Y \leq b)\right)$$

$$= \lim_{b\to\infty} P(X \leq a, Y \leq b)$$

$$= \lim_{b\to\infty} F(a,b)$$

$$= F(a, \infty)$$

Similarly, $F_Y(b) = F(\infty, b)$

$$P(X > a, Y > b) = 1 - P(\{X > a, Y > b\}^c)$$

$$= 1 - P(\{X > a\}^c \cup \{Y > b\}^c)$$

$$= 1 - P(\{X \leq a\} \cup \{Y \leq b\})$$

$$= 1 - [P\{X \leq a\} + P\{Y \leq b\} - P\{X \leq a, Y \leq b\}]$$

$$= 1 - F_X(a) - F_Y(b) + F(a,b) \rightarrow (1)$$

Eqn. 1 is a special ~~ab~~ case of the following eqn.

$$P\{a_1 < X \le a_2,\ b_1 < Y \le b_2\}$$

$$= F(a_2, b_2) + F(a_1, b_1) - F(a_1, b_2) - F(a_2, b_1).$$

where, $a_1 < a_2$, $b_1 < b_2$

In the case when X & Y are both discrete random variables, it is convenient to define the joint probability mass functions of X & Y by:

$$p(x, y) = P\{X = x, Y = y\}$$

The prob. mass func. of X can be obtained from $p(x,y)$ by:

$$p_X(x) = P\{X = x\}$$

$$= \sum_y p(x, y)$$

Similarly, $p_Y(y) = \sum_x p(x, y)$

Example:

Suppose that 15 percent of the families in a certain community have no children, 20 percent have 1 child, 35 percent have 2 children and 30 percent have 1. Suppose further that in each family ~~is~~ each child is equally to be a boy or a girl. If a family is chosen at random from this community, then B, the number of boys and G, the number of girls in this family will have the joint prob. mass func. shown in the table:

$$P\{B=i,\ G=j\}$$

$i \backslash j$	0	1	2	3	Row Sum
0	0.15	0.10	0.0875	0.0375	0.3750
1	0.10	0.175	0.1125	0	0.3875
2	0.875	0.1125	0	0	0.2000
3	0.0375	0	0	0	0.0375
col. sum	0.3750	0.3875	0.2000	0.0375	

The probabilities shown in the above table are obtained as follows:

$P\{B=0, G=0\} = P\{\text{No children}\} = 0.15$

$P\{B=0, G=1\} = P\{1 \text{ girl \& total of } 1 \text{ child}\}$ ~~(0.20)~~

$= P\{1 \text{ child}\} \cdot P\{1 \text{ girl}/1 \text{ child}\}$

$= 0.20 \times \frac{1}{2}$

$P\{B=0, G=2\} = P\{2 \text{ girls and total } 2 \text{ children}\}$

$= P\{2 \text{ children}\} \times P\{2 \text{ girl}/2 \text{ children}\}$

$= 0.35 \times \left(\frac{1}{2}\right)^2$

We say that X & Y are joint continuous if $\exists$ a func. $f(x,y)$ defined for all real (x,y), having the prop. that for every set C of pairs of real numbers,

$$P\{(X,Y) \in C\} = \iint_{(x,y)\in C} f(x,y)\,dx\,dy$$

The func. $f(x,y)$ is called the joint prob. density func. of X & Y. If A & B are any sets of real numbers, then by defining $C = \{(x,y) : x \in A, y \in B\}$ we see from eqn (2) that,

$$P\{X \in A, Y \in B\} = \int_B \int_A f(x,y)\,dx\,dy$$

Now,

$$F(a,b) = P\{X \in (-\infty, a], Y \in (-\infty, b]\}$$

$$= \int_{-\infty}^{b} \int_{-\infty}^{a} f(x,y)\,dx\,dy$$

it follows upon diff. that

$$f(a,b) = \frac{\partial^2}{\partial a\,\partial b} F(a,b).$$

If X & Y are jointly cont., they are individually cont. & their prob. distr. is given as:

$$P\{X \in A\} = P\{X \in A, Y \in (-\infty, \infty)\}$$

$$= \int_A \int_{-\infty}^{\infty} f(x,y)\,dy\,dx$$

$$= \int_A f_X(x)\,dx$$

where, $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ is the prob. density func. of X.

Similarly prob. density func. of Y is,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Example:

The joint density func. of X & Y is given by,

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & 0<x<\infty,\ 0<y<\infty \\ 0 & \text{otherwise} \end{cases}$$

Compute:
ⓐ $P\{X>1, Y<1\}$
ⓑ $P\{X<Y\}$
ⓒ $P\{X<a\}$

$\Rightarrow$ ⓐ $P\{X>1, Y<1\} = \int_0^1 \int_1^\infty 2e^{-x}e^{-2y}\,dx\,dy$

$$= \int_0^1 2e^{-2y}\left(-e^{-x}\Big|_1^\infty\right)dy$$

$$= e^{-1}\int_0^1 2e^{-2y}\,dy$$

$$= e^{-1}(1-e^{-2})$$

ⓑ $P\{X<Y\}$

$$= \iint_{(x,y):x<y} 2e^{-x}e^{-2y}\,dx\,dy$$

$$= \int_0^\infty \int_0^y 2e^{-x}\cdot e^{-2y}\,dx\,dy$$

$$= \int_0^\infty 2e^{-2y}(1-e^{-y})\,dy$$

$$= \int_0^\infty 2e^{-2y}\,dy - \int_0^\infty 2e^{-3y}\,dy$$

$$= 1-\frac{2}{3} = \frac{1}{3}$$

ⓒ $P\{X<a\} = \int_0^a \int_0^\infty 2e^{-2y}e^{-x}\,dy\,dx$

$$= \int_0^a e^{-x}\,dx = 1-e^{-a}$$

Example:

The joint density of X & Y is given by,

$$f(x,y) = \begin{cases} e^{-(x+y)} & 0 < x < \infty,\ 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find the density func. of the r.v. X/Y.

Soln.

$$F_{X/Y}(a) = P\left\{ \frac{X}{Y} \leq a \right\}$$

$$= \iint\limits_{(x,y):\, x/y \leq a} e^{-(x+y)}\, dx\, dy$$

$$= \int_0^\infty \int_0^{ay} e^{-(x+y)}\, dx . dy$$

$$= \int_0^\infty (1 - e^{-ay})\, e^{-y}\, dy$$

$$= \left\{ -e^{-y} + \frac{e^{-(a+1)y}}{a+1} \right\} \Big|_0^\infty$$

$$= 1 - \frac{1}{a+1} \rightarrow (1)$$

Now, to calculate the density func. of X/Y we differentiate (1), which gives, $f_{X/Y}(a) = \frac{1}{(a+1)^2}$

$0 < a < \infty$.

Conditional distributions (Discrete case):

For any two events E & F, the ~~prob. of~~ conditional prob. of E given F is defined as:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

Thus if X & Y are discrete random variables, the conditional prob. mass func. of X given that Y = y, by

$$p_{X|Y}(x|y) = P(X=x \mid Y=y)$$

$$= \frac{P(X=x, Y=y)}{P(Y=y)}$$

$$= \frac{p(x,y)}{p_Y(y)}$$

For all values of y such that $p_Y(y) > 0$. Similarly the conditional prob. distr. func. of X given that Y = y is defined, for all y such that $p_Y(y) > 0$, by

$$F_{X|Y}(x|y) = P\{X \leq x \mid Y = y\}$$

$$= \sum_{a \leq x} p_{X|Y}(a|y)$$

If X & Y are independent,

$$p_{X|Y}(x|y) = P\{X=x \mid Y=y\}$$

$$= \frac{P\{X=x, Y=y\}}{P\{Y=y\}}$$

$$= \frac{P\{X=x\} . P\{Y=y\}}{P\{Y=y\}}$$

$$= P\{X=x\}$$

Example

Suppose that $p(x,y)$, the joint probability mass func. of X & Y is given by

$$p(0,0)=0.4, \quad p(0,1)=0.2, \quad p(1,0)=0.1, \quad p(1,1)=0.3$$

calculate the conditional prob. mass func. of X given that $Y=1$.

$\Rightarrow$ $$p_Y(1)=\sum_x p(x,1)=p(0,1)+p(1,1)$$
$$=0.5$$

Hence, $$p_{X|Y}(0|1)=\frac{p(0,1)}{p_Y(1)}=\frac{2}{5}$$

and $$p_{X|Y}(1|1)=\frac{p(1,1)}{p_Y(1)}=\frac{3}{5}$$

Continuous case:

If X & Y have a joint prob. density func. $f(x,y)$, then the conditional probability density func. of X given that $Y=y$ is defined for all values of y such that $f_Y(y)>0$ by

$$f_{X|Y}(x|y)=\frac{f(x,Y)}{f_Y(y)}$$

~~To motivate~~ If X & Y are jointly cont., then for any set A,

$$P\{X\in A|Y=y\}=\int_A f_{X|Y}(x|y)\,dx$$

Example:

(1) The joint density of X & Y is given by

$$f(x,y)=\begin{cases}\frac{12}{5}x(2-x-y) & 0<x<1,\ 0<y<1\\ 0 & \text{otherwise}\end{cases}$$

Compute the conditional density of X given that $Y=y$, where $0<y<1$.

$\Rightarrow$ For $0<x<1$, $0<y<1$ we have

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$= \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)\,dx}$$

$$= \frac{x(2-x-y)}{\int_0^1 x(2-x-y)\,dx}$$

$$= \frac{x(2-x-y)}{\frac{2}{3}-\frac{y}{2}} = \frac{6x(2-x-y)}{4-3y}$$

② Suppose that the joint density of X & Y are given by,

$$f(x,y) = \begin{cases} \frac{e^{-x/y}\,e^{-y}}{y} & 0<x<\infty,\ 0<y<\infty \\ 0 & \text{otherwise} \end{cases}$$

Find $P\{X>1 \mid Y=y\}$

$\Rightarrow$ $$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$= \frac{e^{-x/y}\cdot e^{-y}/y}{e^{-y}\int_0^{\infty}\frac{1}{y}e^{-x/y}\,dx} = \frac{1}{y}e^{-x/y}$$

Hence, $$P\{X>1 \mid Y=y\} = \int_1^{\infty}\frac{1}{y}e^{-x/y}\,dx$$

$$= -e^{-x/y}\Big|_1^{\infty}$$

$$= e^{-1/y}$$

Uniform random variable

A random variable is said to be uniformly distributed over the interval $(0,1)$ if its prob. density func. is given by:

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \quad \rightarrow (1)$$

Since $f(x) \geq 0$ & $\int_{-\infty}^{\infty} f(x)dx = \int_0^1 dx = 1.$

For any $0 < a < b < 1$,

$$P(a \leq X \leq b) = \int_a^b f(x)dx = b - a.$$

In general, we say that X is a uniform random variable on the interval (α, β) if the probability density func. of X is given by:

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & \text{if } \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases} \quad \rightarrow (2)$$

Since $F(a) = \int_{-\infty}^{a} f(x)dx$, it follows from (2) that the distribution func. of a uniform random variable on the interval (α, β) is given by:

$$F(a) = \begin{cases} 0 & a \leq \alpha \\ \dfrac{a - \alpha}{\beta - \alpha} & \alpha < a < \beta \\ 1 & a \geq \beta \end{cases}$$

Example:

Let X be uniformly distributed over (α, β).

(i) Find $E(X)$

(ii) Find $Var(X)$.

$\Rightarrow$ (i) $E(X) = \int_{-\infty}^{\infty} x f(x)$

$$= \int_{\alpha}^{\beta} \frac{x}{\beta-\alpha}\, dx$$

$$= \frac{\beta^2-\alpha^2}{2(\beta-\alpha)} = \frac{(\beta+\alpha)}{2}$$

(ii) $E(X^2) = \int_{\alpha}^{\beta} \frac{1}{\beta-\alpha} x^2 dx$

$$= \frac{\beta^3-\alpha^3}{3(\beta-\alpha)} = \frac{\beta^2+\alpha\beta+\alpha^2}{3}$$

hence, $Var(X) = E(X^2) - [E(X)]^2$

$$= \frac{(\beta-\alpha)^2}{12}$$

(2) If X is uniformly distr. over $(0,10)$, calculate the prob

(i) $X<3 \Rightarrow P\{X<3\} = \int_0^3 \frac{1}{10} dx = \frac{3}{10}$

(ii) $X>6 \Rightarrow P\{X>6\} = \int_6^{10} \frac{1}{10} dx = \frac{4}{10}$

(iii) $3<X<8 \Rightarrow P\{3<X<8\} = \int_3^8 \frac{1}{10} dx = \frac{1}{2}$