These are theorems that I proved in class that, I think, cover the material in Section 5.3 much more efficiently. These theorems immediately imply Theorem 5.17-5.29 in the textbook.

Theorem 1 If A is an infinite subset of $\mathbb{N}$, then $A \simeq \mathbb{N}$.

Proof Since $A \neq \emptyset$, and $A \subseteq \mathbb{N}$, the Well-Ordering Principle says there is a <u>smallest</u> member $a_1 \in A$.

Since A is not finite, $A - \{a_1\} \neq \emptyset$. Let a_2 be the smallest member of $A - \{a_1\}$

Since A is not finite, we can always continue this process another step. We inductively define a_{n+1} to be the smallest member of $A - \{a_1, ..., a_n\}$.

The sequence $a_1, a_2, ..., a_n, ...$ contains <u>every</u> member of A (*why?*), and the mapping $f : A \to \mathbb{N}$ defined by $f(a_n) = n$ is one-to-one and onto $\mathbb{N}$. Therefore $A \simeq \mathbb{N}$. •

Corollary 2 A set A is countable iff there exists a one-to-one function $f : A \to \mathbb{N}$ (*not necessarily onto!*)

Proof If such a function exists, then $A \simeq \text{range}(f) \subseteq \mathbb{N}$. Therefore A is finite or, by Theorem 1, $A \simeq \mathbb{N}$. Therefore A is countable.

If A is countable, then A is either finite or denumerable so <u>either</u> there is a one-to-one function $f : A \to \mathbb{N}_k \subseteq \mathbb{N}$ <u>or</u> there is a bijection $f : A \to \mathbb{N}$. Either way, we have a one-to-one function from A to $\mathbb{N}$ so A is countable. •

Corollary 3 A subset of a countable set is countable.

Proof Suppose $B \subseteq A$ where A is countable. By Corollary 2, there is a one-to-one function $f : A \to \mathbb{N}$. Then $g = f|B$ is a one-to-one function from B into $\mathbb{N}$ – so B is countable.

Theorem 4 If A and B are countable, then $A \times B$ is countable.

Proof By assumption, there exist one-to-one functions $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$.

For each ordered pair $(a, b) \in A \times B$, define $h(a, b) = 2^{f(a)}3^{g(b)} \in \mathbb{N}$. Then $h : A \times B \to \mathbb{N}$.

(*Example: if, say,* $f(a) = 27$ *and* $g(b) = 113$, *then* $h((a, b)) = 2^{27}3^{113}$.)

We claim that h is one-to-one: Suppose $(a, b) \neq (c, d) \in A \times B$. Then either $a \neq c$ or $b \neq d$.

If $a \neq c$, then $f(a) \neq f(c)$ since f is one-to-one. So $2^{f(a)}3^{g(b)} \neq 2^{f(c)}3^{g(d)}$ by the Fundamental Theorem of Arithmetic.

If $b \neq d$, then $g(b) \neq g(d)$ since g is one-to-one. So $2^{f(a)}3^{g(b)} \neq 2^{f(c)}3^{g(d)}$

Either way, $h(a,b) \neq h(c,d)$, so h is one-to-one.

Therefore, by Corollary 2, $A \times B$ is countable. •

Note: If A,B,C are countable then $A \times B$ is countable so, by Theorem 4, $(A \times B) \times C$ is also countable. But $(A \times B) \times C \simeq A \times B \times C$, so $A \times B \times C$ is countable. Continuing in this way, we can prove by induction that a product of finitely many countable sets is countable.)

Theorem 5 Suppose, for each $n \in \mathbb{N}$, that A_n is countable. Then $\bigcup_{n=1}^{\infty} A_n$ is countable.
(*The union of a countable collection* $\{A_1, ..., A_n, ...\}$ *of countable sets is countable.*)

Proof To show $\bigcup_{n=1}^{\infty} A_n$ is countable, it is sufficient, by to produce a one-to-one map $f\colon \bigcup_{n=1}^{\infty} A_n \to \mathbb{N}$ (by Corollary 2).

1) We begin by showing that we can find pairwise disjoint countable sets B_n such that $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$. (*Then we will just need to show that* $\bigcup_{n=1}^{\infty} B_n$ *is countable.*)

Define
$$\begin{aligned} B_1 &= A_1 \\ B_2 &= A_2 - A_1 \\ B_3 &= A_3 - (A_1 \cup A_2) \\ &\vdots \\ B_n &= A_n - (A_1 \cup ... \cup A_{n-1}) \\ &\vdots \end{aligned}$$

To see that the B_n's are pairwise disjoint:

Consider B_m and B_n where, say, $m < n$. If $s \in B_m = A_m - (A_1 \cup ... \cup A_{m-1})$, then $x \in A_m$. But that implies $x \notin B_n = A_n - (A_1 \cup ... \cup A_m \cup ... \cup A_{n-1})$. So $B_m \cap B_n = \emptyset$.

To see that $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$:

Since $B_n \subseteq A_n$, certainly we have $\bigcup_{n=1}^{\infty} B_n \subseteq \bigcup_{n=1}^{\infty} A_n$.

Conversely, if $x \in \bigcup_{n=1}^{\infty} A_n$, then x is in at least one of the A_n's. Pick the smallest n_0 such that $x \in A_{n_0}$. Then $x \in B_{n_0}$ because x is not in $A_1 \cup ... \cup A_{n_0 - 1}$, the set "discarded" in the definition of B_{n_0}. Therefore $x \in \bigcup_{n=1}^{\infty} B_n$, so $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$.

2) Since each B_n is a subset of a countable set A_n, Corollary 3 tells us that each B_n is countable. We want to show that $\bigcup_{n=1}^{\infty} B_n$ is countable.

There is a one-to-one function $f_n : B_n \to \mathbb{N}$, for each n (since B_n is countable).

List the prime numbers as $p_1 = 2,\ p_2 = 3,\ p_3 = 5, ..., p_n, ...$

Define $f : \bigcup_{n=1}^{\infty} B_n \to \mathbb{N}$ as follows: if $x \in B_n$, then $f(x) = p_n^{fn(x)} \in \mathbb{N}$

For example, if $x \in B_3$, then $f_3 : B_3 \to \mathbb{N}$ and $f_3(x)$ is some natural number. Suppose, to illustrate, that $f_3(x) = 64$. Then $f(x) = p_3^{f_3(x)}$ $= p_3^{64} = 5^{64}$.

Notice that for each $x \in \bigcup_{n=1}^{\infty} B_n$, there is <u>exactly one</u> n for which $x \in B_n$ because the B_n's are pairwise disjoint. This means there's no ambiguity in how f is being defined.

We claim that f is one-to-one. Suppose $x, y \in \bigcup_{n=1}^{\infty} B_n$ and $x \neq y$.

i) if $x \in B_n$ and $y \in B_m$, where $m \neq n$, then

$$f(x) = p_n^{f_n(x)} \neq p_m^{f_m(y)} = f(y) \text{ because } p_n \neq p_m$$

Fundamental Theorem of Arithmetic: $p_n^{f_n(x)}$ can't be factored as a product of p_m's.

ii) if x and y are both in the same B_n, then $f_n(x) \neq f_n(y)$ because the function $f_n : B_n \to \mathbb{N}$ is one-to-one. Therefore

$$f(x) = p_n^{f_n(x)} \neq p_n^{f_n(y)} = f(y)$$

Fundamental Theorem of Arithmetic again: $p_n^{f_n(x)}$ and $p_n^{f_n(y)}$ have different <u>numbers of</u> p_n's in their prime factorizations, so they can't be equal.

Therefore h is one-to-one.

Therefore $\bigcup_{n=1}^{\infty} B_n$ is countable. •

Corollary 6 A union of a finite number of countable sets is countable. (In particular, the union of two countable sets is countable.)
(*This corollary is just a minor "fussy" step from Theorem 5. The way Theorem 5 is stated, it applies to an infinite collection of countable sets $A_1, ..., A_n, ...$ If we have only finitely many, we artificially create the others using $\emptyset$.*)

Proof Suppose $A_1, A_2, ..., A_k$ are countable sets. In order to apply Theorem 5, we define $A_{k+1} = A_{k+2} = A_{k+3} + \ ... \ = \emptyset$.

Then $A_1 \cup ... \cup A_k = A_1 \cup ... \cup A_k \cup \emptyset \cup \emptyset \cup ... = \bigcup_{n=1}^{\infty} A_n$, which is countable. •

Examples i) We proved earlier that the set $\mathbb{Q}^+$ of positive rationals is countable. The set $\mathbb{Q}^-$ of negative rationals is countable since $\mathbb{Q}^- \simeq \mathbb{Q}^+$ (using, for example, the function $f(x) = -x$). The finite set $\{0\}$ is countable.

Therefore $\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+$ is countable, by Corollary 6.

ii) If $B \subseteq A$, and B is uncountable, then A must be uncountable. (*If A were countable, then Corollary 3 would say that B must be countable.*)

For example, $\mathbb{R}$ is uncountable because $(0,1) \subseteq \mathbb{R}$ (*in fact, we already knew $\mathbb{R}$ is uncountable – we proved $\mathbb{R} \simeq (0,1)$*).

$\mathbb{R} \times \{0\} \times \{0\}$ is uncountable (*it's clearly equivalent to* $\mathbb{R}$), and $\mathbb{R} \times \{0\} \times \{0\} \subseteq \mathbb{R}^3$. Therefore $\mathbb{R}^3$ is uncountable.

A similar argument shows that $\mathbb{R}^n$ is uncountable for any integer $n > 1$.

iii) Let $\mathbb{P}$ be the set of irrational numbers. If $\mathbb{P}$ were countable, then $\mathbb{R} = \mathbb{P} \cup \mathbb{Q}$ would be countable (by Corollary 6). Therefore $\mathbb{P}$ is uncountable – so there are "more" irrational numbers than there are rational numbers !!

iv) A harder example: suppose $A = \{a_1, a_2, \dots, a_n, \dots\} \subseteq \mathbb{R}$ and let ϵ be an arbitrary positive number. Let I_n denote the open interval centered at a_n with length $\frac{\epsilon}{2^{n+1}}$, that is $I_n = (a_n - \frac{\epsilon}{2^{n+2}}, a_n + \frac{\epsilon}{2^{n+2}})$. Then $A \subseteq \bigcup_{n=1}^{\infty} I_n$ (because each a_n is in the corresponding I_n); but the total length of all the intervals I_n is $\sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} < \epsilon$.

This says, informally, that any countable subset of $\mathbb{R}$ can be "covered" by a countable collection of open intervals whose total length is $< \epsilon$, where the positive number ϵ can be chosen as small as you like.

So – for a dramatic and counterintuitive example – you can cover the set $\mathbb{Q}$ of all rational numbers with a countable collection of open intervals whose total length is $< 10^{-10}$!!