Markov's Inequality

If X is a random variable that takes only non-negative values, then, for any value $a>0$,

$$P\{X \geq a\} \leq \frac{E(X)}{a}$$

Chebyshev's Inequality

If X is a random variable with finite mean μ & variance σ^2, then, for any value $k>0$

$$P\{|X-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

Example: Suppose that it is known that the number of items produced in a factory during a week is a random variable with mean 50.

(a) What can be said about the prob. that this week's production will exceed 75.

(b) If the variance of a week's production is known to equal 25, then what can be said about the probability that this week's production will be between 40 & 60?

$\Rightarrow$ (a) By Markov's inequality,

$$P\{X > 75\} \leq \frac{E(X)}{75} = \frac{50}{75} = \frac{2}{3}$$

(b) By chebyshev's inequality

$$P\{|X-50| \geq 10\} \leq \frac{\sigma^2}{10^2} = \frac{1}{4}$$

hence, $P\{|X-50| < 10\} = 1-\frac{1}{4} = \frac{3}{4}$

which is the reqd. prob. that week's production will be btn. 40 & 60

☑ The weak law of large numbers

Let $X_1, \ldots, X_n$ be a seqn. of independent & identically distributed random variables, (i.e. if each random variable has the same probability distribution as others and all are mutually independent). each having finite mean $E(X_i)=\mu$, then for each $\epsilon > 0$,

$$P\left\{\left|\frac{X_1+\cdots+X_n}{n} - \mu\right| \geq \epsilon\right\} \to 0 \quad \text{as } n \to \infty$$

☑ Central limit theorem

Let $X_1, X_2, \ldots$ be a seqn. of independent & identically distributed random variables, each having mean μ & variance σ^2. Then the distribution of

$$\frac{X_1+\cdots+X_n - n\mu}{\sigma\sqrt{n}}$$

tends to standard normal as $n \to \infty$. That is for $-\infty < a < \infty$,

$$P\left\{\frac{X_1+\cdots+X_n - n\mu}{\sigma\sqrt{n}} \leq a\right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2}\, dx \quad \text{as } n \to \infty.$$