Pigeonhole Principle (PP) [Rosen]

Ex Among a group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.
(considering a leap year)

Ex For every integer n, there is a multiple of n that has only 0's & 1's in its decimal expansion.

Sol. Let n be a +ve integer.
Let us consider $(n+1)$ integers
$1, 11, 111, 11\ldots\ldots 1$. (where the last integer in the list is the integer with $(n+1)$ [1's] in its decimal expansion)

$\therefore$ There can be n possible remainders when an integer is divided by n.
Because, there are $(n+1)$ integers in the list, by pigeonhole principle, there must be two with the same remainder when divided by n.
The larger of these integers less than the smaller one is a multiple of n, which has a decimal expansion consisting of entirely 0's & 1's

Corollary of the Pigeonhole principle :-

A function f from a set with $(k+1)$ or more elements to a set with k elements is not one-to-one

Proof: Suppose that for each element y in the codomain of f we have a box that contains all elements x of the domain of f such that $y = f(x)$.

Because the domain contains $(k+1)$ or more elements, & the codomain contains only k-elements, the pigeonhole principle tells us that one of these boxes contains two or more elements x of the domain.

$\therefore$ f cannot be one-to-one

Generalized Pigeonhole Principle [Rosen]

If N objects are placed into k boxes, then there must be at least one box containing at least $\lceil N/k \rceil$ objects.

Proof:

Proof by contradiction

Suppose that none of the boxes contains more than $\left(\lceil N/k \rceil - 1\right)$ objects.

$\therefore$ the total no. of objects is at most —

$\lceil N/k \rceil < (N/k) + 1$

$$k\left(\lceil N/k \rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N.$$

$\Rightarrow k\left(\lceil N/k \rceil - 1\right) < N.$

The above is a contradiction because there are a total no. of N objects.

Problem (*) What is the minimum number of objects such that at least r of these objects must be in one of k-boxes when these objects are distributed among the boxes

$\longrightarrow$ When there are N objects, the generalized PP. tells us that there must be at least r-objects in one of the boxes as long as

$$\lceil N/k \rceil \geq r$$

~~eg~~ Among 100 people, there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

~~eg~~ What is the minimum number of students required in a class to be sure that at least 6 will receive the same grade, if there are 5 possible grades A, B, C, D, F.

Ans The minimum no. of students needed to ensure that at least 6 persons receive the same grade is the smallest integer $\lceil N/5 \rceil = 6$. / 8

$\therefore$ If there are 5 grades, then there can minimum of $5 \cdot 5 = 25$ students, where each grade is distributed among 5 people.

If a grade has to be given to at least 6 persons, then the total no. of students should be $25 + 1 = 26$.

Ans $\lceil 26/5 \rceil = 6$

PP (strong & averaging principle) [Richard A Brualdi]

Eg Show that every sequence $a_1, a_2, \dots a_{n^2+1}$ of (n^2+1) real numbers, contains either an increasing subsequence of length $(n+1)$ or a decreasing subsequence of length $(n+1)$

Ans Subsequence :- If $b_1, b_2, \dots b_m$ is a sequence, then $b_{i_1}, b_{i_2}, \dots b_{i_k}$ is a subsequence, provided that $1 \leq i_1 < i_2 < \dots < i_k \leq m$.

$\therefore$ If sequence is $b_1 b_2 b_3 \dots b_8$
then $b_2\, b_4\, b_5\, b_5$ is a subsequence
but $b_2\, b_6\, b_5$ is not X.

Prove by assertion

For each $k = 1, 2, \dots n^2+1$, let m_k be the length of the longest increasing subsequence that begins with a_k.
ie for $k=1$, subsequence starting for a_1 of length 1.
$k=2$, " " " " " " a_2 of length $2^2+1=5$
⋮

Suppose $m_k \leq n$ for each $k = 1, 2 \dots n^2+1$, so that there is no increasing subsequence of length $(n+1)$.

$\therefore$ $m_k \geq 1$ for each $k = 1, 2 \dots n^2+1$, the numbers $m_1, m_2 \dots m_{n^2+1}$ are (n^2+1) integers between 1 to n.

By strong form of PP, $(n+1)$ integers of the numbers
$m_1, m_2 \dots m_{n^2+1}$ are equal.

Let $m_{k_1} = m_{k_2} = \dots m_{k_{n+1}}$

where $1 \leq k_1 < k_2 < \dots < k_{n+1} \leq n^2+1$.

Suppose, for some $i = 1, 2 \dots n$, $a_{k_i} < a_{k_{i+1}}$

Then since $k_i < k_{i+1}$, one could take a longest subsequence beginning with $a_{k_{(i+1)}}$ & put a_{k_i} in front to obtain an increasing subsequence beginning with a_{k_i}

(longest subsequence, length will be $m_{k_{(i+1)}}$)

$\therefore$ the length of the new subsequence starting with a_{k_i} will be $(m_{k_{i+1}} + 1)$

$$\Longrightarrow m_{k_i} > m_{k_{i+1}} \quad \text{i.e.}$$

$$a_{k_i} \geq a_{k_{i+1}}.$$

$\therefore$ This is true for each $i = 1, 2 \dots n$, we have

$$a_{k_1} \geq a_{k_2} \geq \dots \geq a_{k_{n+1}}$$

Thus, $a_{k_1}, a_{k_2}, \dots a_{k_{n+1}}$ is a decreasing subsequence of length $(n+1)$