

Graph Theory - Lecture 3

Walks, Trials, Paths

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1 Walks

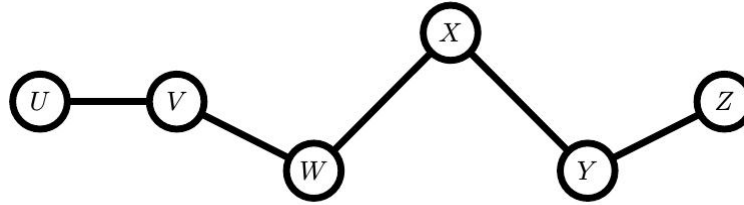


Figure 1: Simple Walk : $UVWXYZ$

Let us consider a graph $G(V, E)$, where $E(G) = \{e_i \in E(G) | i = 1, 2, \dots, n = |E(G)|\}$ and $V(G) = \{v_i \in V(G) | i = 1, 2, \dots, k = |V(G)|\}$

Definition 1.1 (Walk). An ordered sequence of edges such as

$$W(G) = (e_1, e_2, \dots, e_l); e_j \in E(G) \quad (1)$$

where $e_i \in E(G)$ is a walk if there exists a corresponding sequence of vertices (v_0, v_1, \dots, v_l) such that $e_j = (v_{j-1}, v_j)$; the vertices in the walk may not be distinct.

Note : A walk for which $v_0 = v_l$ is a closed walk.

Definition 1.2. The length of a walk is the number of edges in the sequence.

E.g. in [Eq 1], the length of the walk is l .

Definition 1.3 (Distance). The distance $d(u, v)$ between two vertices u and v in a graph $G(V, E)$ is the length of the shortest uv path¹ in G and for any three vertices u, v, w ,

$$d(u, w) + d(w, v) \geq d(u, v). \quad (2)$$

Equation 2 is referred to as the **Triangular Inequality**

Definition 1.4 (Eccentricity). The maximum distance from a vertex $u \in V(G)$ to any other vertex, v in $G(V, E)$ is called the **eccentricity**, $e(u)$.

If w is a vertex such that $e(v) = d(v, w)$, then it is said that the eccentricity of the vertex v is realized by the vertex w .

~~Definition 1.5~~ (Radius). The smallest eccentricity, i.e. $Rad(G) = \min_{v \in V(G)} e(v)$ is the radius of the $G(V, E)$.

~~Definition 1.6~~ (Diameter). The **diameter** $D(G)/Diam(G)$ of $G(V, E)$ is the minimum distance, $d(u, v)$ over all pairs of $\{u, v\}$ pairs of vertices of $V(G)$, i.e.

Maximum

$$Diam(G) = \max_{\{u, v\} \in V(G)} d(u, v)$$

OR

The maximum eccentricity, i.e. $Diam(G) = \max_{v \in V(G)} e(v)$ is the **diameter** of the $G(V, E)$

~~Definition 1.7~~ (Central Vertex). The **Central vertex** is the vertex, for which $e(v) = Rad(G)$.

~~Theorem 1.1.~~ For every non-trivial connected graph $G(V, E)$,

$$Rad(G) \leq Diam(G) \leq 2 \times Rad(G)$$

~~Proof.~~ The inequality $Rad(G) \leq Diam(G)$ is immediate since the smallest eccentricity cannot exceed the largest eccentricity.

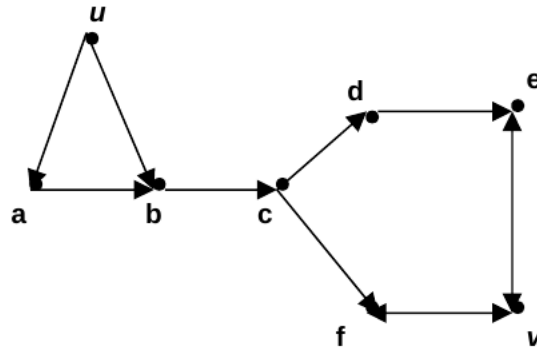
Let u and v be two vertices such that $d(u, v) = Diam(G)$ and let w be a **Central vertex** of $G(V, E)$.

Therefore, $e(w) = Rad(G)$. Hence, the distance between w and any other vertex of $G(V, E)$ is at most $Rad(G)$.

By triangle inequality,

$$\begin{aligned} Diam(G) &= d(u, v) \\ &\leq d(u, w) + d(w, v) \\ &\leq Rad(G) + Rad(G) = 2 \times Rad(G) \end{aligned}$$

□



~~Figure 2:~~ Diameter = ∞ ; Radius = 3

~~Proposition 1.1.~~ If there is a **walk** from vertex u to v , then there is also a **path** from u to v of length $(|V(G)| - 1)$.

~~Proof.~~ If all the vertices are distinct, then there is nothing to prove.

Otherwise, suppose some vertex v_x is repeated in the path. We can remove the segment of the walk between these two v_x 's and obtain another shorter walk from u to v . If this is not a path, we may again repeat the above procedure on this walk.

Clearly, after a finite number of steps, one can obtain a path.

□

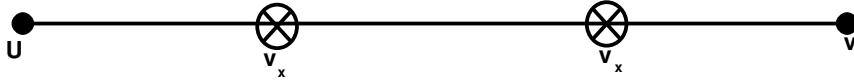


Figure 3: Shorter Path

2 Trial

Definition 2.1 (Trial). A **trial** is a walk in which all the edges e_j are distinct and a **closed trial** is a closed walk, that is also a trial.

A closed trial is also referred to as a Circuit.

3 Path

Definition 3.1 (Path). A **path** is a trial in which all the vertices in the sequence, Eq 1 are distinct

A path is referred to as a P_n , where $n = |P_n|$ is the number of vertices of the path OR order of the path.

Note : A **closed path** is a closed trial in which all the vertices are distinct except $v_0 = v_l$.

4 Cycle

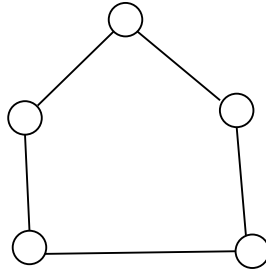


Figure 4: Cycle C_5

Definition 4.1 (Cycle). A **cycle** (C_n , where $n = |C_n|$ is the order of C_n) is a closed path that includes three or more edges.

Note :

- Equivalently, a cycle is a subgraph isomorphic to one of the cycle graphs C_n , e.g. Figure 4
- In set theoretic notations

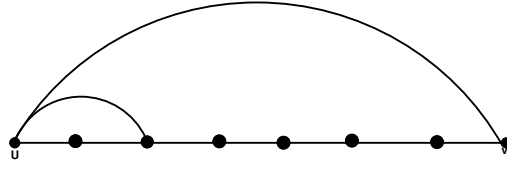
$$Walks \supseteq Trials \supseteq Paths$$

Theorem 4.1. If every vertex of a graph $\mathbf{G}(\mathbf{V}, \mathbf{E})$ has degree at least 2, then $\mathbf{G}(\mathbf{V}, \mathbf{E})$ contains a cycle

Proof. Let $P(G)$ be a maximal path in $\mathbf{G}(\mathbf{V}, \mathbf{E})$, and let $u \in V(G)$ be an endpoint of $P(G)$. Since $P(G)$ cannot be extended, every neighbour of u must already be a vertex of $P(G)$. Since, u has degree at least 2, it has a neighbour $v \in V(P)$ via an edge *not in* $P(G)$

The edge uv completes a cycle with the portion of $P(G)$ from v to u

Note : If $|V(G)| = Z$ and $E(G) = \{ij : |i - j| = 1\}$, then every vertex has degree 2, but $\mathbf{G}(\mathbf{V}, \mathbf{E})$ has no cycle, i.e. no non-extendible path. \square



Lemma 4.2. If every vertex of a (finite) graph $\mathbf{D}(\mathbf{V}, \mathbf{E})$ has out-degree/ $d^+(v)$ (or in-degree/ $d^-(v)$) at least 1, then $\mathbf{D}(\mathbf{V}, \mathbf{E})$ contains a cycle.

Proof. Let $P(D)$ be a maximal path in $\mathbf{D}(\mathbf{V}, \mathbf{E})$, and u be the last vertex on $P(D)$. Since $P(D)$ can not be extended, every successor of u is in $V(D)$. There is at least one successor of u , say v . This edge uv and the path from v to u form a cycle. \square

Abstract

- Walk** : Vertices may REPEAT, edges may REPEAT
- Trail** : Vertices may REPEAT, edges CANNOT REPEAT
- Circuit** : Vertices may REPEAT (Closed Trail)
- Path** : Vertices CANNOT REPEAT, thus edges also CANNOT REPEAT.

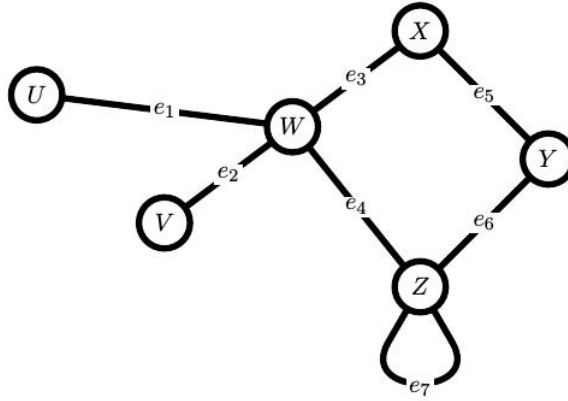


Figure 5: Graph with examples

With respect to Figure 5

1. **Direct walk** between U and Y graph is: $UWXY$
2. A more **roundabout walk** between U and Y could be: $UWVWZZY$
3. Alternative Notation : $Ue_1We_2Ve_2We_4Ze_7Ze_6Y$
4. The walk $Ue_1We_4Ze_7Ze_6Y$ is an **open walk**
5. The walk $Ue_1We_3Xe_5Ye_6Ze_4We_1U$ is **closed walk**
6. The walk $Ue_1We_3Xe_5Ye_6Z$ is a **path**
7. The walk $We_4Ze_7Ze_6Ye_5Xe_3W$ is a **closed trail**
8. The walk $We_3Xe_5Ye_6Ze_4W$ is a **cycle**
9. The walk $Ue_1We_2Ve_2We_4Ze_7Ze_6Ye_7$ is **not a trail or a path**
10. The walk $Ue_1We_4Ze_7Ze_6Y$ is a **trail but not path**