# Graph Theory - Lecture 7
# Planar Graphs

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## 1 Basics

Case Study (GAS-WATER-ELECTRICITY PROBLEM) ;

Three sworn enemies **A-B-C** want to share Gas(G), Water(W) and Electicity(E). To avoid confrontions, the paths of the supplies should not be crossed.

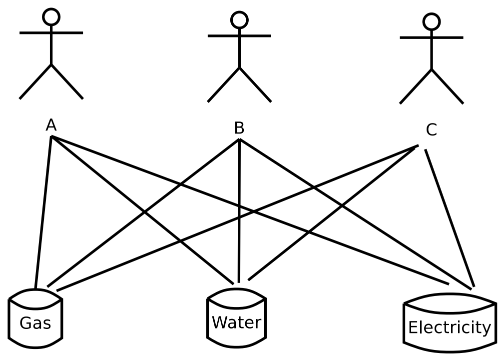


Figure 1: Gas-Water-Electricity Problem

Let $\mathbf{G}(\mathbf{V}, \mathbf{E})$ be a graph and let $\mathbf{S}$ be any surface (e.g. plane, sphere).

Let $P = \{p_1, p_2, \ldots, p_{|V(G)|}\}$ be a set of $|V(G)|$-distinct points of $\mathbf{S}$, $p_i$ corresponding to $v_i \in V(G)$. If $e_i = v_j v_k$, draw an *arc* , $\mathbf{J}_i$ on $\mathbf{S}$ from $p_i$ to $p_k$ such that $\mathbf{J}_i$ does not pass through any other $p_i$. Then $P \bigcup \{J_1, J_2, \ldots, J_m\}$ is called a *drawing* of $\mathbf{G}(\mathbf{V}, \mathbf{E})$ on $\mathbf{S}$ or a diagram representing $\mathbf{G}(\mathbf{V}, \mathbf{E})$ on $\mathbf{S}$.

**Definition 1.1.** *A drawing of a graph $\mathbf{G}(\mathbf{V}, \mathbf{E})$ is a function defined on $V(G) \bigcup E(G)$ that assigns each vertex $v_i$ a point $f(v_i)$ in a plane and assigns each edge with endpoints $v_i, v_j$ a polygonal $f(v_i)f(v_j)$-curve; the images of the vertices being distinct.*

*A point in $f(e_{ij}) \bigcap f(e_{kl})$ that is not a common-point is called a crossing*

**Definition 1.2.** *A graph $\mathbf{G}(\mathbf{V}, \mathbf{E})$ is* **planar** *if it has a drawing without crossings.*

Such a drawing is a **planar embedding** of $\mathbf{G}(\mathbf{V}, \mathbf{E})$ . A *plane graph* is particular planar embedding of planar graph.

*A curve is closed if its first and last points are same. It is simple if it has no repeated points except first and last points.*

**Definition 1.3.** *A* **open set** *in the plane is a set $U \subset \mathbb{R}^2$ such that for every $p_i \in U$, all points within some small distance from $p_i$ blong to $U$.*

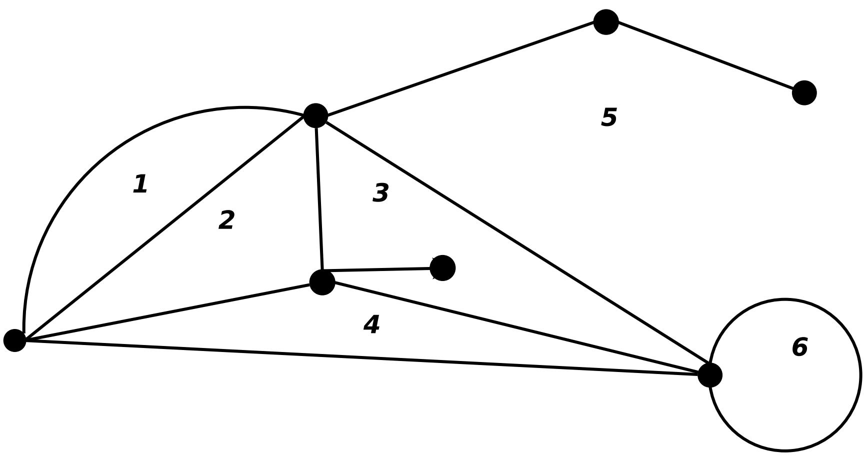


Figure 2: Planar representation of $\mathbf{G(V,E)}$ , where # represents the regions

**Definition 1.4.** *A* **region** *is an open set $U$ that contains a polygonal $v_iv_j$-curve for every pair $v_i, v_j \in U$.*

**Definition 1.5.** *The* **face (f)** *of a plane graph are the maximal regions of the plane that contain no other point used in the embedding.*

- A **plane** representation of a graph divides the plane into *regions* also called *windows, faces and meshes.*
- A *region* is characterised by the set of edges (or th eset of vertices) forming its boundary.
- A *region* **is not** defined in a non-planar graph, or even in a planar graph nor embedded in a plane.
- ⋆⋆ A **region** is a property of the specific plane representation of a graph and not of an abstract graph
- A finite plane graph, $\mathbf{G(V,E)}$ has **one unbounded face** (also called outer face).
- The faces are pairwise disjoint.
- Points $p_i, p_j \in \mathbb{R}^2$ lying in no edge of $\mathbf{G(V,E)}$ are in the same face iff there is a polygonal $p_ip_j$-curve that crosses no edge.

# 2 Euler's Theorem

Euler's Theorem gives the relation between the number of vertices $|V(G)|$, number of edges $|E(G)|$, the number of faces of the planar graph, $|F(G)|$ and the number of components, $K$

**Theorem 2.1.** *If a graph $\mathbf{G(V,E)}$ is a planar graph with $|V(G)|$ vertices, $|E(G)|$ edges, $|F(G)|$ faces and $K$ components, then*

$$\boxed{|V(G)| - |E(G)| + |F(G)| = K + 1} \tag{1}$$

**Theorem 2.2.** *In a connected planar graph with with $|V(G)|$ vertices, $|E(G)|$ edges, $|F(G)|$ faces*

$$\boxed{|V(G)| - |E(G)| + |F(G)| = 2} \tag{2}$$

**Proof 1 of Theorem 2.2 .** Without loss of generality, assume that the planar graph is simple. Since, any simple planar grapg can have a plane representation such that each edge is a straight line, any plananr graph can be drawn such that each region is a polygon (a polygon net). Let the polygon net representing the given graph consists of $|F(G)|$ regions. Let $k_p$ be the number of $p$-sided regions. Since, each edge is on the boundary of exactly two regions,

$$3k_3 + 4k_4 + 5k_5 + \ldots + rk_r = 2|E(G)| \tag{3}$$

where $k_r$ is the number of polygons with $r$ edges.
Also,

$$k_3 + k_4 + k_5 + \ldots + k_r = |F(G)| \tag{4}$$

The sum of all angles subtended at each vertex in the polygon net is

$$2|V(G)|\pi \tag{5}$$

Now, the sum of all interior angles of a $p$-sided polygon is $\pi(p-2)$ and the sum of the exterior angles is $\pi(p+2)$.

Equation (5) is the total sum of all interior angles of $|F(G)|-1$ finite regions plus the sum of the exterior angles of the polygon defining the infinite regions. This sum is

$$\begin{aligned}
\pi(3-2)k_3 + \pi(4-2)k_4 + \ldots + \pi(r-2)k_r + 4\pi \\
= \pi[3k_3 + 4k_4 + \ldots + rk_r - 2(k_3 + k_4 + \ldots + k_r)] + 4\pi \\
= \pi(2|E(G)| - 2|F(G)|) + 4\pi \\
= 2\pi(|E(G)| - |F(G)| + 2)
\end{aligned} \tag{6}$$

Equating equations (5), (6), we get

$$2\pi[|E(G)| - |F(G)| + 2] = 2\pi|V(G)| \tag{7}$$

so that $|F(G)| = |E(G)| - |V(G)| - 2$

□

**Definition 2.1** (Region of a Planar Graph)**.** *Let $\phi$ be a region of a planar graph* $\mathbf{G}(\mathbf{V}, \mathbf{E})$ *. We define the degree of $\phi$, denoted by $d(\phi)$, as the number of edges of the boundary of $\phi$.*

**Proof 2 of Theorem 2.2.** We use *induction* on the number of edges of the planar graph, $|E(G)|$.

If $|E(G)| = 0$, then $\mathbf{G}(\mathbf{V}, \mathbf{E})$ is $K_1$, a complete graph with 1 vertex and 1 region. So, $|V(G)| - |E(G)| + |F(G)| = 1 - 0 + 1 = 2$, and the result is *TRUE*.

If $|E(G)| = 1$, then the number of vertices in $\mathbf{G}(\mathbf{V}, \mathbf{E})$ is either *one* or *two*, the first possibility occuring ehen the edge is a *loop*. These two possibilities give rise respectively to two regions and one region.

Therefore,

$$|V(G)| - |E(G)| + |F(G)| = \begin{cases} 1 - 1 + 2 \textit{ in the loop case} \\ 2 - 1 + 1 \textit{ in the non-loop case} \end{cases}$$

Thus, $|V(G)| - |E(G)| + |F(G)| = 2$ is *TRUE*.

Assume that the result is true for all connected planar graphs with fewer than $|E(G)|$ edges. Let $\mathbf{G}(\mathbf{V}, \mathbf{E})$ have $|E(G)|$ edges.

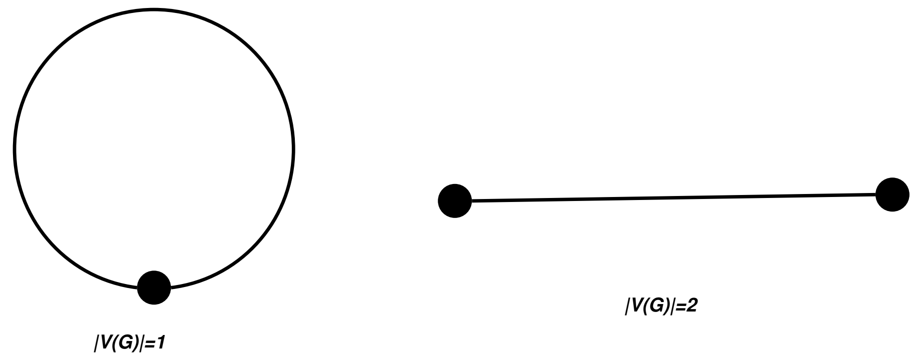


Figure 3: Possibilities for one or two regions

**Case 1 :** Suppose $\mathbf{G}(\mathbf{V},\mathbf{E})$ is a *tree*. Then $|E(G)| = |V(G)| - 1$ and $|F(G)| = 1$, because a planar representation of a tree has one region. Thus, $|V(G)| - |E(G)| + |F(G)| = |V(G)| - (|V(G)| - 1) + 1 = 2$, and the result holds.

**Case 2 :** Suppose $\mathbf{G}(\mathbf{V},\mathbf{E})$ is not a *tree*. Then $\mathbf{G}(\mathbf{V},\mathbf{E})$ has cycles. Let $C(G)$ be a cycle in $\mathbf{G}(\mathbf{V},\mathbf{E})$ . Let $e \in C(G)$ be an edge of of the cycle $C(G)$. The graph $\mathbf{G}(\mathbf{V},\mathbf{E})$ $-e$ has one edge less than the graph $\mathbf{G}(\mathbf{V},\mathbf{E})$ . Also the number of vertices in $V(G-e)$ and $\mathbf{G}(\mathbf{V},\mathbf{E})$ are same. Since, removing $e$ coalesces two , regions in $\mathbf{G}(\mathbf{V},\mathbf{E})$ into one in $\mathbf{G}(\mathbf{V},\mathbf{E})$ $-e$, therefore $\mathbf{G}(\mathbf{V},\mathbf{E})$ $-e$ has one region less that in $\mathbf{G}(\mathbf{V},\mathbf{E})$ .

Thus, by induction hypothesis in $\mathbf{G}(\mathbf{V},\mathbf{E})$ $-e$, we have $|V(G)|-(E(G)-1)+(F(G)-1) = 2$, so that $|V(G)| - |E(G)| + |F(G)| = 2$

Hence, the result.

□

## 3 Upper Bound for the Edges of a Simple Planar Graph

**Theorem 3.1.** *Let graph* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *be a simple planar graph with* $|\mathbf{V}(\mathbf{G})|$ *vertices and* $|\mathbf{E}(\mathbf{G})|$ *edges. where* $|\mathbf{V}(\mathbf{G})| \geq 3$. *Then* $|\mathbf{E}(\mathbf{G})| \leq 3|\mathbf{V}(\mathbf{G})|$ $-6$

*Proof.* First, assume that the planar graph $\mathbf{G}(\mathbf{V},\mathbf{E})$ is connected.

If $|\mathbf{V}(\mathbf{G})| = 3$, then since $\mathbf{G}(\mathbf{V},\mathbf{E})$ is simple. therefore $\mathbf{G}(\mathbf{V},\mathbf{E})$ has at most three edges. Thus, $|\mathbf{E}(\mathbf{G})| \leq 3$, i.e. $|\mathbf{E}(\mathbf{G})| \leq 3 \times 3 - 6$, and the result is *TRUE*.

Now, let $|\mathbf{V}(\mathbf{G})| = 4$.

First, let $\mathbf{G}(\mathbf{V},\mathbf{E})$ be a tree so that $|\mathbf{E}(\mathbf{G})|$ $=$$|\mathbf{V}(\mathbf{G})|$ $-1$. Since, $|\mathbf{V}(\mathbf{G})| \geq 4$, obviously we have $|\mathbf{V}(\mathbf{G})|$ $-1 \leq 3|\mathbf{V}(\mathbf{G})|$ $-6$.

Now, let $\mathbf{G}(\mathbf{V},\mathbf{E})$ *be not a tree*. Then, $\mathbf{G}(\mathbf{V},\mathbf{E})$ has cycles. Clearly, there is a cycle in $\mathbf{G}(\mathbf{V},\mathbf{E})$ , all of whose edges lie on the boundary of the exterior region of $\mathbf{G}(\mathbf{V},\mathbf{E})$ . Then, since $\mathbf{G}(\mathbf{V},\mathbf{E})$ is simple, we have $d(\phi) \geq 3$, for each region $\phi$ of $\mathbf{G}(\mathbf{V},\mathbf{E})$ .

Let

$$b = \sum_{\phi \in F(G)} d(\phi)$$

, where $F(G)$ denotes the set of all regions of $\mathbf{G}(\mathbf{V},\mathbf{E})$ .

Since, each region has at least three edges on its boundary, therfore we have $b \geq 3|F(G)|$, where $|F(G)|$ is the number of regions of $\mathbf{G}(\mathbf{V},\mathbf{E})$ . However, when we sum to get $b$, each edge of $\mathbf{G}(\mathbf{V},\mathbf{E})$ is counted either once or twice (twice when it occurs as a boundary edge for two regions).

Therefore, $b \leq 2|\mathbf{V(G)}|$ , so that $3|F(G)| \leq b \leq 2|\mathbf{V(G)}|$ . In particular, we have $3|F(G)| \leq 2|\mathbf{E(G)}|$ .

Using Euler's formula, $|\mathbf{V(G)}|$ - $|\mathbf{E(G)}|$ +$|F(G)|$, we get $|\mathbf{E(G)}| \leq 3|\mathbf{V(G)}|$ $-6$.

Let graph $\mathbf{G(V,E)}$ be not connected and let $G_1, G_2, \ldots, G_t$ be its connected components. For each $i, 1 \leq i \leq t$, let $|V(G_i)|$ and $|E(G_i)|$ denote the number of vertices and edges in $G_i$. Since each $G_i$ is a planar simple graph, by the above argument, we have $|E(G_i)| \leq 3|V(G_i)| - 6$, for each $i, 1 \leq i \leq t$.

Also, $|\mathbf{V(G)}| = \sum_{i=1}^{t} |V(G_i)|$ and $|\mathbf{E(G)}| = \sum_{i=1}^{t} |E(G_i)|$.

Hence,

$$|\mathbf{E(G)}| = \sum_{i=1}^{t} |V(G_i)| \leq \sum_{i=1}^{t} (3|V(G_i)| - 6) = 3\sum_{i=1}^{t} |V(G_i)| - 6t \leq 3|V(G)| - 6$$

□

# 4 Existence of a vertex of degree less than six in a simple planar graph

**Theorem 4.1.** *If graph* $\mathbf{G(V,E)}$ *is a simple planar graph, then* $\mathbf{G(V,E)}$ *has a vertex* $v$ *of degree less than 6*

*Proof.* If $\mathbf{G(V,E)}$ has only one vertex, the this vertex has degree zero. If $\mathbf{G(V,E)}$ has only two vertices, then both vertices have degree at most one.

Let $|\mathbf{V(G)}| \geq 3$. Assume degree of every vertex in $\mathbf{G(V,E)}$ is at least six.

Then, $\sum_{v \in V(G)} d(v) \geq 6|\mathbf{V(G)}|$

We know, $\sum_{v \in V(G)} d(v) \geq 2|\mathbf{E(G)}|$

Thus, $2|\mathbf{E(G)}| \geq 6|\mathbf{V(G)}|$ , so that $|\mathbf{E(G)}| \geq 3|\mathbf{V(G)}|$ .

But, this is not possible, because by Theorem 3.1, we have $|\mathbf{E(G)}| \leq 3|\mathbf{V(G)}|$ $-6$. Thus, we get a contradiction. Hence, $\mathbf{G(V,E)}$ has at least one vertex of degree less than 6. □

**Corollary 4.2.** $K_5$ *is non-planar.*

*Proof.* Here, $|\mathbf{V(G)}| = 5$ and $|\mathbf{E(G)}| = 10$.

So, $3|\mathbf{V(G)}|$ - 6 = 15 - 6 = 9.

Thus, $|\mathbf{E(G)}| > 3|\mathbf{V(G)}|$ - 6.

Therefore, $K_5$ is non-planar. □

**Corollary 4.3.** $K_{3,3}$ *is non-planar*

*Proof.* Since, $K - 3, 3$ is bipartite, it contains no odd-cycles., and so cycle of length **three** exists. It follows that every region of a plane drawing of $K_{3,3}$ if its exists, has at least **four** boundary edges. We have, thus, $d(\phi) \geq 4.$ for each region $\phi$ of $\mathbf{G(V,E)}$ .

Let $b = \sum_{\phi \in F(G)} d(\phi)$, where $F(G)$ denotes the set of all regions of $\mathbf{G(V,E)}$ . Since, each region has at least **four edges** on its boundary, we have $b \geq 4|F(G)|$.

Now, when we sum up to get $b$, each edge of $\mathbf{G(V,E)}$ is counted either once or twice, and so, $b \leq 2|\mathbf{E(G)}|$ . Thus, $4|F(G)| \leq b \leq 2|\mathbf{E(G)}|$ , so that $2|F(G)| \leq |\mathbf{E(G)}|$ .

For, $K_{3,3}$, we have $|\mathbf{E(G)}|$ =9, and so $2|F(G)| \leq 9$ givig $|F(G)| \leq \frac{9}{2}$. But, by Euler's formula, $|F(G)|$ =$|\mathbf{E(G)}|$ - $|\mathbf{V(G)}|$ + 2 = 9 - 6 + 2 =5, a contradiction.

Hence, $K_{3,3}$ is non-planar. □

# 5 Kuratowski's Graphs

**Definition 5.1.** *The complete graphs $K_5$ and $K_{3,3}$ are non-planar graphs, which are referred to as Kuratowski's Graphs.*

## Observations

1. Both are *regular*
2. Both are non-planar
3. $K_5$ is a non-planar complete graph with the **smallest number of vertices** and $K_{3,3}$ is the non-planar bipartite graph with **smallest number of edges**

The following result is used in proving Kuratowski's theorem.

**Theorem 5.1.**

1. *If* $\mathbf{G}(\mathbf{V},\mathbf{E})\ |e$ *contains a subdivision of* $K_5$ , *then* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *contains a subdivision of* $K_5$ *or* $K_{3,3}$.
2. *If* $\mathbf{G}(\mathbf{V},\mathbf{E})\ |e$ *contains a subdivision of* $K_{3,3}$ , *then* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *contains a subdivision of* $K_{3,3}$.

*Proof.* Let $\mathbf{G}(\mathbf{V},\mathbf{E})^{‘} = \mathbf{G}(\mathbf{V},\mathbf{E})\ |e$ be a graph obtained by contracting the edge $e = v_x v_y$ of $\mathbf{G}(\mathbf{V},\mathbf{E})$ . Let $v_w$ be the vertex of $\mathbf{G}(\mathbf{V},\mathbf{E})^{‘}$ obtained by contracting $e = v_x v_y$.

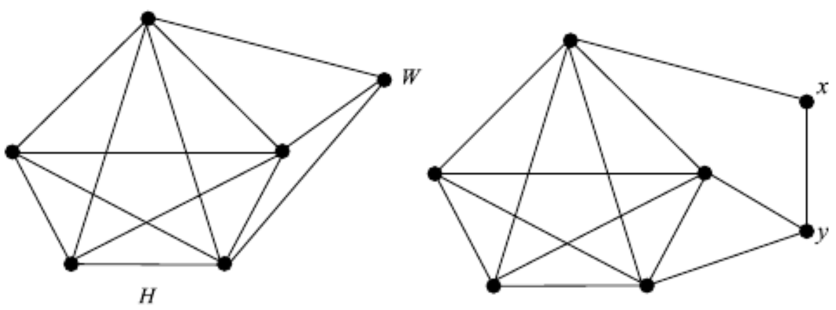


Figure 4:

1. Let $\mathbf{G}(\mathbf{V},\mathbf{E})\ |e$ contains a subdivision of $K_5$, say $H(V,E)$. If $v_w$ is not a branch vertex of $H(V,E)$, then $\mathbf{G}(\mathbf{V},\mathbf{E})$ also contains a subdivision of $K_5$, obtained by expanding $v_w$ back into the edge $v_x v_y$, if necessary (Figure 4).

   Assume $v_w$ is a branch vertex (inner vertex of a tree) of $H(V,E)$ and each of $v_x, v_y$ is incident in $\mathbf{G}(\mathbf{V},\mathbf{E})$ to two of the four edges incident to $v_w$ in $H(V,E)$. Let $u_1$ and $u_2$ be the branch vertices of $H(V,E)$ that are at the other ends of the paths leaving $v_w$ on edges incident to $v_x$ in $\mathbf{G}(\mathbf{V},\mathbf{E})$ . Let $v_1, v_2$ be the branch vertices of $H(V,E)$ that are at the other ends of the paths leaving $v_w$ on edges incident to $v_y$ in $\mathbf{G}(\mathbf{V},\mathbf{E})$ (Figure 5)

   By deleting the $u_1 - u_2$ path and $v_1 - v_2$ path from $H(V,E)$, we obtain a subdivision of $K_{3,3}$ in $\mathbf{G}(\mathbf{V},\mathbf{E})$ , in which $v_y, u_1, u_2$ are branch vertices for one partite set, and $v_x, v_1, v_2$ are branch vertices of the other.

□

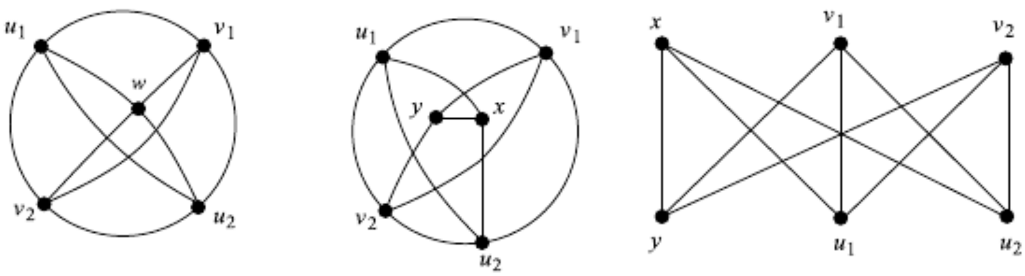


Figure 5: