Relation - A, B

$R \subseteq A \times B$ R is relation from A into B.

~~$A \times A$~~ $A = B$ $R \subseteq A \times A$ R is relation on A.

$A \times A \subseteq A \times A$ → Universal relation $\emptyset \subseteq A \times A$ ← Empty relation.

$(a,b) \in R$, $a \in A$, $b \in B$.

$\{a\} \, a \in A$ and $\exists \, b \in B$ s.t. $(a,b) \in R$ → Domain
$= D(R)$

Image $(I(R))$ $\{b\} \, b \in B$ and $\exists \, a \in A$, $(a,b) \in R\}$

$R : A \to B$

~~$R^{-1} \Rightarrow (a,b) \Rightarrow$~~ R^{-1}: $(b,a) \in R^{-1} \Rightarrow (a,b) \in R$.

$R^{-1} = \{(b,a) \mid (a,b) \in R\}$

R from A to B

L from B to C.

$L \circ R$ from A to C

$(a,c) \in L \circ R$ if $\exists \, b \in B$ s.t. $(a,b) \in R$ & $(b,c) \in L$.

R is a relation ~~of from A into B~~ on A. $R \subseteq A \times A$.

Reflexive

$\forall \, a \in A, \ (a,a) \in R \iff \Delta_A \subseteq R$

Symmetric

$\forall$ ~~(a,b)~~ $a,b \in A$, if $(a,b) \in R$, ~~then~~ $(b,a) \in R$. $R = R^{-1}$.
$\Rightarrow (a,b) \in R^{-1}$

Transitive

$\forall \, a,b,c \in A$, ~~if~~ (a,b) & $(b,c) \in R$, $\Rightarrow$ ~~then~~ $(a,c) \in R$

All 3 → equivalence relation. $\Leftrightarrow R \circ R \subseteq R$.

Δ_A (Diagonal of A) $= \{(a,a) \mid a \in A\}$
→ Reflexive, Symmetric, Transitive ⇒ equivalence.

$\Delta_A \subseteq R$ [Reflexive relation]

Transitive

$R: A \to B$

$L: B \to C$

$(a,b) \in R$ & $(b,d) \in L$

$L \circ R$ $(a,c) \in L \circ R$.

$R \cdot (a,b) \Rightarrow R \circ R \in R$

(a,c), $(c,b) \in R$,

Suppose R be transitive.

(a,d), $(b,c) \in R$.

$(a,c) \in R \circ R$ $\subseteq R$.

$\therefore$ R (Transitive) $\Leftrightarrow$ $R \circ R \subseteq R$.

ρ is a equivalence relation on A iff.

(i) $\Delta_A \subseteq \rho$

(ii) $\rho = \rho^{-1}$

(iii) $\rho\rho \subseteq \rho$.

$(a,b) \in R$

aRb.

$a \in A$.

$[a] \{ b \in A \mid b \rho a \}$

$\Downarrow$

equivalence class of a.

$a \in [a]$. $[a] \neq \emptyset$

$b \in [a] \Rightarrow b \rho a \Rightarrow a \rho b \Rightarrow a \in [b]$.

$x \in [a] \Rightarrow x \rho a \Rightarrow x \rho b \Rightarrow x \in [b]$

$[a] \subseteq [b]$ Similarly $[b] \subseteq [a]$

$[a] = [b]$ $\therefore$ If $a \rho b$ then $[a] = [b]$

(i) $a \in A$, $[a] \in \emptyset$

(ii) If $b \in [a]$, $[a] = [b]$.

(iii) For $a, b \in A$, then $[a] = [b]$ or $[a] \cap [b] = \emptyset$

$a, b \in A$.

$[a]$, $[b]$ Either equal or disjoint.

$c \rho a$, $c \rho b \Rightarrow b \rho c \Rightarrow b \rho a \Rightarrow$ Equal.

Make Equivalent

$R \to$ Useless $\to$ make useful.

$R \cup \Delta_A \to$ Reflex

$R \cup R^{-1} \to$ Symmetry

~~$R \cup R \circ R$~~

$R \circ R = R^2$

$R \circ R^2 = R^3$

$\vdots$

$(a,b), (b,c) \in (R \cup R^2 \cup R^3 \cup \dots) = R^\infty$

~~$\exists$~~ ~~(a,d)~~. $(a,b) \in R^m = R'$

$(b,c) \in R^n = L$.

$(a,c) \in L \circ R' = R^n \circ R^m = R^{n+m} \subseteq R^\infty$

$R = (R \cup \Delta_A \cup R^{-1})^\infty \to$ Equivalence Relation.

Congruence relation

$n \in \mathbb{Z}^+$ $\quad \rho : \equiv_n$ $\quad a, b \in \mathbb{Z}$ $\quad a \equiv_n b$ if $n \mid a-b$

$\downarrow$ Relation

Reflexive $\to$ Yes.

Symmetric $\to$ Yes.

Transitive $\to$ ~~Yes.~~ ~~No~~ Yes.

Equivalence.

$a \equiv b \pmod n$

$\Downarrow$

Equiv Relation.

$a \in \mathbb{Z}$ $\quad [a] = \{ b \mid \cancel{a \equiv_n b} \}$. $\quad b \equiv_n a$

$[a] = \{ b \mid b \equiv a \pmod n \}$

$\hookrightarrow$ Congruence class modulo n.

$m \in \mathbb{Z}$ $\quad \mathbb{Z}_m = \{ [a] \mid a \in \mathbb{Z} \}$ $\parallel$

$= \{ [0], [1], \dots [m-1] \}$

~~$m, k \ \exists \ q, r$.~~

$k = mq + r$

$k \in [r]$

$[k] = [r]$.

$P = \{B \mid B \subset A\} \to$ partition

(i) $B_i, B_j \in P$, either $B_i = B_j$ or $B_i \cap B_j = \emptyset$

(ii) $A = \bigcup_{B \in P} B$.

$A \to$

$a \in A$

$[a] \to$ Equivalence class.

$\{[a] \mid a \in A\} \Rightarrow$ Partition.

Partition P

$a, b \in A$ $\quad a \sim_P b$ if $\exists B \in P$ $a, b \in B$.

Equivalence relation induced by partition.