Mathematics (dt: 16/03/2022)

Bakshi Sir (2nd Class) (Probability)

Exercise:

Q) For any 3 events A, B, & C, show that

$$P(A \cup B | C) = P(A|C) + P(B|C) - P(A \cap B|C)$$

Soln ⇒ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P[(A \cap C) \cup (B \cap C)] = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$

obtained by replacing A with $A \cap C$ & B with $B \cap C$

Dividing both sides by $P(C)$

$$\Rightarrow \frac{P[(A \cap C) \cup (B \cap C)]}{P(C)} = \frac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{P(C)}$$

$$\Rightarrow \frac{P[(A \cup B) \cap C]}{P(C)} = \frac{P(A \cap C)}{P(C)} + \frac{P(B \cap C)}{P(C)} - \frac{P(A \cap B \cap C)}{P(C)}$$

distributive property

$$\Rightarrow P[(A \cup B)|C] = P(A|C) + P(B|C) - P(A \cap B|C)$$

(Hence Proved)

Q Let A_1, A_2 & A_3 be 3 events defined on a sample space S, such that $\underline{A_2 \subset A_3}$ & $P(A_1) > 0$,

Prove that $P(A_2 | A_1) \le P(A_3 | A_1)$

Sol => $P(A_3 | A_1) = \dfrac{P(A_3 \cap A_1)}{P(A_1)}$

X and Y are clearly mutually exclusive

$$= \frac{P[\overbrace{(A_2 \cap A_3 \cap A_1)}^{X} \cup \overbrace{(A_2^c \cap A_3 \cap A_1)}^{Y}]}{P(A_1)}$$

$$= \frac{P(A_2 \cap A_3 \cap A_1)}{P(A_1)} + \frac{P(A_2^c \cap A_3 \cap A_1)}{P(A_1)}$$

A_2 is a subset of A_3 ($\because A_2 \subset A_3$)

$$= P(A_2 \cap A_3 | A_1) + P(A_2^c \cap A_3 | A_1)$$

[as X & Y are mutually exclusive]

$$= P(A_2 | A_1) + \underbrace{P(A_2^c \cap A_3 | A_1)}_{\geq 0}$$

[as $A_2 \subset A_3$]

$\boxed{A_2 \text{ is a subset of } A_3 \Rightarrow A_2 \cap A_3 = A_2}$

since it is a probability & hence have to be in $[0, 1]$.

$\therefore \underline{P(A_3 | A_1) \geq P(A_2 | A_1)}$.

Q In a certain college, 25% of boys & 10% of girls are studying mathematics. The girls constitute 60% of the student body

(i) what is the probability that maths is being studied

(ii) If a student is selected at random & is found to be studying maths, what is the prob. that the student is a girl

G → the student is a girl

B → the student is a boy

M → maths is being studied

$P(G) = 0.6$

$P(B) = 0.4$

$P(M|B) = 0.25$

$P(M|G) = 0.10$

(i) $P(M) = P(G) \cdot P(M|G) + P(B) \cdot P(M|B)$

$= 0.60 \times 0.10 + 0.4 \times 0.25$

$= 0.06 + 0.1$

$= \cancel{1.06}$ $= 0.16$.

(ii) $P(G|M) = \dfrac{P(G)\, P(M|G)}{P(M)}$ $\left(\because P(G|M) = \dfrac{P(G \cap M)}{P(M)} \right.$

now

$P(G \cap M) = P(G)\, P(M|G)$

$= \dfrac{(0.6) \times (0.10)}{0.16}$

as

$P(G)\, P(M|G)$

$= \dfrac{\cancel{P(G)}\, P(M \cap G)}{\cancel{P(G)}}$

Q At an electronics part, it is known from past experience that the probability of a new worker who attended company's program meets the production quota is 0.9. The corresponding probability for a new worker who did not attend the training program is 0.25. It is also known that 80% of all new workers attend the company's training program. Find the probability that a new worker who made the production quota would have attended the company's training program.

Sol^n =>

$E_1 \to$ a new worker who attended company's training

$E_2 \to$ a new worker who didn't attend company's training

$A \to$ a new worker met prod quota.

$P(E_1) = 80\% = 0.8$

$P(E_2) = 0.2$

$P(A|E_1) = 0.9$

$P(A|E_2) = 0.25$

Using Baye's Th.

$$P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

$$= \frac{(0.8)(0.9)}{(0.8)(0.9)+(0.2)(0.25)} \text{ Ans}$$

$P(X \le 1)$

$= 0.3 + 4k$

$\therefore \quad P(X \le 1) \geq 0.32$

$\Rightarrow 0.3 + 4k \geq 0.32$

$\Rightarrow 4k \geq 0.02$

$\Rightarrow k \geq \frac{0.02}{4}$

$\Rightarrow k \geq \frac{\frac{2}{100}}{\cancel{4}\,2}$

$\Rightarrow k \geq \frac{\frac{1}{100}}{2}$

$\Rightarrow k \geq \frac{1}{200}$

$\therefore$ minimum value of $k = \frac{1}{200}$

Continuous R.V

1) Probability density f^n

$f(x) \rightarrow$ p.d.f of a continuous r.v. X

(i) $f(x) \geq 0 \qquad -\infty < x < \infty$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(iii) $P(a < x < b) = \int_a^b f(x)\,dx$.

2) Dist. Func

domain $(-\infty, \infty)$

$F(x) = P(X \le x)$

$= \int_{-\infty}^{x} f(t)\,dt$

range $[0, 1]$

$\boxed{F(-\infty) = 0 \quad F(\infty) = 1}$

Q A random variable X has the following probability dist.

$X=x$	-2	-1	0	1	2	3
$P(x=x)$	0.1	k	0.2	$3k$	$2k$	0.3

(i) Determine k

(ii) $P(X<2)$ & $P(X \geq 2)$

(iii) Find the minimum value of k, such that $P(X \leq 1) \geq 0.32$

Solⁿ

(i) use the property

$$\sum_{i=1}^{n} P_i = 1$$

$\Rightarrow 0.1 + k + 0.2 + 3k + 2k + 0.3 = 1$

$\Rightarrow 0.6 + 6k = 1$

$\Rightarrow 6k = 0.4$

$\Rightarrow k = \frac{0.4}{6} = \frac{1}{15}$

(ii) $P(x<2)$

$= P(x \leq 1)$

$= 0.1 + k + 0.2 + 3k$

$\cancel{= 0.1 + 0.15 + 0.2 + 0.45}$

$\cancel{= 0.3 + 0.6}$

$\cancel{= 0.9}$

$= 0.1 + \frac{1}{15} + 0.2 + \frac{4}{15}$

$= \frac{17}{32}$

$P(x \geq 2)$

$= 1 - P(x<2)$

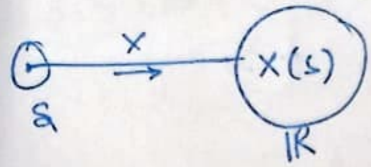
$= 1 - \frac{17}{30}$

$= \frac{13}{30}.$

Random Variable

$X: S \rightarrow \mathbb{R}$

X is the f^n that maps our Sample Space S to $\mathbb{R}$.



Discrete Random Variable

the $X(s)$ points in $\mathbb{R}$ are discrete

Continuous Random Variable

It's an interval where infinite values are possible

Discrete Random Variable

(1) Probability Mass Function

$X \longrightarrow x_1, x_2, x_3, \ldots, x_n$

$\quad\quad p_1 \quad p_2 \quad p_3 \ldots p_n$

$\boxed{p_i \rightarrow \text{probability that } x = x_i}$

then
$$\boxed{\begin{aligned} P(x) = P(x = x_i) &= p_i \quad \text{if } x = x_i \\ &= 0 \quad \text{if } x \neq x_i \end{aligned}}$$

$$\boxed{\sum_{i}^{n} P(x_i) = 1}$$

(2) Distribution Function

$X \rightarrow$ Discrete R.V.

$$\underbrace{F(x)}_{\substack{\downarrow \\ (-\infty,\infty) \\ \text{Domain}}} = P(X \leq x) = \sum_{i=1}^{x} P(x_i)$$

$\hookrightarrow$ range $[0,1]$ (since it is a probability)

$F(-\infty) = 0 \longrightarrow$ no x_i gets considered.

$F(\infty) = 1 \longrightarrow$ all x_i's get considered

Case 1:

$x = 3$

$x_i: \underbrace{0, 1, 2, 3}\ 4$

$\boxed{P = \sum p_i}$

Case 2:

$x = 3$

$x_i: \underbrace{0\ 1\ 2}\ 4$

$P = \sum p_i \leftarrow$ So add all the prob upto floor(x) in x_i's.

Expectation

(i) $X \to$ discrete random variable

(ii) $E(x) = \sum_{i=1}^{n} p_i x_i$ | $X \to \begin{matrix} x_i & \cdots & x_n \\ p_i & \cdots & p_n \end{matrix}$

$= \sum_{i=1}^{n} P(X=x_i) \cdot x_i$ | X takes the value x_i with prob p_i

(iii) $X \to$ cont. r.v

then, $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$.

$f(x)$ the P.D.F is analogous to ~~f(x) the~~ p_i ~~f(x)~~ $(X = x_i)$ the P.M.F

Variance

for discrete r.v

$\sigma^2 = E\left[(x-\mu)^2\right]$

$= \sum_i (x_i - \mu)^2 \cdot P(X = x_i)$ $\longrightarrow$ by using the general formula of $E(x)$ for discrete r.v

for continuous random variable

$\sigma^2 = E\left[(x-\mu)^2\right]$

$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$ $\longrightarrow$ by using the general formula of $E(X)$ for random variable.

$S.D = \sqrt{Var(x)} = \sigma$

$= \pm\sqrt{E(x-\mu)^2}$

$x \geq 2$; $F(x) = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{1} f(t)\,dt + \int_{1}^{2} f(t)\,dt + \int_{2}^{\infty} f(t)\,dt$

$\cancel{= 0 + \frac{1}{2} + \left(2t - \frac{t^2}{2}\right)_1^2 + \int_2^{\infty} 0 \cdot dt}$

$\underbrace{}_{= 0}$

$\cancel{= 0 + \frac{1}{2} + 4 - \frac{1}{2}}$

$\cancel{= 4}$

$= 0 + \frac{1}{2} + \int_1^2 (2 - x)\,dx + 0$

$= \frac{1}{2} + \left(2x - \frac{x^2}{2}\right)_1^2$

$= \frac{1}{2} + (4 - 2) - \left(2 - \frac{1}{2}\right)$

$= \frac{1}{2} + \cancel{2} - \cancel{2} + \frac{1}{2}$

$= 1$

$\therefore$

$$\boxed{\begin{aligned} F(x) &= 0 && x < 0 \\ &= \frac{x^2}{2} && 0 \leq x < 1 \\ &= 2x - \frac{x^2}{2} - 1 && 1 \leq x \leq 2 \\ &= 1 && x > 2 \end{aligned}}$$

Q. The prob. density of a r.v X is

$$f(x) = \begin{cases} 0 & -\infty < x \leq 0 \\ x & 0 \leq x \leq 1 \\ 2-x & 1 \leq x < 2 \\ 0 & x \geq 2 \end{cases}$$

obtain cummulative distribution function of X

Consider the intervals separately

$x<0$; $F(x) = \int_{-\infty}^{0} f(t)\,dt = \int_{-\infty}^{0} 0\,dt = 0$

$$\cancel{F(x) = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{x} f(t)\,dt = 0 + \int_{0}^{1} f(t)\,dt + \int_{1}^{2} f(t)\,dt + \int_{2}^{x} f(t)\,dt}$$

$0 \leq x < 1$; $F(x) = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{x} f(t)\,dt$

$$= 0 + \int_{0}^{x} t\,dt$$

$$= \left(\frac{t^2}{2}\right)_0^x = \frac{x^2}{2}$$

$1 \leq x < 2$ $F(x) = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{1} f(t)\,dt + \int_{1}^{x} f(t)\,dt$

$$= 0 + \int_{0}^{1} t\,dt + \int_{1}^{x} 2-t\,dt$$

$$= 0 + \left(\frac{t^2}{2}\right)_0^1 + \left(2t - \frac{t^2}{2}\right)_1^x$$

$$= 0 + \frac{1}{2} + \left(2t - \frac{t^2}{2}\right)_1^x$$

$$= 0 + \frac{1}{2} + 2x - \frac{x^2}{2} - 2 + \frac{1}{2} = 2x - \frac{x^2}{2} - 1$$