Markov chains

Consider a seqn. of random variable $X_0, X_1, \ldots,$ and suppose that the set of possible values of these random variable is $\{0, 1, \ldots, M\}$. Interprete X_n as the state of system at time n. According to this interpretation, we say that the system is in the state i at the time n, if $X_n = i$.

The seqn. of random variables is said to form a Markov chain if each time the system is in state i, there is a fixed prob. P_{ij}, that the system will move to the state j next.

i.e., for all $i_0, \ldots, i_{n-1}, i, j$

$$P\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\}$$

$$= P_{ij}.$$

The values P_{ij}, $0 \le i \le M$, $0 \le j \le M$ are called transition probabilities of the Markov chain, satisfying

$$P_{ij} \geq 0 \quad \& \quad \sum_{j=0}^{M} P_{ij} = 1 \qquad i = 0, 1, \ldots, M$$

Basically, Markov chain is a seqn. of random variables $X_0, X_1, \ldots,$ with the Markov property, namely that the probability of moving to the next state depends only on the present state & not on the previous states,

i.e.,
$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0)$$

$$= P(X_{n+1} = j \mid X_n = i).$$

$$= P_{ij}$$

It is convenient to arrange the transition probabilities P_{ij} in a square array

$$\begin{bmatrix} P_{00} & P_{01} & \cdots & P_{0M} \\ P_{10} & P_{11} & \cdots & P_{1M} \\ \vdots & & & \\ P_{M0} & P_{M1} & \cdots & P_{MM} \end{bmatrix}$$

This is called the transition matrix.

The joint probability mass function of $X_0, \ldots, X_n$ is given by:

$$P\{X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0\}$$

$$= P\{X_n = i_n \mid X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} \, P\{X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\}$$

$$= P_{i_{n-1} i_n} \cdot P\{X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\}$$

and continual repetition of this argument demonstrates that the preceding is equal to

$$P_{i_{n-1} i_n} \, P_{i_{n-2} i_{n-1}} \cdots P_{i_1 i_2} \cdot P_{i_0 i_1} \, P\{X_0 = i_0\}$$

Example

The husband and wife physicists ~~Paul and Tatyana~~ considered a conceptual model for the movement of molecules in which M molecules are distributed among 2 urns. At each time period one of the molecules is chosen at random and is removed from its urn and placed in the other one. If we let X_n denote the number of molecules in the first urn immediately after the nth exchange, then $\{X_0, X_1, \ldots\}$ is a Markov chain with transition probabilities,

$$P_{i,i+1} = \frac{M-i}{M} \qquad 0 \le i \le M$$

$P_{ij} = 0$ if $|j-i| > 1$

The two step transition $P_{ij}^{(2)}$ that the system presently in state i will be in state j after two additional transition is,

$$P_{ij}^{(2)} = P\{X_{m+2} = i \mid X_m = i\}$$

It can be computed from P_{ij} as:

$$P_{ij}^{(2)} = P\{X_2 = i \mid X_0 = i\}$$

$$= \sum_{k=0}^{M} P\{X_2 = j, X_1 = k \mid X_0 = i\}$$

$$= \sum_{k=0}^{M} P\{X_2 = j \mid X_1 = k, X_0 = i\} \cdot P\{X_1 = k \mid X_0 = i\}$$

$$= \sum_{k=0}^{M} P_{kj} P_{ik}$$

In general we define the n-stage transition probabilities, denoted as $P_{ij}^{(n)}$, by

$$P_{ij}^{(n)} = P\{X_{n+m} = j \mid X_m = i\}$$

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$$P_{ij}^{(n)} = \sum_{k=0}^{M} P_{ik}^{(r)} P_{kj}^{(n-r)} \quad \text{for all } 0 < r < n$$

proof

$$P_{ij}^{(n)} = P\{X_n = j \mid X_0 = i\}$$

$$= \sum_{k} P\{X_n = j, X_r = k \mid X_0 = i\}$$

$$= \sum_k P\{X_n = j \mid X_r = k, X_0 = i\} \cdot P\{X_r = k \mid X_0 = i\}$$

$$= \sum_k P_{kj}^{(n-r)} \cdot P_{ki}^{(r)}$$

For a large number of Markov chains, it turns out that $P_{ij}^{(n)}$ converges to a value π_j as $n \to \infty$. π_j depends only on j. i.e., for large 'n', the prob. of being in state j, after n transitions is approximately equal to π_j, no matter what the initial state was.

The sufficient condition for a Markov chain to possess the above property is that, for some $n > 0$,

$$P_{ij}^{(n)} > 0 \text{ for all } i = 0, 1, \ldots, M. \rightarrow (1)$$

Markov chains, satisfying (1), is said to be Ergodic.

Now, (1) yields
$$P_{ij}^{(n+1)} = \sum_{k=0}^{M} P_{ik}^{(n)} P_{kj} \rightarrow (2)$$

it follows, by letting $n \to \infty$ for ergodic chains,

$$\pi_j = \sum_{k=0}^{M} \pi_k P_{kj} \rightarrow (3)$$

Furthermore, since $1 = \sum_{j=0}^{M} P_{ij}^{(n)}$, we also obtain, by letting $n \to \infty$,

$$\sum_{j=0}^{M} \pi_j = 1 \rightarrow (4)$$

It can be shown that π_j, $0 \leq j \leq M$ are unique non-negative solutions of eqn. (3) & (4).

Theorem:

For an ergodic Markov chain:

$$\pi_j = \lim_{n\to\infty} p_{ij}^{(n)}$$ exists and

the π_j, $0 \le j \le M$ are unique non-negative soln's of

$$\pi_j = \sum_{k=0}^{M} \pi_k P_{kj}$$

$$\sum_{j=0}^{M} \pi_j = 1.$$

Example:

Suppose that whether it rains tomorrow depends on previous weather conditions only through whether it is raining today. Suppose that if it rains today, then it will rain tomorrow with prob. α and if it is not raining today, then it will rain tomorrow with prob. β.

If we say that the system is in state 0 when it rains & state 1 when it does not. Calculate π_0, & π_1.

$\Rightarrow$ From the above theorem, the limiting probabilities π_0 & π_1 of rain and no rain are given by,

$$\pi_0 = \alpha\pi_0 + \beta\pi_1$$

$$\pi_1 = (1-\alpha)\pi_0 + (1-\beta)\pi_1$$

$$\pi_0 + \pi_1 = 1$$

which yields,

$$\pi_0 = \frac{\beta}{1+\beta-\alpha}, \quad \pi_1 = \frac{1-\alpha}{1+\beta-\alpha}$$

For instance, if $\alpha = 0.6$, $\beta = 0.3$, then the limiting prob. of rain on the n th day is $\pi_0 = \frac{3}{}$

Absorbing and transient states:

A state of a Markov chain is called an absorbing state, if once the Markov chain enters the state, it remains there forever.

i.e., $P_{kk} = 1$ & $P_{kj} = 0$ for $j \neq k$ & $0 \leq k \leq M$

A state is called transient if the system, starts from that particular state & have zero prob. of returing to the same state.

If the system returns to the particular state, where it started, ~~is called~~ then the state is called recurrent state.