M.C. Escher – Smaller and Smaller
(1956)

Computer Graphics

Lecture 4

Transformations

Overview

2D Transformations

- Basic 2D transformations
- Matrix representation
- Matrix composition

3D Transformations

- Basic 3D transformations
- Same as 2D

2D Modeling Transformations

Modeling Coordinates

Scale
Translate

Scale
Rotate
Translate

World Coordinates

2D Modeling Transformations

Modeling Coordinates

Let's look at this in detail…

World Coordinates

2D Modeling Transformations

Modeling Coordinates

Initial location at (0, 0) with x- and y-axes aligned

2D Modeling Transformations

Modeling Coordinates

Scale .3, .3
Rotate -90
Translate 5, 3

2D Modeling Transformations

Modeling Coordinates

Scale .3, .3
Rotate -90
Translate 5, 3

2D Modeling Transformations

Modeling Coordinates

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Translate 5, 3

World Coordinates

Scaling

Scaling *a coordinate means multiplying each of its components by a scalar*

Uniform scaling *means this scalar is the same for all components:*

Scaling

Non-uniform scaling: ***different scalars per component:***

How can we represent this in matrix form?

Scaling

Scaling operation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}$$

Or, in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\text{scaling matrix}} \begin{bmatrix} x \\ y \end{bmatrix}$$

2-D Rotation

$$x' = x\cos(\theta) - y\sin(\theta)$$
$$y' = x\sin(\theta) + y\cos(\theta)$$

2-D Rotation

$x = r \cos(\varphi)$

$y = r \sin(\varphi)$

$x' = r \cos(\varphi + \theta)$

$y' = r \sin(\varphi + \theta)$

Trig Identity…

$x' = r \cos(\varphi) \cos(\theta) - r \sin(\varphi) \sin(\theta)$

$y' = r \sin(\varphi) \cos(\theta) + r \cos(\varphi) \sin(\theta)$

Substitute…

$x' = x\,\mathbf{cos}(\theta) - y\,\mathbf{sin}(\theta)$

$y' = x\,\mathbf{sin}(\theta) + y\,\mathbf{cos}(\theta)$

2-D Rotation

This is easy to capture in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Even though sin(θ) and cos(θ) are nonlinear functions of θ,

- ***x' is a linear combination of x and y***
- ***y' is a linear combination of x and y***

Basic 2D Transformations

Translation:

- $x' = x + t_x$
- $y' = y + t_y$

Scale:

- $x' = x * s_x$
- $y' = y * s_y$

Shear:

- $x' = x + h_x * y$
- $y' = y + h_y * x$

Rotation:

- $x' = x*\cos\Theta - y*\sin\Theta$
- $y' = x*\sin\Theta + y*\cos\Theta$

Transformations
can be combined
(with simple algebra)

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$x' = x*s_x$
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- $y' = x*\sin\Theta + y*\cos\Theta$

$$x' = (x*s_x)*\cos\Theta - (y*s_y)*\sin\Theta$$
$$y' = (x*s_x)*\sin\Theta + (y*s_y)*\cos\Theta$$

Basic 2D Transformations

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- $x' = x*\cos\Theta - y*\sin\Theta$
- $y' = x*\sin\Theta + y*\cos\Theta$

$$x' = ((x*s_x)*\cos\Theta - (y*s_y)*\sin\Theta) + t_x$$
$$y' = ((x*s_x)*\sin\Theta + (y*s_y)*\cos\Theta) + t_y$$

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$$y' = ((x*s_x)*\sin\Theta + (y*s_y)*\cos\Theta) + t_y$$

Overview

2D Transformations

- Basic 2D transformations
- Matrix representation
- Matrix composition

3D Transformations

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- Same as 2D

Matrix Representation

Represent 2D transformation by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Multiply matrix by column vector

⇔ apply transformation to point

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

Matrix Representation

Transformations combined by multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrices are a convenient and efficient way to represent a sequence of transformations!

2x2 Matrices

What types of transformations can be represented with a 2x2 matrix?

2D Identity?

$$x' = x$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Scale around (0,0)?

$$x' = s_x * x$$
$$y' = s_y * y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

What types of transformations can be represented with a 2x2 matrix?

2D Rotate around (0,0)?

$$x' = \cos\Theta * x - \sin\Theta * y$$
$$y' = \sin\Theta * x + \cos\Theta * y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Shear?

$$x' = x + sh_x * y$$
$$y' = sh_y * x + y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & sh_x \\ sh_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

What types of transformations can be represented with a 2x2 matrix?

2D Mirror about Y axis?

$$x' = -x$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Mirror over (0,0)?

$$x' = -x$$
$$y' = -y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

What types of transformations can be represented with a 2x2 matrix?

2D Translation?

$$x' = x + t_x$$

$$y' = y + t_y$$

NO !

Only linear 2D transformations can be represented with a 2x2 matrix

Linear Transformations

Linear transformations are combinations of …

- Scale,
- Rotation,
- Shear, and
- Mirror

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Properties of linear transformations:

- Satisfies:

$$T(s_1\mathbf{p}_1 + s_2\mathbf{p}_2) = s_1 T(\mathbf{p}_1) + s_2 T(\mathbf{p}_2)$$

- Origin maps to origin
- Lines map to lines
- Parallel lines remain parallel
- Ratios are preserved
- Associative but not commutative

Homogeneous Coordinates

Q: How can we represent translation as a 3x3 matrix?

$$x' = x + t_x$$

$$y' = y + t_y$$

Homogeneous Coordinates

Homogeneous coordinates

- represent coordinates in 2 dimensions with a 3-vector

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\text{homogeneous coords}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous coordinates seem unintuitive, but they make graphics operations much easier

Homogeneous Coordinates

Q: How can we represent translation as a 3x3 matrix?

$$x' = x + t_x$$

$$y' = y + t_y$$

A: Using the rightmost column:

$$\mathbf{T}\boldsymbol{ranslation} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Translation

Example of translation

Homogeneous Coordinates

α

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

Add a 3rd coordinate to every 2D point

- (x, y, w) represents a point at location (x/w, y/w)
- (x, y, 0) represents a point at infinity
- (0, 0, 0) is not allowed

(2,1, 1) or (4,2,2) or (6,3,3)

Convenient coordinate system to represent many useful transformations

Basic 2D Transformations

Basic 2D transformations as 3x3 matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & sh_x & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear

Affine Transformations

Affine transformations are combinations of …

- Linear transformations, and
- Translations

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Properties of affine transformations:

- Origin does not necessarily map to origin
- Lines map to lines
- Parallel lines remain parallel
- Ratios are preserved
- Associative but not commutative

Overview

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Matrix Composition

Transformations can be combined by matrix multiplication

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$$\mathbf{p}' = T(t_x,t_y) \quad R(\Theta) \quad S(s_x,s_y) \quad \mathbf{p}$$

Matrix Composition

Matrices are a convenient and efficient way to represent a sequence of transformations

- General purpose representation
- Hardware matrix multiply

$$\mathbf{p}' = (T * (R * (S*\mathbf{p}))$$

$$\mathbf{p}' = (T*R*S) * \mathbf{p}$$

Matrix Composition

Be aware: order of transformations matters

– *Matrix multiplication is not commutative*

p’ = T * R * S * **p**

“Global” ⟷ “Local”

Matrix Composition

What if we want to rotate *and* translate?

- Ex: Rotate line segment by 45 degrees about endpoint *a*

  and lengthen

Multiplication Order – Wrong Way

Our line is defined by two endpoints

- Applying a rotation of 45 degrees, R(45), affects both points
- We could try to translate both endpoints to return endpoint *a* to its original position, but by how much?

Multiplication Order - Correct

Isolate endpoint *a* from rotation effects

- First translate line so *a* is at origin: T (-3)
- Then rotate line 45 degrees: R(45)
- Then translate back so *a* is where it was: T(3)

Matrix Composition

Will this sequence of operations work?

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ 1 \end{bmatrix} = \begin{bmatrix} a'_x \\ a'_y \\ 1 \end{bmatrix}$$

p’ = (T * (R * (S***p**)
))
p’ = (T*R*S) * **p**

Matrix Composition

After correctly ordering the matrices Multiply matrices together

What results is one matrix – store it (on stack)!

Multiply this matrix by the vector of each vertex

All vertices easily transformed with one matrix multiply

Reflection about an arbitrary line

Translate the line as well as the object so that the line passes through origin

Rotate the line and the object about the origin until the line is coincident with one of the coordinate axes

Reflect the object through the coordinate axis

Rotate back

Translate back

Reverse Rotations

Q: How do you undo a rotation of θ, $R(\theta)$?

A: Apply the inverse of the rotation… $R^{-1}(\theta) = R(-\theta)$

How to construct $R^{-1}(\theta) = R(-\theta)$

- Inside the rotation matrix: $\cos(\theta) = \cos(-\theta)$
  - *The cosine elements of the inverse rotation matrix are unchanged*
- The sign of the sine elements will flip: $\sin(-\theta) = -\sin(\theta)$

Therefore… $R^{-1}(\theta) = R(-\theta) = R^{T}(\theta)$

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3D Transformations

Same idea as 2D transformations

- Homogeneous coordinates: (x,y,z,w)
- 4x4 transformation matrices

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Basic 3D Transformations

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Identity

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Mirror about Y/Z plane

3-D Rotation About an Axis

Positive rotation is counterclockwise, when looking from positive direction along an axis.

Basic 3D Transformations

Rotate around Z axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 & 0 \\ \sin\Theta & \cos\Theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Rotate around Y axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} \cos\Theta & 0 & \sin\Theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\Theta & 0 & \cos\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Rotate around X axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Theta & -\sin\Theta & 0 \\ 0 & \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Summary

Coordinate systems

- World vs. modeling coordinates

2-D and 3-D transformations

- Trigonometry and geometry
- Matrix representations
- Linear vs. affine transformations

Matrix operations

- Matrix composition