Computer Graphics 12: Spline Representations

Contents

Today we are going to look at Bézier spline curves

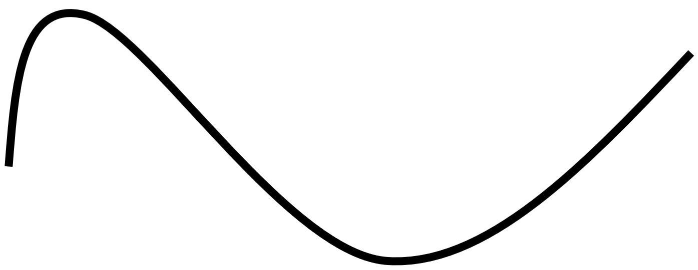
- Introduction to splines
- Bézier curves
- Bézier cubic splines

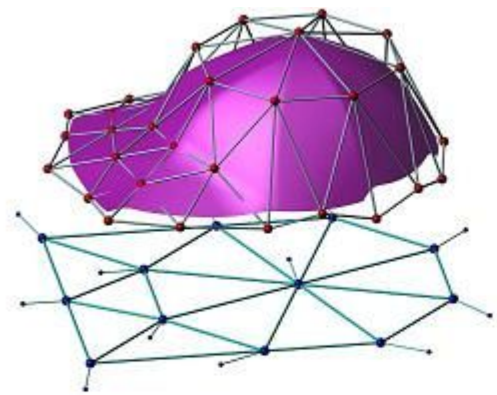
Spline Representations

A spline is a smooth curve defined mathematically using a set of constraints

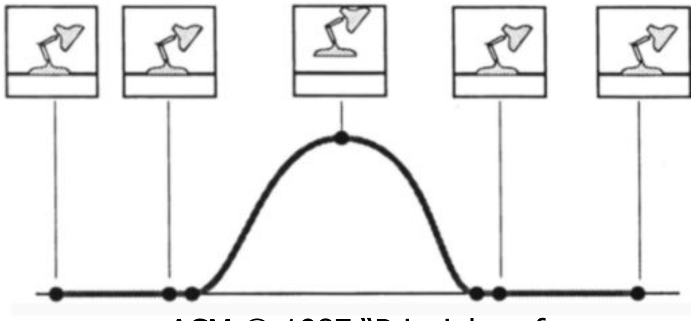
Splines have many uses:

- – 2D illustration
- – Fonts
- – 3D Modelling
- – Animation





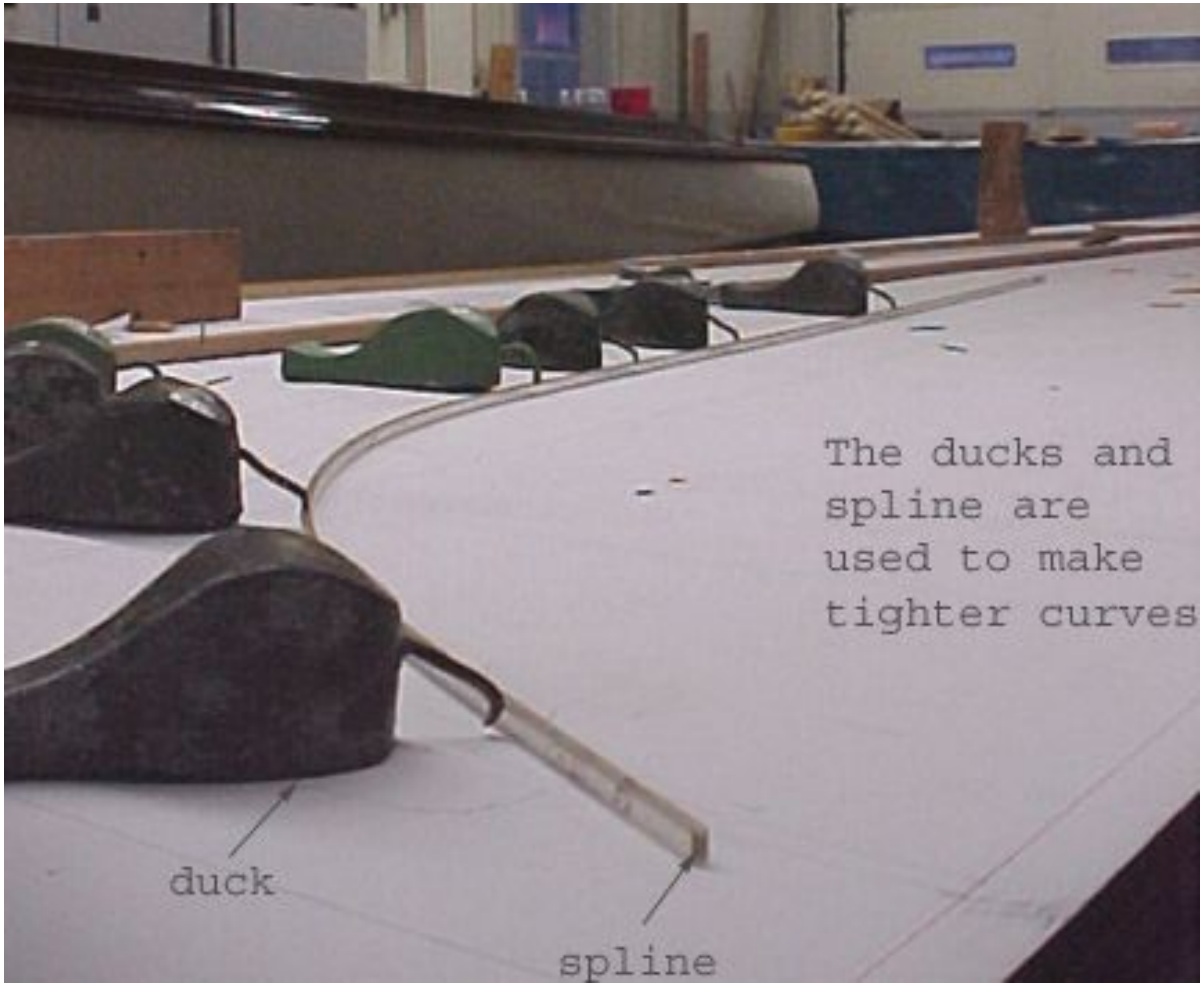
“Manifold Splines”, X. Gu, Y. He & H. Qin, Solid and Physics Modeling 2005.



ACM © 1987 “Principles of traditional animation applied to 3D computer animation”

Physical Splines

Physical splines are used in car/boat design

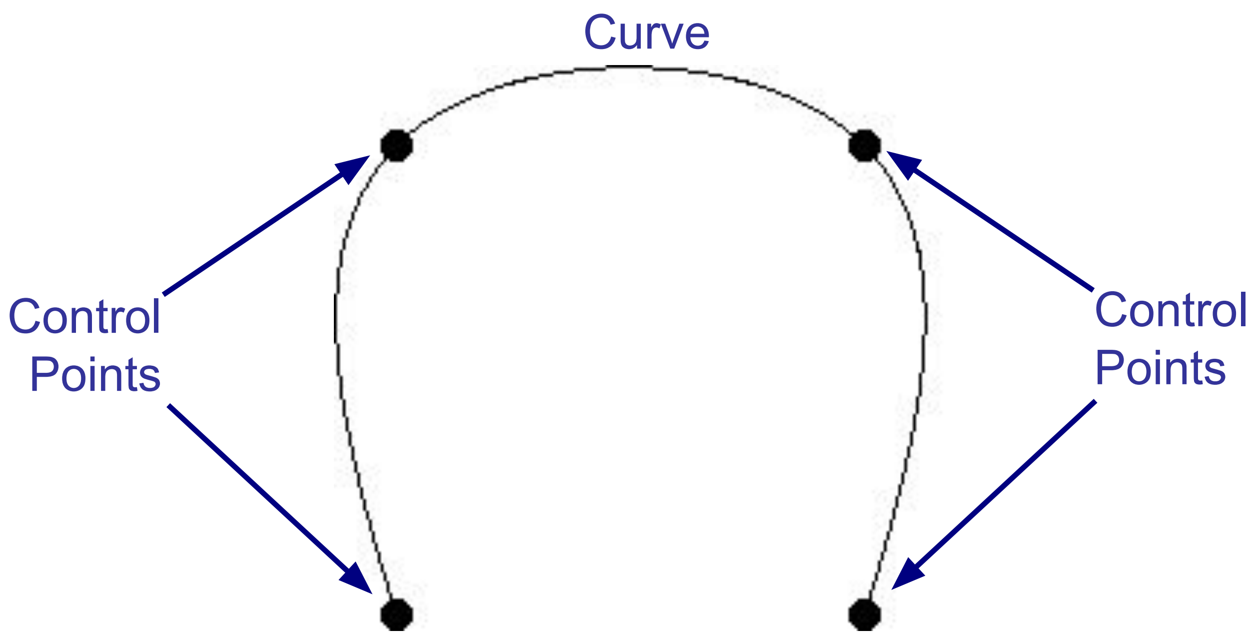


Pierre Bézier

Big Idea

User specifies control points

Defines a smooth curve

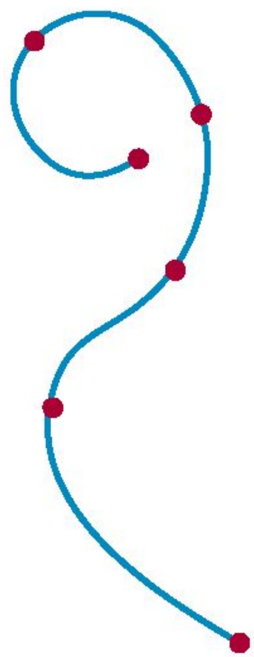


Interpolation Vs Approximation

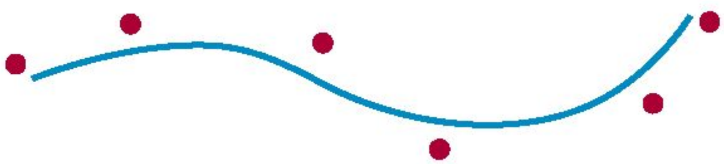
A spline curve is specified using a set of **control points**

There are two ways to fit a curve to these points:

- **Interpolation -** the curve passes through all of the control points



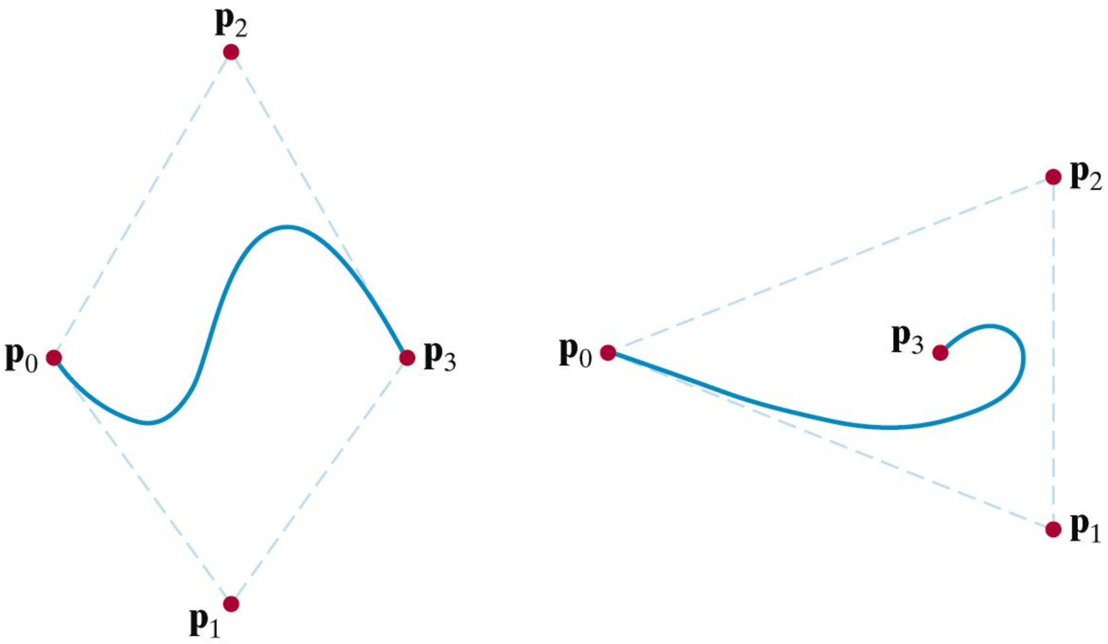
- **Approximation -** the curve does not pass through all of the control points



Convex Hulls

The boundary formed by the set of control points for a spline is known as a **convex hull**

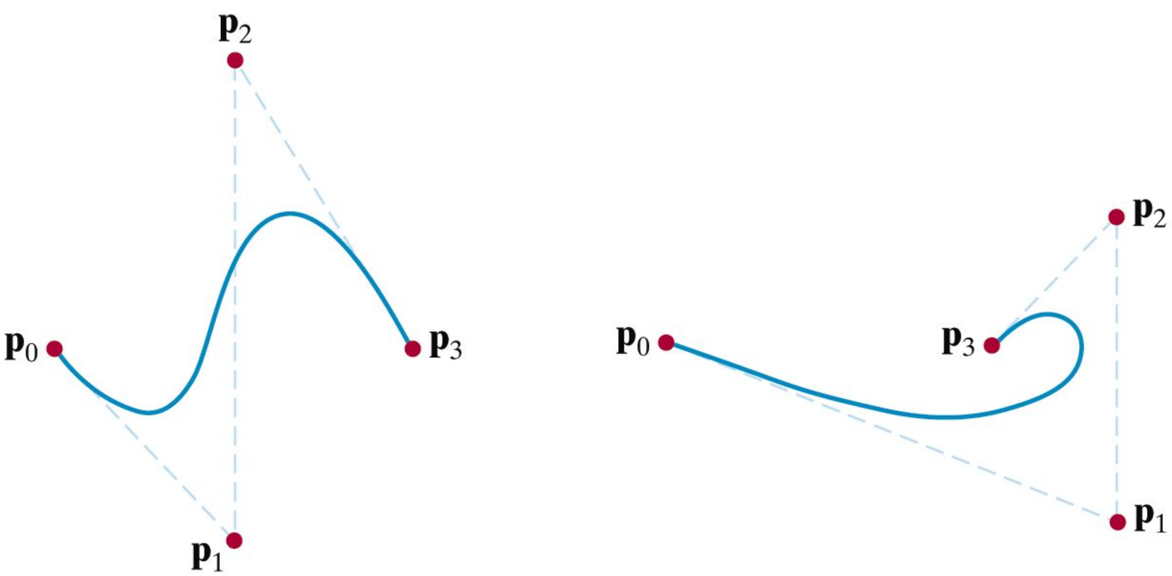
Think of an elastic band stretched around the control points



Control Graphs

A polyline connecting the control points in order is known as a **control graph**

Usually displayed to help designers keep track of their splines



Bézier Spline Curves

A spline approximation method developed by the French engineer Pierre Bézier for use in the design of Renault car bodies

A Bézier curve can be fitted to any number of control points – although usually 4 are used

Bézier Spline Curves (cont…)

Consider the case of $n+1$ control points denoted as $p_k=(x_k, y_k, z_k)$ where k varies from 0 to n

The coordinate positions are blended to produce the position vector $P(u)$ which describes the path of the Bézier polynomial function between p_0 and p_n

$$P(u) = \sum_{k=0}^{n} p_k BEZ_{k,n}(u), \qquad 0 \le u \le 1$$

Bézier Spline Curves (cont…)

The Bézier blending functions $BEZ_{k,n}(u)$ are the *Bernstein polynomials (basis function)*

$$BEZ_{k,n}(u) = C(n,k)u^k(1-u)^{n-k}$$

where parameters $C(n,k)$ are the binomial coefficients

$$C(n,k) = \frac{n!}{k!(n-k)!}$$

Bézier Spline Curves (cont…)

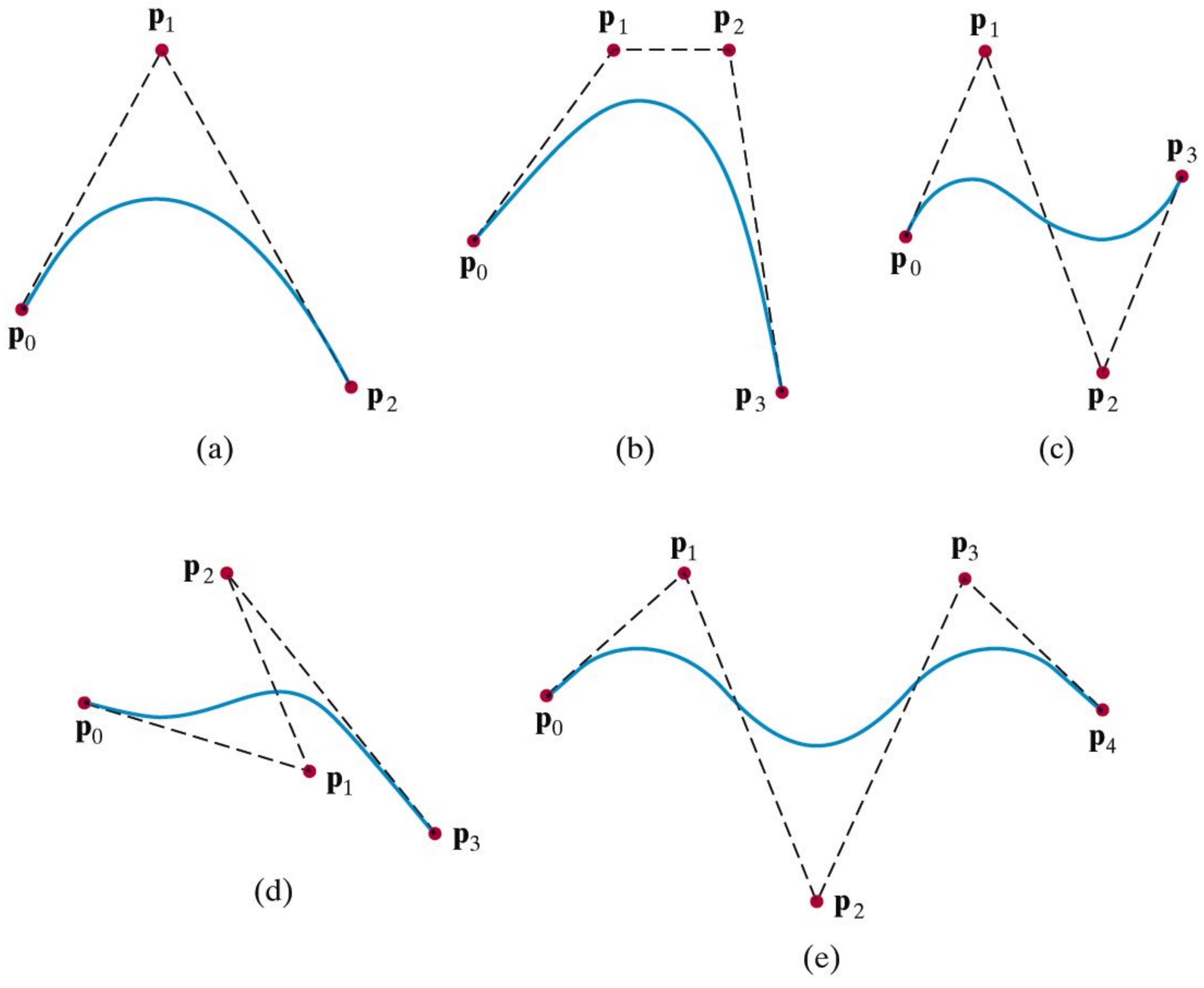
So, the individual curve coordinates can be given as follows

$$x(u) = \sum_{k=0}^{n} x_k BEZ_{k,n}(u)$$

$$y(u) = \sum_{k=0}^{n} y_k BEZ_{k,n}(u)$$

$$z(u) = \sum_{k=0}^{n} z_k BEZ_{k,n}(u)$$

Bézier Spline Curves (cont…)



Important Properties Of Bézier Curves

The first and last control points are the first and last point on the curve

- $P(0) = p_0$
- $P(1) = p_n$

The curve lies within the convex hull as the Bézier blending functions are all positive and sum to 1

$$\sum_{k=0}^{n} BEZ_{k,n}(u) = 1$$

The slope at the beginning and end of the curve are along the along the first two and the last two points respectively

Cubic Bézier Curve

Many graphics packages restrict Bézier curves to have only 4 control points (i.e. n = 3)

The blending functions when n = 3 are simplified as follows:

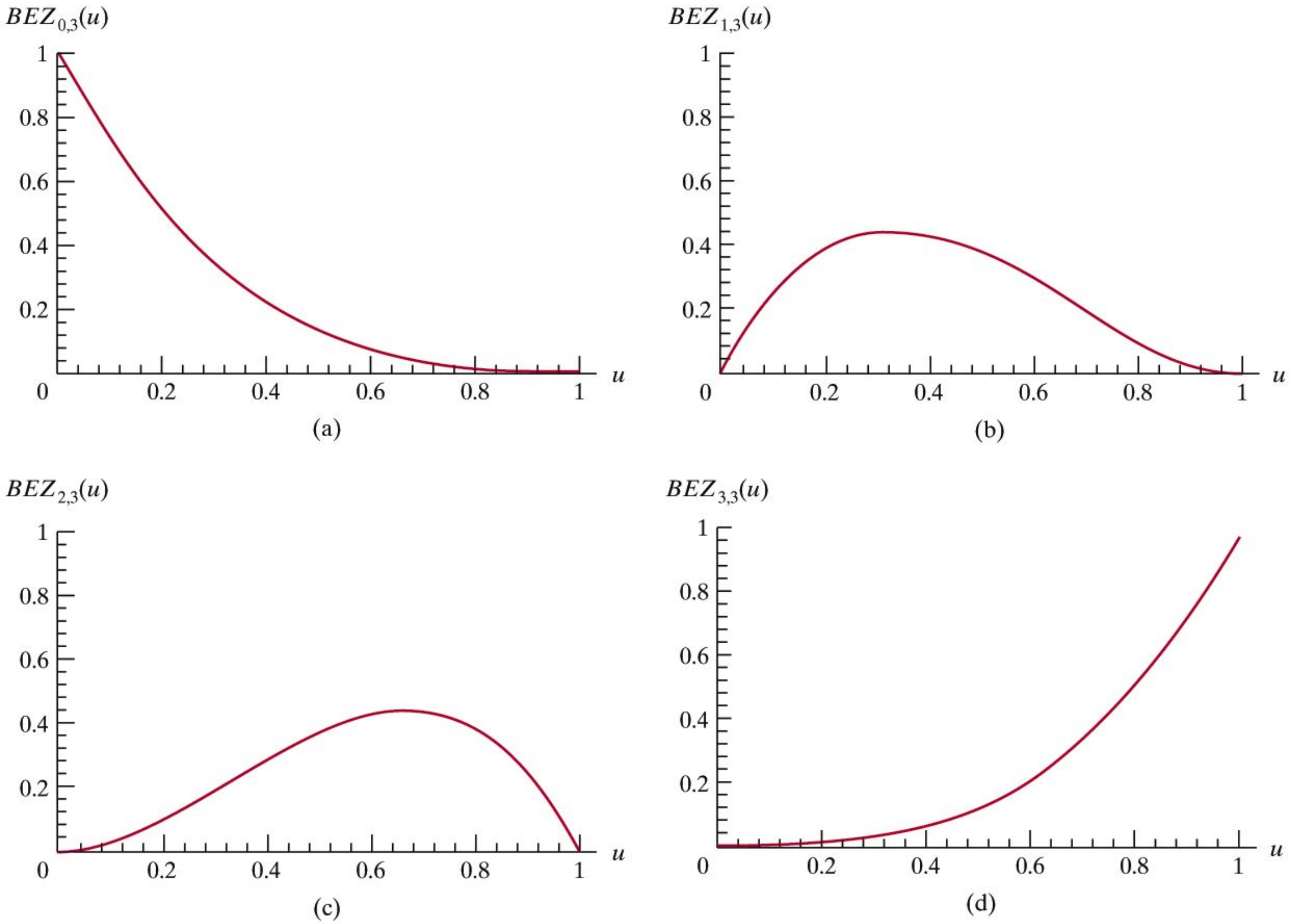
$$BEZ_{0,3} = (1-u)^3$$

$$BEZ_{1,3} = 3u(1-u)^2$$

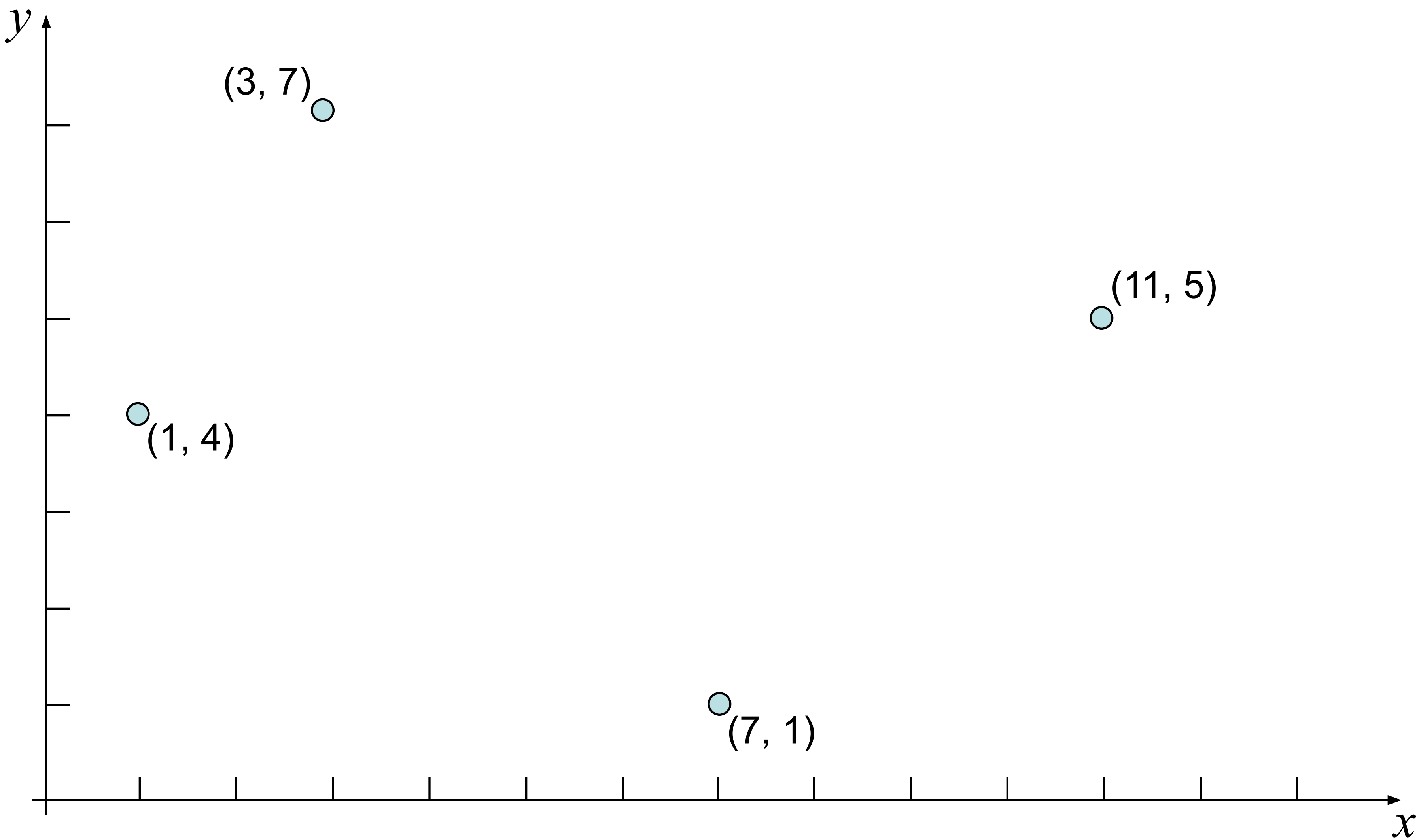
$$BEZ_{2,3} = 3u^2(1-u)$$

$$BEZ_{3,3} = u^3$$

Cubic Bézier Blending Functions



Bézier Spline Curve Exercise

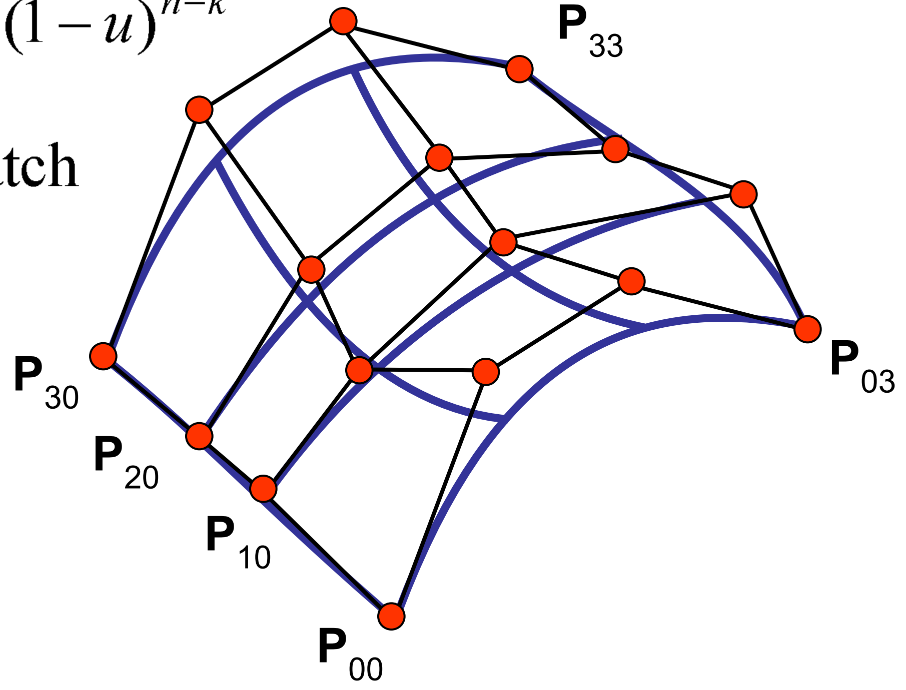


Bézier surface

$$\text{Generic}: \mathbf{P}(u,v) = \sum_{\mathrm{j}=0}^{\mathrm{m}} \sum_{\mathrm{k}=0}^{\mathrm{n}} B_{\mathrm{j}}(v) B_{\mathrm{k}}(u) \mathbf{P}_{\mathrm{j,k}}$$

$$\text{with } B_{\mathrm{k,n}}(u) = \frac{n!}{k!(n-k)!} u^k (1-u)^{n-k}$$

$$\text{Often } m = n = 4: \text{bicubic patch}$$



Limitations of Bézier Curve

- Two characteristics of the Bernstein basis limit the flexibility of the Bezier curves
  - First the number of specified polygon vertices fixes the order of the resulting polynomial which defines the curve.
  - Second the global nature of the Bernstein basis limits the ability to produce a local change within a curve.

B-spline curves

Composed of a series of m-2 curve segments, $Q_3, Q_4, \ldots, Q_m$ controlled by (m + 1) control points P_0, P_1, ..., P_m

each curve segment is controlled by 4 control points over a given knot intervals:

- P_0, P_1, P_2, P_3 --->Q_3 defined on $[u_3, u_4]$
- P_1, P_2, P_3, P_4 --->Q_4 defined on $[u_4, u_5]$
- ...
- P_{m-3}, P_{m-2}, P_{m-1}, P_m --->Q_m defined on $[u_m, u_{m+1}]$

each control point influences *four* curve segments. (*local control property*)

B-spline curves

Entire set of curve segments as one B-spline curve in u:

$$Q(u) = \sum_{i=0}^{m} P_i B_i(u)$$

- $i=[0,m]$, the non-local control point number
- $u=[3, m+1]$, global parameter

Uniform B-splines (Definition)

The joint point on the value of u between segments is called the knot value.

Uniform B-spline means that knots are spaced at equal intervals of the parameter u.

Basis functions are defined over 4 successive knot intervals:

- $B_0(u)$: $[u_0, u_1, u_2, u_3, u_4]$
- $B_1(u)$: $[u_1, u_2, u_3, u_4, u_5]$
- ...
- $B_m(u)$: $[u_{m-3}, u_{m-2}, u_{m-1}, u_m, u_{m+1}]$

(m-2) curve segments

- Q3: P0,P1,P2,P3 u = [3,4]
- Q4: P1,P2,P3,P4u = [4,5]
- ……
- Qm: $P_{m-3}, P_{m-2}, P_{m-1}, P_m$u = [m,m+1]

Uniform B-splines (Basis functions and curve computation)

Basis function for Qi(u): (0≤u≤1)

$$B_i = \frac{1}{6}u^3$$

$$B_{i-1} = \frac{1}{6}(-3u^3 + 3u^2 + 3u + 1)$$

$$B_{i-2} = \frac{1}{6}(3u^3 - 6u^2 + 4)$$

$$B_{i-3} = \frac{1}{6}(1 - u)^3$$

$$Q_i(u) = P_i B_i(u) + P_{i-1}B_{i-1}(u) + P_{i-2}B_{i-2}(u) + P_{i-3}B_{i-3}(u)$$

It is important to note that this definition gives a single segment from each of the 4 B-spline basis functions over the range 0≤u≤1.

It does not define a single B-spline basis function which consists of four segments over the range 0≤u≤4.

Summary

Today we had a look at spline curves and in particular Bézier & B-Spline curves

The whole point is that the spline functions give us an approximation to a smooth curve