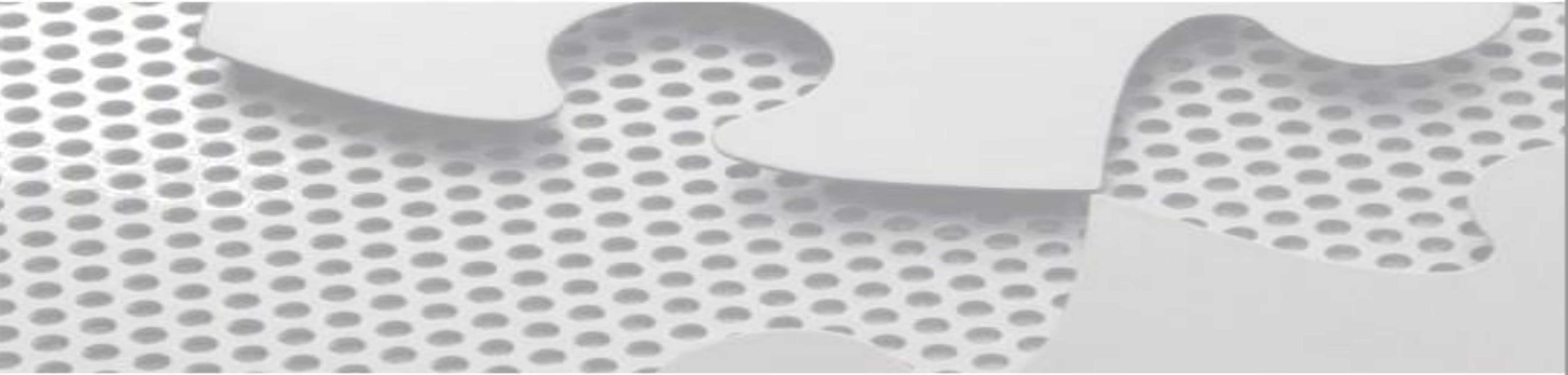
Logic Programming



Chapter 4

Companion slides from the book

Introduction

- **Logic**: the science of reasoning and proof
  - Existed since the time of ancient Greek philosophers
- Logic is closely associated with computers and programming languages
  - Circuits are designed using Boolean algebra
  - Logical statements are used to describe **axiomatic semantics**, the semantics of programming languages
- Logic programming typically restricts itself to well-behaved fragments of logic

https://groups.seas.harvard.edu/courses/cs152/2015sp/lectures/lec23-prolog.pdf

Logic Programming

- We can think of logic programs as having two interpretations.
- In the declarative interpretation, a logic program declares what is being computed.
- In the procedural interpretation, a logic program describes how a computation takes place.
- In the declarative interpretation, one can reason about the correctness of a program without needing to think about underlying computation mechanisms
- This makes declarative programs easier to understand, and to develop.
- A lot of the time, once we have developed a declarative program using a logic programming language, we also have an executable specification
  - A procedural interpretation that tells us how to compute what we described.
  - This is one of the appealing features of logic programming.
  - An understanding of the underlying computational mechanism is needed to make the execution of the declarative program efficient

Logic and Logic Programs

- **First-order predicate calculus**: a way of formally expressing logical statements
- **Logical statements**: statements that are either true or false
- **Axioms**: logical statements that are assumed to be true and from which other true statements can be **proved**

Logic and Logic Programs (cont'd.)

Symbols in statements:

- *Constants (a.k.a. atoms)*
  numbers (e.g., `0`) or names (e.g., `bill`).
- *Predicates*
  Boolean functions (true/false) . Can have arguments. (e.g. `parent(X,Y)`).
- *Functions*
  non-Boolean functions (`successor(X)`).
- *Variables*
  Stands for yet unspecified quantities
  e.g., `X`.
- *Connectives (operations)*
  `and, or, not`
  `implication (→):a→b (b or not a)`
  `equivalence (↔): a↔b (a→b and b→a)`

Logic and Logic Programs (cont'd.)

- Example 1:
  - These English statements are logical statements

    0 is a natur

    2 is a natu

    For all *x*, i al number, then so is the successor of *x*.

    predicate

  - Translation into predi

    constant

    Bound Variable

```
natural(0).
natural(2).
For all x, natural(x) ⟶ natural(successor (x)).
```

  - *x* in the third statement is a variable that stands for an as yet unspecified quantity

Logic and Logic Programs (cont'd.)

- Example 1 (cont'd.)
  - First and third statements are axioms
  - Second statement can be proved since

$$2 = \mathrm{successor}(\mathrm{successor}(0)) \text{ and } \mathrm{natural}(0) \rightarrow \mathrm{natural}(\mathrm{successor}(0)) \rightarrow \mathrm{natural}(\mathrm{successor}(\mathrm{successor}(0)))$$

Logic and Logic Programs (cont'd.)

- **Universal quantifier**: a relationship among predicates is true for all things in the universe named by the variable
  - Ex: *for all x, natural(x) natural(successor(x))*
- **Existential quantifier**: a predicate is true for at least one thing in the universe indicated by the variable
  - Ex: *there exists x, natural(x)*
- A variable introduced by a quantifier is said to be **bound** by the quantifier

Logic and Logic Programs (cont'd.)

- A variable not bound by a quantifier is said to be **free**
- Arguments to predicates and functions can only be **terms**: combinations of variables, constants, and functions
  - Terms cannot contain predicates, quantifiers, or connectives

Quantifier examples

- Examples:
  - $\forall x(\text{speaks}(x,\text{Russian}))$
  - $\exists x(\text{speaks}(x,\text{Russian}))$
  - $\forall x\ \exists y(\text{speaks}(x,y))$
  - $\exists\ x\ \forall\ y(\text{speaks}(x,y))$

Example 2

- *"Every man is mortal.*
- *John is a man. Therefore, John is mortal"*

Every man is mortal	:	$(\forall x)\ (MAN(x) \rightarrow MORTAL(x))$
John is a man	:	$MAN(john)$
John is mortal	:	$MORTAL(john)$

Logic and Logic Programs (cont'd.)

- First-order predicate calculus also has inference rules
- **Inference rules**: ways of deriving or proving new statements from a given set of statements
- Example: from the statements $a \rightarrow b$ and $b \rightarrow c$, one can derive the statement $a \rightarrow c$, written formally as:

$$\frac{(a \rightarrow b) \text{ and } (b \rightarrow c)}{a \rightarrow c}$$

Logic and Logic Programs (cont'd.)

- **Logic programming language**: a notational system for writing logical statements together with specified algorithms for implementing inference rules

- **Logic program**: the set of logical statements that are taken to be axioms

- The statement(s) to be derived can be viewed as the input that initiates the computation
  - Also called **queries** or **goals**

Logic and Logic Programs (cont'd.)

- Logic programming systems are sometimes referred to as **deductive databases**
  - Consist of a set of statements and a deduction system that can respond to queries
  - System can answer queries about facts and queries involving implications
- **Control problem**: specific path or sequence of steps used to derive a statement

Logic and Logic Programs (cont'd.)

- The statements represent the logic of the computation while the deductive systems provide the control by which a new statement is derived
- Logical programming paradigm as a pseudo equation (Kowalski):

  algorithm = logic + control

- Compare this with imperative programming (Wirth):

  algorithms = data structures + programs

- Since logic programs do not express the control, in theory, operations can be carried out in any order or simultaneously
  - Logic programming languages are natural candidates for parallelism

Logic and Logic Programs (cont'd.)

- Automated deduction systems have difficulty handling all of first-order predicate calculus
  - Too many ways of expressing the same statements
  - Too many inference rules
- Most logic programming systems restrict themselves to a particular subset of predicate calculus called Horn clauses

Horn Clauses

- **Horn clause**: a statement of the form

$$a_1 \text{ and } a_2 \text{ and } a_3 \ldots \text{ and } a_n \rightarrow b$$

- The a_i are only allowed to be simple statements
  - No *or* connectives and no quantifiers allowed
- This statement says that a_1 through a_n imply b, or that b is true if all the a_i are true
  - b is the **head** of the clause
  - The $a_1,\ldots,a_n$ is the **body** of the clause
- If there are no a_i's, the clause becomes $\rightarrow b$
  - b is always true and is called a **fact**

Horn Clauses (cont'd.)

- Example 4: first-order predicate calculus:

```
natural(0).
for all x, natural(x) → natural (successor (x)).
```

- Translate these into Horn clauses by dropping the quantifier:

```
natural(0).
natural(x) → natural (successor(x)).
```

1. Or connectives should be broken down to multiple clauses
2. Variables of the universal quantifiers will appear at the head of the clause
3. Variables with the existential quantifier will appear at the body of the clause

Horn Clauses (cont'd.)

- Example 5: logical description for the Euclidian algorithm for greatest common divisor of two positive integers *u* and *v*:

The gcd of *u* and 0 is *u*.
The gcd of *u* and *v*, if *v* is not 0,
is the same as the gcd of *v* and the remainder of dividing *v* into *u*.

- First-order predicate calculus:

```
for all u, gcd(u, 0, u).
for all u, for all v, for all w,
not zero(v) and gcd(v, u mod v, w) → gcd(u, v, w).
```

Horn Clauses (cont'd.)

- Note that `gcd(u,v,w)` is a predicate expressing that `w` is the `gcd` of `u` and `v`
- Translate into Horn clauses by dropping the quantifiers:

```
gcd(u, 0, u).
not zero(v) and gcd(v, u mod v, w) → gcd(u, v, w).
```

Horn Clauses (cont'd.)

x is a grandparent of *y* if *x* is the parent of someone who is the parent of *y*.

Horn Clauses (cont'd.)

- **Procedural interpretation**: Horn clauses can be reversed to view them as a procedure

$$b \leftarrow a_1 \text{ and } a_2 \text{ and } a_3 \ldots \text{ and } a_n$$

- This becomes procedure *b*, wherein the body is the operations indicated by the a_i's

- Most logic programming systems write Horn clauses backward and drop the *and* connectives:

```
gcd(u, 0, u).
gcd(u, v, w) ← not zero(v), gcd(v, u mod v, w).
```

- Similar to standard programming language expression for the gcd:

```
gcd(u, v) = if v = 0 then u else gcd(v, u mod v).
```

Horn Clauses (cont'd.)

- Horn clauses do not supply the algorithms, only the properties that the result must have
- Horn clauses can also be viewed as **specifications** of procedures rather than strictly as implementations
- Example: specification of a sort procedure:

```
sort(x, y) ← permutation(x, y) and sorted(y).
```

Horn Clauses (cont'd.)

- Parsing of natural language was a motivation for the original development of Prolog
- **Definite clause grammars**: the particular kind of grammar rules used in Prolog programs
- Logic programs can be used to directly construct parsers

Definite Clause Grammars

- A grammar is a precise definition of which sequences of words or symbols belong to some language
- Prolog provides a notational extension called DCG (Definite Clause Grammar) that allows the direct implementation of formal grammars

sentence → noun_phrase, verb_phrase

e.g “ The man ran.”

noun_phrase → noun
noun_phrase → determiner, noun
verb_phrase → intransitive_verb
verb_phrase → transitive_verb, noun_phrase

https://info-ruc.github.io/nlp20/dcg.pdf

Definite Clause Grammars

- We can add our own arguments to the non-terminals in DCG rules

```
sentence --> noun(Num), verb_phrase(Num).
verb_phrase(Num) --> verb(Num), noun(_).
noun(singular) --> [bob].
noun(plural) --> [students].
verb(singular) --> [likes].
verb(plural) --> [like].
```

```
| ?- sentence([bob, likes, students], []).

| ?- sentence([students, likes, bob], []).
```

Horn Clauses (cont'd.)

- Variable scope
  - Variables used in the head can be viewed as parameters
  - Variables used only in the body can be viewed as local, temporary variables
- Queries or goal statements: the exact opposite of a fact
  - Are Horn clauses with no head

```
natural(0)← .
natural(2).
```

Resolution and Unification

- **Resolution**: an inference rule for Horn clauses
  - If the head of the first Horn clause matches with one of the statements in the body of the second Horn clause, can replace the head with the body of the first in the body of the second
- Example: given two Horn clauses

$$a \leftarrow a_1, \ldots, a_n.$$
$$b \leftarrow b_1, \ldots, b_m.$$

  - If b_i matches a, then we can infer this clause:

$$b \leftarrow b_1, \ldots, b_i\text{-}1, a_1, \ldots, a_n, b_{i+1}, \ldots, b_m.$$

Resolution and Unification (cont'd.)

- Example: given $b \leftarrow a$ and $c \leftarrow b$
  - Resolution says $c \leftarrow a$

```
natural(0)← .
natural(0).
```

- Example: given $b \leftarrow a$ and $c \leftarrow b$
  - Combine: $b,c \leftarrow a,b$
  - Cancel the b on both sides: $c \leftarrow a$

```
natural(0)← .
natural(0)← natural(0)
```

If the system eventually succeeds in eliminating all goals– thus deriving the empty Horn clause– then the original statement has been proved.

Resolution and Unification (cont'd.)

- A logic processing system uses this process to match a goal and replace it with the body, creating a new list of goals, called **subgoals**
- If all goals are eventually eliminated, deriving the empty Horn clause, then the original statement has been proved
- To match statements with variables, set the variables equal to terms to make the statements identical and then cancel from both sides
    - This process is called **unification**
    - Variables used this way are said to be **instantiated**

Unification

- Unification: process by which variables are instantiated to match during resolution
  - Basic expression whose semantics is determined by unification is equality
- Prolog's unification algorithm:
  - Constant unifies only with itself
  - Uninstantiated variable unifies with anything and becomes instantiated to that thing
  - Structured term (function applied to arguments) unifies with another term only if the same function name and same number of arguments

Unification (cont'd.)

- Examples:

```
?- me = me.

?- me = you.

?- me = X.

?- f(a, X) = f(Y, b).
```

```
?- f(X) = g(X).

?- f(X) = f(a, b).

?- f(a, g(X)) = f(Y, b).

?- f(a, g(X)) = f(Y, g(b)).
```

Resolution and Unification (cont'd.)

Given $b \leftarrow a$ and $c \leftarrow b$

Combine: $b,c \leftarrow a,b$

- Example 10: gcd with resolution and unification

```
gcd(u, 0, u).
gcd(u, v, w) ← not zero(v), gcd(v, u mod v, w).
```

- Goal:

```
← gcd(15, 10, x).
```

- Resolution fails with first clause (10 does not match 0), so use the second clause and unify

```
gcd(15, 10, x) ← not zero(10), gcd(10, 15 mod 10, x),
                 gcd(15, 10, x).
```

Resolution and Unification (cont'd.)

```
not zero(10), gcd(10, 15 mod 10, x)
```

- Example 10 (cont'd.):
  - If *zero(10)* is false, then *not zero(10)* is true
  - Simplify 15 mod 10 to 5, and cancel *gcd (15, 10, x)* from both sides, giving

```
← gcd(10, 5, x).
```

  - Use unification as before

```
gcd(10, 5, x) ← not zero(5), gcd(5, 10 mod 5, x),
                                        gcd(10, 5, x).
```

  - To get this subgoal

```
← gcd(5, 0, x).
```

  - This now matches the first rule, so setting x to 5 gives the empty statement

Resolution and Unification (cont'd.)

- A logic programming system must have a fixed algorithm that specifies:
  - Order in which to attempt to resolve a list of goals
  - Order in which clauses are used to resolve goals

- In some cases, order can have a significant effect on the answers found

- Logic programming systems using Horn clauses and resolution with prespecified orders require that the programmer is aware of the way the system produces answers

The Language Prolog

- **Prolog**: the most widely used logic programming language
  - Uses Horn clauses
  - Implements resolution via a strictly depth-first strategy
- There is now an ISO standard for Prolog
  - Based on the Edinburgh Prolog version developed in the late 1970s and early 1980s

- We can of course **implement** theorem provers in Prolog!
- This is because Prolog is a Turing complete **programming language**, and every theorem prover that can be implemented on a computer can *also* be implemented in Prolog.

Notation and Data Structures

- Prolog notation is almost identical to Horn clauses
  - Implication arrow ← becomes :-
  - Variables are uppercase, while constants and names are lowercase
  - In most implementations, can also denote a variable with a leading underscore
  - Use comma for *and*, semicolon for *or*
  - List is written with square brackets, with items separated by commas
  - Lists may contain terms or variables

Execution in Prolog

- Most Prolog systems are interpreters
- Prolog program consists of:
  - Set of Horn clauses in Prolog syntax, usually entered from a file and stored in a dynamically maintained database of clauses
  - Set of goals, entered from a file or keyboard
- At runtime, the Prolog system will prompt for a query
- Unification, recursion and search are the three basic mechanisms of Prolog

Arithmetic

- Prolog has built-in arithmetic operations
  - Terms can be written in infix or prefix notation

- Prolog cannot tell when a term is arithmetic or strictly data
  - Must use built-in predicate *is* to force evaluation

```
?- write(3 + 5).
3 + 5

?- X is 3 + 5, write(X).
X = 8
```

Notation and Data Structures (cont'd.)

- Can specify head and tail of list using a vertical bar
- Example: `[H|T] = [1, 2, 3]` means
  - `H = 1, T = [2, 3]`
- Example: `[X, Y|Z] = [1, 2, 3]` means
  - `X=1, Y=2, and Z=[3]`
- Built-in predicates include `not`, `=`, and I/O operations `read`, `write`, and `nl` (newline)
- Less than or equal is usually written `=<` to avoid confusion with implication

Unification

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  - Basic expression whose semantics is determined by unification is equality
- Prolog’s unification algorithm:
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Unification (cont'd.)

- Examples:

```
?- me = me.

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?- me = X.

?- f(a, X) = f(Y, b).
```

```
?- f(X) = g(X).

?- f(X) = f(a, b).

?- f(a, g(X)) = f(Y, b).

?- f(a, g(X)) = f(Y, g(b)).
```

Unification (cont'd.)

- Unification causes uninstantiated variables to share memory (to become aliases of each other)
  - Example: two uninstantiated variables are unified

```
?- X = Y.
X = _23
Y = _23
```

Unification for List

- Can specify head and tail of list using a vertical bar
- Example: `[H|T] = [1, 2, 3]` means
  - `H = 1, T = [2, 3]`
- Example: `[X, Y|Z] = [1, 2, 3]` means
  - `X=1, Y=2, and Z=[3]`
- Built-in predicates include `not`, `=`, and I/O operations `read`, `write`, and `nl` (newline)
- Less than or equal is usually written `=<` to avoid confusion with implication

Unifying with a list

- **Pattern-directed invocation**: using a pattern in place of a variable unifies it with a variable used in that place in a goal

- `cons(X,Y,L):- L=[X|Y].`

```
cons(X, Y, [X|Y]).
```

```
?- cons(0, [1, 2, 3], A).
A = [0, 1, 2,3]
```

- There is no need for a prepend function
- `X = [1|[2,3,4,5,8]].`
- No need for a function to extract the head/tail of a list
- `[X|Y] = [1,2,3,4,5].`
- `X = 1,`
- `Y = [2, 3, 4, 5].`

Unification (cont'd.)

- **Append** procedure:

```
append(X, Y, Z) :- X = [], Y = Z.
append(X,Y,Z)   :-
```

- First clause: appending a list to the empty list gives just that list
- Second clause: appending a list whose head is *A* and tail is *B* to a list *Y* gives a list whose head is also *A* and whose tail is *B* with *Y* appended

Unification with pattern

- Append procedure rewritten more concisely:

```
append([], Y, Y).
```

```
append(X, Y, Z) :- X = [], Y = Z.
append(X,Y,Z)    :- X = [A|B], Z = [A|W], append(B, Y, W).
```

Unification with pattern

- Append can also be run backward and find all the ways to append two lists to get a specified list:

```
?- append(X, Y, [1, 2]).
X = []
Y = [1, 2] ->;
```

```
X = [1]
Y = [2] ->;
```

```
X = [1,2].
Y = []
```

Unification (cont'd.)

- **Reverse** procedure:

```
reverse([],[]).
```

- Reverse of an empty list is empty
- Continues to search for solutions setting the subgoals
  - Reversing the non-empty tail part of the list
  - Then inserting the head of the list as the last element of the reversed list

Recursion

- `factorial(0,1).`
  `∀ N and ∀ M, factorial(N-1,M) → factorial(N,N*M).`

- factorial(0,1).
- factorial(N,F) :- N>0, N1 is N-1, factorial(N1,F1), F is N * F1.