1000!=1000*999*998.....1

Multiplication of Two n-digit numbers

Z=x*y; 0(1) is it?

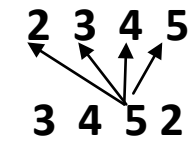
int x[ ], y[], z[];

X[]=2345678912345678

y[]=2145678912245676

Z[]=mul(x*y)

School level algorithm for multplication



..... n muls + n-1 addition

.....x "

$O(n^2)$

Can we do it better?

How?

Yes, using divide and conquer approach

$O(n^{1.59})$

Multiplication of Two n-digit numbers

x: n-bit number

y: n-bit number

x= | x_L | x_R | $=2^{n/2}x_L + x_R$

y= | y_L | y_R | $=2^{n/2}y_L + y_R$

x=10110110, x_L=1011, x_R=0110

x=(1011) * 2^4 + 0110

xy=($2^{n/2}x_L + x_R$) ($2^{n/2}y_L + y_R$)

=$2^n x_L y_L + 2^{n\backslash 2}(x_L y_R + x_R y_L) + x_R\, y_R$

4 products, each one is product of n/2 bit numbers

The original problem is subdivided into four sub-problems and the some problems are of similar type and size of each sub-problem is half of the original problem

T(n): running time of multiplying two n-bit numbers

$T(n)=4T(n/2)+O(n)$ → running time for combining the solutions of sub probs

where T(n/2)= running time for computing product of two n/2 bit numbers

Recurrence relations

How to solve this recurrence equation?

(1) Substitution method

$T(n)=4T(n/2)+O(n)$

$T(n/2)=4T((n/2)/2)+O(n/2)$

$=4T(n/4)+O(n/2)$

$T(n)= 4[4T(n/4)+O(n/2)] +O(n)$

$=4^2T(n/2^2)+4O(n/2)+O(n)$

$=4^2T(n/2^2)+(2^2-1)\,O(n)= 4^2[4T(n/8)+O(n/4)]+3O(n)$

$=4^3\,T(n/2^3)+4^2\,O(n/4) + 3O(n)$

$=4^3\,T(n/2^3)+ (2^3-1)\,O(n)$

.....

.....

$=4^k\,T(n/2^k)+ (2^k-1)\,O(n)$

$=4^{\log_2 n}\,T(1)+ (2^{\log_2 n}-1)\,O(n)$

$=4^{\log_2 n} + (2^{\log_2 n}-1)\,O(n)$ [T(1]=1]

$=n^{\log_2 4}+(n-1)O(n)$

$=n^2 + O(n^2)- O(n)$

$=O(n^2)$

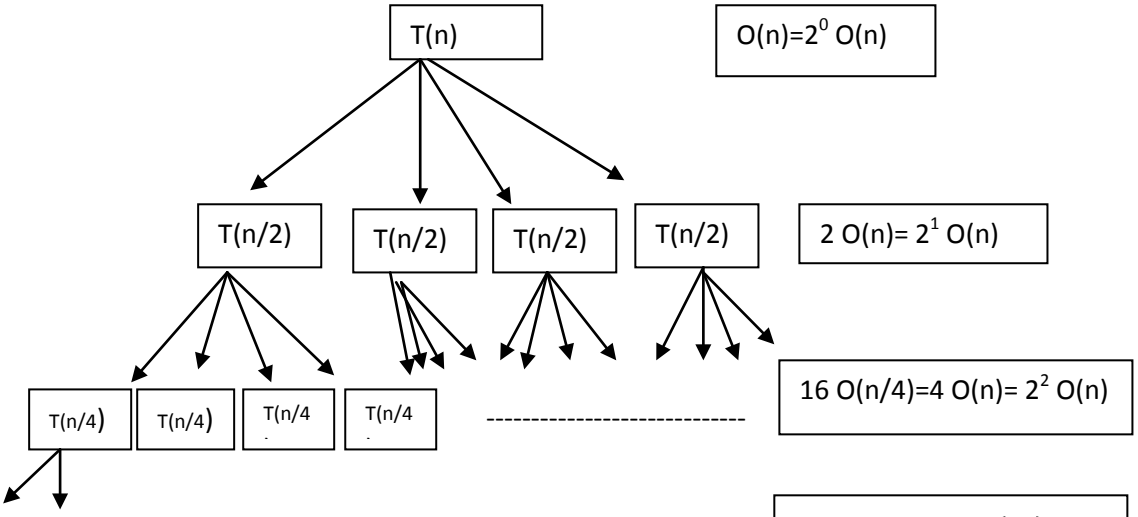
Say at step k, n reduces to 1
$T(n/2^k)=T(1)$
$n/2^k=1$
$2^k=n$
$k=\log_2 n$

No improvement over school level algorithm

How can

(2) Recursion Tree

T(n)=4T(n/2) +O(n)



Say, at level x, prob size reduces to 1, that is, $n/2^x = 1$

x=$log_2 n$ (depth of the tree)

At level 1, number of sub problems=4

At level 2, number of sub-problems=4^2

At level x, number of sub-problems of size 1 =4^x

Time spent at level x =4^x. T(1)

=4^x .1 (assume T[1]=1)

=$4^x . O(\frac{n}{2^x})$ [since 1=$\frac{n}{2^x}$]

=$\left(\frac{4}{2}\right)^x \times O(n) = 2^x \times O(n)$

Say, at level x, size of the sub problem reduces to 1

$$T(n) = \sum_{i=0}^{x} 2^i \, O(n)$$

$$T(n) = \sum_{i=0}^{log_2 n} 2^i \, O(n)$$

$=O(2^{log_2 n} \, O(n))$ [since the last term is the largest]

$= O(n.\, O(n)) = O(n^2)$

Can we have an algorithm which is better than this?

T(n)=4T(n/2) +O(n)

Here dividing the problem into 4 sub-problems

If we want to do better, we need to concentrate on how to reduce number of sub-problems?

1777-1855

C. F gauss

(a + bi) (c+di)=ac-bd +(bc +ad)i (4 different products leading to 4 sub-problems)

bc +ad= (a+b)(c+d)-ac-bd

(a + bi) (c+di)=ac-bd +{(a+b)(c+d)-ac-bd }i (3 products leading to 3 sub-problems)

1 product gain.

How this idea can be used in designing an efficient algorithm for multiplication!!

x: n-bit number

y: n-bit number

x= | x_L | x_R | $=2^{n/2}x_L + x_R$

y= | y_L | y_R | $=2^{n/2}y_L + y_R$

x=10110110, x_L=1011, x_R=0110

x=(1011) * 2^4 + 0110

xy=($2^{n/2}x_L + x_R$) ($2^{n/2}y_L + y_R$)

$$=2^n x_L y_L + 2^{n\backslash 2}(x_L y_R + x_R y_L) + x_R\, y_R$$

4 products, each one is product of n/2 bit numbers

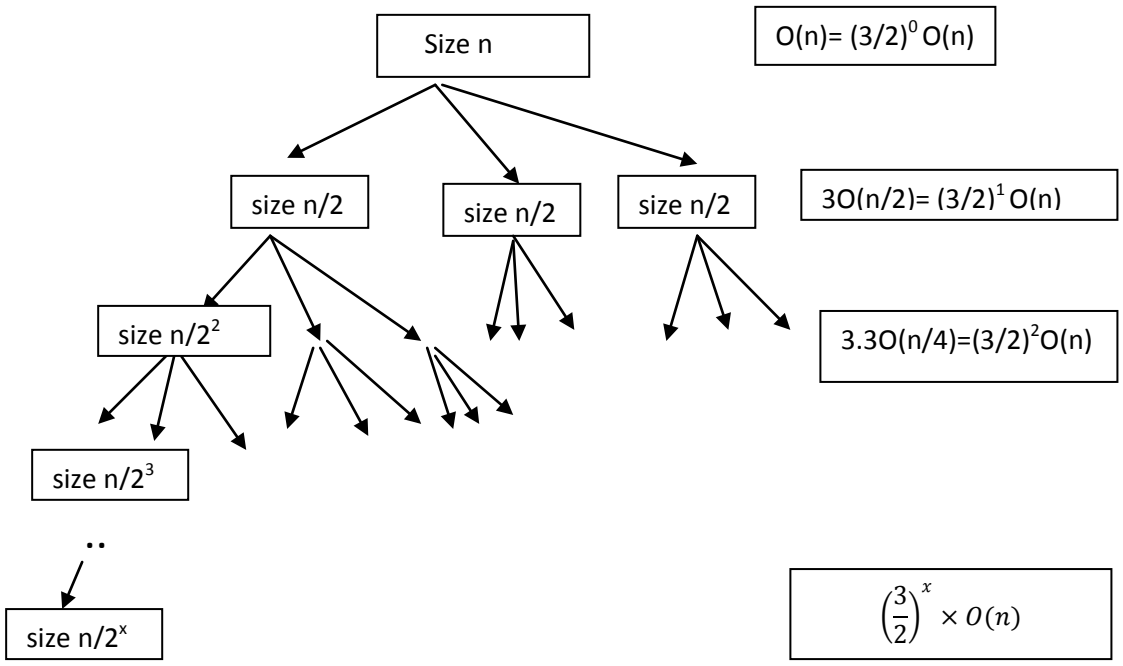
$x_L y_R + x_R y_L$=$(x_L+x_R)(y_L + y_R) - x_L y_L - x_R y_R$=

P3-P1-P2

Function multiply(x,y) → T(n)

Input: n-bit positive integers x and y
output: product
if(n==1): return xy
x_L,x_R=leftmost ceil(n/2), rightmost(floor(n/2))
y_L,y_R=leftmost ceil(n/2), rightmost(floor(n/2))
P1= multiply(x_L, y_L) → T(n/2)
P2= multiply(x_R, y_R) → T(n/2)
P3= multiply(x_L + x_R, y_L + y_R) → T(n/2)
return P1 *2^n + (P3-P1-P2)*$2^{n/2}$ +P2 → O(n)

T(n)=3 T(n/2) + O(n)



Say, at level x, prob size reduces to 1, that is, $n/2^x = 1$

x=$log_2 n$

At level 1, number of sub problems=3

At level 2, number of sub-problems=3^2

At level x, number of sub-problems of size 1 =3^x

Time spent at level x=3^x. T(1)

=3^x .1 (assume T[1]=1)

=$3^x . O(\frac{n}{2^x})$ [since 1=$\frac{n}{2^x}$]

$$=\left(\frac{3}{2}\right)^x \times O(n)$$

$$T(n) = \sum_{i=0}^{log_2 n} [\left(\frac{3}{2}\right)^i \times O(n)]$$

$=O\left(\left(\frac{3}{2}\right)^{log_2 n} \times O(n)\right)$ [since last term of T(n) is the largest]

$=O\left(\frac{3^{log_2 n}}{2^{log_2 n}} \text{ x } O(n)\right)$

$= O(3^{log_2 n}) = O(n^{log_2 3}) = O(n^{1.59})$

Mergesort Algorithm