Correctness of Prim's Algorithm

Lemma: Let (S, A) be a subtree of a MST of an undirected graph $G = (V, E)$, where $S \subset V$ and $A \subset E$. Let $e = \{u, v\}$ be an edge such that

(1) $u \in S$ and $v \in V \setminus S$;
(2) e has lowest weight among all the edges between a vertex in S and a vertex in $V \setminus S$.

Then $(S \cup \{v\}, A \cup \{e\})$ is a subtree of a MST.

Proof: Let T be a MST of G that contains (S, A).
If e is an edge of T, we are done.

Suppose that e is not an edge of T.
There is a unique path from u to v in T. There must be at least one edge $e' = \{u', v'\}$ in the path such that $u' \in S$ and $v' \in V \setminus S$. By (2) above,

$$W(e) \leq W(e'). \qquad (*)$$

Consider the new tree $T' := (T \cup \{e\}) \setminus \{e'\}$. Since T is MST,

$$W(T) \leq W(T') = W(T) - W(e') + W(e)$$

and so $W(e') \leq W(e)$. Combined with $(*)$, this proves that $W(e') = W(e)$, and so $W(T') = W(T)$. Therefore T' is also a MST, and T' contains $(S \cup \{v\}, A \cup \{e\})$.

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We can now prove the correctness of Prim's algorithm by induction.

When the algorithm starts, $(\{r\}, \emptyset)$ is definitely a subtree of a MST of G (why).

At each step the algorithm chooses an edge $e = \{u, v\}$ that satisfies (1) and (2) so, from the lemma, $(S \cup \{v\}, A \cup \{e\})$ remains a subtree of some MST of G.

In particular, when the algorithm ends, $S = V$ and A is a tree on V. We know from above that (S, A) is a subtree of some MST of G but, since A itself is a tree on G, this means that A itself is a MST.