Knapsack problem

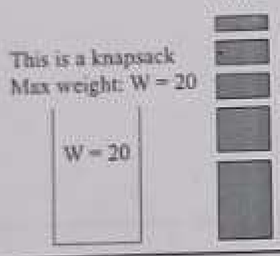
There are two versions of the problem:
1. "0-1 knapsack problem" and
2. "Fractional knapsack problem"

1. Items are indivisible; you either take an item or not. Solved with *dynamic programming*
2. Items are divisible: you can take any fraction of an item. Solved with a *greedy algorithm*.
   - We have already seen this version

0-1 Knapsack problem

- Given a knapsack with maximum capacity W, and a set S consisting of n items
- Each item i has some weight w_i and benefit value b_i (all w_i, b_i and W are integer values)
- Problem: How to pack the knapsack to achieve maximum total value of packed items?

0-1 Knapsack problem: a picture



Weight w_i	Benefit value b_i
2	3
3	4
4	5
5	8
9	10

0-1 Knapsack problem

- Problem, in other words, is to find

$$\max \sum_{i \in T} b_i \text{ subject to } \sum_{i \in T} w_i \le W$$

- The problem is called a "*0-1*" problem, because each item must be entirely accepted or rejected.
- In the "*Fractional Knapsack Problem,*" we can take fractions of items.

0-1 Knapsack problem: brute-force approach

Let's first solve this problem with a straightforward algorithm

- Since there are n items, there are 2^n possible combinations of items.
- We go through all combinations and find the one with maximum value and with total weight less or equal to W
- Running time will be $O(2^n)$

0-1 Knapsack problem: brute-force approach

- Can we do better?
- Yes, with an algorithm based on dynamic programming
- We need to carefully identify the subproblems

Let's try this:
If items are labeled *1..n*, then a subproblem would be to find an optimal solution for
S_k = *{items labeled 1, 2, .. k}*

if we allow 1 to 4 items
and all are taken away
weight = 2+3+4+5 = 14
Benefit = 3+4+5+8 = 20

Solution S4

Defining a Subproblem

If items are labeled *1..n*, then a subproblem would be to find an optimal solution for S_k = {*items labeled 1, 2, .. k*}

- This is a reasonable subproblem definition.
- The question is: can we describe the final solution (S_n) in terms of subproblems (S_k)?
- Unfortunately, we can't do that.

Defining a Subproblem

?

Max weight: W = 20
For S_4:
Total weight: 14;
Maximum benefit: 20

Item	Weight w_i	Benefit b_i
1	2	3
2	3	4
3	4	5
4	5	8
5	9	10

S_4 S_5

For S_5:
Total weight: 20
Maximum benefit: 26

Solution for S_4 is not part of the solution for S_5!!!

~~Solution 5~~ if we allow 1 to 5 items
Solution S5 Take away item1, item3, item4 & item5 (weight = 2+4+5+9 = 20)

Defining a Subproblem (continued)

- As we have seen, the solution for S_4 is not part of the solution for S_5
- So our definition of a subproblem is flawed and we need another one!
- Let's add another parameter: *w*, which will represent the exact weight for each subset of items
- The subproblem then will be to compute $B[k,w]$

So, we cannot represent S_n in terms of subproblem S_k

Recursive Formula for subproblems

Recursive formula for subproblems:

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w \\ \max\{B[k-1,w],\ B[k-1,w-w_k]+b_k\} & \text{else} \end{cases}$$

It means, that the best subset of S_k that has total weight w is:

1) the best subset of S_{k-1} that has total weight w, or
2) the best subset of S_{k-1} that has total weight $w-w_k$ plus the item k

w_k not part of soln.

$B[k-1,w]$ = max. benefit with the subset of k-1 items & capacity w

W_k: weight of object k

Recursive Formula

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w \\ \max\{B[k-1,w],\ B[k-1,w-w_k]+b_k\} & \text{else} \end{cases}$$

max benefit with allowable k items & W capacity

- The best subset of S_k that has the total weight w, either contains item k or not.
- First case: $w_k > w$. Item k can't be part of the solution, since if it was, the total weight would be $> w$, which is unacceptable.
- Second case: $w_k \le w$. Then the item k can be in the solution, and we choose the case with greater value.

0-1 Knapsack Algorithm

```
for w = 0 to W
  B[0,w] = 0
for i = 1 to n
  B[i,0] = 0
for i = 1 to n
  for w = 0 to W
    if w_i <= w // item i can be part of the solution
      if b_i + B[i-1,w-w_i] > B[i-1,w]
        B[i,w] = b_i + B[i-1,w-w_i]
      else
        B[i,w] = B[i-1,w]
    else B[i,w] = B[i-1,w] // w_i > w
```

Example

i	w_i	b_i
1	2	3
2	3	4
3	4	5
4	5	6

Capacity W = 5

if we allow 1 to 2 items
weight = 5
benefit = 7

if we allow 1 to 3 items and if 3rd item is taken, item2 & item3 should be unloaded and we may look because prev. benefit = 7, curr. benefit becomes 5.
So, before taking item 3 we should check whether to take item3 or not.

So, Try to know what was the maximum benefit before taking W_k with weight restriction $W-W_k$

[illegible] Knapsack [illegible] maximum 5 kg
W_k has 4 kg we should try to know
$(5-4) = 1$ kg weight restriction [illegible] K-1
[illegible] item consider [illegible] maximum [illegible]
Benefit [illegible] Benefit [illegible]
[illegible] Benefit [illegible] 1 kg weight restriction [illegible]

Running time

for w = 0 to W — $O(W)$
  B[0,w] = 0
for i = 1 to n
  B[i,0] = 0
for i = 1 to n — Repeat n times
  for w = 0 to W — $O(W)$
    < the rest of the code >

What is the running time of this algorithm?

$O(n*W)$

Remember that the brute-force algorithm takes $O(2^n)$

Example

Let's run our algorithm on the following data:

n = 4 (# of elements)
W = 5 (max weight)
Elements (weight, benefit):
(2,3), (3,4), (4,5), (5,6)

[illegible] item 3-[illegible] consider [illegible] item 1 & item 2 unload [illegible]

Example (2)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1						
2						
3						
4						

for w = 0 to W
  B[0,w] = 0

Example (3)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0					
2	0					
3	0					
4	0					

for i = 1 to n
  B[i,0] = 0

Greedy approach : results : weight - 5, benefit - 6 } item 4

Example (4)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0				
2	0					
3	0					
4	0					

$i=1$
$b_i=3$
$w_i=2$
$w=1$
$w-w_i=-1$

if $w_i <= w$ // item i can be part of the solution
  if $b_i + B[i-1,w-w_i] > B[i-1,w]$
    $B[i,w] = b_i + B[i-1,w-w_i]$
  else
    $B[i,w] = B[i-1,w]$
else $B[i,w] = B[i-1,w]$ // $w_i > w$

B(1,1)

Example (5)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3			
2	0					
3	0					
4	0					

$i=1$
$b_i=3$
$w_i=2$
$w=2$
$w-w_i=0$

if $w_i <= w$ // item i can be part of the solution
  if $b_i + B[i-1,w-w_i] > B[i-1,w]$
    $B[i,w] = b_i + B[i-1,w-w_i]$
  else
    $B[i,w] = B[i-1,w]$
else $B[i,w] = B[i-1,w]$ // $w_i > w$

[illegible] W_k - [illegible] unload [illegible] benefit [illegible] + benefit [illegible] W_k.

Note : [illegible] W_k [illegible] W_k [illegible] (1...(k-1) object-[illegible]) ⇒ [illegible]

$> B(i-1, w-w_i)$ $\rightarrow B(i-1, w)$

Example (6)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3		
2	0					
3	0					
4	0					

$i=1$
$b_i=3$
$w_i=2$
$w=3$
$w-w_i=1$

```
if wi <= w // item i can be part of the solution
    if bi + B[i-1,w-wi] > B[i-1,w]
        B[i,w] = bi + B[i-1,w-wi]
    else
        B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // wi > w
```

Example (7)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	
2	0					
3	0					
4	0					

$i=1$
$b_i=3$
$w_i=2$
$w=4$
$w-w_i=2$

```
if wi <= w // item i can be part of the solution
    if bi + B[i-1,w-wi] > B[i-1,w]
        B[i,w] = bi + B[i-1,w-wi]
    else
        B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // wi > w
```

Example (8)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0					
3	0					
4	0					

$i=1$
$b_i=3$
$w_i=2$
$w=5$
$w-w_i=3$

```
if wi <= w // item i can be part of the solution
    if bi + B[i-1,w-wi] > B[i-1,w]
        B[i,w] = bi + B[i-1,w-wi]
    else
        B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // wi > w
```

Example (9)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0				
3	0					
4	0					

$i=2$
$b_i=4$
$w_i=3$
$w=1$
$w-w_i=-2$

```
if wi <= w // item i can be part of the solution
    if bi + B[i-1,w-wi] > B[i-1,w]
        B[i,w] = bi + B[i-1,w-wi]
    else
        B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // wi > w
```

Example (10)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3			
3	0					
4	0					

$i=2$
$b_i=4$
$w_i=3$
$w=2$
$w-w_i=-1$

```
if wi <= w // item i can be part of the solution
    if bi + B[i-1,w-wi] > B[i-1,w]
        B[i,w] = bi + B[i-1,w-wi]
    else
        B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // wi > w
```

Example (11)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4		
3	0					
4	0					

$i=2$
$b_i=4$
$w_i=3$
$w=3$
$w-w_i=0$

```
if wi <= w // item i can be part of the solution
    if bi + B[i-1,w-wi] > B[i-1,w]
        B[i,w] = bi + B[i-1,w-wi]
    else
        B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // wi > w
```

~~wk [illegible]~~ [illegible]
wk কে বাদ দিয়ে [illegible]
[illegible] 1 to k-1 এর মধ্যে [illegible]
[illegible] maximum হয়,
[illegible] Benefit
[illegible] B(k-1, w-wk). [illegible] add bk ⇒ bk+B(k-1, w-wk)

with only the 1st item the max benefit = 3 with weight res: 5.

Example (12)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	
3	0					
4	0					

$i=2$
$b_i=4$
$w_i=3$
$w=4$
$w-w_i=1$

if $w_i <= w$ // item i can be part of the solution
if $b_i + B[i-1,w-w_i] > B[i-1,w]$
$B[i,w] = b_i + B[i-1,w-w_i]$
else
$B[i,w] = B[i-1,w]$
else $B[i,w] = B[i-1,w]$ // $w_i > w$

Example (13)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0					
4	0					

$i=2$
$b_i=4$
$w_i=3$
$w=5$
$w-w_i=2$

if $w_i <= w$ // item i can be part of the solution
if $b_i + B[i-1,w-w_i] > B[i-1,w]$
$B[i,w] = b_i + B[i-1,w-w_i]$
else
$B[i,w] = B[i-1,w]$
else $B[i,w] = B[i-1,w]$ // $w_i > w$

if we allow 1 & 2 items and item 2 is taken, ~~we have~~ ~~the benefit 3~~ if ~~item~~ weight of item 2 is subtracted from W = 5. we have weight = 2 with this weight restrict we have max benefit = 3.

So, max(3,7) = 7

Example (14)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4		
4	0					

$i=3$
$b_i=5$
$w_i=4$
$w=1..3$

if $w_i <= w$ // item i can be part of the solution
if $b_i + B[i-1,w-w_i] > B[i-1,w]$
$B[i,w] = b_i + B[i-1,w-w_i]$
else
$B[i,w] = B[i-1,w]$
else $B[i,w] = B[i-1,w]$ // $w_i > w$

Example (15)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	
4	0					

$i=3$
$b_i=5$
$w_i=4$
$w=4$
$w-w_i=0$

if $w_i <= w$ // item i can be part of the solution
if $b_i + B[i-1,w-w_i] > B[i-1,w]$
$B[i,w] = b_i + B[i-1,w-w_i]$
else
$B[i,w] = B[i-1,w]$
else $B[i,w] = B[i-1,w]$ // $w_i > w$

Example (16)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0					

$i=3$
$b_i=5$
$w_i=4$
$w=5$
$w-w_i=1$

if $w_i <= w$ // item i can be part of the solution
if $b_i + B[i-1,w-w_i] > B[i-1,w]$
$B[i,w] = b_i + B[i-1,w-w_i]$
else
$B[i,w] = B[i-1,w]$
else $B[i,w] = B[i-1,w]$ // $w_i > w$

Example (17)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	

$i=4$
$b_i=6$
$w_i=5$
$w=1..4$

if $w_i <= w$ // item i can be part of the solution
if $b_i + B[i-1,w-w_i] > B[i-1,w]$
$B[i,w] = b_i + B[i-1,w-w_i]$
else
$B[i,w] = B[i-1,w]$
else $B[i,w] = B[i-1,w]$ // $w_i > w$

Example (18)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	[illegible]

i=4
b_i=6
w_i=5
w= 5
w- w_i=0

```
if wi <= w // item i can be part of the solution
    if bi + B[i-1,w-wi] > B[i-1,w]
        B[i,w] = bi + B[i-1,w-wi]
    else
        B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // wi > w
```

Comments

- This algorithm only finds the max possible value that can be carried in the knapsack
  - I.e., the value in B[n,W]
- To know the items that make this maximum value, an addition to this algorithm is necessary

How to find actual Knapsack Items

- All of the information we need is in the table.
- $B[n,W]$ is the maximal value of items that can be placed in the Knapsack.
- Let i=n and k=W

```
if B[i,k] ≠ B[i−1,k] then
    mark the ith item as in the knapsack
    i = i−1, k = k-wi
else
    i = i−1 // Assume the ith item is not in the knapsack
            // Could it be in the optimally packed knapsack?
```

Finding the Items

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

i=4
k= 5
b_i=6
w_i=5
B[i,k] = 7
B[i−1,k] =7

4th item not taken

```
i=n, k=W
while i,k > 0
    if B[i,k] ≠ B[i−1,k] then
        mark the ith item as in the knapsack
        i = i−1, k = k-wi
    else
        i = i−1
```

Finding the Items (2)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

i=4
k= 5
b_i=6
w_i=5
B[i,k] = 7
B[i−1,k] =7

```
i=n, k=W
while i,k > 0
    if B[i,k] ≠ B[i−1,k] then
        mark the ith item as in the knapsack
        i = i−1, k = k-wi
    else
        i = i−1
```

Finding the Items (3)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

i=3
k= 5
b_i=5
w_i=4
B[i,k] = 7
B[i−1,k] =7

item 3 not taken

```
i=n, k=W
while i,k > 0
    if B[i,k] ≠ B[i−1,k] then
        mark the ith item as in the knapsack
        i = i−1, k = k-wi
    else
        i = i−1
```

Finding the Items (4)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

$i=2$
$k=5$
$b_i=4$
$w_i=3$
$B[i,k]=7$
$B[i-1,k]=3$
$k-w_i=2$

```
i=n, k=W
while i,k > 0
    if B[i,k] ≠ B[i−1,k] then
        mark the i^th item as in the knapsack
        i = i−1, k = k−w_i
    else
        i = i−1
```

Finding the Items (5)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

$i=1$
$k=2$
$b_i=3$
$w_i=2$
$B[i,k]=3$
$B[i-1,k]=0$
$k-w_i=0$

```
i=n, k=W
while i,k > 0
    if B[i,k] ≠ B[i−1,k] then
        mark the i^th item as in the knapsack
        i = i−1, k = k−w_i
    else
        i = i−1
```

Finding the Items (6)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

$i=0$
$k=0$

The optimal knapsack should contain {1, 2}

```
i=n, k=W
while i,k > 0
    if B[i,k] ≠ B[i−1,k] then
        mark the i^th item as in the knapsack
        i = i−1, k = k−w_i
    else
        i = i−1
```

Finding the Items (7)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

The optimal knapsack should contain {1, 2}

```
i=n, k=W
while i,k > 0
    if B[i,k] ≠ B[i−1,k] then
        mark the i^th item as in the knapsack
        i = i−1, k = k−w_i
    else
        i = i−1
```

Review: The Knapsack Problem And Optimal Substructure

- Both variations exhibit optimal substructure
- To show this for the 0-1 problem, consider the most valuable load weighing at most W pounds
  - *If we remove item j from the load, what do we know about the remaining load?*
  - A: remainder must be the most valuable load weighing at most $W - w_j$ that thief could take, excluding item j

Solving The Knapsack Problem

- The optimal solution to the fractional knapsack problem can be found with a greedy algorithm
  - *Do you recall how?*
  - Greedy strategy: take in order of dollars/pound
- The optimal solution to the 0-1 problem cannot be found with the same greedy strategy
  - Example: 3 items weighing 10, 20, and 30 pounds, knapsack can hold 50 pounds
    - *Suppose item 2 is worth $100. Assign values to the other items so that the greedy strategy will fail*

The Knapsack Problem: Greedy Vs. Dynamic

- The fractional problem can be solved greedily
- The 0-1 problem can be solved with a dynamic programming approach

Memoization

- *Memoization* is another way to deal with overlapping subproblems in dynamic programming
  - After computing the solution to a subproblem, store it in a table
  - Subsequent calls just do a table lookup
- With memoization, we implement the algorithm recursively:
  - If we encounter a subproblem we have seen, we look up the answer
  - If not, compute the solution and add it to the list of subproblems we have seen.
- Most useful when the algorithm is easiest to implement recursively
  - Especially if we do not need solutions to all subproblems.

Conclusion

- Dynamic programming is a useful technique of solving certain kind of problems
- When the solution can be recursively described in terms of partial solutions, we can store these partial solutions and re-use them as necessary (memoization)
- Running time of dynamic programming algorithm vs. naïve algorithm:
  - 0-1 Knapsack problem: $O(W*n)$ vs. $O(2^n)$