P & NP- Completeness

P : polynomial Time Solvable problems

Def: A problem is polynomial time solvable if There is an algorithm that correctly solves in $O(n^k)$, for some constant k.

Where n = input size

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The Class P :

The set of polynomial - time problems.

Examples :

class P { Sorting algorithms already seen
~~Single Short~~
Finding minimum spanning tree

etc.

belong to the class P?

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But Does the following belong to the

① Cycle-free shortest path in graph with negative costs

② Knapsack
[running time $= O(nW)$
but input size $\propto \log(W)$]

$W = 2^b$
$O(n\,2^b)$

NO

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NP - Completeness

→ rough litmus test for Computational tractability

Traveling Salesman problem (TSP)

Input: Complete Undirected graph with nonnegative edge costs

Output: a min-cost tour [i.e., a vertex

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Input: Complete Undirected graph with nonnegative edge costs.

Output: a min-cost tour [i.e., a cycle ~~that~~ visits every vertex exactly once]

There is no polynomial time algorithm for TSP problem.

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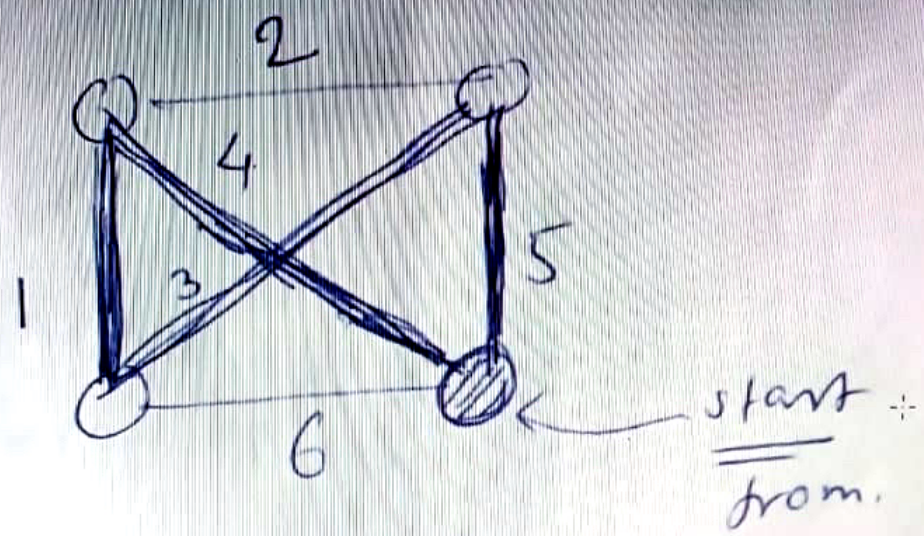
Ex.



Optimal = 13

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Reductions & Completeness

Reduction

Def (informal): problem Π_1 reduces to problem Π_2 if:

Given a polynomial time subroutine for Π_2, can use it to solve Π_1 in polynomial

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Subroutine for Π_2, can ...
it to solve Π_1 in polynomial time.

Ex. Detecting a cycle reduces to DFS
Computing median reduces to Sorting.

... to Π

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= Detecting ... Computing median reduces to ...

- if Π_1 reduces to Π_2

  if Π_1 is not in P, then neither is Π_2, that is, Π_2 is at least as hard as Π_1.

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Completeness.

Def: Let C = a set of problems

The problem Π is C-complete if (i) $\Pi \in C$ (ii) everything in C reduces to Π.

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$\Rightarrow$ π is the hardest problem in all of C

<u>Halting problem</u>

Given a program and an input for it, will it ~~[illegible]~~ eventually halt?

Given a program for it, will it ~~[illegible]~~ eventually halt?

Fact: (Turing '36) no algorithm, however slow, solves the halting problem.

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(2) Purported Solutions can be verified in polynomial time.

Examples –

Is there a TSP tour with length ≤ 1000?

NP – Complete

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The Class NP

N stands for nondeterministic

Def:

A problem is in NP if

(1) Solutions always have length polynomial in the input size

(2) Purported solutions can be verified in polynomial time.

Examples –

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NP – Complete

Def:

A problem Π is NP-Complete if

– (i) $\Pi \in NP$

(ii) Π is NP-hard.

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NP-hard

A problem is called NP-hard if it has the above property, that is, if it has a polynomial time algorithm then ~~exist~~ is one for every problem in NP.

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NP-hard

Cook' theorem: if there exists a polynomial time algorithm for SAT (~~Satisfiab~~ Satisfiability problem), there is one for all problems in NP.

NP-hard

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What is SAT (Satisfiability problem)?

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clause

formula evaluates to T iff every clause evaluates to T.

boolean formula is in CNF is said to be satisfiable if there exists an

[A Clause: OR of literals.

literal: $x, \bar{x}$

Ex. $(x_1 \vee \bar{x}_2 \vee \bar{x}_3 \vee \cdots x_t)$ is clause.]

A clause becomes false if all literals are false otherwise

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To know about SAT, we need to know CNF (Conjunctive normal form).

CNF:

A boolean formula in CNF is an AND of clauses

$$C_1 \wedge C_2 \wedge C_3 \cdots \wedge C_k$$

clause

[A **Clause:** OR of literals

formula evaluates to T

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boolean formula
is in CNF is
said to be satisfiable
if there exists an
assignment to the variables
such that the formula
evaluates to T.

... x_t)
is clause.]

A Clause becomes
false if all
literals are
false otherwise
true

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SAT $\in$ NP

(purported solution)

① A proof is an assignment to the variables.

② The purported solutions can be verified in polynomial time.

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SAT problem

I : boolean formula f in CNF

Q : is f satisfiable?

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SAT $\in$ NP (purported solution)

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SAT is NP-hard problem.

because

If there exists a polynomial time algorithm for SAT there is one for all problems in NP.

NP - Completeness

A problem π is NP-Complete if

① $\pi \in NP$

② π is NP-hard.

(~~every~~ π is called NP-hard if every problem in NP is polynomial time

This implies that ① a problem in NP may not be NP-hard

or

② a problem which is NP-hard may not be in NP

(i.e. if we can find a P such that

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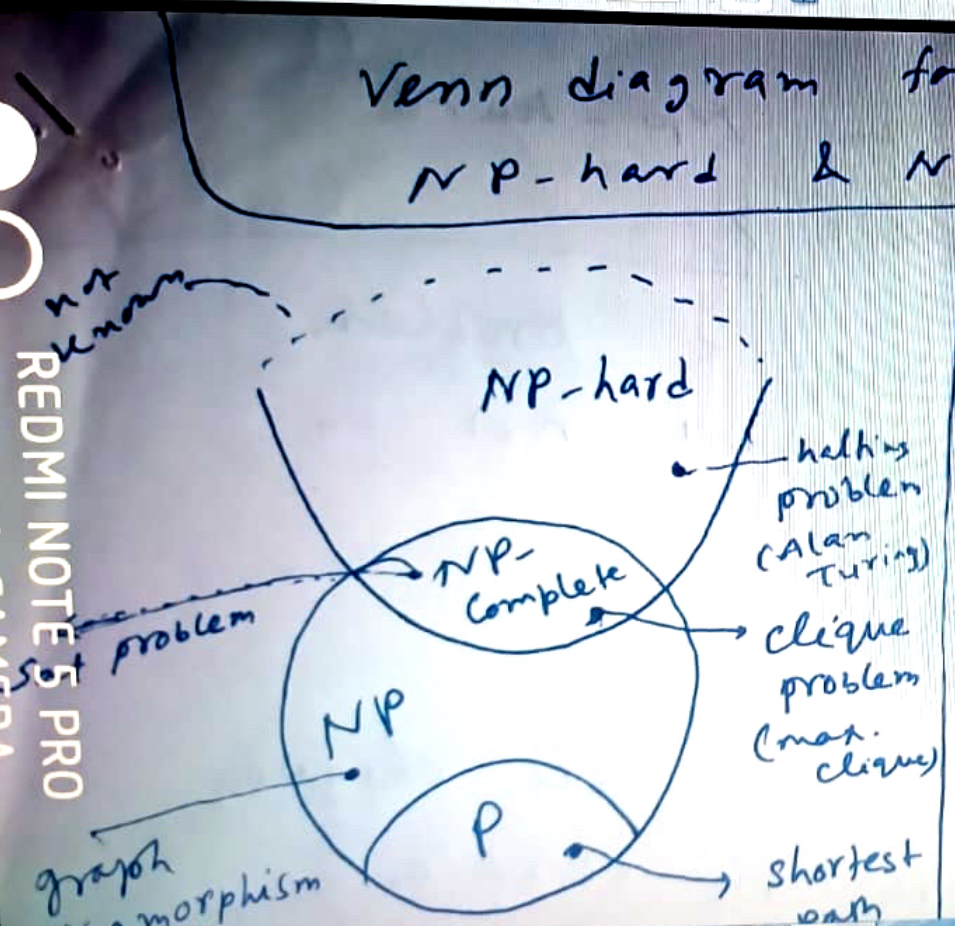
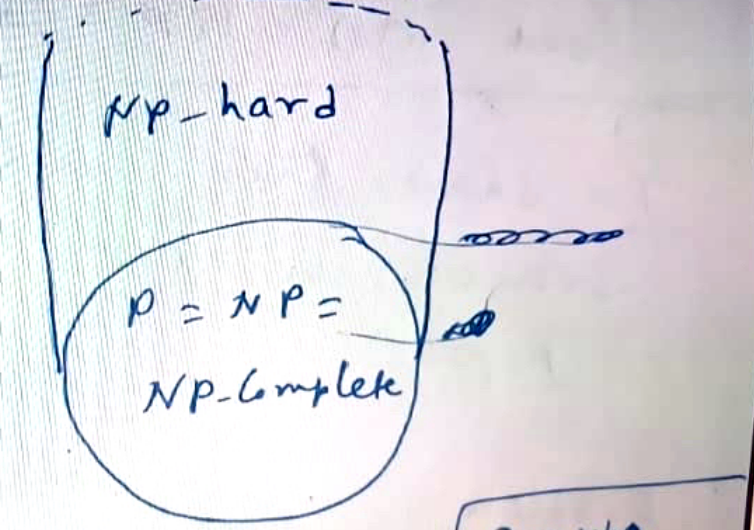
Venn diagram for P, NP, NP-hard & NP-Complete





Assumption:

graph isomorphism is in NP

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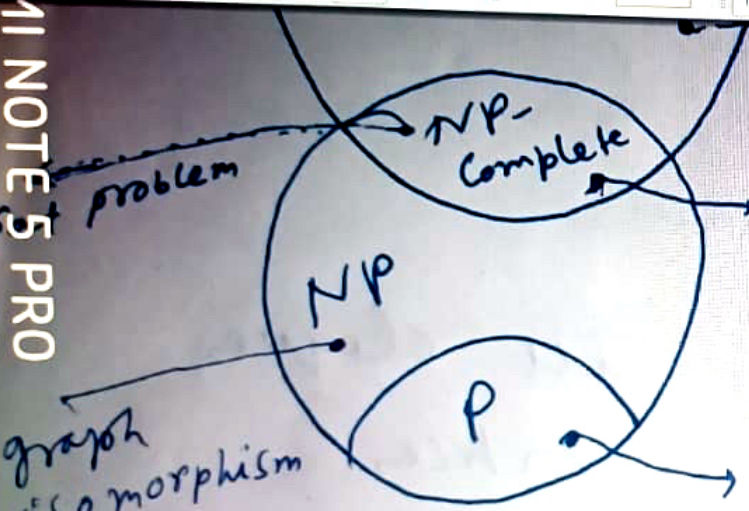


st problem

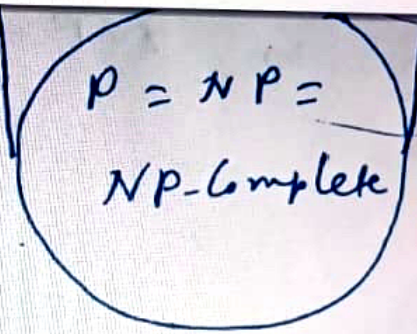
graph isomorphism

problem (Alan Turing)

clique problem (max. clique)

shortest path problem

Assumption: $P \neq NP$



Assumption: $P = NP$

graph Isomorphism is in NP but neither in P nor NP-Complete

problems

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Graph Isomorphism

Isomorphism:

a one-to-one correspondance between two sets that preserves binary relationships between elements of the sets.

For example, a set of natural numbers can be mapped

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of the sets.

For example, a set of natural numbers can be mapped on to the set of even numbers by multiplying each natural number by 2.

Graph isomorphism problem

determine whether two finite

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To determine whether two graphs are isomorphic.

<u>Def</u>:

Two graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ is isomorphic

Bijection $\phi : V_1 \longrightarrow V_2$

$(u, v) \in E_1 \Leftrightarrow (\phi(u), \phi(v)) \in E_2$

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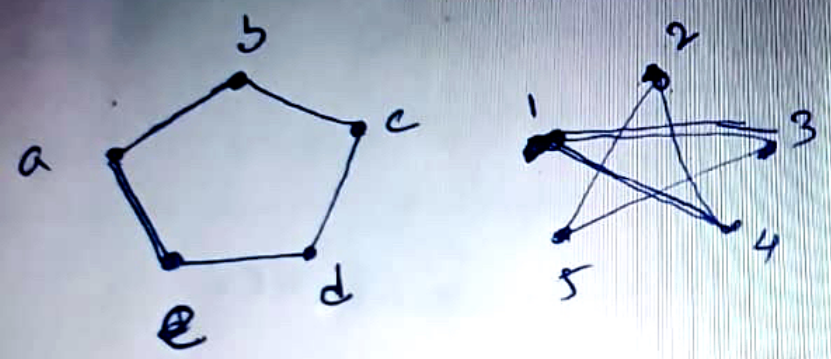
Ex.



Degree of each vertex $\in G_1$ is 2

Degree of each vertex $\in G_2$ is 2

bijection or mapping

$\phi(a) = 1$

$\phi(b) = 3$

$\phi(e) = 4$

$\phi(c) = 5$

Since a is connected to b & e
1 is connected to 3 and 4
(Since b is connected to c)
c is connected to e and c
1 is connected to 4 and 5)

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or mappings.

$\phi(b) = 4$ (since b is connected to c)

$\phi(c) = 5$ (d is connected to e and c)

$\phi(d) = 2$ (2 is connected to 4 and 5)

general.

Graph Isomorphism problem is in NP because it can be verified in polynomial time.

proof is difficult.

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Proof is difficult.

This is not a NP-Complete problem.

& it is not in P.

This is problem which is known

NP-intermediate problem.

(Researchers are working

on this problem).

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Halting problem

(unsolvable)

- Halting problem is not NP Complete but NP-hard

- There exists no algorithm to solve this problem.

- So, it is not in NP

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There exists no algorithm to solve this problem.

So, it is not in NP

So, it is not NP-Complete.

Show that Halting problem is NP-hard.

will show

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Show that Halting is NP- hard.

To show it, we will show that Satisfiablity (NP- hard) problem is a halting problem.

Let us Construct an algorithm A boolean formula X

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Let us Construct an algorithm A whose input is boolean formula. X

Suppose X has n variables

Algorithm A tries out all possible 2^n possible truth assignments and verifies if X is satisfiable

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- if X is satisfied, A stops (halts)
- if X is not satisfiable, A enters an infinite loop.
- Hence A halts on input iff X is satisfiable.
- if we have a polynomial time algorithm for the halting problem, [illegible] satisfiability

even it is harder than SAT, so it is NP-hard.

It is not in NP because input cannot be verified in

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A and X ...

algorithm for the halting problem

Thus the halting problem is NP-hard problem which is not in NP.

So, it is not NP-Complete.

The halting problem is not in

not in NP

So, it is not NP-Complete.

Note: Why halting problem is not in NP?

Ans: all problems in NP are decidable but the halting problem is undecidable.

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