Verifying a process has a law but not necessarily a density

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June 2024

Exercise

X_t is a process which for each fixed t, X_t is a random variable and hence it has a law but not necessarily a density. Verify.

Solution

A stochastic process X_t is a collection of random variables indexed by time t. For each fixed t, X_t is a random variable. This implies:

1. X_t is a mapping from a sample space Ω to the real numbers (or more generally, to some measurable space).

2. X_t has a distribution (or law), which describes the probabilities associated with the possible values of has a distribution (or law), which describes the probabilities associated with the possible values of

The law (or distribution) of a random variable X_t is a probability measure on the space of its possible values. Formally, if X_t maps Ω to R, then the law of X_t is a measure P_{Xt} on R such that for any Borel set $B \subseteq R$,

$$P_{Xt}(B) = P_{Xt} \in B$$

A random variable X_t has a density if its distribution can be described by a probability density function (pdf). This means there exists a non-negative function f_{Xt} such that for any Borel set $B \subseteq R$,

$$P_{Xt}(B) = \int_B f_{Xt} \text{ (x) dx}$$

However, not all random variables have a density. For example:

- A discrete random variable takes on a countable number of values with positive probability, and thus has a probability mass function (pmf) rather than a pdf.

- A continuous random variable has a pdf.
- A random variable could be a mixture of discrete and continuous components.

Given that X_t is a process where for each fixed t,X_t is a random variable, we can verify the statement as follows:

1. X_t is a random variable: By the definition of a stochastic process, for each fixed t, X_t is indeed a random variable.
2. X_t has a law: Every random variable has a distribution (or law). Therefore, X_t has a law for each fixed t X_t does not necessarily have a density.
3. X_t The law of does not necessarily have to be absolutely continuous with respect to the Lebesgue measure. It could be discrete, continuous, or a combination. Hence, X_t may or may not have a density function.