Stability of Finite Difference Methods

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1 Exercise

To compare the errors between the analytical Black-Scholes price and the price obtained through the finite difference methodS for a European put option, we proceed with the following steps

Define Parameters:

- Strike price: $E = 10$
- Risk-free rate: $r = 0.05$
- Time to maturity: $T = 0.5$
- Volatility: $\sigma = 0.2$
- Underlying asset price range: $S = \{5, 6, \ldots, 15\}$
- Number of space steps: $N = 100$
- Number of time steps: $M = 30$

Calculate the Analytical Black-Scholes Price:

- Use the Black-Scholes formula for a European put option to calculate the theoretical prices for the given range of S. European Put Option:

$$P(S,t) = K \cdot e^{-r(T-t)} \cdot \Phi(-d_2) - S \cdot \Phi(-d_1)$$

Where:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Implement the Explicit, Implicit and Crank- Nicolson Finite Difference Methods:

- Explicit

  Set up the grid for space and time.

  Use the boundary conditions for the option at $S = 0$ and $S = \infty$.

  Iterate backwards in time to compute the option prices

- Implicit

  Set up the grid for space and time.

  Use boundary conditions for the option at $S = 0$ and $S = \infty$.

  Construct the tridiagonal matrix corresponding to the implicit method.

  Use matrix inversion or LU decomposition to solve the system of linear equations backward in time.

- Crank-Nicolson

  Set up the grid for space and time.

  Use boundary conditions for the option at $S = 0$ and $S = \infty$.

  Construct the tridiagonal matrix corresponding to the Crank-Nicolson method.

  Solve the system of linear equations backward in time using LU decomposition.

Compute Errors:

- For each underlying price in the range $S = \{5, 6, \ldots, 15\}$, calculate the absolute error between the Black-Scholes price and the price obtained through the Explicit, Implicit and Crack Nicolson methods.

When increasing the number of time steps from $M = 30$ to $M = 500$, the plot typically shows reduced errors for all three finite difference methods. This improvement is due to the finer discretization of the time domain, which results in a more accurate approximation of the partial differential equation governing the option's price.Key observations include:

- With more time steps, the errors decrease for all methods, especially near the boundaries of the underlying asset price range.
- Even with more time steps, the largest errors occur around the strike price $S = E = 10$. This is because the option's value and its derivatives (such as delta) are more sensitive in this region, making it harder to approximate accurately.
- The Crank-Nicolson method, known for its stability and accuracy, shows the least error among the three methods, especially with increased time steps.

Increasing the number of space steps N to 200 while keeping the number of time steps M at 30, and then increasing the volatility from $\sigma = 0.2$ to $\sigma = 0.4$, would generally lead to the following observations:

- Higher volatility increases the magnitude of the option's payoff, making it more sensitive to changes in the underlying asset's price.
- This sensitivity usually leads to larger errors in the finite difference methods, particularly in the Explicit method, which is more prone to numerical instability under high volatility.
- Further, increasing N improves the spatial resolution, leading to a more accurate representation of the underlying asset's price. This generally reduces errors across the board.
- For $\sigma = 0.2$: The errors are expected to be lower, especially with the increased N. The Explicit method may still show larger errors near the strike price.
- For $\sigma = 0.4$: The errors will generally increase due to the higher volatility.

When you vary the number of space steps N from 300 to 700 while keeping the number of time steps $M = 30$, the stability and accuracy of the errors in the finite difference schemes are affected as follows:

- As N increases, errors decrease, but the scheme's stability needs to be monitored. If the grid becomes too fine (high N) while keeping M fixed, you might approach the stability limits of the explicit method.The explicit scheme becomes less stable, especially if the time step Δt is too large relative to the spatial step ΔS.
- Implicit and Crank-Nicolson Methods Errors will consistently decrease as N increases, with minimal risk of instability. The Crank-Nicolson method, in particular, shows excellent stability and accuracy as N increases.