Brownian Motion and SDE

Sheila Tonui

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Exercise

Generate the sample realisations of the following linear SDE with multiplicative noise:

$$\mathrm{dX}_t = (a_1(t)X_t + a_2(t))dt + (b_1(t)X_t + b_2(t))dW_t$$

How does this compare with the general SDE of the form:

$$\mathrm{dX}_t = a_1(t)X_t dt + b_1(t)X_t dW_t$$

Solution

We discretize the equation and simulate it using a numerical method, the Euler-Maruyama method.

For the given SDE:

$$\mathrm{dX}_t = (a_1(t)X_t + a_2(t))dt + (b_1(t)X_t + b_2(t))dW_t$$

using Euler-Maruyama, the discretized form is:

$$\mathrm{X}_{t+\Delta t} = X_t + (a_1(t)X_t + a_2(t))\Delta t + (b_1(t)X_t + b_2(t))\Delta W_t$$

where:

- ΔW_t is a Wiener process increment, which can be simulated as $\sqrt{\Delta t}$.N(0,1) with N(0,1) being a standard normal variable.
- Time discretization parameters are:
  T - time horizon
  N - number of time steps
  dt - time step size

$a_1(t)$, $a_2(t)$,$b_1(t)$, and $b_2(t)$ are as constant functions. The presence of $a_2(t)$ and $b_2(t)$ introduces an additional drift and diffusion component independent of X_t which can significantly alter the dynamics of X_t.

The linear SDE with mulitiplicative noice is simulated with both $a_2(t)$ and $b_2(t)$ terms while the general SDE without $a_2(t)$ and $b_2(t)$ terms. The two sample paths visually helps to understand how these additional terms impact the behavior of the solution.