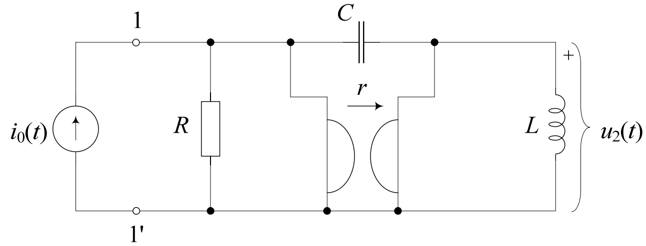
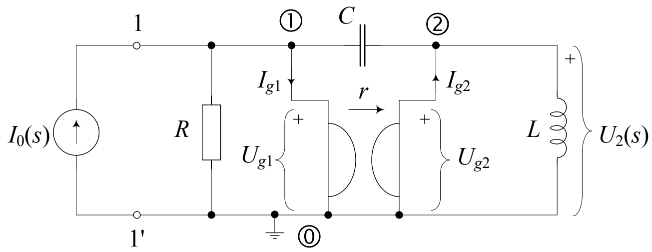
DRUGI MEĐUISPIT IZ ELEKTRIČNIH KRUGOVA – Rješenja – 2010

1. Za električni krug na slici s pobudom $i_0(t)$ i normiranim vrijednostima elemenata: $R=2$, $L=1$ i $C=1$, te konstantom giratora $r=2$ odrediti: *a*) ulaznu impedanciju $Z_{ul}(s)$ gledanu s priključnica 1-1'; *b*) prijenosnu impedanciju $Z_{21}(s)=U_2(s)/I_0(s)$; *c*) polove i nule funkcije $Z_{21}(s)$ i njihov prikaz u s-ravnini; *d*) odziv $u_2(t)$ ako je zadana pobuda $i_0(t)=S(t)$, *e*) odziv $u_2(t)$ ako je $i_0(t)$ stacionarna sinusna pobuda valnoga oblika $i_0(t)=5\cos(t)$.



Rješenje: Jednadžbe čvorišta



$$U_1(s)=U_{g1}(s)$$
$$U_2(s)=U_{g2}(s)$$

(1) $$I_0(s)-I_{g1}(s)=\left(\frac{1}{R}+sC\right)U_1(s)-sCU_2(s)$$

(2) $$I_{g2}(s)=-sCU_1(s)+\left(sC+\frac{1}{sL}\right)U_2(s)$$

$$U_{g1}(s)=-rI_{g2}(s)\Rightarrow I_{g2}(s)=-\frac{1}{r}U_1(s)$$

$$U_{g2}(s)=-rI_{g1}(s)\Rightarrow I_{g1}(s)=-\frac{1}{r}U_2(s)$$

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(1) $$I_0(s)+\frac{1}{r}U_2(s)=\left(\frac{1}{R}+sC\right)U_1(s)-sCU_2(s)$$

(2) $$-\frac{1}{r}U_1(s)=-sCU_1(s)+\left(sC+\frac{1}{sL}\right)U_2(s)$$

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(1) $$I_0(s)=\left(\frac{1}{R}+sC\right)U_1(s)-\left(sC+\frac{1}{r}\right)U_2(s)$$

(2) $$0=-\left(sC-\frac{1}{r}\right)U_1(s)+\left(sC+\frac{1}{sL}\right)U_2(s)\quad\Rightarrow\quad U_1(s)=\frac{sC+1/(sL)}{sC-1/r}U_2(s)\quad\rightarrow(1)$$

$$\Rightarrow\ I_0(s)=\left(\frac{1}{R}+sC\right)\frac{sC+1/(sL)}{sC-1/r}U_2(s)-\left(sC+\frac{1}{r}\right)U_2(s)$$

$$\Rightarrow\ I_0(s)\left(sC-\frac{1}{r}\right)=\left(\frac{1}{R}+sC\right)\left(sC+\frac{1}{sL}\right)U_2(s)-\left(sC+\frac{1}{r}\right)\left(sC-\frac{1}{r}\right)U_2(s)$$

$$\Rightarrow\ I_0(s)\left(sC-\frac{1}{r}\right)=\left(\frac{sC}{R}+\frac{1}{RsL}+\frac{sC}{sL}+\frac{1}{r^2}\right)U_2(s)\ \Rightarrow\ U_2(s)=\frac{\left(sC-\frac{1}{r}\right)I_0(s)}{\frac{sC}{R}+\frac{1}{RsL}+\frac{sC}{sL}+\frac{1}{r^2}}$$

$$U_2(s)=\frac{rRLs(srC-1)}{s^2r^2LC+sR(r^2C+L)+r^2}I_0(s)$$

$$a)\ Z_{ul}(s)=\frac{U_1}{I_0}=\frac{U_1}{U_2}\cdot\frac{U_2}{I_0}=\frac{sC+\frac{1}{sL}}{\left(sC-\frac{1}{r}\right)}\cdot\frac{\left(sC-\frac{1}{r}\right)I_0(s)}{\frac{sC}{R}+\frac{1}{RsL}+\frac{sC}{sL}+\frac{1}{r^2}}=\frac{\left(sC+\frac{1}{sL}\right)I_0(s)}{\frac{sC}{R}+\frac{1}{RsL}+\frac{sC}{sL}+\frac{1}{r^2}}$$

$$Z_{ul}(s)=\frac{U_1}{I_0}=\frac{r^2R\left(s^2LC+1\right)}{s^2r^2LC+sR(r^2C+L)+r^2}$$ (uvrstimo R=2, L=1, C=1, r=2)

$$Z_{ul}(s)=\frac{U_1}{I_0}=\frac{8\left(s^2+1\right)}{4s^2+10s+4}=2\frac{s^2+1}{s^2+\frac{5}{2}s+1}$$ (1 bod)

$$b)\ Z_{21}(s)=\frac{U_2}{I_0}=\frac{rRLs(srC-1)}{s^2r^2LC+sR(r^2C+L)+r^2}$$ (uvrstimo R=2, L=1, C=1, r=2)

$$Z_{21}(s)=\frac{4s(2s-1)}{4s^2+10s+4}=\frac{2s(2s-1)}{2s^2+5s+2}=\frac{s(2s-1)}{s^2+\frac{5}{2}s+1}=2\frac{s\left(s-\frac{1}{2}\right)}{s^2+\frac{5}{2}s+1}$$ (1 bod)

c) $Z_{21}(s)$: nule: $s(s-1/2)=0 \Rightarrow s_{o1}=0,\ s_{o2}=1/2$,

polovi: $s^2+\frac{5}{2}s+1=0 \Rightarrow s_{p1,2}=-\frac{5}{4}\pm\sqrt{\left(\frac{5}{4}\right)^2-1}=-\frac{5}{4}\pm\sqrt{\frac{25-16}{16}}=-\frac{5}{4}\pm\frac{3}{4}$

$s_{p1}=-1/2,\ s_{p2}=-2$, (1 bod)

$$d)\ U_2(s)=Z_{21}(s)\cdot I_0(s)=2\frac{s\left(s-\frac{1}{2}\right)}{s^2+\frac{5}{2}s+1}\cdot\frac{1}{s}=\frac{2s-1}{s^2+\frac{5}{2}s+1}=\frac{2s-1}{\left(s+\frac{1}{2}\right)(s+2)}$$

Rastav na parcijalne razlomke:

$$U_2(s)=\frac{2s-1}{(s+1/2)(s+2)}=\frac{A}{s+1/2}+\frac{B}{s+2}=\frac{A(s+1)+B(s+1/2)}{(s+1/2)(s+2)}=\frac{(A+B)s+2A+B/2}{(s+1/2)(s+2)}$$

$$\begin{aligned}&A+B=2\\&\underline{2A+B/2=-1}\end{aligned}\qquad\begin{aligned}&A=2-B\Rightarrow 2(2-B)+B/2=-1\\&\Rightarrow 4-2B+B/2=-1\Rightarrow 2B-B/2=5\\&\Rightarrow 3B=10\Rightarrow B=\frac{10}{3},\ A=\frac{6-10}{3}=-\frac{4}{3}\end{aligned}$$

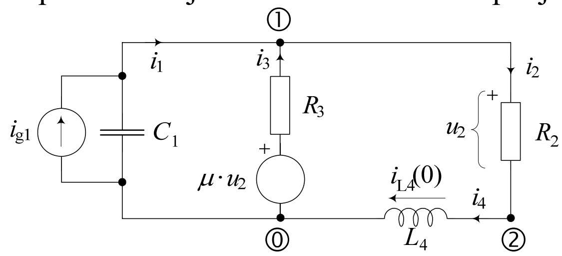
$$U_2(s)=\frac{-4/3}{s+1/2}+\frac{10/3}{s+2}\ \Rightarrow\ u_2(t)=\left(-\frac{4}{3}\cdot e^{-\frac{t}{2}}+\frac{10}{3}\cdot e^{-2t}\right)S(t)$$ (1 bod)

e) Fazori $i_0(t)=5\cdot\cos(t)$ $I_0(j\omega)=5\cdot e^{j0}=5$, $\omega=1$ $U_2(j\omega)=Z_{21}(j\omega)I_0(j\omega)=\frac{-2\omega^2-j\omega}{-\omega^2+(5/2)j\omega+1}\cdot 5$

$$U_2(j1)=\frac{-2-j}{(5/2)j}\cdot 5=\left(-\frac{2}{j}-1\right)2=(2j-1)2=-2+4j=2\sqrt{5}e^{j\varphi}$$ φ=arctan(–2)=116.56° (II kvadrant)

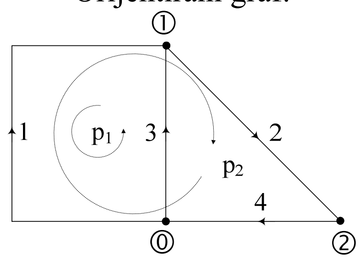
$$u_2(t)=2\sqrt{5}\cdot\cos(t+116.56°)$$ (1 bod)

2. Za električni krug prikazan slikom i pridruženim orijentacijama grana te čvorovima (grane stabla: 1, 2) treba odrediti temeljni sustav jednadžbi petlji topološkom analizom. Napisati: a) spojnu matricu **S**, b) matricu impedancija grana $\mathbf{Z}_b$ i vektor početnih uvjeta i nezavisnih izvora grana $\mathbf{U}_{0b}$, c) matricu impedancija petlji $\mathbf{Z}_p$ i d) vektor početnih uvjeta i nezavisnih izvora petlji $\mathbf{U}_{0p}$.



Rješenje:

Orijentirani graf:



Spojna matrica:

$$\mathbf{S} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

(1 bod)

Naponsko – strujne relacije grana: $\mathbf{U}_b = \mathbf{Z}_b \cdot \mathbf{I}_b + \mathbf{U}_{0b}$

(1) $U_1 = \dfrac{1}{sC_1} \cdot I_1 - \dfrac{1}{sC_1} \cdot I_{g1}$

(2) $U_2 = R_2 \cdot I_2$

(3) $U_3 = R_3 \cdot I_3 - \mu U_2 = R_3 \cdot I_3 - \mu R_2 I_2$

(3) $\underline{U_4 = sL_4 \cdot I_4 - L_4 i_{L4}(0)}$

$$\mathbf{Z}_b = \begin{bmatrix} \frac{1}{sC_1} & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ 0 & -\mu R_2 & R_3 & 0 \\ 0 & 0 & 0 & sL_4 \end{bmatrix}, \quad \mathbf{U}_{0b} = \begin{bmatrix} -\frac{I_{g1}}{sC_1} \\ 0 \\ 0 \\ -L_4 i_{L4}(0) \end{bmatrix}$$

(1 bod)

Matrica $\mathbf{Z}_b$ je regularna.

Temeljni sustav jednadžbi petlji u matričnom obliku: $\mathbf{Z}_p \cdot \mathbf{I}_p = \mathbf{U}_{0p}$

$$\mathbf{S} \cdot \mathbf{Z}_b = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{sC_1} & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ 0 & -\mu R_2 & R_3 & 0 \\ 0 & 0 & 0 & sL_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{sC_1} & -\mu R_2 & R_3 & 0 \\ \frac{1}{sC_1} & R_2 & 0 & sL_4 \end{bmatrix}$$

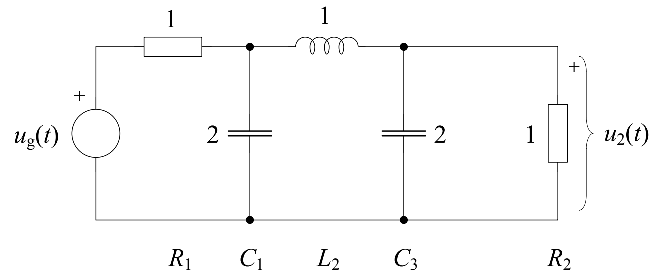
$$\mathbf{Z}_p = \mathbf{S} \cdot \mathbf{Z}_b \cdot \mathbf{S}^T = \begin{bmatrix} -\frac{1}{sC_1} & -\mu R_2 & R_3 & 0 \\ \frac{1}{sC_1} & R_2 & 0 & sL_4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{sC_1} + R_3 & -\frac{1}{sC_1} - \mu R_2 \\ -\frac{1}{sC_1} & \frac{1}{sC_1} + R_2 + sL_4 \end{bmatrix}$$

(1 bod)

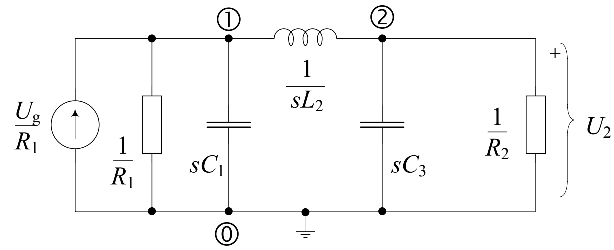
$$\mathbf{U}_{0p} = -\mathbf{S} \cdot \mathbf{U}_{0b} = -\begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{I_{g1}}{sC_1} \\ 0 \\ 0 \\ -L_4 \cdot i_{L4}(0) \end{bmatrix} = \begin{bmatrix} -\frac{I_{g1}}{sC_1} \\ \frac{I_{g1}}{sC_1} + L_4 \cdot i_{L4}(0) \end{bmatrix}$$

(1 bod)

3. Za mrežu na slici izračunati naponsku prijenosnu funkciju $H(s)=U_2(s)/U_g(s)$ ako su zadane normalizirane vrijednosti elemenata: $R_1=R_2=1$, $C_1=C_3=2$ i $L_2=1$. (Koristiti metodu napona čvorova) Napisati: a) Jednadžbe za čvorove (1) i (2) za izračun $H(s)$; b) $H(s)$ kao funkciju elemenata; c) $H(s)$ s uvrštenim elementima.



Rješenje:



$$1)\ U_1\left(\frac{1}{R_1}+sC_1+\frac{1}{sL_2}\right)-U_2\left(\frac{1}{sL_2}\right)=\frac{U_g}{R_1}$$

$$2)\ -U_1\frac{1}{sL_2}+U_2\left(\frac{1}{sL_2}+sC_3+\frac{1}{R_2}\right)=0$$ (1 bod)

$$2)\Rightarrow U_1=U_2 sL_2\left(\frac{1}{sL_2}+sC_3+\frac{1}{R_2}\right)\Rightarrow U_1=U_2\left(s^2L_2C_3+s\frac{L_2}{R_2}+1\right)$$

$$2)\rightarrow 1)\Rightarrow\ U_2\left(s^2L_2C_3+s\frac{L_2}{R_2}+1\right)\left(\frac{1}{R_1}+sC_1+\frac{1}{sL_2}\right)-U_2\left(\frac{1}{sL_2}\right)=\frac{U_g}{R_1}$$

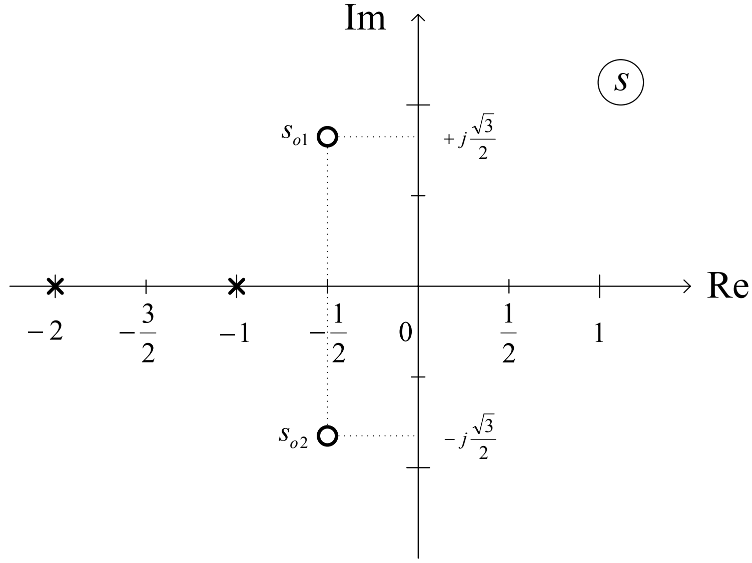
$$U_2\left(s^2\frac{L_2C_3}{R_1}+s\frac{L_2}{R_1R_2}+\frac{1}{R_1}+s^3L_2C_1C_3+s^2\frac{L_2C_1}{R_2}+sC_1+sC_3+\frac{1}{R_2}+\frac{1}{sL_2}-\frac{1}{sL_2}\right)=\frac{U_g}{R_1}\Big/\cdot R_1R_2$$

$$H(s)=\frac{U_2}{U_g}=\frac{R_2}{s^3L_2C_1C_3R_1R_2+s^2(L_2C_3R_2+L_2C_1R_1)+s(L_2+R_1R_2C_1+R_1R_2C_3)+R_1+R_2}$$ (1 bod)

Uz uvrštene vrijednosti elemenata:

$$H(s)=\frac{U_2(s)}{U_g(s)}=\frac{1}{4s^3+4s^2+5s+2}=\frac{\frac{1}{4}}{s^3+s^2+\frac{5}{4}s+\frac{1}{2}}$$ (1 bod)

4. Zadan je raspored polova i nula prema slici prijenosne funkcije $H(s)=U_{iz}(s)/U_{ul}(s)$ nekog električnog kruga. Odrediti: a) prijenosnu funkciju $H(s)$ ako se traži da je $H(0)=1/2$; b) kompleksnu frekvencijsku karakteristiku $H(j\omega)$; c) odziv $u_{iz}(t)$ za pobudu $u_{ul}(t)=\sin(t)$; $\quad -\infty < t < \infty$.



Rješenje:

a) Opći oblik prijenosne funkcije (funkcije mreža) napisan pomoću nula i polova:

$$H(s)=k\cdot\frac{\prod_i (s-s_{0i})}{\prod_j (s-s_{pj})}$$

Nule: $s_{01}=-\frac{1}{2}+j\frac{\sqrt{3}}{2}$; $\quad s_{02}=-\frac{1}{2}-j\frac{\sqrt{3}}{2}$

Polovi: $s_{p1}=-1$; $s_{p2}=-2$

$$H(s)=k\cdot\frac{\left(s+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\left(s+\frac{1}{2}+-j\frac{\sqrt{3}}{2}\right)}{(s+1)(s+2)}=k\cdot\frac{\left(s+\frac{1}{2}\right)^2-\left(j\frac{\sqrt{3}}{2}\right)^2}{s^2+3s+2}=$$

$(a-b)(a+b)=a^2-b^2$ (Upotrijebili smo pravilo: 'Razlika kvadrata')

$$=k\cdot\frac{s^2+s+\frac{1}{4}+\frac{3}{4}}{s^2+3s+2}=k\cdot\frac{s^2+s+1}{s^2+3s+2}$$

Konstanta *k* u općem obliku prijenosne funkcije slijedi iz:

$$H(0)=k\cdot\left.\frac{s^2+s+1}{s^2+3s+2}\right|_{s=0}=k\cdot\frac{1}{2}=\frac{1}{2}\Rightarrow k=1$$

Konačno je: $H(s)=\frac{s^2+s+1}{s^2+3s+2}$ (1 bod)

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b) Kompleksna frekvencijska karakteristika prijenosne funkcije:

$$H(j\omega)=\frac{-\omega^2+j\omega+1}{-\omega^2+3j\omega+2}$$ (1 bod)

c) Odziv na pobudu sve-vremenskom sinus funkcijom frekvencije ω=1:

$$u_{ul}(t)=\sin(t)\Rightarrow U_{ul}(j\omega)=1\angle 0°$$

$$U_{iz}(j\omega)=H(j\omega)\cdot U_{ul}(j\omega)=H(j\omega)\cdot 1\angle 0°$$

$$H(j1)=\frac{-1+j+1}{-1+3j+2}=\frac{j}{1+3j}$$

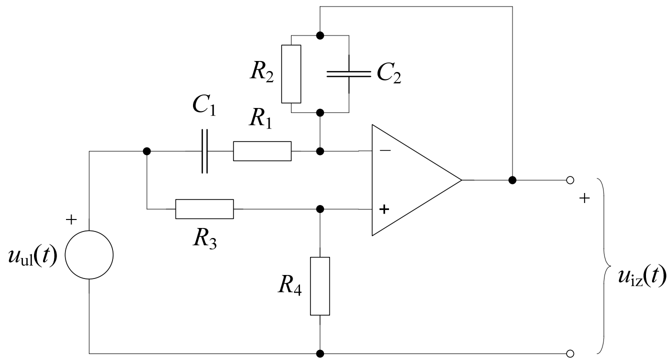
$$|H(j1)|=\frac{1}{\sqrt{3^2+1}}=\frac{1}{\sqrt{10}}=0.3162$$

$$H(j1)=\frac{j}{(1+3j)}\cdot\frac{1-3j}{1-3j}=\frac{3+j}{10}$$

$\angle H(j1)=\arctan(1/3)=18.43°$ (jer je fazor u prvom kvadrantu)

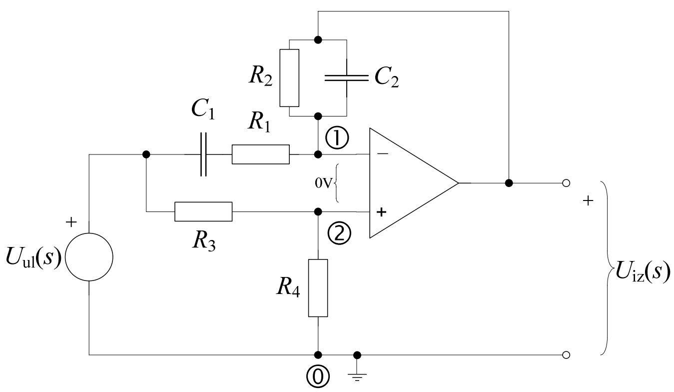
$u_{iz}(t)=0.3162\sin(t+18.43°)$ (1 bod)

5. Za električni krug prikazan slikom: a) Odrediti prijenosnu funkciju napona $H(s)=U_{iz}(s)/U_{ul}(s)$; b) Izračunati polove i nule prijenosne funkcije; c) Izračunati A-F karakteristiku $|H(j\omega)|$; d) Izračunati F-F karakteristiku $\varphi(\omega)=\arg H(j\omega)$; e) Izračunati logaritamsku mjeru pojačanja $\alpha(\omega)$. Zadano je: $R_1=R_2=1$, $C_1=C_2=1$, $R_3=R_4=1$, $A\to\infty$ (A je pojačanje operacijskog pojačala).



Rješenje:

a) Određivanje prijenosne funkcije napona $H(s)=U_{iz}(s)/U_{ul}(s)$:



$$Z_1=\frac{1}{sC_1}+R_1=\frac{1+sR_1C_1}{sC_1}$$

$$Z_2=\frac{1}{1/R_2+sC_2}=\frac{R_2}{1+sR_2C_2}$$

1) $U_1\left(\frac{1}{Z_1}+\frac{1}{Z_2}\right)=U_{ul}\frac{1}{Z_1}+U_{iz}\frac{1}{Z_2}$

2) $U_2\left(\frac{1}{R_3}+\frac{1}{R_4}\right)=\frac{U_{ul}}{R_3}\ \Rightarrow\ U_2=\frac{U_{ul}}{R_3\left(\frac{1}{R_3}+\frac{1}{R_4}\right)}=\frac{R_4}{R_4+R_3}U_{ul}$

3) $U_1=U_2$

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2), 3) $\rightarrow$ 1) $\Rightarrow\ \frac{R_4}{R_4+R_3}U_{ul}\left(\frac{1}{Z_1}+\frac{1}{Z_2}\right)=U_{ul}\frac{1}{Z_1}+U_{iz}\frac{1}{Z_2}\Big/\cdot Z_2$

$$\Rightarrow\ \frac{R_4}{R_4+R_3}U_{ul}\left(\frac{Z_2}{Z_1}+1\right)-U_{ul}\frac{Z_2}{Z_1}=U_{iz}\ \Rightarrow U_{ul}\left[\frac{R_4}{R_4+R_3}\left(\frac{Z_2}{Z_1}+1\right)-\frac{Z_2}{Z_1}\right]=U_{iz}$$

$$H(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{R_4}{R_4+R_3}\left(\frac{Z_2}{Z_1}+1\right)-\frac{Z_2}{Z_1}=\frac{R_4}{R_4+R_3}\frac{Z_2}{Z_1}+\frac{R_4}{R_4+R_3}-\frac{Z_2}{Z_1}=$$

$$=\left(\frac{R_4}{R_4+R_3}-1\right)\frac{Z_2}{Z_1}+\frac{R_4}{R_4+R_3}=\frac{-R_3}{R_4+R_3}\frac{Z_2}{Z_1}+\frac{R_4}{R_4+R_3}=\frac{R_4}{R_4+R_3}\cdot\frac{Z_1-(R_3/R_4)Z_2}{Z_1}=$$

$$=\frac{R_4}{R_4+R_3}\cdot\frac{\frac{1+sR_1C_1}{sC_1}-\frac{R_3}{R_4}\frac{R_2}{1+sR_2C_2}}{\frac{1+sR_1C_1}{sC_1}}=\frac{R_4}{R_4+R_3}\cdot\frac{(1+sR_1C_1)(1+sR_2C_2)-(R_3/R_4)sC_1R_2}{(1+sR_1C_1)(1+sR_2C_2)}$$

Uz uvrštene vrijednosti elemenata:

$$H(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=\frac{1}{2}\cdot\frac{(1+s)^2-s}{(1+s)^2}=\frac{1}{2}\cdot\frac{s^2+2s+1-s}{s^2+2s+1}=\frac{1}{2}\cdot\frac{s^2+s+1}{s^2+2s+1}$$

(1 bod)

b) Izračun polova i nula:

Polovi: $(s+1)^2=0$

$s_{p1,2}=-1$

Nule: $s^2+s+1=0$

$$s_{o1,2}=-\frac{1}{2}\pm\sqrt{\left(\frac{1}{2}\right)^2-1}=-\frac{1}{2}\pm j\frac{\sqrt{3}}{2}$$

(1 bod)

c) Amplitudno-frekvencijska karakteristika:

Uvrstimo $s=j\omega$ u $H(s)$

$$H(j\omega)=\frac{1}{2}\cdot\frac{-\omega^2+j\omega+1}{(j\omega+1)^2}\Rightarrow|H(j\omega)|=\frac{\sqrt{(1-\omega^2)^2+\omega^2}}{2(1+\omega^2)}$$

(1 bod)

d) Fazno-frekvencijska karakteristika:

$$\varphi(\omega)=\arctan\left(\frac{\omega}{1-\omega^2}\right)-2\arctan(\omega)$$

(1 bod)

e) Logaritamska mjera pojačanja:

$$\alpha(\omega)=20\log|H(j\omega)|=20\log 0.5+10\log\left(\omega^2+(1-\omega^2)^2\right)-20\log\left(\omega^2+1\right)=$$

$$=-6.0206+10\log\left(\omega^4-\omega^2+1\right)-20\log\left(\omega^2+1\right)[\mathrm{dB}]$$

(1 bod)

6. Teoretska pitanja: (5 bodova)

a) Ako električna mreža ima N_v=5 čvorova i N_b=7 grana, načiniti redoslijed metoda kojima je moguće riješiti mrežu počevši od one s najmanjim brojem jednadžbi prema većem.

b) Koji element može biti razlog da je električni krug nerecipročan?

c) Što je to struja temeljne petlje?

d) Što je to napon temeljnog reza?

e) Koliko dB ima jedan Neper?

Rješenje:
a) metodom petlji, metodom napona čvorova, Kirchhoffovim zakonima i naponsko-strujnim jednadžbama grana;
Ukupan broj jednadžbi za: petlje $N_b - N_v + 1 = 7 - 5 + 1 = 3$, za čvorove i rezove $N_v - 1 = 5 - 1 = 4$, za KZ $2N_b = 14$.
Redoslijed je 3, 4, 14.

b) girator;

c) Struja koja prolazi kroz grane stabla i jednu sponu;

d) Napon grane stabla obuhvaćene rezom;

e) 20log(e) ili 20/ln(10).