Električni krugovi

Funkcije mreža

Funkcije mreža

- Funkcija mreže je omjer L-transformacije odziva i pobude kad su svi početni uvjeti jednaki nuli.

$$u_i(0)=0$$

$$i_i(0)=0$$

$$H(s)=\frac{Y(s)}{X(s)}\rightarrow$$ **Funkcija mreže**

- Funkcija mreže $H(s)$ je omjer dvaju polinoma u s

$$H(s)=\frac{Y(s)}{X(s)}=\frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0}=\frac{P(s)}{Q(s)}$$

$P(s)$ i $Q(s) \rightarrow$ polinomi kompleksne varijable s

$a_i, b_i \rightarrow$ realni koeficijenti

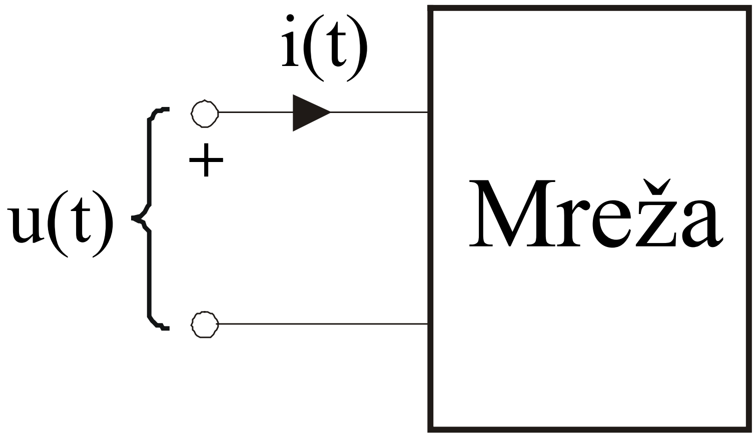
Funkcije mreža

FUNKCIJE MREŽA

ULAZNE FUNKCIJE
ili
FUNKCIJE IMITANCIJE

PRIJENOSNE
ili
TRANSFER FUNKCIJE

- **FUNKCIJE IMITANCIJE** –
  - **odnose se na par priključnica ili jedan prilaz**

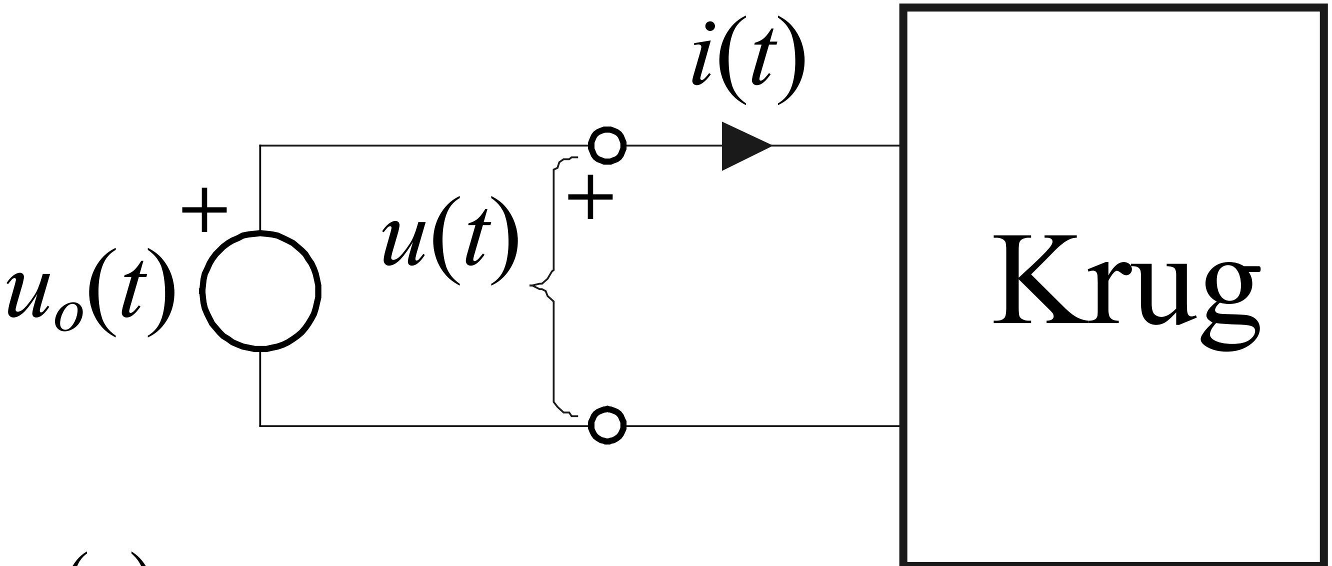


Omjer L-transformacije struje i napona promatranih na paru priključnica.

1. Slučaj

Pobuda → napon $u(t) \circ\!\!-\!\!\bullet U(s)$

Odziv → struja $i(t) \circ\!\!-\!\!\bullet I(s)$

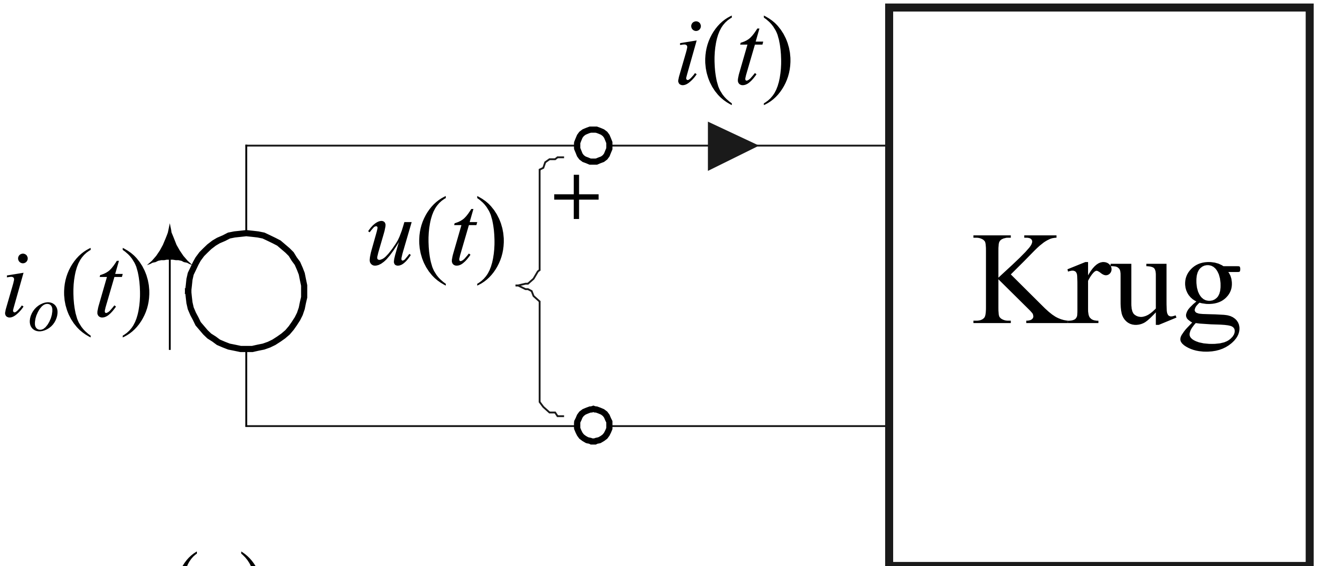


$$H(s) = \frac{I(s)}{U(s)} = Y(s) \rightarrow$$ **Funkcija admitancije**

2. slučaj

Pobuda → struja $i(t) \circ\!\!-\!\!\bullet I(s)$

Odziv → napon $u(t) \circ\!\!-\!\!\bullet U(s)$

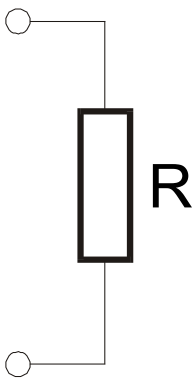


$$H(s) = \frac{U(s)}{I(s)} = Z(s) \rightarrow$$ **Funkcija impedancije**

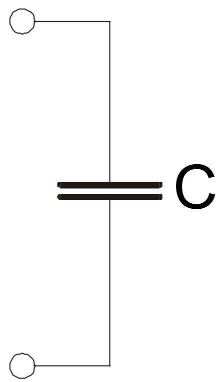
$$\left.\begin{array}{l}\text{funkcije impedancije } Z(s)\\ \text{funkcije admitancije } Y(s)\end{array}\right\}$$

funkcije imitancije
ili ulazne funkcije!!!

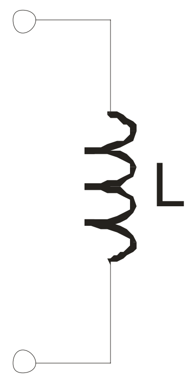
Vrijedi: $$\boxed{Z(s)=\frac{1}{Y(s)}}$$



$$Z(s)=R$$
$$Y(s)=\frac{1}{R}$$



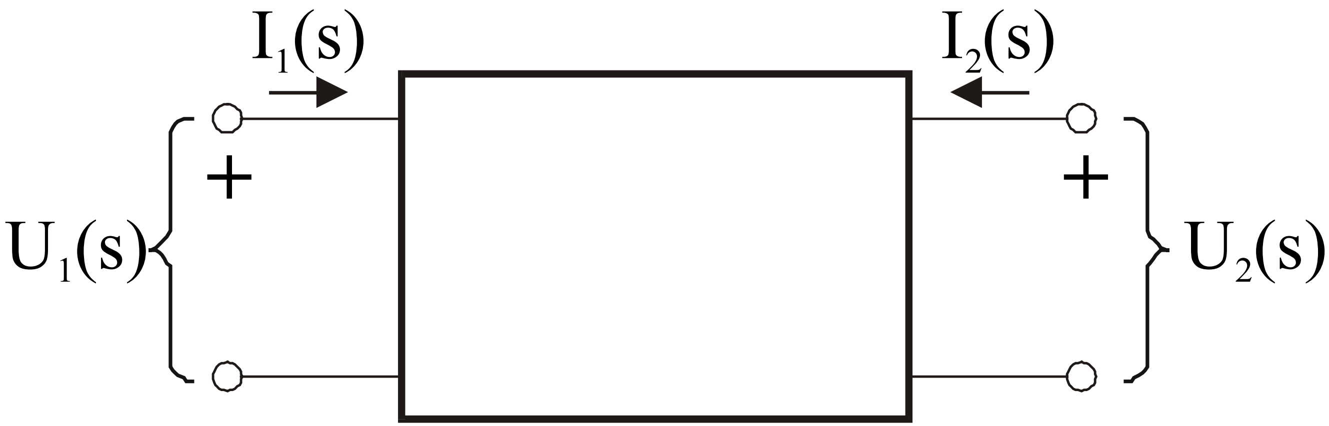
$$Z(s)=\frac{1}{sC}$$
$$Y(s)=sC$$



$$Z(s)=sL$$
$$Y(s)=\frac{1}{sL}$$

PRIJENOSNE FUNKCIJE

Odnose se na različite parove priključnica neke mreže



Pobuda → na prilazu 1

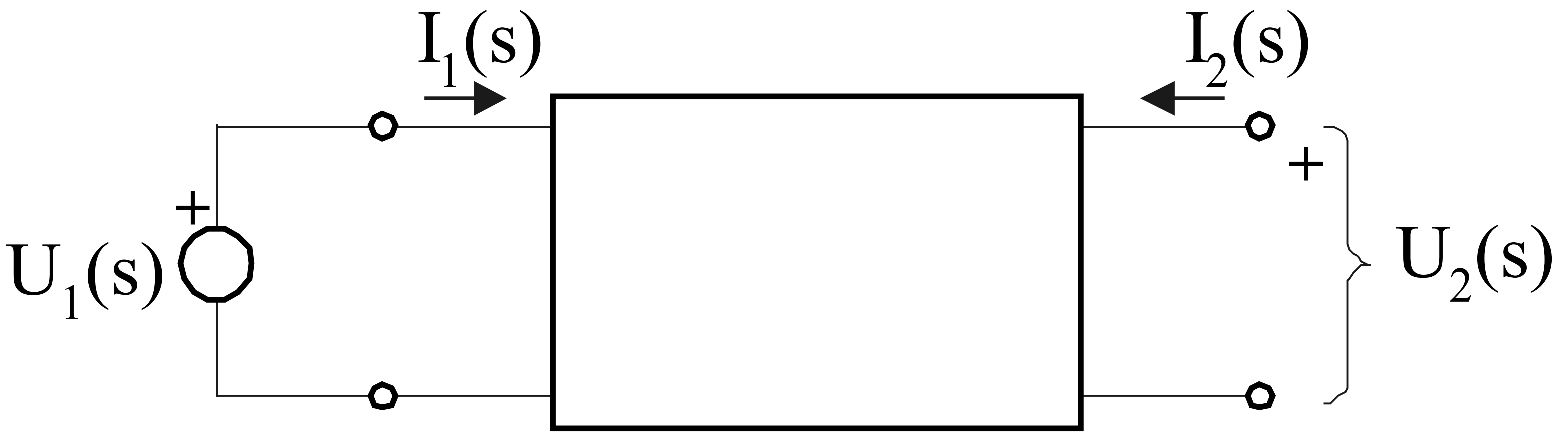
Napon **Struja**

Odziv → na prilazu 2

Napon **Struja**

4-tipa prijenosnih funkcija

1) Prijenosna funkcija (prijenosni omjer) napona



Odziv → napon na prilazu 2

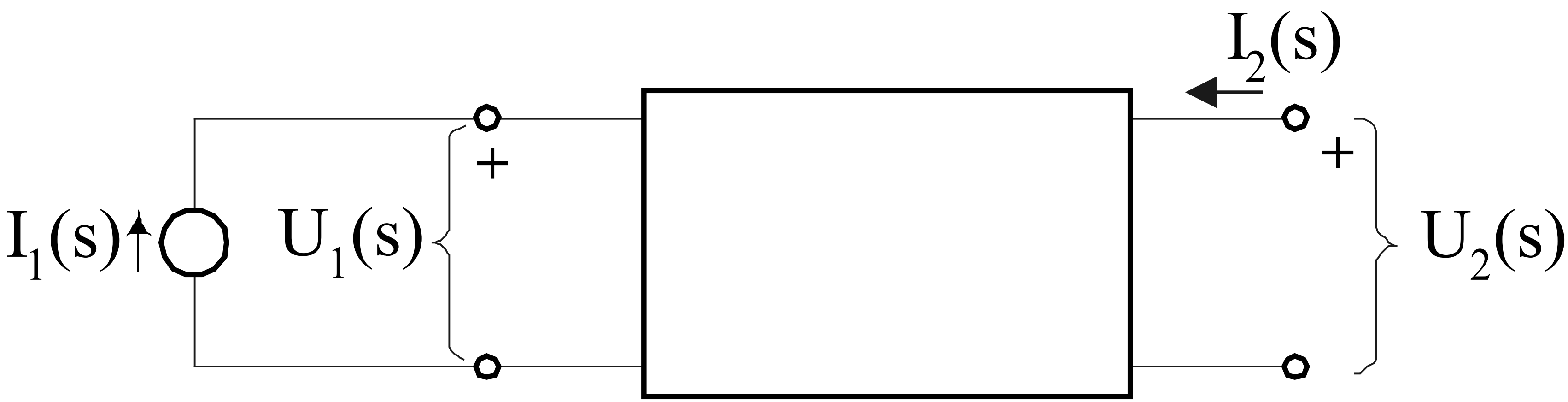
Pobuda → napon na prilazu 1

$$H(s) = \frac{U_2(s)}{U_1(s)}$$

odziv

pobuda

2) Prijenosna funkcija (prijenosni omjer) struja



Odziv → struja na prilazu 2

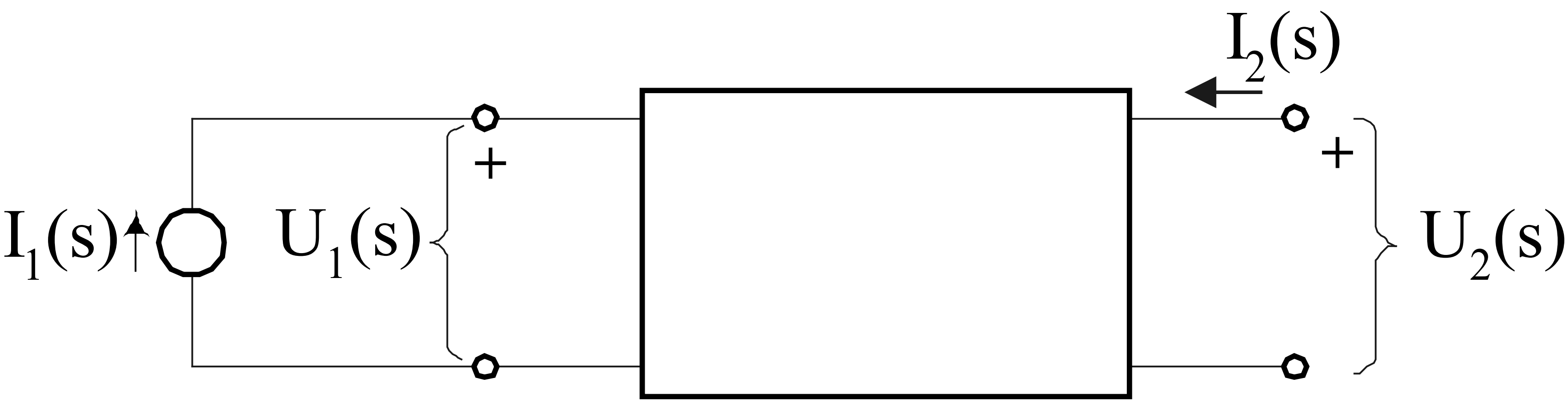
Pobuda → struja na prilazu 1

$$H(s)=\frac{I_2(s)}{I_1(s)}$$

odziv

pobuda

3) Prijenosna impedancija



Odziv → napon na prilazu 2

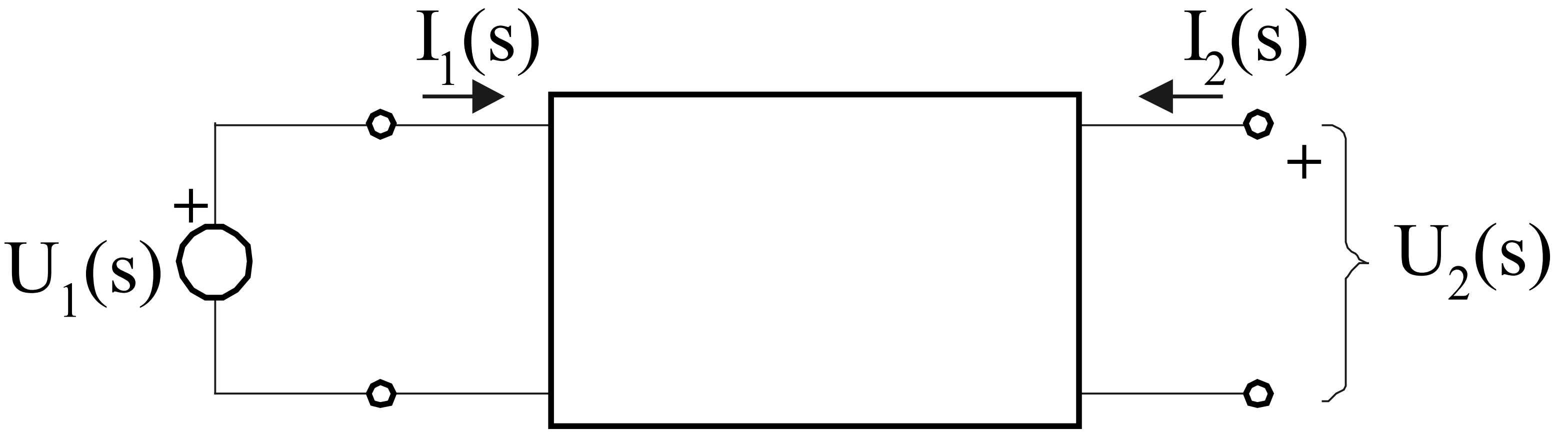
Pobuda → struja na prilazu 1

$$Z_{21}(s) = \frac{U_2(s)}{I_1(s)}$$

odziv

pobuda

4) Prijenosna admitancija



Odziv → struja na prilazu 2

Pobuda → napon na prilazu 1

$$Y_{21}(s) = \frac{I_2(s)}{U_1(s)}$$

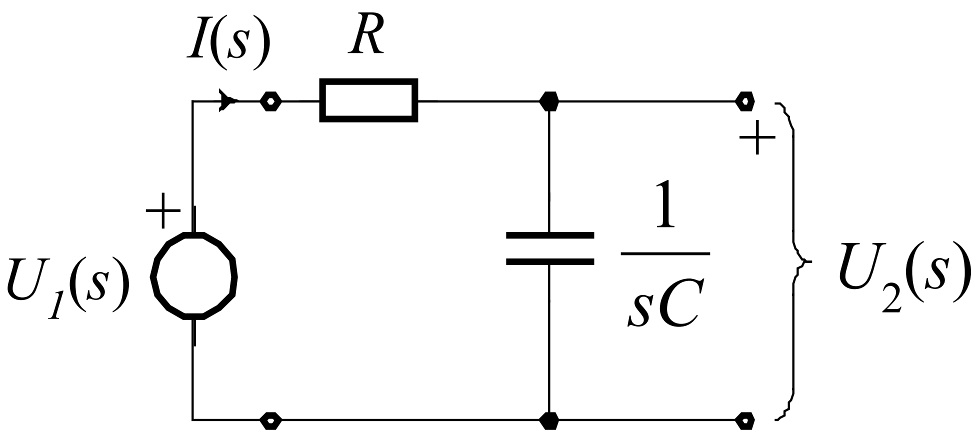
odziv

pobuda

Važno:

$$\boxed{Z_{21}(s) \neq \frac{1}{Y_{21}(s)}}$$

- <u>Primjer 1.: RC mreža</u>



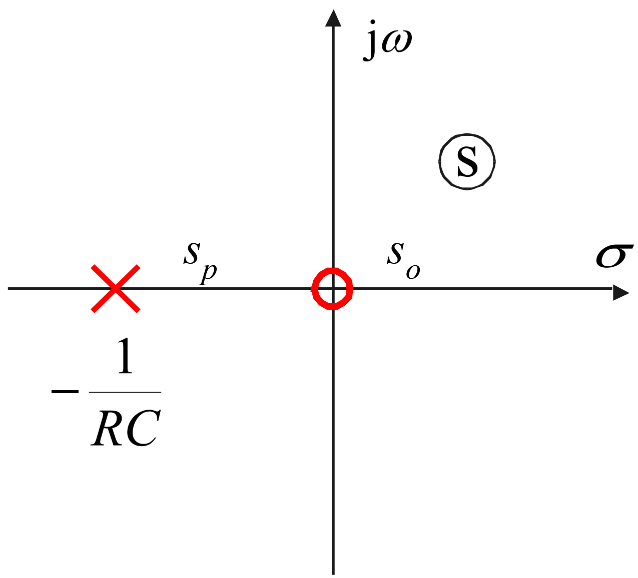
- Ulazna admitancijaY(s)=?
- Prijenosna funkcija napona H(s)=?

- Ulazna funkcija - admitancija

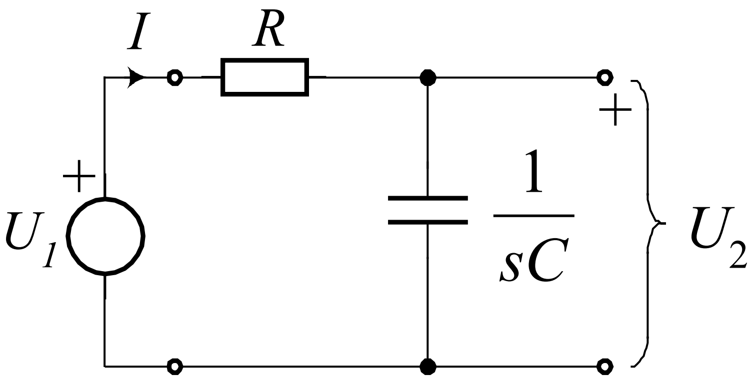
$$Y(s)=\frac{I(s)}{U_1(s)}=\frac{1}{R}\cdot\frac{s}{s+\frac{1}{RC}}$$

$$s_o=0$$

$$s_p=-\frac{1}{RC}$$



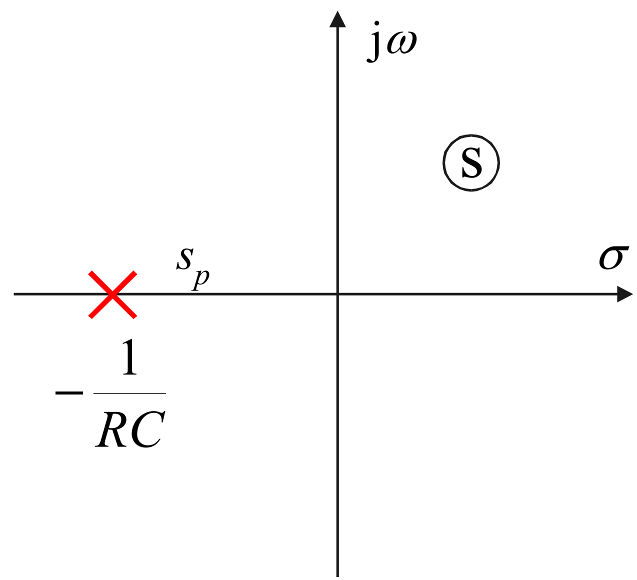
- Prijenosna funkcija napona – naponski djelitelj



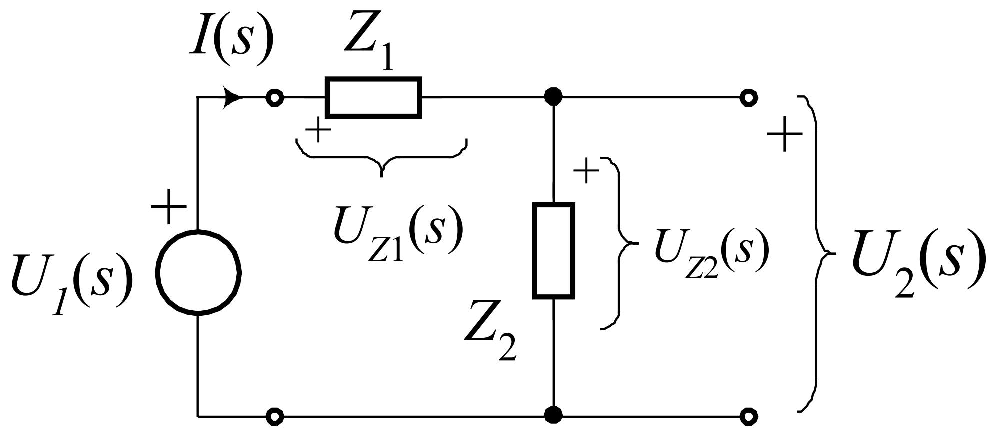
$$s_o = \infty$$

$$s_p = -\frac{1}{RC}$$

$$H(s) = \frac{U_2(s)}{U_1(s)} = \frac{\dfrac{1}{RC}}{s + \dfrac{1}{RC}}$$



- Mreža oblika



je ***naponski djelitelj*** jer napon U_1(s) raspodjeljuje na:

- napon na impedanciji Z_1 → U_{Z1} i
- napon na impedanciji Z_2 → $U_{Z2}=U_2$

$$U_1(s) = U_{Z1}(s) + U_{Z2}(s)$$

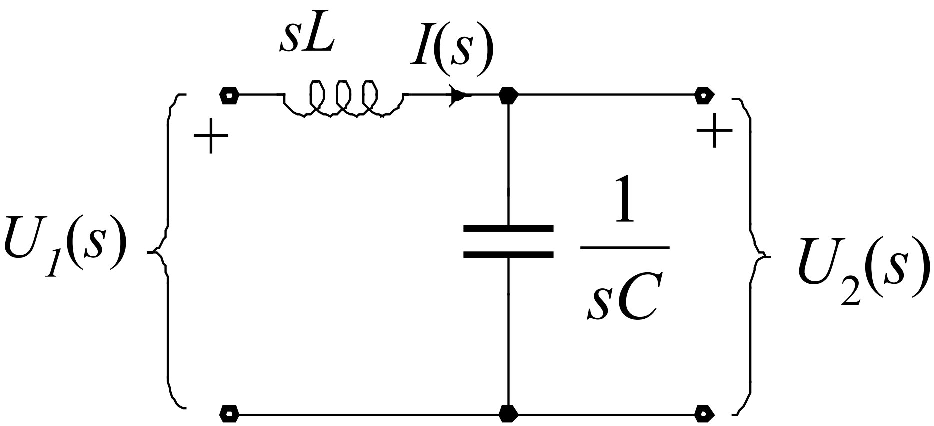
- Omjer napona $U_2(s)$ prema $U_1(s)$ je

$$H(s)=\frac{U_{Z2}(s)}{U_{Z1}(s)+U_{Z2}(s)}=\frac{I(s)\cdot Z_2(s)}{I(s)\cdot Z_1(s)+I(s)\cdot Z_2(s)}$$

$$H(s)=\frac{Z_2(s)}{Z_1(s)+Z_2(s)}$$

- Napon je raspodijeljen proporcionalno s impedancijama.

Primjer 2.: LC mreža



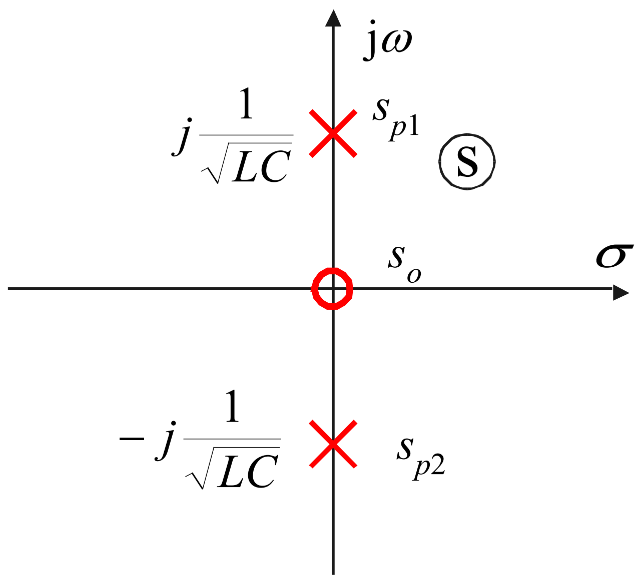
- Ulazna funkcija - admitancija

$$Y(s)=\frac{I(s)}{U_1(s)}=\frac{1}{L}\cdot\frac{s}{s^2+\frac{1}{LC}}$$

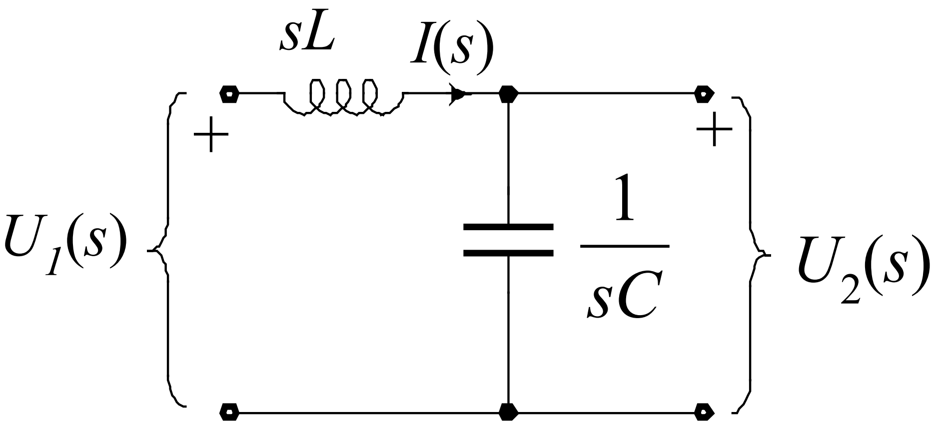
$$s_{o1}=0$$

$$s_{o2}=\infty$$

$$s_{p1,2}=\pm j\frac{1}{\sqrt{LC}}$$

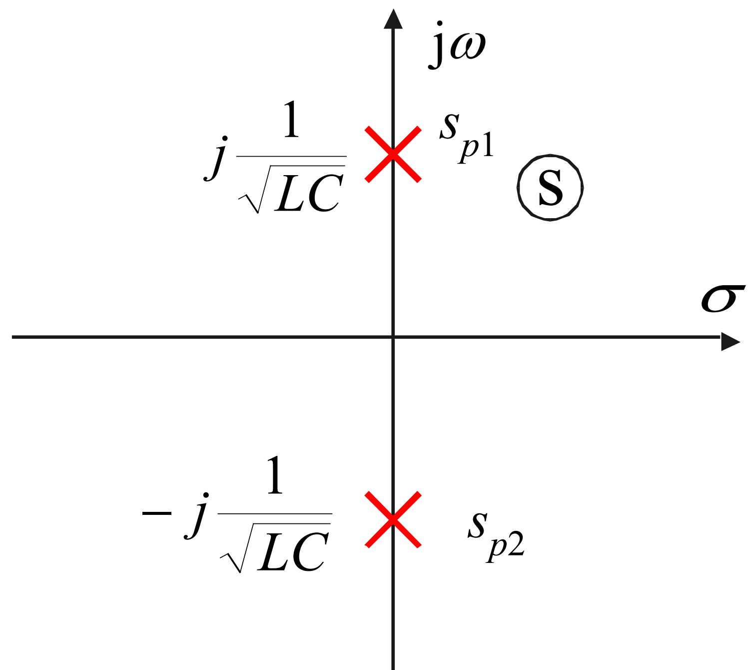


Prijenosna funkcija napona – naponski djelitelj



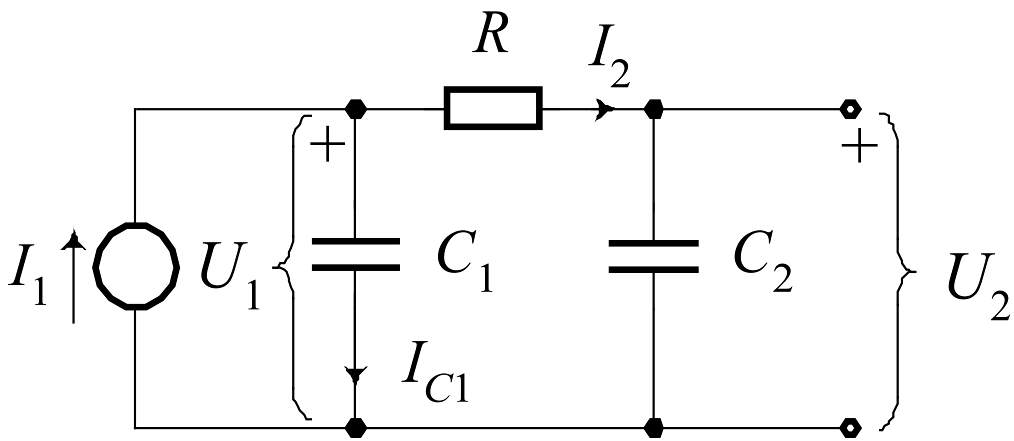
$$s_{o1,2} = \infty$$

$$s_p = \pm j \frac{1}{\sqrt{LC}}$$



$$H(s) = \frac{U_2(s)}{U_1(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{1}{LC}}$$

- <u>Primjer 3.:</u> Odrediti prijenosne funkcije $H(s)=I_2/I_1$ i $Z_{21}(s)=U_2/I_1$



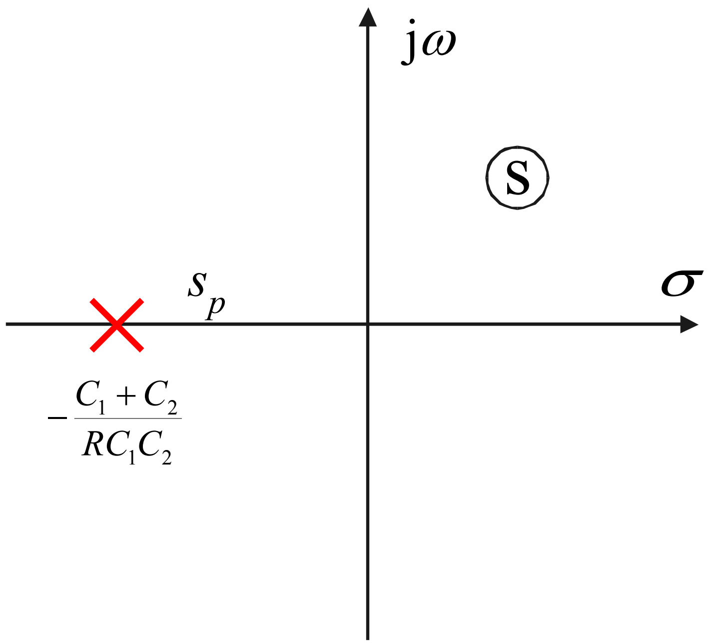
- Krug na slici → strujni djelitelj

$$H(s)=\frac{1}{C_1R}\cdot\frac{1}{s+(C_1+C_2)/RC_1C_2}$$

- Polovi i nule

$$s_o=\infty$$

$$s_p=-\frac{C_1+C_2}{RC_1C_2}$$



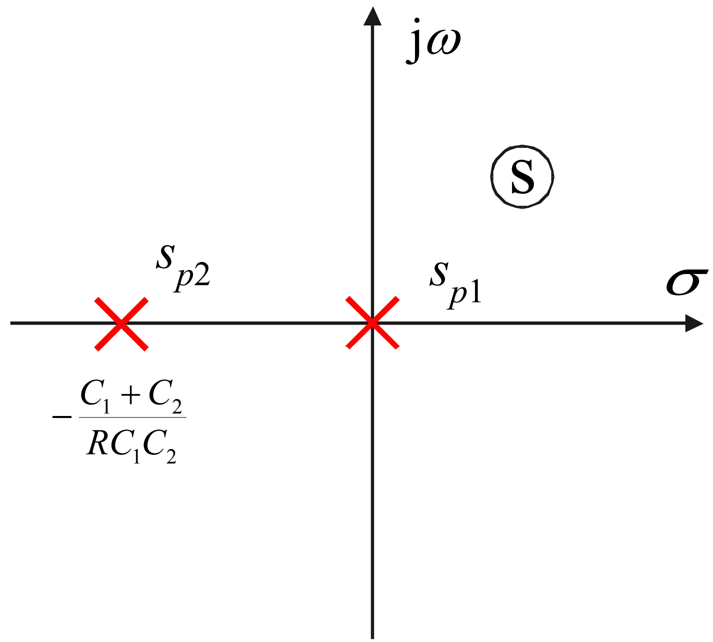
- Prijenosna impedancija

$$Z_{21}(s)=\frac{U_2(s)}{I_1(s)}=\frac{1/C_1C_2R}{s\left(s+(C_1+C_2)/RC_1C_2\right)}$$

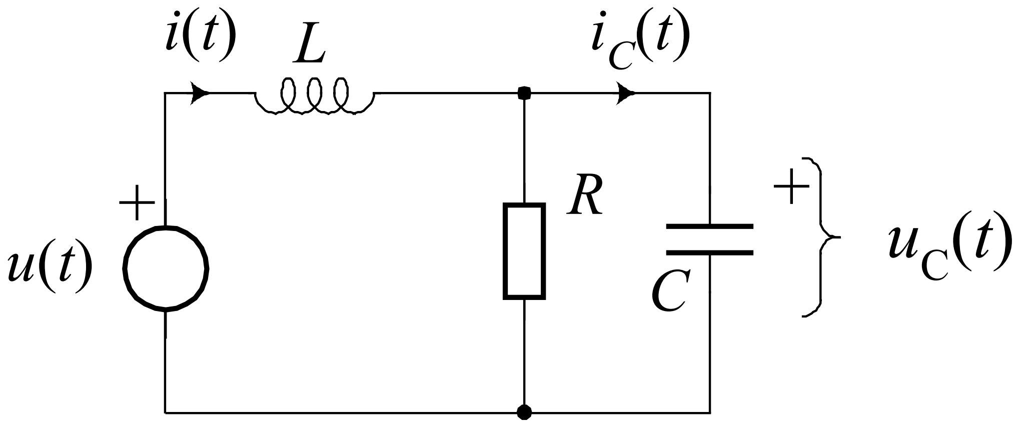
$$s_{o1,2}=\infty$$

$$s_{p1}=0$$

$$s_{p2}=-\frac{C_1+C_2}{RC_1C_2}$$



- Primjer 4.: RLC krug



$R = 2$

$C = 1$

$L = 2$

- Odziv → struja $I(s)$ → Funkcija admitancije

- Odziv → napon $U_C(s)$ → Prijenosna funkcija

- Odziv → struja $I_C(s)$ → Prijenosna funkcija

- Odziv → struja $I(s)$ → Funkcija admitancije

$$Y(s) = \frac{I(s)}{U(s)} = \frac{s + 1/2}{s^2 + s/2 + 1}$$

- Odziv → napon $U_C(s)$ → Prijenosna funkcija

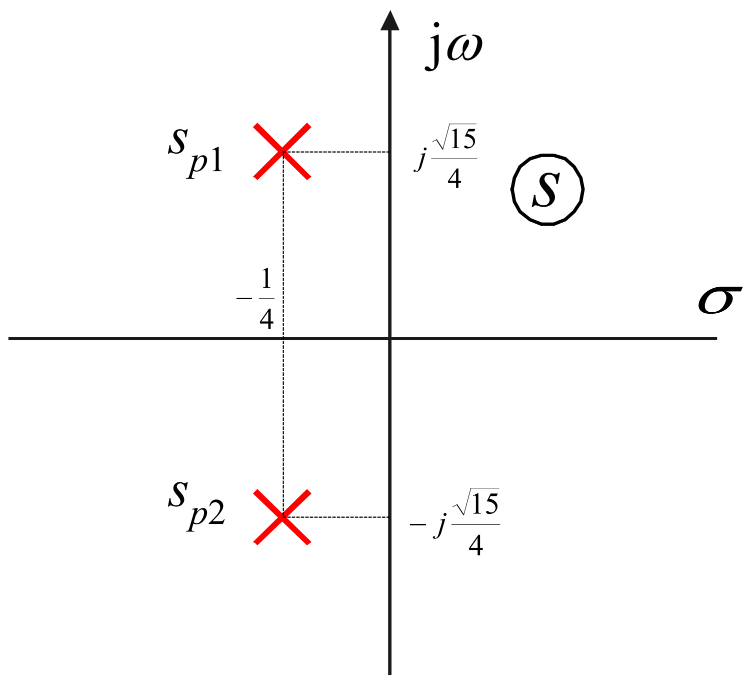
$$H(s) = \frac{U_C(s)}{U(s)} = \frac{1}{s^2 + s/2 + 1}$$

- Nule prijenosne funkcije

$$s_{o1,2} = \infty$$

- Polovi prijenosne funkcije

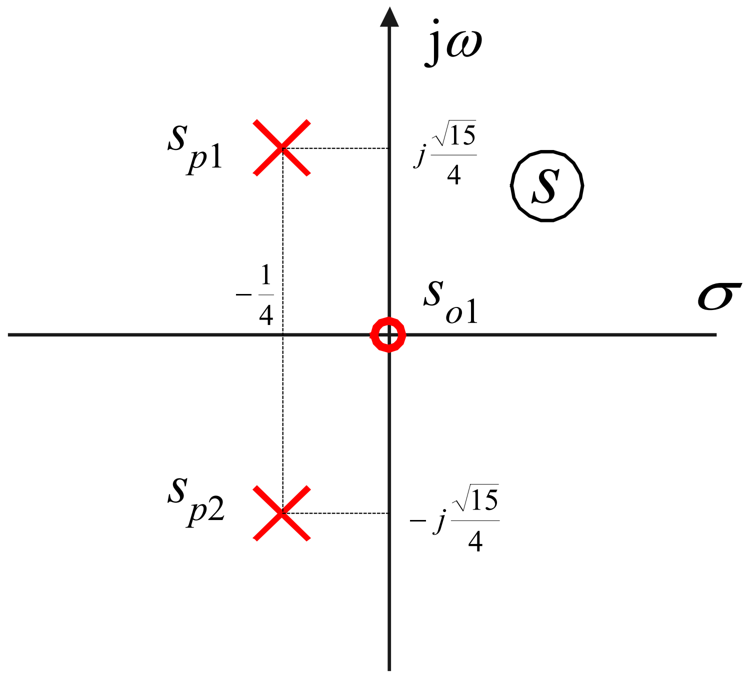
$$s_{p1,2} = -\frac{1}{4} \pm j\frac{\sqrt{15}}{4}$$



- Odziv → struja $I_C(s)$ → Prijenosna funkcija

- Prijenosna admitancija

$$H(s) = Y_{21}(s) = \frac{I_C(s)}{U(s)} = \frac{2s}{2s^2 + s + 2}$$



Recipročna vrijednost funkcije imitancije je također ***funkcija imitancije*** dakle i funkcija mreže.

Recipročna vrijednost prijenosne funkcije ***nije funkcija mreže*** (nema njena svojstva).

Za sve funkcije mreža vrijedi:

- razlomljena realna racionalna f. od "*s*"
- polovi samo u lijevoj poluravnini i na jω osi
- polovi na jω osi jednostruki

Za prijenosne funkcije vrijedi:

– nule mogu biti i u desnoj poluravnini

Za funkcije imitancije vrijedi:

– nule ne smiju biti u desnoj poluravnini →

– za njih vrijedi isto što i za polove

Funkcije mreža i stacionarno stanje sinusne pobude

Funkcija mreže H(s) je omjer Laplace-ovih transformacija odziva Y(s) i pobude X(s) uz početne uvjete jednake nuli.

$$H(s)=\frac{Y(s)}{X(s)}$$

U stacionanome stanju sinusne pobude

→ Pobuda i odziv → <u>fazori</u>

Fazor - definicije

$$X(j\omega)=\left|X(j\omega)\right|\angle\varphi_x(\omega)=\left|X(j\omega)\right|e^{j\varphi_x(\omega)}$$

$$Y(j\omega)=\left|Y(j\omega)\right|\angle\varphi_y(\omega)=\left|Y(j\omega)\right|e^{j\varphi_y(\omega)}$$

Pobuda djeluje od $t=-\infty$

Prijelazne pojave završene → STACIONARNO STANJE.

Naponi i struje u krugu imaju sinusni oblik frekvencije ω.

$$x(t)=X\sin(\omega t+\varphi_x)\rightarrow X\angle\varphi_x$$

Razlikuju se po iznosu amplituda i faznog pomaka.

- Ako u funkciji H(s) zamijenimo varijablu *s* sa jω

→ H(jω)=omjer fazora odziva i fazora pobude.

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{|Y(j\omega)|e^{j\varphi_y(\omega)}}{|X(j\omega)|e^{j\varphi_x(\omega)}}=\left|\frac{Y(j\omega)}{X(j\omega)}\right|\cdot e^{j(\varphi_y-\varphi_x)}$$

$H(j\omega)\rightarrow$ općenito kompleksni broj

$$H(j\omega)=\mathrm{Re}[H(j\omega)]+j\,\mathrm{Im}[H(j\omega)]$$

U polarnim koordinatama

$$H(j\omega) = |H(j\omega)| \cdot e^{j\varphi(\omega)} \longrightarrow$$ kompleksna frekvencijska karakteristika

$$|H(j\omega)| = \frac{|Y(j\omega)|}{|X(j\omega)|} \longrightarrow$$ amplitudno frekvencijska karakteristika

$$\varphi(\omega) = \varphi_y(\omega) - \varphi_x(\omega) \longrightarrow$$ fazno frekvencijska karakteristika

Ako je poznata funkcija $H(j\omega)$, tada je:

$$H^2(j\omega) = \underbrace{H(j\omega)\cdot H(-j\omega)}_{|H(j\omega)|^2}\cdot\frac{H(j\omega)}{H(-j\omega)} = |H(j\omega)|^2\cdot e^{2j\varphi(\omega)}$$

$$|H(j\omega)| = \sqrt{H(j\omega)\cdot H(-j\omega)}$$

$$\varphi(\omega) = \frac{1}{2j}\ln\left|\frac{H(j\omega)}{H(-j\omega)}\right|$$

ili: $\left|H(j\omega)\right| = \sqrt{\mathrm{Re}^2\left[H(j\omega)\right] + \mathrm{Im}^2\left[H(j\omega)\right]}$

Moguće je pisati i:

$$H(j\omega) = \left|H(j\omega)\right| \cdot e^{j\varphi(\omega)} = \left|H(j\omega)\right| \cdot \left[\cos(\varphi(\omega)) + j\sin(\varphi(\omega))\right]$$

$$\left.\begin{aligned} \mathrm{Re}\left[H(j\omega)\right] &= \left|H(j\omega)\right| \cos(\varphi(\omega)) \\ \mathrm{Im}\left[H(j\omega)\right] &= \left|H(j\omega)\right| \sin(\varphi(\omega)) \end{aligned}\right\} \quad \mathrm{tg}\,\varphi(\omega) = \frac{\mathrm{Im}\left[H(j\omega)\right]}{\mathrm{Re}\left[H(j\omega)\right]}$$

$$\varphi(\omega) = \mathrm{arctg}\frac{\mathrm{Im}\left[H(j\omega)\right]}{\mathrm{Re}\left[H(j\omega)\right]}$$

$$H(s)=\frac{P(s)}{Q(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_0}{a_n s^n+a_{n-1}s^{n-1}+\ldots+a_0}$$

$$H(s)=k\frac{(s-s_{01})(s-s_{02})\cdots(s-s_{0m})}{(s-s_{p1})(s-s_{p2})\cdots(s-s_{pn})}=k\frac{\prod_{i=1}^{m}(s-s_{0i})}{\prod_{j=1}^{n}(s-s_{pj})}$$

$s_{0i}=\sigma_{0i}+j\omega_{0i}$ $\longrightarrow$ nule funkcije mreže

$s_{pj}=\sigma_{pj}+j\omega_{pj}$ $\longrightarrow$ polovi funkcije mreže

$$H(j\omega)=\frac{P(j\omega)}{Q(j\omega)}=\frac{b_m(j\omega)^m+b_{m-1}(j\omega)^{m-1}+\ldots+b_0}{a_n(j\omega)^n+a_{n-1}(j\omega)^{n-1}+\ldots+a_0}$$

$$H(j\omega)=k\frac{(j\omega-s_{01})(j\omega-s_{02})\cdots(j\omega-s_{0m})}{(j\omega-s_{p1})(j\omega-s_{p2})\cdots(j\omega-s_{pn})}$$

$$(j\omega-s_{0i})=\left|j\omega-s_{0i}\right|\cdot e^{j\varphi_{oi}} \qquad \varphi_{oi}=\mathrm{arctg}\left(\frac{\omega-\omega_{0i}}{\sigma_{0i}}\right)$$

$$\left|H(j\omega)\right| = k\frac{\left|j\omega - s_{01}\right|\cdot\left|j\omega - s_{02}\right|\cdots\left|j\omega - s_{0m}\right|}{\left|j\omega - s_{p1}\right|\cdot\left|j\omega - s_{p2}\right|\cdots\left|j\omega - s_{pn}\right|}$$

$$\left|H(j\omega)\right| = k\frac{\left|j\omega - \sigma_{01} - j\omega_{01}\right|\cdots\left|j\omega - \sigma_{0m} - j\omega_{0m}\right|}{\left|j\omega - \sigma_{p1} - j\omega_{p1}\right|\cdots\left|j\omega - \sigma_{pn} - j\omega_{pn}\right|}$$

$$\left|H(j\omega)\right| = k\frac{\sqrt{\sigma_{01}^2 + (\omega - \omega_{01})^2}\cdots\sqrt{\sigma_{0m}^2 + (\omega - \omega_{0m})^2}}{\sqrt{\sigma_{p1}^2 + (\omega - \omega_{p1})^2}\cdots\sqrt{\sigma_{pn}^2 + (\omega - \omega_{pn})^2}}$$

$$\varphi(\omega) = \left[\varphi_{01}(\omega) + \varphi_{02}(\omega) + \ldots + \varphi_{0m}(\omega)\right] - \\ - \left[\varphi_{p1}(\omega) + \varphi_{p2}(\omega) + \ldots + \varphi_{pn}(\omega)\right]$$

$$\varphi(\omega) = \sum_{i=1}^{m} \operatorname{arctg}\left(\frac{\omega - \omega_{0i}}{\sigma_{0i}}\right) - \sum_{j=1}^{n} \operatorname{arctg}\left(\frac{\omega - \omega_{pj}}{\sigma_{pj}}\right)$$

Za prijenosne funkcije često se koristi logaritamska mjera.

$$\ln H(j\omega) = \ln\left[\left|H(j\omega)\right| \cdot e^{j\varphi(\omega)}\right] = \ln\left|H(j\omega)\right| + j\varphi(\omega)$$

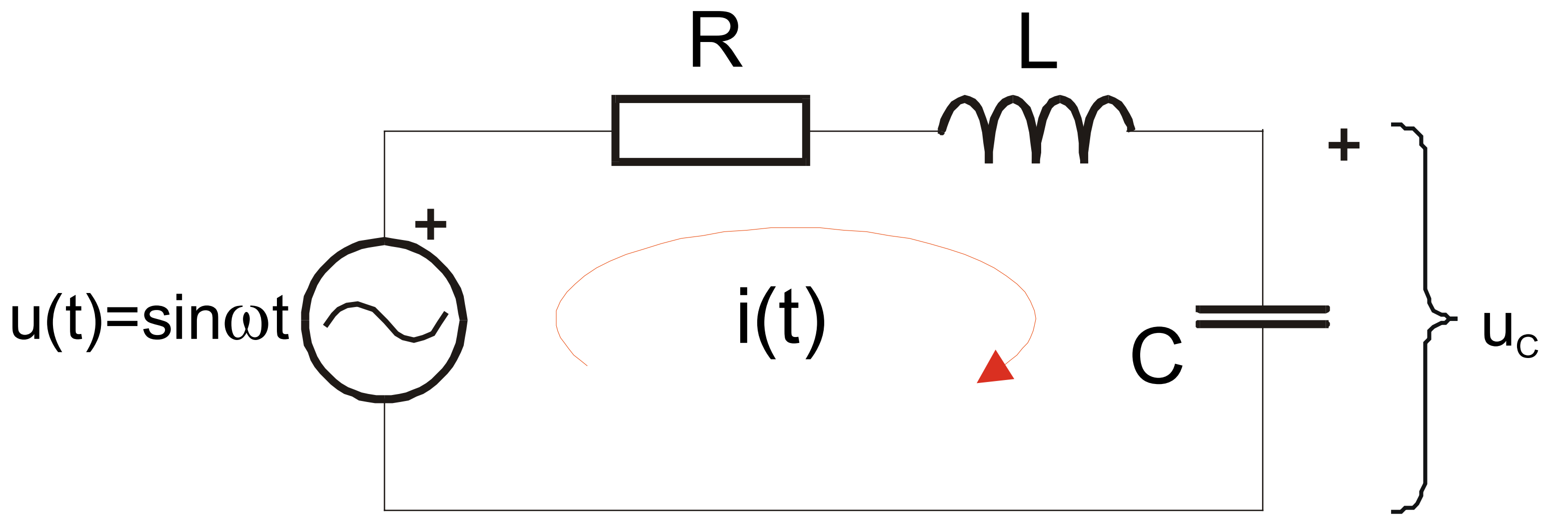
$$\alpha_N(\omega) = \ln\left|H(j\omega)\right| \rightarrow$$ Logaritamska mjera pojačanja u Neperima [N].

Često umjesto prirodnog → dekadski log, pa je uobičajena mjera:

$$\alpha(\omega) = 20\log\left|H(j\omega)\right| \rightarrow$$ logaritamska mjera pojačanja u decibelima [dB]

$$\alpha(\omega)[dB] \cong 8.68\,\alpha_N(\omega)[N]$$

Primjer: RLC krug



$$\omega = 1$$
$$R = 2$$
$$C = 1$$
$$L = 2$$

$$U_C = ?$$

$$u(t) = \cos(\omega t); \; -\infty < t < \infty$$

$$H(s) = \frac{U_C(s)}{U(s)} = \frac{1}{s^2 LC + sRC + 1}$$

$$H(s)=\frac{U_C(s)}{U(s)}=\frac{1}{s^2LC+sRC+1}$$

Sinusna pobuda $\rightarrow$ s=jω $\rightarrow$ frekvencijska karakteristika

$$H(j\omega)=\frac{U_C(j\omega)}{U(j\omega)}=\frac{1}{-\omega^2LC+j\omega RC+1}$$

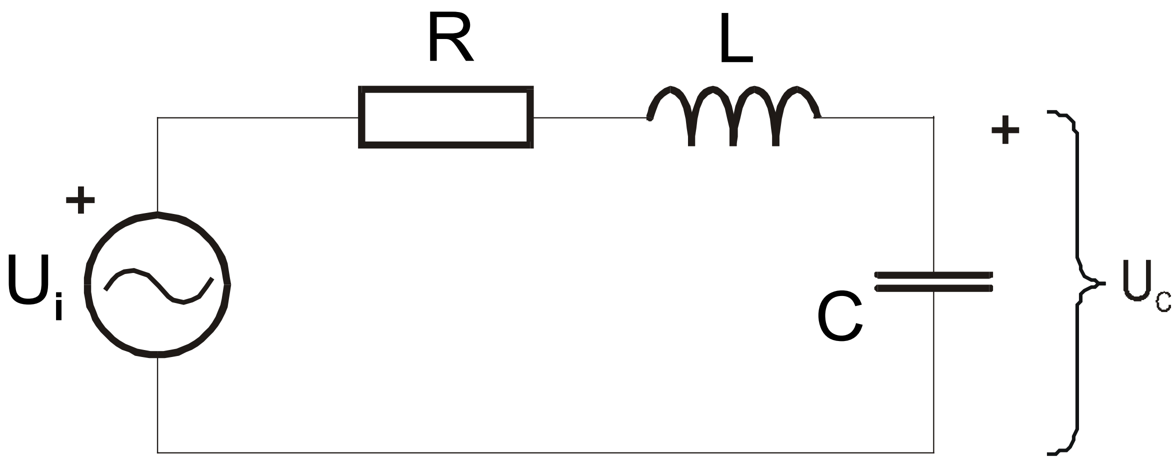
Fazor odziva $\rightarrow$ U_C(jω)

$$U_C(j\omega)=U(j\omega)\cdot H(j\omega)=\frac{U(j\omega)}{-\omega^2LC+j\omega RC+1}$$

$$U(j\omega)=1\angle 0°=1$$

$$U_C(j\omega)=-\frac{1+2j}{5}=-\frac{\sqrt{5}}{5}\angle \mathrm{arctg}(2)$$

<u>Primjer: RLC krug</u> → Frekvencijski odziv ili karakteristika



$$H(s)=\frac{U_C(s)}{U_i(s)} \rightarrow$$ Prijenosna funkcija napona

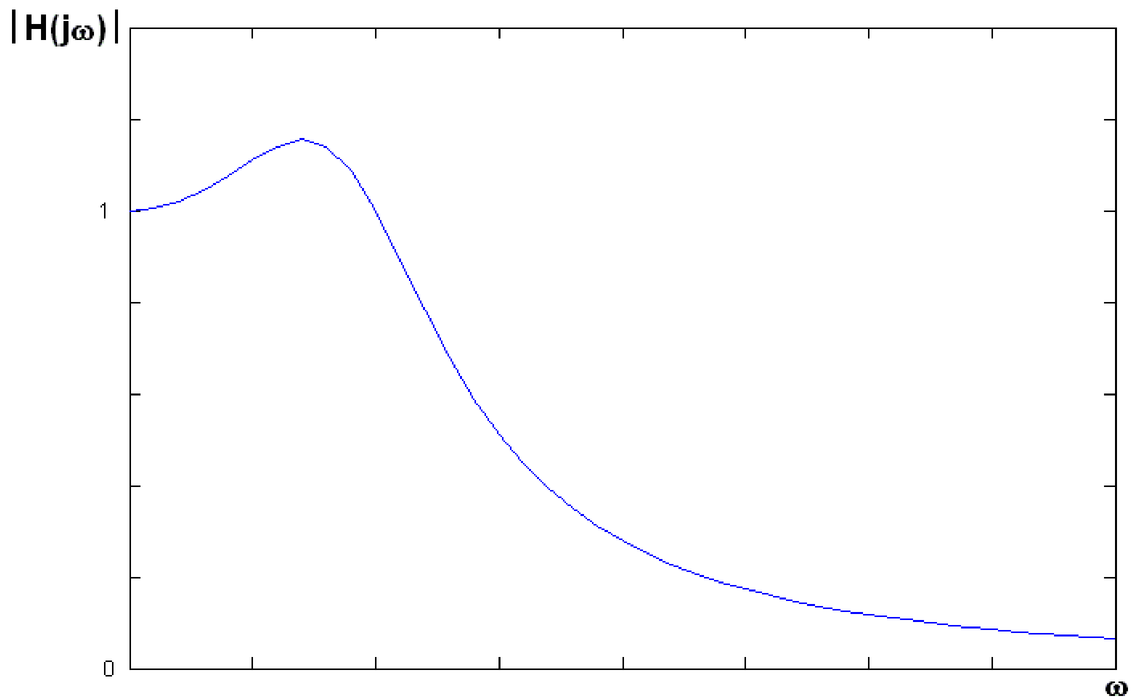
$$H(s)=\frac{1}{s^2LC+sRC+1}$$

- Amplitudno frekvencijska karakteristika

$$\left|H(j\omega)\right| = \frac{1}{\sqrt{\left(1-\omega^2 LC\right)^2 + (RC\omega)^2}}$$

Za R=L=C=1 →

$$\left|H(j\omega)\right| = \frac{1}{\sqrt{1-\omega^2+\omega^4}}$$

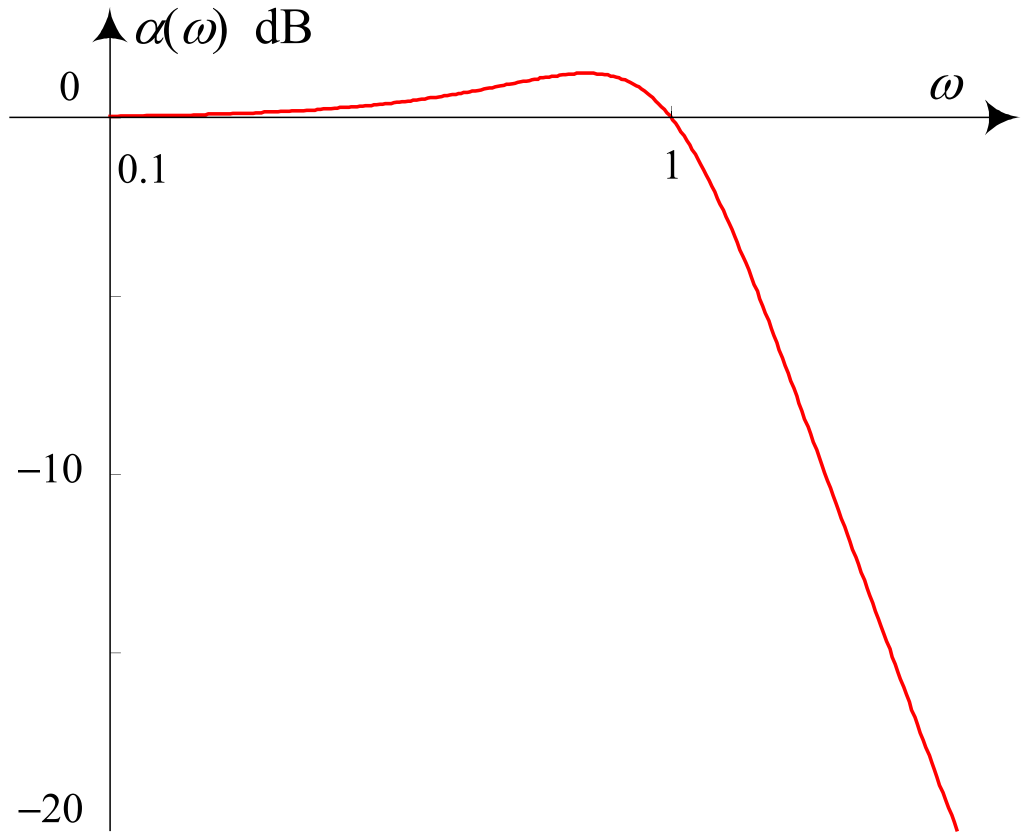


- Logaritamska mjera pojačanja

$$\alpha(\omega) = 20 \cdot \log\left|H(j\omega)\right| =$$

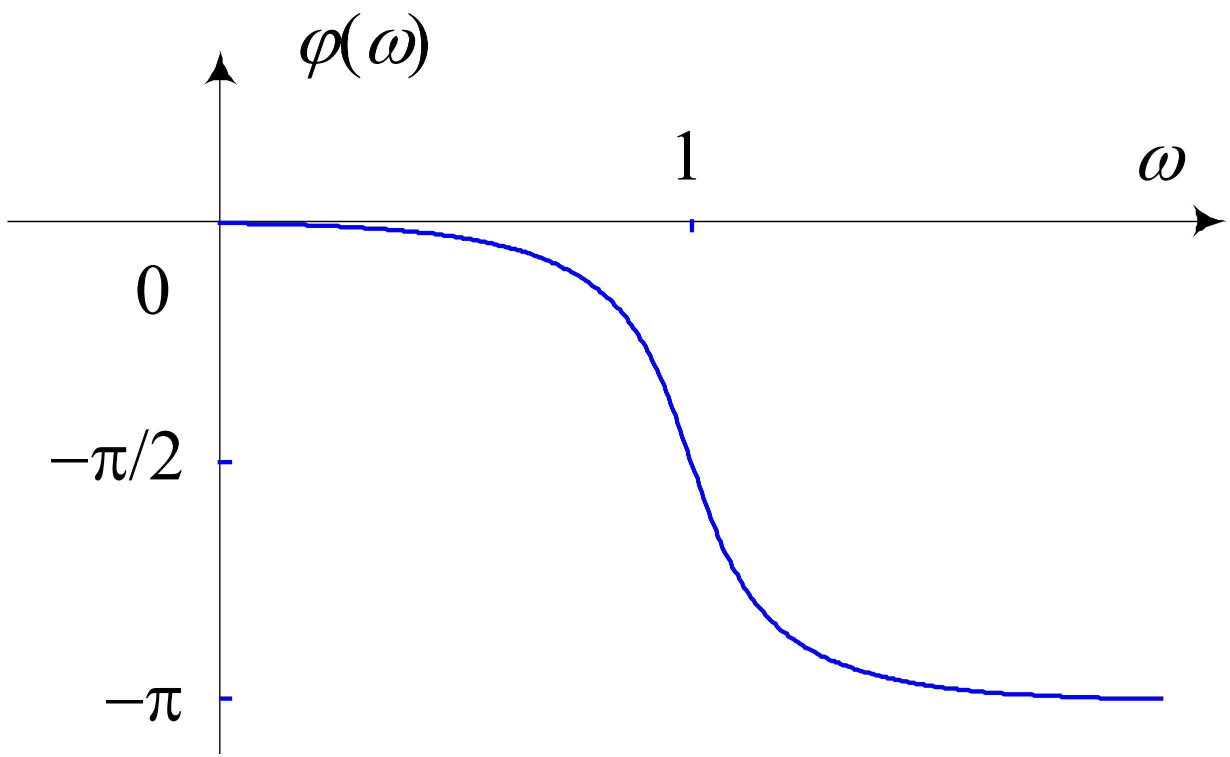
$$= 20 \cdot \log \frac{1}{\sqrt{1-\omega^2+\omega^4}} =$$

$$= -10\log\left(1-\omega^2+\omega^4\right)$$



- Fazna frekvencijska karakteristika

$$\varphi(\omega) = -\mathrm{arctg}\frac{\mathrm{RC}\omega}{1-\omega^2 LC} \qquad \varphi(\omega) = -\mathrm{arctg}\frac{\omega}{1-\omega^2}$$



Funkciju H(jω) moguće je napisati u obliku

$$H(j\omega)=\frac{b_m(j\omega)^m+b_{m-1}(j\omega)^{m-1}+\ldots+b_0}{a_n(j\omega)^n+a_{n-1}(j\omega)^{n-1}+\ldots+a_0}$$

odnosno

$$H(j\omega)=\frac{\left(b_0-b_2\omega^2+b_4\omega^4\mp\ldots\right)+j\omega\left(b_1-b_3\omega^2+b_5\omega^4\mp\ldots\right)}{\left(a_0-a_2\omega^2+a_4\omega^4\mp\ldots\right)+j\omega\left(a_1-a_3\omega^2+a_5\omega^4\mp\ldots\right)}$$

vrijedi

$$H(j\omega)=\frac{\left(polinom\ od\ \omega^2\right)+j\omega\left(polinom\ od\ \omega^2\right)}{\left(polinom\ od\ \omega^2\right)+j\omega\left(polinom\ od\ \omega^2\right)}$$

$$H(-j\omega)=\frac{\left(polinom\ od\ \omega^2\right)-j\omega\left(polinom\ od\ \omega^2\right)}{\left(polinom\ od\ \omega^2\right)-j\omega\left(polinom\ od\ \omega^2\right)}$$

Očito je: $\boxed{H^*(j\omega) = H(-j\omega)}$

pa pošto vrijedi za kompleksne brojeve:

$$\left.\begin{aligned}
\mathrm{Re}[H^*(j\omega)] &= \mathrm{Re}[H(j\omega)] \\
\mathrm{Im}[H^*(j\omega)] &= -\mathrm{Im}[H(j\omega)] \\
|H^*(j\omega)| &= |H(j\omega)| \\
H^*(-j\omega) &= H(j\omega)
\end{aligned}\right\}
\begin{aligned}
\mathrm{Re}[H(-j\omega)] &= \mathrm{Re}[H(j\omega)] \\
\mathrm{Im}[H(-j\omega)] &= -\mathrm{Im}[H(j\omega)] \\
|H(-j\omega)| &= |H(j\omega)| \\
\varphi(-\omega) &= -\varphi(\omega)
\end{aligned}$$