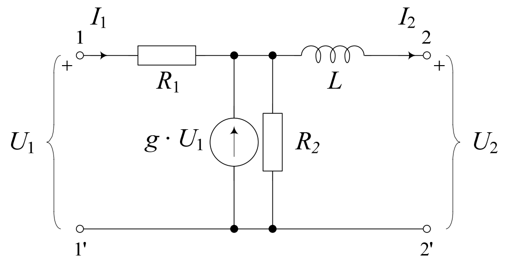
ZAVRŠNI ISPIT IZ PREDMETA ELEKTRIČNI KRUGOVI 2009/10

Rješenja i bodovi (svaki zadatak je bodovan od 0 do 5 bodova):

1. Za četveropol na slici izračunati: *a*) [**z**]-parametre i napisati ih u matričnom obliku. Da li je četveropol: *b*) recipročan; *c*) simetričan ? Obrazložiti odgovore.



Rješenje:

a) izračun [**z**] parametara

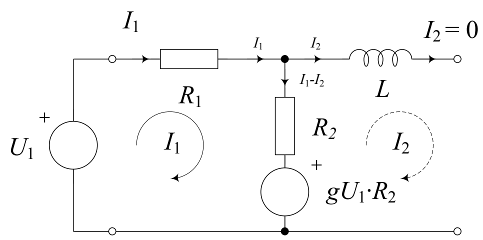
$$U_1 = z_{11} \cdot I_1 - z_{12} \cdot I_2$$
$$U_2 = z_{21} \cdot I_1 - z_{22} \cdot I_2$$

Najjednostavnije je izračunati [**z**]-parametre pomoću jednadžbi petlji (pretpostavimo izvore U_1 i U_2):

$$(1)\ I_1 \cdot (R_1 + R_2) - I_2 \cdot R_2 = U_1 - gU_1 \cdot R_2$$
$$(2)\ -I_1 \cdot R_2 + I_2 (sL + R_2) = gU_1 R_2 - U_2$$

$I_2 = 0$ (na prilazu 2-2' prazni hod)

$$U_1 = z_{11} \cdot I_1 \Rightarrow z_{11} = U_1 / I_1$$
$$U_2 = z_{21} \cdot I_1 \Rightarrow z_{21} = U_2 / I_1$$



$$(1) \Rightarrow I_1 \cdot (R_1 + R_2) = U_1 (1 - g \cdot R_2) \Rightarrow U_1 = \frac{R_1 + R_2}{1 - g \cdot R_2} \cdot I_1 \Rightarrow z_{11} = \frac{U_1}{I_1} = \frac{R_1 + R_2}{1 - g \cdot R_2}$$

$$(2) \Rightarrow -I_1 \cdot R_2 = gU_1 R_2 - U_2 \Rightarrow U_2 = gU_1 R_2 + I_1 \cdot R_2$$

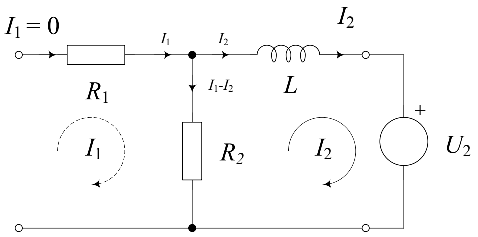
$$(1) \rightarrow (2) \Rightarrow U_2 = gR_2 \frac{R_1 + R_2}{1 - g \cdot R_2} \cdot I_1 + R_2 \cdot I_1 \Rightarrow z_{21} = \frac{U_2}{I_1} = gR_2 \frac{R_1 + R_2}{1 - g \cdot R_2} + R_2$$

Sređivanje z_{21}

$$z_{21} = \frac{U_2}{I_1} = \frac{gR_1 R_2 + gR_2^2 + R_2 - gR_2^2}{1 - g \cdot R_2} = \frac{(1 + gR_1) R_2}{1 - g \cdot R_2}$$

$I_1 = 0$ (na prilazu 1-1' prazni hod)

$$U_1 = -z_{12} \cdot I_2 \Rightarrow z_{12} = -U_1 / I_2$$
$$U_2 = -z_{22} \cdot I_2 \Rightarrow z_{22} = -U_2 / I_2$$



$$(1) \Rightarrow -I_2 \cdot R_2 = U_1 - gU_1 \cdot R_2 \Rightarrow U_1 = -\frac{R_2}{1-gR_2} \cdot I_2 \Rightarrow z_{12} = -\frac{U_1}{I_2} = \frac{R_2}{1-gR_2}$$

$$(2) \Rightarrow I_2(sL + R_2) = gU_1R_2 - U_2 \Rightarrow U_2 = gU_1R_2 - I_2(sL + R_2)$$

$$(1)\rightarrow(2) \Rightarrow U_2 = -gR_2 \frac{R_2}{1-gR_2} \cdot I_2 - (sL + R_2) \cdot I_2 \Rightarrow z_{22} = -\frac{U_2}{I_2} = \frac{gR_2^2}{1-g \cdot R_2} + sL + R_2$$

Slijedi sređivanje z_{22}

$$z_{22} = -\frac{U_2}{I_2} = \frac{gR_2^2 + sL + R_2 - g \cdot R_2 sL - gR_2^2}{1-g \cdot R_2} = \frac{sL + R_2(1 - g \cdot sL)}{1-g \cdot R_2}$$

Matrica [**z**]-parametara:

$$[\mathbf{z}] = \begin{bmatrix} \dfrac{R_1 + R_2}{1-g \cdot R_2} & -\dfrac{R_2}{1-g \cdot R_2} \\ \dfrac{(1+gR_1)R_2}{1-g \cdot R_2} & -\dfrac{R_2(1-g \cdot sL) + sL}{1-g \cdot R_2} \end{bmatrix}$$

(do sada: maksimum 4 boda – ako su sva 4 parametra točna)

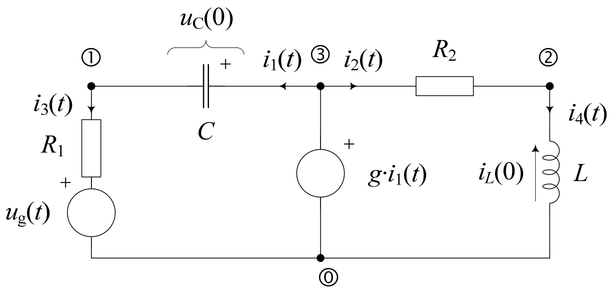
b) Da li je četveropol recipročan ?

Ne, jer za recipročnost mora vrijediti z_{12}=z_{21}. To očigledno ne vrijedi, a razlog tomu je zavisni izvor.

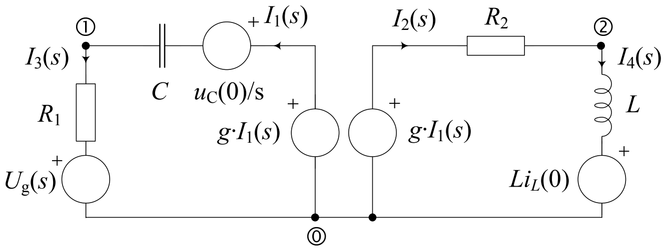
c) Da li je četveropol simetričan ?

Ne, jer za simetričnost mora vrijediti z_{11}=z_{22}. (1 bod)

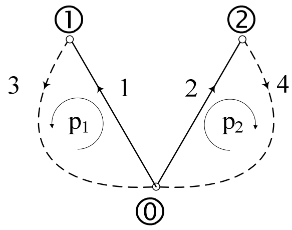
2. Za električni krug prikazan slikom, poštujući oznake čvorišta i grana, odrediti: *a*) orijentirani graf i temeljni sustav petlji; *b*) spojnu matricu **S**; *c*) naponsko-strujne jednadžbe grana u matričnom obliku; *d*) temeljni sustav jednadžbi petlji u matričnom obliku (matrice $\mathbf{Z}_p$ i $\mathbf{U}_{0p}$ preko matrica $\mathbf{Z}_b$ i $\mathbf{U}_{0b}$). Matrica $\mathbf{Z}_b$ mora biti regularna.



Rješenje: Primjena $\mathcal{L}$-transformacije i posmicanje naponskog izvora



a) Orijentirani graf:



b) Spojna matrica:

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

(1bod)

c) Naponsko-strujne jednadžbe grana:

(2 boda: 1 za jednadžbe i 1 za sliku u $\mathcal{L}$-domeni s posmicanjem izvora)

$$U_1 = I_1 \cdot \frac{1}{sC} + \frac{u_C(0)}{s} - gI_1(s)$$
$$U_2 = I_2 \cdot R_2 - gI_1(s)$$
$$U_3 = I_3 \cdot R_1 + U_g(s)$$
$$\underline{U_4 = I_4 \cdot sL + L \cdot i_L(0)}$$

U matričnom obliku: $\mathbf{U}_b = \mathbf{Z}_b \cdot \mathbf{I}_b + \mathbf{U}_{0b}$

$$\underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}}_{\mathbf{U}_b} = \underbrace{\begin{bmatrix} \frac{1}{sC}-g & 0 & 0 & 0 \\ -g & R_2 & 0 & 0 \\ 0 & 0 & R_1 & 0 \\ 0 & 0 & 0 & sL \end{bmatrix}}_{\mathbf{Z}_b} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}}_{\mathbf{I}_b} + \underbrace{\begin{bmatrix} \frac{u_C(0)}{s} \\ 0 \\ U_g(s) \\ Li_L(0) \end{bmatrix}}_{\mathbf{U}_{0b}}$$

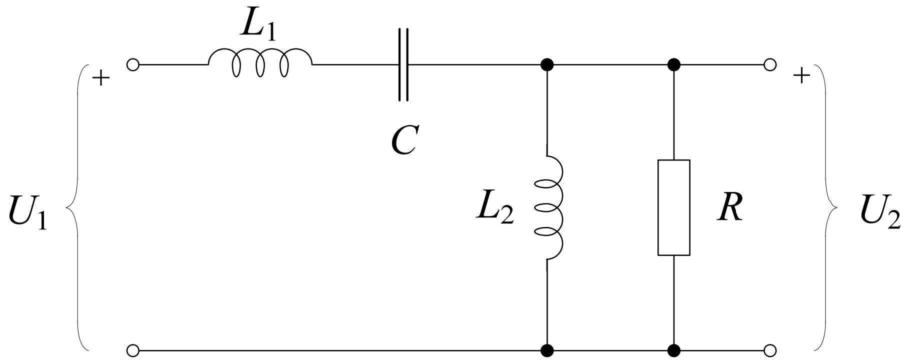
d) Temeljni sustav jednadžbi petlji u matričnom obliku $\mathbf{Z}_p \cdot \mathbf{I}_p = \mathbf{U}_{0p}$, gdje su:

$$\mathbf{Z}_p = \mathbf{S} \cdot \mathbf{Z}_b \cdot \mathbf{S}^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{sC}-g & 0 & 0 & 0 \\ -g & R_2 & 0 & 0 \\ 0 & 0 & R_1 & 0 \\ 0 & 0 & 0 & sL \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{sC}-g & 0 & R_1 & 0 \\ -g & R_2 & 0 & sL \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{sC}-g+R_1 & 0 \\ -g & R_2+sL \end{bmatrix}$$ (1 bod)

$$\mathbf{U}_{0p} = -\mathbf{S} \cdot \mathbf{U}_{0b} = -\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{u_C(0)}{s} \\ 0 \\ U_g(s) \\ Li_L(0) \end{bmatrix} = \begin{bmatrix} -\frac{u_C(0)}{s} - U_g(s) \\ -Li_L(0) \end{bmatrix}$$ (1 bod)

3. Za četveropol na slici zadane su normalizirane vrijednosti elemenata $L_1 = 2/3$, $L_2 = 2$, $R = 1$, $C = 3/4$. a) Odrediti prijenosnu funkciju $T(s)=U_2(s)/U_1(s)$; b) polove i nule $T(s)$; c) Napisati izraz za amplitudno-frekvencijsku karakteristiku i skicirati ju; d) Napisati izraz za fazno-frekvencijsku karakteristiku (skica poželjna, nije neophodna)



Rješenje:

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a) Prijenosna funkcija:

$$U_2 = \frac{Z_2}{Z_1 + Z_2} U_1$$

$$Z_1 = sL_1 + \frac{1}{sC};\ Z_2 = sL_2 \| R = \frac{sL_2 R}{sL_2 + R}$$

$$T(s) = \frac{U_2}{U_1} = \frac{Z_2}{Z_2 + Z_1} = \frac{\dfrac{sL_2R}{sL_2+R}}{\dfrac{sL_2R}{sL_2+R} + sL_1 + \dfrac{1}{sC}} \quad \begin{matrix} /\cdot(sL_2+R) \\ /\cdot(sL_2+R) \end{matrix}$$

$$= \frac{sL_2R}{sL_2R + (sL_2+R)(sL_1 + 1/sC)} = \frac{s^2L_2RC}{s^3L_1L_2C + s^2L_2CR + s^2L_1CR + sL_2 + R} =$$

$$= \frac{s^2L_2RC}{s^3L_1L_2C + s^2RC(L_1+L_2) + sL_2 + R}$$

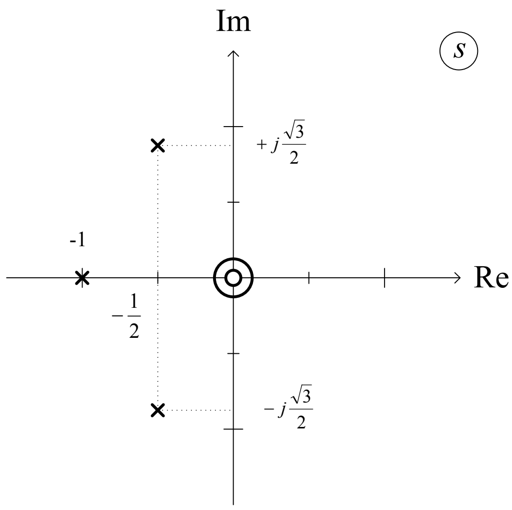
$$T(s) = \frac{\frac{3}{2}s^2}{s^3 + 2s^2 + 2s + 1} = \frac{\frac{3}{2}s^2}{(s+1)(s^2+s+1)}$$ (2 boda)

b) Polovi i nule:

Nule: $s^2 = 0 \quad \Rightarrow \quad s_{o_{1,2}} = 0$

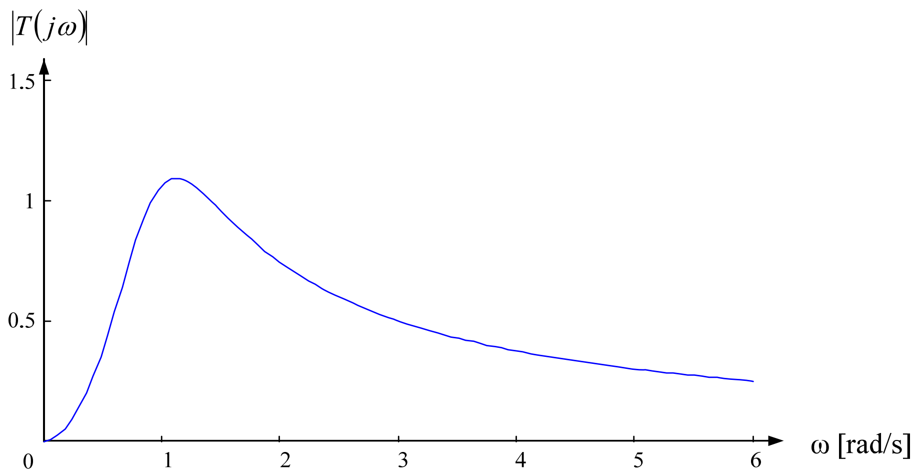
u brojniku s^2 u nazivniku $s^3 \quad \Rightarrow \quad s_{o_3} = \infty$

Polovi: $(s+1)(s^2+s+1) = 0 \quad \Rightarrow \quad s_{p_1} = -1$, $\quad s_{p_{2,3}} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$ (1 bod)



c) A-F karakteristika:

$$T(j\omega)=\frac{-\frac{3}{2}\omega^2}{(j\omega+1)(-\omega^2+j\omega+1)} \quad \Rightarrow \quad |T(j\omega)|=\frac{\frac{3}{2}\omega^2}{\sqrt{\left((1-\omega^2)^2+\omega^2\right)\cdot(\omega^2+1)}}$$

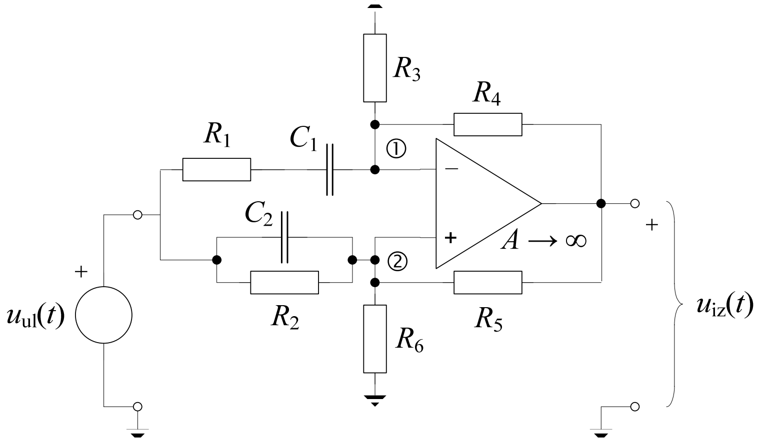


(1 bod)

d) F-F karakteristika:

$$\varphi(\omega)=\pi-\left[\arctan(\omega)+\arctan\left(\frac{\omega}{1-\omega^2}\right)\right]$$ (1 bod)

4. Za električni krug na slici zadane su normalizirane vrijednosti elemenata: $R_1 = R_2 = R_3 = R_4 = R_5 = 1$, $R_6 = 1/2$, $C_1 = C_2 = 1$, $A \to \infty$ i poticaj $u_{ul}(t) = \delta(t)$. Odrediti: a) jednadžbe čvorišta; b) prijenosnu funkciju: $T(s)=U_{iz}(s)/U_{ul}(s)$; c) polove i nule prijenosne funkcije $T(s)$; te d) odziv $U_{iz}(s)$ i odziv $u_{iz}(t)$.



Rješenje:

a) Jednadžbe čvorišta:

Ako za početni račun pojednostavimo:

$$Z_1 = R_1 + \frac{1}{sC_1}$$

$$Z_2 = R_2 \,\|\, \frac{1}{sC_2} = \frac{R_2 \frac{1}{sC_2}}{R_2 + \frac{1}{sC_2}} = \frac{R_2}{sR_2C_2 + 1}$$

Tada:

1) $A \to \infty \Rightarrow U_1 = U_2$

2) $U_1\left(\frac{1}{Z_1} + \frac{1}{R_3} + \frac{1}{R_4}\right) - U_{ul}\frac{1}{Z_1} - U_{iz}\frac{1}{R_4} = 0$ (1 bod)

3) $U_2\left(\frac{1}{Z_2} + \frac{1}{R_5} + \frac{1}{R_6}\right) - U_{ul}\frac{1}{Z_2} - U_{iz}\frac{1}{R_5} = 0$ (1 bod)

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b) Prijenosna funkcija:

$$2) \Rightarrow U_1 = \frac{U_{ul}\frac{1}{Z_1} + U_{iz}\frac{1}{R_4}}{\frac{1}{Z_1} + \frac{1}{R_3} + \frac{1}{R_4}}$$

$$3) \Rightarrow U_2 = \frac{U_{ul}\frac{1}{Z_2} + U_{iz}\frac{1}{R_5}}{\frac{1}{Z_2} + \frac{1}{R_5} + \frac{1}{R_6}}$$

Uz uvrštene $\frac{1}{Z_1} = Y_1$, $\frac{1}{Z_2} = Y_2$, $\frac{1}{R_1} = G_1$ ... $\frac{1}{R_6} = G_6$ slijedi

1) $U_1 = U_2 \Rightarrow \frac{U_{ul}Y_1 + U_{iz}G_4}{Y_1 + G_3 + G_4} = \frac{U_{ul}Y_2 + U_{iz}G_5}{Y_2 + G_5 + G_6}$

$$\frac{U_{ul}Y_1}{Y_1 + G_3 + G_4} - \frac{U_{ul}Y_2}{Y_2 + G_5 + G_6} = \frac{U_{iz}G_5}{Y_2 + G_5 + G_6} - \frac{U_{iz}G_4}{Y_1 + G_3 + G_4}$$

$$\frac{U_{iz}}{U_{ul}} = \frac{\frac{Y_1}{Y_1 + G_3 + G_4} - \frac{Y_2}{Y_2 + G_5 + G_6}}{\frac{G_5}{Y_2 + G_5 + G_6} - \frac{G_4}{Y_1 + G_3 + G_4}}$$

Nakon supstitucije Z_1 i Z_2:

$$Y_1 = \frac{1}{Z_1} = \frac{1}{1 + \frac{1}{s}} = \frac{s}{s+1}, \qquad Y_2 = \frac{1}{Z_2} = s + 1$$

$$\frac{U_{iz}}{U_{ul}} = \frac{\frac{s/(s+1)}{s/(s+1) + 2} - \frac{s+1}{s+1+3}}{\frac{1}{s+1+3} - \frac{1}{s/(s+1)+2}} = \frac{\frac{s}{s+2(s+1)} - \frac{s+1}{s+4}}{\frac{1}{s+4} - \frac{s+1}{s+2(s+1)}} = \frac{\frac{s}{3s+2} - \frac{s+1}{s+4}}{\frac{1}{s+4} - \frac{s+1}{3s+2}} =$$

$$= \frac{s(s+4) - (s+1)(3s+2)}{3s+2-(s+1)(s+4)} = \frac{s^2+4s-3s^2-5s-2}{3s+2-s^2-5s-4} = \frac{-2s^2-s-2}{-s^2-2s-2} = \frac{2s^2+s+2}{s^2+2s+2}$$

$$T(s) = \frac{U_{iz}(s)}{U_{ul}(s)} = \frac{2s^2+s+2}{s^2+2s+2}$$ (1 bod)

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c) Polovi, nule i odziv:

Nule: $2s^2 + s + 2 = 0 \Rightarrow s_{o_{1,2}} = -\frac{1}{4} \pm j\frac{\sqrt{15}}{4}$

$s_{o_3} = \infty$

Polovi: $s^2 + 2s + 2 = 0 \Rightarrow s_{p_{1,2}} = -1 \pm j$

(1 bod)

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d) Odziv $u_{iz}(t)$:

Odziv:

$$U_{iz}(s) = T(s) \cdot U_{ul}(s);\ u_{ul}(t) = \delta \Rightarrow U_{ul}(s) = 1 \ \Rightarrow U_{iz}(s) = \frac{2s^2+s+2}{s^2+2s+2}$$

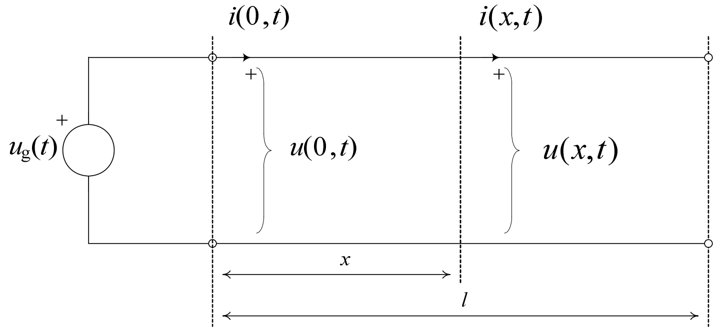
$$(2s^2+s+2):(s^2+2s+2) = 2 - \frac{3s+2}{(s+1)^2+1} = 2 - \frac{3s+3-1}{(s+1)^2+1} = 2 - 3\cdot\frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$$

$\underline{-2s^2 - 4s - 4}$

$= \quad -3s - 2$

$$\Rightarrow u_{iz}(t) = \left(2\delta - 3e^{-t}\cos(t) + e^{-t}\sin(t)\right)\cdot S(t)$$ (1 bod)

5. Na ulazu linije bez gubitaka s L=4mH/km, C=400nF/km, duljine $l = 2.5\lambda$ km, djeluje stacionarni sinusni izvor napona $u_g = 5\cos(2\pi \cdot f_0 \cdot t)$ uz f_0=6,25kHz. Izlaz linije je u praznom hodu. Odrediti: *a*) valnu duljinu λ signala na liniji; *b*) duljinu l linije, *c*) karakterističnu impedanciju Z_0, faktor prijenosa γ te brzinu širenja vala po liniji v; *d*) ulaznu impedanciju Z_{ul1}; *e*) napon i struju na sredini linije (x=l/2)?



Rješenje:

a) $\omega_0 = 2\pi f_0 = 39{,}2699$ rad/s

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega_0\sqrt{LC}} = \frac{2\pi}{2\pi \cdot 6{,}25 \cdot 10^3 \sqrt{4\cdot 10^{-3} \cdot 400 \cdot 10^{-9}}} = 4\text{km} \quad \text{(1bod)}$$

b) $l = 2.5\lambda = 10\text{km}$ (1bod)

c) Za liniju bez gubitaka vrijedi:

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{4\cdot 10^{-3}}{400\cdot 10^{-9}}} = \frac{1}{10}\cdot 10^3\,\Omega = 100\Omega\,,$$

$$\gamma = j\beta;\ \alpha = 0\,, \qquad \beta = \frac{2\pi}{\lambda} = \frac{\pi}{2}\left[\frac{\text{rad}}{\text{km}}\right] \qquad \beta l = \beta \cdot 10 = 5\pi$$

$$v = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}} = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4\cdot 10^{-3}\cdot 400\cdot 10^{-9}}} = \frac{1}{40\cdot 10^{-6}} = 25\cdot 10^3\,\text{km/s} \quad \text{(1bod)}$$

d)
$$Z_{ul} = Z_0 \frac{Z_2\cosh(\gamma l) + Z_0\sinh(\gamma l)}{Z_2\sinh(\gamma l) + Z_0\cosh(\gamma l)} = Z_0\,\text{cth}(\gamma l) = Z_0 j\text{ctg}(\beta l)$$

$$\text{ctg}(\beta l) = \text{ctg}(5\pi) = \infty \quad \Rightarrow \quad Z_{ul} = Z_0 j\text{ctg}(\beta l) = \infty \quad \text{(1bod)}$$

e) Napon i struja na mjestu x=2.5 km

$$\beta x = \frac{5\pi}{2}\,, \qquad U(0) = 5\angle 0°\,, \qquad I(0) = \frac{U(0)}{Z_{ul}} = 0$$

Prijenosne jednadžbe linije:

$$U(x) = U(0)\cdot \text{ch}(\gamma x) - I(0)\cdot Z_0 \cdot \text{sh}(\gamma x) = U(0)\cdot \cos(\beta x) = 5\cdot\cos\left(\frac{\pi}{2}\right) = 0$$

$$I(x) = -\frac{U(0)}{Z_0}\cdot \text{sh}(\gamma x) + I(0)\cdot \text{ch}(\gamma x) = -j\frac{U(0)}{Z_0}\cdot \sin(\beta x) = -j\frac{5}{100}\cdot\sin\left(\frac{\pi}{2}\right) = -j0{,}05 = 0{,}05\angle -90°$$

$i(x,t) = 5\cos(2\pi f_0 t - 90°)V$ (1bod)