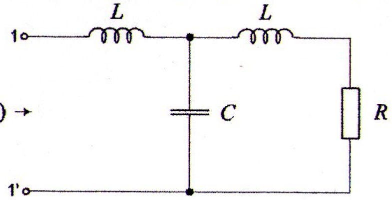
PRVI MEĐUISPIT IZ ELEKTRIČNIH KRUGOVA – Rješenja – 2010

1. Zadan je dvopol sastavljen od normiranih elemenata R=1, C=2, L=1.
   a) Izračunati ulaznu admitanciju $Y(s)$ na priključnicama 1-1' tog dvopola;
   b) Denormirati elemente dvopola na frekvenciju $\omega_0=10^6$ rad/s i na otpor R_0=1000 Ω;
   c) Odrediti denormiranu ulaznu admitanciju $Y(s)$;
   d) Koliki je iznos denormirane ulazne admitancije $Y(s)$ na frekvenciji nula $Y(0)$?



Rješenje:

a) Ulazna admitancija:

$$Y_n(s)=\frac{1}{sL+\dfrac{1}{sC+\dfrac{1}{sL+R}}}=\frac{1}{sL+\dfrac{sL+R}{sC(sL+R)+1}}=\frac{sC(sL+R)+1}{sL\left[sC(sL+R)+1\right]+sL+R}=$$

$$=\frac{s^2LC+sCR+1}{sL\left[s^2LC+sCR+1\right]+sL+R}=\frac{s^2LC+sCR+1}{s^3L^2C+s^2LCR+s2L+R}=\frac{2s^2+2s+1}{2s^3+2s^2+2s+1}$$ (1 bod)

b) Denormiranje elemenata: (1 bod)

$$R=R_0\cdot R_n=1000\cdot 1=1000\Omega=1k\Omega$$

$$Z_{C_n}=\frac{1}{sCR_0}=\frac{1}{\dfrac{s}{\omega_0}\underbrace{\omega_0 CR_0}_{C_n}}\quad\Rightarrow\quad C=\frac{C_n}{\omega_0R_0}=\frac{2}{10^6\cdot 10^3}=2\cdot 10^{-9}F=2nF$$

$$Z_{L_n}=\frac{sL}{R_0}=\frac{s}{\omega_0}\cdot\underbrace{\frac{\omega_0 L}{R_0}}_{L_n}\quad\Rightarrow\quad L=\frac{L_nR_0}{\omega_0}=\frac{1\cdot 10^3}{10^6}=1\cdot 10^{-3}H=1mH$$

c) Uz uvrštene denormirane vrijednosti elemenata admitancija dvopola glasi:

$$Y(s)=\frac{s^2LC+sCR+1}{s^3L^2C+s^2LCR+s2L+R}=\frac{s^2\cdot 10^{-3}\cdot 2\cdot 10^{-9}+s\cdot 2\cdot 10^{-9}\cdot 10^3+1}{s^3\cdot(10^{-3})^2\cdot 2\cdot 10^{-9}+s^2\cdot 10^{-3}\cdot 2\cdot 10^{-9}\cdot 10^3+s\cdot 2\cdot 10^{-3}+10^3}=$$

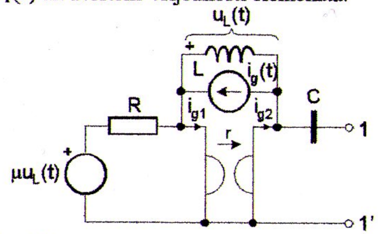
$$=\frac{s^2\cdot 2\cdot 10^{-12}+s\cdot 2\cdot 10^{-6}+1}{s^3\cdot 2\cdot 10^{-15}+s^2\cdot 2\cdot 10^{-9}+s\cdot 2\cdot 10^{-3}+10^3}=\frac{s^2\cdot\dfrac{2\cdot 10^{-12}}{2\cdot 10^{-15}}+s\cdot\dfrac{2\cdot 10^{-6}}{2\cdot 10^{-15}}+\dfrac{1}{2\cdot 10^{-15}}}{s^3+s^2\cdot\dfrac{2\cdot 10^{-9}}{2\cdot 10^{-15}}+s\cdot\dfrac{2\cdot 10^{-3}}{2\cdot 10^{-15}}+\dfrac{10^3}{2\cdot 10^{-15}}}=$$

$$=\frac{s^2\cdot 10^3+s\cdot 10^9+5\cdot 10^{14}}{s^3+s^2\cdot 10^6+s\cdot 10^{12}+5\cdot 10^{17}}$$ (1 bod)

d) Uz uvrštene denormirane vrijednosti elemenata admitancija dvopola na frekvenciji nula glasi:

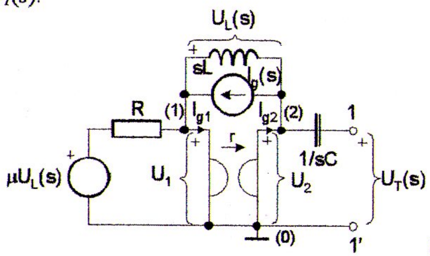
$$Y(0)=\frac{5\cdot 10^{14}}{5\cdot 10^{17}}=10^{-3}$$ (1 bod)

2. Za mrežu prikazanu slikom odrediti nadomjesne parametre mreže po Theveninu $U_T(s)$ i $Z_T(s)$ s obzirom na polove 1–1'. Koristiti metodu napona čvorova u proračunu. Zadane su normalizirane vrijednosti elemenata: L=1, C=1, R=1, μ=2, r=2 te izvor $i_g(t)=S(t)$. Napisati:
   a) Jednadžbu za čvor (1) za izračun $U_T(s)$;
   b) Jednadžbu za čvor (2) za izračun $U_T(s)$;
   c) Theveninov napon $U_T(s)$ uz uvrštene vrijednosti elemenata;
   d) Theveninovu impedanciju $Z_T(s)$ uz uvrštene vrijednosti elemenata.



Rješenje: Metodom napona čvorova:

Theveninov napon $U_T(s)$:



$$U_T(s)=U_2(s)$$

$$(1)\qquad U_1\left(\frac{1}{R}+\frac{1}{sL}\right)-U_2\frac{1}{sL}=\mu\frac{U_L(s)}{R}+I_g(s)-I_{g1}(s)$$ (1 bod)

$$(2)\qquad -U_1\frac{1}{sL}+U_2\frac{1}{sL}=-I_g(s)+I_{g2}(s)$$ (1 bod)

$$U_2=-r\cdot I_{g1}$$

$$U_1=-r\cdot I_{g2}\qquad U_L(s)=U_1(s)-U_2(s)$$

$$(1)\qquad U_1\left(\frac{1}{R}+\frac{1}{sL}\right)-U_2\frac{1}{sL}=\mu\frac{U_1(s)-U_2(s)}{R}+I_g(s)+\frac{U_2}{r}$$

$$(2)\qquad -U_1\frac{1}{sL}+U_2\frac{1}{sL}=-I_g(s)-\frac{U_1}{r}$$

$$(2)\Rightarrow U_2\frac{1}{sL}+I_g(s)=U_1\left(\frac{1}{sL}-\frac{1}{r}\right)\quad\Rightarrow\quad U_1=\frac{U_2\dfrac{1}{sL}+I_g(s)}{\dfrac{1}{sL}-\dfrac{1}{r}}\rightarrow(1)$$

$$(1)\Rightarrow U_1\left(\frac{1-\mu}{R}+\frac{1}{sL}\right)=U_2\left(\frac{1}{sL}-\frac{\mu}{R}+\frac{1}{r}\right)+I_g$$

$$(1),(2)\Rightarrow \frac{U_2\frac{1}{sL}+I_g(s)}{\frac{1}{sL}-\frac{1}{r}}\left(\frac{1-\mu}{R}+\frac{1}{sL}\right)=U_2\left(\frac{1}{sL}-\frac{\mu}{R}+\frac{1}{r}\right)+I_g$$

$$\left[U_2\frac{1}{sL}+I_g(s)\right]\left(\frac{1-\mu}{R}+\frac{1}{sL}\right)=U_2\left(\frac{1}{sL}-\frac{\mu}{R}+\frac{1}{r}\right)\left(\frac{1}{sL}-\frac{1}{r}\right)+I_g\left(\frac{1}{sL}-\frac{1}{r}\right)$$

$$U_2\frac{1}{sL}\left(\frac{1}{R}-\frac{\mu}{R}+\frac{1}{sL}\right)-U_2\frac{1}{sL}\left(\frac{1}{sL}-\frac{\mu}{R}+\frac{1}{r}\right)+U_2\frac{1}{r}\left(\frac{1}{sL}-\frac{\mu}{R}+\frac{1}{r}\right)=$$

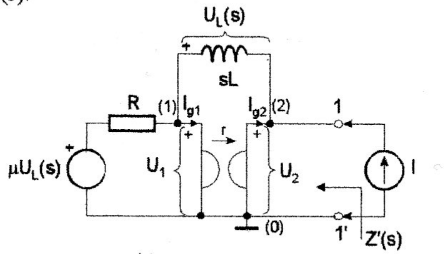
$$=I_g\left(\frac{1}{sL}-\frac{1}{r}\right)-I_g(s)\left(\frac{1}{R}-\frac{\mu}{R}+\frac{1}{sL}\right)$$

$$U_2\left(\frac{1}{sL}\frac{1}{R}+\frac{1}{r^2}-\frac{\mu}{rR}\right)=-I_g(s)\left(\frac{1}{r}+\frac{1-\mu}{R}\right)$$

$$U_2\left(\frac{1}{s}+\frac{1}{4}-1\right)=-I_g(s)\left(\frac{1}{2}-1\right)$$

$$U_2\left(\frac{1}{s}-\frac{3}{4}\right)=\frac{1}{2}I_g(s)\ \Rightarrow\ U_T(s)=U_2=\frac{\frac{1}{2}}{\frac{1}{s}-\frac{3}{4}}I_g(s)=\frac{\frac{1}{2}}{\frac{1}{s}-\frac{3}{4}}\cdot\frac{1}{s}=\frac{2}{4-3s}$$ (1 bod)

Theveninova impedancija $Z_T(s)$:



$$(1)\qquad U_1\left(\frac{1}{R}+\frac{1}{sL}\right)-U_2\frac{1}{sL}=\mu\frac{U_L(s)}{R}-I_{g1}(s)$$

$$(2)\qquad -U_1\frac{1}{sL}+U_2\frac{1}{sL}=I(s)+I_{g2}(s)$$

$$U_2=-r\cdot I_{g1}$$

$$U_1=-r\cdot I_{g2}\qquad U_L(s)=U_1(s)-U_2(s)$$

$$(1)\qquad U_1\left(\frac{1}{R}+\frac{1}{sL}\right)-U_2\frac{1}{sL}=\mu\frac{U_1(s)-U_2(s)}{R}+\frac{U_2(s)}{r}$$

$$(2)\qquad -U_1\frac{1}{sL}+U_2\frac{1}{sL}=I(s)-\frac{U_1(s)}{r}$$

$$(1)\qquad U_1\left(\frac{1-\mu}{R}+\frac{1}{sL}\right)-U_2\left(\frac{1}{sL}-\frac{\mu}{R}+\frac{1}{r}\right)=0$$

$$(2)\qquad -U_1\left(\frac{1}{sL}-\frac{1}{r}\right)+U_2\frac{1}{sL}=I(s)$$

$$(1)\Rightarrow U_1(s)=\frac{\frac{1}{sL}-\frac{\mu}{R}+\frac{1}{r}}{\frac{1-\mu}{R}+\frac{1}{sL}}U_2(s)$$

$$(1),(2)\Rightarrow -\frac{\frac{1}{sL}-\frac{\mu}{R}+\frac{1}{r}}{\frac{1-\mu}{R}+\frac{1}{sL}}U_2\left(\frac{1}{sL}-\frac{1}{r}\right)+U_2\frac{1}{sL}=I(s)$$

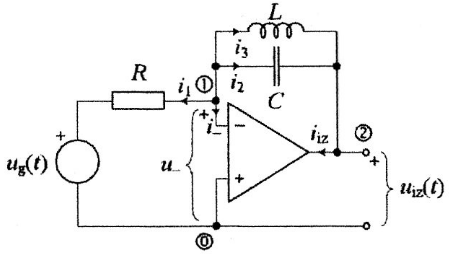
$$Z'(s)=\frac{U_2(s)}{I(s)}=\frac{\frac{1-\mu}{R}+\frac{1}{sL}}{\left(\frac{1}{sL}-\frac{\mu}{R}+\frac{1}{r}\right)\left(\frac{1}{r}-\frac{1}{sL}\right)+\frac{1}{sL}\left(\frac{1-\mu}{R}+\frac{1}{sL}\right)}=\frac{-1+\frac{1}{s}}{\left(\frac{1}{s}-2+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{s}\right)+\frac{1}{s}\left(-1+\frac{1}{s}\right)}=$$

$$=\frac{-1+\frac{1}{s}}{\frac{1}{2s}-\frac{1}{s^2}-1+\frac{2}{s}+\frac{1}{4}-\frac{1}{2s}-\frac{1}{s}+\frac{1}{s^2}}=\frac{-1+\frac{1}{s}}{-\frac{3}{4}+\frac{1}{s}}=\frac{4s-4}{3s-4}\quad\Rightarrow$$

$$Z_T(s)=Z'(s)+\frac{1}{sC}=\frac{4s-4}{3s-4}+\frac{1}{s}=\frac{4s^2-s-4}{s(3s-4)}$$ (1 bod)

3. Za električni krug na slici izračunati napon $u_{iz}(t)$ ako su zadane normalizirane vrijednosti elemenata $R=1$, $L=1$, $C=1$, uz početne uvjete jednake nuli te $u_g(t)=S(t)$. Koristiti KZS i KZN te oznake grana i čvorova prema slici. Napisati:

a) Broj neovisnih jednadžbi KZS i KZN (mreža ima 5 grana i 3 čvora);
b) Jednadžbe KZS;
c) Jednadžbe KZN;
d) Naponsko-strujne jednadžbe za grane;
e) Napon na izlazu $u_{iz}(t)$.



Rješenje:

$N_b=5$ (broj grana)
$N_v=3$ (broj čvorova)
Broj jednadžbi $\text{KZS} = N_v - 1 = 3 - 1 = 2$
Broj jednadžbi $\text{KZN} = N_b - N_v + 1 = 5 - 3 + 1 = 3$ (1 bod)

Jednadžbe Kirchhoffovih zakona (5 jednadžbi):
1) $I_1 + I_2 + I_3 + I_- = 0$ KZS
2) $I_{iz} - I_2 - I_3 = 0$ KZS(1 bod)
3) $U_1 - U_- = 0$ KZN
4) $-U_2 + U_3 = 0$ KZN
5) $-U_- + U_2 + U_{iz} = 0$ KZN(1 bod)

Naponsko–strujne jednadžbe grana (5 jednadžbi):
1) $U_1 = I_1 \cdot R + U_g$
2) $U_2 = I_2 \cdot \frac{1}{sC}$
3) $U_3 = I_3 \cdot sL$ (1 bod)
4) $U_- = 0$
5) $I_- = 0$

Sustav ima ukupno $2N_b=10$ jednadžbi i 10 nepoznanica (sve struje i svi naponi grana)
Naponsko – strujne jednadžbe grana uvrstimo u jednadžbe (1)–(5):

1) $\frac{U_1}{R} - \frac{U_g}{R} + sCU_2 + \frac{1}{sL}U_3 = 0$

2) $I_{iz} = sCU_2 + \frac{1}{sL}U_3$

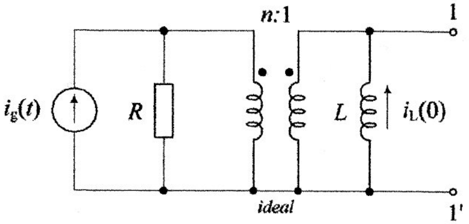
3) $U_1 = U_- = 0$
4) $U_2 = U_3$
5) $U_{iz} = -U_2$

$$1) \Rightarrow \left(sC + \frac{1}{sL}\right)U_2 = \frac{U_g}{R} \Rightarrow U_{iz} = -U_2 = -\frac{1}{R}\frac{U_g}{\left(sC + \frac{1}{sL}\right)}$$

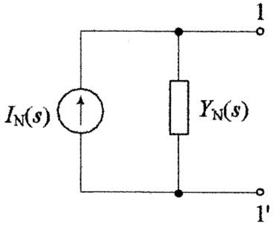
$$U_{iz}(s) = -\frac{\frac{1}{s}}{s + \frac{1}{s}} = -\frac{1}{s^2+1}$$

$u_{iz}(t) = -\sin(t)\cdot S(t)$ (1 bod)

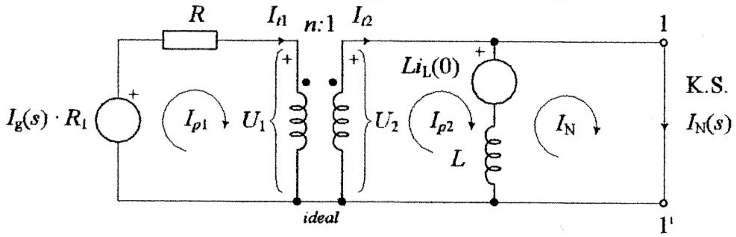
4. Za električni krug na slici izračunati parametre nadomjesnog kruga po Nortonu s obzirom na polove 1–1': $I_N(s)$ i $Y_N(s)$. Zadane su normalizirane vrijednosti elemenata: R=1, L=1, $i_L(0)$=1, n=2, $i_g(t)$=$S(t)$. (Koristiti bilo koju metodu u izračunu; preporučuje se metoda petlji.) Napisati:
a) Nortonovu struju $I_N(s)$ uz uvrštene vrijednosti elemenata;
b) Nortonovu admitanciju $Y_N(s)$ uz uvrštene vrijednosti elemenata;
c) Struju kroz otpor R.



Rješenje:



a) Nortonova struja $I_N(s)$ primjenom $\mathscr{L}$-transformacije na električni krug:



Jednadžbe idealnog transformatora:

$$U_1 = n\cdot U_2 \Rightarrow U_2 = \frac{U_1}{n}$$

$$I_{t1} = \frac{1}{n}\cdot I_{t2} \Rightarrow I_{t2} = n\cdot I_{t1}$$

1) $I_{p1}R = -U_1 + I_gR$ $\qquad I_{p1} = I_{t1}$

2) $(I_{p2} - I_N)sL + Li_L(0) = U_2$ $\qquad I_{p2} = I_{t2}$

Nakon sređivanja jednadžbe glase:

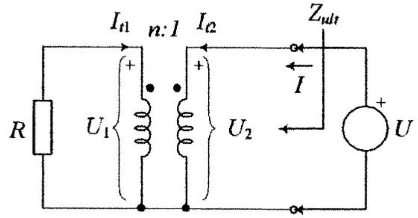
$$\left.\begin{aligned} &1)\ I_{p1}R + nU_2 = I_gR \\ &2)\ I_N = \frac{1}{sL}(I_{p2}sL + Li_L(0) - U_2) \end{aligned}\right\} \quad \left.\begin{aligned} &1)\ I_{p1}R + nU_2 = I_gR \\ &2)\ I_N = \frac{1}{sL}\left[nI_{p1}sL + Li_L(0) - U_2\right] \end{aligned}\right\}$$

U_2=0

$$\left.\begin{aligned} &1)\ I_{p1}R = I_gR \\ &2)\ I_N = \frac{1}{sL}\left[nI_{p1}sL + Li_L(0)\right] \end{aligned}\right\} \quad I_N = \frac{1}{sL}\left[nI_g sL + Li_L(0)\right]$$

$$I_N = \frac{1}{s}\left[2\frac{1}{s}s + 1\right] = \frac{3}{s}$$ (1 bod)

b) Nortonova admitancija $Y_N(s)$:
Najjednostavnije je izračunati ulaznu impedanciju u transformator zaključen s R. Označimo je s Z_{ulr}.



$$Z_{ulr} = \frac{1}{n^2}R$$

$$I_{t1} = -\frac{U_1}{R_1} \Rightarrow \frac{U_1}{I_{t1}} = -R$$

$$U = U_2$$

$$I = I_{t2}$$

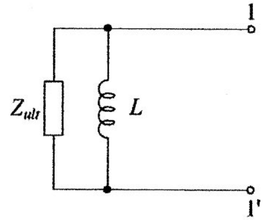
Jednadžbe transformatora su:

$$U_1 = n\cdot U_2 \Rightarrow U_2 = \frac{U_1}{n}$$

$$I_{t1} = -\frac{1}{n}\cdot I_{t2} \Rightarrow I_{t2} = -n\cdot I_{t1}$$

$$Z_{ulr} = \frac{U}{I} = \frac{U_2}{I_{t2}} = \frac{\frac{U_1}{n}}{-n\cdot I_{t1}} = -\frac{\frac{U_1}{I_{t1}}}{n^2} = -\frac{-R}{n^2} = \frac{R}{n^2}$$

Tada je Nortonova admitancija:



$$\leftarrow Y_N(s) = \frac{1}{Z_{ulr}} + \frac{1}{sL}$$

$$Y_N(s) = \frac{n^2}{R} + \frac{1}{sL} = 4 + \frac{1}{s}$$ (1 bod)

c) Za izračun Nortonove struje $I_N(s)$, polovi dvopola 1–1' trebaju biti kratko spojeni.
Struja kroz R je nula jer sva struja iz izvora I_g teče kroz primar idealnog transformatora, odn. I_g=I_{t1}. To je zato jer kratki spoj na sekundaru transformatora uzrokuje kratki spoj na primaru transformatora, a cjelokupna struja uvijek teče kroz kratki spoj, a ništa ne teče kroz R. (1 bod)

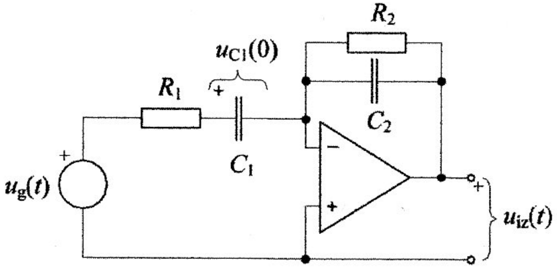
5. Zadan je električni krug prema slici. Odrediti napon na izlazu $u_{iz}(t)$ ako je zadano: $R_1=R_2=1$, $C_1=C_2=1$, $u_g(t)=S(t)$. Početni napon na kapacitetu C_1 je $u_{C1}(0)=2$, a na C_2 je jednak nuli. Koristiti bilo koju metodu u izračunu odziva. Koristeći princip superpozicije, izračunati:

a) Odziv $U_{iz1}(s)$ uslijed naponskog izvora $u_g(t)$;

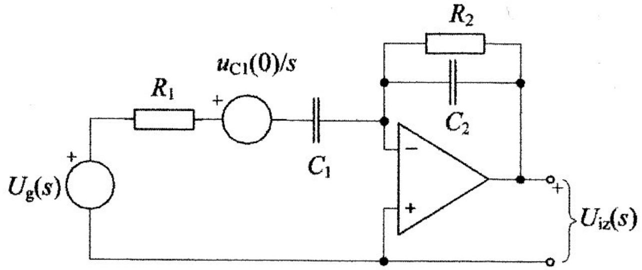
b) Odziv $U_{iz2}(s)$ uslijed početnog uvjeta $u_{C1}(0)$;

c) Ukupni odziv $U_{iz}(s)$;

d) Ukupni odziv $u_{iz}(t)$.

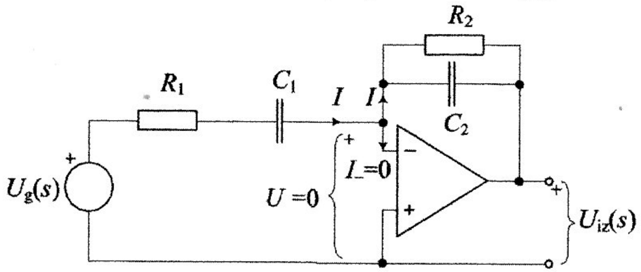


Rješenje: Primjena $\mathcal{L}$-transformacije:



Primjena principa superpozicije:

a) Isključen početni uvjet $u_{C1}(0)=0$, ostaje kao poticaj naponski izvor $U_g(s)$:



1) $I=U_g(s)\cdot\dfrac{1}{R_1+\dfrac{1}{sC_1}}$

2) $I=-U_{iz}(s)\cdot\left(\dfrac{1}{R_2}+sC_2\right)$

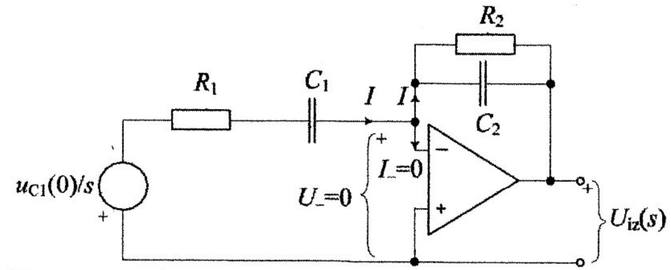
$$\Rightarrow\quad U_{iz}(s)=\frac{-U_g(s)}{\left(\dfrac{1}{R_2}+sC_2\right)\left(R_1+\dfrac{1}{sC_1}\right)}=-U_g(s)\cdot\frac{s\dfrac{1}{R_1C_2}}{\left(s+\dfrac{1}{R_1C_1}\right)\left(s+\dfrac{1}{R_2C_2}\right)}=$$

$$=-U_g(s)\cdot\frac{s}{(s+1)(s+1)}=-U_g(s)\cdot\frac{s}{(s+1)^2}$$

Uvrstimo vrijednosti:

$$U_{iz}(s)=-\frac{1}{s}\cdot\frac{s}{(1+s)^2}=-\frac{1}{(1+s)^2}\ \text{(1 bod)}$$

b) Isključen naponski izvor $U_g(s)=0$, ostaje kao poticaj početni uvjet $u_{C1}(0)$:



$$U_{iz}(s)=\frac{u_{C1}(0)}{s}\cdot\frac{1}{\left(\dfrac{1}{R_2}+sC_2\right)\left(R_1+\dfrac{1}{sC_1}\right)}$$

Uvrstimo vrijednosti:

$$U_{iz}(s)=2\frac{1}{(1+s)^2}\ \text{(1 bod)}$$

c) Ukupni odziv:

$$U_{iz}(s)=-\frac{1}{(1+s)^2}+\frac{2}{(1+s)^2}=\frac{1}{(1+s)^2}\ \text{(1 bod)}$$

d) Inverzna Laplaceova transformacija:

$$u_{iz}(t)=(te^{-t})\cdot S(t)\ \text{(1 bod)}$$