

# Električni krugovi

Funkcije mreža

# Funkcije mreža

- Funkcija mreže je omjer L-transformacije odziva i pobude kad su svi početni uvjeti jednaki nuli.



$$u_i(0) = 0$$

$$i_i(0) = 0$$

$$H(s) = \frac{Y(s)}{X(s)} \rightarrow \textbf{\underline{Funkcija mreže}}$$

- Funkcija mreže  $H(s)$  je omjer dvaju polinoma u  $s$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0} = \frac{P(s)}{Q(s)}$$

$P(s)$  i  $Q(s) \rightarrow$  polinomi kompleksne varijable  $s$

$a_i, b_i \rightarrow$  realni koeficijenti

**FUNKCIJE MREŽA**



**ULAZNE FUNKCIJE**

**ili**

**FUNKCIJE IMITANCije**

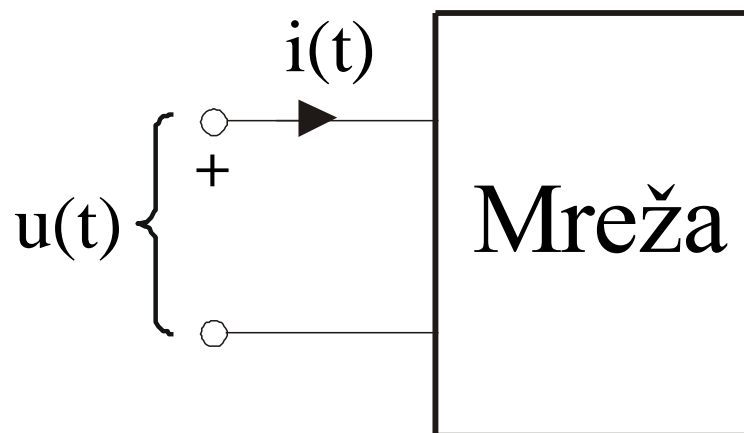
**PRIJENOSNE**

**ili**

**TRANSFER FUNKCIJE**

## ■ FUNKCIJE IMITANCIJE –

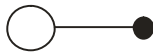
- odnose se na par priključnica ili jedan prilaz



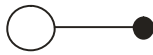
Omjer L-transformacije struje i napona promatranih na paru priključnica.

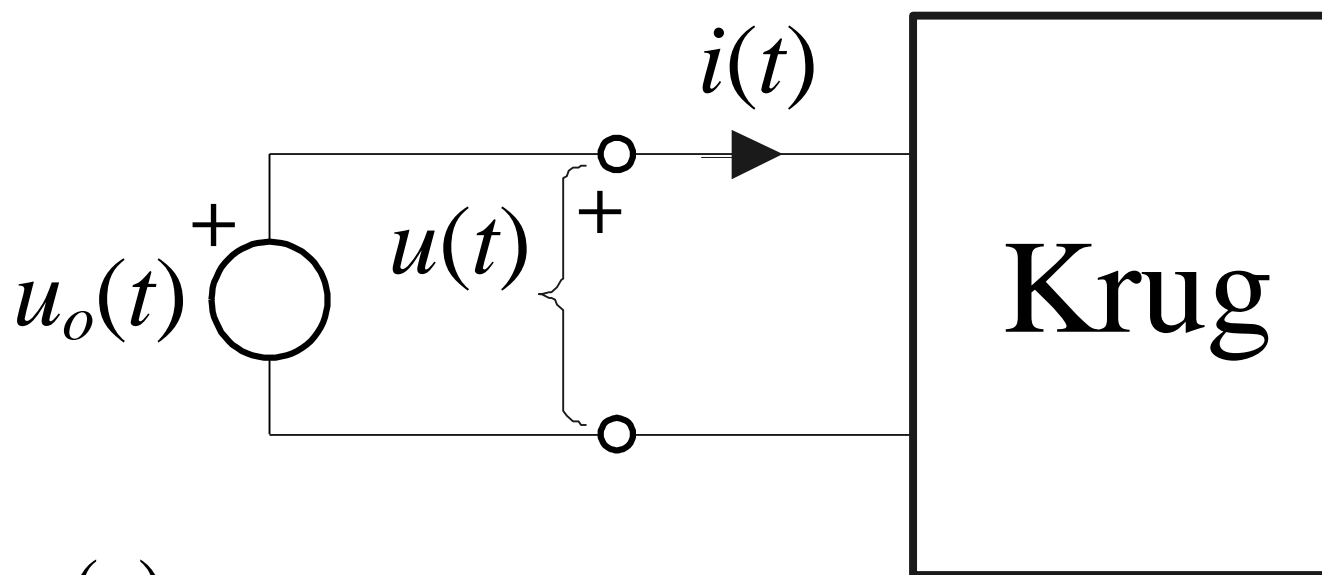
## 1. Slučaj

**Pobuda  $\rightarrow$  napon**

$u(t)$    $U(s)$

**Odziv  $\rightarrow$  struja**

$i(t)$    $I(s)$



$$H(s) = \frac{I(s)}{U(s)} = Y(s) \rightarrow \text{Funkcija admitancije}$$

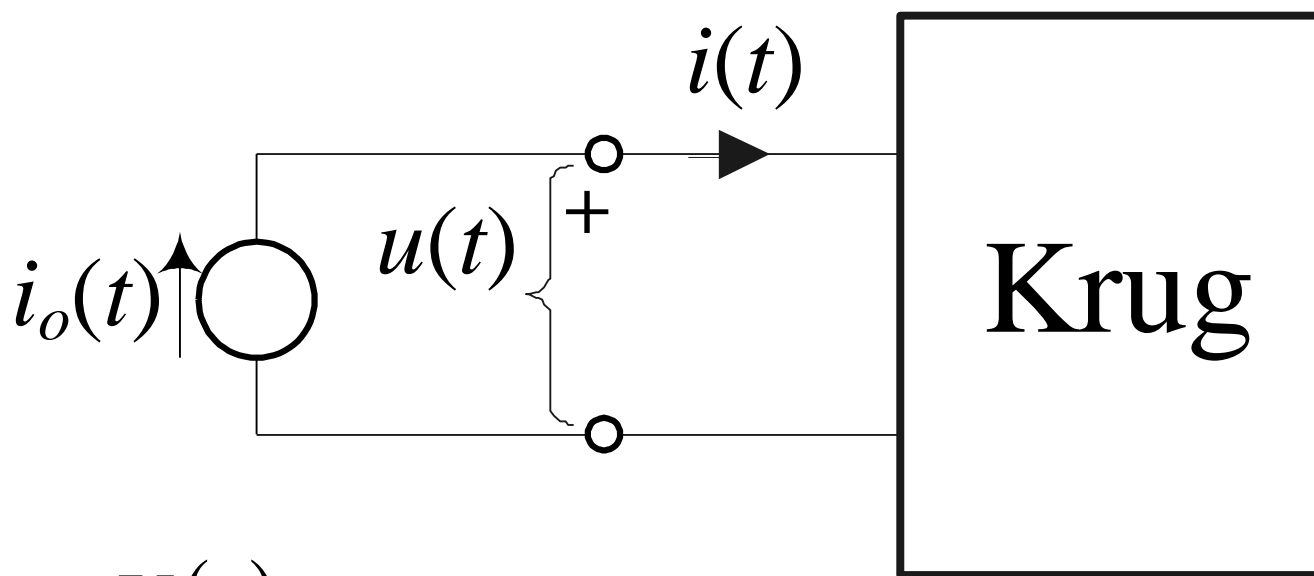
## 2. slučaj

**Pobuda  $\rightarrow$  struja**

$$i(t) \quad \text{---} \bullet \quad I(s)$$

**Odziv  $\rightarrow$  napon**

$$u(t) \quad \text{---} \bullet \quad U(s)$$



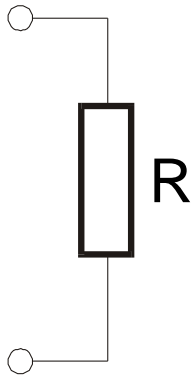
$$H(s) = \frac{U(s)}{I(s)} = Z(s) \rightarrow \text{Funkcija impedancije}$$

funkcije impedancije  $Z(s)$   
 funkcije admitancije  $Y(s)$

funkcije imitancije  
 ili ulazne funkcije!!!

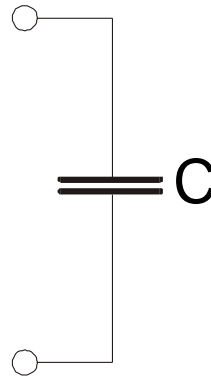
Vrijedi:

$$Z(s) = \frac{1}{Y(s)}$$



$$Z(s) = R$$

$$Y(s) = \frac{1}{R}$$



$$Z(s) = \frac{1}{sC}$$

$$Y(s) = sC$$



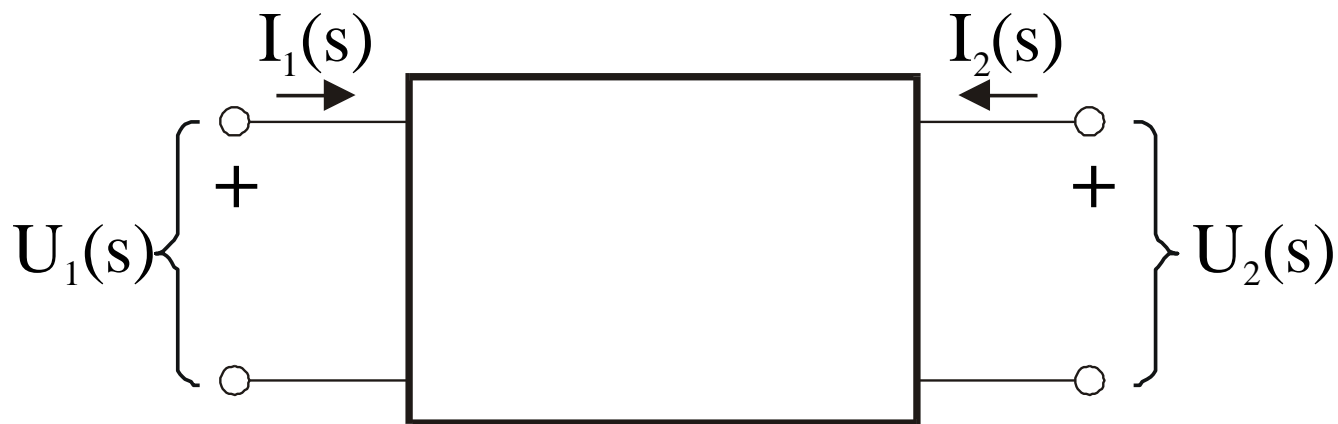
$$Z(s) = sL$$

$$Y(s) = \frac{1}{sL}$$



# PRIJENOSNE FUNKCIJE

Odnose se na različite parove priključnica neke mreže



**Pobuda  $\rightarrow$  na prilazu 1**



**Napon**



**Struja**

**Odziv  $\rightarrow$  na prilazu 2**



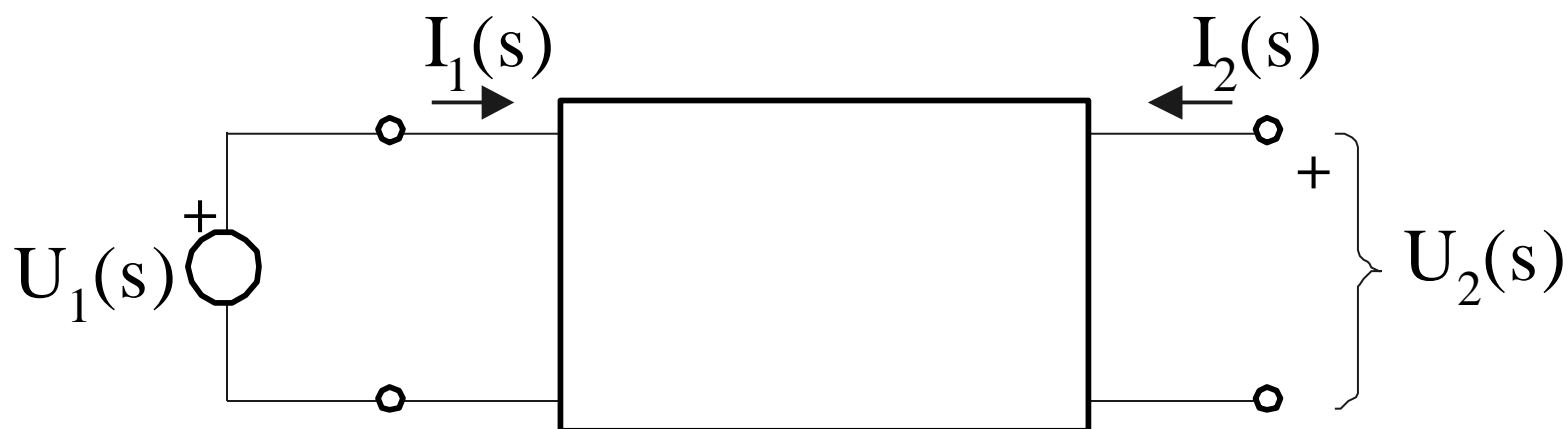
**Napon**



**Struja**

## 4-tipa prijenosnih funkcija

### 1) Prijenosna funkcija (prijenosni omjer) napona



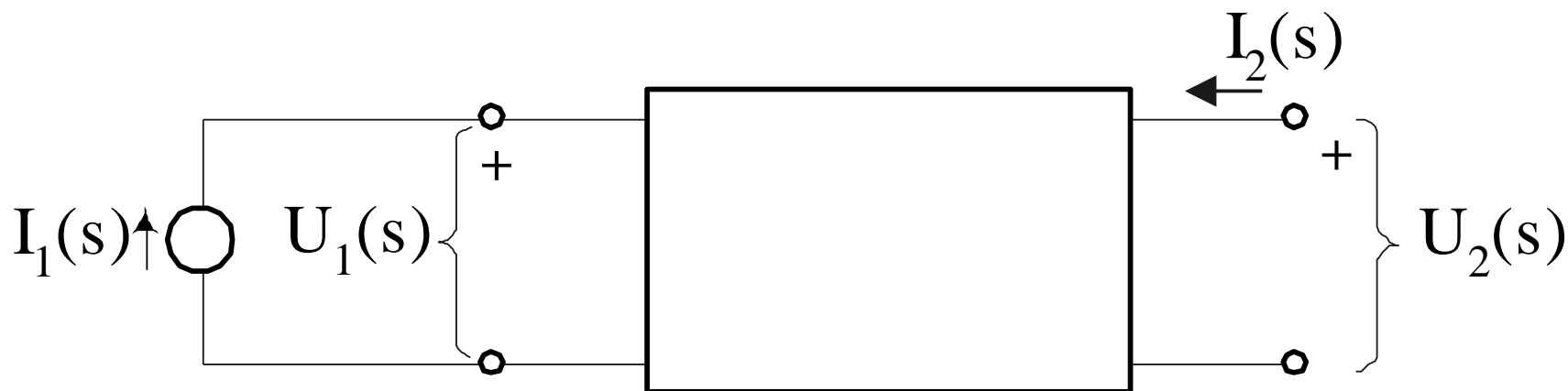
**Odziv**  $\rightarrow$  napon na prilazu 2

**Pobuda**  $\rightarrow$  napon na prilazu 1

$$H(s) = \frac{U_2(s)}{U_1(s)}$$

$\rightarrow$  **odziv**  
 $\rightarrow$  **pobuda**

## 2) Prijenosna funkcija (prijenosni omjer) struja



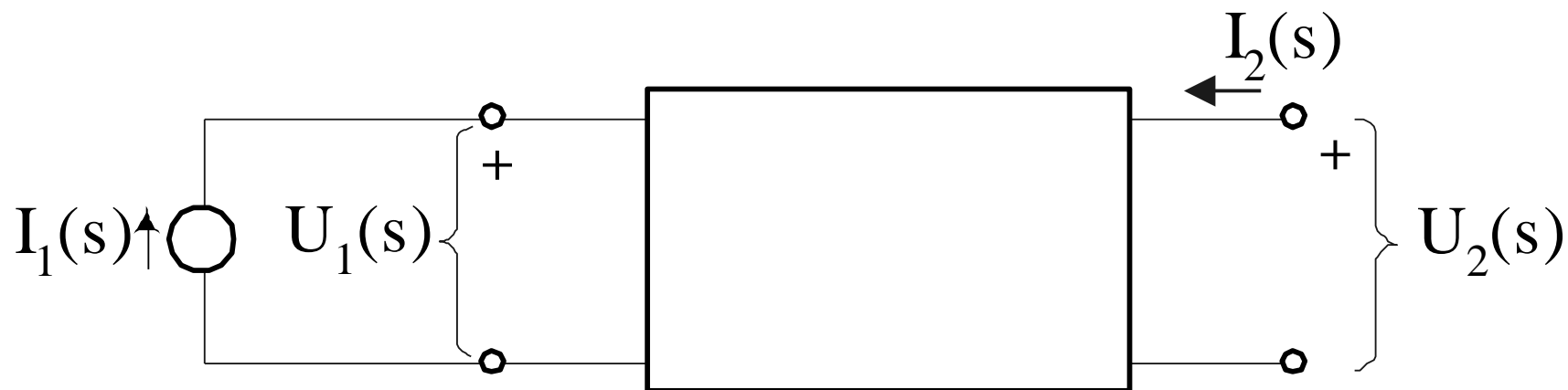
**Odziv  $\rightarrow$  struja na prilazu 2**

**Pobuda  $\rightarrow$  struja na prilazu 1**

$$H(s) = \frac{I_2(s)}{I_1(s)}$$

$\rightarrow$  **odziv**  
 $\rightarrow$  **pobuda**

### 3) Prijenosna impedancija



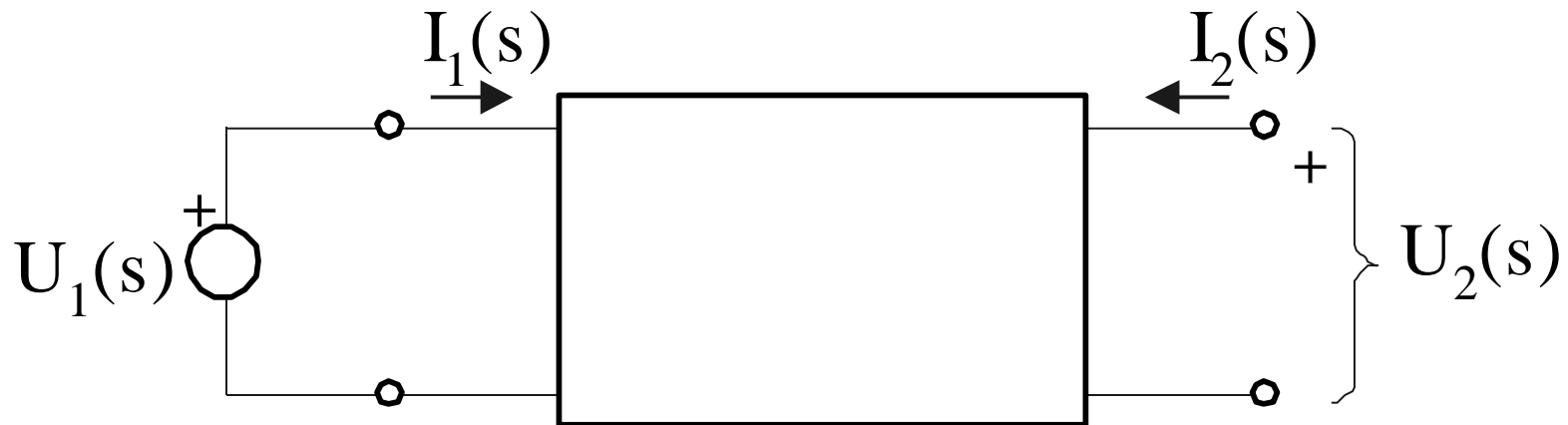
**Odziv  $\rightarrow$  napon na prilazu 2**

**Pobuda  $\rightarrow$  struja na prilazu 1**

$$Z_{21}(s) = \frac{U_2(s)}{I_1(s)}$$

$\rightarrow$  **odziv**  
 $\rightarrow$  **pobuda**

## 4) Prijenosna admitancija



**Odziv  $\rightarrow$  struja na prilazu 2**

**Pobuda  $\rightarrow$  napon na prilazu 1**

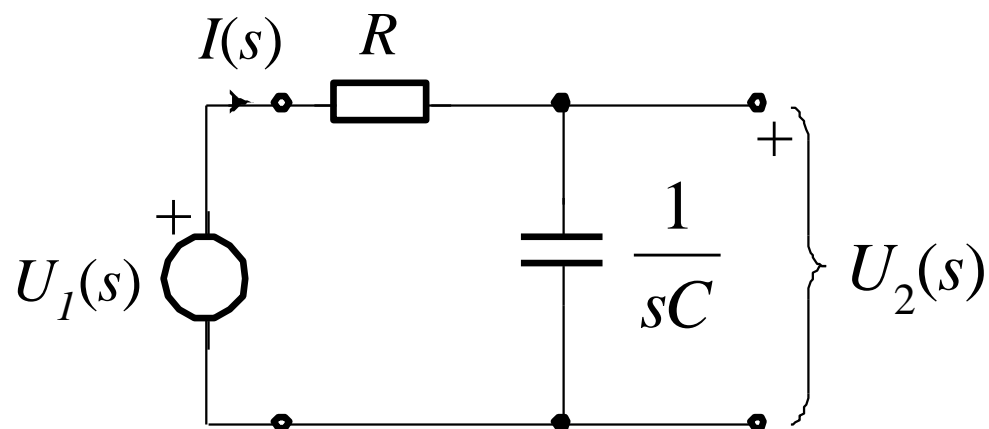
$$Y_{21}(s) = \frac{I_2(s)}{U_1(s)}$$

$\nearrow$  **odziv**  
 $\searrow$  **pobuda**

**Važno:**

$$Z_{21}(s) \neq \frac{1}{Y_{21}(s)}$$

## ■ Primjer 1.: RC mreža



■ Ulazna admitancija  $Y(s)=?$

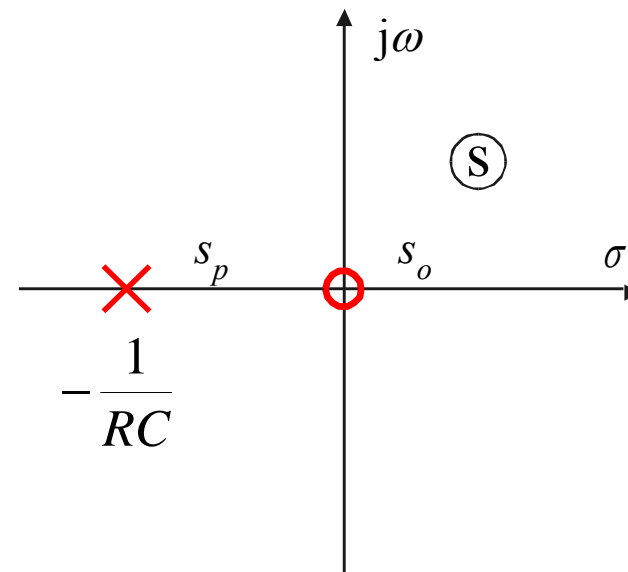
■ Prijenosna funkcija napona  $H(s)=?$

## ■ Ulazna funkcija - admitancija

$$Y(s) = \frac{I(s)}{U_1(s)} = \frac{1}{R} \cdot \frac{s}{s + \frac{1}{RC}}$$

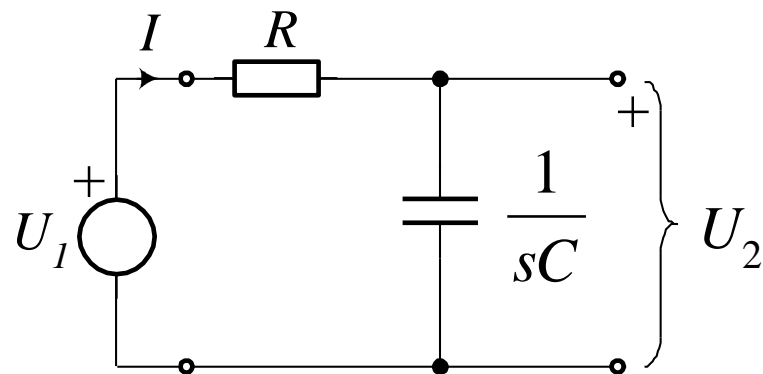
$$s_o = 0$$

$$s_p = -\frac{1}{RC}$$





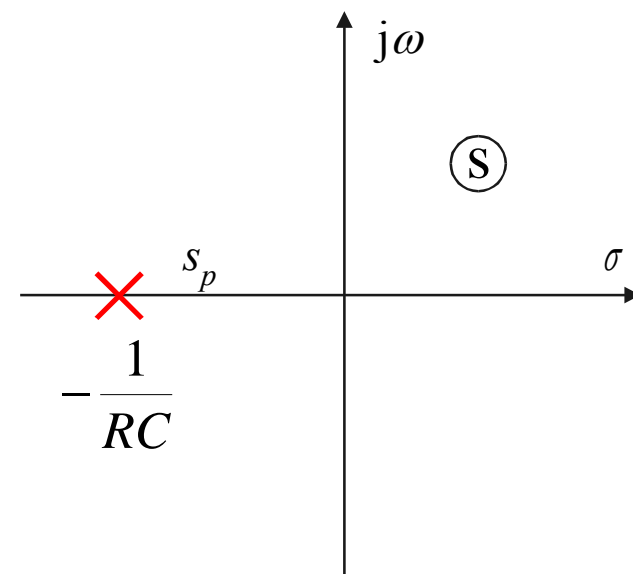
## ■ Prijenosna funkcija napona – naponski djelitelj



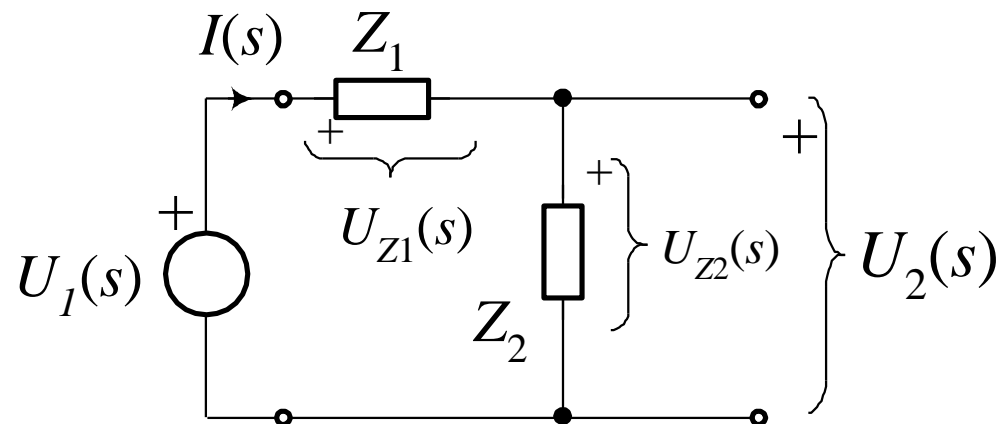
$$H(s) = \frac{U_2(s)}{U_1(s)} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$$

$$s_o = \infty$$

$$s_p = -\frac{1}{RC}$$



## ■ Mreža oblika



je ***naponski djelitelj*** jer napon  $U_1(s)$  raspodjeljuje na:

- napon na impedanciji  $Z_1 \rightarrow U_{Z1}$  i
- napon na impedanciji  $Z_2 \rightarrow U_{Z2}=U_2$

$$U_1(s) = U_{Z1}(s) + U_{Z2}(s)$$

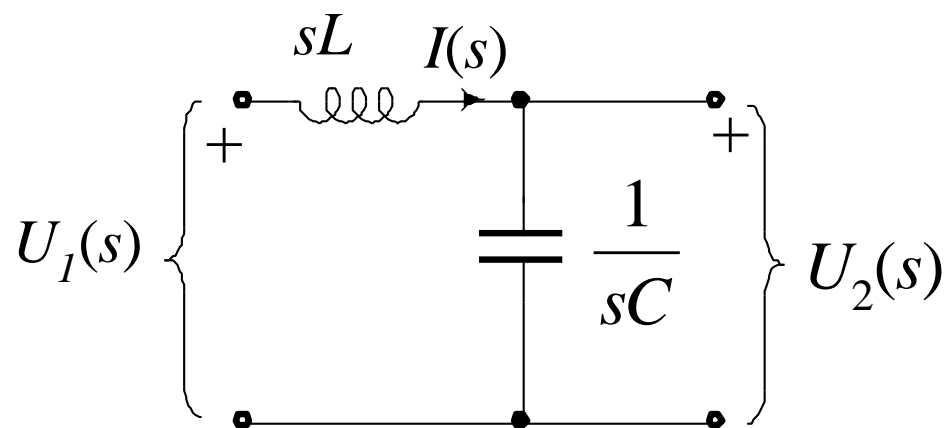
- Omjer napona  $U_2(s)$  prema  $U_1(s)$  je

$$H(s) = \frac{U_{Z_2}(s)}{U_{Z_1}(s) + U_{Z_2}(s)} = \frac{I(s) \cdot Z_2(s)}{I(s) \cdot Z_1(s) + I(s) \cdot Z_2(s)}$$

$$H(s) = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$

- Napon je raspodijeljen proporcionalno s impedancijama.

## ■ Primjer 2.: LC mreža



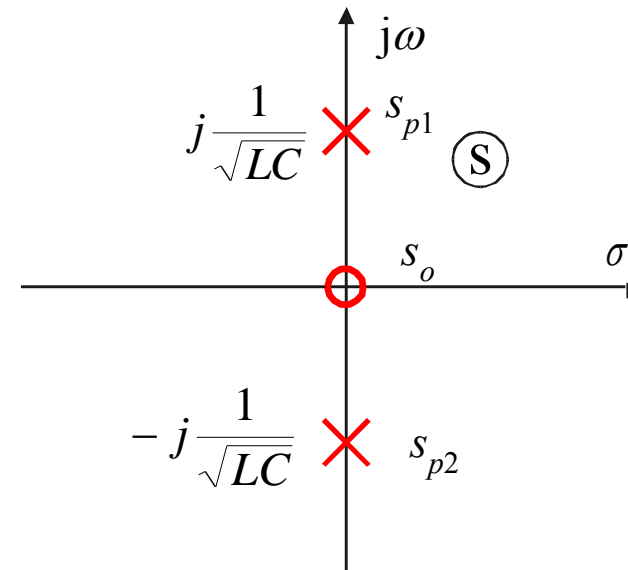
## ■ Ulazna funkcija - admitancija

$$Y(s) = \frac{I(s)}{U_1(s)} = \frac{1}{L} \cdot \frac{s}{s^2 + \frac{1}{LC}}$$

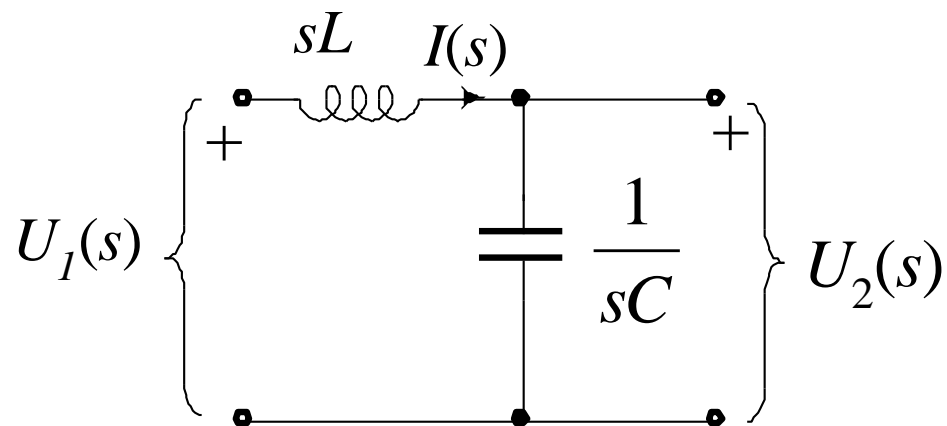
$$s_{o1} = 0$$

$$s_{o2} = \infty$$

$$s_{p1,2} = \pm j \frac{1}{\sqrt{LC}}$$



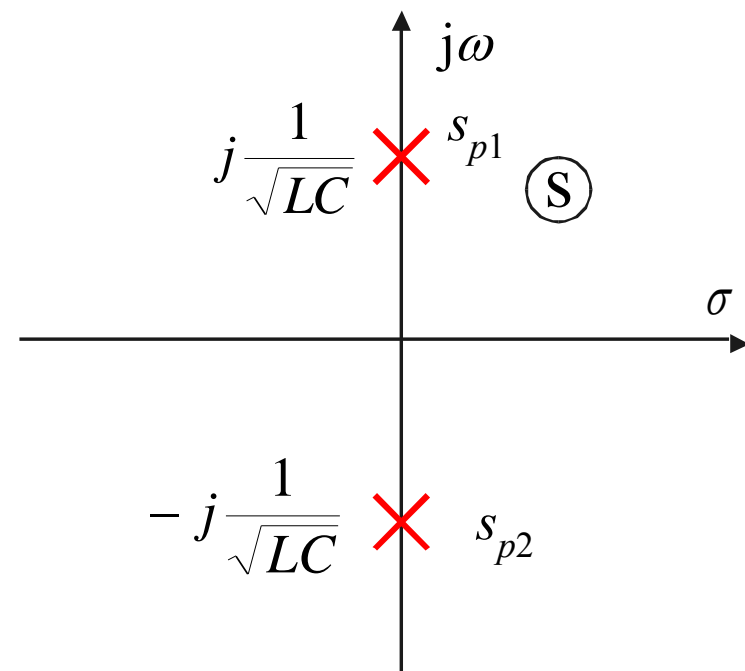
## ■ Prijenosna funkcija napona – naponski djelitelj



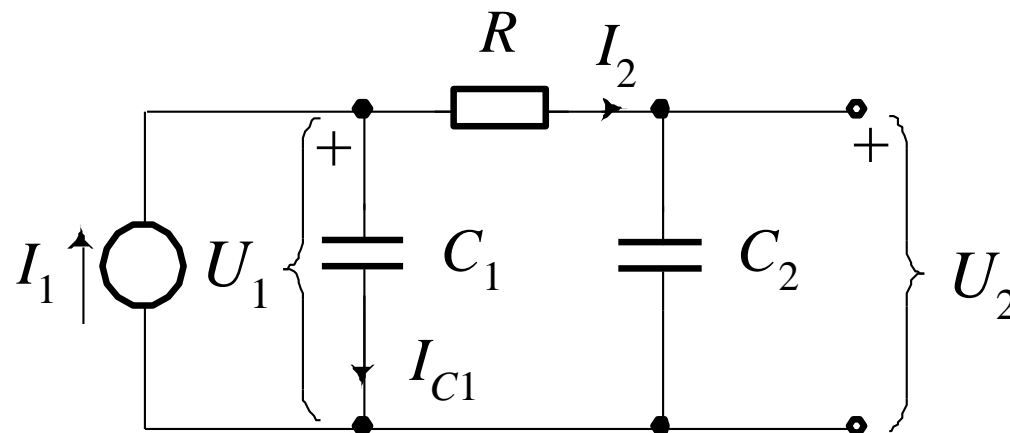
$$H(s) = \frac{U_2(s)}{U_1(s)} = \frac{1}{s^2 + \frac{1}{LC}}$$

$$s_{o1,2} = \infty$$

$$s_p = \pm j \frac{1}{\sqrt{LC}}$$



- Primjer 3.: Odrediti prijenosne funkcije  $H(s)=I_2/I_1$  i  $Z_{21}(s)=U_2/I_1$



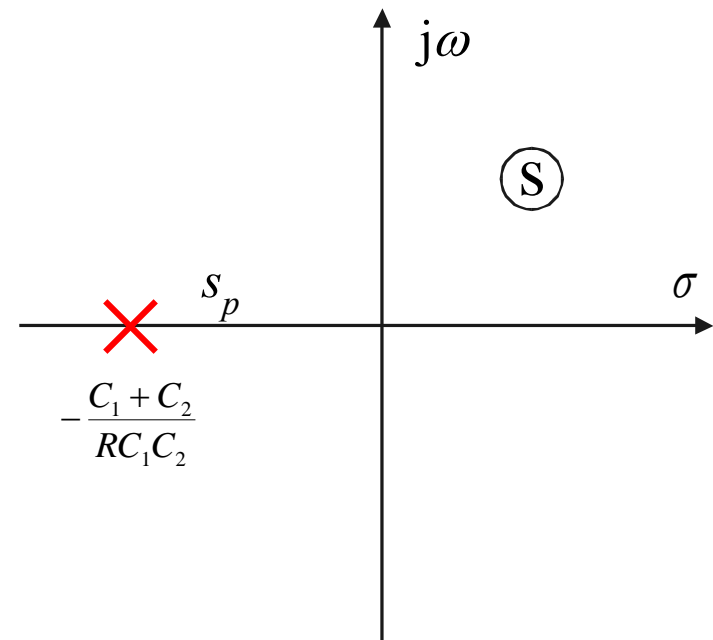
- Krug na slici  $\rightarrow$  strujni djelitelj

$$H(s) = \frac{1}{C_1 R} \cdot \frac{1}{s + (C_1 + C_2)/RC_1 C_2}$$

## ■ Polovi i nule

$$s_o = \infty$$

$$s_p = -\frac{C_1 + C_2}{RC_1 C_2}$$





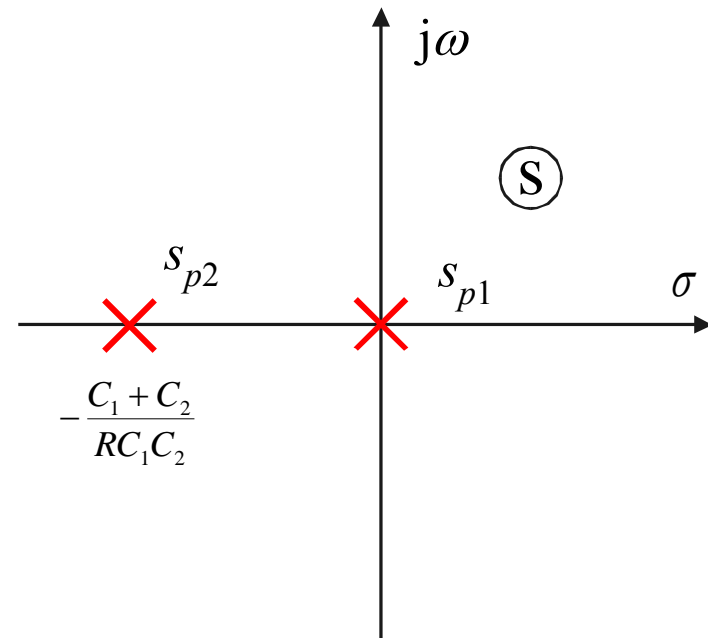
## ■ Prijenosna impedancija

$$Z_{21}(s) = \frac{U_2(s)}{I_1(s)} = \frac{1/C_1 C_2 R}{s(s + (C_1 + C_2)/RC_1 C_2)}$$

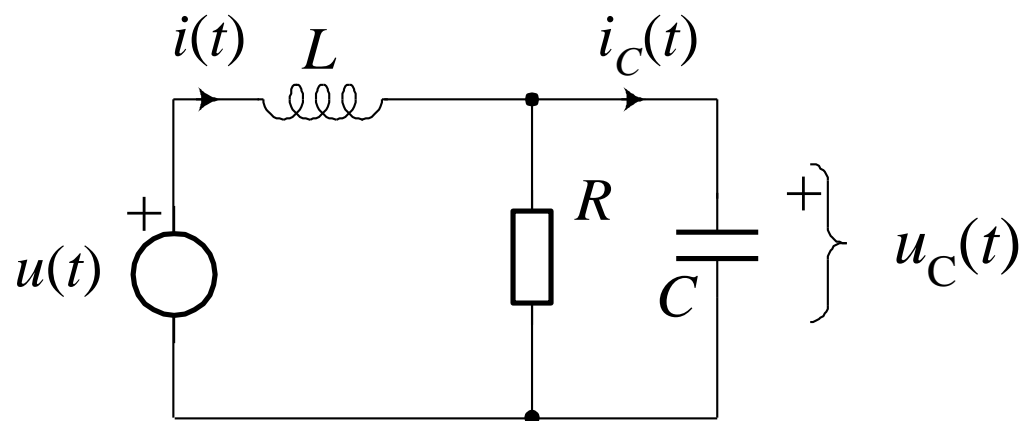
$$s_{o1,2} = \infty$$

$$s_{p1} = 0$$

$$s_{p2} = -\frac{C_1 + C_2}{RC_1 C_2}$$



## ■ Primjer 4.: RLC krug



$$R = 2$$

$$C = 1$$

$$L = 1$$

- Odziv  $\rightarrow$  struja  $I(s) \rightarrow$  Funkcija admitancije
- Odziv  $\rightarrow$  napon  $U_C(s) \rightarrow$  Prijenosna funkcija
- Odziv  $\rightarrow$  struja  $I_C(s) \rightarrow$  Prijenosna funkcija

- Odziv  $\rightarrow$  struja  $I(s) \rightarrow$  Funkcija admitancije

$$Y(s) = \frac{I(s)}{U(s)} = \frac{s + 1/2}{s^2 + s/2 + 1}$$

- Odziv  $\rightarrow$  napon  $U_C(s) \rightarrow$  Prijenosna funkcija

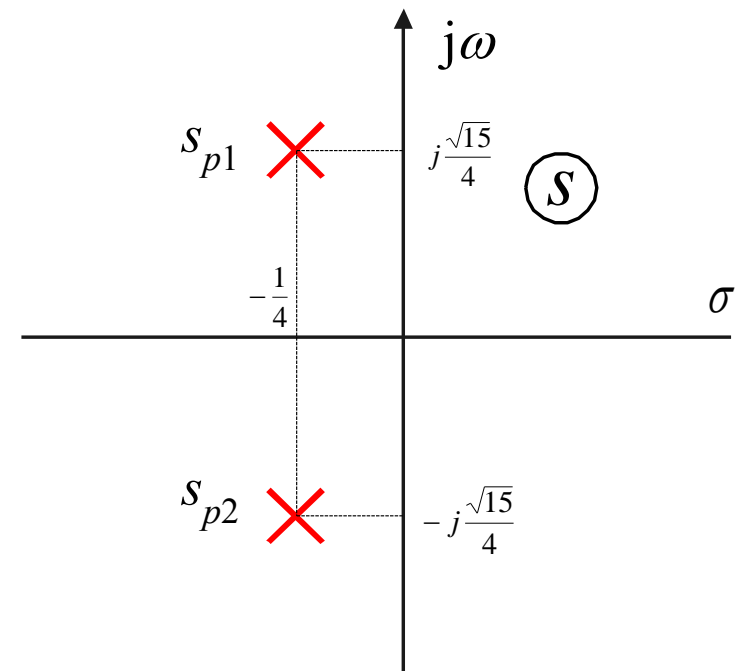
$$H(s) = \frac{U_C(s)}{U(s)} = \frac{1}{s^2 + s/2 + 1}$$

- Nule prijenosne funkcije

$$s_{o1,2} = \infty$$

- Polovi prijenosne funkcije

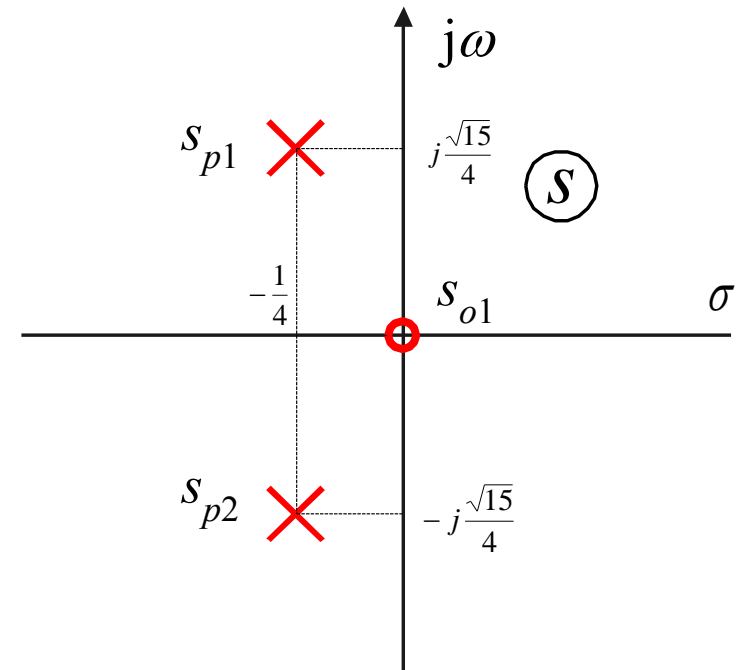
$$s_{p1,2} = -\frac{1}{4} \pm j\frac{\sqrt{15}}{4}$$



■ Odziv  $\rightarrow$  struja  $I_C(s) \rightarrow$  Prijenosna funkcija

■ Prijenosna admitancija

$$H(s) = Y_{21}(s) = \frac{I_C(s)}{U(s)} = \frac{2s}{2s^2 + s + 2}$$



Recipročna vrijednost funkcije imitancije je također *funkcija imitancije* dakle i funkcija mreže.

Recipročna vrijednost prijenosne funkcije *nije funkcija mreže* (nema njena svojstva).

Za sve funkcije mreža vrijedi:

- razlomljena realna racionalna f. od “ $s$ ”
- polovi samo u lijevoj poluravnini i na  $j\omega$  osi
- polovi na  $j\omega$  osi jednostruki

Za prijenosne funkcije vrijedi:

- nule mogu biti i u desnoj poluravnini

Za funkcije imitancije vrijedi:

- nule ne smiju biti u desnoj poluravnini →
- za njih vrijedi isto što i za polove

# Funkcije mreža i stacionarno stanje sinusne pobude

Funkcija mreže  $H(s)$  je omjer Laplace-ovih transformacija odziva  $Y(s)$  i pobude  $X(s)$  uz početne uvjete jednake nuli.

$$H(s) = \frac{Y(s)}{X(s)}$$

U stacionanome stanju sinusne pobude

→ Pobuda i odziv → fazori



## Fazor - definicije

$$X(j\omega) = |X(j\omega)| \angle \varphi_x(\omega) = |X(j\omega)| e^{j\varphi_x(\omega)}$$

$$Y(j\omega) = |Y(j\omega)| \angle \varphi_y(\omega) = |Y(j\omega)| e^{j\varphi_y(\omega)}$$

Pobuda djeluje od  $t = -\infty$

Prijelazne pojave završene  $\rightarrow$  STACIONARNO STANJE.  
Naponi i struje u krugu imaju sinusni oblik frekvencije  $\omega$ .

$$x(t) = X \sin(\omega t + \varphi_x) \rightarrow X \angle \varphi_x$$

Razlikuju se po iznosu amplituda i faznog pomaka.

- Ako u funkciji  $H(s)$  zamijenimo varijablu  $s$  sa  $j\omega$   
→  $H(j\omega)$ =omjer fazora odziva i fazora pobude.

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{|Y(j\omega)|e^{j\varphi_y(\omega)}}{|X(j\omega)|e^{j\varphi_x(\omega)}} = \left| \frac{Y(j\omega)}{X(j\omega)} \right| \cdot e^{j(\varphi_y - \varphi_x)}$$

$H(j\omega) \rightarrow$  općenito kompleksni broj

$$H(j\omega) = \operatorname{Re}[H(j\omega)] + j \operatorname{Im}[H(j\omega)]$$

U polarnim koordinatama

$H(j\omega) = |H(j\omega)| \cdot e^{j\varphi(\omega)} \longrightarrow$  kompleksna frekvencijska karakteristika

$|H(j\omega)| = \frac{|Y(j\omega)|}{|X(j\omega)|} \longrightarrow$  amplitudno frekvencijska karakteristika

$\varphi(\omega) = \varphi_y(\omega) - \varphi_x(\omega) \longrightarrow$  fazno frekvencijska karakteristika

Ako je poznata funkcija  $H(j\omega)$  , tada je:

$$H^2(j\omega) = \underbrace{H(j\omega) \cdot H(-j\omega)}_{|H(j\omega)|^2} \cdot \frac{H(j\omega)}{H(-j\omega)} = |H(j\omega)|^2 \cdot e^{2j\varphi(\omega)}$$

$$|H(j\omega)| = \sqrt{H(j\omega) \cdot H(-j\omega)}$$

$$\varphi(\omega) = \frac{1}{2j} \ln \left| \frac{H(j\omega)}{H(-j\omega)} \right|$$

$$\text{ili: } |H(j\omega)| = \sqrt{\text{Re}^2[H(j\omega)] + \text{Im}^2[H(j\omega)]}$$

Moguće je pisati i:

$$H(j\omega) = |H(j\omega)| \cdot e^{j\varphi(\omega)} = |H(j\omega)| \cdot [\cos(\varphi(\omega)) + j \sin(\varphi(\omega))]$$

$$\left. \begin{array}{l} \text{Re}[H(j\omega)] = |H(j\omega)| \cos(\varphi(\omega)) \\ \text{Im}[H(j\omega)] = |H(j\omega)| \sin(\varphi(\omega)) \end{array} \right\} \quad \text{tg } \varphi(\omega) = \frac{\text{Im}[H(j\omega)]}{\text{Re}[H(j\omega)]}$$

$$\varphi(\omega) = \text{arctg} \frac{\text{Im}[H(j\omega)]}{\text{Re}[H(j\omega)]}$$

$$H(s) = \frac{P(s)}{Q(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0}$$

$$H(s) = k \frac{(s - s_{01})(s - s_{02}) \cdots (s - s_{0m})}{(s - s_{p1})(s - s_{p2}) \cdots (s - s_{pn})} = k \frac{\prod_{i=1}^m (s - s_{0i})}{\prod_{j=1}^n (s - s_{pj})}$$

$$s_{0i} = \sigma_{0i} + j\omega_{0i} \longrightarrow \text{nule funkcije mreže}$$

$$s_{pj} = \sigma_{pj} + j\omega_{pj} \longrightarrow \text{polovi funkcije mreže}$$

$$H(j\omega) = \frac{P(j\omega)}{Q(j\omega)} = \frac{b_m(j\omega)^m + b_{m-1}(j\omega)^{m-1} + \dots + b_0}{a_n(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \dots + a_0}$$

$$H(j\omega) = k \frac{(j\omega - s_{01})(j\omega - s_{02}) \cdots (j\omega - s_{0m})}{(j\omega - s_{p1})(j\omega - s_{p2}) \cdots (j\omega - s_{pn})}$$

$$(j\omega - s_{0i}) = |j\omega - s_{0i}| \cdot e^{j\varphi_{oi}} \quad \varphi_{oi} = \operatorname{arctg} \left( \frac{\omega - \omega_{0i}}{\sigma_{0i}} \right)$$

$$|H(j\omega)| = k \frac{|j\omega - s_{01}| \cdot |j\omega - s_{02}| \cdots |j\omega - s_{0m}|}{|j\omega - s_{p1}| \cdot |j\omega - s_{p2}| \cdots |j\omega - s_{pn}|}$$

$$|H(j\omega)| = k \frac{|j\omega - \sigma_{01} - j\omega_{01}| \cdots |j\omega - \sigma_{0m} - j\omega_{0m}|}{|j\omega - \sigma_{p1} - j\omega_{p1}| \cdots |j\omega - \sigma_{pn} - j\omega_{pn}|}$$

$$|H(j\omega)| = k \frac{\sqrt{\sigma_{01}^2 + (\omega - \omega_{01})^2} \cdots \sqrt{\sigma_{0m}^2 + (\omega - \omega_{0m})^2}}{\sqrt{\sigma_{p1}^2 + (\omega - \omega_{p1})^2} \cdots \sqrt{\sigma_{pn}^2 + (\omega - \omega_{pn})^2}}$$



$$\varphi(\omega) = [\varphi_{01}(\omega) + \varphi_{02}(\omega) + \dots + \varphi_{0m}(\omega)] - \\ - [\varphi_{p1}(\omega) + \varphi_{p2}(\omega) + \dots + \varphi_{pn}(\omega)]$$

$$\varphi(\omega) = \sum_{i=1}^m \operatorname{arctg}\left(\frac{\omega - \omega_{0i}}{\sigma_{0i}}\right) - \sum_{j=1}^n \operatorname{arctg}\left(\frac{\omega - \omega_{pj}}{\sigma_{pj}}\right)$$

Za prijenosne funkcije često se koristi logaritamska mjera.

$$\ln H(j\omega) = \ln \left[ |H(j\omega)| \cdot e^{j\varphi(\omega)} \right] = \ln |H(j\omega)| + j\varphi(\omega)$$

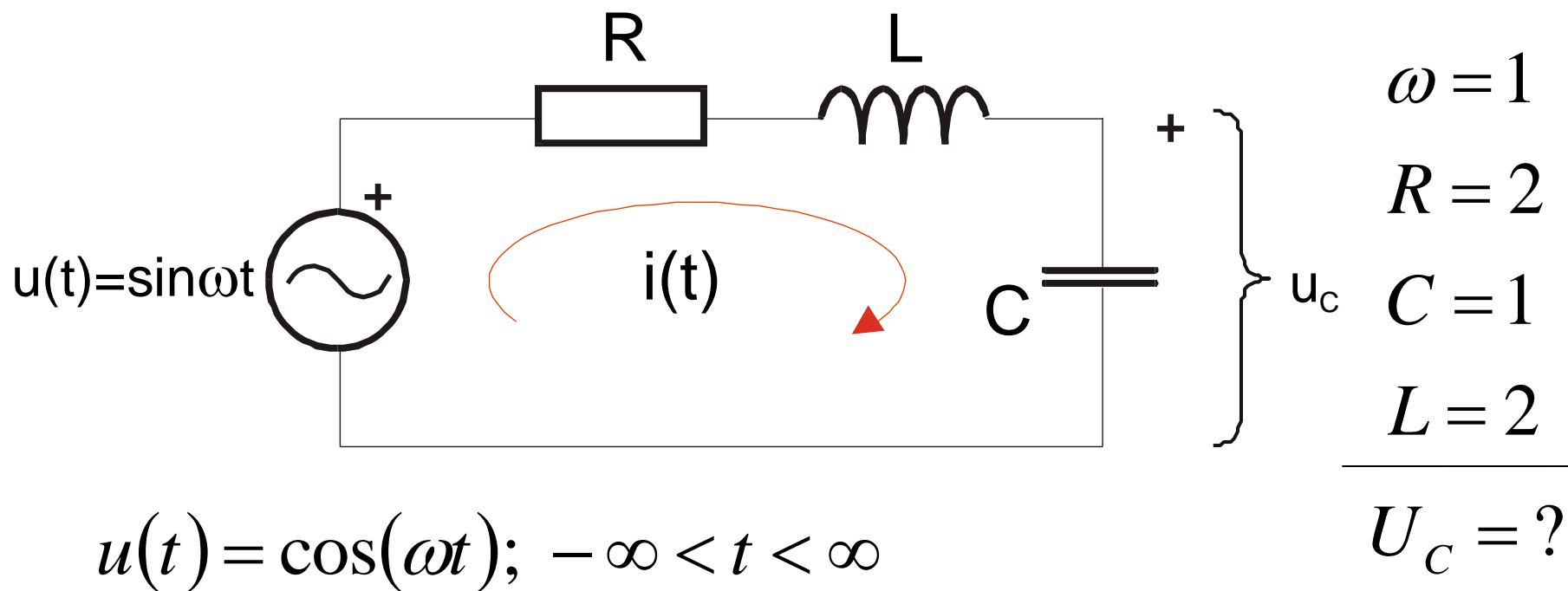
$\alpha_N(\omega) = \ln |H(j\omega)| \rightarrow$  Logaritamska mjera pojačanja u **Neperima** [N].

Često umjesto prirodnog  $\rightarrow$  dekadski log, pa je uobičajena mjera:

$\alpha(\omega) = 20 \log |H(j\omega)| \rightarrow$  logaritamska mjera pojačanja  
u decibelima [dB]

$$\alpha(\omega)[dB] \cong 8.68 \alpha_N(\omega)[N]$$

## Primjer: RLC krug



$$u(t) = \cos(\omega t); -\infty < t < \infty$$

$$H(s) = \frac{U_c(s)}{U(s)} = \frac{1}{s^2 LC + sRC + 1}$$

$$H(s) = \frac{U_c(s)}{U(s)} = \frac{1}{s^2 LC + sRC + 1}$$

Sinusna pobuda  $\rightarrow s=j\omega \rightarrow$  frekvencijska karakteristika

$$H(j\omega) = \frac{U_c(j\omega)}{U(j\omega)} = \frac{1}{-\omega^2 LC + j\omega RC + 1}$$

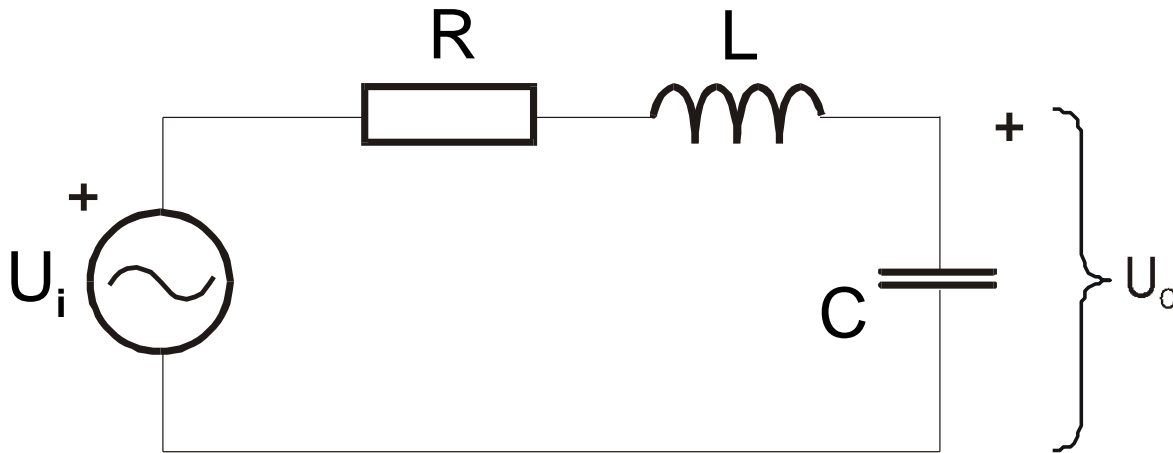
Fazor odziva  $\rightarrow U_c(j\omega)$

$$U_c(j\omega) = U(j\omega) \cdot H(j\omega) = \frac{U(j\omega)}{-\omega^2 LC + j\omega RC + 1}$$

$$U(j\omega) = 1 \angle 0^\circ = 1$$

$$U_c(j\omega) = -\frac{1+2j}{5} = -\frac{\sqrt{5}}{5} \angle \arctg(2)$$

Primjer: RLC krug → Frekvencijski odziv ili karakteristika



$$H(s) = \frac{U_c(s)}{U_i(s)} \rightarrow \text{Prijenosna funkcija napona}$$

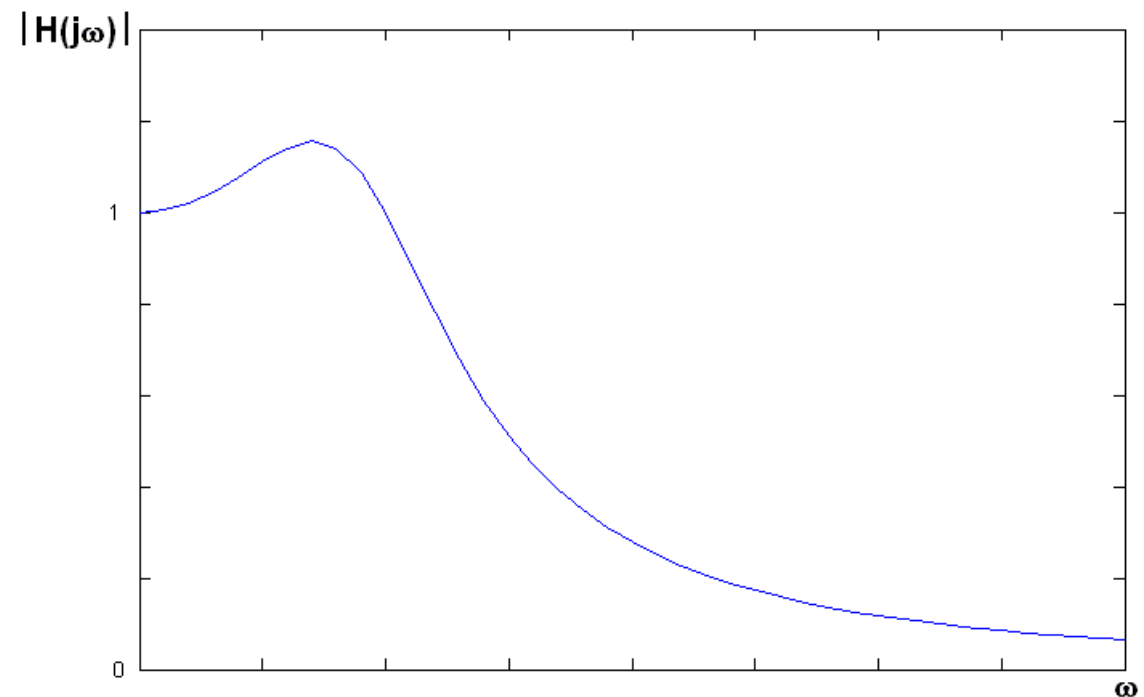
$$H(s) = \frac{1}{s^2 LC + sRC + 1}$$

## ■ Amplitudno frekvencijska karakteristika

$$|H(j\omega)| = \frac{1}{\sqrt{(1 - \omega^2 LC)^2 + (RC\omega)^2}}$$

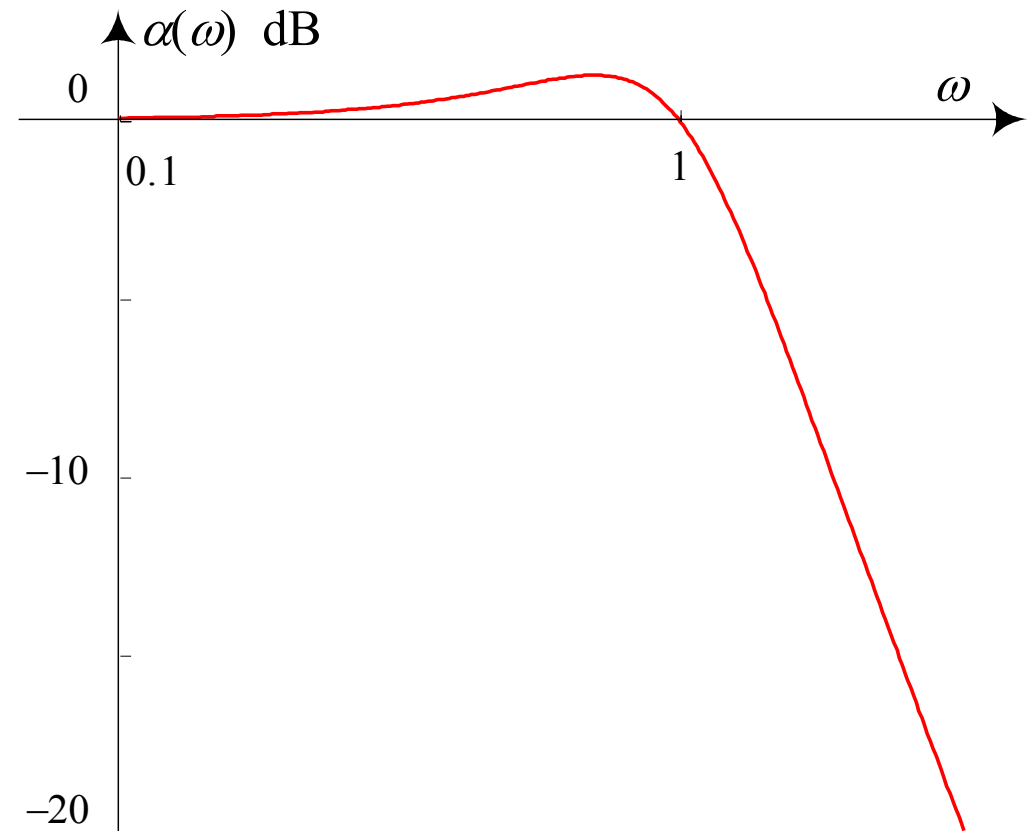
Za  $R=L=C=1 \rightarrow$

$$|H(j\omega)| = \frac{1}{\sqrt{1 - \omega^2 + \omega^4}}$$



## ■ Logaritamska mjera pojačanja

$$\begin{aligned}\alpha(\omega) &= 20 \cdot \log |H(j\omega)| = \\ &= 20 \cdot \log \frac{1}{\sqrt{1 - \omega^2 + \omega^4}} = \\ &= -10 \log(1 - \omega^2 + \omega^4)\end{aligned}$$

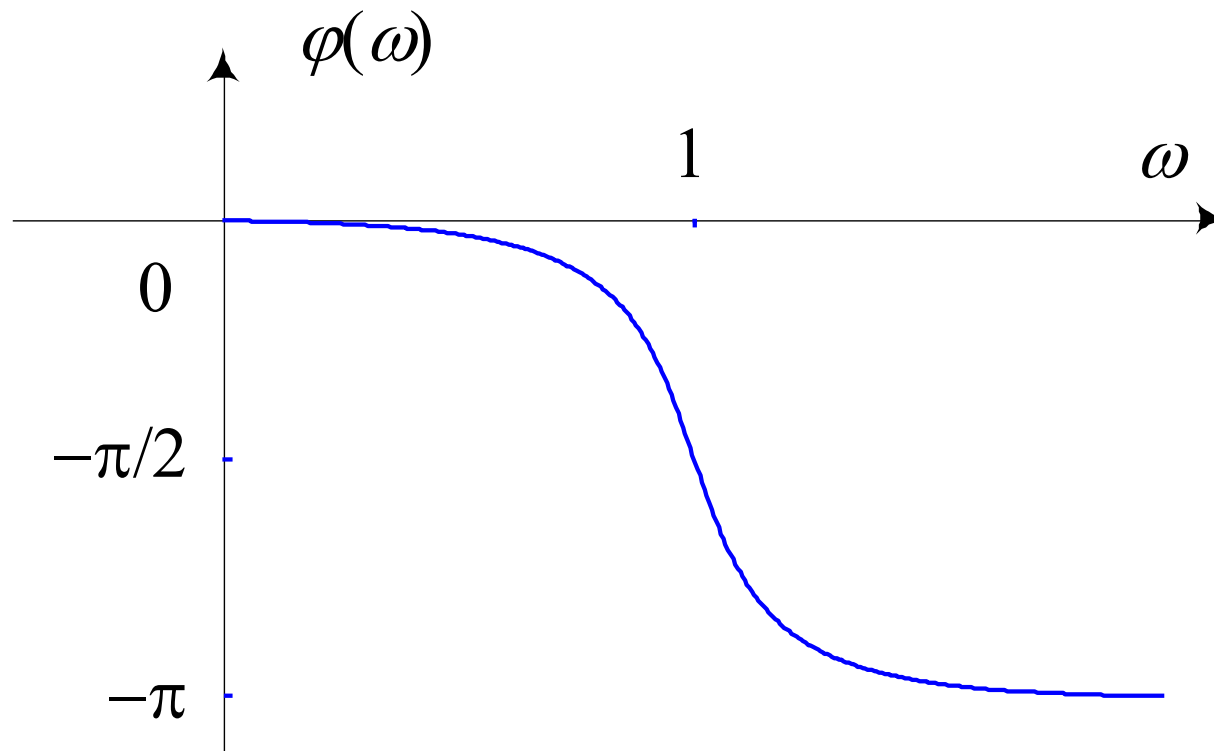




## ■ Fazna frekvencijska karakteristika

$$\varphi(\omega) = -\operatorname{arctg} \frac{RC\omega}{1 - \omega^2 LC}$$

$$\varphi(\omega) = -\operatorname{arctg} \frac{\omega}{1 - \omega^2}$$



Funkciju  $H(j\omega)$  moguće je napisati u obliku

$$H(j\omega) = \frac{b_m (j\omega)^m + b_{m-1} (j\omega)^{m-1} + \dots + b_0}{a_n (j\omega)^n + a_{n-1} (j\omega)^{n-1} + \dots + a_0}$$

odnosno

$$H(j\omega) = \frac{(b_0 - b_2\omega^2 + b_4\omega^4 \mp \dots) + j\omega(b_1 - b_3\omega^2 + b_5\omega^4 \mp \dots)}{(a_0 - a_2\omega^2 + a_4\omega^4 \mp \dots) + j\omega(a_1 - a_3\omega^2 + a_5\omega^4 \mp \dots)}$$

vrijedi

$$H(j\omega) = \frac{(\text{polinom od } \omega^2) + j\omega(\text{polinom od } \omega^2)}{(\text{polinom od } \omega^2) + j\omega(\text{polinom od } \omega^2)}$$

$$H(-j\omega) = \frac{(\text{polinom od } \omega^2) - j\omega(\text{polinom od } \omega^2)}{(\text{polinom od } \omega^2) - j\omega(\text{polinom od } \omega^2)}$$

Očito je:  $H^*(j\omega) = H(-j\omega)$

pa pošto vrijedi za kompleksne brojeve:

$$\left. \begin{array}{l} \operatorname{Re}[H^*(j\omega)] = \operatorname{Re}[H(j\omega)] \\ \operatorname{Im}[H^*(j\omega)] = -\operatorname{Im}[H(j\omega)] \\ |H^*(j\omega)| = |H(j\omega)| \\ H^*(-j\omega) = H(j\omega) \end{array} \right\} \begin{array}{l} \operatorname{Re}[H(-j\omega)] = \operatorname{Re}[H(j\omega)] \\ \operatorname{Im}[H(-j\omega)] = -\operatorname{Im}[H(j\omega)] \\ |H(-j\omega)| = |H(j\omega)| \\ \varphi(-\omega) = -\varphi(\omega) \end{array}$$