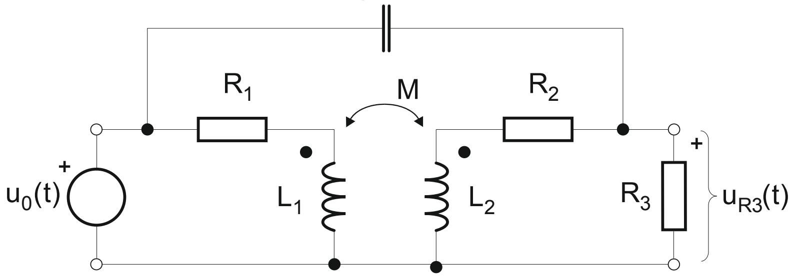
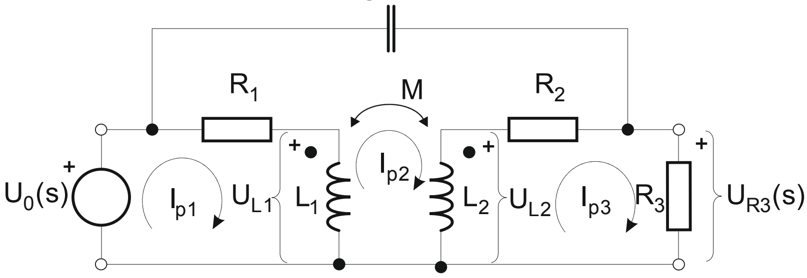
MEĐUISPIT IZ ELEKTRIČNIH KRUGOVA 2015-2016 – Rješenja

1. Za električni krug na slici zadane su normalizirane vrijednosti elemenata R_1=R_2=R_3=1, C=1, L_1=L_2=1, M=1, početne struje kroz induktivitete i početni napon na kapacitetu jednaki su nuli, te pobuda $u_0(t)$=$S(t)$. Primjenom Laplaceove transformacije i koristeći metodu petlji izračunati napon $u_{R3}(t)$ na otporu R_3. Napisati sustav jednadžbi petlji u matričnom obliku.



<u>Rješenje:</u> Primjena Laplaceove transformacije



a) Jednadžbe petlji: (1 bod)

1) $I_{p1}R_1 - I_{p2}R_1 = U_0 - U_{L1}$;

2) $-I_{p1}R_1 + I_{p2}\left(R_1 + R_2 + \dfrac{1}{sC}\right) - I_{p3}R_2 = U_{L1} - U_{L2}$;

3) $-I_{p2}R_2 + I_{p3}(R_2 + R_3) = U_{L2}$;

Jednadžbe vezanih induktiviteta:

4) $U_{L1} = (I_{p1} - I_{p2})sL_1 - (I_{p3} - I_{p2})sM$;

5) $U_{L2} = (I_{p1} - I_{p2})sM - (I_{p3} - I_{p2})sL_2$;

Nakon uvrštavanja 4) i 5) u 1), 2) i 3) te malo sređivanja: (1 bod)

1) $I_{p1}(R_1 + sL_1) - I_{p2}(R_1 + sL_1 - sM) - I_{p3}sM = U_0$;

2) $-I_{p1}(R_1 + sL_1 - sM) + I_{p2}\left(R_1 + R_2 + \dfrac{1}{sC} + sL_1 + sL_2 - 2sM\right) - I_{p3}(R_2 + sL_2 - sM) = 0$;

3) $-I_{p1}sM - I_{p2}(R_2 + sL_2 - sM) + I_{p3}(R_2 + R_3 + sL_2) = 0$;

U matričnom obliku:

$$\begin{bmatrix} R_1 + sL_1 & -(R_1 + sL_1 - sM) & -sM \\ -(R_1 + sL_1 - sM) & R_1 + R_2 + \dfrac{1}{sC} + sL_1 + sL_2 - 2sM & -(R_2 + sL_2 - sM) \\ -sM & -(R_2 + sL_2 - sM) & R_2 + R_3 + sL_2 \end{bmatrix} \begin{bmatrix} I_{p1} \\ I_{p2} \\ I_{p3} \end{bmatrix} = \begin{bmatrix} U_0 \\ 0 \\ 0 \end{bmatrix}$$

Na ovom mjestu se može napraviti provjera postupka ako je dobivena matrica impedancija petlji simetrična oko dijagonale. To vrijedi za recipročne mreže koje smiju sadržavati međuinduktivitete.

Uz uvrštene vrijednosti elemenata: (1 bod)

$$\begin{bmatrix} 1+s & -1 & -s \\ -1 & 2+\dfrac{1}{s} & -1 \\ -s & -1 & 2+s \end{bmatrix} \begin{bmatrix} I_{p1} \\ I_{p2} \\ I_{p3} \end{bmatrix} = \begin{bmatrix} U_0 \\ 0 \\ 0 \end{bmatrix}$$

Jadan od načina izračunavanja struje I_{p3}: $I_{p3}(s) = \dfrac{\Delta_3}{\Delta}$

Determinanta matrice npr. razvojem po prvom stupcu:

$$\Delta = \begin{vmatrix} 1+s & -1 & -s \\ -1 & 2+\dfrac{1}{s} & -1 \\ -s & -1 & 2+s \end{vmatrix} = (1+s)\cdot \begin{vmatrix} 2+\dfrac{1}{s} & -1 \\ -1 & 2+s \end{vmatrix} + 1\cdot \begin{vmatrix} -1 & -s \\ -1 & 2+s \end{vmatrix} - s\cdot \begin{vmatrix} -1 & -s \\ 2+\dfrac{1}{s} & -1 \end{vmatrix} =$$

$$= (1+s)\cdot\left[\left(2+\frac{1}{s}\right)(2+s)-1\right] + 1\cdot\left[-(2+s)-s\right] - s\cdot\left[1+s\left(2+\frac{1}{s}\right)\right] =$$

$$= (1+s)\left(2+\frac{1}{s}\right)(2+s) - (1+s) - (2+s) - s - s - s(2s+1) =$$

$$= (1+s)\left(5+2s+\frac{2}{s}\right) - 3 - 5s - 2s^2 =$$

$$= 5+2s+\frac{2}{s}+5s+2s^2+2-3-5s-2s^2 = 4+2s+\frac{2}{s} = 2\left(2+s+\frac{1}{s}\right)$$

$$\Delta_3 = \begin{vmatrix} 1+s & -1 & U_0 \\ -1 & 2+\dfrac{1}{s} & 0 \\ -s & -1 & 0 \end{vmatrix} = U_0\cdot \begin{vmatrix} -1 & -s \\ 2+\dfrac{1}{s} & -1 \end{vmatrix} = U_0\cdot\left[1+s\left(2+\frac{1}{s}\right)\right] = U_0\cdot 2(s+1)$$

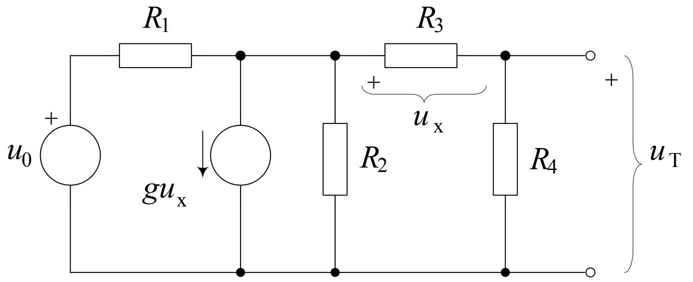
uz R_3=1

$$U_{R3}(s) = I_{p3}(s)R_3 = \frac{\Delta_3}{\Delta} = U_0(s)\cdot\frac{s+1}{2+s+\dfrac{1}{s}} = \frac{1}{s}\cdot\frac{s+1}{2+s+\dfrac{1}{s}} = \frac{s+1}{s^2+2s+1}$$ (1 bod)

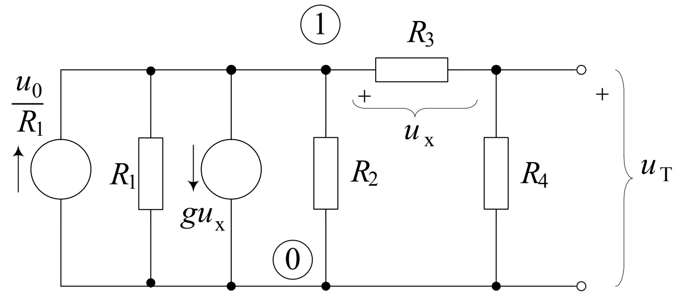
Inverzna Laplaceova transformacija izlaznog napona: (1 bod)

$$U_{R3}(s) = \frac{s+1}{s^2+2s+1} = \frac{s+1}{(s+1)^2} = \frac{1}{s+1} \quad\Rightarrow\quad \underline{u_{R3}(t) = e^{-t}\cdot S(t)}$$

2. Za električni krug na slici odrediti ekvivalentni dvopol po Theveninu. Zadane su vrijednosti elemenata: $R_1 = 1/6\ \text{k}\Omega$, $R_2 = 1/4\ \text{k}\Omega$, $R_3 = 1/3\ \text{k}\Omega$, $R_4 = 1\ \text{k}\Omega$, $g = 2\ \text{mS}$, te $u_0 = 2{,}5\ V$. Odrediti: a) Theveninov napon u_T i b) Theveninov otpor R_T, c) napon u_x, d) napon na otporu R_2 i e) struju kroz R_2.



Rješenje: a) Theveninov napon



$$\frac{u_0}{R_1} - g u_x = u_1\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) - u_T \frac{1}{R_3}$$

$$\frac{u_T}{R_4} = \frac{u_x}{R_3} \Rightarrow u_x = \frac{u_T R_3}{R_4}$$

$$u_1 = u_x + u_T = u_T\left(1 + \frac{R_3}{R_4}\right)$$

$$\frac{u_0}{R_1} - g\frac{u_T R_3}{R_4} = u_T\left(1 + \frac{R_3}{R_4}\right)\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) - u_T \frac{1}{R_3}$$

$$\frac{u_0}{R_1} = u_T\left(g\frac{R_3}{R_4} + \left(1 + \frac{R_3}{R_4}\right)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{1}{R_4}\right)$$

$$u_T = \frac{\frac{1}{R_1}u_0}{g\frac{R_3}{R_4} + \left(1 + \frac{R_3}{R_4}\right)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{1}{R_4}} = \frac{6 \cdot 2{,}5}{\frac{2}{3} + \left(1 + \frac{1}{3}\right)(6+4) + 1} = \frac{15}{\frac{2}{3} + \frac{4}{3}10 + 1} = 1\ \text{V}$$

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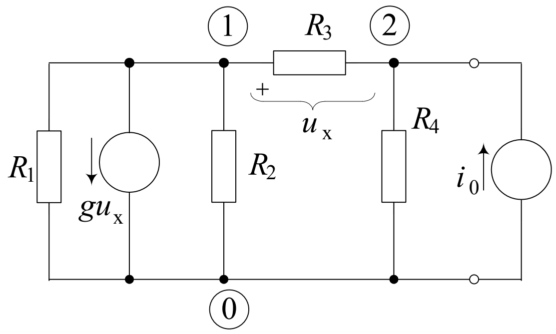
b) Theveninov otpor



$$-gu_x = u_1\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)-u_2\frac{1}{R_3} \qquad u_x = u_1 - u_2$$

$$i_0 = -u_1\frac{1}{R_3}+u_2\left(\frac{1}{R_3}+\frac{1}{R_4}\right)$$

$$-g(u_1-u_2) = u_1\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)-u_2\frac{1}{R_3} \quad \Rightarrow \quad u_1 = u_2\frac{g+\frac{1}{R_3}}{g+\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}}$$

$$i_0 = -u_2\frac{g+\frac{1}{R_3}}{g+\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}}\frac{1}{R_3}+u_2\left(\frac{1}{R_3}+\frac{1}{R_4}\right) = u_2\frac{\frac{1}{R_3}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+\frac{1}{R_4}\left(g+\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)}{g+\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}}$$

$$R_T = \frac{u_2}{i_0} = \frac{g+\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}}{\frac{1}{R_3}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+\frac{1}{R_4}\left(g+\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)} = \frac{2+6+4+3}{3(6+4)+1(2+6+4+3)} = \frac{1}{3}\ \text{k}\Omega$$

Bodovi: a) + b) = (3 boda)

c) napon u_x

$$u_x = \frac{u_T R_3}{R_4} = \frac{1}{3}\ V$$

d) napon na R_2

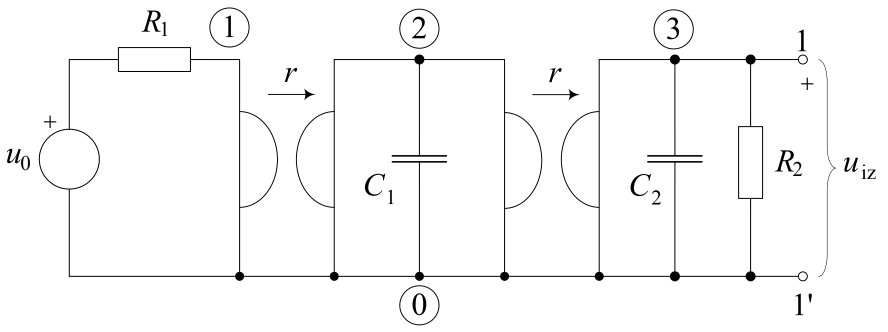
$$u_{R2} = u_1 = u_x + u_T = \frac{4}{3}\ V$$

e) struja kroz R_2

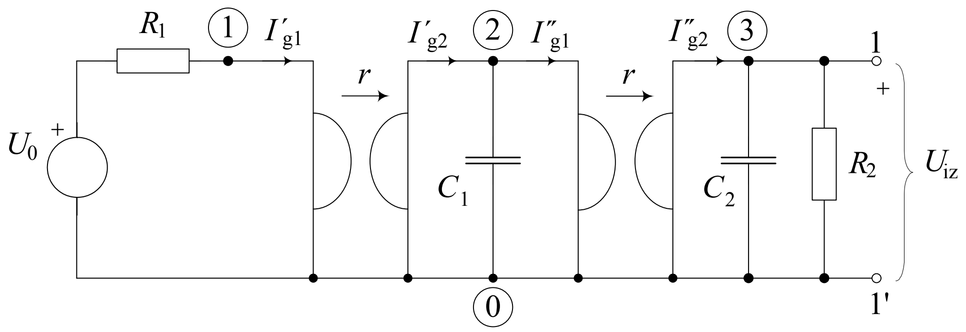
$$i_{R2} = \frac{u_{R2}}{R_2} = \frac{4}{3}\cdot 4 = \frac{16}{3}\ mA$$

Bodovi: c) + d) + e) = (2 boda)

3. Za električni krug na slici zadane su normalizirane vrijednosti elemenata $C_1 = C_2 = \sqrt{2}$, $R_1 = R_2 = 1$, te $r = 1$. Odrediti: nadomjesne parametre mreže po Theveninu obzirom na polove 1–1': a) Theveninov napon U_T i b) Theveninovu impedanciju Z_T. Također izračunati: c) napon u_1; d) napon u_2. Koristiti metodu napona čvorišta (čvorište 0 je referentno).



Rješenje: a) Jednadžbe napona za čvorišta 1, 2 i 3 (1 bod)



a) Theveninov napon

$$\frac{U_0}{R_1} - I'_{g1} = \frac{U_1}{R_1} \qquad I'_{g1} = -\frac{U_2}{r}, \quad I'_{g2} = -\frac{U_1}{r}$$

$$I'_{g2} - I''_{g1} = U_2 s C_1 \qquad I''_{g1} = -\frac{U_3}{r}, \quad I''_{g2} = -\frac{U_2}{r}$$

$$I''_{g2} = U_3\left(sC_2 + \frac{1}{R_2}\right)$$

$$\frac{U_0}{R_1} = \frac{U_1}{R_1} - \frac{U_2}{r}$$

$$0 = \frac{U_1}{r} + U_2 s C_1 - \frac{U_3}{r} \quad \Rightarrow \quad U_1 = -rU_2 sC_1 + U_3$$

$$0 = \frac{U_2}{r} + U_3\left(sC_2 + \frac{1}{R_2}\right) \quad \Rightarrow \quad U_2 = -rU_3\left(sC_2 + \frac{1}{R_2}\right)$$

$$U_1 = -r\left(-rU_3\left(sC_2 + \frac{1}{R_2}\right)\right)sC_1 + U_3 = \left(r^2\left(sC_2 + \frac{1}{R_2}\right)sC_1 + 1\right)U_3$$

$$\frac{U_0}{R_1}=\frac{U_1}{R_1}-\frac{U_2}{r}=\frac{U_3}{R_1}\cdot\left(r^2\left(sC_2+\frac{1}{R_2}\right)sC_1+1\right)+U_3\left(sC_2+\frac{1}{R_2}\right)$$

$$U_3=\frac{U_0}{\left(r^2\left(sC_2+\frac{1}{R_2}\right)sC_1+1\right)+R_1\left(sC_2+\frac{1}{R_2}\right)}$$

$$U_T=U_3=\frac{U_0R_2}{s^2r^2R_2C_1C_2+s\left(r^2C_1+R_1R_2C_2\right)+R_1+R_2}$$

$$U_T=\frac{1}{2}\cdot\frac{U_0}{s^2+\sqrt{2}s+1}\quad \text{(1 bod)}$$

b) Theveninova impedancija

$$Z_{iz1}=\frac{r^2}{R_1}$$

$$Z_{iz2}=\frac{r^2\left(Z_{iz1}+\frac{1}{sC_2}\right)}{Z_{iz1}\frac{1}{sC_2}}=\frac{r^2\left(Z_{iz1}sC_2+1\right)}{Z_{iz1}}=r^2sC_2+R_1$$

$$Z_T=\frac{Z_{iz2}R_2}{Z_{iz2}\left(sC_3R_2+1\right)+R_2}=\frac{\left(r^2sC_1+R_1\right)R_2}{s^2r^2C_1C_2R_2+s\left(R_1R_2C_2+r^2C_1\right)+R_1+R_2}=\frac{\sqrt{2}s+1}{2\left(s^2+\sqrt{2}s+2\right)}\quad \text{(1 bod)}$$

c) Napon U_1

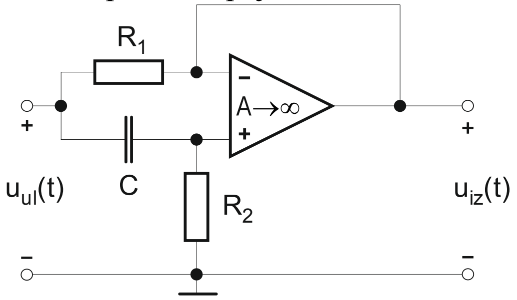
$$U_1=\left(r^2\left(sC_2+\frac{1}{R_2}\right)sC_1+1\right)U_3=\frac{1}{R_2}\left(r^2\left(sR_2C_2+1\right)sC_1+R_2\right)U_3$$

$$U_1=\frac{U_0\left(r^2\left(sC_2R_2+1\right)sC_1+R_2\right)}{s^2r^2R_2C_1C_2+s\left(r^2C_1+R_1R_2C_2\right)+R_1+R_2}=\frac{U_0\left(2s^2+\sqrt{2}s+1\right)}{2s^2+2\sqrt{2}s+2}\quad \text{(1 bod)}$$

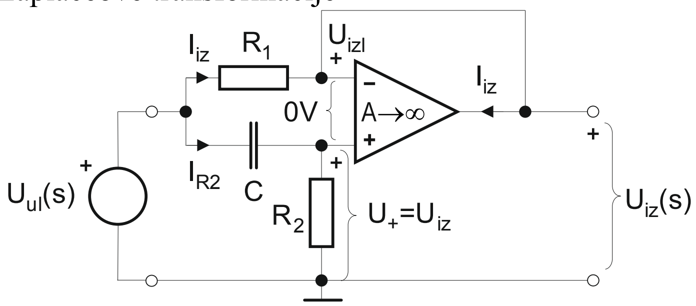
d) Napon U_2

$$U_2=\frac{-r\left(sC_2R_2+1\right)U_0}{s^2r^2C_1C_2R_1+s\left(r^2C_1+C_2R_1R_2\right)+R_1+R_2}=\frac{-\left(s\sqrt{2}+1\right)U_0}{2s^2+2\sqrt{2}s+2}\quad \text{(1 bod)}$$

4. Odrediti odziv $u_{iz}(t)$ za mrežu prikazanu slikom ako je zadano: R_1=1kΩ, R_2=20kΩ, C=250nF te kao poticaj jedinična Step funkcija $u_{ul}(t)=S(t)$. Koji se ekvivalentni element može upotrijebiti umjesto kapaciteta C otpora R_2 i pojačala i koliko on iznosi ?



<u>Rješenje:</u> Primjena Laplaceove transformacije



$$U_+\cdot\left(\frac{1}{R_2}+sC\right)-U_{ul}\cdot sC=0 \Rightarrow U_+=U_{ul}\cdot\frac{sC}{\frac{1}{R_2}+sC}=U_{ul}\cdot\frac{sR_2C}{1+sR_2C}=U_{ul}\cdot\frac{R_2}{\frac{1}{sC}+R_2}$$

$$\frac{U_{iz}-U_{ul}}{R_1}=I_{iz}$$

$$(U_+-U_{iz})\cdot A=U_{iz} \Rightarrow U_+=U_{iz}$$

$$U_{ul}-U_{iz}\cdot R_1=U_{iz}$$

$$U_{ul}-I_{R2}\cdot\frac{1}{sC}=U_+$$

$$I_{R2}=\frac{U_{iz}}{\frac{1}{sC}+R_2}$$

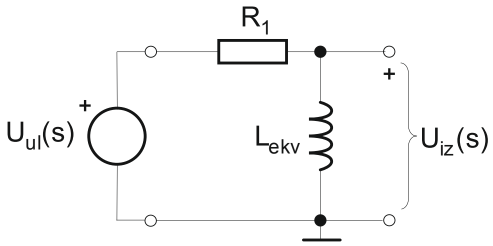
$$I_{R2}\cdot R_2=U_+=U_{iz}$$

$$U_{iz}=U_{ul}\cdot\frac{R_2}{\frac{1}{sC}+R_2}=U_{ul}\cdot\frac{1}{\frac{1}{sR_2C}+1}=U_{ul}\cdot\frac{s}{s+\frac{1}{R_2C}}$$ (2 boda)

Što je isto kao i $U_{iz}=U_{ul}\cdot\frac{sL_{ekv}}{R_1+sL_{ekv}}=U_{ul}\cdot\frac{s}{s+\frac{R_1}{L_{ekv}}}$ (vidi sliku)

Uspoređujući $\frac{R_1}{L_{ekv}}=\frac{1}{R_2C}$ slijedi $L_{ekv}=R_1R_2C=1[\text{k}\Omega]\cdot 20[\text{k}\Omega]\cdot 250[\text{nF}]=5[\text{H}]$

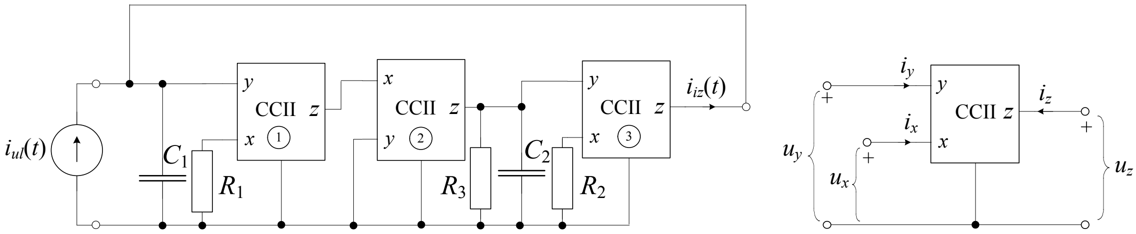
Dakle ekvivalentni element je induktivitet. (2 boda)



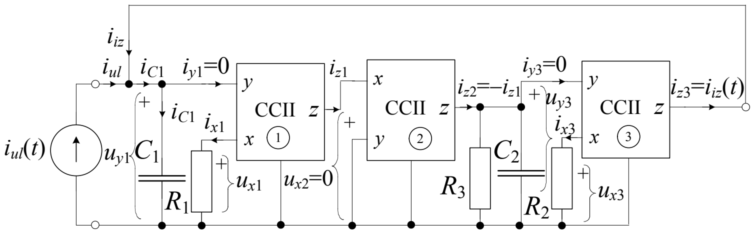
Odziv na poticaj $u_{ul}(t)=S(t)\Rightarrow U_{ul}(s)=\frac{1}{s}$ glasi: (1 bod)

$$U_{iz}(s)=U_{ul}\cdot\frac{s}{s+\frac{R_1}{L_{ekv}}}=U_{ul}\cdot\frac{s}{s+\frac{10^3}{5}}=\frac{1}{s}\cdot\frac{s}{s+200}=\frac{1}{s+200}\Rightarrow \underline{u_{iz}(t)=e^{-200t}\cdot S(t)[\mathrm{V}]}$$

5. Za električni krug prikazan slikom izračunati valni oblik struje $i_{iz}(t)$ za $t>0$ kao odziv, ako je zadana pobuda $i_{ul}(t)=\delta(t)$[A]. Zadane su normalizirane vrijednosti elemenata $R_1=1$, $R_2=1$, $R_3=4$, $C_1=1$, $C_2=1/16$. Za strujni prijenosnik druge generacije (CCII) vrijede slijedeće definicijske jednadžbe: $u_x=u_y$, $i_y=0$, $i_z=i_x$ uz referentna usmjerenja struja i napona prilaza prikazana na slici.



Rješenje:



a) Za prvi CCII vrijedi: (1 bod)

$$U_{x1}=U_{y1}=U_{C1}=I_{C1}\cdot\frac{1}{sC_1},\ I_{y1}=0$$

$$I_{z1}=I_{x1}=\frac{U_{x1}}{R_1}=I_{C1}\frac{1}{sR_1C_1}\Rightarrow\boxed{I_{z1}=I_{C1}\frac{1}{sR_1C_1}}$$

b) Za drugi CCII vrijedi: (1 bod)

$$I_{z2}=I_{x2}=-I_{z1}\Rightarrow\boxed{I_{z2}=-I_{z1}=-I_{C1}\frac{1}{sR_1C_1}}$$

$$U_{x2}=0,\ U_{y2}=0$$

c) Za treći CCII vrijedi: (1 bod)

$$U_{y3}=I_{z2}\cdot\frac{R_3\cdot 1/(sC_2)}{R_3+1/(sC_2)}=I_{z2}\cdot\frac{R_3}{sR_3C_2+1},$$

$$I_{iz}=I_{z3}=I_{x3}=\frac{U_{x3}}{R_2}=\frac{U_{y3}}{R_2}=I_{z2}\cdot\frac{R_3}{R_2}\cdot\frac{1}{sR_3C_2+1}\Rightarrow\boxed{I_{z3}=I_{z2}\cdot\frac{R_3}{R_2}\cdot\frac{1}{sR_3C_2+1}}$$

d) Nakon sređivanja do sada napisanih izraza: (1 bod)

$$I_{C1}=I_{ul}+I_{iz};\quad I_{iz}=I_{z3}=-(I_{ul}+I_{iz})\cdot\frac{1}{sR_1C_1}\cdot\frac{R_3}{R_2}\cdot\frac{1}{sR_3C_2+1}$$

$$I_{iz}\cdot\left[1+\frac{R_3}{sR_1C_1\cdot R_2\cdot(sR_3C_2+1)}\right]=-I_{ul}\cdot\frac{R_3}{sR_1C_1\cdot R_2\cdot(sR_3C_2+1)}$$

$$I_{iz}=-I_{ul}\cdot\frac{R_3}{s^2R_1R_2R_3C_1C_2+sR_1R_2C_1+R_3}=-I_{ul}\cdot\frac{1}{s^2R_1R_2C_1C_2+sR_1R_2C_1/R_3+1}$$

e) Uz uvrštene vrijednosti elemenata: (1 bod)

$$I_{iz}(s)=-I_{ul}\cdot\frac{16}{s^2+4s+16}=-1\cdot\frac{16}{(s+2)^2+12}=-\frac{16}{\sqrt{12}}\cdot\frac{\sqrt{12}}{(s+2)^2+(\sqrt{12})^2}$$

$$\Rightarrow i_{iz}(t)=-\frac{8\sqrt{3}}{3}\cdot e^{-2t}\cdot\sin(2\sqrt{3})\cdot S(t)[A]$$