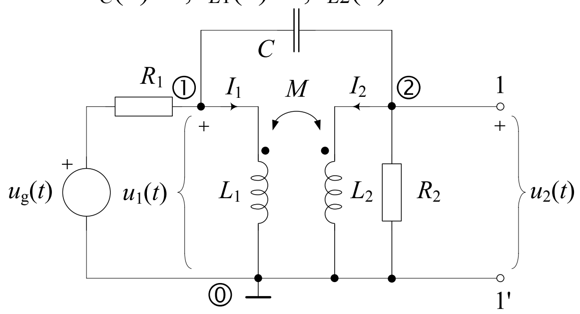
ZAVRŠNI ISPIT IZ ELEKTRIČNIH KRUGOVA – Rješenja – 2011

1. Za električni krug na slici izračunati odziv $u_2(t)$ na prilazu 1–1', ako je zadan poticaj $u_g(t)=e^{-t}\cdot S(t)$. Zadane su normalizirane vrijednosti elemenata: $R_1=R_2=1$, $L_1=L_2=1$, $M=1$, $C=1$. Početni uvjeti su jednaki nula: $u_C(0)=0$, $i_{L1}(0)=0$, $i_{L2}(0)=0$.



Rješenje:

Napomena: ako se odmah uvrste numeričke vrijednosti (što se jednako priznaje za točno rješenje) tada je postupak znatno jednostavniji i kraći.

Postavimo jednadžbe čvorova:

$$(1)\ U_1\cdot\left(\frac{1}{R_1}+sC\right)-U_2\cdot sC=\frac{U_g}{R_1}-I_1$$

$$(2)\ -U_1\cdot sC+U_2\cdot\left(\frac{1}{R_2}+sC\right)=-I_2$$

$$\begin{aligned}U_1&=sL_1\cdot I_1+sM\cdot I_2\\ U_2&=sM\cdot I_1+sL_2\cdot I_2\end{aligned}\quad\text{(jednadžbe vezanih induktiviteta)}$$

(2 boda)

Uvrstimo vrijednosti elemenata:

$$(1)\ U_1\cdot(1+s)-U_2\cdot s=\frac{1}{s+1}-I_1$$

$$(2)\ -U_1\cdot s+U_2\cdot(1+s)=-I_2$$

$$(3)\ U_1=s\cdot I_1+s\cdot I_2$$

$$(4)\ U_2=s\cdot I_1+s\cdot I_2$$

(1 bod)

$$(3),\,(4)\Rightarrow U_1=U_2$$

$$(1)\ I_1=\frac{1}{s+1}-U_1\cdot(1+s)+U_1\cdot s=\frac{1}{s+1}-U_1$$

$$(2)\ I_2=-U_2$$

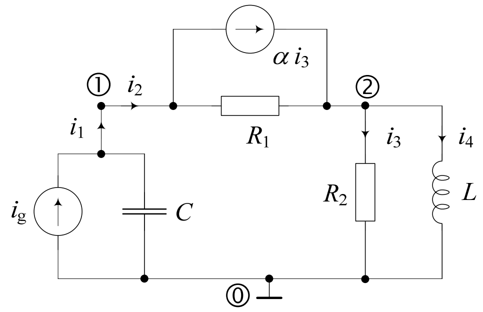
$$(4)\ U_2=s\cdot\overbrace{\left(\frac{1}{s+1}-U_1\right)}^{I_1}+s\cdot I_2=s\cdot\left(\frac{1}{s+1}-U_2\right)-s\cdot U_2=\frac{s}{s+1}-2s\cdot U_2$$

$$U_2(1+2s)=\frac{s}{s+1}$$

$$U_2(s)=\frac{s}{(s+1)(2s+1)}=\frac{s}{2(s+1)(s+1/2)}=\frac{A}{s+1}+\frac{B}{s+1/2}\Rightarrow A=1,\quad B=-\frac{1}{2}$$ (1 bod)

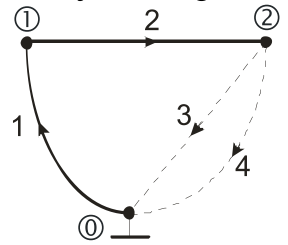
$$\Rightarrow u_2(t)=\left(e^{-t}-\frac{1}{2}e^{-\frac{t}{2}}\right)\cdot S(t)$$ (1 bod)

2. Za električni krug prikazan slikom i pridružene orijentacije grana i čvorove napisati: a) reduciranu matricu incidencija **A**, temeljnu spojnu matricu **S** i temeljnu rastavnu matricu **Q**; b) matricu admitancija grana $\mathbf{Y}_b$; c) vektor početnih uvjeta i nezavisnih strujnih izvora grana $\mathbf{I}_{0b}$; d) matricu admitancija čvorova $\mathbf{Y}_v$; e) vektor početnih uvjeta i nezavisnih strujnih izvora čvorova $\mathbf{I}_{0v}$.



Rješenje:

Orijentirani graf:



a) reducirana matrica incidencija **A**, temeljna spojna matrica **S**, temeljna rastavna matrica **Q**: (1 bod)

Reducirana matrica incidencija:

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

Temeljna spojna matrica:

$$\mathbf{S} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Temeljna rastavna matrica:

$$\mathbf{Q} = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & -1 \end{bmatrix}$$

Naponsko – strujne relacije grana: $\mathbf{I}_b = \mathbf{Y}_b \cdot \mathbf{U}_b + \mathbf{I}_{0b}$

$$I_1 = sC \cdot U_1 + I_g(s)$$

$$I_2 = \frac{1}{R_1} \cdot U_2 + \alpha \cdot I_3 = \frac{1}{R_1} \cdot U_2 + \frac{\alpha}{R_2} \cdot U_3$$

$$I_3 = \frac{1}{R_2} \cdot U_3$$

$$I_4 = \frac{1}{sL} \cdot U_4$$

Sustav jednadžbi grana u matričnom obliku:

$$I_b = \begin{bmatrix} sC & 0 & 0 & 0 \\ 0 & \frac{1}{R_1} & \frac{\alpha}{R_2} & 0 \\ 0 & 0 & \frac{1}{R_2} & 0 \\ 0 & 0 & 0 & \frac{1}{sL} \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} + \begin{bmatrix} I_g \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

b) matrica admitancija grana $\mathbf{Y}_b$: (1 bod)

$$\mathbf{Y}_b = \begin{bmatrix} sC & 0 & 0 & 0 \\ 0 & \frac{1}{R_1} & \frac{\alpha}{R_2} & 0 \\ 0 & 0 & \frac{1}{R_2} & 0 \\ 0 & 0 & 0 & \frac{1}{sL} \end{bmatrix}$$

c)vektor početnih uvjeta i nezavisnih strujnih izvora grana $\mathbf{I}_{0b}$: (1 bod)

$$\mathbf{I}_{0b} = \begin{bmatrix} I_g \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Sustav jednadžbi čvorova

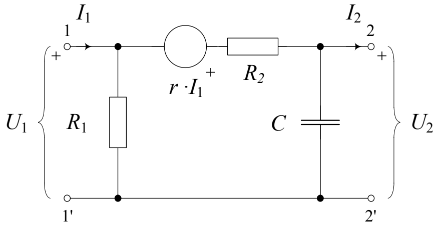
d) matrica $\mathbf{Y}_v$: (1 bod)

$$\mathbf{Y}_v = \mathbf{A} \cdot \mathbf{Y}_b \cdot \mathbf{A}^T = \begin{bmatrix} sC + \frac{1}{R_1} & \frac{\alpha}{R_2} - \frac{1}{R_1} \\ -\frac{1}{R_1} & \frac{1}{R_1} + \frac{1}{sL} + \frac{1-\alpha}{R_2} \end{bmatrix}$$

e) matrica $\mathbf{I}_{0v}$: (1 bod)

$$\mathbf{I}_{0v} = -\mathbf{A} \cdot \mathbf{I}_{0b} = \begin{bmatrix} I_g \\ 0 \end{bmatrix}$$

3. Za četveropol na slici izračunati: *a*) prijenosne [*a*]-parametre i napisati ih u matričnom obliku. *b*) Ako je na izlaz četveropola spojena impedancija zaključenja $Z_2=sL$ izračunati ulaznu impedanciju četveropola $Z_{ul1}(s)$. *c*) Da li je četveropol: recipročan, simetričan? Obrazložiti odgovore. Zadane su normalizirane vrijednosti $R_1=R_2=1$, $C=1$, $L=1$, $r=2$.



Rješenje:

Napomena: ako se odmah uvrste numeričke vrijednosti (što je ovdje učinjeno i jednako se priznaje za točno rješenje) tada je postupak znatno jednostavniji i kraći.

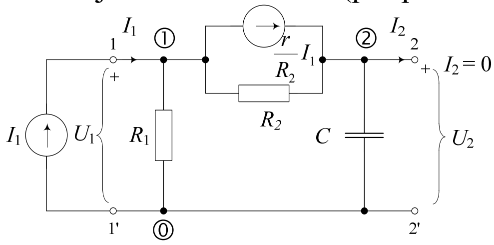
a) izračun [*a*]-parametara

$$U_1 = A\cdot U_2 + B\cdot I_2$$
$$I_1 = C\cdot U_2 + D\cdot I_2$$
prijenosne jednadžbe četveropola

$I_2 = 0$ (na prilazu 2-2' prazni hod)

$$A = \left.\frac{U_1}{U_2}\right|_{I_2=0} \quad ; \quad C = \left.\frac{I_1}{U_2}\right|_{I_2=0}$$

Izračunajmo[*a*]-parametre pomoću jednadžbi čvorova (pretpostavimo izvor I_1):



$$(1)\ U_1\cdot\left(\frac{1}{R_1}+\frac{1}{R_2}\right) - U_2\cdot\frac{1}{R_2} = I_1 - I_1\cdot\frac{r}{R_2}$$

$$(2)\ -U_1\cdot\frac{1}{R_2} + U_2\cdot\left(sC+\frac{1}{R_2}\right) = I_1\cdot\frac{r}{R_2}$$

uvrstimo vrijednosti elemenata:

$$(1)\ U_1\cdot(1+1) - U_2\cdot 1 = I_1 - I_1\cdot 2 \Rightarrow U_1\cdot 2 - U_2 = -I_1 \Rightarrow U_1 = \frac{U_2}{2} - \frac{I_1}{2}$$

$$(2)\ -U_1\cdot 1 + U_2\cdot(s+1) = I_1\cdot 2 \Rightarrow -U_1 + U_2\cdot(s+1) = I_1\cdot 2 \Rightarrow I_1 = -\frac{U_1}{2} + U_2\frac{s+1}{2}$$

Supstitucija I_1: (2) → (1) $\Rightarrow U_1 = \frac{U_2}{2} - \frac{1}{2}\left(-\frac{U_1}{2} + U_2\frac{s+1}{2}\right) \Big/ \cdot 4$

$$4U_1 = 2U_2 - \left[-U_1 + U_2(s+1)\right]$$

$$3U_1 = 2U_2 - U_2 s - U_2 \quad \Rightarrow A = \left.\frac{U_1}{U_2}\right|_{I_2=0} = \frac{1-s}{3}$$

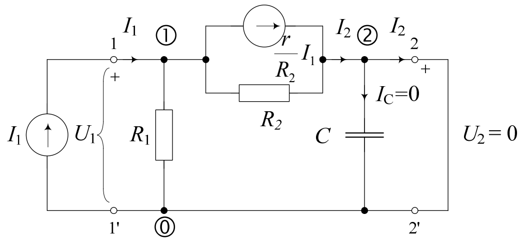
Supstitucija U_1: (1) → (2) ⇒ $I_1 = -\frac{1}{2}\left(\frac{U_2}{2} - \frac{I_1}{2}\right) + U_2\frac{s+1}{2} \Big/ \cdot 4$

$$4I_1 = -(U_2 - I_1) + U_2(2s+2)$$

$$3I_1 = -U_2 + 2U_2 + 2sU_2 \Rightarrow C = \left.\frac{I_1}{U_2}\right|_{I_2=0} = \frac{1+2s}{3}$$

$U_2 = 0$ (na prilazu 2-2' kratki spoj)

$$B = \left.\frac{U_1}{I_2}\right|_{U_2=0} \quad ; \quad D = \left.\frac{I_1}{I_2}\right|_{U_2=0}$$



$$(1)\ U_1 \cdot \frac{1}{R_1} = I_1 - I_2;$$

$$(2)\ I_2 = \frac{r}{R_2}I_1 + \frac{U_1}{R_2} \Rightarrow U_1 = R_2 I_2 - rI_1$$

uvrstimo vrijednosti elemenata:

(1) $U_1 = I_1 - I_2$;

(2) $U_1 = I_2 - 2I_1$

$$(2) \to (1) \Rightarrow I_2 - 2I_1 = I_1 - I_2 \Rightarrow 2I_2 = 3I_1 \Rightarrow D = \left.\frac{I_1}{I_2}\right|_{U_2=0} = \frac{2}{3}$$

$$\to (2) \Rightarrow U_1 = I_2 - 2I_1 = I_2 - 2\frac{2}{3}I_2 = -\frac{1}{3}I_2 \Rightarrow B = \left.\frac{U_1}{I_2}\right|_{U_2=0} = -\frac{1}{3}$$

Matrica prijenosnih [a]-parametara glasi: $[a] = \frac{1}{3}\begin{bmatrix} 1-s & -1 \\ 2s+1 & 2 \end{bmatrix}$

(do sada: maksimum 3 boda, svaki neispravan parametar 1 bod manje)

b) $Z_{ul1}(s) = \frac{U_1}{I_1} = \frac{AU_2 + BI_2}{CU_2 + DI_2} = \frac{AU_2/I_2 + B}{CU_2/I_2 + D} = \frac{AZ_2 + B}{CZ_2 + D}$; $Z_2 = \frac{U_2}{I_2} = sL$

$$Z_{ul1}(s) = \frac{(1-s)s - 1}{(2s+1)s + 2} = \frac{-s^2 + s - 1}{2s^2 + s + 2}$$ (1 bod)

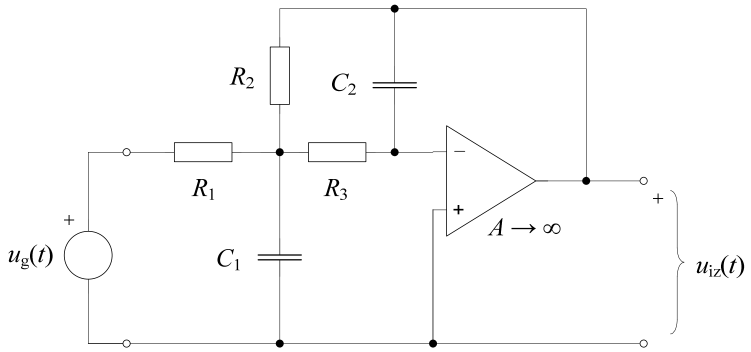
c) Da li je četveropol recipročan ? Ne, jer za recipročnost mora vrijediti det[a]=1.

$$\det[a] = \begin{vmatrix} \frac{1-s}{3} & -\frac{1}{3} \\ \frac{2s+1}{3} & \frac{2}{3} \end{vmatrix} = \frac{2(1-s)}{9} + \frac{2s+1}{9} = \frac{3}{9} = \frac{1}{3} \neq 1$$

To očigledno ne vrijedi, a razlog tomu je zavisni izvor.

Da li je četveropol simetričan ? Ne, jer za simetričnost mora vrijediti A=D. (1 bod)

4. Zadan je aktivni filtar prikazan slikom i njegova prijenosna funkcija $T(s)=U_{iz}(s)/U_g(s)$. *a*) Usporedbom s odgovarajućim općim oblikom prijenosne funkcije filtra 2. stupnja odrediti parametre k, ω_0, Q. O kojem se tipu filtra radi (NP, VP, PP ili PB)? *b*) Ako su zadane normalizirane vrijednosti parametara $\omega_0=1$, $Q=1/\sqrt{2}$ i $|k|=1$ te ako je $R_2=R_3=1$, izračunati normalizirane vrijednosti otpora R_1 i kapaciteta C_1 i C_2. *c*) Prikazati raspored polova i nula u kompleksnoj ravnini. *d*) Nacrtati amplitudno-frekvencijsku karakteristiku.



$$T(s)=\frac{U_{iz}(s)}{U_g(s)}=-\frac{R_2}{s^2R_1R_2R_3C_1C_2+s\left[C_2R_1(R_2+R_3)+C_2R_2R_3\right]+R_1}$$

<u>Rješenje:</u>

a) $T(s)=\dfrac{k\cdot\omega_0^{\,2}}{s^2+\dfrac{\omega_0}{Q}\cdot s+\omega_0^{\,2}}$ Opći oblik NP

(uobičajeno je kod el. filtara da je pojačanje k zadano s apsolutnom vrijednosti)
prepišimo $T(s)$ tako da najvišu potenciju od s u nazivniku množi jedinica

$$T(s)=\frac{U_{iz}(s)}{U_g(s)}=\frac{-\dfrac{R_2}{R_1}\cdot\dfrac{R_1}{R_1R_2R_3C_1C_2}}{s^2+s\cdot\dfrac{C_2R_1(R_2+R_3)+C_2R_2R_3}{R_1R_2R_3C_1C_2}+\dfrac{R_1}{R_1R_2R_3C_1C_2}}$$

-o kojem se tipu filtra radi (NP, VP, PP ili PB)? $\Rightarrow$ NP (niski propust)
-parametri k, ω_0, Q: (1 bod)

$$\omega_0=\frac{1}{\sqrt{R_2R_3C_1C_2}}\qquad \frac{\omega_0}{Q}=\frac{C_2R_1(R_2+R_3)+C_2R_2R_3}{R_1R_2R_3C_1C_2}=\frac{R_1R_2+R_1R_3+R_2R_3}{R_1R_2R_3C_1}$$

$$\Rightarrow\qquad Q=\frac{R_2R_3C_1C_2}{C_2(R_2+R_3)+C_2R_2R_3/R_1}\omega_0=\frac{\sqrt{R_2R_3C_1C_2}}{C_2(R_2+R_3)+C_2R_2R_3/R_1},\qquad k=\frac{R_2}{R_1}$$

b) ako su zadane vrijednosti parametara $\omega_0=1$, $Q=1/\sqrt{2}$ i $k=1$ te ako je $R_2=R_3=1$, izračunati normalizirane vrijednosti otpora R_1 i kapaciteta C_1 i C_2. (2 boda)

uz $R_2=R_3=1\Rightarrow\omega_0=\dfrac{1}{\sqrt{C_1C_2}}=1;\Rightarrow C_1=\dfrac{1}{C_2}$ (*)

$$Q=\frac{\sqrt{R_2R_3C_1C_2}}{C_2(R_2+R_3)+k\cdot C_2R_3}=\frac{\sqrt{C_1C_2}}{2C_2+k\cdot C_2}=\sqrt{\frac{C_1}{C_2}}\cdot\frac{1}{2+k};\ Q=\frac{1}{3}\cdot\sqrt{\frac{C_1}{C_2}}\Rightarrow\frac{C_1}{C_2}=9Q^2=\frac{9}{2}\qquad(**)$$

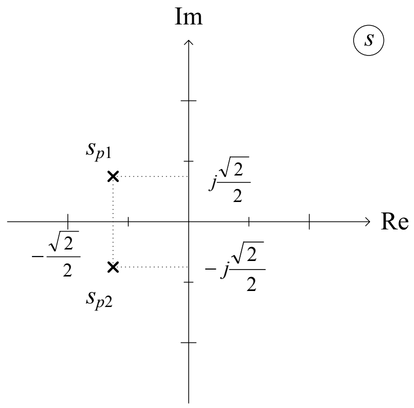
$$k=\frac{R_2}{R_1}=1\Rightarrow R_1=R_2=1;\ (*),(**)\Rightarrow C_1^2=\frac{9}{2}\ \Rightarrow\ \boxed{C_1=\frac{3}{\sqrt{2}},\ C_2=\frac{\sqrt{2}}{3};\ R_1=R_2=R_3=1.}$$

c) raspored polova i nula u kompleksnoj ravnini: (1bod)

$$T(s) = \frac{1}{s^2 + \sqrt{2} \cdot s + 1}$$

nule $\quad s_{o1,2} = \infty$ (dvije nule su u neizmjerno)

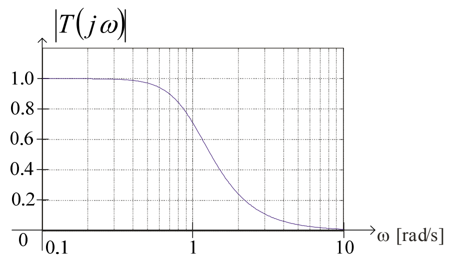
polovi $\quad s^2 + \sqrt{2} \cdot s + 1 = 0 \quad \Rightarrow \quad s_{p1,2} = -\frac{\sqrt{2} \pm \sqrt{2-4}}{2} = -\frac{\sqrt{2}}{2} \pm j\frac{\sqrt{2}}{2}$



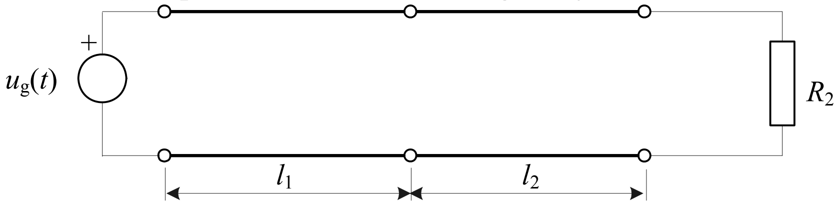
d) amplitudno-frekvencijska karakteristika: (1bod)

$$T(j\omega) = \frac{1}{-\omega^2 + j\omega \cdot \sqrt{2} + 1} \quad \Rightarrow$$

$$|T(j\omega)| = \frac{1}{\sqrt{(1-\omega^2)^2 + (\omega \cdot \sqrt{2})^2}} = \frac{1}{\sqrt{1 - 2\omega^2 + \omega^4 + 2\omega^2}} = \frac{1}{\sqrt{1+\omega^4}}$$



5. Na ulazu linije bez gubitaka, duljine $l_1=\lambda_1/2$, s primarnim parametrima L_1=0.2 mH/km i C_1=80nF/km, djeluje napon $u_g(t)=2\cos(10^6 t)$. Na izlaz je priključena linija bez gubitaka duljine $l_2=\lambda_2/4$, zadana sa L_2=4,5 mH/km i C_2=800 nF/km, zaključena otporom R_2. Izračunati: a) valne impedancije obiju linija Z_{01} i Z_{02}, te koeficijente prijenosa γ_1 i γ_2; b) duljine obiju linija l_1 i l_2; c) vrijednost otpora R_2 da bi prva linija bila prilagođena na svome izlazu; d) napon $u_{II}(0,t)$ na ulazu; e) napon $u_{II}(l_2,t)$ na izlazu druge linije.



Rješenje:

a) $Z_{01}=\sqrt{L_1/C_1}=\sqrt{2\cdot10^{-4}/8\cdot10^{-8}}=50\Omega$, $Z_{02}=\sqrt{L_2/C_2}=\sqrt{4{,}5\cdot10^{-3}/8\cdot10^{-7}}=75\Omega$

$$\gamma_{01}=j\omega\sqrt{L_1\cdot C_1}=j10^6\sqrt{2\cdot10^{-4}\cdot8\cdot10^{-8}}=j4\text{ rad/s/km}\Rightarrow\beta_1=4\text{ rad/s/km}$$

$$\gamma_{02}=j\omega\sqrt{L_2\cdot C_2}=j10^6\sqrt{4{,}5\cdot10^{-3}\cdot8\cdot10^{-7}}=j60\text{ rad/s/km}\Rightarrow\beta_2=60\text{ rad/s/km}$$

(1 bod)

b) $$l_1=\frac{\lambda_1}{2}=\frac{2\pi}{2\beta_1}=\frac{\pi}{\omega\sqrt{L_1C_1}}=\frac{\pi}{10^6\sqrt{2\cdot10^{-4}\cdot8\cdot10^{-8}}}=\frac{\pi}{4}km$$

$$l_2=\frac{\lambda_2}{4}=\frac{2\pi}{4\beta_2}=\frac{\pi}{2\omega\sqrt{L_2C_2}}=\frac{\pi}{2\cdot10^6\sqrt{4{,}5\cdot10^{-3}\cdot8\cdot10^{-7}}}=\frac{\pi}{120}km$$ (1 bod)

c) $$\beta_1l_1=\beta_1\frac{\lambda_1}{2}=\beta_1\frac{2\pi}{2\beta_1}=\pi \qquad \beta_2l_2=\beta_2\frac{\lambda_2}{4}=\beta_2\frac{2\pi}{4\beta_2}=\frac{\pi}{2}$$

$$Z_{ul2}=\frac{R_2ch(\gamma_2l_2)+Z_{02}sh(\gamma_2l_2)}{\frac{R_2}{Z_{02}}sh(\gamma_2l_2)+ch(\gamma_2l_2)}=Z_{02}\frac{R_2\cos(\beta_2l_2)+jZ_{02}\sin(\beta_2l_2)}{R_2j\sin(\beta_2l_2)+Z_{02}\cos(\beta_2l_2)}=\frac{Z_{02}^2}{R_2}$$

$$Z_{ul2}=Z_{01}=\frac{Z_{02}^2}{R_2}\qquad\Rightarrow\qquad R_2=\frac{Z_{02}^2}{Z_{01}}=\frac{75^2}{50}=112{,}5\Omega$$ (1 bod)

d) $$U_2(0)=U_1(0)e^{-j\beta_1l_1}=U_1(0)e^{-j\pi}=-U_1(0)\qquad\Rightarrow\qquad u_2(0,t)=-2\cos(\omega t)$$ (1 bod)

$$I_2(0)=\frac{U_2(0)}{Z_{ul2}}=\frac{U_2(0)}{Z_{02}^2}R_2$$

e) $$U_2(l)=U_2(0)ch(j\beta_2l_2)-I_2(0)Z_{02}sh(j\beta_2l_2)=-U_1(0)\left(\cos\left(\frac{\pi}{2}\right)-j\frac{R_2}{Z_{02}}\sin\left(\frac{\pi}{2}\right)\right)=1{,}5jU_1(0)$$

$$u_2(l,t)=3\cos\left(\omega t+\frac{\pi}{2}\right)$$ (1 bod)