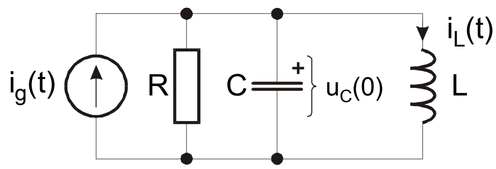
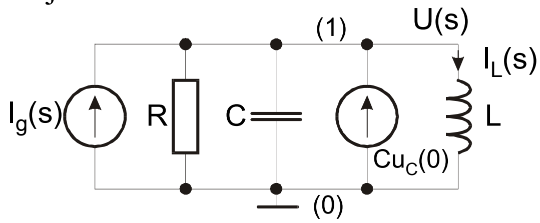
ZAVRŠNI ISPIT IZ ELEKTRIČNIH KRUGOVA

1. U električnom krugu na slici odrediti odziv struje kroz induktivitet $i_L(t)$ ako su zadane normalizirane vrijednosti elemenata: C=2, R=1/2, L=1/4, početni uvjeti $i_L(0)$=0, $u_C(0)$=1 te struja pobude $i_g(t)$=$e^{-2t}\cdot S(t)$.



<u>Rješenje:</u>
Primjena Laplaceove transformacije:



$$U(s)\cdot\left(sC+\frac{1}{R}+\frac{1}{sL}\right)-C\cdot u_C(0)-I_g(s)=0$$

$$U(s)=\frac{I_g(s)+C\cdot u_C(0)}{sC+\frac{1}{R}+\frac{1}{sL}}$$

$$I_L(s)=U(s)\cdot\frac{1}{sL}=\frac{I_g(s)+C\cdot u_C(0)}{s^2LC+s\frac{L}{R}+1}$$

$$i_g(t)=e^{-2t}\cdot S(t)\ \ \Rightarrow\ \ I_g(s)=\frac{1}{s+2}$$

$$I_L(s)=\frac{\frac{1}{s+2}+2}{\frac{s^2}{2}+\frac{s}{2}+1}=\frac{2(1+2s+4)}{(s+2)(s^2+s+2)}=\frac{4s+10}{(s+2)(s^2+s+2)}$$

Slijedi rastav na parcijalne razlomke izraza $\dfrac{4s+10}{(s+2)(s^2+s+2)}=\dfrac{A}{s+2}+\dfrac{Bs+C}{s^2+s+2}$

$$As^2+As+2A+Bs^2+2Bs+Cs+2C=4s+10$$

$$(A+B)s^2+(A+2B+C)s+(2A+2C)=4s+10$$

(1) $\quad A+B=0$

(2) $\quad A+2B+C=4$

(3) $\quad 2(A+C)=10$

(1)$\Rightarrow$ $B=-A$, (3)$\Rightarrow$ $C=5-A$, (2) $\Rightarrow$ $A+2B+5-A=4$

$$\Rightarrow \quad 2B = -1 \Rightarrow \quad B = -\frac{1}{2},\ A = \frac{1}{2},\ C = 5 - A = 5 - \frac{1}{2} = \frac{10}{2} - \frac{1}{2} = \frac{9}{2}$$

$$\Rightarrow I_L(s) = \frac{4s+10}{(s+2)(s^2+s+2)} = \frac{1}{2}\cdot\frac{1}{s+2} - \frac{1}{2}\cdot\frac{s}{s^2+s+2} + \frac{9}{2}\cdot\frac{1}{s^2+s+2} =$$

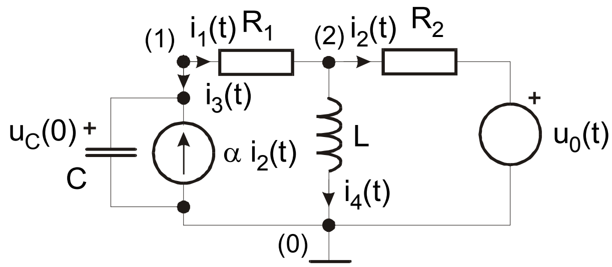
$$= \frac{1}{2}\cdot\frac{1}{s+2} - \frac{1}{2}\cdot\frac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} + \frac{9}{2}\cdot\frac{1}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} =$$

$$= \frac{1}{2}\cdot\frac{1}{s+2} - \frac{1}{2}\cdot\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} + \frac{19}{4}\cdot\frac{\frac{2}{\sqrt{7}}\cdot\frac{\sqrt{7}}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} =$$

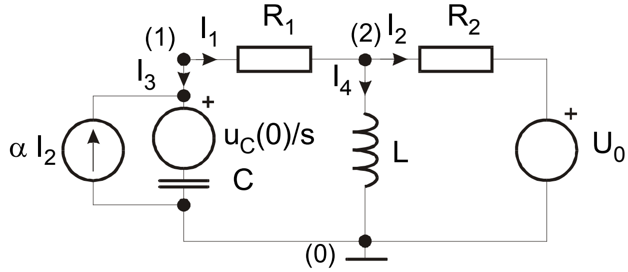
$$= \frac{1}{2}\cdot\frac{1}{s+2} - \frac{1}{2}\cdot\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} + \frac{19}{2\sqrt{7}}\cdot\frac{\frac{\sqrt{7}}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} =$$

$$i_L(t) = \mathscr{L}\{I_L(s)\} \quad \Rightarrow \quad \underline{i_L(t) = \left(\frac{1}{2}e^{-2t} - \frac{1}{2}e^{-\frac{t}{2}}\cos\frac{\sqrt{7}}{2}t + \frac{19}{2\sqrt{7}}e^{-\frac{t}{2}}\sin\frac{\sqrt{7}}{2}t\right)\cdot S(t)}$$

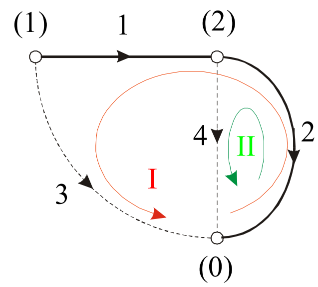
2. Zadan je električni krug prema slici. Poštujući oznake grana i čvorišta nacrtati pripadni orijentirani graf. Napisati naponsko-strujne jednadžbe grana u matričnom obliku te ispisati matricu impedancija grana $\mathbf{Z}_b$ i vektor izvora grana $\mathbf{U}_{0b}$. Napisati temeljni sustav jednadžbi petlji, odnosno odrediti matrice impedancija petlji $\mathbf{Z}_p$ i vektor izvora petlji $\mathbf{U}_{0p}$. (Uputa: grane 1 i 2 su grane stabla.)



Rješenje: Uz primjenu Laplaceove transformacije :



Električni krug



orijentirani graf i temeljni sustav petlji

Spojna matrica: $S = \begin{array}{c} \\ \mathrm{I} \\ \mathrm{II} \end{array}\!\!\begin{array}{c} \overbrace{\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}}^{grane} \\ \begin{bmatrix} -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$

$\uparrow$

temeljne petlje

Naponsko strujne relacije grana:

$$\begin{aligned} U_1 &= I_1 R_1 \\ U_2 &= I_2 R_2 + U_0(s) \\ U_3 &= (I_3 + \alpha I_2)\frac{1}{sC} + \frac{u_C(0)}{s} \\ U_4 &= I_4 sL \end{aligned} \quad \Rightarrow \quad \begin{aligned} U_1 &= I_1 R_1 \\ U_2 &= I_2 R_2 + U_0(s) \\ U_3 &= \frac{1}{sC} I_3 + \alpha \frac{1}{sC} I_2 + \frac{u_C(0)}{s} \\ U_4 &= I_4 sL \end{aligned}$$

Naponsko-strujne relacije grana u matričnom obliku:

$$\underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}}_{\mathbf{U}_b} = \underbrace{\begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ 0 & \dfrac{\alpha}{sC} & \dfrac{1}{sC} & 0 \\ 0 & 0 & 0 & sL \end{bmatrix}}_{\mathbf{Z}_b} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}}_{\mathbf{I}_b} + \underbrace{\begin{bmatrix} 0 \\ U_0 \\ \dfrac{u_C(0)}{s} \\ 0 \end{bmatrix}}_{\mathbf{U}_{0b}}$$

Temeljni sustav jednadžbi petlji :

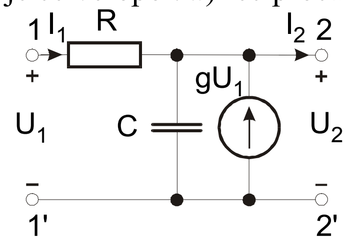
$$\mathbf{Z}_p = \mathbf{S}\mathbf{Z}_b\mathbf{S}^T = \begin{bmatrix} -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ 0 & \dfrac{\alpha}{sC} & \dfrac{1}{sC} & 0 \\ 0 & 0 & 0 & sL \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} -R_1 & -R_2 + \dfrac{\alpha}{sC} & \dfrac{1}{sC} & 0 \\ 0 & -R_2 & 0 & sL \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 + R_2 - \dfrac{\alpha}{sC} + \dfrac{1}{sC} & R_2 - \dfrac{\alpha}{sC} \\ R_2 & R_2 + sL \end{bmatrix} =$$

$$= \begin{bmatrix} R_1 + R_2 + \dfrac{1-\alpha}{sC} & R_2 - \dfrac{\alpha}{sC} \\ R_2 & R_2 + sL \end{bmatrix}$$

$$\mathbf{U}_{0p} = -\mathbf{S}\mathbf{U}_{0b} = -\begin{bmatrix} -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ U_0 \\ \dfrac{u_C(0)}{s} \\ 0 \end{bmatrix} = \begin{bmatrix} U_0 - \dfrac{u_C(0)}{s} \\ U_0 \end{bmatrix}$$

3. Za četveropol prikazan slikom izračunati z-parametre i napisati matricu z-parametara. Pomoću poznatih z-parametara izračunati naponsku prijenosnu funkciju četveropola $T(s)=U_2(s)/U_1(s)$ ako je izlazni prilaz (2–2') otvoren. Da li je četveropol: *a*) recipročan, *b*) simetričan ? Obrazložiti odgovor.



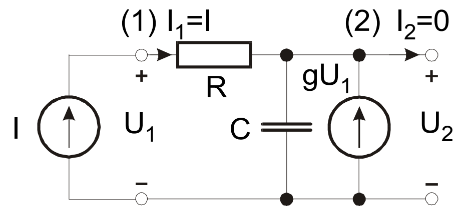
Rješenje:

[z]-parametri:

$$U_1 = z_{11}I_1 - z_{12}I_2$$

$$U_2 = z_{21}I_1 - z_{22}I_2$$

a) $I_2=0$



$$(1) \qquad U_1\frac{1}{R} - U_2\frac{1}{R} = I$$

$$(2) \qquad -U_1\frac{1}{R} + U_2\left(\frac{1}{R}+sC\right) = gU_1$$

$$(1) \Rightarrow U_1 = U_2 + I\cdot R, \quad U_2 = U_1 - I\cdot R$$

$$(2) \Rightarrow -U_1\frac{1}{R} + (U_1 - I\cdot R)\left(\frac{1}{R}+sC\right) = gU_1$$

$$-U_1\frac{1}{R} + U_1\left(\frac{1}{R}+sC\right) - gU_1 = I\cdot R\left(\frac{1}{R}+sC\right)$$

$$U_1(sC-g) = I(1+sRC), \quad I = I_1$$

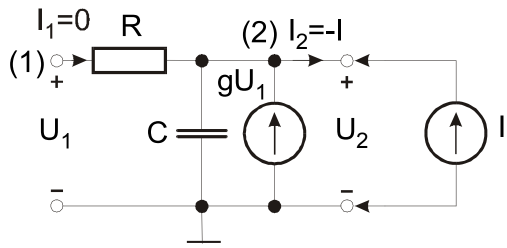
$$z_{11} = \left.\frac{U_1}{I_1}\right|_{I_2=0} = \frac{1+sRC}{sC-g} = \frac{R+\dfrac{1}{sC}}{1-\dfrac{g}{sC}}$$

$$(2) \Rightarrow -U_1\left(\frac{1}{R}+g\right) + U_2\left(\frac{1}{R}+sC\right) = 0 \;\Rightarrow -(U_2 + I\cdot R)\left(\frac{1}{R}+g\right) + U_2\left(\frac{1}{R}+sC\right) = 0$$

$$-U_2\frac{1}{R} - U_2 g - I - I\cdot R\cdot g + U_2\frac{1}{R} + sCU_2 = 0 \Rightarrow U_2(sC-g) = I(1+Rg)$$

$$z_{21} = \left.\frac{U_2}{I_1}\right|_{I_2=0} = \frac{1+Rg}{sC-g} = \frac{1}{sC}\left(1+g\frac{R+\dfrac{1}{sC}}{1-\dfrac{g}{sC}}\right) = g\frac{sRC+1}{sC-g} - R \quad \text{i slično}$$

b) I_1=0



$$U_2 sC = gU_1 + I$$

$$\underline{I_1 = 0 \rightarrow U_1 = U_2, \qquad I = -I_2}$$

$$U_2 sC = gU_2 - I_2$$

$$U_2 (sC - g) = -I_2, \;\; U_1 = U_2$$

$$z_{22} = -\left.\frac{U_2}{I_2}\right|_{I_1=0} = \frac{1}{sC_2 - g}$$

$$z_{12} = -\left.\frac{U_1}{I_2}\right|_{I_1=0} = \frac{1}{sC_2 - g}$$

$$[z] = \begin{bmatrix} \dfrac{1+sRC}{sC_2 - g} & -\dfrac{1}{sC_2 - g} \\ \dfrac{1+Rg}{sC_2 - g} & -\dfrac{1}{sC_2 - g} \end{bmatrix}$$

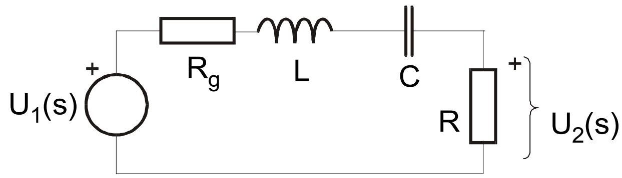
$$[z] = \begin{bmatrix} z_{11} & -z_{12} \\ z_{21} & -z_{22} \end{bmatrix}$$

Prijenosna funkcija napona:

$$T(s) = \frac{z_{21}}{z_{11}} = \left.\frac{\dfrac{U_2}{I_1}}{\dfrac{U_1}{I_1}}\right|_{I_2=0} = \left.\frac{U_2}{U_1}\right|_{I_2=0} = \frac{1+Rg}{1+sRC}$$

Odgovori na pitanja:
a) Četveropol nije recipročan jer sadrži naponsko ovisni strujni izvor i parametri z_{12} i z_{21} su stoga različiti;
b) Četveropol nije električki simetričan jer se parametri z_{11} i z_{22} razlikuju.

4. Za pojasno propusni električni filtar prikazan slikom izračunati naponsku prijenosnu funkciju $T(s)=U_2(s)/U_1(s)$, ako su zadane vrijednosti elemenata L=1mH, R_g=50Ω, C=100nF i R=150Ω. Izračunati i skicirati amplitudno-frekvencijsku karakteristiku $|T(j\omega)|$. Kolika je širina pojasa propuštanja B, Q-faktor i centralna frekvencija ω_0 , pojačanje u području propuštanja k te gornja i donja granična frekvencija ω_g i ω_d ?



Rješenje: Naponska prijenosna funkcija:

$$T(s)=\frac{U_2(s)}{U_1(s)}=\frac{Z_2}{Z_1+Z_2}$$

$$Z_1(s)=sL+\frac{1}{sC}+R_g,\quad Z_2(s)=R$$

$$T(s)=\frac{U_2(s)}{U_1(s)}=\frac{R}{sL+\frac{1}{sC}+R_g+R}=\frac{\frac{R}{L}\cdot s}{s^2+\frac{R+R_g}{L}\cdot s+\frac{1}{LC}}=\frac{\frac{R}{R+R_g}\cdot\frac{R+R_g}{L}\cdot s}{s^2+\frac{R+R_g}{L}\cdot s+\frac{1}{LC}}$$

$$T(s)=\frac{U_2(s)}{U_1(s)}=\frac{k\cdot\frac{\omega_p}{q_p}\cdot s}{s^2+\frac{\omega_p}{q_p}s+\omega_p^2}\quad\Rightarrow\quad\omega_p=\frac{1}{\sqrt{LC}},\quad k=\frac{R}{R+R_g}$$

$$\frac{\omega_p}{q_p}=\frac{R+R_g}{L}\quad\Rightarrow\quad q_p=\omega_p\cdot\frac{L}{R+R_g}=\frac{1}{\sqrt{LC}}\cdot\frac{L}{R+R_g}=\frac{1}{R+R_g}\cdot\sqrt{\frac{L}{C}}$$

$$T(s)=\frac{U_2(s)}{U_1(s)}=\frac{1.5\cdot10^5\cdot s}{s^2+2\cdot10^5\cdot s+10^{10}}$$

$$\Rightarrow\ T(j\omega)=\frac{1.5\cdot10^5\cdot j\omega}{-\omega^2+2\cdot10^5\cdot j\omega+10^{10}}$$

$$\Rightarrow\ |T(j\omega)|=\frac{1.5\cdot10^5\cdot\omega}{\sqrt{(10^{10}-\omega^2)^2+(2\cdot10^5\cdot\omega)^2}}$$

$$\omega_p=\sqrt{10^{10}}=10^5\text{ rad/s},\quad\frac{\omega_p}{q_p}=2\cdot10^5\quad\Rightarrow\quad q_p=\frac{10^5}{2\cdot10^5}=\frac{1}{2},\ k=\frac{3}{4}$$

Širina pojasa propuštanja $B=\frac{\omega_p}{q_p}=2\cdot10^5$ [rad/s]

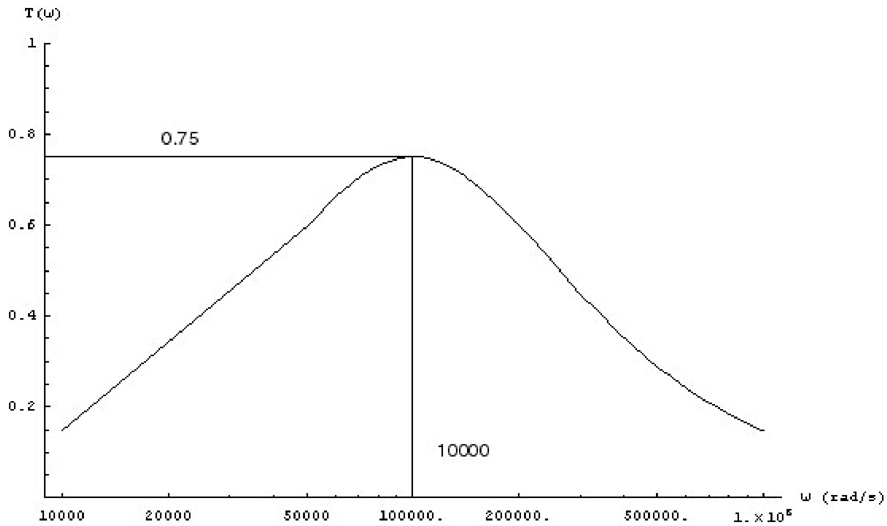
Gornja i donja granična frekvencija su:

$$\omega_{g,d} = \omega_p \sqrt{1+\frac{1}{4q_p^2}} \pm \frac{\omega_p}{2q_p} = 10^5 \sqrt{1+\frac{1}{4\cdot 0.25}} \pm \frac{2\cdot 10^5}{2} = 10^5(\sqrt{2}\pm 1)\ \text{[rad/s]}$$

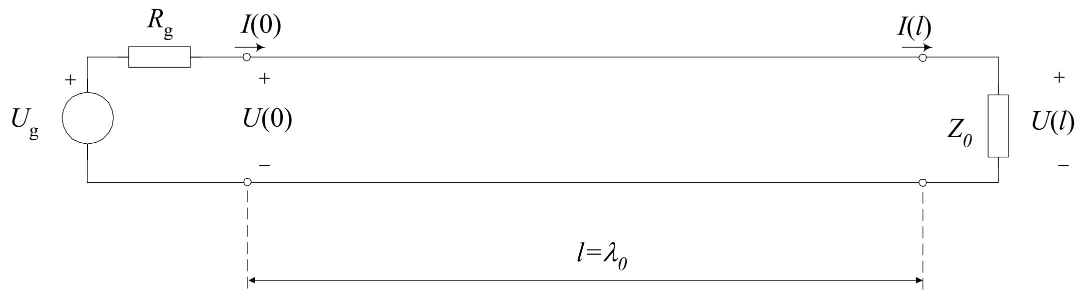
ω_g=241421 [rad/s], ω_d=41421 [rad/s]

$B=\omega_g-\omega_d$=241421–41421=20 000= $2\cdot 10^5$ [rad/s]

Amplitudno-frekvencijska karakteristika:



5. Zadana je linija bez gubitaka s L=10 μH/km i C=100 nF/km. Na ulaz linije je priključen naponski izvor $u_g(t)$ = 2 cos(ω_0t) u seriji s otporom R_g=50Ω. Duljina linije je $l=\lambda_0$, gdje je λ_0 valna duljina signala pri frekvenciji $\omega_0=10^6$. Izlaz linije je zaključen karakterističnom impedancijom.
   a) Odrediti karakterističnu impedanciju Z_0 i koeficijent prijenosa γ linije.
   b) Kolika je duljina linije u km?
   c) Odrediti napon i struju na polovici linije.
   d) Odrediti napon i struju na kraju linije.



Rješenje:

a) Linija bez gubitaka $\rightarrow$ $R=0$, $G=0$ $\Rightarrow$ $Z_0=\sqrt{L/C}$, $\gamma=s\sqrt{LC}$

Stac. sinusna pobuda $\rightarrow$ $s=j\omega$ $\Rightarrow$ $\gamma=j\omega\sqrt{LC}=j\beta$

$$Z_0=\sqrt{L/C}=\sqrt{10^{-5}/10^{-7}}=10\Omega$$

$$\gamma=j\omega\sqrt{LC}=j10^6\sqrt{10^{-5}\cdot10^{-7}}=j$$

b) $$l=\lambda_0=\frac{2\pi}{\beta}=\frac{2\pi}{\omega\sqrt{LC}}=2\pi=6{,}28\text{km}$$

c) $$U(x)=U(0)\cdot ch\gamma x-I(0)Z_0 sh\gamma x$$

$$I(x)=-\frac{U(0)}{Z_0}sh\gamma x+I(0)ch\gamma x$$

$$Z_{ul}=Z_0\ \Rightarrow\ U(0)=Z_0I(0)\quad U(0)=\frac{Z_0}{Z_0+R_g}U_g(s)=\frac{10\Omega}{10\Omega+50\Omega}U_g(s)=\frac{1}{6}U_g(s)$$

$$U(x)=U(0)\cdot(ch\gamma x-sh\gamma x)=U(0)\cdot e^{-\gamma x}$$

$$I(x)=\frac{U(0)}{Z_0}(-sh\gamma x+ch\gamma x)=\frac{U(0)}{Z_0}e^{-\gamma x}$$

$$U(l/2)=U(0)\cdot(ch(\gamma l/2)-sh(\gamma l/2))=U(0)\cdot e^{-\gamma l/2}=U(0)\cdot e^{-j\pi}=-U(0)$$

$$I(l/2)=\frac{U(0)}{Z_0}(-sh(\gamma l/2)+ch(\gamma l/2))=\frac{U(0)}{Z_0}e^{-\gamma l/2}=\frac{U(0)}{Z_0}e^{-j\pi}=-\frac{U(0)}{Z_0}$$

$$U(0)=\frac{1}{6}U_g(s)\quad \underline{u(l/2,t)=-\frac{1}{3}\cos(\omega_0 t)},\ \underline{i(l/2,t)=-\frac{1}{30}\cos(\omega_0 t)}$$

d) $$U(l)=U(0)\cdot(ch(\gamma l)-sh(\gamma l))=U(0)\cdot e^{-\gamma l}=U(0)\cdot e^{-j2\pi}=U(0)$$

$$I(l)=\frac{U(0)}{Z_0}(-sh(\gamma l)+ch(\gamma l))=\frac{U(0)}{Z_0}e^{-\gamma l}=\frac{U(0)}{Z_0}e^{-j2\pi}=\frac{U(0)}{Z_0}$$

$$\underline{u(l,t)=\frac{1}{3}\cos(\omega_0 t)},\ \underline{i(l,t)=\frac{1}{30}\cos(\omega_0 t)}$$