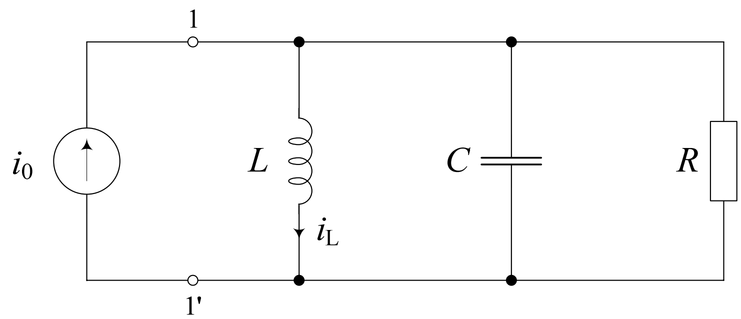
## DRUGI MEĐUISPIT IZ ELEKTRIČNIH KRUGOVA – Rješenja – 2008

1. Zadan je električni krug prema slici s pobudom $i_0(t)$. Odrediti: a) Ulaznu admitanciju $Y(s)$ gledano s priključnica 1 - 1'; b) Polove i nule $Y(s)$; c) Prijenosnu funkciju struje $H(s)=I_L(s)/I_0(s)$; d) Fazor odziva $I_L$, i odziv $i_L(t)$, ako su zadane vrijednosti elemenata: $R$ = 1, L = 1, C = 1/2 i $i_0(t)=2\cos\left(\sqrt{2}\cdot t\right)$




Rješenje:

$$U=\frac{I_0}{\frac{1}{sL}+sC+\frac{1}{R}}=\frac{I_0\cdot sL}{s^2\cdot LC+s\frac{L}{R}+1}=I_0\frac{s\frac{1}{C}}{s^2+\frac{s}{RC}+\frac{1}{LC}}$$

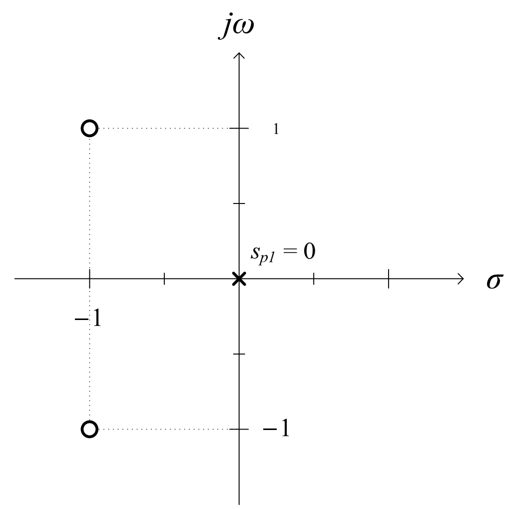
a) $Y(s)=\dfrac{I_0}{U}=C\dfrac{s^2+\frac{s}{RC}+\frac{1}{LC}}{s}$

b) Polovi i nule:

$$s_{01,2}=-\frac{1}{2RC}\pm\sqrt{\left(\frac{1}{2RC}\right)^2-\frac{1}{LC}}$$

$$s_{p1}=0$$

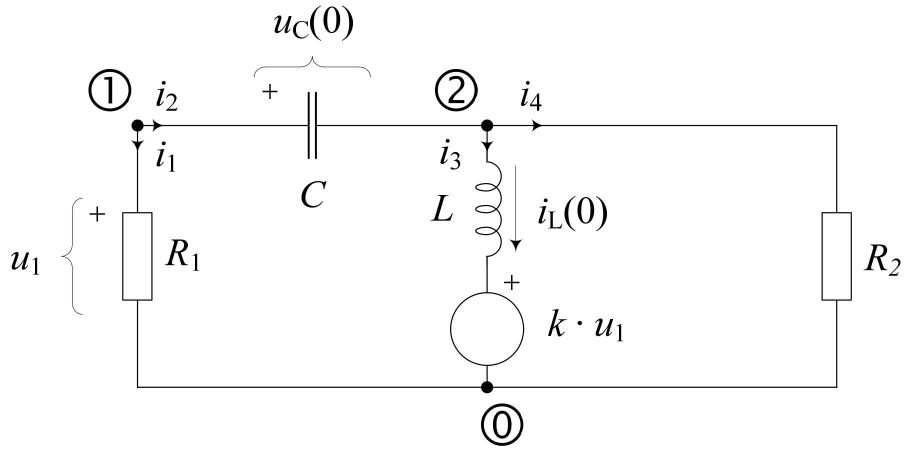
$$s_{p2}\to\infty$$




c) $I_L(s)=U\cdot\dfrac{1}{sL}=\dfrac{I_0\cdot\frac{s}{C}\cdot\frac{1}{sL}}{s^2+\frac{s}{RC}+\frac{1}{LC}}\Rightarrow H(s)=\dfrac{I_0}{I}=\dfrac{\frac{1}{LC}}{s^2+\frac{s}{RC}+\frac{1}{LC}}$

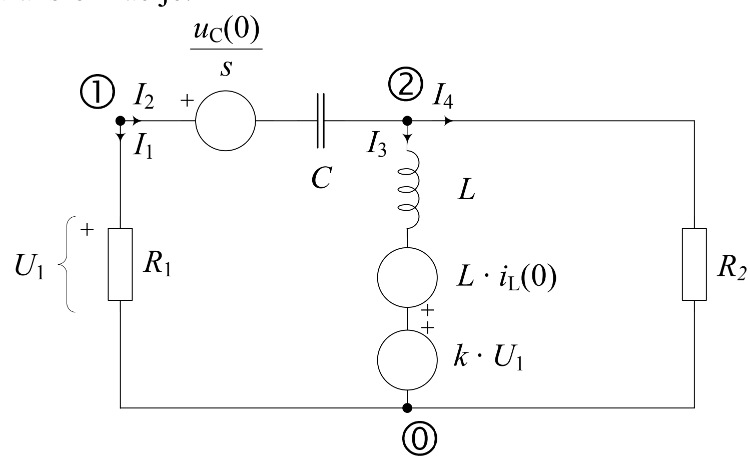
d) $I_0=2e^{j\phi}\Rightarrow I_L=\dfrac{I_0\cdot\frac{1}{LC}}{-\omega^2+\frac{j\omega}{RC}+\frac{1}{LC}}=\dfrac{2\cdot 2}{-2+j\cdot\sqrt{2}\cdot 2+2}=-j\sqrt{2}$, $i_L(t)=\sqrt{2}\cos\left(\sqrt{2}\cdot t-90°\right)$

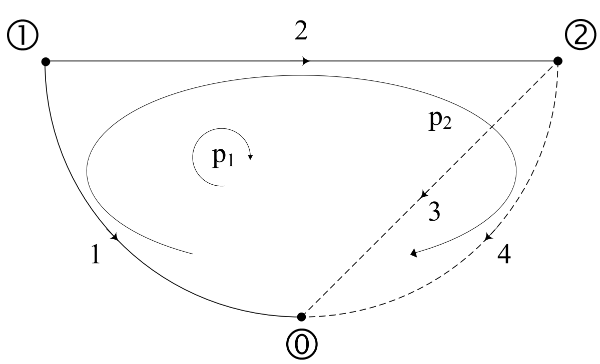
2. Za krug na slici i pridružene oznake čvorova i grana, napisati temeljni sustav jednadžbi petlji u matričnom obliku topološkom analizom (odrediti matrice $\mathbf{Z}_p$ i $\mathbf{U}_{0p}$ preko matrica $\mathbf{Z}_b$ i $\mathbf{U}_{0b}$). Matrica $\mathbf{Z}_b$ mora biti regularna. Nacrtati orijentirani graf. (Uputa: grane stabla: 1, 2).




Rješenje:

Primjena $\mathcal{L}$ − transformacije:




Orijentirani graf i temeljni sustav petlji

1, 2: stablene grane

3, 4: spone

Spojna matrica:

$$\mathbf{S} = \begin{array}{c} \\ p_1 \\ p_2 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cc:cc} -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right] \end{array}$$

Petljama $p_1$, $p_2$ teku struje petlji $I_{p_1}$, $I_{p_2}$

U-I jednadžbe grana:

$$U_1 = I_1 \cdot R_1$$
$$U_2 = I_2 \cdot \frac{1}{sC} + \frac{u_C(0)}{s}$$
$$U_3 = I_3 \cdot sL - L \cdot i_L(0) + k \cdot U_1$$
$$U_4 = I_4 \cdot R_2$$

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$$U_1 = I_1 \cdot R_1$$
$$U_2 = I_2 \cdot \frac{1}{sC} + \frac{u_C(0)}{s}$$
$$U_3 = I_1 \cdot k \cdot R_1 + I_3 \cdot sL - L \cdot i_L(0)$$
$$U_4 = I_4 \cdot R_2$$

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$$\underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}}_{\mathbf{U}_b} = \underbrace{\begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & \frac{1}{sC} & 0 & 0 \\ k \cdot R_1 & 0 & sL & 0 \\ 0 & 0 & 0 & R_2 \end{bmatrix}}_{\mathbf{Z}_b} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}}_{\mathbf{I}_b} + \underbrace{\begin{bmatrix} 0 \\ \frac{u_C(0)}{s} \\ -L \cdot i_L(0) \\ 0 \end{bmatrix}}_{\mathbf{U}_{0b}}$$

Da bi bila regularna, matrica impedancija grana $\mathbf{Z}_b$ ne smije sadržavati niti redak niti stupac sa svim nulama!

$$\mathbf{Z}_p = \mathbf{S} \cdot \mathbf{Z}_b \cdot \mathbf{S}^T = \begin{bmatrix} -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & \frac{1}{sC} & 0 & 0 \\ k \cdot R_1 & 0 & sL & 0 \\ 0 & 0 & 0 & R_2 \end{bmatrix} \cdot \mathbf{S}^T =$$

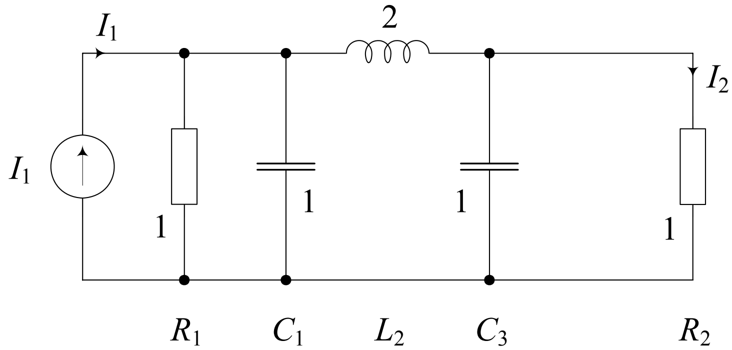
$$= \begin{bmatrix} -R_1 + k \cdot R_1 & \frac{1}{sC} & sL & 0 \\ -R_1 & \frac{1}{sC} & 0 & R_2 \end{bmatrix} \cdot \begin{bmatrix} -1 & -1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1(1-k) + \frac{1}{sC} + sL & R_1(1-k) + \frac{1}{sC} \\ R_1 + \frac{1}{sC} & R_1 + \frac{1}{sC} + R_2 \end{bmatrix}$$

$$\mathbf{U}_{0p} = -\mathbf{S} \cdot \mathbf{U}_{0b} = -\begin{bmatrix} -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \frac{u_C(0)}{s} \\ -L \cdot i_L(0) \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{u_C(0)}{s} + L \cdot i_L(0) \\ -\frac{u_C(0)}{s} \end{bmatrix}$$

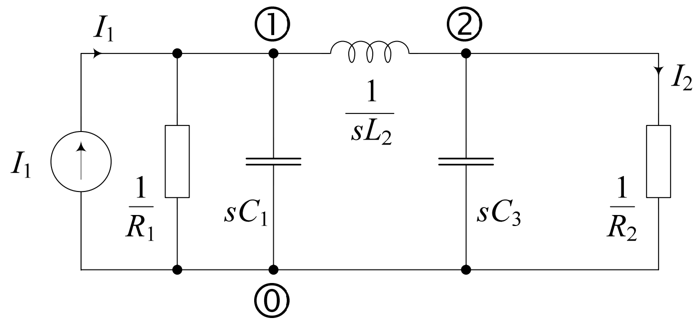
Vektor struja petlji: $\mathbf{I}_p = \begin{bmatrix} I_{p_1} \\ I_{p_2} \end{bmatrix}$

$\mathbf{Z}_p \cdot \mathbf{I}_p = \mathbf{U}_{0p}$ Temeljni sustav petlji u matričnom obliku.

3. Za mrežu na slici izračunati prijenosni omjer struja $H_i(s) = I_2(s)/I_1(s)$ ako su zadane normalizirane vrijednosti elemenata: $R_1$=$R_2$=1, $C_1$=$C_3$=1 i $L_2$=2. (Koristiti metodu napona čvorova)




Rješenje:




1) $U_1\left(\frac{1}{R_1} + sC_1 + \frac{1}{sL_2}\right) - U_2\left(\frac{1}{sL_2}\right) = I_1$

2) $-U_1\frac{1}{sL_2} + U_2\left(\frac{1}{sL_2} + sC_3 + \frac{1}{R_2}\right) = 0$

3) $I_2 = \frac{U_2}{R_2}$

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Riješiti po $U_1$ i $U_2$. Naći $\frac{U_2}{I_1}$, pa se uz $\frac{U_2}{R_2} = I_2$ lako dobije $I(s) = \frac{I_2(s)}{I_1(s)}$.

1) $U_1\left(\frac{1}{R_1} + sC_1 + \frac{1}{sL_2}\right) - \frac{I_2 \cdot R_2}{sL_2} = I_1$

2) $-U_1\frac{1}{sL_2} + \frac{I_2 \cdot R_2}{sL_2} + I_2 \cdot R_2 \cdot sC_3 + I_2 = 0$

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2) $\Rightarrow U_1 = \frac{\frac{I_2 \cdot R_2}{sL_2}}{\frac{1}{sL_2}} + \frac{I_2 \cdot R_2 \cdot sC_3}{\frac{1}{sL_2}} + \frac{I_2}{\frac{1}{sL_2}} \Rightarrow U_1 = I_2 \cdot R_2 + s^2 \cdot I_2 \cdot R_2 \cdot L_2 \cdot C_3 + s \cdot I_2 \cdot L_2 =$

$= I_2\left(R_2 + s^2 \cdot R_2 \cdot L_2 \cdot C_3 + sL_2\right)$

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2) $\Rightarrow$ 1):

$$I_2\left(R_2 + s^2 \cdot R_2 \cdot L_2 \cdot C_3 + sL_2\right)\left(\frac{1}{R_1} + sC_1 + \frac{1}{sL_2}\right) - I_2 \frac{R_2}{sL_2} = I_1$$

Uz uvrštene vrijednosti elemenata:

$$I_2\left(1 + 2s^2 + 2s\right)\left(1 + s + \frac{1}{2s}\right) - \frac{I_2}{2s} = I_1$$

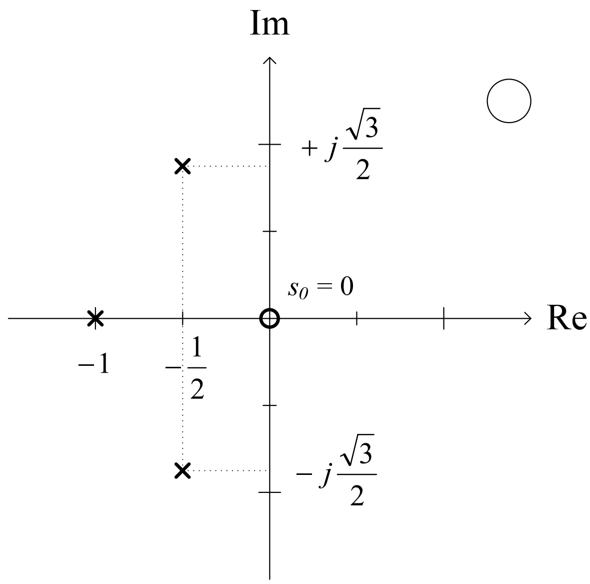
$$I_2\left[\left(1 + 2s^2 + 2s\right)\left(1 + s + \frac{1}{2s}\right) - \frac{1}{2s}\right] = I_1$$

$$H_i(s) = \frac{I_2}{I_1} = \frac{1}{\left(1 + 2s^2 + 2s\right)\left(1 + s + \frac{1}{2s}\right) - \frac{1}{2s}} = \frac{1}{1 + s + \frac{1}{2s} + 2s^2 + 2s^3 + s + 2s + 2s^2 + 1 - \frac{1}{2s}} =$$

$$= \frac{1}{2s^3 + 4s^2 + 4s + 2} = \frac{\frac{1}{2}}{s^3 + 2s^2 + 2s + 1} =$$

$$H_i(s) = \frac{\frac{1}{2}}{(s+1)\left(s^2 + s + 1\right)}$$

4. Zadan je raspored polova i nula prijenosne funkcije $H(s)=U_{iz}(s)/U_{ul}(s)$ nekog električnog kruga prema slici. Odrediti prijenosnu funkciju $H(s)$ ako se traži da bude $H(1)=1/2$. Odrediti odziv $u_{iz}(t)$ za pobudu $u_{ul}(t)=\delta(t)$.




Rješenje:

$$H(s)=k\cdot\frac{\prod_i (s-s_{zi})}{\prod_j (s-s_{pj})}$$ ← Opći oblik prijenosne funkcije (funkcije mreža) napisan pomoću nula i polova.

Nule: $s_{z1}=0$

Polovi: $s_{p1}=-1$

$$s_{p2}=-\frac{1}{2}+j\frac{\sqrt{3}}{2}$$

$$s_{p3}=-\frac{1}{2}-j\frac{\sqrt{3}}{2}$$

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$$H(s)=k\cdot\frac{(s-0)}{(s+1)\left(s+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\left(s+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)}=k\cdot\frac{s}{(s+1)\left(\left(s+\frac{1}{2}\right)^2-\left(j\frac{\sqrt{3}}{2}\right)^2\right)}=$$

$a^2-b^2=(a-b)(a+b)\rightarrow$ Razlika kvadrata

$$=k\cdot\frac{s}{(s+1)\left(s^2+s+\frac{1}{4}+\frac{3}{4}\right)}=k\cdot\frac{s}{(s+1)(s^2+s+1)}$$

$$H(1)=\frac{1}{2}=k\cdot\frac{1}{(1+1)(1^2+1+1)}=\frac{k}{6}\Rightarrow k=3$$

Konačno je: $H(s)=\dfrac{3\cdot s}{(s+1)(s^2+s+1)}$

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$u_{ul}(t)=\delta(t)\Rightarrow U_{ul}(s)=1$

$$U_{iz}(s)=H(s)\cdot U_{ul}(s)=H(s)\cdot 1$$

$$U_{iz}(s)=H(s)=\frac{3\cdot s}{(s+1)(s^2+s+1)}$$

Slijedi rastav na parcijalne razlomke:

$$U_{iz}(s)=\frac{A}{(s+1)}+\frac{Bs+C}{(s^2+s+1)}=\frac{As^2+As+A+Bs^2+Bs+Cs+C}{(s+1)(s^2+s+1)}=$$

$$=\frac{s^2(A+B)+s(A+B+C)+(A+C)}{(s+1)(s^2+s+1)}$$

$$\begin{aligned}
&A+B=0 &&\Rightarrow && A=-B\\
&A+B+C=3 &&\Rightarrow && C=3\\
&A+C=0 &&\Rightarrow && A=-C=-3\\
& && && B=-A=3
\end{aligned}$$

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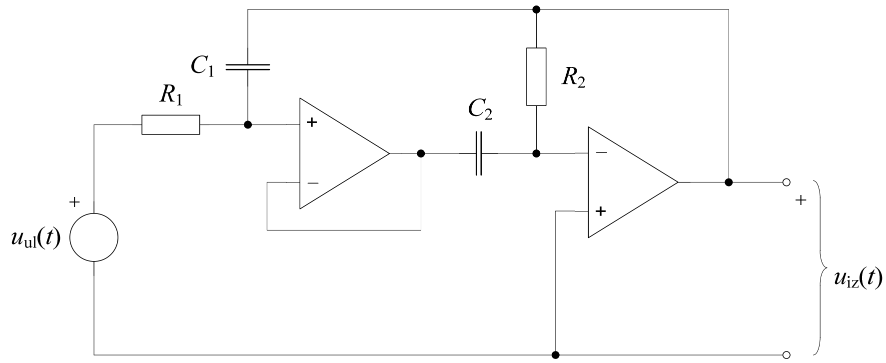
Vrijedi: $s^2+s+1=\left(s+\dfrac{1}{2}\right)^2+\left(\dfrac{\sqrt{3}}{2}\right)^2$

$$U_{iz}(s)=\frac{-3}{s+1}+\frac{3s+3}{s^2+s+1}=3\left(\frac{-1}{s+1}+\frac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)=$$

$$=3\left(\frac{-1}{s+1}+\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\cdot\left(\frac{\sqrt{3}}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)=$$

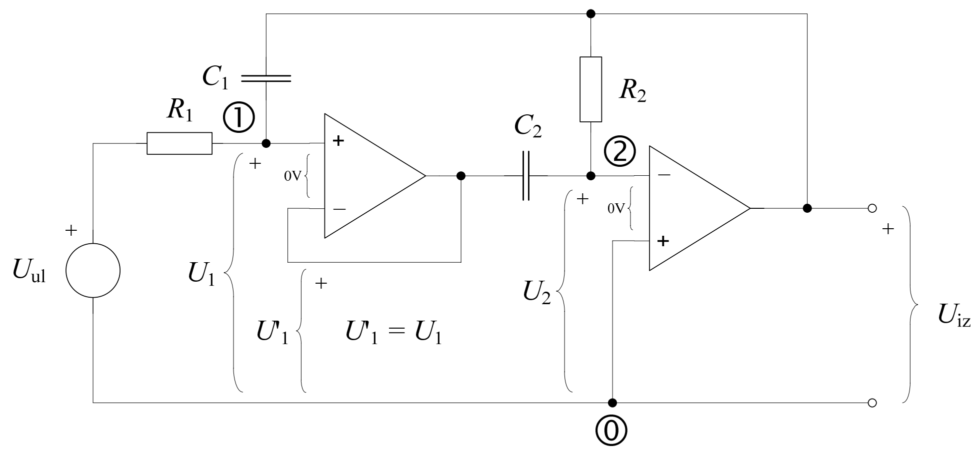
$$u_{iz}(t)=3\left(-e^{-t}+e^{-\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t\right)\cdot S(t)$$

5. Za električni krug prikazan slikom: a) Odrediti prijenosnu funkciju napona $H(s)=U_{iz}(s)/U_{ul}(s)$; b) Izračunati polove i nule prijenosne funkcije i prikazati njihov raspored u $s$-ravnini; c) Izračunati i skicirati A-F karakteristiku $|H(j\omega)|$; d) Izračunati logaritamsku mjeru pojačanja $\alpha(\omega)$. Zadano je: $R_1$=$R_2$=1, $C_1$=$C_2$=1, $A \to \infty$ ($A$ je pojačanje operacijskih pojačala).




Rješenje:

a) Određivanje prijenosne funkcije napona $H(s)=U_{iz}(s)/U_{ul}(s)$:




1) $U_1\left(\dfrac{1}{R_1}+sC_1\right)=\dfrac{U_{ul}}{R_1}+U_{iz}\cdot sC_1$

2) $U_2\left(\dfrac{1}{R_2}+sC_2\right)=U_1\cdot sC_2+\dfrac{U_{iz}}{R_2}$

3) $U_2=0$

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2) $\Rightarrow U_1\cdot sC_2+\dfrac{U_{iz}}{R_2}=0 \Rightarrow U_1=-\dfrac{U_{iz}}{sC_2R_2}$

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2) $\to$ 1) $\Rightarrow$

$$-\frac{U_{iz}}{sC_2R_2}\left(\frac{1}{R_1}+sC_1\right)=\frac{U_{ul}}{R_1}+U_{iz}\cdot sC_1$$

$$-\frac{U_{iz}}{sC_2R_2}\left(\frac{1}{R_1}+sC_1\right)-U_{iz}\cdot sC_1=\frac{U_{ul}}{R_1}$$

$$U_{iz}\left[-\frac{1}{sC_2R_2}\left(\frac{1}{R_1}+sC_1\right)-sC_1\right]=\frac{U_{ul}}{R_1}\Rightarrow\frac{U_{iz}}{U_{ul}}=\frac{1}{-\frac{R_1}{sC_2R_2}\left(\frac{1}{R_1}+sC_1\right)-sC_1R_1}$$

$$H(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=-\frac{1}{\frac{1}{sC_2R_2}+\frac{sC_1R_1}{sC_2R_2}+sC_1R_1}=-\frac{sC_2R_2}{1+sC_1R_1+s^2C_1C_2R_1R_2}=-\frac{s}{s^2+s+1}$$

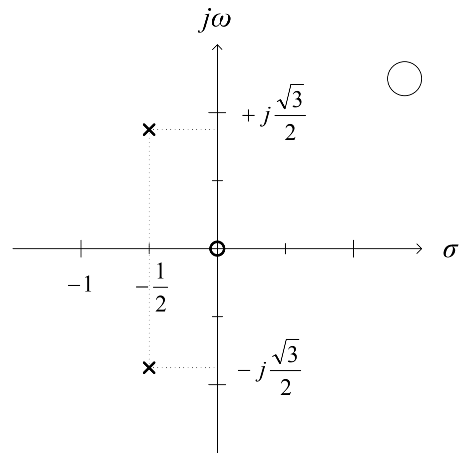
b) Raspored polova i nula:

Polovi: $s^2+s+1=0$

$$s_{p1,2}=\frac{-1\pm\sqrt{1-4}}{2}=-\frac{1}{2}\pm j\frac{\sqrt{3}}{2}$$

Nule: $s_{01}=0$

$$s_{02}=\infty\Rightarrow\lim_{s\to\infty}-\frac{s}{s^2+s+1}=0$$

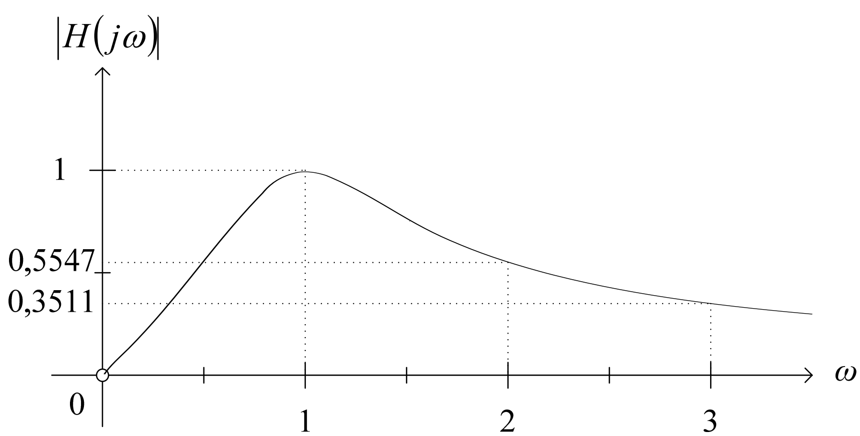



c) Amplitudno-frekvencijska karakteristika:

Uvrstimo $s=j\omega$ u $H(s)$

$$H(j\omega)=-\frac{j\omega}{(j\omega)^2+j\omega+1}=-\frac{j\omega}{-\omega^2+j\omega+1}=-\frac{j\omega}{j\omega+(1-\omega^2)}$$

$$|H(j\omega)|=\frac{\omega}{\sqrt{\omega^2+(1-\omega^2)^2}}$$




Pomoćne točke za skiciranje a-f karakteristike:

| $\omega$ | $\lvert H(j\omega)\rvert$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | $\frac{2}{\sqrt{13}}=0.5547$ |
| 3 | $\frac{3}{\sqrt{73}}=0.3511$ |

d) Logaritamska mjeru pojačanja: $\alpha(\omega)=20\log|H(j\omega)|=20\log\omega-10\log(\omega^4-\omega^2+1)$ [dB]