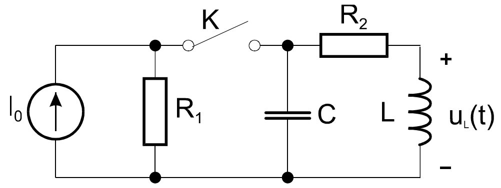
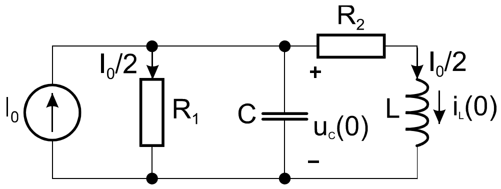
PISMENI ISPIT IZ ELEKTRIČNIH KRUGOVA Jesen 2014 – Rješenja

1. Odrediti odziv $u_L(t)$ u mreži nakon što se otvori sklopka. Skicirati dobiveni odziv. Zadano je $i_0(t)=5, R_1=R_2=L=C=1$.



Rješenje:

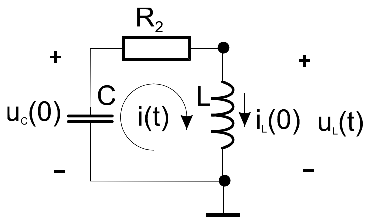
$a) \, t<0$



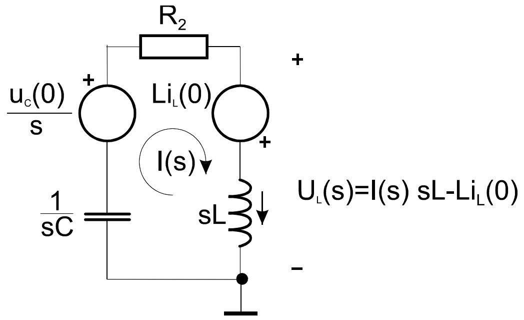
Početni uvjeti: $i_L(0)=\frac{5}{2}=2{,}5\ [\mathrm{A}], \quad u_C(0)=\frac{I_0}{2}R_1=\frac{5}{2}\cdot 1=2{,}5\ [\mathrm{V}]$

$b) \, t\geq 0$

Za $t\geq 0$ mreža izgleda ovako:



U vremenskoj domeni s početnim uvjetima



U frekvencijskoj domeni (primjenom Laplaceove transformacije)

$$I(s)\left(\frac{1}{sC}+\frac{1}{R_2}+sL\right)=\frac{u_C(0)}{s}+Li_L(0)$$

$$\Rightarrow \quad I(s)=\frac{\frac{u_C(0)}{s}+Li_L(0)}{\frac{1}{sC}+\frac{1}{R_2}+sL}=\frac{\frac{2{,}5}{s}+2{,}5}{\frac{1}{s}+1+s}=2{,}5\frac{s+1}{s^2+s+1}$$

$s^2+s+1=0$, $\Delta=b^2-4ac=1-4=-3$,

$$s_{1,2}=\frac{-b\pm\sqrt{\Delta}}{2a}=-\frac{1}{2}\pm j\frac{\sqrt{3}}{2}$$

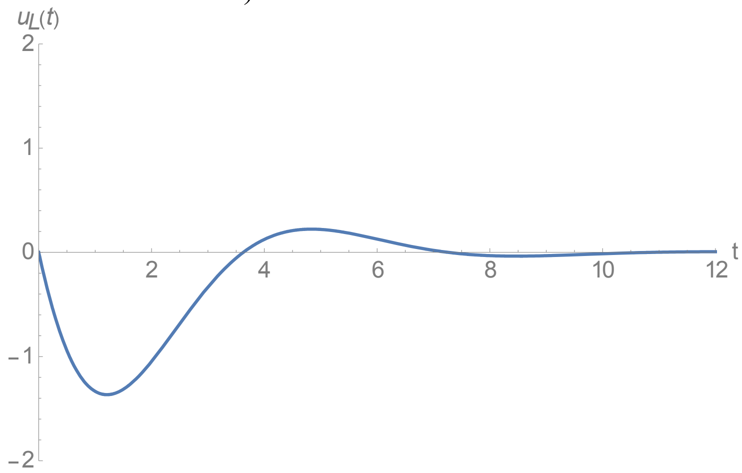
$$U_L(s)=I(s)\cdot sL-Li_L(0)=2{,}5\frac{(1+s)s}{s^2+s+1}-2{,}5=2{,}5\left(\frac{s^2+s+1-1}{s^2+s+1}-1\right)=$$

$$U_L(s)=2{,}5\left(1-\frac{1}{s^2+s+1}-1\right)=2{,}5\left(-\frac{1}{s^2+s+\frac{1}{4}+\frac{3}{4}}\right)=-\frac{5}{2}\cdot\frac{\frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

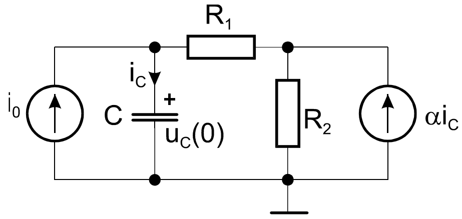
Rješenje:

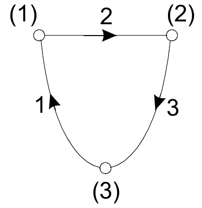
$$u_L(t) = -\frac{5}{\sqrt{3}} e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2} t\right) S(t)$$

Skica odziva (Wolfram Mathematica):



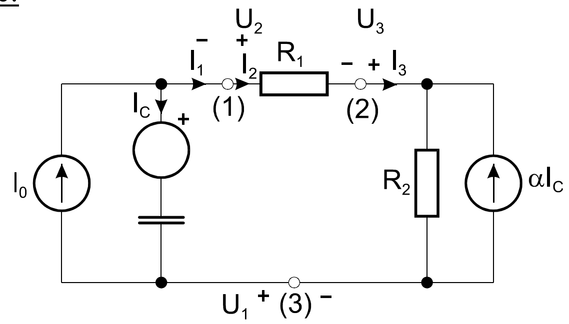
2. Zadana je mreža prema slici i njoj pripadni orijentirani graf. Odrediti sustav jednadžbi čvorova topološkom analizom.



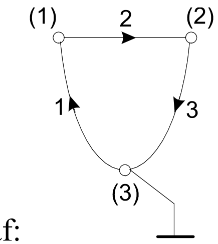


Rješenje:

Mreža:



Graf:



Matrica incidencije:

$$\mathbf{A}=\begin{array}{c} \\ (1) \\ (2) \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$$

Naponsko-strujne relacije grana:

$$U_1=-I_C\frac{1}{sC}-\frac{u_C(0)}{s}=(I_1-I_0)\frac{1}{sC}-\frac{u_C(0)}{s}\text{, gdje je } I_0=I_C+I_1\Rightarrow I_C=I_0-I_1$$

$$U_2=R_1I_2$$

$$\underline{U_3=(\alpha I_C+I_3)R_2=(\alpha I_0-\alpha I_1+I_3)R_2}$$

$$U_1=I_1\frac{1}{sC}-I_0\frac{1}{sC}-\frac{u_C(0)}{s}$$

$$U_2=R_1I_2$$

$$\underline{U_3=-\alpha R_2I_1+R_2I_3+\alpha R_2I_0}$$

$$\mathbf{U}_b=\mathbf{I}_b\mathbf{Z}_b+\mathbf{U}_{0b}$$

$$\mathbf{Z}_b=\begin{bmatrix} \frac{1}{sC} & 0 & 0 \\ 0 & R_1 & 0 \\ -\alpha R_2 & 0 & R_2 \end{bmatrix} \qquad \mathbf{U}_{0b}=\begin{bmatrix} -I_0\frac{1}{sC}-\frac{u_C(0)}{s} \\ 0 \\ \alpha R_2I_0 \end{bmatrix}$$

Strujno-naponske relacije grana:

$$U_1=I_1\frac{1}{sC}-I_0\frac{1}{sC}-\frac{u_C(0)}{s}\Big/\cdot sC$$

$$U_2=R_1I_2\Big/\cdot\frac{1}{R_1}$$

$$\underline{U_3=-\alpha R_2I_1+R_2I_3+\alpha R_2I_0\Big/\cdot\frac{1}{R_2}}$$

$$sCU_1 = I_1 - I_0 - Cu_C(0)$$
$$\frac{1}{R_1}U_2 = I_2$$
$$\frac{1}{R_2}U_3 = -\alpha I_1 + I_3 + \alpha I_0$$

$$I_1 = sCU_1 + I_0 + Cu_C(0)$$
$$I_2 = \frac{1}{R_1}U_2$$
$$I_3 = \frac{1}{R_2}U_3 + \alpha I_1 - \alpha I_0 = \frac{1}{R_2}U_3 + \alpha(sCU_1 + I_0 + Cu_C(0)) - \alpha I_0 =$$
$$= \frac{1}{R_2}U_3 + \alpha sCU_1 + \alpha I_0 + \alpha Cu_C(0) - \alpha I_0 = \alpha sCU_1 + \frac{1}{R_2}U_3 + \alpha Cu_C(0)$$

$$\mathbf{I}_b = \mathbf{Y}_b\mathbf{U}_b + \mathbf{I}_{0b}$$

$$\mathbf{Y}_b = \mathbf{Z}_b^{-1} = \begin{bmatrix} sC & 0 & 0 \\ 0 & \frac{1}{R_1} & 0 \\ \alpha sC & 0 & \frac{1}{R_2} \end{bmatrix} \qquad \mathbf{I}_{0b} = \begin{bmatrix} I_0 + Cu_C(0) \\ 0 \\ \alpha Cu_C(0) \end{bmatrix}$$

Sustav jednadžbi čvorova glasi $\mathbf{Y}_v(s)\cdot\mathbf{U}_v(s) = \mathbf{I}_{0v}(s)$, gdje su:

$$\mathbf{Y}_v = \begin{bmatrix} sC + \frac{1}{R_1} & -\frac{1}{R_1} \\ -\alpha sC - \frac{1}{R_1} & \frac{1}{R_1} + \frac{1}{R_2} \end{bmatrix} \qquad \mathbf{I}_{0v}(s) = \begin{bmatrix} I_0 + Cu_C(0) \\ -\alpha Cu_C(0) \end{bmatrix}$$

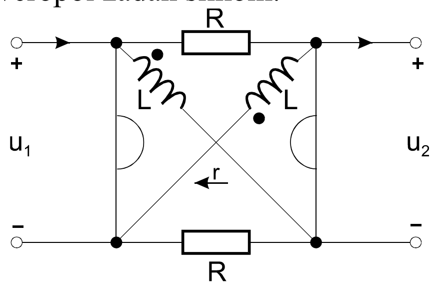
$$\mathbf{Y}_v = \mathbf{A}\cdot\mathbf{Y}_b(s)\cdot\mathbf{A}^T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} sC & 0 & 0 \\ 0 & \frac{1}{R_1} & 0 \\ \alpha sC & 0 & \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} -sC & \frac{1}{R_1} & 0 \\ \alpha sC & -\frac{1}{R_1} & \frac{1}{R_2} \end{bmatrix}}_{\mathbf{A}\cdot\mathbf{Y}_b(s)} \underbrace{\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}}_{\mathbf{A}^T} =$$

$$\mathbf{I}_{0v}(s) = \mathbf{A}\cdot\mathbf{Y}_b(s)\cdot\mathbf{U}_{0b} = \begin{bmatrix} -sC & \frac{1}{R_1} & 0 \\ \alpha sC & -\frac{1}{R_1} & \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} -I_0\frac{1}{sC} - \frac{u_C(0)}{s} \\ 0 \\ \alpha R_2 I_0 \end{bmatrix} = \begin{bmatrix} I_0 + Cu_C(0) \\ -\alpha I_0 - \alpha Cu_C(0) + \alpha I_0 \end{bmatrix}$$

Rješenje $\mathbf{I}_{0v}(s)$ na drugi način:

$$\mathbf{I}_{0v}(s) = -\mathbf{A}\cdot\mathbf{I}_{0b} = -\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} I_0 + Cu_C(0) \\ 0 \\ \alpha Cu_C(0) \end{bmatrix} = \begin{bmatrix} I_0 + Cu_C(0) \\ -\alpha Cu_C(0) \end{bmatrix}$$

3. Odrediti *y*-parametre za četveropol zadan slikom.



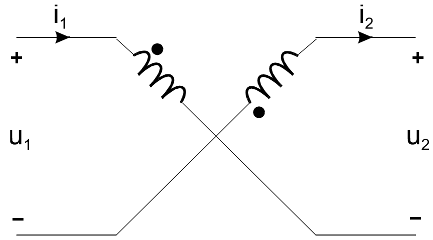
<u>Rješenje:</u> Strujne jednadžbe četveropola:

$$I_1 = y_{11}U_1 - y_{12}U_2 \qquad \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & -y_{12} \\ y_{21} & -y_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$I_2 = y_{21}U_1 - y_{22}U_2$$

Paralelni spoj 3 četveropola:

1. četveropol



$$U_1 = I_1 sL - U_2 + I_1 sL - I_1 2sM$$

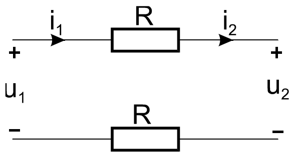
$$U_1 = -I_2 sL - U_2 - I_2 sL + I_2 2sM$$

$$I_1 = \frac{1}{2s(L-M)}U_1 + \frac{1}{2s(L-M)}U_2$$

$$I_2 = -\frac{1}{2s(L-M)}U_1 - \frac{1}{2s(L-M)}U_2$$

$$[\mathrm{y}]_I = \begin{bmatrix} \dfrac{1}{2s(L-M)} & \dfrac{1}{2s(L-M)} \\ -\dfrac{1}{2s(L-M)} & -\dfrac{1}{2s(L-M)} \end{bmatrix}$$

2. četveropol



$$U_1 = I_1 R + U_2 + I_1 R$$

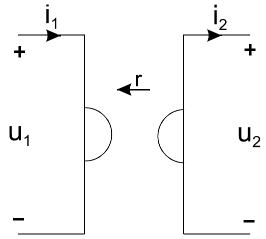
$$U_1 = I_2 R + U_2 + I_2 R$$

$$I_1 = \frac{1}{2R}U_1 - \frac{1}{2R}U_2$$

$$I_2 = \frac{1}{2R}U_1 - \frac{1}{2R}U_2$$

$$[\mathrm{y}]_{II} = \begin{bmatrix} \dfrac{1}{2R} & -\dfrac{1}{2R} \\ \dfrac{1}{2R} & -\dfrac{1}{2R} \end{bmatrix}$$

3. četveropol



$$U_2 = rI_1$$
$$U_1 = rI_2$$

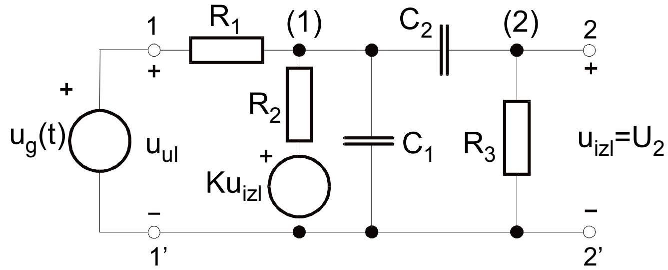
$$I_1 = 0U_1 + \frac{1}{r}U_2$$
$$I_2 = \frac{1}{r}U_1 + 0U_2$$

$$[\mathrm{y}]_{III} = \begin{bmatrix} 0 & \frac{1}{r} \\ \frac{1}{r} & 0 \end{bmatrix}$$

Ukupni y-parametri:

$$[y] = [y]_I + [y]_{II} + [y]_{III} = \begin{bmatrix} \frac{1}{2s(L-M)} + \frac{1}{2R} & \frac{1}{2s(L-M)} - \frac{1}{2R} + \frac{1}{r} \\ -\frac{1}{2s(L-M)} + \frac{1}{2R} + \frac{1}{r} & -\frac{1}{2s(L-M)} - \frac{1}{2R} \end{bmatrix}$$

4. Odrediti prijenosnu funkciju $T(s)=U_{iz}(s)/U_{ul}(s)$ za mrežu prikazanu slikom (koristiti metodu čvorova ili metodu petlji). Izračunati omjer amplituda te razliku u fazi napona na ulazu i izlazu mreže ako je zadano: napon generatora na ulazu $u_g(t)$= 10 sin t, i normirane vrijednosti elemenata R_1=R_2=2, R_3=1, C_1=C_2=1 i K=2.



Rješenje: Metoda čvorišta:

$$(1)\qquad U_1\left(\frac{1}{R_1}+\frac{1}{R_2}+sC_1+sC_2\right)-U_2sC_2=\frac{U_g}{R_1}+\frac{KU_{iz}}{R_2}$$

$$(2)\qquad -U_1sC_2+U_2\left(sC_2+\frac{1}{R_3}\right)=0$$

$$(2)\Rightarrow U_1=U_2\left(1+\frac{1}{sR_3C_2}\right)\rightarrow(1)$$

$$(1)\qquad U_2\left[\left(1+\frac{1}{sR_3C_2}\right)\left(\frac{1}{R_1}+\frac{1}{R_2}+sC_1+sC_2\right)-sC_2-\frac{K}{R_2}\right]=\frac{U_g}{R_1}$$

$$U_2\left[\left(\frac{1}{R_1}+\frac{1}{R_2}+sC_1+sC_2\right)+\frac{1}{sR_1R_3C_2}+\frac{1}{sR_2R_3C_2}+\frac{C_1}{R_3C_2}+\frac{1}{R_3}-sC_2-K\frac{1}{R_2}\right]=\frac{U_g}{R_1}\,/\,R_1R_2R_3C_2s$$

$$U_2\left[sR_2R_3C_2+sR_1R_3C_2+s^2C_1C_2R_1R_2R_3+R_2+R_1+sR_1R_2C_1+sR_1R_2C_2-sKR_1R_3C_2\right]=U_gR_2R_3C_2s$$

Prijenosna funkcija napona:

$$T(s)=\frac{U_2}{U_g}=\frac{sR_2R_3C_2}{s^2C_1C_2R_1R_2R_3+s\left[R_2R_3C_2+R_1R_2(C_1+C_2)+R_1R_3C_2(1-K)\right]+R_1+R_2}$$

Uz uvrštene vrijednosti:

$$T(s)=\frac{s\cdot 2}{s^2 4+s[2+8-2]+4}=\frac{1}{4}\cdot\frac{2\cdot s}{s^2+2s+1}$$

Signal: $\omega_g=1 \quad U_g=10\angle 0^o$

$$\left|T(j\omega)\right|_{\omega=1}=\frac{1}{2}\frac{|j\omega|}{\left|-\omega^2+2j\omega+1\right|}=\frac{1}{2}\frac{\omega}{\sqrt{(1-\omega^2)^2+(2\omega)^2}}$$

$$\left|T(j\omega)\right|_{\omega=1}=\frac{1}{2}\frac{1}{\sqrt{0+2^2}}=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$$

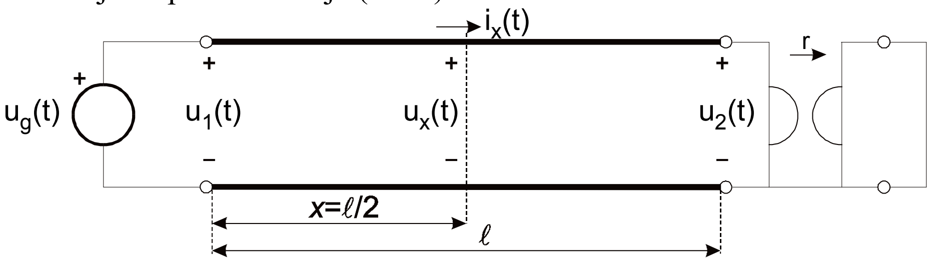
$$T(j\omega)=\frac{1}{2}\cdot\frac{\left[(1-\omega^2)-j(2\omega)\right]j\omega}{(1-\omega^2)^2+(2\omega)^2}=\frac{1}{2}\cdot\frac{2\omega^3+j\omega(1-\omega^2)}{(1-\omega^2)^2+4\omega^2}$$

$$\varphi(\omega)=\arctan\left[\frac{\mathrm{Im}\{T(j\omega)\}}{\mathrm{Re}\{T(j\omega)\}}\right]=\arctan\left[\frac{\omega(1-\omega^2)}{2\omega^3}\right]=\arctan\left[\frac{1-\omega^2}{2\omega^2}\right]$$

$$\varphi(1)=\arctan\left(\frac{1-1}{2\cdot 1}\right)=\arctan(0)=0^o$$

Odgovor: Omjer amplituda iz-ul signala je: 1:4. Razlika u fazi iz-ul signala je: 0

5. Zadana je linija bez gubitaka prema slici s primarnim parametrima L=4mH/km i C=8nF/km i duljine $l=\sqrt{2}$/16km. Na ulazu linije spojen je generator sinusnog valnog oblika $u_g(t)=10\sin 2\pi\,10^6\,t$, a na izlazu linije girator koji je na kraju kratko spojen. Odrediti valni oblik napona i struje na polovini linije ($x=l/2$).



Rješenje:

$L = 4[\text{mH/km}];\ C = 8[\text{nF/km}] \Rightarrow$

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{4\cdot 10^{-3}}{8\cdot 10^{-9}}} = \frac{\sqrt{2}}{2}\cdot 10^3[\Omega]$$

$$\gamma = j\beta = j\omega\sqrt{LC} = j\omega\sqrt{4\cdot 10^{-3}\cdot 8\cdot 10^{-9}} = j\omega\cdot 4\sqrt{2}\cdot 10^{-6}[/\text{km}]$$

$$g = \gamma l = j\omega_0\cdot\sqrt{LC}\cdot l = j\cdot 2\pi\cdot 10^6\cdot 4\sqrt{2}\cdot 10^{-6}\cdot\left[\frac{1}{\text{km}}\right]\cdot\frac{\sqrt{2}}{16}[\text{km}] = j\pi$$

$$\lambda_0 = \frac{2\pi}{\omega_0\sqrt{LC}} = \frac{2\pi}{2\pi\cdot 10^6\cdot 4\sqrt{2}\cdot 10^{-6}} = \frac{\sqrt{2}}{8}[\text{km}] \quad\Rightarrow\quad l = \frac{\lambda_0}{2}$$

a) na izlazu je prazni hod, tj. $i_2(t)=0$, a zbog $l=\dfrac{\lambda_0}{2}$ na ulazu je također $i_1(t)=0$

te je $u_1(t)=u_g(t)$. Također vrijedi:

$$shjx = j\sin x$$
$$\underline{chjx = \cos x}$$

za $x = l;\quad g = j\pi$

$$shj\pi = j\sin\pi = 0$$
$$\underline{ch\,j\pi = \cos\pi = 1}$$

Pa je:

$$Z_{ul} = \frac{U_1}{I_1} = \frac{U_2 ch\,g + I_2 Z_0 sh\,g}{U_2\dfrac{sh\,g}{Z_0} + I_2 ch\,g} = \frac{Z_2 ch\,g + Z_0 sh\,g}{\dfrac{Z_2}{Z_0}sh\,g + ch\,g} = Z_2;$$

$Z_2 = \infty$ je ulazni otpor u girator kojemu je izlaz u kratkom spoju

$$Z_{ul} = Z_2 = \infty \quad\Rightarrow\quad I_1 = 0,\ U_1 = U_g$$

b) na mjestu $x = \dfrac{l}{2} = \dfrac{\lambda_0}{4}$ je $u_x(t)=0$, te još treba izračunati $i_x(t)$

$$g_x = \gamma\frac{l}{2} = j\frac{\pi}{2}$$

$$U(x) = U_1 ch\,yx - I_1 Z_0 sh\,yx$$

$$I(x) = -\frac{U_1}{Z_0}sh\,yx + I_1 ch\,yx$$

$$U\left(x=\frac{l}{2}=\frac{\lambda_0}{4}\right)=U_1 ch\, j\frac{\pi}{2}-I_1 Z_0 sh\, j\frac{\pi}{2}$$

$$I\left(x=\frac{l}{2}=\frac{\lambda_0}{4}\right)=-\frac{U_1}{Z_0} sh\, j\frac{\pi}{2}+I_1 ch\, j\frac{\pi}{2}$$

$$shj\frac{\pi}{2}=j\sin\frac{\pi}{2}=j$$

$$chj\frac{\pi}{2}=\cos\frac{\pi}{2}=0$$

$$U(x)=U_1\cdot 0-I_1 Z_0 j$$

$$I(x)=-\frac{U_1}{Z_0}j+I_1 0$$

$$U_1=10\angle 0°,\quad I_1=0$$

Pa je stoga:

$$U(x)=0$$

$$I(x)=-\frac{U_1}{Z_0}j=-\frac{10}{\sqrt{2}/2}\cdot 10^{-3}j=-\sqrt{2}\cdot 10^{-2}j=\sqrt{2}\cdot 10^{-2}\angle -180°+90°=\sqrt{2}\cdot 10^{-2}\angle -90°$$

Rješenje:

$$u_x(t)=0$$

$$i_x(t)=\sqrt{2}\cdot 10^{-2}\cdot\sin(2\pi 10^6 t-90°)$$