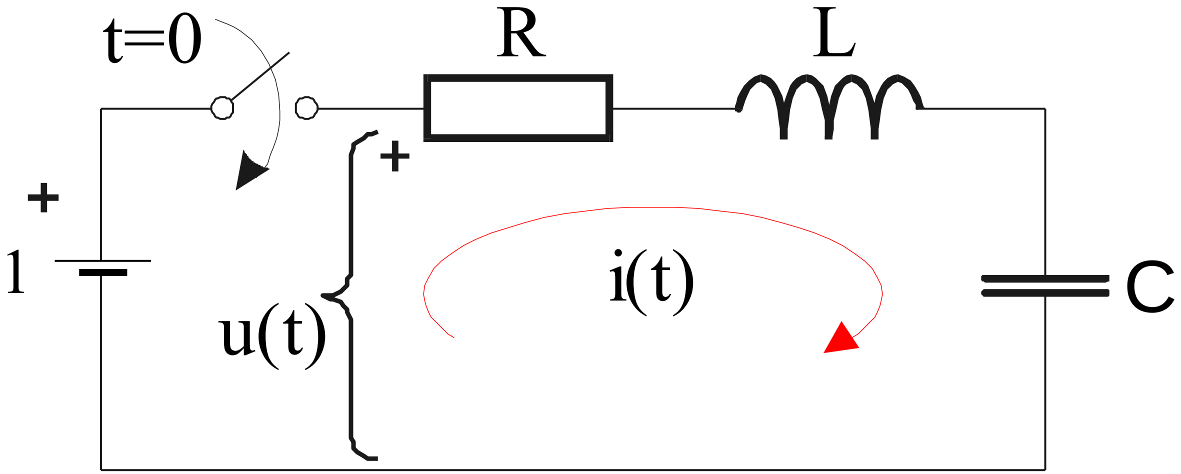
RLC-krug –karakterističan krug drugog reda



u(t)=S(t)

$$u(t) = S(t) \rightarrow U(s) = \frac{1}{s}$$

$$I(s) = \frac{\frac{1}{L} \cdot s}{s^2 + \frac{R}{L} \cdot s + \frac{1}{LC}} \cdot U(s) = \frac{1}{L} \cdot \frac{1}{s^2 + \frac{R}{L} \cdot s + \frac{1}{LC}}$$

Polovi:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \quad \Rightarrow \quad s_{1,2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

Neka je $\quad \frac{R}{2L} = \alpha \quad \frac{1}{LC} = \omega_0^2$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

4 karakteristična slučaja odziva RLC-kruga:

1) Nadkritično prigušeni odziv

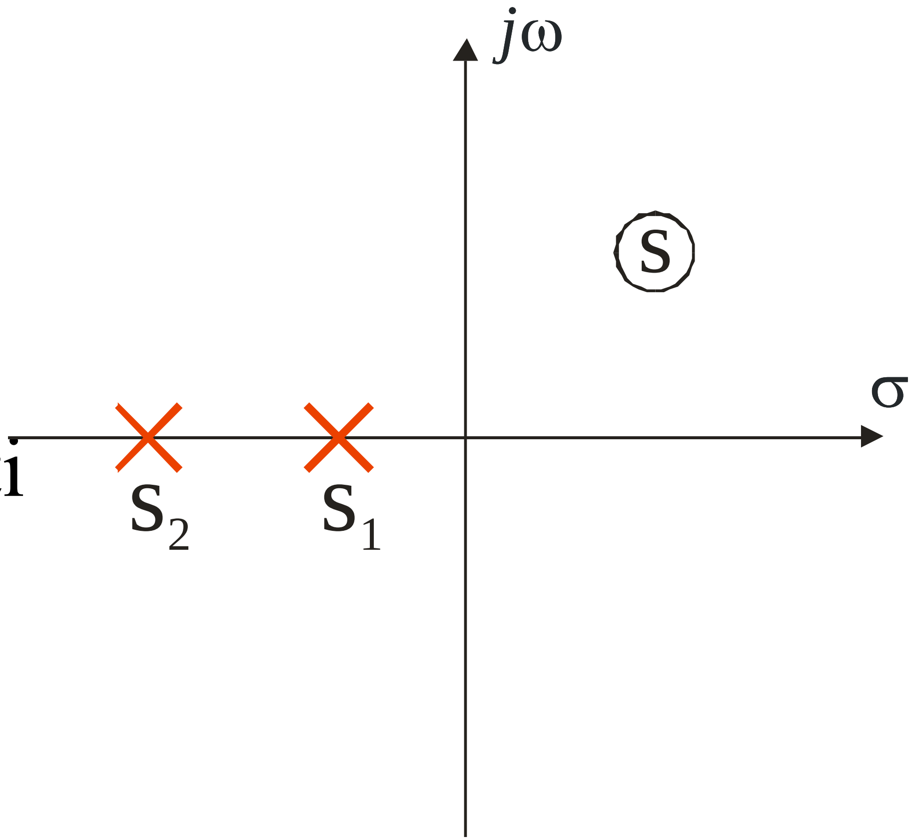
$$\alpha > \omega_0 \Rightarrow R > 2\sqrt{\frac{L}{C}}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

s_1 i $s_2 \rightarrow$ realni i različiti

$$s_1 = -a_1$$

$$s_2 = -a_2$$



$$I(s)=\frac{1}{L}\frac{1}{(s+a_1)\cdot(s+a_2)}=\frac{k_1}{s+a_1}+\frac{k_2}{s+a_2}$$

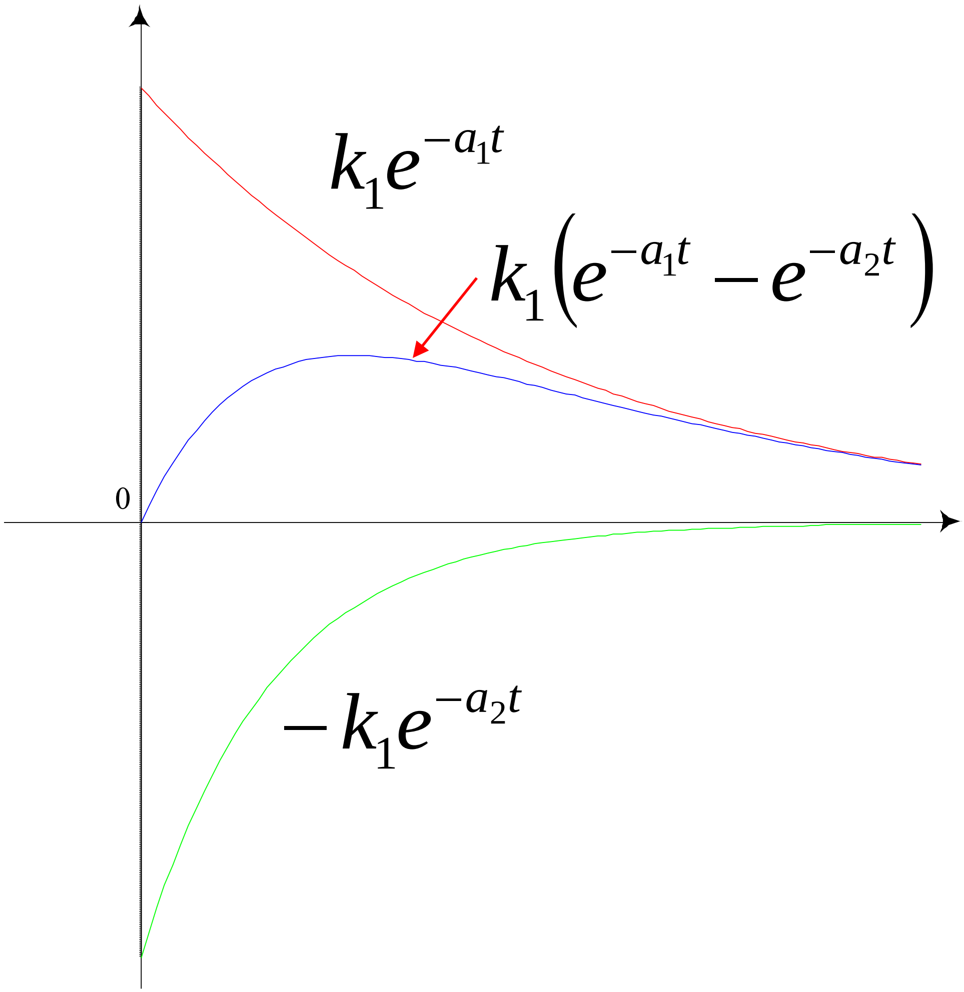
$$\frac{1}{L}=k_1(s+a_2)+k_2(s+a_1) \quad \Longrightarrow \quad \begin{cases} k_1+k_2=0 \\ k_1a_2+k_2a_1=\dfrac{1}{L} \end{cases}$$

$$k_1=-k_2=\frac{1}{L}\cdot\frac{1}{a_2-a_1}$$

- Odziv

$$i(t) = k_1 e^{-a_1 t} + k_2 e^{-a_2 t}$$

$$i(t) = k_1 \left(e^{-a_1 t} - e^{-a_2 t} \right)$$

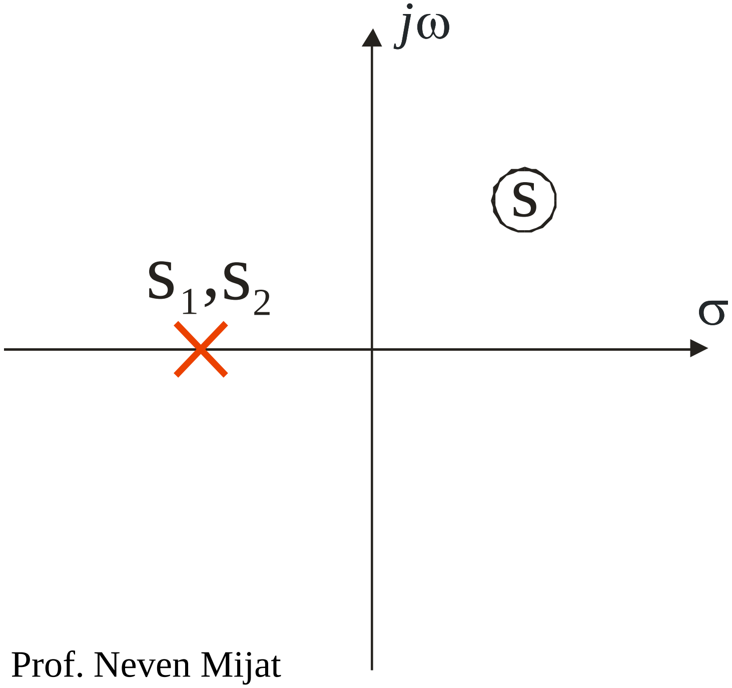


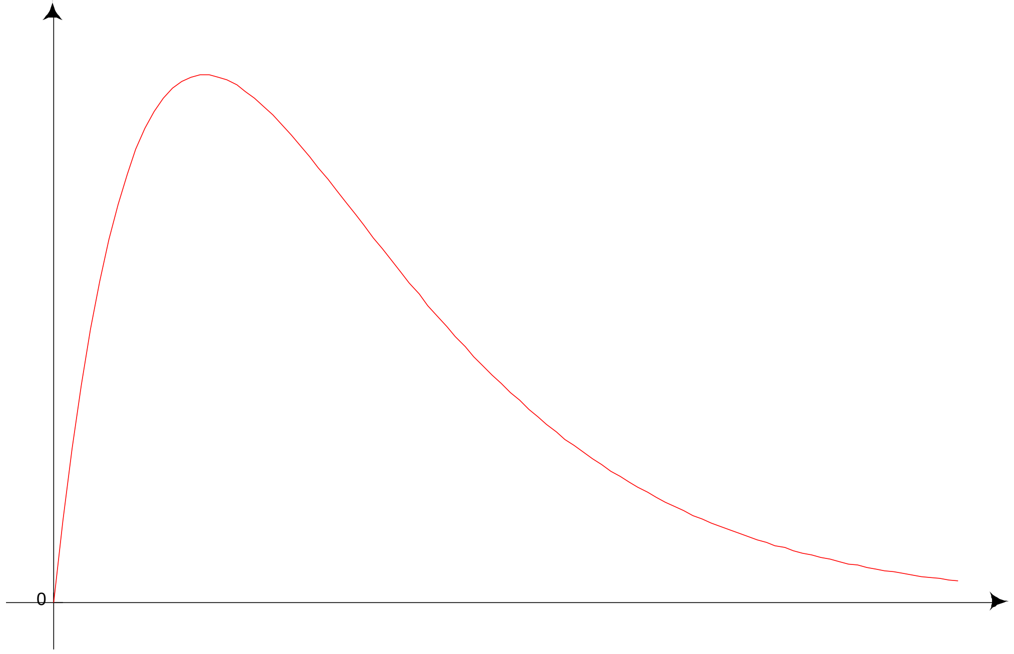
2) Kritično prigušeni odziv

$$\alpha = \omega_0 \Rightarrow R = 2\sqrt{\frac{L}{C}} \qquad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$s_1 = s_2 = -\alpha$$

$$I(s) = \frac{k}{(s+\alpha)^2} \longrightarrow i(t) = k \cdot t \cdot e^{-\alpha t}$$





3) Podkritično prigušeni odziv

$$\alpha < \omega_0 \Rightarrow R < 2\sqrt{\frac{L}{C}}$$

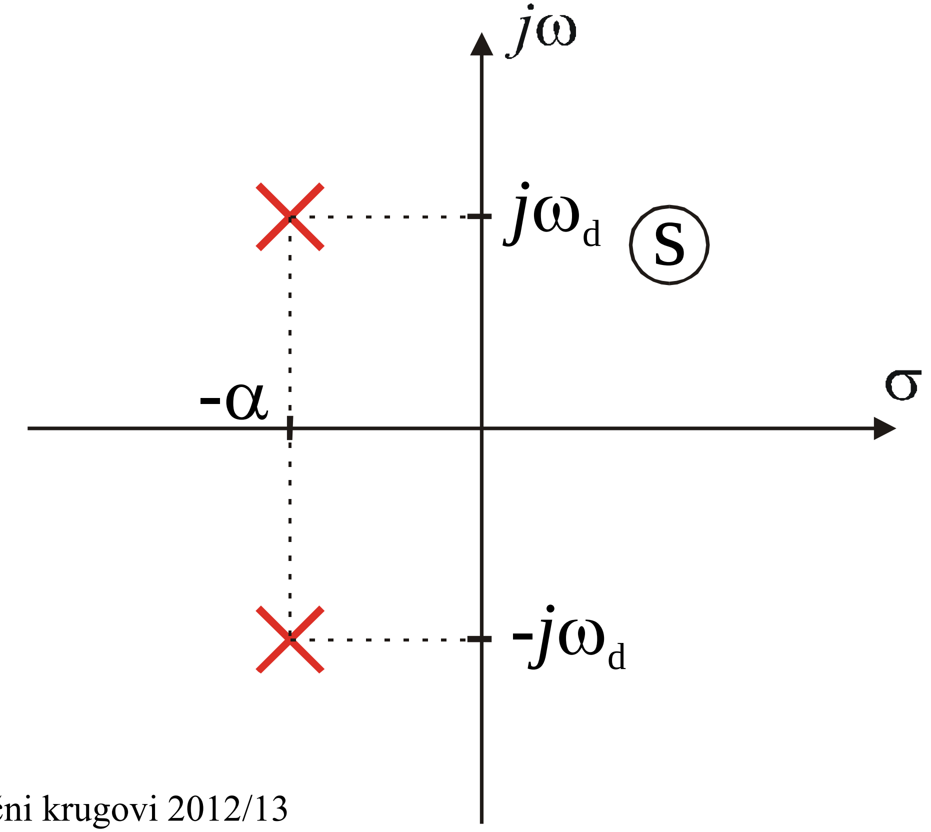
$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha^2 - \omega_0^2 < 0$$ → konjugirano kompleksni korijeni

$$s_1 = -\alpha + j\omega_d$$

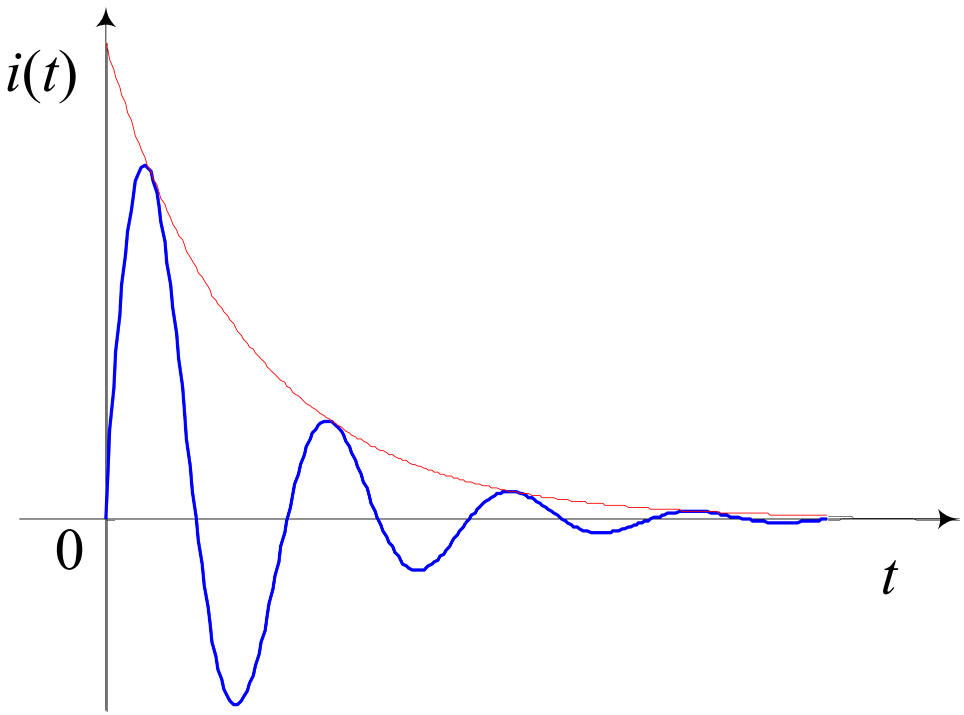
$$s_2 = -\alpha - j\omega_d$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$



$$I(s) = \frac{k}{s+\alpha-j\omega_d} + \frac{k^*}{s+\alpha+j\omega_d}$$

$$i(t) = k \cdot e^{-\alpha t}\sin(\omega_d t + \varphi)$$

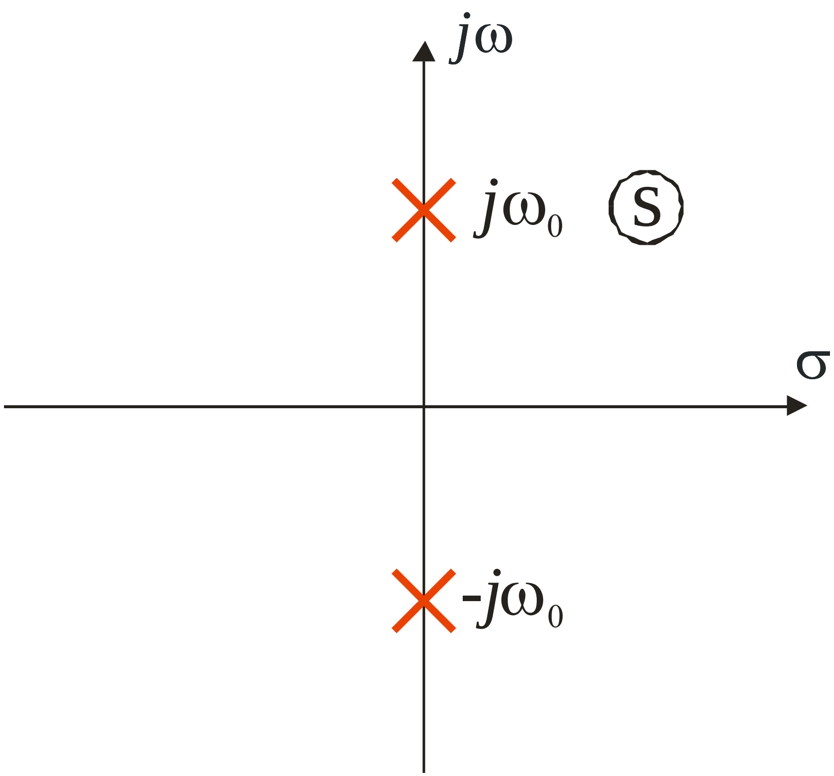


4) Neprigušeni odziv

$$\alpha = 0 \Rightarrow R = 0;$$

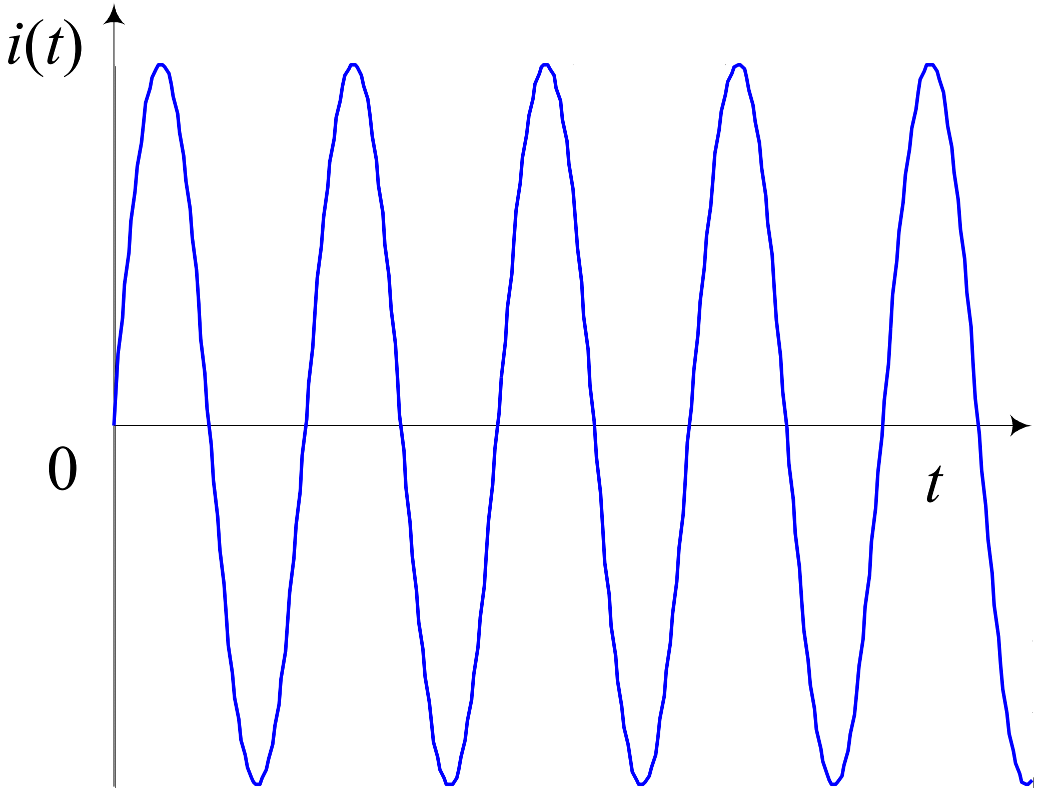
$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

Imaginarni polovi: $s_{1,2} = \pm j\omega_0$



$$I(s) = \frac{k}{s - j\omega_0} + \frac{k^*}{s + j\omega_0}$$

$$i(t) = k \cdot \sin(\omega_0 t + \varphi)$$



Električni krugovi

Jednadžbe krugova u uvjetima stacionarne sinusne pobude

Jednadžbe krugova u uvjetima stacionarne sinusne pobude

- Poseban režim rada električnog kruga je u uvjetima ***stacionarnoga sinusnog signala***.
- Pobudni signal → sinusnog valnog oblika
- Početak djelovanja → $-\infty$.
- Sve prijelazne pojave nakon uključenja → završene.

- U takvim uvjetima:
- svi naponi i struje u krugu imaju sinusni valni oblik;
- frekvencije svih signala jednake su frekvenciji izvora
- razlikuju se samo po **amplitudi** i **faznim pomacima**.

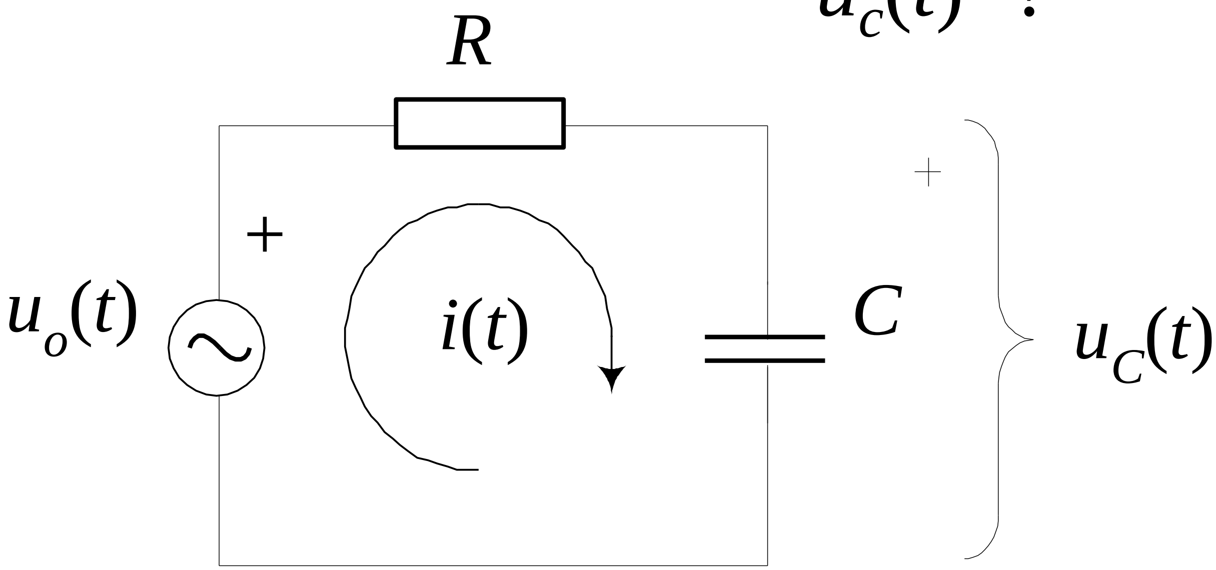
- Jednadžbe krugova u tom slučaju moguće je znatno pojednostavniti primjenom koncepta

fazora.

- Primjer: serijski RC krug

$u_o(t)=U_m\cos(\omega t)$

$u_c(t)=?$



- U vremenskoj domeni → diferencijala jednadžba

$$u_o(t)=R\cdot i(t)+u_C(t)$$

$$u_o(t)=R\cdot C\cdot\frac{du_C(t)}{dt}+u_C(t)$$

- Pobuda $u_0(t) \longrightarrow$ sinusni valni oblik
- Pretpostavka: $u_C(t)$ ima također sinusni valni oblik

$$u_C(t) = U_C \cos(\omega t + \varphi)$$

$$u_C(t) = U_C \cos(\varphi)\cos(\omega t) - U_C \sin(\varphi)\sin(\omega t) =$$

$$= A\cos(\omega t) + B\sin(\omega t)$$

$$u_o(t) = R \cdot C \cdot \frac{du_C(t)}{dt} + u_C(t)$$

- Uvrštenjem u diferencijalnu jednadžbu

$$U_m \cos(\omega t) = RC \frac{d}{dt}\left(U_C \cos(\omega t + \varphi)\right) + U_C \cos(\omega t + \varphi)$$

$$U_m \cos(\omega t) = RC\left(-A\omega \cdot \sin(\omega t) + B\omega \cdot \cos(\omega t)\right) + A\cos(\omega t) + B\sin(\omega t)$$

- Izjednačenjem koeficijenata uz sin i cos funkcije

$$-RCA\omega + B = 0$$
$$RCB\omega + A = U_m$$

$\Rightarrow$ rješenja za A i B

$$A = U_C \cos(\varphi) = \frac{U_m}{(RC\omega)^2 + 1}$$

$$B = U_C \sin(\varphi) = \frac{RC\omega U_m}{(RC\omega)^2 + 1}$$

- Napon $u_C(t)$ je

$$u_C(t) = \frac{U_m}{(RC\omega)^2 + 1}\cos(\omega t) - \frac{\omega RCU_m}{(RC\omega)^2 + 1}\sin(\omega t)$$

- U_C i faza φ $$U_C = \frac{U_m}{\sqrt{(RC\omega)^2 + 1}}$$

$$\varphi = -\mathrm{arctg}(\omega RC)$$

- Napon $u_C(t)$ $$u_C(t) = \frac{U_m}{\sqrt{(RC\omega)^2 + 1}} \cos(\omega t - arctg(\omega RC))$$

- Iz primjera je vidljivo:
  - analiza jednostavnog kruga prilično je složena;
  - krug sa više grana i čvorišta bit će puno kompliciraniji za analizu.

- Drugi način : → umjesto sinusne pobude uvodimo
    - ***eksponencijalnu funkciju*** → $U \cdot e^{j\omega t}$

$$U \cdot e^{j\omega t} = U \cdot \cos(\omega t) + j \cdot U \cdot \sin(\omega t)$$

- Stvarna pobuda → realni dio

$$u(t) = U \cos(\omega t) = \mathrm{Re}\left[U \cdot e^{j\omega t}\right]$$

- Odziv → eksponencijalna funkcija → $U_c e^{j(\omega t + \varphi)}$
- Stvarni odziv → realni dio

$$u_C(t) = U_C \cos(\omega t + \varphi) = \mathrm{Re}\left[U_C \cdot e^{j(\omega t + \varphi)}\right]$$

- Diferencijalna jednadžba

$$u_o(t) = R \cdot C \cdot \frac{du_C(t)}{dt} + u_C(t)$$

$$U_m \cos(\omega t) = RC \frac{d}{dt}\left(U_C \cos(\omega t + \varphi)\right) + U_C \cos(\omega t + \varphi)$$

$$\mathrm{Re}\left[U_m e^{\mathrm{j}\omega t}\right] = \mathrm{Re}\left[RC \frac{d}{dt}\left(U_C e^{\mathrm{j}(\omega t + \varphi)}\right) + U_C e^{\mathrm{j}(\omega t + \varphi)}\right]$$

$$U_m e^{\mathrm{j}\omega t} = RC\mathrm{j}\omega \cdot U_C e^{\mathrm{j}(\omega t + \varphi)} + U_C e^{\mathrm{j}(\omega t + \varphi)} \quad /: e^{\mathrm{j}\omega t}$$

$$U_m = RC\mathrm{j}\omega \cdot U_C e^{\mathrm{j}\varphi} + U_C e^{\mathrm{j}\varphi}$$

- Rezultat:

$$U_C e^{j\varphi} = \frac{U_m}{1 + j\omega RC}$$

- Množenjem s $e^{j\omega t} \longrightarrow$ odziv na pobudu $U_m e^{j\omega t}$

$$U_C e^{j(\omega t + \varphi)} = \frac{U_m e^{j\omega t}}{1 + j\omega RC} = \frac{U_m}{\sqrt{(RC\omega)^2 + 1}} e^{j(\omega t - \mathrm{arctg}(RC\omega))}$$

$$U_C = \frac{U_m}{\sqrt{(RC\omega)^2 + 1}} \qquad \varphi = -\mathrm{arctg}(\omega RC)$$

- Vremenski odziv $u_C(t) \longrightarrow$ realni dio

$$u_C(t) = \frac{U_m}{\sqrt{(RC\omega)^2 + 1}} \cos(\omega t - arctg(\omega RC))$$

- Zaključak:
- Umjesto opće funkcije $f(t)=A\cos(\omega t+\varphi)$ koristimo funkciju $f(t)=A\mathrm{e}^{\mathrm{j}(\omega t+\varphi)}$.

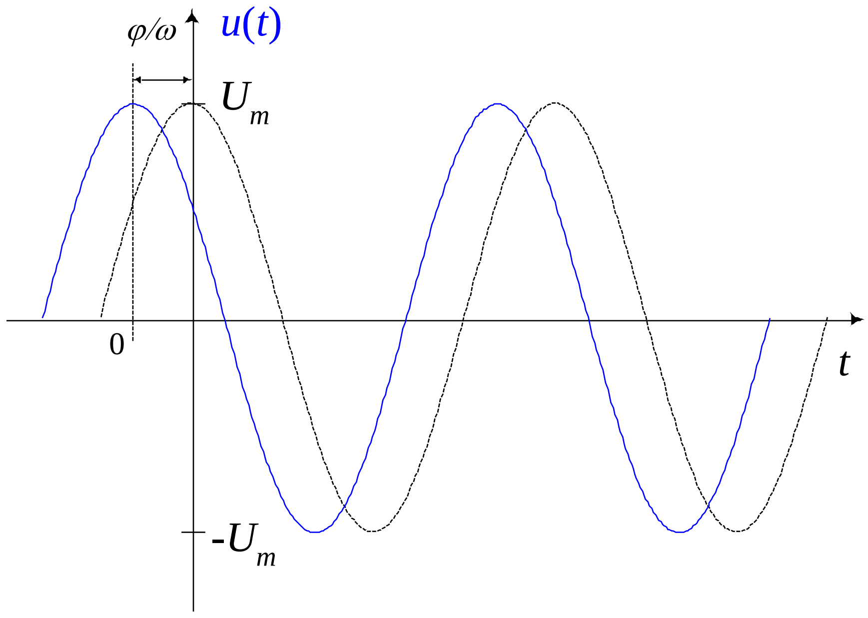
$$f\left(t\right)=A\cdot e^{j\left(\omega t+\varphi\right)}=A\cdot e^{j\varphi}e^{j\omega t}=\boldsymbol{A}\cdot e^{j\omega t}$$

- Veličina $\boldsymbol{A}=Ae^{j\varphi}$ je ***fazor*** sinusnoga signala
- ***Fazor*** sadrži sve informacije o amplitudi i fazi sinusnoga signala.

•Fazor

- Fazor je kompleksni broj pridružen sinusnoj veličini, koji sadrži informaciju o njezinoj amplitudi i faznom pomaku.
- Neka je npr. $u(t)$ naponski signal definiran izrazom

$$u(t) = U_m \cos(\omega t + \varphi)$$



- Taj signal je moguće prikazati kao

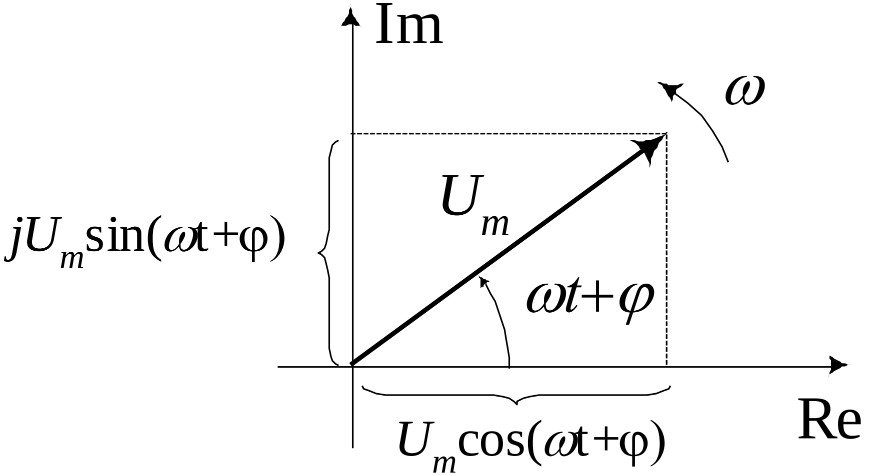
$$u(t) = \mathrm{Re}\left[U_m e^{\mathrm{j}(\omega t+\varphi)}\right]$$

- $U_m \mathrm{e}^{\mathrm{j}(\omega t+\varphi)} \rightarrow$ kompleksni broj
- Trigonometrijski oblik

$$U_m e^{\mathrm{j}(\omega t+\varphi)} = U_m \cos(\omega t+\varphi) + \mathrm{j}U_m \sin(\omega t+\varphi)$$

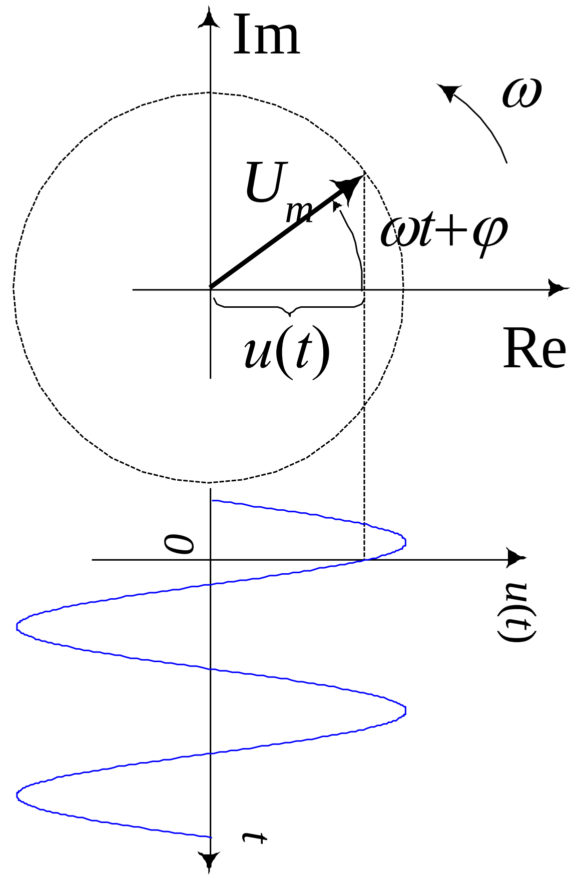
- Signal $u(t) \rightarrow$ realni dio tog kompleksnog broja
- $U_m \rightarrow$ modul
- $\omega t+\varphi \rightarrow$ faza

- U kompleksnoj ravnini → vektor duljine U_m, koji rotira oko ishodišta kutnom brzinom ω



- φ → kut kojeg vektor čini s realnom osi u t=0.

- Taj se vektor naziva ***rotirajućim fazorom***.
- Njegova projekcija na realnu os je signal $u(t)$.



- Svi signali u krugu → sinusoidalni s frekvencijom ω
- Svi signali → rotirajući fazori
- Svaki ima definiran:
  - duljinu ili modul → amplituda sinusnoga signala
  - kut φ → fazni pomak sinusnoga signala.
- Vektori rotiraju istom kutnom brzinom
- Ako umjesto fazora rotira koordinatni sustav u suprotnome smjeru,
- Vektori su nepomični i nazivaju se ***fazorima***.
- Rotirajući fazor → $U_m e^{\mathrm{j}(\omega t+\varphi)} = \boldsymbol{U} e^{\mathrm{j}\omega t}$
- Fazor → rotirajući vektor u trenutku t=0 $\quad \boldsymbol{U} = U_{\mathrm{m}} e^{\mathrm{j}\varphi}$

- Fazor → kompleksni broj
- U ***trigonometrijskome*** obliku

$$\boldsymbol{U} = U_m e^{\mathrm{j}\varphi} = U_m \cos(\varphi) + \mathrm{j} U_m \sin(\varphi)$$

- U ***algebarskome obliku*** → $\boldsymbol{U} = \mathrm{Re}[\boldsymbol{U}] + j\,\mathrm{Im}[\boldsymbol{U}]$

$$\mathrm{Re}[\boldsymbol{U}] = U_m \cdot \cos(\varphi)$$

$$\mathrm{Im}[\boldsymbol{U}] = U_m \cdot \sin(\varphi)$$

$$U_m = \sqrt{\mathrm{Re}^2[\boldsymbol{U}] + \mathrm{Im}^2[\boldsymbol{U}]}$$

$$\varphi = \mathrm{arctg}\left(\frac{\mathrm{Im}[\boldsymbol{U}]}{\mathrm{Re}[\boldsymbol{U}]}\right)$$

- U ***polarnome obliku***. $\boldsymbol{U} = U_m \cdot e^{j\varphi}$

uobičajena oznaka $\boldsymbol{U} = U_m \angle \varphi$

- U analizi mreže nužno je koristiti sve oblike fazora.
- Za zbrajanje ili oduzimanje → algebarski oblik.
- Npr.

$$\boldsymbol{X}_1 = A_1 + jB_1$$

$$\boldsymbol{X}_2 = A_2 + jB_2$$

$$\boldsymbol{X}_1 + \boldsymbol{X}_2 = A_1 + A_2 + j\left(B_1 + B_2\right)$$

- Polarni oblik → kod množenja i dijelenja.
- Npr. umnožak dvaju fazora u polarnome obliku

$$\boldsymbol{X}_1 = X_1 e^{j\varphi_1}$$

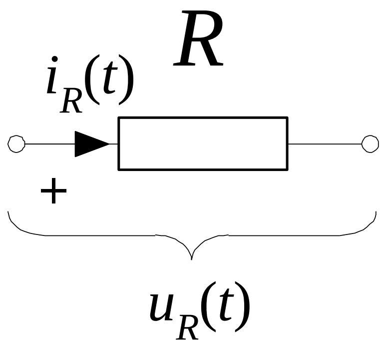
$$\boldsymbol{X}_2 = X_2 e^{j\varphi_2}$$

$$\boldsymbol{X}_1 \cdot \boldsymbol{X}_2 = X_1 \cdot X_2 e^{j\left(\varphi_1 + \varphi_2\right)}$$

u algebarskome obliku

$$\boldsymbol{X}_1 \cdot \boldsymbol{X}_2 = A_1 \cdot A_2 - B_1 \cdot B_2 + j\left(A_1 \cdot B_2 + A_2 \cdot B_1\right)$$

- Odnosi između napona i struja na elementima krugova
- Otpor



$$u_R(t) = R \cdot i_R(t)$$

- Ako je struja sinusnoga valnog oblika $i_R(t) = I\cos(\omega t + \varphi)$

tada je napon $u_R(t) = Ri_R(t) = RI\cos(\omega t + \varphi)$

- U fazorskom prikazu

$$\boldsymbol{I}_R = I\,e^{\mathrm{j}\varphi}$$

$$\boldsymbol{U}_R = R \cdot I\,e^{j\varphi} = R \cdot \boldsymbol{I}_R$$

Stacionarna sinusna pobuda

- Ako je napon na otporu dan izrazom

$$u_R(t) = U\cos(\omega t + \varphi)$$

tada je struja

$$i_R(t) = \frac{1}{R}u_R(t) = \frac{1}{R}U\cos(\omega t + \varphi)$$

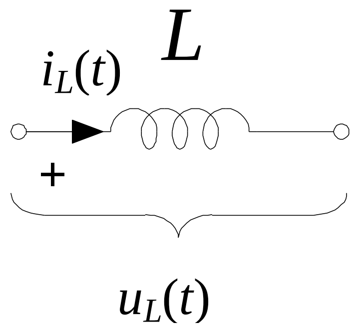
- U fazorskom prikazu

$$\boldsymbol{U}_R = U\,e^{\mathrm{j}\varphi}$$

$$\boldsymbol{I}_R = \frac{1}{R}Ue^{j\varphi} = \frac{1}{R}\boldsymbol{U}_R$$

Stacionarna sinusna pobuda

- Induktivitet



$$u_L(t) = L\,\frac{\mathrm{d}i_L(t)}{\mathrm{d}t}$$

- Ako je struja sinusnoga valnog oblika $\quad i_L(t) = I\cos(\omega t + \varphi)$

tada je napon

$$u_L(t) = L\,\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} = L\,\frac{\mathrm{d}}{\mathrm{d}t}\{I\cos(\omega t + \varphi)\} = \omega L\, I\cos(\omega t + \varphi + \pi/2)$$

- U fazorskom prikazu $\quad \boldsymbol{I}_L = I\,e^{\mathrm{j}\varphi}$

$$\boldsymbol{U}_L = \omega L I\, e^{\mathrm{j}(\varphi + \pi/2)} = \mathrm{j}\omega L\, I\, e^{j\varphi} = \mathrm{j}\omega L\,\boldsymbol{I}_L$$

- Ako je napon na induktivitetu dan izrazom

$$u_L(t) = U\cos(\omega t + \varphi)$$

tada je struja

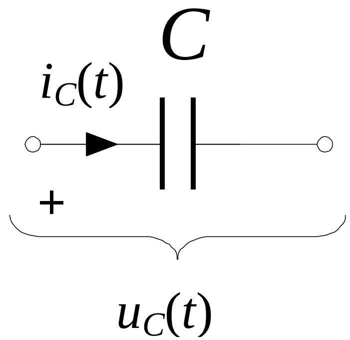
$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} u_L(\tau)d\tau = \frac{1}{L}\int_{-\infty}^{t}(U\cos(\omega\tau + \varphi))d\tau = \frac{1}{\omega L}U\cos(\omega t + \varphi - \pi/2)$$

- U fazorskom prikazu

$$\boldsymbol{U}_L = U\,e^{\mathrm{j}\varphi}$$

$$\boldsymbol{I}_L = \frac{1}{\omega L}U\,e^{\mathrm{j}(\varphi - \pi/2)} = \frac{1}{j\omega L}U\,e^{j\varphi} = \frac{1}{j\omega L}\boldsymbol{U}_L$$

- Kapacitet



$$u_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)d\tau$$

- Ako je struja sinusnoga valnog oblika $i_C(t) = I\cos(\omega t + \varphi)$

tada je napon

$$u_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)d\tau = \frac{1}{C}\int_{-\infty}^{t} (I\cos(\omega\tau + \varphi))d\tau = \frac{1}{\omega C} I\cos(\omega t + \varphi - \pi/2)$$

- U fazorskom prikazu $\boldsymbol{I}_C = I\,e^{\mathrm{j}\varphi}$

$$\boldsymbol{U}_C = \frac{1}{\omega C} I\,e^{\mathrm{j}(\varphi - \pi/2)} = \frac{1}{j\omega C} I\,e^{j\varphi} = \frac{1}{j\omega C}\boldsymbol{I}_C$$

- Ako je napon na kapacitetu dan izrazom

$$u_C(t) = U\cos(\omega t + \varphi)$$

tada je struja

$$i_C(t) = C\frac{\mathrm{d}u_C(t)}{\mathrm{d}t} = C\frac{\mathrm{d}}{\mathrm{d}t}\{U\cos(\omega t + \varphi)\} = \omega C\,U\cos(\omega t + \varphi + \pi/2)$$

- U fazorskom prikazu

$$\boldsymbol{U}_C = U\,e^{\mathrm{j}\varphi}$$

$$\boldsymbol{I}_C = \omega C U\,e^{\mathrm{j}(\varphi+\pi/2)} = j\omega C\,U\,e^{j\varphi} = j\omega C\,\boldsymbol{U}_C$$

- Zaključak:
- Derivaciji sinusnoga signala u vremenskoj domeni u fazorskome prikazu odgovara množenje fazora s jω.
- Integriranju sinusnoga signala u vremenskoj domeni u fazorskome prikazu odgovara dijelenje fazora s jω.
- Fazori odzivnih signala $\longrightarrow$ općenito funkcije frekvencije ω.
- Moduli faze $\varphi \longrightarrow$ zavisne varijable ovisne o frekvencije ω.
- Zbog toga se analiza kruga primjenom fazora često naziva ***analizom u frekvencijskoj domeni***.

- Odnos napona i struje na dvopolu u fazorskome prikazu

$$\boldsymbol{U} = Z \cdot \boldsymbol{I}$$

Z → ***impedancija*** .

- Isto tako vrijedi

$$\boldsymbol{I} = Y \cdot \boldsymbol{U}$$

Y → ***admitancija*** .

- Impedancija i admitancija → kompleksne veličine
- U općem slučaju → funkcije frekvencije ω.

- Za tri osnovna pasivna dvopola

$$Z_R = R$$

$$Z_L = j\omega L = \omega L \cdot e^{j90^\circ} = \omega L\angle 90^\circ$$

$$Z_C = \frac{1}{j\omega C} = \frac{1}{\omega C} \cdot e^{-j90^\circ} = \frac{1}{\omega C}\angle -90^\circ$$

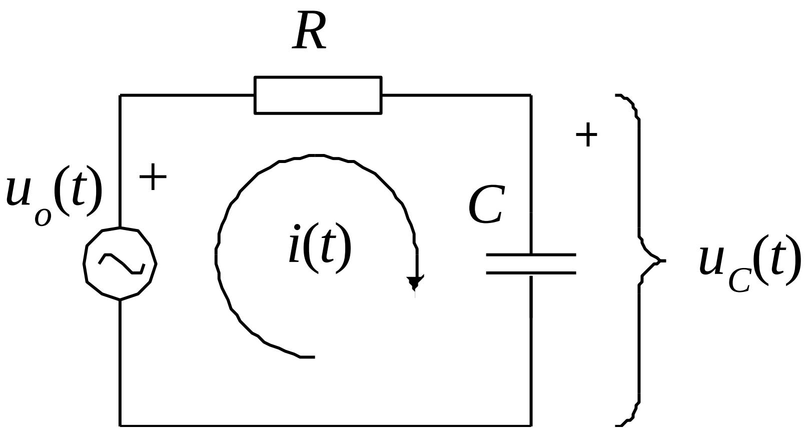
- Admitancije

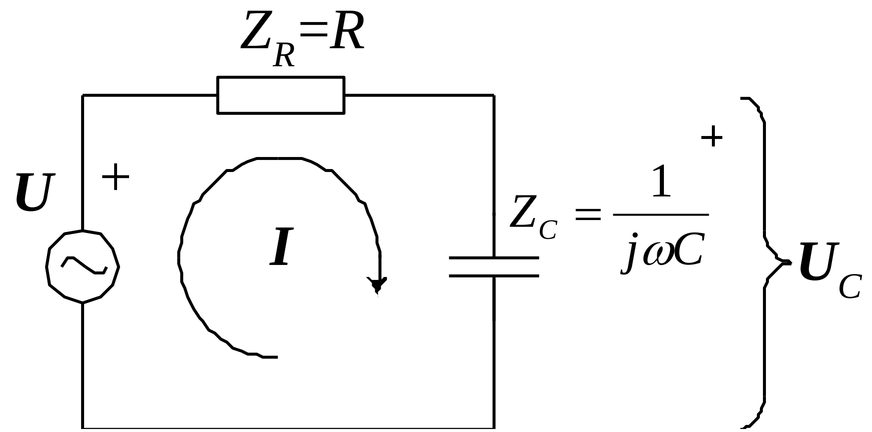
$$Y_R = \frac{1}{R}$$

$$Y_L = \frac{1}{j\omega L} = \frac{1}{\omega L} \cdot e^{-j90^\circ} = \frac{1}{\omega L}\angle -90^\circ$$

$$Y_C = j\omega C = \omega C e^{j90^\circ} = \omega C\angle 90^\circ$$

- Primjer: Izračunati pomoću fazora napon $u_c(t)$ ako je $u_o(t)=U_m\cos(\omega t)$.





- U frekvencijskoj domeni primjenom fazora

$$\boldsymbol{U}_C=\frac{\boldsymbol{U}}{1+j\omega RC}=\frac{1-j\omega RC}{1+(\omega RC)^2}\boldsymbol{U}=\frac{U\angle\text{-arctg}(\omega RC)}{\sqrt{1+(\omega RC)^2}}$$

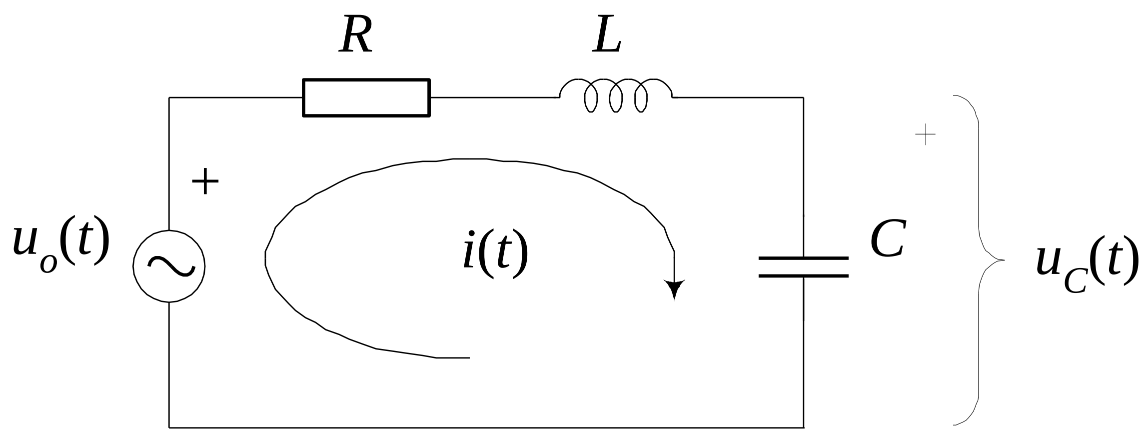
- Primjer: Za RLC krug na slici odrediti napon $u_C(t)$.

$$R = 2\Omega \qquad L = 1H$$

$$C = 1F$$

$$u_0(t) = 2\cos\left(t + \frac{\pi}{4}\right)$$

$$\omega = 1rad/s$$

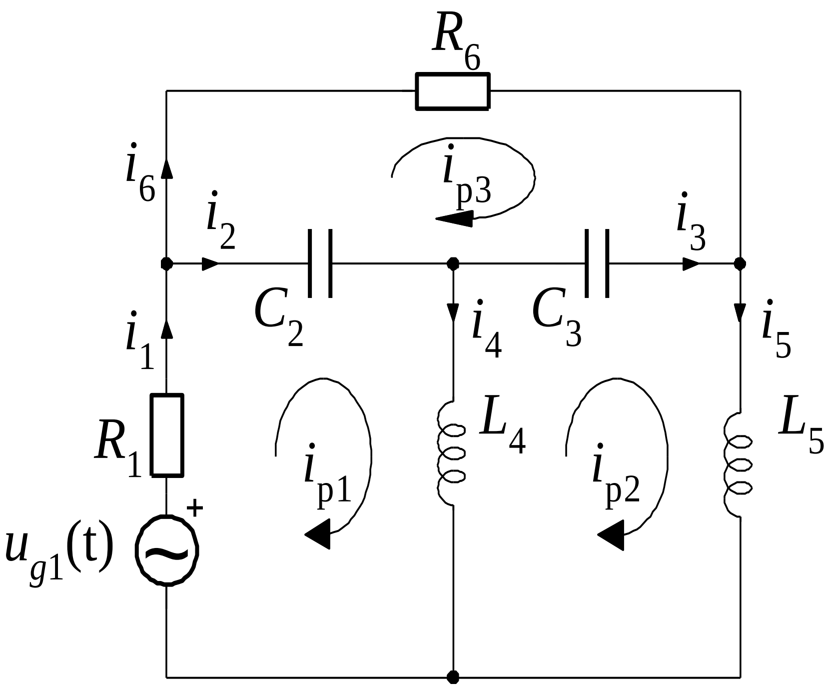


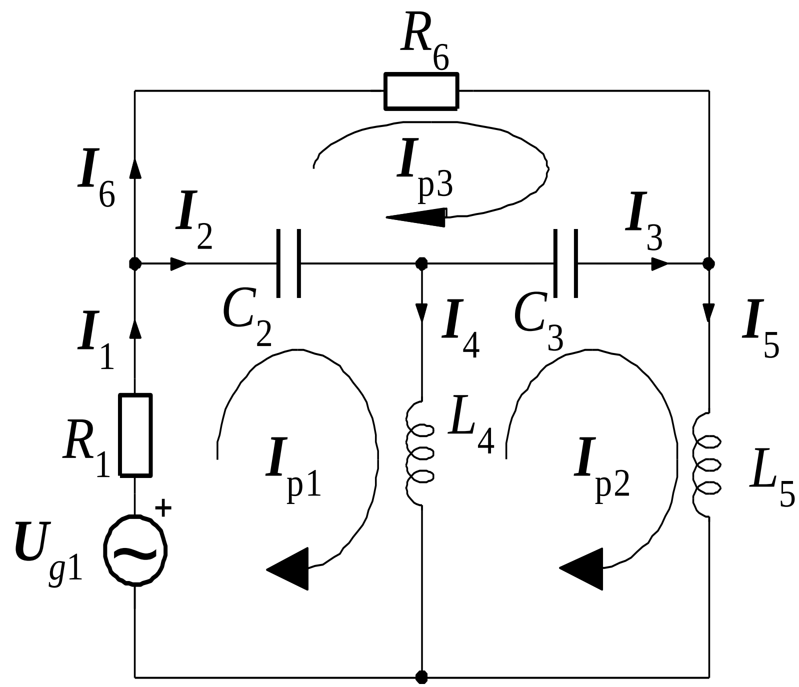
$$\boldsymbol{U}_C = \frac{2 \cdot e^{j(\pi/4)}}{1+2j-1} = \frac{2}{2j} \cdot \left(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} \right) = \left(\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} \right)$$

$$\boldsymbol{U}_C = 1 \cdot e^{-j\operatorname{arctg}(1)} = 1 \cdot e^{-j(\pi/4)} \quad \Longrightarrow \quad u_C(t) = \cos(t - \pi/4)$$

- Jednadžbe krugova dobivene primjenom fazora imaju isti oblik kao i jednadžbe dobivene primjenom Laplaceove transformacije, uz napomenu:
  - početni uvjeti za napone i struje jednaki su nuli i
  - umjesto kompleksne varijable s kao varijabla se pojavljuje jω.

- Tako npr. za mrežu na slici, u slučaju da je $u_{g1}(t)$ stacionarna sinusna pobuda, sustav jednadžbi petlji se može pisati u fazorskom obliku.





$$-\mathrm{U}_{g1}+R_1\boldsymbol{I}_{p1}+\frac{1}{\mathrm{j}\omega C_2}\left(\boldsymbol{I}_{p1}-\boldsymbol{I}_{p3}\right)+\mathrm{j}\omega L_4\left(\boldsymbol{I}_{p1}-\boldsymbol{I}_{p2}\right)=0$$

$$\mathrm{j}\omega L_4\left(\boldsymbol{I}_{p2}-\boldsymbol{I}_{p1}\right)+\frac{1}{\mathrm{j}\omega C_3}\left(\boldsymbol{I}_{p2}-\boldsymbol{I}_{p3}\right)+\mathrm{j}\omega L_5\boldsymbol{I}_{p2}=0$$

$$\frac{1}{\mathrm{j}\omega C_2}\left(\boldsymbol{I}_{p3}-\boldsymbol{I}_{p1}\right)+R_6\boldsymbol{I}_{p3}+\frac{1}{\mathrm{j}\omega C_3}\left(\boldsymbol{I}_{p3}-\boldsymbol{I}_{p2}\right)=0$$