ELEKTRIČNI KRUGOVI — Zadaci sa rješenjima za vježbu

Električni filtri

1. Zadana je prijenosna funkcija $T(s)=U_{iz}(s)/U_{ul}(s)$ električnog filtra. Nacrtati raspored nula i polova u kompleksnoj s-ravnini i amplitudno-frekvencijsku karakteristiku funkcije $T(s)$. Izračunati vrijednost faktora dobrote q_p, frekvencije ω_p i pojačanja u području propuštanja K. O kakvoj se vrsti filtra radi (NP, VP, PP ili PB)?

$$T(s)=\frac{0.6s^2}{s^2+1.2s+0.4}$$

Rješenje: izjednačimo zadanu prijenosnu funkciju sa općim oblikom prijenosne funkcije:

$$T(s)=\frac{0.6s^2}{s^2+1.2s+0.4}=\frac{K\cdot s^2}{s^2+\frac{\omega_p}{q_p}s+\omega_p^2}$$

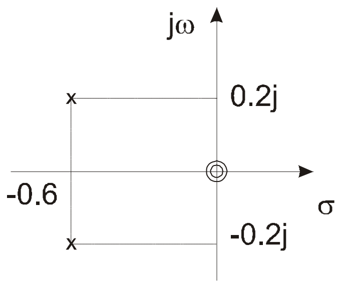
Odatle slijede parametri filtra:

$$\omega_p=\sqrt{0.4}\cong 0.63,\quad q_p=\frac{\omega_p}{2\sigma_p}=\frac{\sqrt{0.4}}{1.2}\cong 0.53,\quad K=0.6$$

Zatim slijede polovi: $s^2+1.2s+0.4=0 \Rightarrow s_{p1,2}=\frac{-1.2\pm\sqrt{1.2^2-4\cdot 0.4}}{2}=-0.6\pm 0.2j$

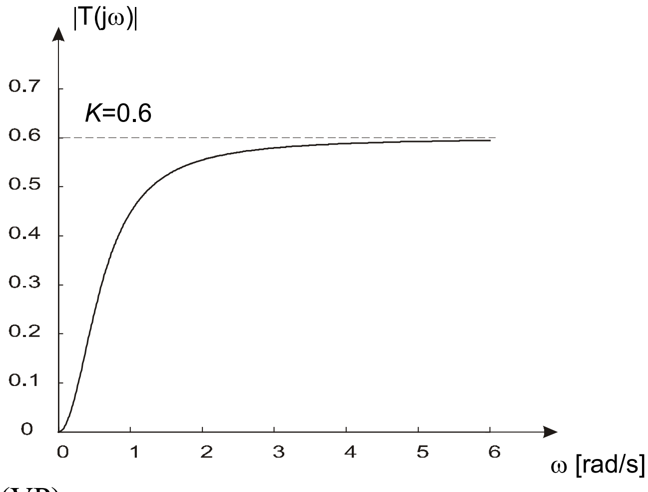
i nule: $s^2=0 \Rightarrow s_{01,2}=0$ dvostruka nula u ishodištu

raspored nula i polova u s-ravnini:



amplitudno-frekvencijska karakteristika slijedi ako uvrstimo s=jω

$$|T(j\omega)|=\left|\frac{0.6\omega^2}{-\omega^2+1.2j\omega+0.4}\right|=\frac{0.6\omega^2}{\sqrt{(0.4-\omega^2)^2+(1.2\omega)^2}}$$



ovo je visoki propust (VP)

2. Zadana je prijenosna funkcija $T(s)=U_{iz}(s)/U_{ul}(s)$ električnog filtra. Izračunati vrijednost faktora dobrote q_p, frekvencije ω_p i pojačanja u području propuštanja K. O kakvoj se vrsti filtra radi (NP, VP, PP ili PB)? Koliko iznosi širina pojasa propuštanja filtra B te gornja i donja granična frekvencija f_g i f_d ? U kakvoj vezi su granične frekvencije sa centralnom frekvencijom ? Nacrtati raspored nula i polova u kompleksnoj s-ravnini i amplitudno-frekvencijsku karakteristiku funkcije $T(s)$.

$$T(s)=\frac{31.4159\cdot s}{s^2+31.4159\cdot s+98696}$$

<u>Rješenje:</u> izjednačimo zadanu prijenosnu funkciju sa općim oblikom prijenosne funkcije:

$$T(s)=\frac{31.4159\cdot s}{s^2+31.4159\cdot s+98696}=\frac{K\cdot\dfrac{\omega_p}{q_p}\cdot s}{s^2+\dfrac{\omega_p}{q_p}s+\omega_p^2}$$

Odatle slijede parametri filtra:

$$\omega_p=\sqrt{98696}=314.159=2\pi\cdot 50\,[\text{rad/s}]\Rightarrow f_p=50[\text{Hz}]$$

$$q_p=\frac{\omega_p}{2\sigma_p}=\frac{314.159}{31.4159}=10$$

$$K=1$$

(gdje je $2\sigma_p$ član koji množi s u nazivniku prijenosne funkcije)

Ovo je pojasni propust (PP)

Širina pojasa propuštanja : $B=\dfrac{\omega_p}{q_p}=\dfrac{314.159}{10}=31.4159\ [\text{rad/s}]$

Gornja i donja granična frekvencija su :

$$\omega_{g,d}=\omega_p\sqrt{1+\frac{1}{4q_p^2}}\pm\frac{\omega_p}{2q_p}=314.159\sqrt{1+\frac{1}{4\cdot 100}}\pm\frac{314.159}{2\cdot 10}=314.551\pm 15.708\ [\text{rad/s}]$$

ω_g=330.259 [rad/s], ω_d=298.844 [rad/s] ili

$f_g=\omega_g/2\pi$=330.259/2π=52.5624 [Hz], $f_d=\omega_d/2\pi$=298.844 /2π=47.5625 [Hz],

$B=\omega_g-\omega_d$=330.259–298.844=31.415 [rad/s] ili

f_g-f_d=31.415/2π=5 [Hz]

centralna frekvencija $\omega_0=\omega_p=314.159=2\pi\cdot 50\,[\text{rad/s}]$ ili

f_0=50 [Hz]

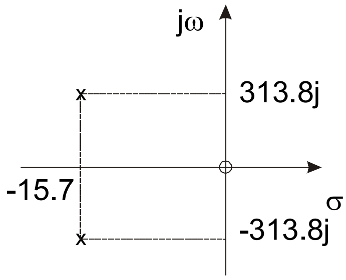
$\omega_0^2=\omega_d\cdot\omega_g\rightarrow$ ω_0 je geometrijska sredina od ω_d i ω_g ili

$f_0^2=f_d\cdot f_g\rightarrow$ f_0 je geometrijska sredina od f_d i f_g

Zatim slijede polovi: $s^2+31.4159\cdot s+98696=0\ \Rightarrow$

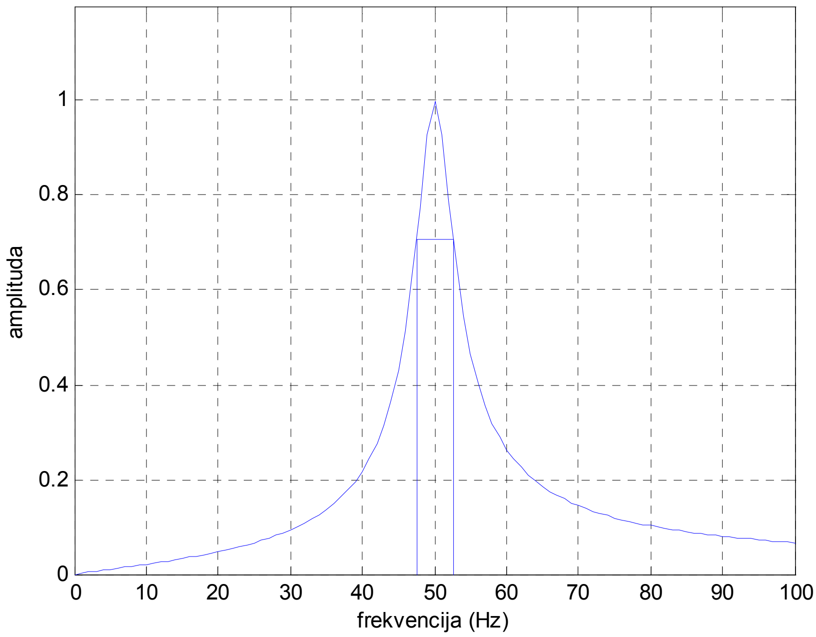
$$s_{p1,2}=\frac{-31.4159\pm\sqrt{31.4159^2-4\cdot 98696}}{2}=-15.708\pm 313.766j$$

i nule: $s=0\ \Rightarrow\ s_{01}=0,\ s_{02}=\infty$ jedna nula u ishodištu, druga u beskonačnosti raspored nula i polova u s-ravnini:

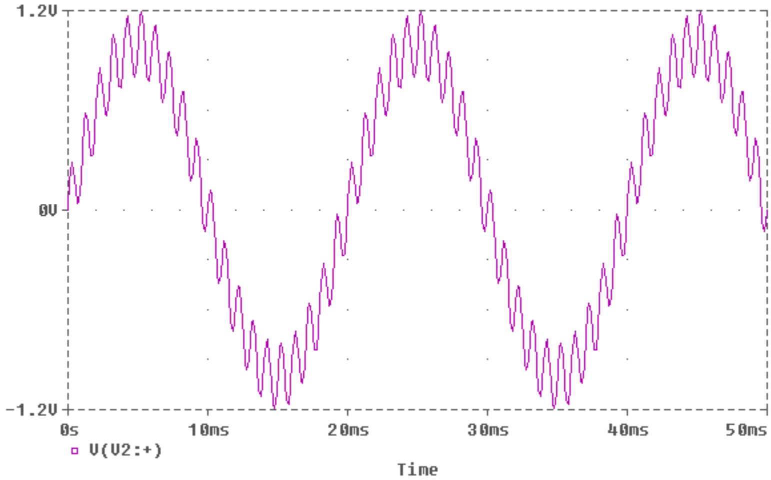


amplitudno-frekvencijska karakteristika slijedi ako uvrstimo $s=j\omega$

$$|T(j\omega)| = \frac{|31.4159 \cdot \omega|}{\sqrt{(98696-\omega^2)^2 + (31.4159 \cdot \omega)^2}}, \ \omega=2\pi f$$



3. Telefonski prijenosni sistem sadrži osim korisnoga signala i smetnju od gradske mreže frekvencije 50 Hz. Za ilustraciju neka je ulazni napon sastavljen od sinusnoga signala frekvencije 1000 Hz i smetnje frekvencije 50 Hz : $u_{ul}(t) = \sin(2\pi \cdot 50t) + 0.2 \cdot \sin(2\pi \cdot 1000t)$ i prikazan je na slijedećoj slici.



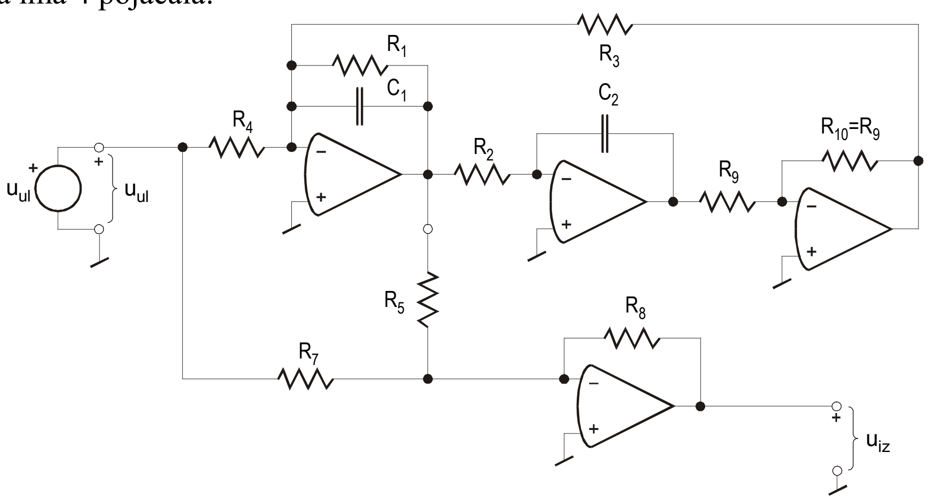
Treba projektirati filtar koji eliminira smetnju. Takav filtar bi trebao biti pojasna brana s frekvencijom nule f_z=50Hz, i sa velikim Q-faktorom polova q_p=10 koji osigurava visoku selektivnost filtra (odn. uski pojas gušenja).

Napomena: navedeni primjer je prvenstveno edukativnog karaktera mada dijelovi ovog dosta opširnog zadatka mogu biti kao primjer ispitnog zadatka. Na predavanjima je prikazan isti zadatak koji je riješen pomoću primjera pasivnog RLC filtra.

Ciljevi:

1. Upoznavanje sa osnovnim principima rada filtara,
2. korištenje metode superpozicije u izračunavanju odziva i fazorskog računa u predstavljanju signala pobude,
3. dobivanje predodžbe o prijenosnoj funkciji, značenju nula prijenosne funkcije,
4. dobivanje predodžbe o a-f i f-f karakteristici filtra,
5. postupak proračuna naočigled kompliciranog filtarskog električnog kruga,
6. uočavanje da navedeni električni krug sadrži operacijska pojačala, te otpore i kapacitete, a ne sadrži induktivitete (kažemo da je to aktivni-RC filtar).

Rješenje: U ovom zadatku ćemo za realizaciju koristiti slijedeću aktivnu RC filtarsku sekciju 2. reda koja ima 4 pojačala:



Prijenosna funkcija se da lako izračunati, npr. pomoću metode napona čvorova, i ona za navedeni električni krug (filtar) glasi:

$$T(s)=\frac{U_{iz}(s)}{U_{ul}(s)}=-\frac{R_8}{R_7}\cdot\frac{s^2+s\frac{\omega_p}{q_p}\left(1-\frac{R_7}{R_5}\cdot\frac{R_1}{R_4}\right)+\omega_p^2}{s^2+\frac{\omega_p}{q_p}s+\omega_p^2}=-k\cdot\frac{s^2+s\frac{\omega_z}{q_z}+\omega_z^2}{s^2+\frac{\omega_p}{q_p}s+\omega_p^2},$$

gdje su:

$$\omega_p=\frac{1}{\sqrt{R_2R_3C_1C_2}},\ q_p=R_1\sqrt{\frac{C_1}{R_2R_3C_2}},\ \omega_z=\omega_p,\ k=\frac{R_8}{R_7}.$$

Srednji član u brojniku je: $\frac{\omega_z}{q_z}=\frac{\omega_p}{q_p}\left(1-\frac{R_7}{R_5}\cdot\frac{R_1}{R_4}\right)$ i on mora biti jednak nuli da bismo ostvarili prijenosnu karakteristiku pojasne brane na frekvenciji $\omega_z=\omega_p=2\pi\cdot 50=100\pi$.

Da bismo izračunali elemente filtra provedimo slijedeći postupak proračuna:

1. Odaberimo vrijednosti elemenata $R_2=R_3=R_4=R_7=R_9=R_{10}=R$, $C_1=C_2=C$. Tada gornje jednadžbe poprimaju jednostavan oblik:

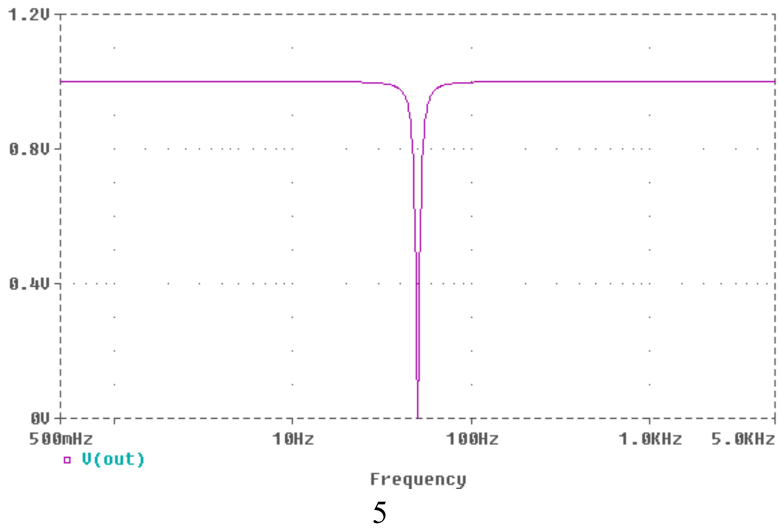
$$\omega_p=\frac{1}{RC},\ q_p=\frac{R_1}{R},\ \frac{q_p}{\omega_p}=\frac{1}{R_1C},\ \omega_z=\omega_p,\ k=\frac{R_8}{R}$$

2. Ako odaberemo kapacitete vrijednosti C=1μF izračunajmo $R=1/(C\,\omega_p)=1/(10^{-6}\,100\pi)=$ 3183.1 Ω=3.183 kΩ
3. Iz $\omega_p/q_p=(R_1C)^{-1}$ izračunajmo $R_1=q_p/(\omega_pC)=10/(10^{-6}\,100\pi)=$ 31831 Ω=31.831 kΩ
4. Izjednačimo $R_5=R_1$ =31.831 kΩ, da bismo ostvarili $q_z=\infty$,
5. Izračunajmo $R_8=k\cdot R$ =1·3.183 kΩ=3.183 kΩ da bismo realizirali jedinično pojačanje u području propuštanja.

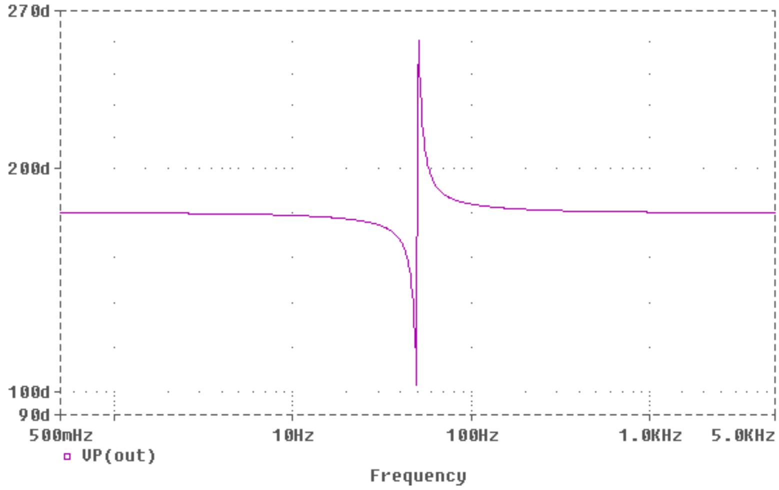
Naponska prijenosna funkcija glasi: $T(s)=-\frac{s^2+98696}{s^2+31.4159\cdot s+98696}$

Za sinusoidalno stacionarno stanje (uvrstimo $s=j\omega$): $T(j\omega)=\frac{98696-\omega^2}{-\omega^2+31.4159\cdot j\omega+98696}$

a-f karakteristika : $|T(j\omega)|=\frac{|98696-\omega^2|}{\sqrt{(98696-\omega^2)^2+(31.4159\cdot\omega)^2}}$



f-f karakteristika : $\varphi(\omega) = \pi \cdot S(\omega - 100\pi) - \arctan \dfrac{31.4159 \cdot \omega}{98696 - \omega^2}$



Koliki je izlazni napon ?

Ulazni napon je $u_{ul}(t) = \sin(2\pi \cdot 50t) + 0.2 \cdot \sin(2\pi \cdot 1000t)$, dakle ima dvije komponente : $u_{ul}(t) = u_{ul1}(t) + u_{ul2}(t)$, gdje su

$$u_{ul1}(t) = \sin(2\pi \cdot 50t);$$
$$u_{ul2}(t) = 0.2 \cdot \sin(2\pi \cdot 1000t).$$

Tim komponentama su pridruženi fazori :

$$U_{ul1}(j\omega) = 1\angle 0°, \text{ uz frekvenciju } \omega = 2\pi \cdot 50 \text{ rad/s};$$
$$U_{ul2}(j\omega) = 0.2\angle 0°, \text{ uz frekvenciju } \omega = 2\pi \cdot 1000 \text{ rad/s}.$$

<u>Metodom superpozicije</u> možemo izračunati valni oblik izlaznog signala:

$$u_{iz}(t) = u_{iz1}(t) + u_{iz2}(t).$$

Pritom je svaka komponenta u izlaznom signalu :

i) Odziv $u_{iz1}(t)$ uslijed prvog signala (poticaja) $u_{ul1}(t)$:

Amplituda signala $u_{iz1}(t)$:

$$|U_{iz1}(j\omega)| = |T(j\omega)| \cdot |U_{ul1}(j\omega)| = \frac{|98696 - \omega^2|}{\sqrt{(98696 - \omega^2)^2 + (31.4159 \cdot \omega)^2}} \cdot 1,$$ te uz uvrštenu

frekvenciju ω=2π·50 rad/s dobivamo da je $|U_{iz1}(j\omega)| = 4.45924 \cdot 10^{-6}$ ≈0.

Fazni kut signala $u_{iz1}(t)$: $\varphi_{iz1}(\omega) = \varphi(\omega) + \varphi_{ul1}(\omega)$, gdje je :

$$\varphi(\omega) = \pi \cdot S(\omega - 100\pi) - \arctan \frac{31.4159 \cdot \omega}{98696 - \omega^2}.$$

Fazni kut signala na izlazu jednak je kutu prijenosne funkcije jer je kut signala na ulazu jednak nula.

Dakle, za frekvenciju ω=2π·50 rad/s dobivamo dva rješenja odnosno da je fazni kut $\varphi_{iz1}(\omega)$=180°–89.9997°≈90° ili $\varphi_{iz1}(\omega)$=–89.9997°≈–90° (odnosno +270°). Ili, jednostavnije, možemo prikazati kut izlaznog signala kao $\varphi_{iz1}(\omega)$=±90°. No taj signal ima amplitudu približno jednaku nuli pa njegov doprinos možemo zanemariti, stoga niti njegov fazni pomak nije od velikog interesa.

ii) Odziv $u_{iz2}(t)$ uslijed drugog signala $u_{ul2}(t)$:

Amplituda signala $u_{iz2}(t)$:

$$|U_{iz2}(j\omega)| = |T(j\omega)| \cdot |U_{ul2}(j\omega)| = \frac{|98696 - \omega^2|}{\sqrt{(98696 - \omega^2)^2 + (31.4159 \cdot \omega)^2}} \cdot 0.2,$$

te uz uvrštenu frekvenciju ω=2π·1000 rad/s dobivamo da je

$|U_{iz2}(j\omega)| = 0.999987 \cdot 0.2 = 0.199997 \approx 0.2.$

Fazni kut signala $u_{iz2}(t)$:

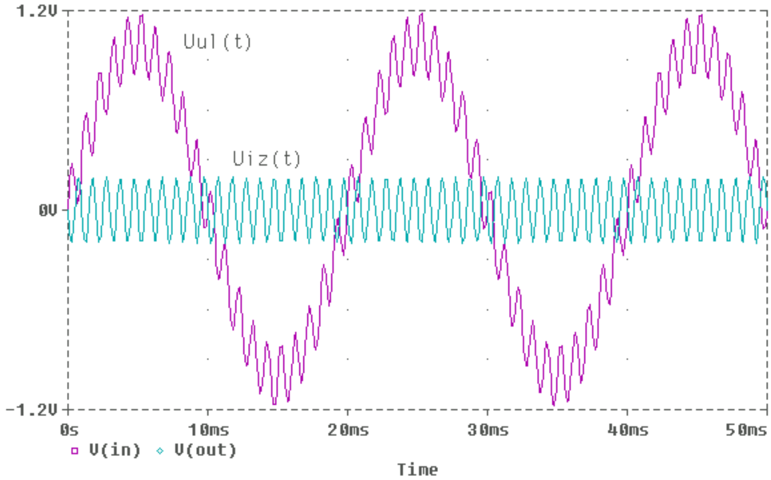
Za frekvenciju ω=2π·1000 rad/s dobivamo koristeći iste izraze za fazni kut kao u prvom slučaju $\varphi_{iz2}(\omega)$= 180°–0.287194°≈180°.

iii) Ukupni odziv $u_{iz}(t)$:

Konačno, izlazni napon ne sadrži više komponentu od 50 Hz pa je:

$$u_{iz}(t) = 0.2 \cdot \sin(2\pi \cdot 1000t + 180°) = -0.2 \cdot \sin(2\pi \cdot 1000t)$$

Dakle, cijeli odziv je jednak izlaznom naponu jedino uslijed poticaja signala $u_{ul2}(t)$. Poticaj i odziv su prikazani na slijedećoj slici:



Sve slike su realizirane pomoću programa PSpice.

Četveropoli

4. Koliki moraju biti α, β i γ (α, β, γ>0) da bi četveropol zadan prijenosnim parametrima bio recipročan i simetričan. Naći ekvivalentni *T*-spoj.

$$[a]=\begin{bmatrix} \dfrac{1+\alpha s}{1-2s} & \dfrac{4}{1-2s} \\ \dfrac{\beta s}{1-2s} & \dfrac{\gamma+\alpha s}{1-2s} \end{bmatrix}$$

Rješenje:

Uvjet simetrije: $A = D$

$A = D \Rightarrow \gamma = 1$

Uvjet recipročnosti: $\Delta = AD - BC = 1 \quad (\Delta = \det[a])$

$$AD - BC = 1 \quad \Rightarrow \quad \frac{(1+\alpha s)^2}{(1-2s)^2} - \frac{4\beta s}{(1-2s)^2} = 1$$

$$(1+\alpha s)^2 - 4\beta s = (1-2s)^2$$

$$1 + 2\alpha s + \alpha^2 s^2 - 4\beta s = 1 - 4s + 4s^2$$

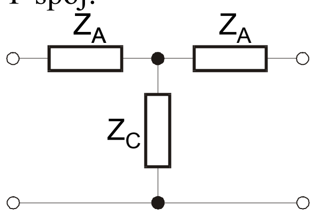
$$1 + (2\alpha - 4\beta)s + \alpha^2 s^2 = 1 - 4s + 4s^2$$

$$\alpha^2 = 4 \quad \Rightarrow \quad \alpha = 2$$

$$\underline{2\alpha - 4\beta = -4} \quad \Rightarrow \quad 4 - 4\beta = -4 \quad \Rightarrow \quad \beta = \frac{8}{4} = 2$$

$$[a]=\begin{bmatrix} \dfrac{1+2s}{1-2s} & \dfrac{4}{1-2s} \\ \dfrac{2s}{1-2s} & \dfrac{1+2s}{1-2s} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

T-spoj:



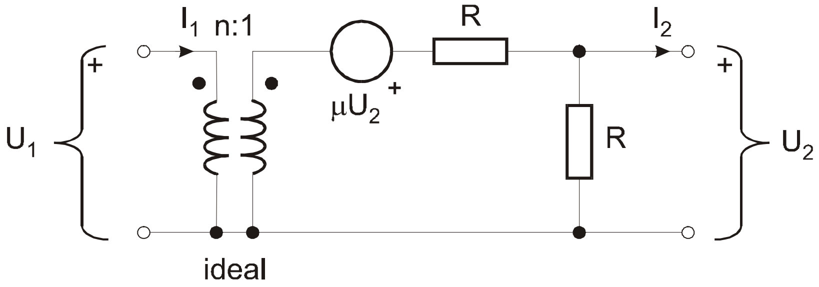
$$z_{11} = \frac{A}{C} = \frac{1+2s}{2s}$$

$$z_{12} = \frac{AD}{C} - B = \frac{(1+2s)^2}{2s(1-2s)} - \frac{4}{1-2s} = \frac{1-2s}{2s}$$

$$Z_A = z_{11} - z_{12} = 2$$

$$Z_C = z_{12} = \frac{1-2s}{2s} = \frac{1}{2s} - 1$$

5. Za četveropol prikazan slikom izračunati $[y]$ parametre. Odrediti vrijednost za n kako bi četveropol bio simetričan.



Rješenje:

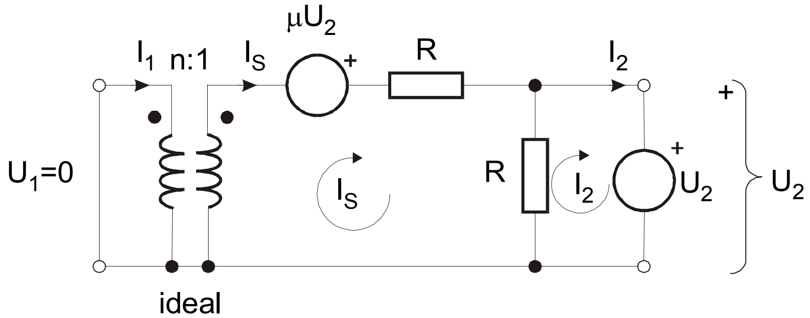
$$I_1 = y_{11} \cdot U_1 - y_{12} \cdot U_2$$
$$I_2 = y_{21} \cdot U_1 - y_{22} \cdot U_2$$

$$y_{11} = \left.\frac{I_1}{U_1}\right|_{U_2=0} \quad y_{21} = \left.\frac{I_2}{U_1}\right|_{U_2=0} \quad y_{12} = -\left.\frac{I_1}{U_2}\right|_{U_1=0} \quad y_{22} = -\left.\frac{I_2}{U_2}\right|_{U_1=0}$$

jednadžbe transformatora :

$$U_S = \frac{1}{n} \cdot U_1$$
$$I_S = n \cdot I_1$$

a) $U_1 = 0 \qquad \Rightarrow \qquad U_S = 0$

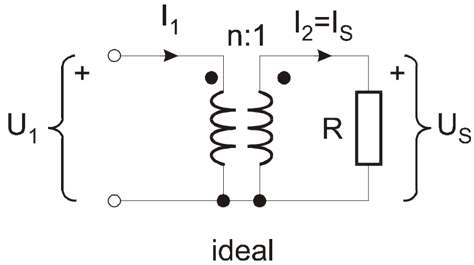


$$I_S \cdot R + U_2 = \mu \cdot U_2 \quad \Rightarrow \quad I_S \cdot R = (\mu - 1) \cdot U_2 \quad \Rightarrow \quad nI_1 \cdot R = (\mu - 1) \cdot U_2 \quad \Rightarrow \quad I_1 = -\frac{1-\mu}{n \cdot R} \cdot U_2$$

$$y_{12} = -\left.\frac{I_1}{U_2}\right|_{U_1=0} = \frac{1-\mu}{n \cdot R}$$

$$U_2 = I_S \cdot R - I_2 \cdot R \quad \Rightarrow \quad U_2 = \frac{\mu - 1}{R} \cdot U_2 \cdot R - I_2 \cdot R \quad \Rightarrow \quad y_{22} = -\left.\frac{I_2}{U_2}\right|_{U_1=0} = \frac{2-\mu}{R}$$

$U_2 = 0$



iz jednadžbi transformatora slijedi $\Rightarrow I_1 = \frac{1}{n} \cdot I_2 \quad I_2 = n \cdot I_1$

$$U_S = I_2 \cdot R \quad \Rightarrow \frac{1}{n} \cdot U_1 = n \cdot I_1 \cdot R \quad \Rightarrow \quad y_{11} = \left. \frac{I_1}{U_1} \right|_{U_2=0} = \frac{1}{n^2 R}$$

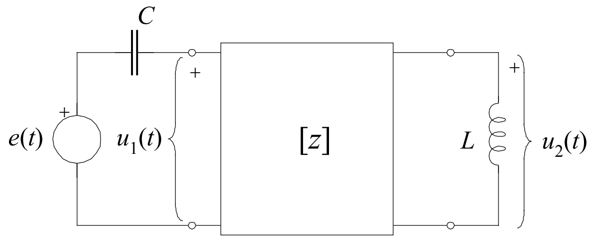
$$y_{21} = \left. \frac{I_2}{U_1} \right|_{U_2=0} = \frac{n \cdot I_1}{U_1} = n \cdot \frac{1}{n^2 R} = \frac{1}{nR}$$

$$[y] = \begin{bmatrix} \frac{1}{n^2 R} & -\frac{1-\mu}{nR} \\ \frac{1}{nR} & -\frac{2-\mu}{R} \end{bmatrix} = \begin{bmatrix} y_{11} & -y_{12} \\ y_{21} & -y_{22} \end{bmatrix}$$

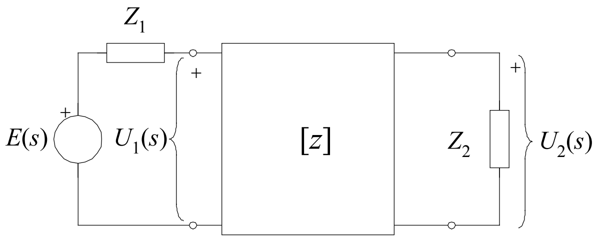
$\frac{1-\mu}{nR} = \frac{1}{nR} \Rightarrow \qquad \mu = 0$ za recipročnost

$\frac{1}{n^2 R} = \frac{2-\mu}{R} \Rightarrow \qquad n = \frac{1}{\sqrt{2-\mu}}$ za simetričnost

6. Naći naponsku prijenosnu funkciju $T(s)=U_2(s)/E(s)$ četveropola na slici. Odrediti napon $u_2(t)$ na izlazu četveropola ako je zadano $e(t)=S(t)$, $C=1$, $L=2$.



$$[\mathbf{z}]=\begin{bmatrix} 2s+1 & -2s \\ 2s & -2s \end{bmatrix}$$



Rješenje:

Prijenosna funkcija: $T(s)=\dfrac{U_2(s)}{E(s)}$

Jednadžbe četveropola

$$U_1 = z_{11}I_1 - z_{12}I_2$$

$$U_2 = z_{11}I_1 - z_{12}I_2$$

$$I_1(s)=\frac{E(s)-U_1(s)}{Z_1} \quad ; \quad I_2(s)=\frac{U_2(s)}{Z_2}$$

$$T(s)=\frac{U_2(s)}{E(s)}=\frac{Z_2\, z_{21}}{(Z_1+z_{11})(Z_2+z_{22})-(z_{12}\, z_{21})}$$

$$T(s)=\frac{s^2}{s^2+s+1}$$

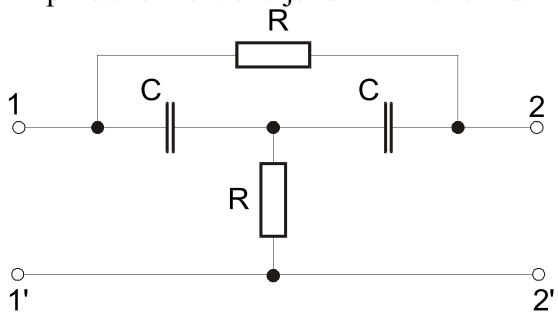
$$U_2(s)=T(s)\cdot E(s)=\frac{s^2}{s^2+s+1}\cdot\frac{1}{s}=\frac{s}{s^2+s+1}$$

$$U_2(s)=\frac{s}{s^2+s+1} \qquad s_{p1,2}=\frac{-1\pm\sqrt{1-4}}{2}=-\frac{1}{2}\pm j\frac{\sqrt{3}}{2}$$

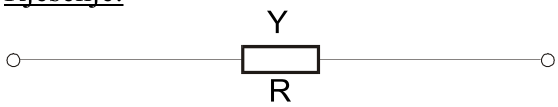
$$U_2(s)=\frac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}-\frac{1}{2}\frac{2}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

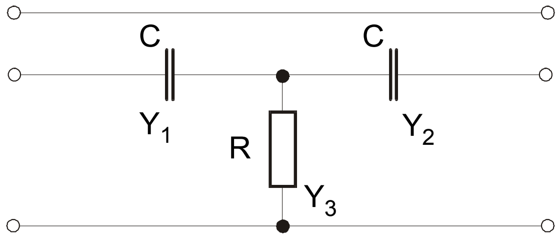
$$u_2(t)=e^{-\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t-\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)\cdot S(t)$$

7. Za četveropol prikazan slikom odrediti matricu [**y**]-parametara i prijenosnu funkciju $T(s)=U_{iz}(s)/U_{ul}(s)$, za prazni hod na izlazu. Nacrtati raspored polova i nula u kompleksnoj s-ravnini i konstruirati amplitudno-frekvencijsku karakteristiku. Zadano je R=1, C=1.



Rješenje:





$$[\mathbf{y}]^{I} = \begin{bmatrix} \frac{1}{R} & -\frac{1}{R} \\ \frac{1}{R} & -\frac{1}{R} \end{bmatrix}$$

$$[\mathbf{y}]^{I} = \begin{bmatrix} Y & -Y \\ Y & -Y \end{bmatrix}$$

$$[\mathbf{y}]^{II} = \begin{bmatrix} \frac{Y_1(Y_2+Y_3)}{Y_1+Y_2+Y_3} & -\frac{Y_1Y_2}{Y_1+Y_2+Y_3} \\ \frac{Y_1Y_2}{Y_1+Y_2+Y_3} & -\frac{Y_2(Y_1+Y_3)}{Y_1+Y_2+Y_3} \end{bmatrix}$$

$$Y_1+Y_2+Y_3 = 2sC+\frac{1}{R}$$

$$Y_1(Y_2+Y_3) = sC\left(sC+\frac{1}{R}\right) = s^2C^2+\frac{sC}{R}$$

$$Y_1Y_2 = s^2C^2$$

$$[\mathbf{y}]^{II} = \begin{bmatrix} \frac{sC(sRC+1)}{2sRC+1} & -\frac{s^2C^2R}{2sRC+1} \\ \frac{s^2C^2R}{2sRC+1} & -\frac{sC(sRC+1)}{2sRC+1} \end{bmatrix}$$

$$[\mathbf{y}]=[\mathbf{y}]^{I}+[\mathbf{y}]^{II}=\begin{bmatrix} \dfrac{sC(sRC+1)}{2sRC+1}+\dfrac{1}{R} & -\dfrac{s^2C^2R}{2sRC+1}-\dfrac{1}{R} \\ \dfrac{s^2C^2R}{2sRC+1}+\dfrac{1}{R} & -\left[\dfrac{sC(sRC+1)}{2sRC+1}+\dfrac{1}{R}\right] \end{bmatrix}=$$

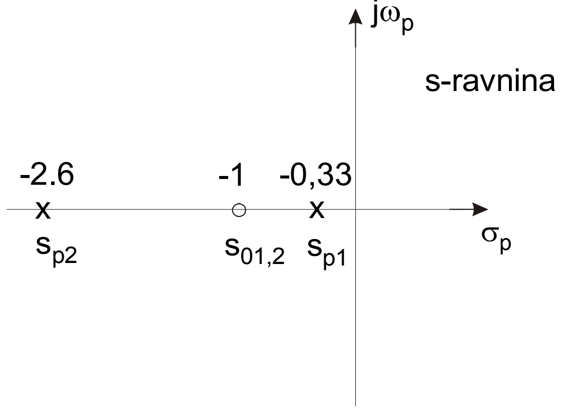
$$=\frac{1}{2s+1}\begin{bmatrix} s^2+3s+1 & -(s^2+2s)+1 \\ s^2+2s+1 & -(s^2+3s+1) \end{bmatrix}$$

$$T(s)=\frac{y_{21}}{y_{22}}=\frac{U_2}{U_1}=\frac{s^2+2s+1}{s^2+3s+1}$$

nule: $(s+1)^2=0 \quad S_{0_{1,2}}=-1$

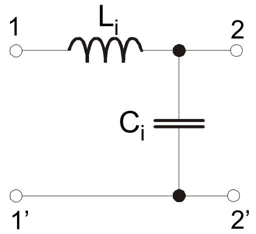
polovi: $s^2+3s+1=0$

$$S_{P_{12}}=\frac{-3\pm\sqrt{9-4}}{2}=\frac{-3\pm\sqrt{5}}{2}=\begin{cases}-2.618\\-0.382\end{cases}$$

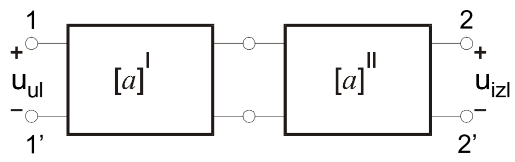


$$|T(j\omega)|=\frac{\sqrt{4\omega^2-(1-\omega^2)^2}}{\sqrt{9\omega^2+(1-\omega^2)^2}}$$

8. Za četveropol na slici **(a)** naći matricu prijenosnih parametara [a]. Za kaskadu dva takva četveropola, koja je prikazana na slici **(b),** naći ukupnu matricu prijenosnih parametara [a]. Pomoću matrice prijenosnih parametara izračunati naponsku prijenosnu funkciju kaskade $T(s)=U_{iz}(s)/U_{ul}(s)$, ako je na izlazu otpornik R=1. Zadano je C_i=L_i=1 (i=1, 2).



(a)



(b)

Rješenje:

prijenosne jednadžbe četeveropola

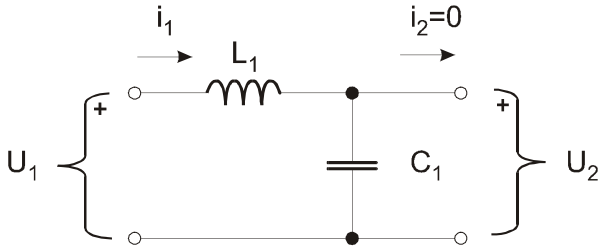
(1) $\quad U_1 = A \cdot U_2 + B \cdot I_2$

(2) $\quad I_1 = C \cdot U_2 + D \cdot I_2$

iz njih slijede prijenosni parametri

$$A = \frac{U_1}{U_2}\Big|_{I_2=0}; \quad B = \frac{U_1}{I_2}\Big|_{U_2=0}; \quad C = \frac{I_1}{U_2}\Big|_{I_2=0}; \quad D = \frac{I_1}{I_2}\Big|_{U_2=0}$$

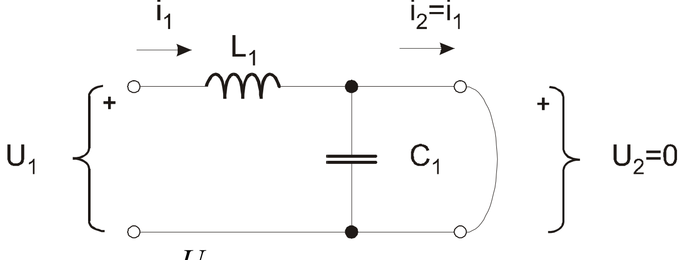
$I_2 = 0$



$$\frac{U_2}{U_1} = \frac{\frac{1}{sC_1}}{sL_1 + \frac{1}{sC_1}} = \frac{1}{s^2 L_1 C_1 + 1} \Rightarrow \qquad A = \frac{U_1}{U_2} = s^2 L_1 C_1 + 1$$

$$U_2 = I_1 \cdot \frac{1}{sC_1} \Rightarrow \qquad C = \frac{I_1}{U_2} = sC_1$$

$U_2 = 0$



$$U_1 = I_1 \cdot sL_1 = I_2 \cdot sL_1 \Rightarrow \qquad B = \frac{U_1}{I_2} = sL_1$$

$$I_1 = I_2 \Rightarrow \qquad D = \frac{I_1}{I_2} = 1$$

Konačno matrica prijenosnih parametara $[a]$ glasi

$[a]^{I} = \begin{bmatrix} 1+s^2L_1C_1 & sL_1 \\ sC_1 & 1 \end{bmatrix}$ analogno tome slijedi i $[a]^{II} = \begin{bmatrix} 1+s^2L_2C_2 & sL_2 \\ sC_2 & 1 \end{bmatrix}$

Spoj u kaskadu ili lanac:

$$[a] = [a]^{I} \cdot [a]^{II} = \begin{bmatrix} 1+s^2L_1C_1 & sL_1 \\ sC_1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1+s^2L_2C_2 & sL_2 \\ sC_2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1+s^2L_1C_1)(1+s^2L_2C_2)+sL_1sC_2 & (1+s^2L_1C_1)sL_2+sL_1 \\ sC_1(1+s^2L_2C_2)+sC_2 & sC_1sL_2+1 \end{bmatrix}$$

$$[a] = \begin{bmatrix} s^4L_1L_2C_1C_2 + s^2(L_1C_1+L_2C_2+L_1C_2)+1 & s^3L_1C_1L_2 + s(L_1+L_2) \\ s^3L_2C_2C_1 + s(C_1+C_2) & s^2L_2C_1+1 \end{bmatrix}$$

uz uvrštene vrijednosti:

$$[a] = \begin{bmatrix} s^4+3s^2+1 & s^3+2s \\ s^3+2s & s^2+1 \end{bmatrix}$$

zaključenje $Z_2 = \dfrac{U_2}{I_2}$

(1) $\qquad U_1 = A \cdot U_2 + \dfrac{B}{Z_2} \cdot U_2 = \left(A + \dfrac{B}{Z_2} \right) \cdot U_2$

naponska prijenosna funkcija glasi:

$T(s) = \dfrac{U_2}{U_1} = \dfrac{1}{A + \dfrac{B}{Z_2}}$; $\; Z_2 = R$, odnosno:

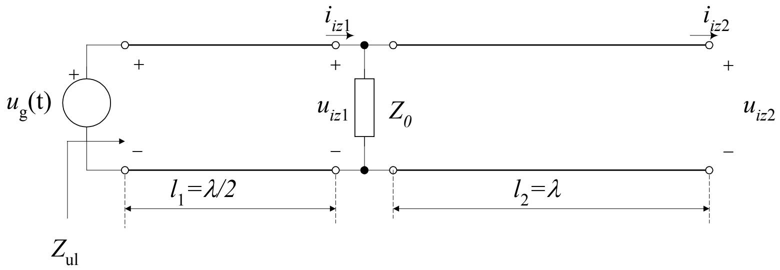
$$T(s) = \frac{1}{s^4L_1L_2C_1C_2 + s^2(L_1C_1+L_2C_2+L_1C_2)+1+s^3\dfrac{L_1C_1L_2}{R}+s\dfrac{L_1+L_2}{R}}$$

uz uvrštene vrijednosti:

$$T(s) = \frac{1}{s^4+s^3+3s^2+2s+1}$$

Linije

9. Zadan je spoj dviju linija bez gubitaka s primarnim parametrima C=1nF/km, L=1mH/km, duljina $\ell_1=\lambda/2$ i $\ell_2=\lambda$ prema slici. Odrediti valne oblike napona i struje na krajevima linija u_{izli} i i_{izli}; (i=1,2) ako je zadano $u_g(t)=\cos(2\pi\ 10^3 t)$; $-\infty<t<\infty$.



$$Z_0=\sqrt{\frac{L}{C}}=\sqrt{\frac{1\cdot 10^{-3}}{1\cdot 10^{-9}}}=\sqrt{10^6}=10^3\,\Omega \qquad \gamma=\alpha+j\beta;\quad \alpha=0,\ \ \beta=\omega_0\sqrt{LC}$$

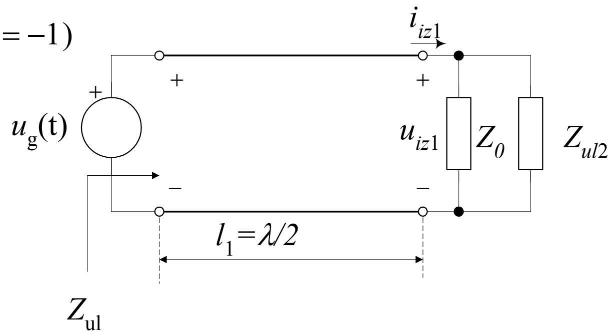
ω_0 - frekvencija sinusne pobude; $\lambda=2\pi/\beta$ - valna duljina signala

Prijenosne jednadžbe linije :

$$U(0)=U(l)\mathrm{ch}(\gamma l)+Z_0 I(l)\mathrm{sh}(\gamma l)=U(l)\cos(\beta l)+Z_0 I(l) j\sin(\beta l)$$

$$I(0)=\frac{U(l)}{Z_0}j\mathrm{sh}(\gamma l)+I(l)\mathrm{ch}(\gamma l)=\frac{U(l)}{Z_0}j\sin(\beta l)+I(l)\cos(\beta l)$$

($\mathrm{ch}\, jx=\cos x$, $\mathrm{sh}\, jx=j\sin x$, $\sin\pi=0$, $\cos\pi=-1$)



a) prva linija: $l_1=\dfrac{\lambda}{2}=\dfrac{\pi}{\beta}\ \Rightarrow\ \beta\cdot l_1=\pi$

$$U^I(0)=-U^I(l_1)$$
$$\underline{I^I(0)=-I^I(l_1)}$$
$$U^I(l_1)=(Z_0\,\|\,Z_{ul2})I^I(l_1)$$

b) druga linija: $l_2=\lambda=\dfrac{2\pi}{\beta}\ \Rightarrow\ \beta\cdot l_2=2\pi$

$$Z_{ul2}=\frac{U^{II}(0)}{I^{II}(0)}=\frac{U^{II}(l)\mathrm{ch}(\gamma l_2)+Z_0 I^{II}(l)\mathrm{sh}(\gamma l_2)}{\frac{U^{II}(l)}{Z_0}\mathrm{sh}(\gamma l_2)+I^{II}(l)\mathrm{ch}(\gamma l_2)}=\frac{Z_2\cos(\beta l_2)+jZ_0\sin(\beta l_2)}{j\frac{Z_2}{Z_0}\sin(\beta l_2)+\cos(\beta l_2)}=Z_2$$

($\mathrm{ch}\, jx=\cos x$, $\mathrm{sh}\, jx=j\sin x$, $\sin 2\pi=0$, $\cos 2\pi=1$)

$Z_2=\infty$ pa je $Z_{ul2}=\infty$

$$U^I(l_1)=Z_0 I^I(l_1)$$

$$U^I(l_1)=-U^I(0)=-1=1\cdot e^{j\pi}\ \Rightarrow\ \underline{u_l^I(t)=u_{izl1}(t)=-\cos(2\pi\cdot 10^3 t)}$$

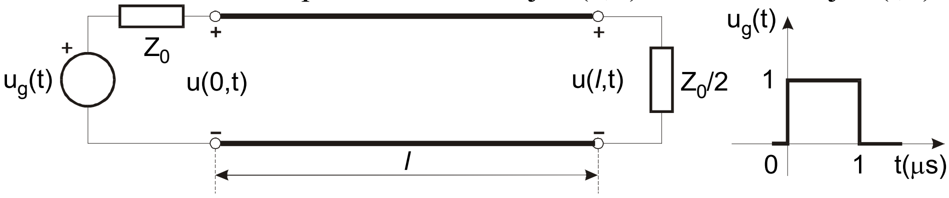
$$\underline{I^I(l_1)=\frac{U^I(l_1)}{Z_0}}=-10^{-3}=10^{-3}\cdot e^{j\pi}\ \Rightarrow\ \underline{i_l^I(t)=i_{izl1}(t)=-10^{-3}\cos(2\pi\cdot 10^3 t)}$$

$$Z_{ul}^I=\frac{U^I(0)}{I^I(0)}=Z_0$$

$$U^{II}(l_2)=U^{II}(0)\cdot\underbrace{\cos 2\pi}_{1}-I^{II}(0)\cdot Z_0 j\underbrace{\sin 2\pi}_{0}=U^{II}(0)=U^I(l_1)$$

$$\underline{u_{izl2}(t)=u_{izl1}(t)=-\cos(2\pi\cdot 10^3 t)} \qquad \underline{i_{izl2}(t)=0}$$

10. Zadana je linija s primarnim parametrima R=0.5Ω/km, L=10μH/km, G=0,2mS/km, C=4nF/km, duljine l=1km. Na liniju je spojen generator $u_g(t)$ s unutarnjim otporom jednakim zrcalnoj impedanciji linije Z_0 i valnim oblikom prema slici, a linija je zaključena s $Z_0/2$. Odrediti i nacrtati valne oblike napona na ulazu linije $u(0, t)$ i na izlazu linije $u(l, t)$.



Rješenje:

$$\frac{R}{L}=\frac{0.5\Omega}{10\mu H}=\frac{0.5}{10\cdot 10^{-6}}=5\cdot 10^{4} \qquad \frac{G}{C}=\frac{0,2mS}{4nF}=5\cdot 10^{4} \Rightarrow \frac{R}{L}=\frac{G}{C}$$

Linija bez distorzije:

$$Z_0=\sqrt{\frac{L}{C}}=\sqrt{\frac{10\cdot 10^{-6}}{4\cdot 10^{-9}}}=50\Omega \qquad \gamma=s\sqrt{LC}+\sqrt{RG}=20\cdot 10^{-6}s+0,01$$

Polazni val na izlazu: $U_p(l)=\underbrace{U(0)}_{\frac{U_g}{2}}\cdot e^{-\gamma\cdot l}=\underbrace{U(0)}_{\frac{U_g}{2}}\cdot e^{-(20\cdot 10^{-6}\cdot s)}\cdot e^{-0,01}$

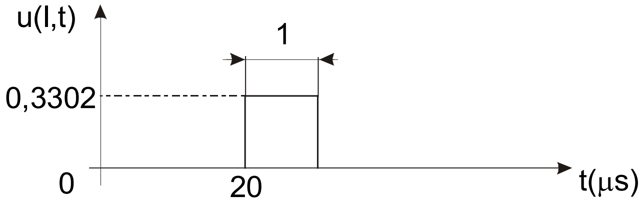
faktor refleksije na izlazu: $\Gamma_2=\frac{Z_2-Z_0}{Z_2+Z_0}=\frac{\frac{Z_0}{2}-Z_0}{\frac{Z_0}{2}+Z_0}=-\frac{1}{3}$

reflektirani val na izlazu: $U_r(l)=\Gamma_2\cdot U_p(l)=\Gamma_2\cdot\frac{U_g(s)}{2}\cdot e^{-(20\cdot 10^{-6}\cdot s)}\cdot e^{-0,01}$

ukupni napon na izlazu:

$$U_{izl}=U_p(l)+U_r(l)=(1+\Gamma_2)\cdot U_p(l)=(1+\Gamma_2)\cdot\frac{U_g(s)}{2}\cdot e^{-(20\cdot 10^{-6}\cdot s)}\cdot e^{-0,01}=\frac{2}{3}\cdot\frac{U_g(s)}{2}\cdot e^{-(20\cdot 10^{-6}\cdot s)}\cdot e^{-0,01}$$

$$\Rightarrow u_{izl}(t)=\frac{1}{3e^{0,01}}\cdot u_g(t-20\cdot 10^{-6}), \qquad e^{0,01}=1,01005$$



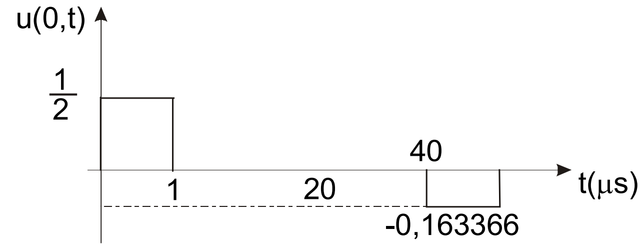
Rreflektirani val na ulazu:

$$U_r(0)=U_r(l)\cdot e^{-(20\cdot 10^{-6}\cdot s+0,01)}=\Gamma_2\cdot U_p(l)\cdot e^{-(20\cdot 10^{-6}\cdot s+0,01)}=-\frac{1}{3}\cdot\frac{U_g(s)}{2e^{0,01}}\cdot e^{-(20\cdot 10^{-6}\cdot s)}\cdot e^{-(20\cdot 10^{-6}\cdot s)}\cdot e^{-0,01}$$

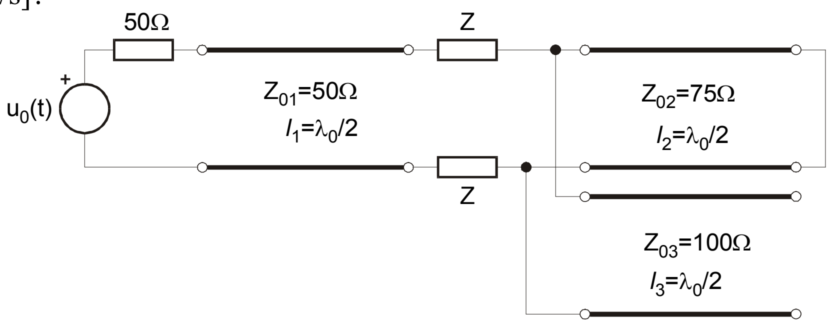
$$U_r(0)=-\frac{1}{6e^{0,02}}\cdot U_g(s)\cdot e^{-(40\cdot 10^{-6}\cdot s)} \Rightarrow u_r(0,t)=-\frac{1}{6e^{0,02}}\cdot u_g(t-40\cdot 10^{-6})$$

Ukupni napon na ulazu :

$$u(0,t)=\frac{u_g(t)}{2}-\frac{1}{6e^{0,02}}\cdot u_g(t-40\cdot 10^{-6}), \qquad e^{0,02}=1,0202$$



11. Zadan je sustav linija bez gubitaka prikazan slikom. Odrediti impedanciju Z da bi prva linija bila prilagođena po zrcalnim impedancijama. Odrediti napone na kraju svake linije ako je u_0=sin (4π 10^5 t)[mV]. Koliko su duge linije ako je brzina širenja vala na linijama v=4·10^5[km/s]?



Rješenje:
linija bez gubitaka uz sinusoidalnu pobudu

$$Z_0 = \sqrt{\frac{L}{C}} \qquad \gamma = \alpha + j\beta \Rightarrow \alpha = 0,\ \beta = \omega_0\sqrt{LC} \qquad \lambda = \frac{v}{f_0} = \frac{2\cdot\pi\cdot v}{\omega_0}$$

$f_0 = \dfrac{\omega_0}{2\pi}$ frekvencija signala

$v = \dfrac{\omega}{\beta}$ brzina širenja vala duž linije

$\lambda = \dfrac{2\pi}{\beta}$ valna dužina

$$\Rightarrow \lambda_0 = \frac{2\pi}{\omega_0\sqrt{LC}} \Rightarrow \lambda_0 = \frac{2\pi\cdot(4\cdot 10^{-5})}{4\pi\cdot 10^5} = 2km$$

$$l_1 = l_2 = l_3 = \frac{\lambda_0}{2} = 1km$$

prijenosne jednadžbe linije:

$$U(0) = U(x)\cdot ch(\gamma x) + I(x)Z_0 sh(\gamma x) = U(x)\cdot\cos(\beta x) + jI(x)Z_0\sin(\beta x)$$

$$I(0) = \frac{U(x)}{Z_0}sh(\gamma x) + I(x)ch(\gamma x) = j\frac{U(x)}{Z_0}\sin(\beta x) + I(x)\cos(\beta x)$$

---

ako je $x = l \Rightarrow \gamma x = \gamma l = g$ i ako je linija zaključena impedancijom $Z_2 = \dfrac{U(l)}{I(l)}$ tada je ulazna

impedancija $Z_{ul} = \dfrac{U(0)}{I(0)} = \dfrac{Z_2 ch\, g + Z_0 sh\, g}{\dfrac{Z_2}{Z_0}sh\, g + ch\, g}$

općenito vrijedi: $ch\, jx = \cos x$
$sh\, jx = j \sin x$

ako je: $l = \dfrac{\lambda_0}{2} \Rightarrow g = \gamma \cdot l = j\beta \cdot \dfrac{1}{2} \cdot \dfrac{2\pi}{\beta} = j\dfrac{1}{2} \cdot 2\pi = j\pi$

$ch\, g = ch\, j\pi = \cos \pi = -1$

$sh\, g = sh\, j\pi = j \sin \pi = 0$

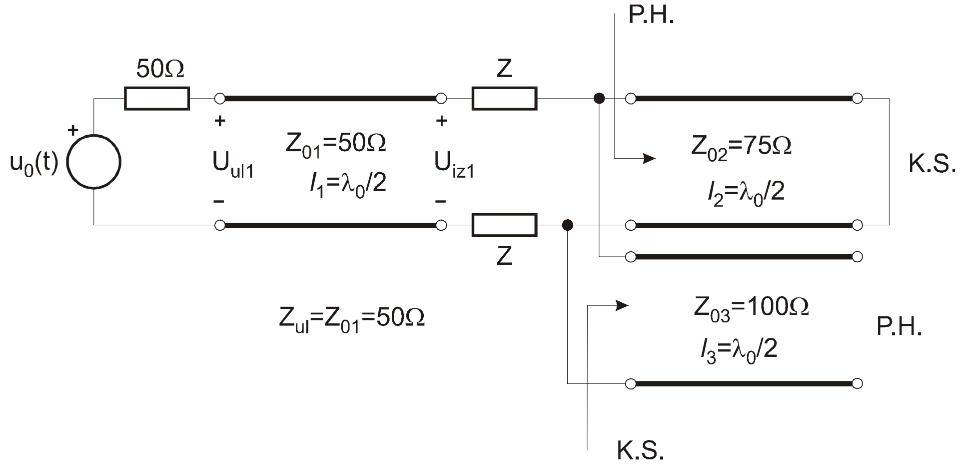
$$Z_{ul} = \frac{Z_2 \cdot (-1) + Z_0 \cdot 0}{\dfrac{Z_2}{Z_0} \cdot 0 + (-1)} = Z_2$$

imamo slijedeće slučajeve:

a) $Z_2 = \infty \Rightarrow \quad Z_{ul} = \infty$

b) $Z_2 = 0 \Rightarrow \quad Z_{ul} = 0$

c) $Z_2 = Z_0 \Rightarrow \quad Z_{ul} = Z_0$



$$Z = \frac{Z_0}{2} = 25\Omega$$

prijenosna funkcija na prvoj liniji uz zaključenje $Z_2 = Z_0 = \dfrac{U_{iz1}}{I_{iz1}} = 50\Omega$

$$U(0) = U(l) ch\gamma l + \underbrace{I(l) \cdot Z_0}_{U(l)} \cdot sh\gamma l = U(l) \cdot (ch\gamma l + sh\gamma l) = U(l) \cdot e^{g}$$

$T(s) = \dfrac{U(l)}{U(0)} = e^{-g}; \qquad g = j\pi$, dolazi do zakreta faze za $-\pi$, a gušenja nema

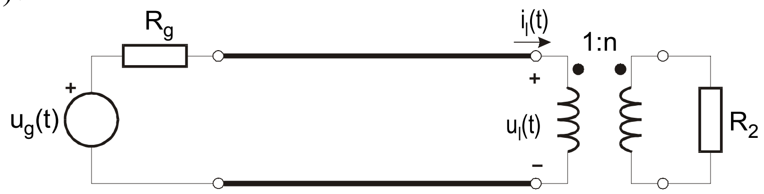
$$sh\, x = \frac{e^{-x} - e^{-x}}{2}; \qquad ch\, x = \frac{e^{x} + e^{-x}}{2} \Rightarrow \qquad sh\, x + ch\, x = e^{x}$$

$$U_{ul1} = \frac{U_0}{2}$$

$$U_{iz1} = U_{ul1} \cdot e^{-j\pi} = \frac{U_0}{2} \cdot e^{-j\pi}$$

$$u_{iz1}(t) = \frac{1}{2} \sin(4\pi \cdot 10^5 t - \pi) [mV] \qquad u_{iz2}(t) = u_{iz3}(t) = 0 [V]$$

12. Zadana je linija bez gubitaka s primarnim parametrima L=1mH/km i C=400nF/km duljine l=314m prema slici. Na ulaz je spojen naponski izvor $u_g(t)=10 \sin 10^6 t$ unutrašnjeg otpora jednakog karakterističnoj impedanciji linije. Koliki mora biti omjer transformatora n da bi na izlazu bilo postignuto prilagođenje ako je R_2=800Ω. Koliki su napon i struja na izlazu linije $u_l(t)$ i $i_l(t)$?



Rješenje:

Linija bez gubitaka: $R = 0;\ \ G = 0$

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{1\cdot 10^{-3}}{400\cdot 10^{-9}}} = 50\Omega$$

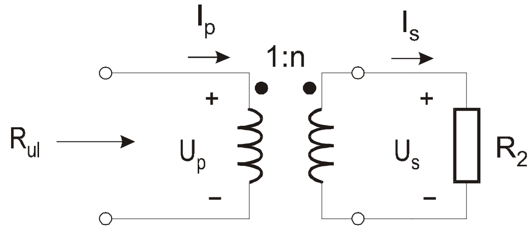
$$\gamma = s\cdot\sqrt{LC} = s\cdot\sqrt{10^{-3}\cdot 400\cdot 10^{-9}} = s\cdot 2\cdot 10^{-5}/km$$

$$\gamma = \alpha + j\beta = 0 + j\omega_0\sqrt{LC}$$

$$\nu = \frac{\omega_0}{\beta}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega_0\sqrt{LC}} = \frac{2\pi}{20}\cdot km = 0.314\,km$$

$$\beta = \omega_0\cdot\sqrt{LC} = 10^6\cdot 2\cdot 10^{-5} = 20/km$$



$$U_s = n\cdot U_p$$

$$I_s = \frac{1}{n}\cdot I_p$$

$$U_s = I_s\cdot R_2$$

$$R_{ul} = \frac{U_p}{I_p} = \frac{\frac{U_s}{n}}{I_s\cdot n} = \frac{\frac{U_s}{I_s}}{n^2} = \frac{R_2}{n^2}$$

$$R_2 = n^2\cdot R_{ul}$$

$$n^2 = \frac{R_2}{R_{ul}} = \frac{800}{50} = 16 \Rightarrow \qquad n = 4$$

$g = j\omega\sqrt{LC}\cdot l = j\cdot 10^{6}\cdot 2\cdot 10^{-5}\cdot 0.314 = j\cdot 2\pi$

prijenosne jednadžbe linije:

$$U(x) = U(0)\cdot ch\gamma x - I(0)\cdot Z(0)\cdot sh\gamma x$$
$$I(x) = -\frac{U(0)}{Z_0}\cdot sh\gamma x + I(0)\cdot ch\gamma x$$

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na mjestu $x = l$

$$U(l) = U(0)\cdot ch\, g - I(0)\cdot Z(0)\cdot sh\, g$$

na ulazu linije $I(0) = \dfrac{U(0)}{Z_0}$ jer je $Z_{ul} = Z_0 = \dfrac{R_2}{n^2} = 50\Omega$ (prilagođenje)

$$U(l) = U(0)\cdot (ch\, g - sh\, g) = U(0)\cdot e^{-g}$$
$$U(0) = \frac{U_g}{2} = 5\angle 0°$$
$$U(l) = 5\cdot e^{-j2\pi} = 5\cdot(\cos 2\pi - j\sin 2\pi) = -j5$$

$$I(l) = -\frac{U(0)}{Z_0} sh\, g + I(0) ch\, g$$
$$I(0) = \frac{U(0)}{Z_0}$$

$$I(l) = -\frac{U_0}{Z_0} sh\, g + \frac{U_0}{Z_0} ch\, g = \frac{U(0)}{Z_0}(ch\, g - sh\, g) = \frac{U(0)}{Z_0}\cdot e^{-g}$$
$$I(l) = \frac{U(l)}{Z_0} = \frac{-j5}{50} = -0.1j$$

$$u(l,t) = 5\sin(10^6 t - 90°)$$
$$i(l,t) = 0.1\sin(10^6 t - 90°)$$