## Table with the $\underline{\text{del}}$ operator in cylindrical, spherical and parabolic cylindrical coordinates

Operation	<u>Cartesian coordinates</u> (x,y,z)	<u>Cylindrical coordinates</u> (ρ,φ,z)	<u>Spherical coordinates</u> (r,θ,φ)
Definition of coordinates	$ \rho = \sqrt{x^2 + y^2}  \phi = \arctan(y/x)  z = z $	$\begin{array}{rcl} x & = & \rho \cos \phi \\ y & = & \rho \sin \phi \\ z & = & z \end{array}$	$x = r \sin \theta \cos \phi$ $y = r \sin \theta \sin \phi$ $z = r \cos \theta$
	$r = \sqrt{x^2 + y^2 + z^2}$ $\theta = \arccos(z/r)$ $\phi = \arctan(y/x)$	$r = \sqrt{\rho^2 + z^2}$ $\theta = \arctan(\rho/z)$ $\phi = \phi$	$ \rho = r \sin(\theta)  \phi = \phi  z = r \cos(\theta) $
Definition of unit vectors	$egin{array}{lcl} \hat{oldsymbol{ ho}} &=& rac{x}{\sqrt{x^2+y^2}} \hat{f x} + rac{y}{\sqrt{x^2+y^2}} \hat{f y} \ \hat{oldsymbol{\phi}} &=& -rac{y}{\sqrt{x^2+y^2}} \hat{f x} + rac{x}{\sqrt{x^2+y^2}} \hat{f y} \ \hat{f z} &=& \hat{f z} \end{array}$	$\begin{array}{rcl} \hat{\mathbf{x}} & = & \cos\phi  \hat{\boldsymbol{\rho}} - \sin\phi  \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} & = & \sin\phi  \hat{\boldsymbol{\rho}} + \cos\phi  \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} & = & \hat{\mathbf{z}} \end{array}$	$ \hat{\mathbf{x}} = \sin \theta \cos \phi \hat{\mathbf{r}} + \cos \theta \cos \phi \hat{\boldsymbol{\theta}} - \sin \phi \hat{\boldsymbol{\phi}}  \hat{\mathbf{y}} = \sin \theta \sin \phi \hat{\mathbf{r}} + \cos \theta \sin \phi \hat{\boldsymbol{\theta}} + \cos \phi \hat{\boldsymbol{\phi}}  \hat{\mathbf{z}} = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}} $
	$ \hat{\mathbf{r}} = \frac{x^3 + y^2}{\sqrt{x^2 + y^2 + z^2} \sqrt{x^2 + y^2}} \hat{\mathbf{x}} + \frac{x^2 + y^3}{\sqrt{x^2 + y^2 + z^2} \sqrt{x^2 + y^2}} \hat{\mathbf{y}} + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \hat{\mathbf{z}} $ $ \hat{\boldsymbol{\theta}} = \frac{xz}{\sqrt{x^2 + y^2 + z^2} \sqrt{x^2 + y^2}} \hat{\mathbf{x}} + \frac{yz}{\sqrt{x^2 + y^2 + z^2} \sqrt{x^2 + y^2}} \hat{\mathbf{y}} - \frac{\sqrt{x^2 + y^2 + z^2}}{\sqrt{x^2 + y^2 + z^2}} \hat{\mathbf{z}} $ $ \hat{\boldsymbol{\phi}} = -\frac{y}{\sqrt{x^2 + y^2}} \hat{\mathbf{x}} + \frac{x}{\sqrt{x^2 + y^2}} \hat{\mathbf{y}} $	$egin{array}{lcl} \hat{\mathbf{r}} &=& rac{ ho}{\sqrt{ ho^2+z^2}}\hat{oldsymbol{ ho}}+rac{z}{\sqrt{ ho^2+z^2}}\hat{\mathbf{z}}\ \hat{oldsymbol{ heta}} &=& rac{z}{\sqrt{ ho^2+z^2}}\hat{oldsymbol{ ho}}-rac{ ho}{\sqrt{ ho^2+z^2}}\hat{\mathbf{z}}\ \hat{oldsymbol{\phi}} &=& \hat{oldsymbol{\phi}} \end{array}$	$\hat{oldsymbol{ ho}} = \sin \theta \hat{f r} + \cos \theta \hat{oldsymbol{ heta}}$ $\hat{oldsymbol{\phi}} = \hat{oldsymbol{\phi}}$ $\hat{f z} = \cos \theta \hat{f r} - \sin \theta \hat{oldsymbol{ heta}}$
A vector field A	$A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}}$	$A_{ ho}m{\hat{ ho}} + A_{\phi}m{\hat{\phi}} + A_{z}m{\hat{z}}$	$A_r \hat{m{r}} + A_{m{ heta}} \hat{m{ heta}} + A_{m{\phi}} \hat{m{\phi}}$
Gradient $\nabla f$	$\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$rac{\partial f}{\partial  ho} \hat{m{ ho}} + rac{1}{ ho} rac{\partial f}{\partial \phi} \hat{m{\phi}} + rac{\partial f}{\partial z} \hat{m{z}}$	$\frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}$
$rac{ ext{Divergence}}{ abla \cdot \mathbf{A}}$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial \left(\rho A_{\rho}\right)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_{\phi}}{\partial \phi} + \frac{\partial A_{z}}{\partial z}$	$\frac{1}{r^2} \frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_{\theta} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial A_{\phi}}{\partial \phi}$

$_{ ext{Curl}} abla imes\mathbf{A}$	$ \begin{pmatrix} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \end{pmatrix} \hat{\mathbf{x}} + \\ \begin{pmatrix} \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \end{pmatrix} \hat{\mathbf{y}} + \\ \begin{pmatrix} \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{pmatrix} \hat{\mathbf{z}} $		$ \frac{1}{r \sin \theta} \left( \frac{\partial}{\partial \theta} (A_{\phi} \sin \theta) - \frac{\partial A_{\theta}}{\partial \phi} \right) \hat{\boldsymbol{r}} + \frac{1}{r} \left( \frac{1}{\sin \theta} \frac{\partial A_{r}}{\partial \phi} - \frac{\partial}{\partial r} (r A_{\phi}) \right) \hat{\boldsymbol{\theta}} + \frac{1}{r} \left( \frac{\partial}{\partial r} (r A_{\theta}) - \frac{\partial A_{r}}{\partial \theta} \right) \hat{\boldsymbol{\phi}} $
$rac{ ext{Laplace}}{ ext{operator}} \ \Delta f =  abla^2 f$	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$
$rac{ ext{Vector}}{ ext{Laplacian}} \Delta  extbf{A} =  abla^2  extbf{A}$	$\Delta A_x \mathbf{\hat{x}} + \Delta A_y \mathbf{\hat{y}} + \Delta A_z \mathbf{\hat{z}}$	$ \left(\Delta A_{\rho} - \frac{A_{\rho}}{\rho^{2}} - \frac{2}{\rho^{2}} \frac{\partial A_{\phi}}{\partial \phi}\right) \hat{\boldsymbol{\rho}} + \left(\Delta A_{\phi} - \frac{A_{\phi}}{\rho^{2}} + \frac{2}{\rho^{2}} \frac{\partial A_{\rho}}{\partial \phi}\right) \hat{\boldsymbol{\phi}} + \left(\Delta A_{z}\right) \hat{\boldsymbol{z}} $	$ \left(\Delta A_r - \frac{2A_r}{r^2} - \frac{2}{r^2 \sin \theta} \frac{\partial (A_\theta \sin \theta)}{\partial \theta} - \frac{2}{r^2 \sin \theta} \frac{\partial A_\phi}{\partial \phi}\right) \hat{\boldsymbol{r}} + \left(\Delta A_\theta - \frac{A_\theta}{r^2 \sin^2 \theta} + \frac{2}{r^2} \frac{\partial A_r}{\partial \theta} - \frac{2\cos \theta}{r^2 \sin^2 \theta} \frac{\partial A_\phi}{\partial \phi}\right) \hat{\boldsymbol{\theta}} + \left(\Delta A_\phi - \frac{A_\phi}{r^2 \sin^2 \theta} + \frac{2}{r^2 \sin \theta} \frac{\partial A_r}{\partial \phi} + \frac{2\cos \theta}{r^2 \sin^2 \theta} \frac{\partial A_\theta}{\partial \phi}\right) \hat{\boldsymbol{\phi}} $
Material derivative $(\mathbf{A}\cdot abla)\mathbf{B}$	$ \begin{pmatrix} A_{x}\frac{\partial B_{x}}{\partial x} + A_{y}\frac{\partial B_{x}}{\partial y} + A_{z}\frac{\partial B_{x}}{\partial z} \end{pmatrix} \hat{\boldsymbol{x}} + \\ \begin{pmatrix} A_{x}\frac{\partial B_{y}}{\partial x} + A_{y}\frac{\partial B_{y}}{\partial y} + A_{z}\frac{\partial B_{y}}{\partial z} \end{pmatrix} \hat{\boldsymbol{y}} + \\ \begin{pmatrix} A_{x}\frac{\partial B_{z}}{\partial x} + A_{y}\frac{\partial B_{z}}{\partial y} + A_{z}\frac{\partial B_{z}}{\partial z} \end{pmatrix} \hat{\boldsymbol{z}} $	$ \left(A_{\rho}\frac{\partial B_{\rho}}{\partial \rho} + \frac{A_{\phi}}{\rho}\frac{\partial B_{\rho}}{\partial \phi} + A_{z}\frac{\partial B_{\rho}}{\partial z} - \frac{A_{\phi}B_{\phi}}{\rho}\right)\hat{\boldsymbol{\rho}} + \left(A_{\rho}\frac{\partial B_{\phi}}{\partial \rho} + \frac{A_{\phi}}{\rho}\frac{\partial B_{\phi}}{\partial \phi} + A_{z}\frac{\partial B_{\phi}}{\partial z} + \frac{A_{\phi}B_{\rho}}{\rho}\right)\hat{\boldsymbol{\phi}} + \left(A_{\rho}\frac{\partial B_{z}}{\partial \rho} + \frac{A_{\phi}}{\rho}\frac{\partial B_{z}}{\partial \rho} + A_{z}\frac{\partial B_{z}}{\partial z}\right)\hat{\boldsymbol{z}} $	$ \left(A_r \frac{\partial B_r}{\partial r} + \frac{A_{\theta}}{r} \frac{\partial B_r}{\partial \theta} + \frac{A_{\phi}}{r \sin(\theta)} \frac{\partial B_r}{\partial \phi} - \frac{A_{\theta} B_{\theta} + A_{\phi} B_{\phi}}{r}\right) \hat{r} + \left(A_r \frac{\partial B_{\phi}}{\partial r} + \frac{A_{\theta}}{r} \frac{\partial B_{\theta}}{\partial \theta} + \frac{A_{\phi}}{r \sin(\theta)} \frac{\partial B_{\theta}}{\partial \phi} - \frac{A_{\theta} B_r}{r} - \frac{A_{\phi} B_{\phi} \cot(\theta)}{r}\right) \hat{\theta} + \left(A_r \frac{\partial B_{\phi}}{\partial r} + \frac{A_{\theta}}{r} \frac{\partial B_{\phi}}{\partial \theta} + \frac{A_{\phi}}{r \sin(\theta)} \frac{\partial B_{\phi}}{\partial \phi} + \frac{A_{\theta} B_r}{r} + \frac{A_{\phi} B_{\theta} \cot(\theta)}{r}\right) \hat{\phi} $
Differential displacement	$d\mathbf{l} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$	$d\mathbf{l} = d ho\hat{oldsymbol{ ho}} +  ho d\phi\hat{oldsymbol{\phi}} + dz\hat{oldsymbol{z}}$	$d\mathbf{l} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}}$
Differential normal area	$d\mathbf{S} = dy dz \hat{\mathbf{x}} +  dx dz \hat{\mathbf{y}} +  dx dy \hat{\mathbf{z}}$	$d\mathbf{S} =  ho  d\phi  dz  \hat{oldsymbol{ ho}} + \ d ho  dz  \hat{oldsymbol{\phi}} + \  ho  d ho d\phi  \hat{\mathbf{z}}$	$d\mathbf{S} = r^2 \sin \theta  d\theta  d\phi  \hat{\mathbf{r}} + r \sin \theta  dr  d\phi  \hat{\boldsymbol{\theta}} + r  dr  d\theta  \hat{\boldsymbol{\phi}}$
Differential volume	dV = dx  dy  dz	$dV = \rho  d\rho  d\phi  dz$	$dV = r^2 \sin\theta  dr  d\theta  d\phi$

## Non-trivial calculation rules:

- 1. div grad  $f = \nabla \cdot (\nabla f) = \nabla^2 f = \Delta f_{\text{(Laplacian)}}$

- 1. div grad  $f = \nabla \times (\nabla f) = \mathbf{0}$ 2. curl grad  $f = \nabla \times (\nabla f) = \mathbf{0}$ 3. div curl  $\mathbf{A} = \nabla \cdot (\nabla \times \mathbf{A}) = \mathbf{0}$ 4. curl curl  $\mathbf{A} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) \nabla^2 \mathbf{A}$  (using Lagrange's formula for the cross product)
- 5.  $\Delta fg = f\Delta g + 2\nabla f \cdot \nabla g + g\Delta f$