ZZV, Elektrostatika 2/3

EMP – Elektrostatika 2/3

~Wolfman

(1.)

$R = 5\,m$

$\rho = 1{,}2\,(5-2r)\ \mu C/m^3$

$r' = 100\,m$

$$\oiint \vec{D}\,d\vec{S} = \iiint \rho\,dV$$

$$D \cdot 4r^2\pi = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\,d\theta \int_0^{R} r^2(5-2r)\cdot 1{,}2\,\mu C/m^2 \cdot dr$$

$$D \cdot 4r^2\pi = 2\pi \cdot 2 \cdot 1{,}2\frac{\mu C}{m^2} \int_0^R (5r^2 - 2r^3)\,dr$$

$$D \cdot \cancel{4r^2\pi} = \cancel{4\pi} \cdot 1{,}2\frac{\mu C}{m^2}\left(-\frac{625}{6}\right)$$

$$D = \frac{1{,}2\cdot(-104{,}1667)}{100^2}\frac{\mu C}{m^2} = -0{,}0125\frac{\mu C}{m^2} = -12{,}5\frac{nC}{m^2}$$

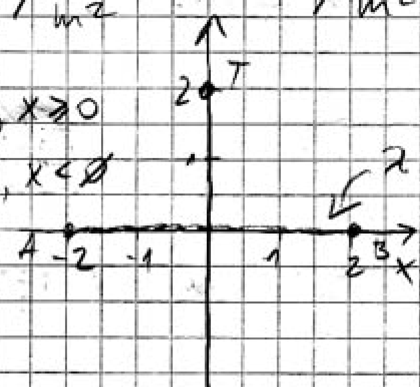
(2.)

$$\lambda = |x|\ nC/m$$

$$= 10^{-9}|x|\ C/m = \begin{cases} 10^{-9}x, & x \geq 0 \\ 10^{-9}(-x), & x < 0 \end{cases}$$

$A(-2,0,0)$; $B(2,0,0)$

$T(0,2,0)$



$$\frac{x+2}{4} = \frac{y}{0} = \frac{z}{0}$$

$x = 4t-2$

$y = 0$

$z = 0$

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\cdot\int\frac{\lambda\,dl}{|\vec{r}-\vec{r}'|}$$

$\vec{r} = y\,\vec{a_y}$

$\vec{r}' = (4t-2)\vec{a_x}$, $t \in [0,1]$

$dl = \sqrt{4^2}\,dt = 4dt$

$$\varphi = \frac{1}{4\pi\varepsilon_0}\cdot\int_0^1 \frac{\lambda\,4dt}{(y^2+(4t-2)^2)^{1/2}} = \frac{1}{\pi\varepsilon_0}\left(\int_0^{1/2}\frac{10^{-9}(2-4t)\,dt}{(y^2+(4t-2)^2)^{1/2}} + \int_{1/2}^{1}\frac{10^{-9}(4t-2)\,dt}{(y^2+(4t-2)^2)^{1/2}}\right)$$

$$\int_a^b \frac{(4t-2)\,dt}{(y^2+(4t-2)^2)^{1/2}} = \left|\begin{matrix} u = 4t-2 \\ du = 4dt \\ u_1 = 4b-2 \\ u_2 = 4a-2 \end{matrix}\right| = \int_{4a-2}^{4b-2}\frac{\frac{1}{4}u\,du}{(y^2+u^2)^{1/2}} = \frac{1}{4\cdot 2}\int_{4a-2}^{4b-2}\frac{2u\,du}{(u^2+y^2)^{1/2}} \left|\begin{matrix} x = u^2+y^2 \\ dx = 2u\,du \end{matrix}\right|$$

$$= \frac{1}{8}\int_{(4a-2)^2+y^2}^{(4b-2)^2+y^2} x^{-\frac{1}{2}}\,dx = \frac{1}{8}\cdot 2\sqrt{x}\,\Big|_{(4a-2)^2+y^2}^{(4b-2)^2+y^2} = \frac{1}{4}\left(\sqrt{(4b-2)^2+y^2} - \sqrt{(4a-2)^2+y^2}\right)$$

$$\varphi = \frac{10^{-9}}{\pi\varepsilon_0}\left(\frac{1}{4}\left(\sqrt{(4\cdot 0-2)^2+y^2} - \sqrt{(4\cdot\frac{1}{2}-2)^2+y^2}\right) + \frac{1}{4}\left(\sqrt{(4-2)^2+y^2} - \sqrt{(4\cdot\frac{1}{2}-2)^2+y^2}\right) = \right.$$

$$\varphi = \frac{10^{-9}}{4\pi\varepsilon_0}\left[\left(\sqrt{4+y^2}-y\right)+\left(\sqrt{4+y^2}-y\right)\right]$$

$$= \frac{10^{-9}}{4\pi\varepsilon_0}\cdot\left(2\sqrt{4+y^2}-2y\right) = \frac{10^{-9}}{2\pi\varepsilon_0}\left(\sqrt{4+y^2}-y\right)$$

$y = 2$

$$\varphi(2) = \frac{10^{-9}}{2\pi\varepsilon_0}\cdot\left(2\sqrt{2}-2\right) = \frac{10^{-9}}{\pi\varepsilon_0}\left(\sqrt{2}-1\right) =$$

$$= 14{,}89\,V$$

3.

$$\vec{E} = -\nabla\varphi$$

$$\varphi = \frac{10^{-9}}{2\pi\varepsilon_0}\left(\sqrt{4+y^2}-y\right)$$

$$\nabla\varphi = \frac{10^{-9}}{2\pi\varepsilon_0}\cdot\left[\frac{\cancel{2}y}{\cancel{2}\sqrt{4+y^2}}-1\right]\cdot\vec{a_y}$$

$$\vec{E} = \frac{10^{-9}}{2\pi\varepsilon_0}\left(\frac{-y}{\sqrt{4+y^2}}+1\right)\cdot\vec{a_y}$$

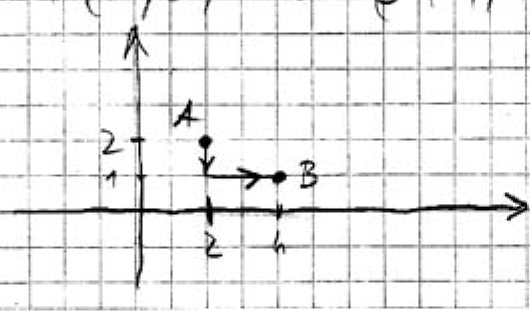
$y = 2$

$$E = \frac{10^{-9}}{2\pi\varepsilon_0}\cdot\left(1-\frac{2}{\sqrt{8}}\right) = 5{,}26\ V/m$$

4.

$$\vec{E} = 4x\,\vec{a_x} + 2\vec{a_y}\quad\left[V/m\right]$$

A (2,2) do B (4,1)



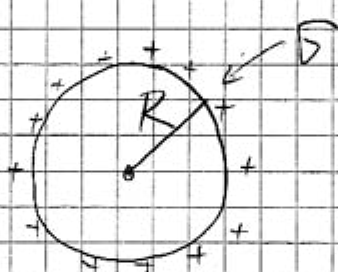
$$W(a) - W(b) = -\int_b^a q \cdot \vec{E}\, d\vec{l} =$$

$$= -q \int_{(4,1)}^{(2,2)} 4x\,dx + 2dy = -q \left(\int_2^1 2dy + \int_2^4 4x\,dx \right) =$$

$$= -q \left(2 \cdot (1-2) + 2(16-4) \right) = -q(-2 + 24) =$$

$$= -22q = -22 eV = -3{,}5244 \cdot 10^{-18} J$$

5.



$$, \quad r > R \quad , \quad S = 4\pi R^2$$

Gaussov zakon:

$$\oint \vec{D} \cdot d\vec{S} = Q = \sigma \cdot 4\pi R^2$$

$$\varepsilon_0 \cdot E \cdot \cancel{4\pi} r^2 = \sigma \cdot \cancel{4\pi} R^2$$

$$\vec{E} = \frac{\sigma}{\varepsilon_0} \cdot \left(\frac{R}{r}\right)^2 \cdot \vec{r}$$

6.

$$\vec{E} = \frac{\sigma}{\varepsilon_0} \cdot \left(\frac{R}{r}\right)^2 \cdot \vec{r} \quad , \quad r \geq R$$

$$\vec{E} = \vec{0} \quad , \quad r < R$$

$$1^\circ \quad \varphi(r) = -\int_\infty^r \frac{\sigma}{\varepsilon_0} \cdot \left(\frac{R}{r}\right)^2 dr = -\frac{\sigma}{\varepsilon_0} \cdot R^2 \int_\infty^r r^{-2} dr =$$

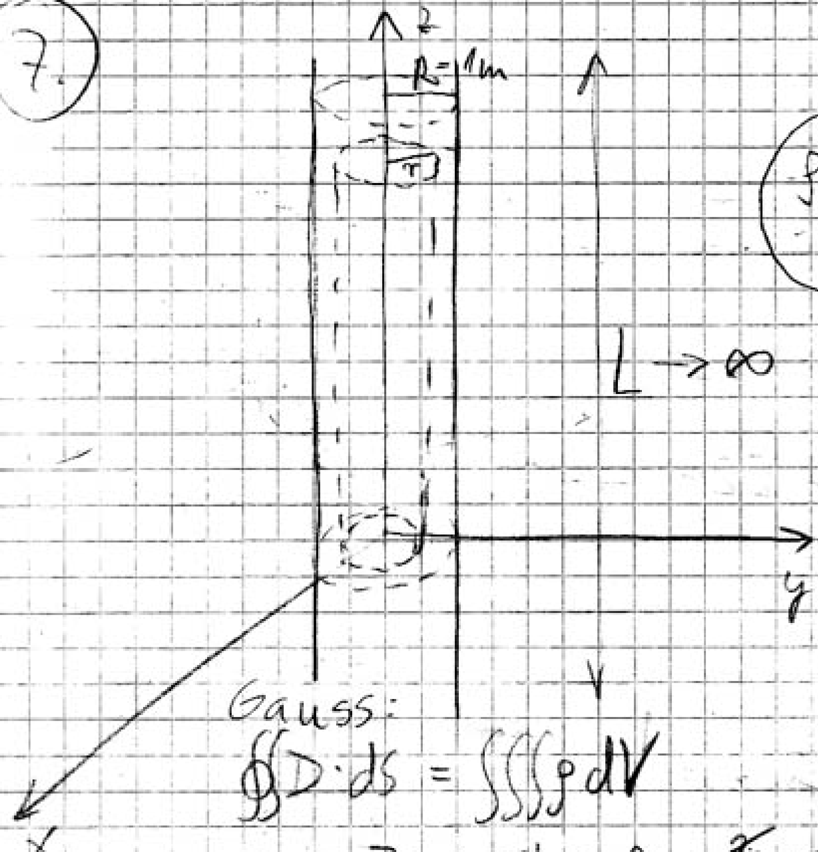
$$= +\frac{\sigma}{\varepsilon_0} \cdot R^2 \cdot \frac{1}{r} \Big|_\infty^r = \frac{\sigma}{\varepsilon_0} R^2 \left(\frac{1}{r} - 0 \right) = \frac{\sigma \cdot R^2}{\varepsilon_0 r} \;, \; r \geq R$$

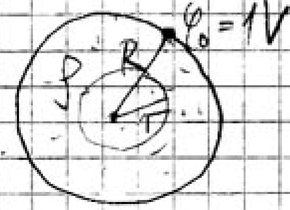
$$2^\circ \quad \varphi(R) = \frac{\sigma \cdot R}{\varepsilon_0}$$

, Unutar sfere nema naboja pa je $\vec{E}$ za $r<R$ nula, iz čega slijedi da je potencijal unutar kugle konstantan!

$$\Rightarrow \varphi(r) = \frac{\sigma \cdot R}{\varepsilon_0} \;, \; r < R$$

(7.)





Gauss:

$$\oiint D \cdot ds = \iiint \rho \, dV$$

$$\varepsilon_0 \cdot E \cdot 2 \cancel{r} \cancel{\pi} \cdot \cancel{L} = \rho \cdot r^{\cancel{2}} \cancel{\pi} \cdot \cancel{L}$$

$$\vec{E} = \frac{\rho}{2\varepsilon_0} \cdot r \, \vec{a}_R$$

$$\varphi(1) - \varphi(r) = -\int_r^1 \frac{\rho}{2\varepsilon_0} \cdot r \, dr = -\frac{\rho}{2\varepsilon_0} \cdot \frac{r^2}{2}\Big|_r^1 =$$

$$= -\frac{\rho}{4\varepsilon_0}\left(1 - r^2\right)$$

$$\varphi(r) = \varphi(1) + \frac{\rho}{4\varepsilon_0}\left(1 - r^2\right)$$

$$= 1 + \frac{\rho}{4\varepsilon_0}\left(1 - r^2\right)$$

(8.) $\lambda = 5\ \mu C/m$

$A(1, \pi, 4)$ i $B(3, \pi, 4)$ => cilindrični sustav

$\vec{r} = x\vec{a_x} + y\vec{a_y} + z\vec{a_z}$ $\qquad \frac{\vec{r}}{r} = \frac{\frac{x\vec{a_x} + y\vec{a_y}}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}$

$\vec{r'} = z'\vec{a_z}$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} \lambda\, dz' \cdot \frac{x\vec{a_x} + y\vec{a_y} + (z - z')\vec{a_z}}{\left(x^2 + y^2 + (z-z')^2\right)^{3/2}}$$

$E_z = \emptyset$

$$E_x = \frac{\lambda x}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} \frac{dz}{(x^2+y^2+z^2)^{3/2}} =$$

$$= \frac{\lambda x}{4\pi\varepsilon_0} \cdot \frac{z}{(x^2+y^2)(x^2+y^2+z^2)^{1/2}}\Bigg|_{-\infty}^{\infty} =$$

$$= \frac{\lambda x}{4\pi\varepsilon_0}\,\frac{2}{x^2+y^2} = \frac{\lambda}{2\pi\varepsilon_0}\,\frac{x}{x^2+y^2} = \frac{\frac{x}{\sqrt{x^2+y^2}}}{\frac{x^2+y^2}{\sqrt{x^2+y^2}}} \cdot \frac{\lambda}{2\pi\varepsilon_0}$$

$$E_y = \frac{\lambda}{2\pi\varepsilon_0} \cdot \frac{\frac{y}{\sqrt{x^2+y^2}}}{\frac{x^2+y^2}{\sqrt{x^2+y^2}}} \Rightarrow \qquad \vec{r} = x\vec{a_x} + y\vec{a_y}$$

$$\Rightarrow \vec{E} = \frac{\lambda}{2\pi\varepsilon_0} \cdot \frac{\vec{r}}{r}$$

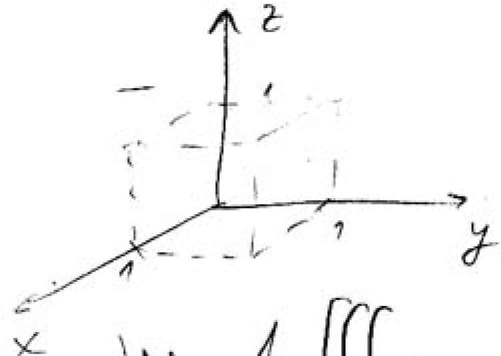
$$\varphi(A) - \varphi(B) = -\int_B^A \vec{E}\, d\vec{r} = -\int_3^1 \frac{\lambda}{2\pi\varepsilon_0 r}\, dr =$$

$$= \frac{-\lambda}{2\pi\varepsilon_0} \ln r \Big|_3^1 = \frac{-\lambda}{2\pi\varepsilon_0}(\ln 1 - \ln 3) =$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \cdot \ln\left|\frac{1}{3}\right|^{-1} = \frac{\lambda}{2\pi\varepsilon_0}\ln 3 = 98{,}74\,kV$$

9.) $\varphi = 2x + 4y \;,\; [V]$

$V \ldots$ jedinična kocka , $V = 1 \cdot 1 \cdot 1 = 1$



$$W = \frac{1}{2} \iiint\limits_V \varepsilon_0 \cdot E^2 \, dV$$

$$\vec{E} = -\nabla \varphi = -\left(2\vec{a_x} + 4\vec{a_y}\right) = -2\vec{a_x} - 4\vec{a_y}$$

$$E^2 = 2^2 + 4^2 = 4 + 16 = 20$$

$$W = \frac{1}{2} \cdot \iiint \varepsilon_0 \cdot 20 \, dV = 10\,\varepsilon_0 \cdot \iiint dV =$$

$$= 10\,\varepsilon_0 \cdot V = 10\,\varepsilon_0 \; J$$

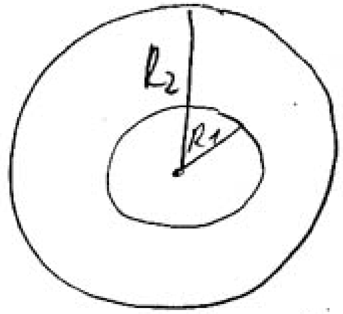
$$\varepsilon_0 = \frac{1}{c^2 \mu} = \frac{1}{c^2 \cdot 4\pi \cdot 10^{-7}} = \frac{1}{9 \cdot 10^{16} \cdot 4\pi \cdot 10^{-7}} = \frac{1}{36\pi \cdot 10^{9}}$$

$$W = \frac{10^{-8}}{36\pi} \; J$$

(10.) $\vec{E} = \frac{10^5}{r} \cdot \vec{a}_R$

$R_1 = 0{,}01$

$R_2 = 0{,}05$



$$W = \frac{1}{2} \iiint_V \varepsilon_0 \cdot E^2 dV = \frac{1}{2} \cdot \int_0^{2\pi} d\varphi \int_{R_1}^{R_2} r \cdot \varepsilon_0 \cdot \frac{10^{10}}{r^2} \cdot dr \int_0^{0,5} dz =$$

$$= \frac{1}{\cancel{2}} \cdot \cancel{2}\pi \cdot \varepsilon_0 \cdot 10^{10} \cdot \ln r \Big|_{R_1}^{R_2} \cdot \frac{1}{2} =$$

$$= \frac{\pi \cdot \varepsilon_0 \cdot 10^{10}}{2} \cdot \ln\left(\frac{0{,}05}{0{,}01}\right) = 0{,}224\,J$$

11. $\vec{E} = -5\, e^{-\frac{r}{a}}\, a_r$

$0 \le r \le 2a$ $\quad E^2 = 25 \cdot e^{-\frac{2r}{a}}$

$0 \le z \le 5a$

$$W = \frac{1}{2} \iiint_V \varepsilon_0 \cdot E^2\, dV$$

$$= \frac{1}{2} \int_0^{2\pi} d\varphi \int_0^{2a} r \cdot \varepsilon_0 25 \cdot e^{-\frac{2r}{a}} dr \int_0^{5a} dz =$$

$$= \frac{1}{\cancel{2}} \cdot \cancel{2}\pi \cdot 25 \cdot 5a\varepsilon_0 \int_0^{2a} r \cdot e^{-\frac{2r}{a}} dr$$

$$= 125\pi\varepsilon_0 a \int_0^{2a} r \cdot e^{-\frac{2r}{a}} dr \left| \begin{matrix} u = r & dv = e^{-\frac{2r}{a}} dr \\ du = dr & v = e^{-\frac{2r}{a}} \cdot \left(-\frac{a}{2}\right) \end{matrix} \right|$$

$$= 125\pi\varepsilon_0 a \left(-\frac{a}{2} \cdot e^{-\frac{2r}{a}} \cdot r \Big|_0^{2a} + \frac{a}{2} \int e^{-\frac{2r}{a}} dr \right) =$$

$$= 125\pi\varepsilon_0 a \left(-a^2 \cdot e^{-4} + \frac{a}{2} \cdot e^{-\frac{2r}{a}} \cdot \left(-\frac{a}{2}\right) \Big|_0^{2a} \right) =$$

$$= 125\pi a\varepsilon_0 \left(-a^2 \cdot e^{-4} - \frac{a^2}{4} \cdot e^{-\frac{2r}{a}} \Big|_0^{2a} \right) =$$

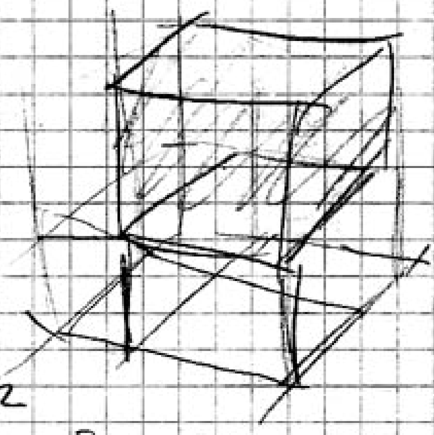
$$= 125\pi a\varepsilon_0 \left(-a^2 e^{-4} - \frac{a^2}{4} \left(e^{-4} - 1 \right) \right)$$

$$= a^3\, 125\pi\varepsilon_0 \left(-e^{-4} - \frac{e^{-4}}{4} + \frac{1}{4} \right) = a^3 \cdot 125\pi\, \varepsilon_0 \cdot 0{,}2271$$

$$= 7{,}89 \cdot 10^{-10}\, a^3$$

(12.) $\varphi = 3x^2 + 4y^2$

$1 < x, y, z \leq 3$



$$\vec{E} = -\nabla\varphi = -\left(6x\vec{a}_x + 8y\vec{a}_y\right)$$

$$\vec{E}\cdot\vec{E} = E^2 = 36x^2 + 64y^2$$

$$W = \frac{1}{2}\iiint \varepsilon_0 \, E^2 \, dV = \frac{\varepsilon_0}{2}\int_1^3 dx \int_1^3 dy \int_1^3 \left(36x^2 + 64y^2\right) dz$$

$$= \varepsilon_0 \int_1^3 dx \int_1^3 \left(36x^2 + 64y^2\right) dy = \varepsilon_0 \cdot \frac{5200}{3} = 15{,}34\,\mathrm{nJ}$$