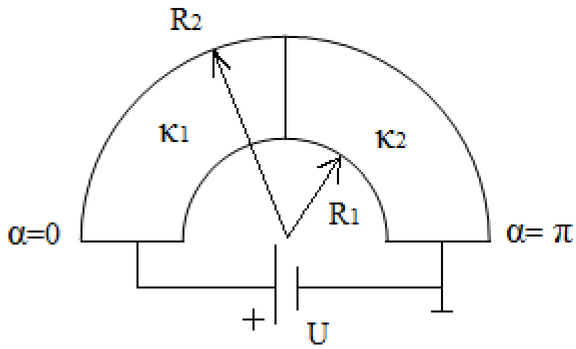
Međuispit iz Elektromagnetskih polja
26.4.2014.

1. Vodič pravokutnog poprečnog presjeka, polukružno savijen prema slici, sastoji se od materijala dielektričnosti ε_0 i provodnosti κ_1=56 MS u području α=[0, π/2] i dielektričnosti ε_0 i provodnosti κ_2=37 MS u području α=[π/2, π]. Vodič je spojen na naponski izvor tako da je za kut α=0 potencijal φ=U, a za kut α= π potencijal φ=0. Dubina vodiča je d=1mm, polumjeri vodiča su R_1=5 mm, R_2=6 mm, a U=12 V.
Odredite gustoću struje kroz pravokutni vodič.
Odredite otpor vodiča.



Rješenje:
Zbirka prof. Berberovića, primjer 7.4.1
Izvedena je formula za otpor četvrtine kružnog vijenca:

$$R = \frac{\pi}{2\kappa d \ln\left(\frac{r_2}{r_1}\right)}$$

Naš vodič možemo promatrati kao 2 otpora spojena u seriji, a svaki dio je četvrtina kružnog vijenca.

$$R_1' = \frac{\pi}{2\kappa_1 d \ln\left(\frac{R_2}{R_1}\right)} = 153.85\ \mu\Omega$$

$$R_2' = \frac{\pi}{2\kappa_2 d \ln\left(\frac{R_2}{R_1}\right)} = 232.85\ \mu\Omega$$

$$R_{uk}' = R_1' + R_2' = 386.7\ \mu\Omega$$

$$I = \frac{U}{R} = 31.03\ \text{kA} \rightarrow \vec{J} = \frac{I}{S}\vec{a}_\alpha$$

$$J = \frac{I}{d\cdot(R_2 - R_1)} = 31.03\cdot 10^9\ \text{A/m}^2$$

2. Ravnina y=2 dijeli prostor u 2 područja: područje (1), y<2, dielektričnosti $\varepsilon_1=3\varepsilon_0$ i područje (2), y>2, dielektričnosti $\varepsilon_2=2\varepsilon_0$. U području (1) jakost električnog polja iznosi $\mathbf{E_1}=3\mathbf{a_x}+2\mathbf{a_y}+\mathbf{a_z}$ [V/m]. Koordinate točaka A i B su A(1;4;1) i B(4;3;7).
Odredite napon U_{AB} između točaka A i B.

Rješenje:

$$\vec{n}\left(\vec{D}_2-\vec{D}_1\right)=\sigma_s=0$$

$$\vec{n}=\vec{a}_y$$

$$D_{1y}=D_{2y}\Rightarrow\varepsilon_1E_{1y}=\varepsilon_2E_{2y}$$

$$E_{2y}=\frac{\varepsilon_1}{\varepsilon_2}E_{1y}=3\,\frac{\mathrm{V}}{\mathrm{m}}$$

$$\vec{n}\times\left(\vec{E}_2-\vec{E}_1\right)=0$$

$$\vec{a}_y\times\left[\left(E_{2x}-3\right)\vec{a}_x+\left(3-2\right)\vec{a}_y+\left(E_{2z}-1\right)\vec{a}_z\right]=0$$

$$-\left(E_{2x}-3\right)\vec{a}_z+\left(E_{2z}-1\right)\vec{a}_x=0$$

$$E_{2x}=3\,\frac{\mathrm{V}}{\mathrm{m}}\qquad E_{2z}=1\,\frac{\mathrm{V}}{\mathrm{m}}$$

$$\vec{E}_2=3\vec{a}_x+3\vec{a}_y+\vec{a}_z$$

Obje točke se nalaze u području 2 (y>2).

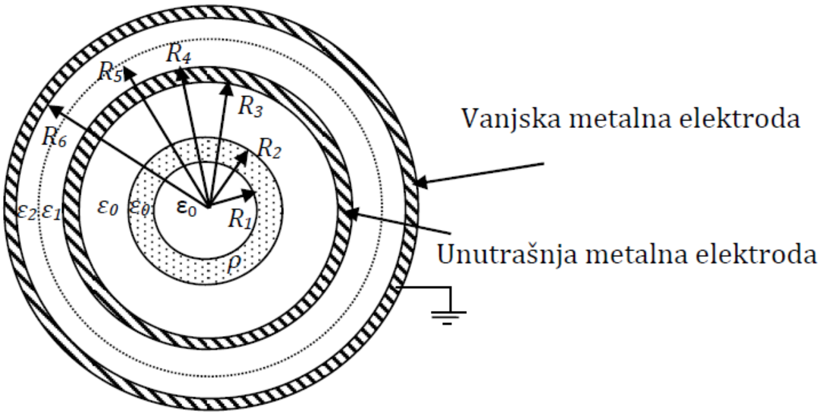
$$U_{AB}=-\int_B^A\vec{E}_2\cdot d\vec{l}$$

$$d\vec{l}=dx\cdot\vec{a}_x+dy\cdot\vec{a}_y+dz\cdot\vec{a}_z$$

$$U_{AB}=-\int_{x_B=4}^{x_A=1}\int_{y_B=3}^{y_A=4}\int_{z_B=7}^{z_A=1}\left(3dx+3dy+dz\right)$$

$$U_{AB}=12\ \mathrm{V}$$

3. Kuglasti kondenzator prema slici u području $R_1<r<R_2$ sadrži naboj jednolike gustoće $\rho=5$ C/m³. Zadano je $\varepsilon_{r1}=4$, $\varepsilon_{r2}=2$, $R_1=1$ cm, $R_2=3$ cm, $R_3=4.5$ cm, $R_4=5$ cm, $R_5=10$ cm i $R_6=20$ cm. Odredite energiju sadržanu u području $R_2<r<R_5$.

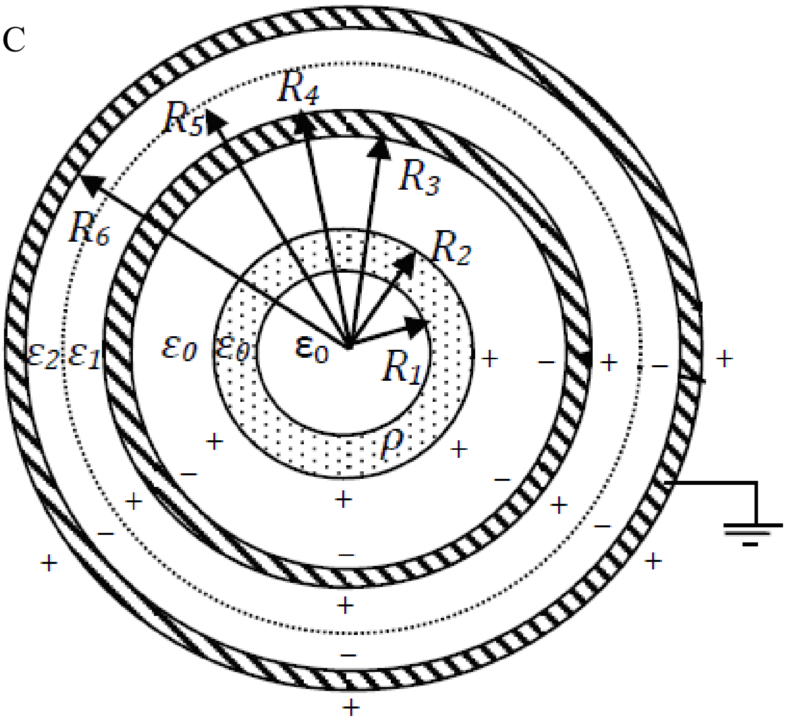


Rješenje:

Slika je ista kao i u zadatku 2.57 iz Trkuljine zbirke, no ovdje je riječ o kuglastom kondenzatoru, a ne cilindričnom.

$$Q_{uk} = \rho \cdot \frac{4}{3} \cdot \left(R_2^3 - R_1^3\right) \cdot \pi = 5.45 \cdot 10^{-4}\ \text{C}$$

$$Q_{\text{inf}} = Q_{uk}$$



$$R_2 \le r \le R_3 \Rightarrow \vec{E}_{23}$$

$$R_3 \le r \le R_4 \Rightarrow \vec{E}_{34} = 0$$

$$R_4 \le r \le R_5 \Rightarrow \vec{E}_{45}$$

$$\text{Gaussov zakon}: \oiint_{S_G} \vec{E} \cdot \vec{n} \cdot dS_G = \frac{Q}{\varepsilon}$$

$$Q = Q_{uk} - Q_{\text{inf}} + Q_{\text{inf}} - Q_{\text{inf}} + Q_{\text{inf}} = Q_{uk}$$

$$E_{23} \cdot 4r^2\pi = \frac{Q_{uk}}{\varepsilon_0} \Rightarrow E_{23} = \frac{Q_{uk}}{4\pi\varepsilon_0 r^2}$$

$$E_{45} \cdot 4r^2\pi = \frac{Q_{uk}}{\varepsilon_1} \Rightarrow E_{45} = \frac{Q_{uk}}{4\pi\varepsilon_1 r^2}$$

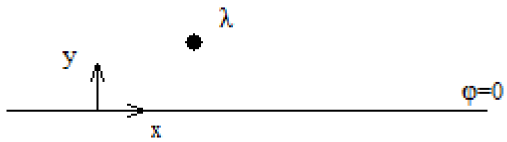
$$W = \frac{1}{2}\varepsilon \iiint |\vec{E}|^2\, dV; \quad dV = 4r^2\pi \cdot dr$$

$$W_{23} = \left(\frac{Q_{uk}}{4\pi\varepsilon_0}\right)^2 \cdot \frac{1}{2}\varepsilon_0 \cdot 4\pi \cdot \int_{R_2}^{R_3} \frac{r^2 \cdot dr}{r^4} = \frac{Q_{uk}^2}{8\pi\varepsilon_0}\left(\frac{1}{R_2} - \frac{1}{R_3}\right) = 14.83\ \text{kJ}$$

$$W_{45} = \left(\frac{Q_{uk}}{4\pi\varepsilon_1}\right)^2 \cdot \frac{1}{2}\varepsilon_0 \cdot 4\pi \cdot \int_{R_4}^{R_5} \frac{r^2 \cdot dr}{r^4} = \frac{Q_{uk}^2}{8\pi\varepsilon_1}\left(\frac{1}{R_4} - \frac{1}{R_5}\right) = 3.33\ \text{kJ}$$

$$W = W_{23} + W_{45} = 18.16\ \text{kJ}$$

4. Linijski naboj λ=15 nC/m postavljen je u zraku u položaj (x=1 m, y=3 m) iznad uzemljene vodljive ravnine, paralelno s osi z, prema slici.
Odredite jakost električnog polja u točki (2 m; 3 m; 4 m).
Odredite gustoću naboja u uzemljenoj ravnini.
Odredite ukupni naboj po jedinici duljine na uzemljenoj ravnini.



Rješenje:

a)

$$\vec{r} = 2\vec{a}_x + 3\vec{a}_y$$

$$\text{original}: \vec{r}' = \vec{a}_x + 3\vec{a}_y \Rightarrow \vec{R} = \vec{r} - r' = \vec{a}_x$$

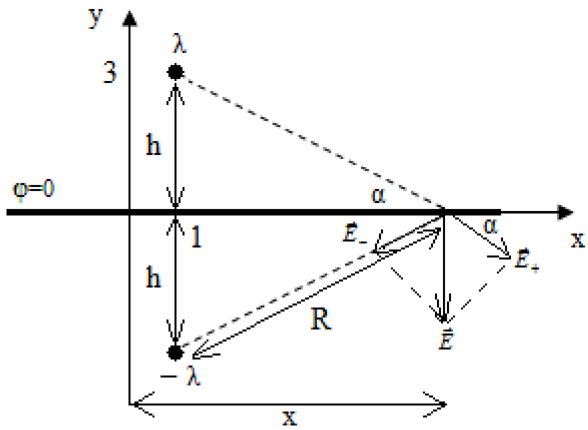
$$|\vec{R}| = 1$$

$$\text{slika}: \vec{r}_1' = \vec{a}_x - 3\vec{a}_y \Rightarrow \vec{R}_1 = r - \vec{r}_1' = \vec{a}_x + 6\vec{a}_y$$

$$|\vec{R}_1| = \sqrt{37}$$

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{\vec{R}}{|\vec{R}|^2} + \frac{-\lambda}{2\pi\varepsilon_0}\frac{\vec{R}_1}{|\vec{R}_1|^2} = \frac{269.63\vec{a}_x}{1} - \frac{269.63\vec{a}_x + 1617.79\vec{a}_y}{37}$$

$$\vec{E} = 262.34\vec{a}_x - 43.72\vec{a}_y \left[\frac{\text{V}}{\text{m}}\right]$$



b)

$$D = \sigma$$

$$E(x) = 2 \cdot \frac{\lambda}{2\pi\varepsilon_0 R} \cdot \sin(\alpha)$$

$$R = \sqrt{h^2 + (x-1)^2} \qquad \sin(\alpha) = \frac{h}{R}$$

$$\sigma = \varepsilon_0 E = -\frac{\lambda}{\pi} \cdot \frac{x-1}{h^2 + (x-1)^2}$$ (moguća greška)

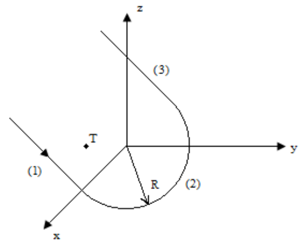
c)

$$\frac{Q_{ind}}{l} = \frac{1}{l}\iint \sigma \cdot dS$$

$$dS = dx \cdot l$$

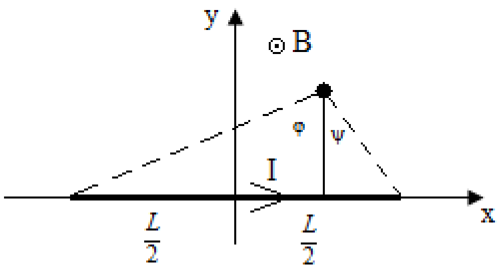
$$\frac{Q_{ind}}{l} = \lambda_{ind} = \int_{-\infty}^{\infty} \sigma \cdot dx = -\lambda$$

5. Strujnica kojom teče struja I= 2 A zadana je slikom, a sastoji se od dva beskonačno dugačka ravna vodiča paralelna s osi y i polukružnice koji leže u xy ravnini.
Odredite magnetsku indukciju u točki T(0; -1; 0).

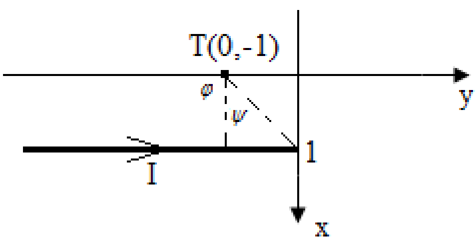


Rješenje:

$$\vec{B} = \frac{\mu_0 \cdot I}{4\pi \cdot r}\left[\sin(\varphi) + \sin(\psi)\right]$$



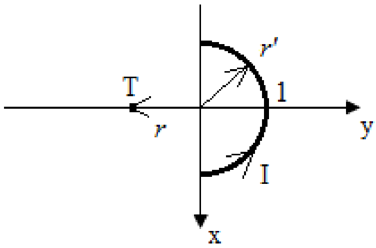
-dijelovi (1) i (3):



$$\varphi = \frac{\pi}{2}, \psi = \frac{\pi}{4}$$

$$B_1 = B_3 = \frac{\mu_0 \cdot I}{4\pi \cdot 1}\left(1 + \frac{\sqrt{2}}{2}\right) = 0.341\vec{a}_z\ \mu\mathrm{T}$$

-dio (2):



$$\vec{r} = -1 \cdot \vec{a}_y = -\left[\sin(\alpha)\vec{a}_r + \cos(\alpha)\vec{a}_\alpha\right]$$
$$\vec{r}' = 1 \cdot \vec{a}_r$$
$$d\vec{l} = \vec{a}_\alpha \cdot 1 \cdot d\alpha$$

$$\vec{R} = \vec{r} - \vec{r}' = -(\sin(\alpha) + 1)\vec{a}_r - \cos(\alpha)\vec{a}_\alpha$$
$$|\vec{R}| = \sqrt{(1 + \sin(\alpha))^2 + \cos^2(\alpha)} = \sqrt{1 + 2\sin(\alpha) + \sin^2(\alpha) + \cos^2(\alpha)}$$
$$|\vec{R}| = \sqrt{2} \cdot \sqrt{1 + \sin(\alpha)}$$

$$\vec{B}_2 = \frac{\mu_o I}{4\pi}\int \frac{d\vec{l} \times \vec{R}}{|\vec{R}|^3} \quad ; \quad d\vec{l} \times \vec{R} = d\alpha \times \left[-(\sin(\alpha) + 1)\vec{a}_r - \cos(\alpha)\vec{a}_\alpha\right] = +\vec{a}_z(1 + \sin(\alpha))d\alpha$$

$$\vec{B}_2 = \frac{\mu_0 I}{4\pi}\int \frac{(1 + \sin(\alpha))\vec{a}_z}{2\sqrt{2} \cdot (1 + \sin(\alpha))^{3/2}} d\alpha = \frac{\mu_0 I}{4\pi} \cdot \frac{1}{2\sqrt{2}}\int_0^\pi \frac{d\alpha}{\sqrt{1 + \sin(\alpha)}}\vec{a}_z = 0.176\vec{a}_z\ \mu\mathrm{T}$$

$$\vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3$$

$$\vec{B} = 0.858\vec{a}_z\ \mu\mathrm{T}$$