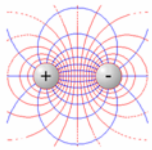


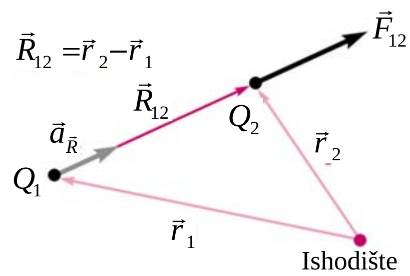
Elektromagnetska polja

STATIČKO ELEKTRIČNO POLJE U VAKUUMU

Charles Augustin de Coulomb (1736 - 1806)

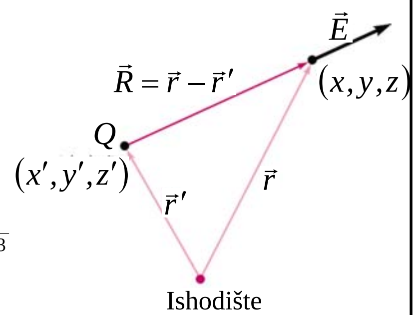
- Građevinski inženjer u francuskoj vojsci
- Zbog bolesti napustio vojsku 1781.
- Koristeći torzionu vagu dokazao je da se sila između točkastih naboja mijenja po inverznom kvadratnom zakonu

Coulombov zakon



$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{\left|\vec{R}_{12}\right|^2}\vec{a}_{\vec{R}}$$

$$\varepsilon_0 = 8{,}854\cdot 10^{-12}\ \frac{\mathrm{As}}{\mathrm{Vm}} \approx \frac{1}{36\pi}\cdot 10^{-9}\ \frac{\mathrm{F}}{\mathrm{m}}$$



Jakost polja točkastog naboja
$Q_1 = Q$, $Q_2 = Q_p$:

$$\vec{E} = \lim_{Q_p \to 0}\frac{\vec{F}}{Q_p} = \frac{Q}{4\pi\varepsilon_0\left|\vec{R}\right|^2}\frac{\vec{R}}{\left|\vec{R}\right|} = \frac{Q}{4\pi\varepsilon_0}\frac{\vec{r} - \vec{r}\,'}{\left|\vec{r} - \vec{r}\,'\right|^3}$$

- Električno polje skupine točkastih naboja
  - Načelo superpozicije

$$\vec{E}(\vec{r}) = \sum_{i=1}^{N}\vec{E}_i(\vec{r}) = \sum_{i=1}^{N}\frac{Q_i}{4\pi\varepsilon_0}\frac{\vec{r} - \vec{r}_i\,'}{\left|\vec{r} - \vec{r}_i\,'\right|^3}$$

- Električno polje kontinuiranih raspodjela naboja
  - Volumni naboj

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_V \mathrm{d}Q\,\frac{\vec{r} - \vec{r}\,'}{\left|\vec{r} - \vec{r}\,'\right|^3} = \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\vec{r} - \vec{r}\,'}{\left|\vec{r} - \vec{r}\,'\right|^3}\rho(\vec{r}\,')\,\mathrm{d}V$$

- Plošni naboj

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \iint_S \frac{\vec{r}-\vec{r}\,'}{\left|\vec{r}-\vec{r}\,'\right|^3} \sigma(\vec{r}\,')\mathrm{d}S$$

- Linijski naboj

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_l \frac{\vec{r}-\vec{r}\,'}{\left|\vec{r}-\vec{r}\,'\right|^3} \lambda(\vec{r}\,')\mathrm{d}l$$

- Tok $\vec{E}$ točkastog naboja po zatvorenoj plohi

$$\oint\!\!\!\oint_S \vec{E}\cdot\vec{n}\mathrm{d}S = \oint\!\!\!\oint_S \frac{Q}{4\pi\varepsilon_0}\frac{\vec{R}}{\left|\vec{R}\right|^3}\cdot\vec{n}\mathrm{d}S = -\oint\!\!\!\oint_S \frac{Q}{4\pi\varepsilon_0}\nabla\left(\frac{1}{\left|\vec{R}\right|}\right)\cdot\vec{n}\mathrm{d}S =$$

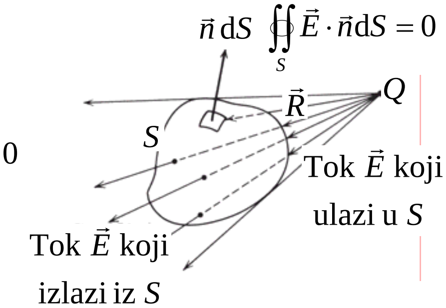
$$= -\iiint_V \frac{Q}{4\pi\varepsilon_0}\nabla\cdot\left[\nabla\left(\frac{1}{\left|\vec{R}\right|}\right)\right]\mathrm{d}V$$

$$\nabla\cdot\left[\nabla\left(\frac{1}{\left|\vec{R}\right|}\right)\right] = \begin{cases} 0 & \text{za } \left|\vec{R}\right| \neq 0 \\ \text{nedefiniran za} & \left|\vec{R}\right| = 0 \end{cases}$$

- Dva slučaja: a) Q izvan V
  b) Q unutar V

- a) Q izvan V

$$\left|\vec{R}\right| \neq 0 \quad \Rightarrow \quad \oint\!\!\!\oint_S \vec{E} \cdot \vec{n} \mathrm{d}S = 0$$

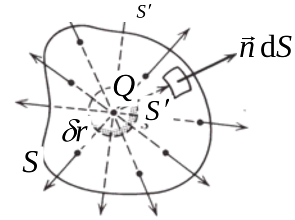


- b) Q unutar V

$$\oint\!\!\!\oint_S \varepsilon_0 \vec{E} \cdot \vec{n} \mathrm{d}S = \oint\!\!\!\oint_{S'} \varepsilon_0 \vec{E} \cdot \vec{n} \mathrm{d}S = Q$$

$$\oint\!\!\!\oint_S \varepsilon_0 \vec{E} \cdot \vec{n} \mathrm{d}S = \lim_{S' \to 0} \oint\!\!\!\oint_{S'} \varepsilon_0 \vec{E} \cdot \vec{n} \mathrm{d}S =$$

$$= \varepsilon_0 \frac{Q}{4\pi\varepsilon_0 (\delta r)^2} 4(\delta r)^2 \pi = Q$$



- Ako unutar S postoji više Q :

$$\oint\!\!\!\oint_S \vec{E} \cdot \vec{n} \mathrm{d}S = \frac{\sum Q}{\varepsilon_0} = \frac{1}{\varepsilon_0} \iiint_V \rho_s \mathrm{d}V$$

- Primjena Gaussova teorema → Gaussov zakon

$$\oint\!\!\!\oint_S \vec{E} \cdot \vec{n} \mathrm{d}S = \iiint_V \nabla \cdot \vec{E}\, \mathrm{d}V = \frac{1}{\varepsilon_0} \iiint_V \rho_s\, \mathrm{d}V \Rightarrow$$

$$\Rightarrow \varepsilon_0 \nabla \cdot \vec{E} = \rho_s$$

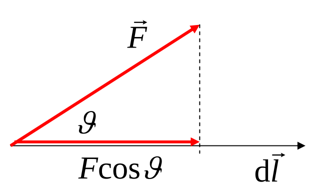
Carl Friedrich Gauss (1777-1855)

- Njemački matematičar, fizičar i astronom
- Jedan od najvećih matematičara u povijesti
- Proučavao magnetizam
- Niz doprinosa poslije smrti nakon otkrića tajnog dnevnika

Potencijal u električnom polju

- Električno polje $\vec{E}$ na točkasti naboj q djeluje silom $\vec{F} = q\vec{E}$
- Pomaci naboja u električnom polju → obavljanje rada → promjena energije naboja
- Na diferencijalnom dijelu puta $\mathrm{d}\vec{l}$ sila $\vec{F}$ obavi rad

$$A = \vec{F} \cdot \mathrm{d}\vec{l}$$



- Naboj se miče suprotno smjeru električnog polja djelovanjem vanjske mehaničke sile $\vec{F}_m$
  - U ravnoteži s električnom (nema promjene brzine)

$$\vec{F}_m + \vec{F}_e = 0$$

  - Na putu od točke P do točke Q obavi se rad

$$A = \int_{\mathrm{P}}^{\mathrm{Q}} \vec{F}_m \cdot \mathrm{d}\vec{l} = -\int_{\mathrm{P}}^{\mathrm{Q}} q\vec{E} \cdot \mathrm{d}\vec{l}$$

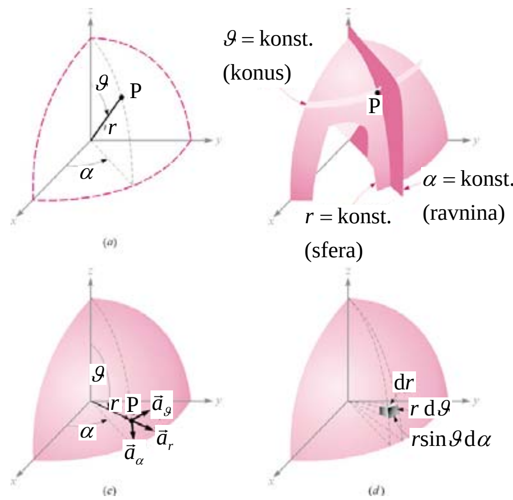
  - Time se za *A* promijenila potencijalna energija naboja
  - Uvodimo (skalarni) električni potencijal φ u točki P

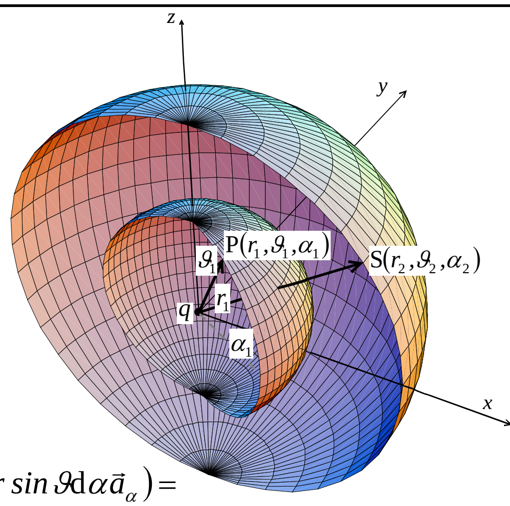
$$\varphi(\mathrm{P}) = \frac{A(\mathrm{P})}{q} = -\int_{\mathrm{T_{referntno}}}^{\mathrm{P}} \vec{E} \cdot \mathrm{d}\vec{l}$$

- Električni potencijal u točki P je omjer rada koji je potrebno obaviti da bi se naboj *q* iz referentne točke nultog potencijala doveo u P i naboja *q*.
- Jedinica za potencijal je volt (V) $1\mathrm{V} = \frac{1\mathrm{J}}{1\mathrm{C}}$
- Mjeriti možemo samo razlike potencijala

Potencijal točkastog naboja

- Sferni koordinatni sustav





$$\varphi(\mathrm{P})-\varphi(\mathrm{S})=-\int_{\mathrm{S}}^{\mathrm{P}}\vec{E}\cdot\mathrm{d}\vec{l}=$$

$$=-\frac{q}{4\pi\varepsilon_0}\int_{\mathrm{S}}^{\mathrm{P}}\frac{\vec{a}_r}{r^2}\cdot\left(\mathrm{d}r\vec{a}_r+r\mathrm{d}\vartheta\vec{a}_\vartheta+r\,sin\,\vartheta\mathrm{d}\alpha\vec{a}_\alpha\right)=$$

$$=-\frac{q}{4\pi\varepsilon_0}\int_{r_2}^{r_1}\frac{\mathrm{d}r}{r^2}=-\frac{q}{4\pi\varepsilon_0}\left(-\frac{1}{r}\right)\Bigg|_{r_2}^{r_1}=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

za q>0:
φ(P)−φ(S)>0

- Točku referentnog potencijala odabiremo beskonačno daleko

$$r_2 \to \infty \quad \Rightarrow \quad \frac{1}{r_2} \to 0$$

- Potencijal točkastog naboja q na udaljenosti r je:

$$\varphi(r) = \frac{q}{4\pi\varepsilon_0 r}$$

- Potencijal skupine N točkastih naboja u točkama

$$\vec{r}_i' \quad ; \quad i = 1,\ldots N$$

$$\varphi(\vec{r}) = \sum_{i=1}^{N} \varphi(\vec{r}) = \sum_{i=1}^{N} \frac{q_i}{4\pi\varepsilon_0} \frac{1}{|\vec{r} - \vec{r}_i'|}$$

Potencijal kontinuiranih raspodjela naboja

- Volumni naboj $\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \iiint_V \frac{\mathrm{d}q}{|\vec{r} - \vec{r}'|} = \frac{1}{4\pi\varepsilon_0} \iiint_V \frac{\rho(\vec{r}')\mathrm{d}V}{|\vec{r} - \vec{r}'|}$
- Plošni naboj $\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \iint_S \frac{\sigma(\vec{r}')\mathrm{d}S}{|\vec{r} - \vec{r}'|}$
- Linijski naboj $\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_l \frac{\lambda(\vec{r}')\mathrm{d}l}{|\vec{r} - \vec{r}'|}$

Slika statičkog električnog polja

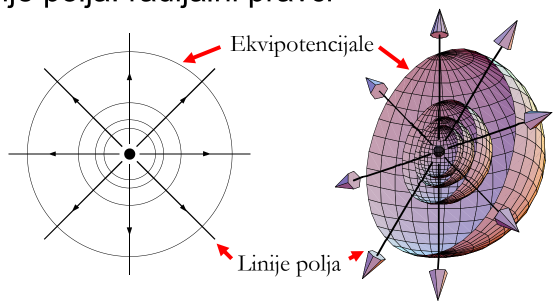
- Pomoć pri razumijevanju polja
- Linije polja (silnice): $\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z}$
- Ekvipotencijalne plohe: $\varphi = \text{konst.}$
- Dogovor
  - Ekvipotencijalne plohe (linije) crtamo tako da je razlika potencijala susjednih ploha jednaka
  - Linije polja crtamo tako da je električni tok između susjednih linija jednak

Slika polja točkastog naboja

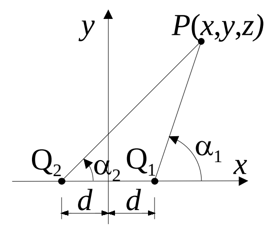
- Potencijal i polje točkastog naboja

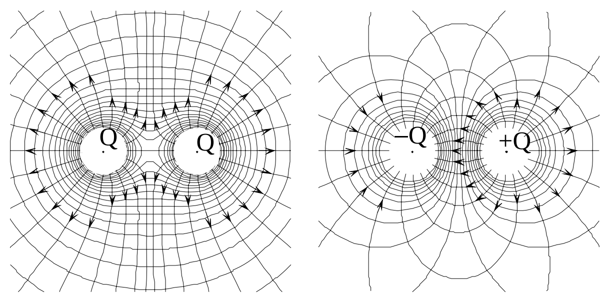
$$\varphi = \frac{1}{4\pi\varepsilon_0 |\vec{r}|} \quad ; \quad \vec{E} = \frac{1}{4\pi\varepsilon_0 |\vec{r}|^2} \frac{\vec{r}}{|\vec{r}|}$$

  - Ekvipotencijalne plohe: kugle
  - Linije polja: radijalni pravci

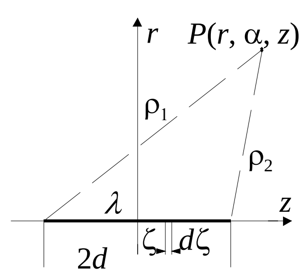


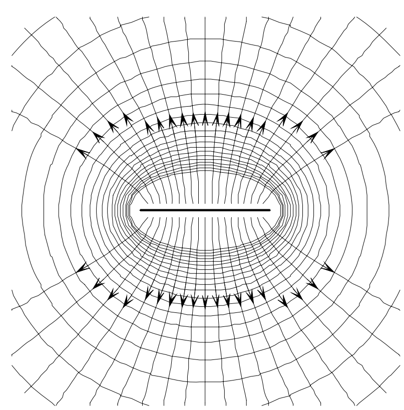
1. Odredite električno polje i potencijal dva točkasta naboja!



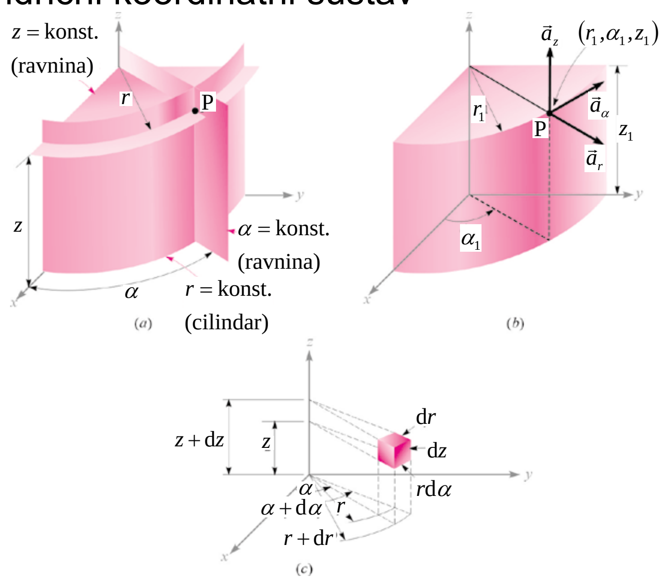


2. Odredite električno polje i potencijal jednoliko nabijene dužine!

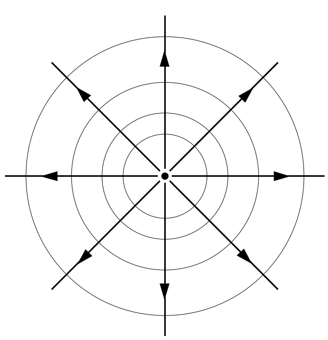


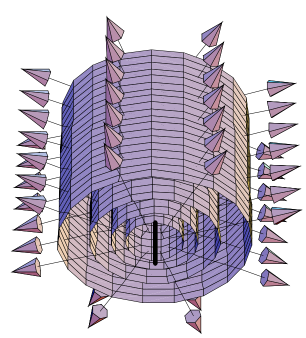


- Cilindrični koordinatni sustav

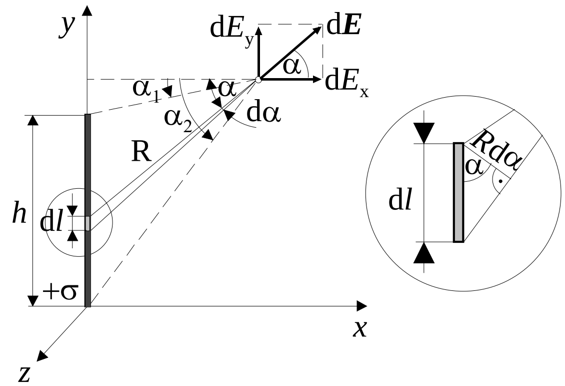


3. Odredite električno polje beskonačno dugog linijskog naboja!





4. Odredite električno polje beskonačno duge trake nabijene jednoliko nabojem gustoće σ.



5. U vakuumu je raspoređen naboj čija se prostorna gustoća mijenja kao

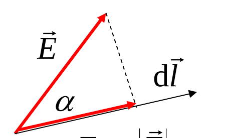
$$\rho_s = \begin{cases} kr^n & ; \quad r \le R \\ 0 & ; \quad r > R \end{cases}$$

gdje je *R* polumjer zamišljene kugle oko ishodišta. Odredite omjer $\varphi(r=0)/\varphi(r=R)$!

Veza jakosti polja i potencijala

- Definicija potencijala

$$\mathrm{d}\varphi = -\vec{E}\cdot \mathrm{d}\vec{l} = -\left|\vec{E}\right|\left|\mathrm{d}\vec{l}\right|\cos\alpha = -E_l \mathrm{d}l$$



$E_l = \left|\vec{E}\right|\cos\alpha$

- Slijedi

$$\frac{\mathrm{d}\varphi}{\mathrm{d}l} = -\left|\vec{E}\right|\cos\alpha = -E_l$$

- Komponenta polja u smjeru $\mathrm{d}\vec{l}$ je negativna derivacija potencijala u tom smjeru

Gradijent skalarnog polja

- Vektorsko polje
- Veličina gradijenta u svakoj točki jednaka je najvećoj brzini promjene polja
- Smjer gradijenta je u smjeru najveće brzine promjene polja
- Vektorski diferencijalni operator nabla ∇ (del)

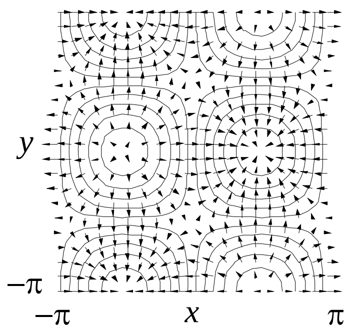
$$\nabla = \frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z$$

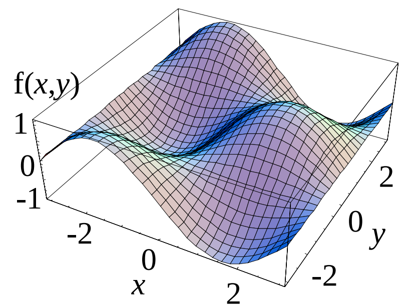
- Gradijent računamo kao:

$$\nabla \mathrm{f}(x,y,z) = \frac{\partial \mathrm{f}(x,y,z)}{\partial x}\vec{a}_x + \frac{\partial \mathrm{f}(x,y,z)}{\partial y}\vec{a}_y + \frac{\partial \mathrm{f}(x,y,z)}{\partial z}\vec{a}_z$$

- Primjer:

$$\mathrm{f}(x,y)=\sin x\cos y \Rightarrow \nabla \mathrm{f}=\cos x\cos y\,\vec{a}_x-\sin x\sin y\,\vec{a}_y$$





- Najveća vrijednost usmjerene derivacije

$$\max\left\{\frac{\mathrm{d}\varphi}{\mathrm{d}l}=-\left|\vec{E}\right|\cos\alpha=-E_l\right\}\Rightarrow\alpha=\pi\Rightarrow\max\left\{\frac{\mathrm{d}\varphi}{\mathrm{d}l}\right\}=E_l$$

- Smjer suprotan smjeru polja $\mathrm{d}\vec{l}=-\vec{a}_E\mathrm{d}l\ ;\ \vec{a}_E=\frac{\vec{E}}{\left|\vec{E}\right|}$

$$\left.\frac{\mathrm{d}\varphi}{\mathrm{d}l}\right|_{\max}=\left.\frac{\mathrm{d}\varphi}{\mathrm{d}l}\right|_{\mathrm{d}\vec{l}=-\vec{a}_E\mathrm{d}l}=\left|\vec{E}\right|$$

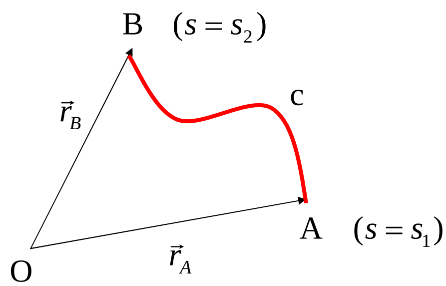
- Slijedi: $-\vec{a}_E\left.\frac{\mathrm{d}\varphi}{\mathrm{d}l}\right|_{max}=-\left|\vec{E}\right|\vec{a}_E=-\vec{E}$

- Vrijedi: $-\vec{a}_E\left.\frac{\mathrm{d}\varphi}{\mathrm{d}l}\right|_{\max}=\vec{a}_{-E}\left.\frac{\mathrm{d}\varphi}{\mathrm{d}l}\right|_{\max}=\nabla\varphi \quad\Rightarrow\quad \vec{E}=-\nabla\varphi$

- Uvrstimo $\vec{E} = -\nabla\varphi$ u izraz za potencijal između točaka A i B

$$\varphi(\mathrm{A}) - \varphi(\mathrm{B}) = -\int_{\mathrm{B}}^{\mathrm{A}} \vec{E}\cdot \mathrm{d}\vec{l} = \int_{\mathrm{B}}^{\mathrm{A}} \nabla\varphi \cdot \mathrm{d}\vec{l}$$

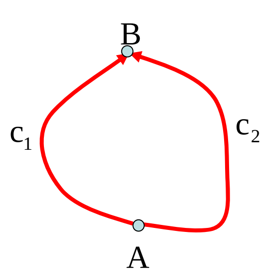
- Neka je $r(s)$; $s_1 \le s \le s_2$ parametarska jednadžba krivulje *c* između točaka A i B



- Slijedi

$$-\int_c \vec{E}[\vec{r}(t)]\cdot \mathrm{d}\vec{r} = \int_{s_1}^{s_2} \nabla\varphi(\vec{r})\cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}s}\mathrm{d}s = \int_{s_1}^{s_2} \frac{\mathrm{d}}{\mathrm{d}s}[\varphi(\vec{r})]\mathrm{d}s = \varphi(\vec{r}_B) - \varphi(\vec{r}_A)$$

- Očito, integral ovisi samo o razlici potencijala, a ne o obliku puta između A i B



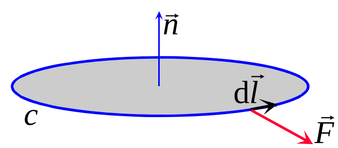
$$\int_{\mathrm{A}-\mathrm{c}_1-\mathrm{B}} \vec{E}\cdot \mathrm{d}\vec{l} = \int_{\mathrm{A}-\mathrm{c}_2-\mathrm{B}} \vec{E}\cdot \mathrm{d}\vec{l} = -\int_{\mathrm{B}-\mathrm{c}_2-\mathrm{A}} \vec{E}\cdot \mathrm{d}\vec{l}$$

$$\left.\begin{aligned} \varphi(\mathrm{B}) - \varphi(\mathrm{A}) &= -\int_{\mathrm{A}-\mathrm{c}_1-\mathrm{B}} \vec{E}\cdot \mathrm{d}\vec{l} \\ \varphi(\mathrm{A}) - \varphi(\mathrm{B}) &= -\int_{\mathrm{B}-\mathrm{c}_2-\mathrm{A}} \vec{E}\cdot \mathrm{d}\vec{l} \end{aligned}\right\} \Rightarrow \oint_c \vec{E}\cdot \mathrm{d}\vec{l} = 0$$

Linijski integral i rotor vektorskog polja

- Linijski integral $\vec{F}$ po krivulji c jest $\int_c \vec{F} \cdot \mathrm{d}\vec{l}$
- Cirkulacija $\vec{F}$ je linijski integral $\vec{F}$ po zatvorenoj krivulji *c*:

$$C = \oint_c \vec{F} \cdot \mathrm{d}\vec{l}$$

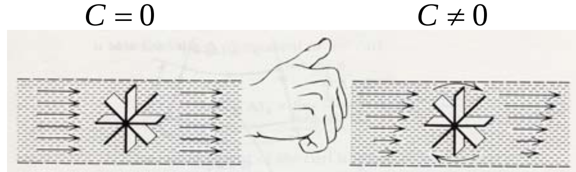


- Rotor $\vec{F}$ je cirkulacija vektorskog polja po jedinici površine ΔS koju obuhvaća krivulja *c* kad ta površina teži ništici:

$$\nabla \times \vec{F} = \vec{n} \lim_{\Delta S \to 0} \frac{\oint_c \vec{F} \cdot \mathrm{d}\vec{l}}{\Delta S} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} =$$

$$= \left[\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right]\vec{a}_x + \left[\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right]\vec{a}_y + \left[\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right]\vec{a}_z$$

$C = 0$ $\qquad\qquad$ $C \neq 0$



- Stokesov teorem

$$\iint_S \left(\nabla \times F\right) \cdot \vec{n}\, \mathrm{d}S = \oint_c \vec{F} \cdot \mathrm{d}\vec{l}$$

- Primjena Stokesova teorema

$$\oint_c \vec{E} \cdot \mathrm{d}\vec{l} = \iint_S \left(\nabla \times \vec{E}\right) \cdot \vec{n}\, \mathrm{d}S = 0 \quad \Rightarrow \nabla \times \vec{E} = 0$$