Signali i sustavi – zadaci za aktivnost – tjedan 18.

Akademska školska godina 2006./2007.

1. Kontinuirani sustav zadan je matricama:

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix};\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\ C = \begin{bmatrix} 1 & 0 \end{bmatrix};\ D = \begin{bmatrix} 0 \end{bmatrix}.$$

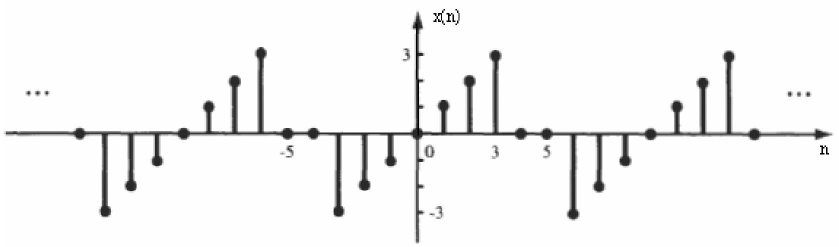
   Nađite amplitudno-frekvencijsku i fazno-frekvencijsku karakteristiku sustava, te odziv nepobuđenog i mirnog sustava ako se na ulaz dovede pobuda $u(t) = \cos 3t$, uz početne uvjete $y(0^-) = 2$ i $y'(0^-) = 0$.

2. Nađite vremenski diskretnu Fourierovu transformaciju diskretnog signala:

$$x(n) = \begin{cases} n, & |n| \le 3 \\ 0, & \text{inace} \end{cases}.$$

   Kakav je spektar dobivenog signala? Nađite energiju ovog signala.

3. Odredite diskretan Fourierov red periodičnog diskretnog signala prikazanog na slici 1. Nađite snagu ovog signala.



Slika 1.

4. Zadan je periodičan vremenski kontinuirani signal $x(t) = 2\cos(200\pi t) + 3\cos(500\pi t)$. Ako se ovaj signal otipkava s frekvencijom otipkavanja $F_s = 1\text{kHz}$, nađite koeficijente Fourierovog reda za diskretne periodične signale. Nacrtajte dobiveni red.

5. Linearni sustav ima dva pola $p_1=p_2=-1$ i nema nula. Impulsni odziv sustava ima maksimalnu vrijednost $h_{\max} = \frac{5}{e}$. Odredite prijenosnu funkciju diskretnog sustava koji bi imao isti impulsni odziv kao i zadani kontinuirani sustav u točkama $t=nT_s$, uz $T_s=1s$.

1.

$$A=\begin{bmatrix}0 & 1\\ -3 & -2\end{bmatrix}$$

$$B=\begin{bmatrix}0\\ 1\end{bmatrix}$$

$$C=\begin{bmatrix}1 & 0\end{bmatrix}$$

$$D=[0]$$

$$sI-A=\begin{bmatrix}s & -1\\ 3 & s+2\end{bmatrix}$$

$$(sI-A)^{-1}=\left[\begin{array}{cc|cc}s & -1 & 1 & 0\\ 3 & s+2 & 0 & 1\end{array}\right]\sim\left[\begin{array}{cc|cc}1 & -\frac{1}{s} & \frac{1}{s} & 0\\ 3 & s+2 & 0 & 1\end{array}\right]$$

$$\sim\left[\begin{array}{cc|cc}1 & -\frac{1}{s} & \frac{1}{s} & 0\\ 0 & \underbrace{s+2+\frac{3}{s}}_{\frac{s^2+2s+3}{s}} & -\frac{3}{s} & 1\end{array}\right]\sim$$

$$\frac{s^2+2s+\cancel{3}-\cancel{3}}{s(s^2+2s+3)}=\frac{s+2}{s^2+2s+3}$$

$$\sim\left[\begin{array}{cc|cc}1 & 0 & \frac{1}{s}-\frac{3}{s^3+2s^2+3s} & \frac{1}{s^2+2s+3}\\ 0 & 1 & -\frac{3}{s^2+2s+3} & \frac{s}{s^2+2s+3}\end{array}\right]$$

$$C(sI-A)^{-1}=\begin{bmatrix}\frac{s+2}{s^2+2s+3} & \frac{1}{s^2+2s+3}\end{bmatrix}$$

$$C(sI-A)^{-1}B=\frac{1}{s^2+2s+3}$$

$$H(s)=C(sI-A)^{-1}B+D$$

$$=\frac{1}{s^2+2s+3}$$

$$H(j\omega)=\frac{1}{-\omega^2+3+2j\omega}$$

AMPLITUDNO-FREKVENCIJSKA KARAKTERISTIKA

$$\boxed{|H(j\omega)|=\sqrt{\frac{1}{(3-\omega^2)^2+4\omega^2}}=\sqrt{\frac{1}{\omega^4-2\omega^2+9}}}$$

FAZNO-FREKVENCIJSKA KARAKTERISTIKA

$$\boxed{\measuredangle H(j\omega)=-\mathrm{arctg}\,\frac{2\omega}{3-\omega^2}}$$

POČETNI UVJETI

$b_0=0$

$b_1=0$

$y(0^-)=y(0^+)$

$y'(0^-)=y'(0^+)$

POBUDA

$u(t)=\cos 3t$

$\hookrightarrow \Omega=3,\ U=1$

PRISILNI ODZIV = PARTIKULARNO RJEŠENJE

$$y_p(t)=U\cdot|H(j\Omega)|\cos(\Omega t+\measuredangle H(j\Omega))$$

$$|H(j\Omega)|=\sqrt{\frac{1}{3^4-2\cdot 3^2+9}}=\frac{1}{\sqrt{72}}=\frac{1}{6\sqrt{2}}=\frac{\sqrt{2}}{12}$$

$$\measuredangle H(j\Omega)=-\mathrm{arctg}\,\frac{2\cdot 3}{3-9}=-\mathrm{arctg}\,\frac{6}{-6}=-\mathrm{arctg}(-1)=\frac{\pi}{4}$$

$$\boxed{y_p(t)=\frac{\sqrt{2}}{12}\cos\left(3t+\frac{\pi}{4}\right)}$$

HOMOGENO RJEŠENJE

$$y''(t)+2y'(t)+3y(t)=u(t)$$

$$s^2+2s+3=0$$

$$s_{1,2}=\frac{-2\pm\sqrt{4-4\cdot 3}}{2}=\frac{-2\pm j2\sqrt{2}}{2}=-1\pm j\sqrt{2}$$

$$y_h=C_1e^{(-1-j\sqrt{2})t}\mu(t)+C_2e^{(-1+j\sqrt{2})t}\mu(t)$$

NEPOBUĐENI

$y_0(t)=y_h(t)$ uz poč. uvj.

$$=C_1e^{(-1-j\sqrt{2})t}+C_2e^{(-1+j\sqrt{2})t}$$

$$y_0'(t)=(-1-j\sqrt{2})C_1e^{(-1-j\sqrt{2})t}+(-1+j\sqrt{2})C_2e^{(-1+j\sqrt{2})t}$$

$$y_0(0)=C_1+C_2=2$$

$$y_0'(0)=(-1-j\sqrt{2})C_1+(-1+j\sqrt{2})C_2=$$

1.

$$C_1 + C_2 = 2 \quad / \ (1+j\sqrt{2})$$

$$(-1-j\sqrt{2})C_1 + (-1+j\sqrt{2})C_2 = 0$$
$$(1+j\sqrt{2})C_1 + (1+j\sqrt{2})C_2 = 2(1+j\sqrt{2})$$

$$C_1 = \frac{2+j\sqrt{2}}{2}$$
$$C_2 = \frac{-j\sqrt{2}+2}{2}$$

$$y_0 = \left[\frac{2-j\sqrt{2}}{2} e^{(-1-j\sqrt{2})t} + \frac{2+j\sqrt{2}}{2} e^{(-1+j\sqrt{2})t}\right]\mu(t)$$

$$e^{-t}(\cos\sqrt{2}t - j\sin\sqrt{2}t) \qquad e^{-t}(\cos\sqrt{2}t + j\sin\sqrt{2}t)$$

$$\boxed{y_0(t) = e^{-t}\left[2\cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t\right]\mu(t)}$$

MIRNI

HOMOGENO

$$y_h = \left[C_1 e^{(-1-j\sqrt{2})t} + C_2 e^{(-1+j\sqrt{2})t}\right]\mu(t)$$
$$= e^{-t}\left(B_1\cos\sqrt{2}t + B_2\sin\sqrt{2}t\right)\mu(t)$$

MIRNI

$$y_m = e^{-t}(B_1\cos\sqrt{2}t + B_2\sin\sqrt{2}t) + \frac{\sqrt{2}}{12}\cos(3t+\frac{\pi}{4})$$

$$y_m' = (-B_1\sqrt{2}\sin\sqrt{2}t + \sqrt{2}B_2\cos\sqrt{2}t)e^{-t} - e^{-t}(B_1\cos\sqrt{2}t + B_2\sin\sqrt{2}t)$$
$$+ \frac{\sqrt{2}}{12}(-3)\sin(3t+\frac{\pi}{4}))$$

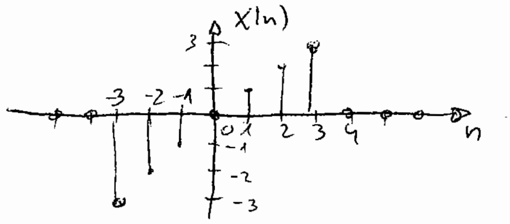
$$y_m(0) = B_1 + \frac{\sqrt{2}}{12}\cdot\frac{\sqrt{2}}{2} = B_1 + \frac{1}{12} \quad \rightarrow \quad B_1 = -\frac{1}{12}$$

$$y_m'(0) = \sqrt{2}B_2 - B_1 - \frac{\sqrt{2}\cdot 3\cdot\frac{\sqrt{2}}{2}}{12} = \sqrt{2}B_2 + \frac{1}{12} - \frac{3}{12}$$

$$\sqrt{2}B_2 = \frac{2}{12}$$
$$B_2 = \frac{2}{\sqrt{2}\cdot 12} = \frac{2\sqrt{2}}{2\cdot 12}$$
$$B_2 = \frac{\sqrt{2}}{12}$$

$$\boxed{y_m = e^{-t}\left(-\frac{1}{2}\cos\sqrt{2}t + \frac{\sqrt{2}}{12}\sin\sqrt{2}t\right) + \frac{\sqrt{2}}{12}\cos(3t+\frac{\pi}{4})}$$

2. $x(n) = \begin{cases} n, & |n| \leq 3 \\ 0, & \text{inače} \end{cases}$



$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= -3e^{+j\omega \cdot 3} - 2e^{2j\omega} - e^{j\omega} + e^{-j\omega} + 2e^{-j\omega \cdot 2} + 3e^{-3j\omega}$$

$$= 3(-\cancel{\cos 3\omega} - j\sin 3\omega + \cancel{\cos 3\omega} - j\sin 3\omega)$$
$$+ 2(-\cancel{\cos 2\omega} - j\sin 2\omega + \cancel{\cos 2\omega} - j\sin 2\omega)$$
$$+ \cancel{\cos\omega} - j\sin\omega - \cancel{\cos\omega} - j\sin\omega$$
$$= -2j(3\sin 3\omega + 2\sin 2\omega + \sin\omega)$$

KONTINUIRANI PERIODIČAN SPEKTAR

ENERGIJA

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \ldots + |-3|^2 + |-2|^2 + |-1|^2 + 0^2 + 1^2 + 2^2 + 3^2 + 0 + \ldots$$
$$= 9 + 4 + 1 + 1 + 4 + 9 = \boxed{28}$$

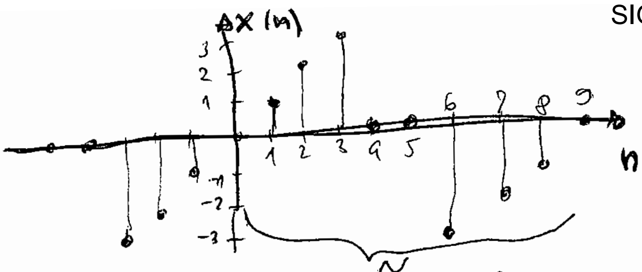
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |-2j(3\sin 3\omega + 2\sin 2\omega + \sin\omega)|^2 d\omega =$$

$$= \frac{1}{\cancel{2}\pi}\int_{-\pi}^{\pi} \cancel{4}2 (3\sin 3\omega + 2\sin 2\omega + \sin\omega)^2 d\omega$$

$$9\sin^2 3\omega + 4\sin^2 2\omega + \sin^2\omega + 6\sin 3\omega \cdot \sin 2\omega + 3\sin 3\omega \sin\omega$$
$$+ 6\sin 3\omega \sin 2\omega + 3\sin 3\omega \sin\omega$$
$$+ 2\sin 2\omega \sin\omega + 2\sin 2\omega \sin\omega$$

$$= \ldots = \boxed{28}$$

3.



$N = 9$

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-jk\frac{2\pi}{N}\cdot n}$$

$$X_k = \frac{1}{9}\sum_{n=0}^{8} x(n)\, e^{-jk\frac{2\pi}{9}\cdot n}$$

$$= \frac{1}{9}\left[0 + e^{-jk\frac{2\pi}{9}} + 2e^{-jk\frac{4\pi}{9}} + 3e^{-jk\frac{6\pi}{9}} - 3e^{-jk\frac{12\pi}{9}} - 2e^{-jk\frac{14\pi}{9}} - e^{-jk\frac{16\pi}{9}}\right]$$

$$= \frac{1}{9}\Big[\cancel{\cos\frac{2\pi}{9}k} - j\sin\frac{2\pi}{9}k + \cancel{2\cos\frac{4\pi}{9}k} - 2j\sin\frac{4\pi}{9}k + \cancel{3\cos\frac{6\pi}{9}k} - 3j\sin\frac{6\pi}{9}k$$

$$-3\left(\cancel{\cos\frac{12\pi}{9}k} - j\sin\frac{12\pi}{5}k\right) - 2\left(\cancel{\cos\frac{14\pi}{9}k} - j\sin\frac{14\pi}{9}k\right) - \left(\cancel{\cos\frac{16\pi}{9}k} - j\sin\frac{16\pi}{9}\right.$$

$\Rightarrow$

$$\cos k\frac{2\pi}{9} = \cos\frac{16\pi}{9}k$$

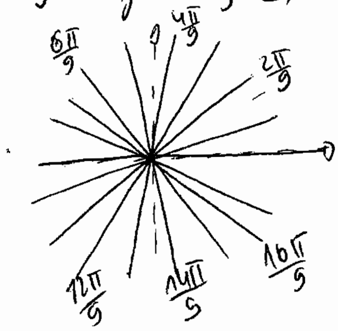
$$\cos\frac{4\pi}{9}k = \cos\frac{14\pi}{9}k$$

$$\cos\frac{6\pi}{9}k = \cos\frac{12\pi}{9}k$$

$$\sin\frac{2\pi}{9}k = -\sin\frac{16\pi}{9}k$$

$$\sin\frac{4\pi}{9}k = -\sin\frac{14\pi}{9}k$$

$$\sin\frac{6\pi}{9}k = -\sin\frac{12\pi}{9}k$$



$$X_k = \frac{1}{9}\left[-2j\sin\frac{2\pi}{9}k - 4j\sin\frac{4\pi}{9}k + 6j\sin\frac{12\pi}{9}k\right]$$

$$= -\frac{2j}{9}\left[\sin\frac{2\pi}{9}k + 2\sin\frac{4\pi}{9}k + 3\sin\frac{6\pi}{9}k\right]$$

DISKRETAN PERIODIČAN SPEKTAR

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1}|x(n)|^2 = \sum_{k=0}^{N-1}|X_k|^2$$

$$= \frac{1}{9}\sum_{n=0}^{8}|x(n)|^2 = \frac{1}{9}\left[0 + 1^2 + 2^2 + 3^2 + 0^2 + 0^2 + (-3)^2 + (-2)^2 + (-1)^2\right]$$

$$= \frac{1}{9}\left[1+4+9+9+4+1\right] = \underline{\frac{28}{9}}$$

4.

$$x(t) = 2\cos(200\pi t) + 3\cos(500\pi t)$$

$$F_s = 1000\,Hz$$

$$T_s = \frac{1}{F_s} = 10^{-3}\,s$$

otipkani signal

$$x(n) = 2\cos(200\pi \cdot nT_s) + 3\cos(500\pi \cdot nT_s)$$

$$= 2\cos\left(\frac{\pi}{5} n\right) + 3\cos\left(\frac{\pi}{2} n\right)$$

periodičan s periodom $N_1 = 10$

$N_2 = 4$

suma je periodična sa $N = 20$

$$x(n) = 2\cos\left(\frac{2\pi}{20} \cdot 2n\right) + 3\cos\left(\frac{2\pi}{20} \cdot 5n\right)$$

$$= \frac{2}{2} e^{j\frac{2\pi}{20}2n} + \frac{2}{2} e^{-j\frac{2\pi}{20}\cdot 2n} + \frac{3}{2} e^{j\frac{2\pi}{20}5n} + \frac{3}{2} e^{-j\frac{2\pi}{20}5n}$$

$$e^{-j\frac{2\pi}{20}\cdot 2n} = e^{j\frac{2\pi}{20}\cdot 18n}$$

$$e^{-j\frac{2\pi}{20}5n} = e^{j\frac{2\pi}{20}15n}$$

$$x(n) = e^{j\frac{2\pi}{20}2n} + e^{j\frac{2\pi}{20}18n} + \frac{3}{2} e^{j\frac{2\pi}{20}5n} + \frac{3}{2} e^{j\frac{2\pi}{20}15n}$$

FOURIEROV RED

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} = \frac{1}{20}\sum_{k=0}^{19} x(n) e^{-j\frac{2\pi}{20}kn}$$

$$x(n) = \sum_{k=0}^{N-1} X_k e^{j\frac{2\pi}{N}kn} = \sum_{k=0}^{19} X_k e^{j\frac{2\pi}{20}kn}$$

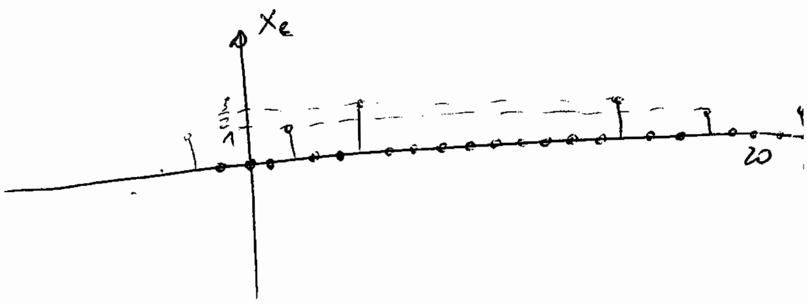
za $k = 2$ $X_k = 1$

$k = 18$ $X_k = 1$

$k = 5$ $X_k = \frac{3}{2}$

$k = 15$ $X_k = \frac{3}{2}$

za ostale $k \to X_k = 0$



5.

$p_1 = p_2 = -1$

$H(s) = \frac{A}{(s+1)^2}$

$h(t) = At\,e^{-t}\mu(t)$

$h'(t) = Ae^{-t}\mu(t) + A(-1)\,t\,e^{-t}\mu(t) + A\underbrace{te^{-t}\delta(t)}_{0}$

$\cancel{A}\,\cancel{e^{-t}}\,\cancel{\mu(t)} - \cancel{A}\,t\,\cancel{e^{-t}}\,\cancel{\mu(t)} = 0$

$t = 1$

$h(1) = Ae^{-1} = \frac{5}{e}$

$\boxed{A = 5}$

$\boxed{H(s) = \frac{5}{(s+1)^2}}$

$h(t) = 5t\,e^{-t}$

$h(n) = 5 \cdot nT_s\,e^{-nT_s}$

$h(n) = 5n\,e^{-n} = 5n(e^{-1})^n$

$\boxed{H(z) = 5 \cdot \frac{e^{-1} \cdot z}{(z - e^{-1})^2}}$