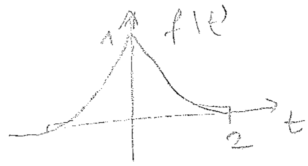
1. $f(t) = e^{-2|t|}\left(\mu(t+2) - \mu(t-2)\right)$




a) CTFT

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = \int_{-2}^{2} e^{-2|t|}\, e^{-j\omega t}\, dt$$

$$= \int_{-2}^{0} e^{2t} e^{-j\omega t}\, dt + \int_{0}^{2} e^{-2t} e^{-j\omega t}\, dt$$

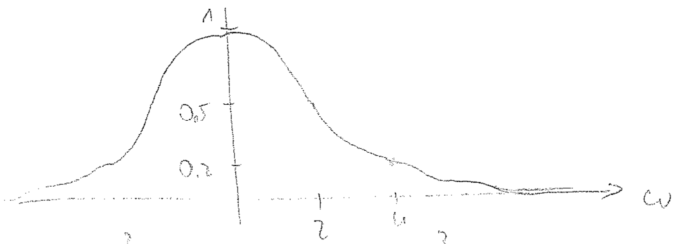
$$= \frac{e^{(2-j\omega)t}}{2-j\omega}\Bigg|_{-2}^{0} + \frac{e^{-(2+j\omega)t}}{-(2+j\omega)}\Bigg|_{0}^{2}$$

$$= \frac{1}{2-j\omega} - \frac{e^{-2(2-j\omega)}}{2-j\omega} + \frac{e^{-2(2+j\omega)}}{-(2+j\omega)} + \frac{1}{+(2+j\omega)}$$

$$= \frac{2+j\omega+2-j\omega}{4+\omega^2} - \frac{(2+j\omega)\, e^{-4} e^{2j\omega} + (2-j\omega)\, e^{-4} e^{-2j\omega}}{4+\omega^2}$$

$$= \frac{1}{\omega^2+4}\left(4 - 2e^{-4}\left(e^{2j\omega} + e^{-2j\omega}\right) - j\omega e^{-4}\left(e^{2j\omega} - e^{-2j\omega}\right)\right)$$

$$= \frac{1}{\omega^2+4}\left(4 - 2e^{-4}\cdot 2\cos 2\omega + 2\omega e^{-4}\sin 2\omega\right)$$




b) $E = \int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-2}^{2} |e^{-2|t|}|^2\, dt = \int_{-2}^{2} e^{-4|t|}\, dt$

$$= \int_{-2}^{0} e^{4t}\, dt + \int_{0}^{2} e^{-4t}\, dt = \frac{e^{4t}}{4}\Bigg|_{-2}^{0} + \frac{e^{-4t}}{-4}\Bigg|_{0}^{2}$$

$$= \frac{1}{4} - \frac{e^{-2\cdot 4}}{4} + \frac{e^{-4\cdot 2}}{-4} + \frac{1}{+4} = \frac{1}{2} - \frac{1}{2}e^{-8} \approx \frac{1}{2}$$

c) Ne, iz $F(j\omega)$ je vidljivo kako ne postoji maksimalna frekvencija zadanog signala. Stoga nije moguće očitati signal dvostrukom max frekvencijom kako pri rekonstrukciji idealnim interpolatorom ne bi došlo do aliasinga.

2. $f(n) = \{2, 1, 4, 1\}$

a) $DFT_4$

$$F(k) = \sum_{n=0}^{N-1} f(n)\, W_N^{kn} = \sum_{n=0}^{N-1} f(n)\, e^{-\frac{2\pi}{N} j nk}$$

$$F(k) = \sum_{n=0}^{3} f(n)\, e^{-j\frac{2\pi}{4} nk} = 2 + e^{-j\frac{2\pi}{4}\cdot k} + 4e^{-j\frac{2\pi}{4}\cdot 2k} + e^{-j\frac{2\pi}{4}\cdot 3k}$$

$$= 2 + e^{-j\frac{2\pi}{4}\cdot 2k}\left(e^{j\frac{2\pi}{4}k} + 4 + e^{-j\frac{2\pi}{4}k}\right)$$

$$= 2 + e^{-j\pi k}\left(4 + 2\cos\frac{\pi}{2}k\right)$$

$$F(0) = 2 + (4 + 2\cos\tfrac{\pi}{2}\cdot 0) = 2 + 4 + 2 = 8$$

$$F(1) = 2 + e^{-j\pi}(4 + 2\cos\tfrac{\pi}{2}) = 2 - (4) = -2$$

$$F(2) = 2 + e^{-j\pi\cdot 2}(4 + 2\cos\tfrac{\pi}{2}\cdot 2) = 2 + 4 - 2 = 4$$

$$F(3) = 2 + e^{-j\pi\cdot 3}(4 + 2\cos\tfrac{\pi}{2}\cdot 3) = 2 - 4 = -2$$

$F(k) = \{8, -2, 4, -2\}$

b) DTFT

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

Parseval:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})\, X^*(e^{j\Omega})\, d\Omega$$

Izvod:

$$E = \sum_{n=-\infty}^{\infty} x(n)\cdot x^*(n) = \sum_{n=-\infty}^{\infty} x(n)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\Omega})\, e^{-j\Omega n}\, d\Omega\right] =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\Omega})\left[\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}\right] d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\Omega})\, X(e^{j\Omega})\, d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\Omega})|^2\, d\Omega$$

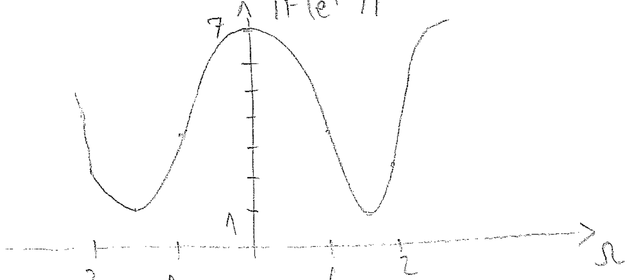
c) DTFT

$$F(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} f(n)\, e^{-j\Omega n} = 2 + e^{-j\Omega} + 4e^{-j\Omega\cdot 2} + e^{-j\Omega\cdot 3}$$

$$= 2 + e^{-j\Omega\cdot 2}\left(e^{j\Omega} + e^{-j\Omega} + 4\right)$$

$$= 2 + e^{-j2\Omega}(\cos 2\Omega + 4) = 2 + \cos^2 2\Omega + 4\cos 2\Omega - j\sin 2\Omega\cos 2\Omega - j4\sin 2\Omega$$

$$|F(e^{j\Omega})| = \sqrt{(2 + 4\cos 2\Omega + \cos^2 2\Omega)^2 + \sin^2 2\Omega\,(\cos 2\Omega + 4)^2}$$



3. $s_1 = -1 - j$

$s_2 = -1 + j$

a) $H(s) = \dfrac{B}{s^2 + a_1 s + a_2}$

$(s - s_1)(s - s_2) = (s + 1 + j)(s + 1 - j) = s^2 + s(1+j) + s(1-j) + (1-j)(1+j)$

$= s^2 + 2s + 2$

$H(s) = \dfrac{B}{s^2 + 2s + 2}$

$H(j\omega) = \dfrac{B}{(j\omega)^2 + 2j\omega + 2} = \dfrac{B}{2 - \omega^2 + 2j\omega}$

$\omega = 0 \longrightarrow |H(j0)| = \left|\dfrac{B}{2}\right| = 2$

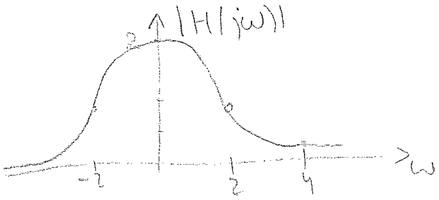
$B = 4$

$H(s) = \dfrac{4}{s^2 + 2s + 2} = \dfrac{Y(s)}{U(s)}$

$y''(t) + 2y'(t) + 2y(t) = 4u(t)$

b) $H(j\omega) = \dfrac{4}{2 - \omega^2 + 2j\omega}$

$|H(j\omega)| = \dfrac{4}{\sqrt{(2-\omega^2)^2 + 4\omega^2}} = \dfrac{4}{\sqrt{4 - 4\omega^2 + \omega^4 + 4\omega^2}} = \dfrac{4}{\sqrt{\omega^4 + 4}}$




c) $u(t) = 2e^{-2t}$

$y_p(t) = 2 \cdot |H(2)| e^{-2t}$

$|H(2)| = \dfrac{4}{2^2 + 2 \cdot 2 + 2} = \dfrac{4}{4 + 4 + 2} = \dfrac{4}{10} = \dfrac{2}{5}$

$y_p(t) = \dfrac{4}{5} e^{-2t}$

4. $y(n) - \frac{9}{20} y(n-1) + \frac{1}{20} y(n-2) = 4u(n) - u(n-1)$

$u(n) = 4\left(\frac{1}{3}\right)^n \mu(n)$

a) u vremenskoj domeni

$q^2 - \frac{9}{20} q + \frac{1}{20} = 0$

$\left(q - \frac{1}{4}\right)\left(q - \frac{1}{5}\right) = 0$

$q_1 = \frac{1}{4} \qquad q_2 = \frac{1}{5}$

$y_h(n) = C_1 \left(\frac{1}{4}\right)^n + C_2 \left(\frac{1}{5}\right)^n$

$y_p(n) = K \left(\frac{1}{3}\right)^n$

$K\left(\frac{1}{3}\right)^n - \frac{9}{20} K \left(\frac{1}{3}\right)^{n-1} + \frac{1}{20} K \left(\frac{1}{3}\right)^{n-2} = 4 \cdot 4 \left(\frac{1}{3}\right)^n - 4\left(\frac{1}{3}\right)^{n-1}$

$K \cdot \frac{1}{9} - \frac{9}{20} \cdot K \cdot \frac{1}{3} + \frac{1}{20} \cdot K = 4 \cdot 4 \cdot \frac{1}{9} - 4 \cdot \frac{1}{3}$

$\frac{20 - 9 \cdot 3 + 9}{20 \cdot 9} \cdot K = \frac{16 - 12}{9} = \frac{4}{9}$

$\frac{1}{90} K = \frac{4}{9} \quad / \cdot 90$

$K = \frac{4 \cdot \cancel{90}^{10}}{\cancel{9}}$

$K = 40$

$y_p(n) = 40\left(\frac{1}{3}\right)^n$

$y_m(n) = C_1 \left(\frac{1}{4}\right)^n + C_2 \left(\frac{1}{5}\right)^n + 40\left(\frac{1}{3}\right)^n$

$y(-1) = 0$

$y(-2) = 0$

$y(0) = 4u(0) - u(-1) + \frac{9}{20} y(-1) - \frac{1}{20} y(-2) = 4 \cdot 4 \left(\frac{1}{3}\right)^0 = 16$

$y(1) = 4u(1) - u(0) + \frac{9}{20} y(0) - \frac{1}{20} y(-1) = 4 \cdot 4 \cdot \frac{1}{3} - 4 \cdot \left(\frac{1}{3}\right)^0 + \frac{9}{20} \cdot 16 = \frac{128}{15}$

$$\left.\begin{aligned} y_m(0) &= C_1 + C_2 + 40 = 16 \\ y_m(1) &= C_1 \frac{1}{4} + C_2 \frac{1}{5} + 40 \cdot \frac{1}{3} = \frac{128}{15} \end{aligned}\right\} \quad \left.\begin{aligned} C_1 + C_2 &= -24 \\ \frac{1}{4} C_1 + \frac{1}{5} C_2 &= \frac{-72}{15} \end{aligned}\right\} \quad \begin{aligned} C_1 &= 0 \\ C_2 &= -24 \end{aligned}$$

$$\boxed{y_m(n) = \left[-24 \left(\frac{1}{5}\right)^n + 40 \left(\frac{1}{3}\right)^n\right] \mu(n)}$$

b) $z$ - transformacija

$$Y(z) - \frac{9}{20} Y(z) z^{-1} + \frac{1}{20} Y(z) z^{-2} = 4U(z) - U(z) z^{-1}$$

$$Y(z) = \frac{4 - z^{-1}}{1 - \frac{9}{20} z^{-1} + \frac{1}{20} z^{-2}} \cdot U(z)$$

$$= \frac{4z^2 - z}{z^2 - \frac{9}{20} z + \frac{1}{20}} \cdot U(z)$$

$$= \frac{z(4z-1)}{(z - \frac{1}{4})(z - \frac{1}{5})} U(z)$$

$$u(n) = 4 \left(\frac{1}{3}\right)^n \mu(n) \longrightarrow U(z) = 4 \cdot \frac{z}{z - \frac{1}{3}}$$

$$Y(z) = \frac{4z\cancel{(z - \frac{1}{4})}}{\cancel{(z - \frac{1}{4})}(z - \frac{1}{5})} \cdot \frac{4z}{z - \frac{1}{3}} = \frac{16z^2}{(z - \frac{1}{5})(z - \frac{1}{3})}$$

$$\frac{Y(z)}{z} = \frac{16z}{(z - \frac{1}{5})(z - \frac{1}{3})} = \frac{A}{z - \frac{1}{5}} + \frac{B}{z - \frac{1}{3}}$$

$$\left.\begin{aligned} A + B &= 16 \\ -\frac{1}{3} A - \frac{1}{5} B &= 0 \end{aligned}\right\} \qquad -\frac{1}{3} A = \frac{1}{5} B \qquad A = -\frac{3}{5} B$$

$$-\frac{3}{5} B + B = 16 \qquad \frac{2}{5} B = 16 \qquad B = 40$$

$$A = -\frac{3}{5} \cdot 40 \qquad A = -24$$

$$Y(z) = \frac{-24z}{z - \frac{1}{5}} + \frac{40z}{z - \frac{1}{3}}$$

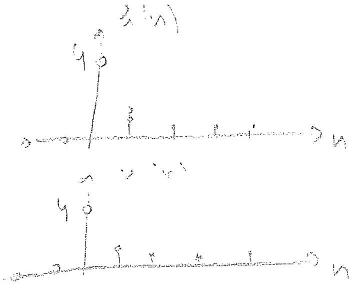
$$\boxed{y(n) = \left[-24\left(\frac{1}{5}\right)^n + 40\left(\frac{1}{3}\right)^n\right] \mu(n)}$$

c) Impulzni odziv

$$y(n) - \frac{9}{20}y(n-1) + \frac{1}{20}y(n-2) = 4u(n) - u(n-1)$$

$$H(z) = \frac{4 - z^{-1}}{1 - \frac{9}{20}z^{-1} + \frac{1}{20}z^{-2}} = \frac{4z^2 - z}{z^2 - \frac{9}{20}z + \frac{1}{20}} = \frac{4z\cancel{\left(z - \frac{1}{4}\right)}}{\cancel{\left(z - \frac{1}{4}\right)}\left(z - \frac{1}{5}\right)} = \frac{4z}{z - \frac{1}{5}}$$

$$h(n) = 4 \cdot \left(\frac{1}{5}\right)^n \mu(n)$$




$$u(n) = 4\left(\frac{1}{3}\right)^n \mu(n)$$

$$y(n) = h(n) * u(n) = \sum_{m=-\infty}^{\infty} h(m)\,u(n-m)$$

$$= \sum_{m=0}^{n} 4\left(\frac{1}{5}\right)^m \cdot 4\left(\frac{1}{3}\right)^{n-m}$$

$$= 16 \cdot \left(\frac{1}{3}\right)^n \cdot \sum_{m=0}^{n} \left(\frac{1}{5}\right)^m \cdot 3^m = 16\left(\frac{1}{3}\right)^n \cdot \sum_{m=0}^{n} \left(\frac{3}{5}\right)^m$$

$$= 16\left(\frac{1}{3}\right)^n \cdot \frac{1 - \left(\frac{3}{5}\right)^{n+1}}{1 - \frac{3}{5}} = \overset{8}{\cancel{16}}\left(\frac{1}{3}\right)^n \cdot \frac{1 - \frac{3}{5} \cdot \left(\frac{3}{5}\right)^n}{\frac{\cancel{2}}{5}}$$

$$= 40\left(\frac{1}{3}\right)^n \cdot \left(1 - \frac{3}{5}\left(\frac{3}{5}\right)^n\right)$$

$$= 40\left(\frac{1}{3}\right)^n - \overset{8}{\cancel{40}} \cdot \frac{3}{\cancel{5}} \left(\frac{1}{\cancel{3}}\right)^n \cdot \left(\frac{\cancel{3}}{5}\right)^n$$

$$\boxed{y(n) = \left[40\left(\frac{1}{3}\right)^n - 24\left(\frac{1}{5}\right)^n\right]\mu(n)}$$

5.

$$y''(t) + 5y'(t) + 4y(t) = u(t)$$

$$u(t) = \begin{cases} 20 \cos 2t & , \ t < 0 \\ 40 \cos 2t & , \ t \geq 0 \end{cases}$$

$$s^2 + 5s + 4 = 0$$

$$s^2 + 4s + s + 4 = 0$$

$$s(s+4) + (s+4) = 0$$

$$(s+1)(s+4) = 0$$

$$s_1 = -1 \qquad s_2 = -4$$

$$y_h(t) = C_1 e^{-t} + C_2 e^{-4t}$$

$Re\{s_1\} < 0$ Sustav je stabilan

$Re\{s_2\} < 0$

$$H(s) = \frac{1}{s^2 + 5s + 4}$$

$$H(j\omega) = \frac{1}{4 - \omega^2 + 5j\omega}$$

$t < 0$

$$y_p(t) = 20 \cdot |H(j\omega)| \cos(2t + \measuredangle H(j2))$$

$$H(j2) = \frac{1}{4 - 4 + 5j \cdot 2} = \frac{1 \cdot j}{10 jj} = \frac{j}{-10} = -\frac{j}{10}$$

$$|H(j \cdot 2)| = \frac{1}{10} \qquad \measuredangle H(j2) = \operatorname{arctg} \frac{-\frac{1}{10}}{0} = -\pi/2$$

$$y_p(t) = 20 \cdot \frac{1}{10} \cos(2t - \pi/2) = 2 \cos(2t - \pi/2)$$

$$y(t) = 2 \cos(2t - \tfrac{\pi}{2})$$

$$y'(t) = -4 \sin(2t - \tfrac{\pi}{2})$$

$$y(0^-) = 2 \cos(2 \cdot 0 - \tfrac{\pi}{2}) = 2 \cos(-\tfrac{\pi}{2}) = 0$$

$$y'(0^-) = -4 \sin(-\tfrac{\pi}{2}) = 4$$

$t \geq 0$

$$y(0^+) = 0$$

$$y'(0^+) = 4$$

$$y_p(t) = 40 \cdot |H(j2)| \cos(2t + \measuredangle H(j \cdot 2))$$

$$y_p(t) = 40 \cdot \frac{1}{10} \cos(2t - \tfrac{\pi}{2}) = 4 \cos(2t - \tfrac{\pi}{2})$$

$$y(t) = c_1 e^{-t} + c_2 e^{-4t} + 4\cos(2t - \tfrac{\pi}{2})$$

$$y'(t) = -c_1 e^{-t} - 4c_2 e^{-4t} - 8\sin(2t - \tfrac{\pi}{2})$$

$$y(0^+) = c_1 + c_2 + 4\cos(-\tfrac{\pi}{2}) = c_1 + c_2 = 0$$

$$y'(0^+) = -c_1 - 4c_2 - 8\sin(-\tfrac{\pi}{2}) = -c_1 - 4c_2 + 8 = 4$$

$$c_1 = -c_2$$

$$c_2 - 4c_2 = 4 - 8 = -4$$

$$-3c_2 = -4$$

$$c_2 = \frac{4}{3} \qquad c_1 = -\frac{4}{3}$$

$$y(t) = -\frac{4}{3} e^{-t} + \frac{4}{3} e^{-4t} + 4\cos(2t - \tfrac{\pi}{2})$$

$$\boxed{y(t) = \begin{cases} 2\cos(2t - \frac{\pi}{3}) & , \quad t < 0 \\ -\frac{4}{3} e^{-t} + \frac{4}{3} e^{-4t} + 4\cos(2t - \frac{\pi}{2}) & , \quad t \geq 0 \end{cases}}$$