# Signali i sustavi

**Pismeni ispit – 24. travnja 2014.**

1. **(9 bodova)** Zadan je vremenski kontinuiran signal $f(t) = t^2(\mu(t+7) - \mu(t-7))$.

   a) **(2 boda)** Izračunajte energiju signala.

   b) **(2 boda)** Izračunajte i skicirajte prvu derivaciju signala.

   c) **(2 boda)** Očitajte signal i njegovu prvu derivaciju s periodom očitavanja $T_s = 3$.

   d) **(3 boda)** Iz očitaka signala izračunajte prvu derivaciju signala pomoću aproksimacije derivacije silaznom diferencijom.

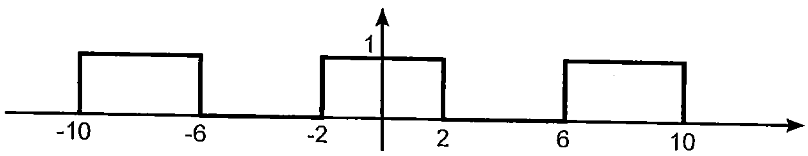
2. **(9 bodova)** Vremenski kontinuiran periodičan signal zadan je slikom.

   a) **(5 bodova)** Odredite i skicirajte amplitudni i fazni spektar signala za $k \in \{-3, -2, -1, 0, 1, 2, 3\}$.

   b) **(2 boda)** Objasnite Gibbsovu pojavu. Navedite primjer signala kod kojeg se javlja i primjer signala kod kojeg se ne javlja Gibbsova pojava.

   c) **(2 boda)** Pokažite da za vremenski kontinuirane realne signale $f(t)$ za koje postoji CTFS vrijedi

$$f(t) = \sum_{k=-\infty}^{\infty} F_k e^{jk\omega_0 t} = F_0 + \sum_{k=1}^{\infty} 2|F_k| \cos(k\omega_0 t + \angle F_k).$$




3. **(9 bodova)** Spektar vremenski kontinuiranog signala $f(t)$ je $F(j\omega) = e^{-4|\omega|}$.

   a) **(3 boda)** Odredite signal $f(t)$.

   b) **(3 boda)** Izračunajte energiju signala.

   c) **(3 boda)** Signal $f(t)$ očitali smo s periodom očitavanja $T_s = 1\,\text{ms}$. Koliko točaka očitanog signala moramo uzeti ako želimo numerički odrediti spektar s rezolucijom od $f_0 = 10\,\text{Hz}$?

4. **(9 bodova)** Zadan je vremenski diskretan signal $f(n) = \begin{cases} 4^{-n}, & \text{za } n > 1 \\ 0, & \text{inače} \end{cases}$.

   a) **(4 boda)** Odredite amplitudni i fazni spektar signala (nije potrebno skicirati).

   b) **(2 boda)** Izračunajte vrijednost amplitudnog i faznog spektra za $\Omega = \frac{\pi}{2}$.

   c) **(3 boda)** Pokažite da je spektar vremenski diskretnog aperiodičnog signala periodičan s osnovnim periodom $2\pi$.

5. **(9 bodova)** Vremenski kontinuiran signal $f(t)$ očitan je u osam točaka s frekvencijom očitavanja $f_s = 1\,\text{kHz}$, te je dobiven vremenski diskretan signal $f(n) = \{\underline{-4}, -2, 2, 4, -4, -2, 2, 4\}$.

   a) **(5 bodova)** Izračunajte DFT u osam točaka vremenski diskretnog signala $f(n)$.

   b) **(2 boda)** Odredite frekvenciju $\Omega$ na kojoj amplitudni spektar DFT-a vremenski diskretnog signala $f(n)$ poprima maksimum.

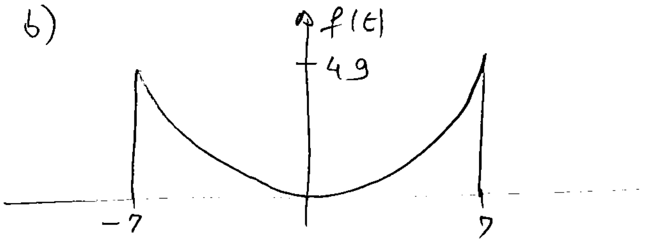
   c) **(2 boda)** Odredite dominantnu spektralnu komponentu vremenski kontinuiranog signala $f(t)$.

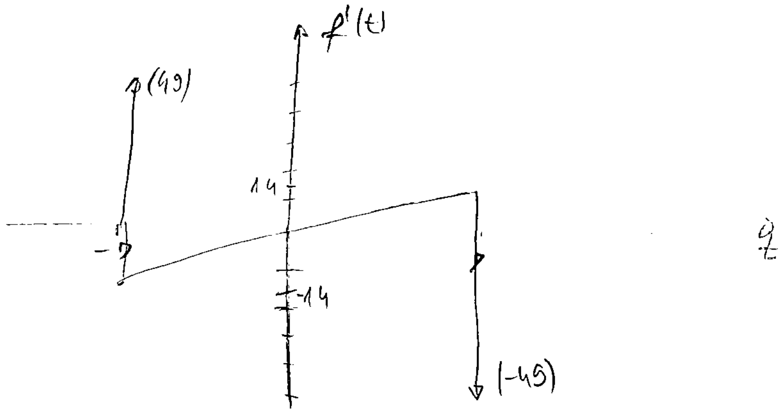
B

1. $f(t) = t^2 \left(\mu(t+7) - \mu(t-7)\right)$

a) $E = \int\limits_{-\infty}^{\infty} |f(t)|^2 dt = \int\limits_{-7}^{7} t^4 dt = \frac{t^5}{5}\Big|_{-7}^{7} = \frac{1}{5}\left(7^5 - (-7)^5\right) = \frac{1}{5}\left(2 \cdot 7^5\right)$

b)




$$\begin{aligned} f_1(t) = f'(t) &= 2t\left(\mu(t+7) - \mu(t-7)\right) \\ &\quad + t^2\left(\delta(t+7) - \delta(t-7)\right) \\ &= 2t\left(\mu(t+7) - \mu(t-7)\right) \\ &\quad + 49\left(\delta(t+7) - \delta(t-7)\right) \end{aligned}$$

c) $T_s = 3$

$$\begin{aligned} f(n) &= (3n)^2 \left(\mu(3n+7) - \mu(3n-7)\right) \\ &= 9n^2 \left(\mu(3n+7) - \mu(3n-7)\right) \\ &= \begin{cases} 9n^2 & \text{za } n = -2, -1, 0, 1, 2 \\ 0 & \text{inače} \end{cases} \\ &= \{\dots 0, 36, 9, \underline{0}, 9, 36, 0, \dots\} \end{aligned}$$

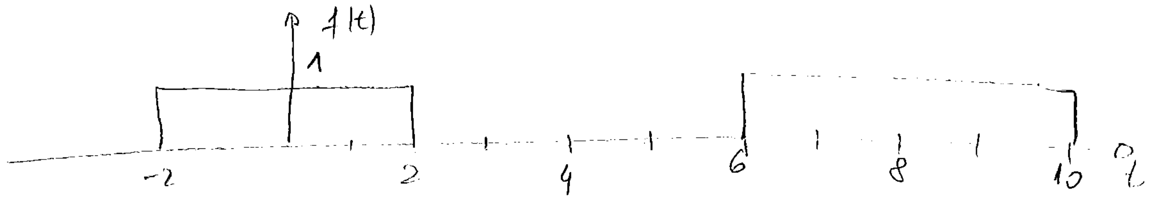
$$\begin{aligned} f_1(n) = f_1(nT_s) &= \begin{cases} 6n & \text{za } n = -2, -1, 0, 1, 2 \\ 0 & \text{inače} \end{cases} \\ &= \{\dots 0, -12, -6, \underline{0}, 6, 12, 0, \dots\} \end{aligned}$$

d)

$$\begin{aligned} f_1(n) &= \frac{1}{T_s}\{f(n) - f(n-1)\} \\ &= \frac{1}{3}\{9n^2 - 9(n-1)^2\} = \frac{1}{3}\left(\cancel{9n^2} - \cancel{9n^2} + 2 \cdot 9n - 9\right) \\ &= \frac{1 \cdot 9}{3}(2n-1) = 3(2n-1) = 6n - 3 \qquad \text{za } n = -2, -1, 0, 1, 2 \\ &= \{\dots 0, -15, -9, \underline{-3}, 3, 9, 0, \dots\} \end{aligned}$$

B

2.




a) CTFS

$$F_k = \frac{1}{T_0}\int_{T_0} f(t)\, e^{-j\frac{2\pi}{8}kt}\, dt$$

$$= \frac{1}{8}\int_{-2}^{2} 1\, e^{-j\frac{2\pi}{8}kt}\, dt = \frac{1}{8}\; \frac{e^{-j\frac{2\pi}{8}kt}\Big|_{-2}^{2}}{-\frac{2\pi}{8}kj}$$

$$= \frac{1}{8\cdot(-2\pi)kj\frac{1}{8}}\cdot\left(e^{-j\frac{2\pi}{8}\cdot 2\cdot k} - e^{j\frac{2\pi}{8}2k}\right)$$

$$= \frac{1}{+2\pi k j}\left(e^{j\frac{\pi}{2}k} - e^{-j\frac{\pi}{2}k}\right) = \frac{1}{\cancel{2}\pi \cancel{j} k}\cdot \cancel{2}\cancel{j}\sin\frac{\pi}{2}k$$

$$= \frac{\sin\frac{\pi}{2}k}{\pi k} = \frac{1}{2}\;\frac{\sin\frac{\pi}{2}k}{\frac{\pi}{2}\cdot k}$$

$$F_0 = \frac{1}{8}\int_{-2}^{2} 1\, dt = \frac{t}{8}\Big|_{-2}^{2} = \frac{2-(-2)}{8} = \frac{1}{2}$$

$$|F_k| = \frac{1}{2}\left|\frac{\sin\frac{\pi}{2}k}{\frac{\pi}{2}k}\right|$$

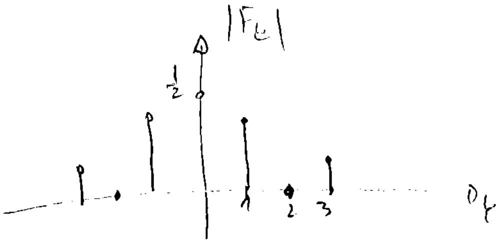
$$\measuredangle F_k = \measuredangle\,\frac{\sin\frac{\pi}{2}k}{\frac{\pi}{2}k} = \pm\pi \quad \text{kad je} \quad \frac{\sin\frac{\pi}{2}k}{\frac{\pi}{2}k} < 0$$

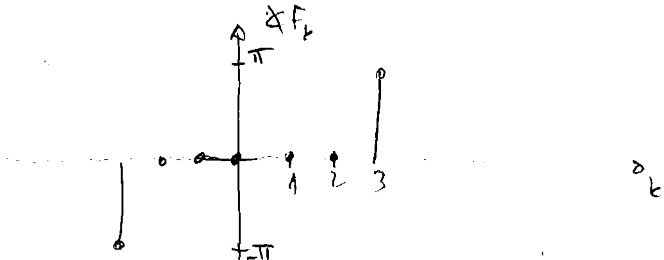
$$k \in \{-3,-2,-1,0,1,2,3\}$$

$F_{-3} = \frac{1}{2}\cdot\frac{2}{-3\pi} = -\frac{1}{3\pi}$ $\qquad$ $F_3 = \frac{1}{2}\;\frac{-2}{3\pi} = -\frac{1}{3\pi}$

$F_{-2} = 0$ $\qquad$ $F_2 = 0$

$F_{-1} = \frac{1}{2}\cdot\frac{2}{\pi} = \frac{1}{\pi}$ $\qquad$ $F_1 = \frac{2}{\pi}\cdot\frac{1}{2} = \frac{1}{\pi}$



B

2, b) Gibbsove pojave se javlja kod postavne kontinuiranih signala s diskontinuitetom u CTFS u obliku nelovitosti na mjestima diskontinuiteta. Energija greške teži u nule, ali Fourierove reprezentacija signala u točkama diskontinuiteta ne teži prema vrijednosti signala.

Primjer kod pravokutnika koji se ponavlja.

Gibbsove pojave nema kod npr. sinusnog signala.

c) $$\sum_{k=-\infty}^{\infty} F_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{-1} F_k e^{jk\omega_0 t} + F_0 + \sum_{k=1}^{\infty} F_k e^{jk\omega_0 t}$$

Za realni signal → koeficijenti spektra su konjugirano simetrični

$$F_k = |F_k| \cdot e^{j\measuredangle F_k}$$

$$F_{-k} = |F_k| e^{-j\measuredangle F_k}$$

$$f(t) = F_0 + \sum_{k=1}^{\infty} F_k e^{-jk\omega_0 t} + \sum_{k=1}^{\infty} F_k e^{jk\omega_0 t}$$

$$= F_0 + \sum_{k=1}^{\infty} |F_k| e^{-j\measuredangle F_k} e^{-jk\omega_0} + \sum_{k=1}^{\infty} |F_k| e^{j\measuredangle F_k} e^{jk\omega_0 t}$$

$$= F_0 + \sum_{k=1}^{\infty} |F_k| \left( e^{-j(k\omega_0 t + \measuredangle F_k)} + e^{j(k\omega_0 t + \measuredangle F_k)} \right)$$

$$\cos(k\omega_0 t + \measuredangle F_k) - j\cancel{\sin(k\omega_0 t + \measuredangle F_k)} + \cos(k\omega_0 t + \measuredangle F_k) + j\cancel{\sin(k\omega_0 t + \measuredangle F_k)}$$

$$= F_0 + \sum_{k=1}^{\infty} |F_k| \cdot 2\cos(k\omega_0 t + \measuredangle F_k)$$

✓

B

3. $F(j\omega) = e^{-4|\omega|}$

a) $$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-4|\omega|} e^{j\omega t} d\omega =$$

$$= \frac{1}{2\pi} \left[ \int_{-\infty}^{0} e^{4\omega} e^{j\omega t} d\omega + \int_{0}^{\infty} e^{-4\omega} e^{j\omega t} d\omega \right]$$

$$= \frac{1}{2\pi} \frac{e^{(4+jt)\omega}}{4+jt} \Big|_{-\infty}^{0} + \frac{1}{2\pi} \frac{e^{-(4-jt)\omega}}{-(4-jt)} \Big|_{0}^{\infty}$$

$$= \frac{1}{2\pi} \frac{1}{4+jt} + \frac{1}{2\pi} \frac{-1}{-(4-jt)} = \frac{1}{2\pi} \left( \frac{1}{4+jt} + \frac{1}{4-jt} \right)$$

$$= \frac{1}{2\pi} \cdot \frac{4-\cancel{jt}+4+\cancel{jt}}{16+t^2} = \frac{4}{\pi(16+t^2)}$$

b) $$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-8|\omega|} d\omega$$

$$= \frac{1}{2\pi} \left[ \int_{-\infty}^{0} e^{8\omega} d\omega + \int_{0}^{\infty} e^{-8\omega} d\omega \right] =$$

$$= \frac{1}{2\pi} \left[ \frac{e^{8\omega}}{8} \Big|_{-\infty}^{0} + \frac{e^{-8\omega}}{-8} \Big|_{0}^{\infty} \right] = \frac{1}{2\pi} \left[ \frac{1}{8} + \frac{-1}{-8} \right] = \frac{1}{2\pi} \frac{2}{8} = \frac{1}{8\pi}$$

c) $T_s = 1\,ms = 10^{-3}s \longrightarrow f_s = 1000\,Hz$ odtipkan signal

$f_0 = 10\,Hz \longrightarrow T_p = \frac{1}{f_0} = \frac{1}{10} = 0{,}1\,s$ odtipkan spektar

$N_{\omega_0} = N_1 = T_p \cdot f_s = \frac{1}{10} \cdot 1000 = 100$ točaka odtipkanog signala moramo uzeti

B

4. $f(n) = \begin{cases} 4^{-n} & , n > 1 \\ 0 & , \text{inače} \end{cases}$

a) DTFT

$$F(e^{i\Omega}) = \sum_{n=-\infty}^{\infty} f(n) e^{-i\Omega n}$$

$$= \sum_{n=2}^{\infty} 4^{-n} e^{-i\Omega n}$$

$$= 4^{-2} e^{-i\Omega \cdot 2} + 4^{-3} e^{-i\Omega \cdot 3} + 4^{-4} e^{-i\Omega 4} + \dots$$

$$= 4^{-2} e^{-i\Omega 2} \left(1 + 4^{-1} e^{-i\Omega} + 4^{-2} e^{-i\Omega \cdot 2} + 4^{-3} e^{-i\Omega \cdot 3} + \dots\right)$$

$$= \frac{1}{16} e^{-i\Omega \cdot 2} \sum_{n=0}^{\infty} \left(4^{-1} e^{-i\Omega}\right)^n$$

$$= \frac{1}{16} e^{-i\Omega \cdot 2} \cdot \frac{1}{1 - \frac{1}{4} e^{-i\Omega}} = \frac{\frac{1}{16} 4 e^{-i\Omega 2}}{4 - e^{-i\Omega}} = \frac{1}{4} \cdot \frac{e^{-i\Omega \cdot 2}}{4 - e^{-i\Omega}}$$

$$= \frac{1}{4} \frac{\cos 2\Omega - i \sin 2\Omega}{4 - \cos\Omega + i \sin\Omega}$$

$$|F(e^{i\Omega})| = \frac{1}{4} \cdot \frac{1}{\sqrt{(4-\cos\Omega)^2 + \sin^2\Omega}} = \frac{1}{4} \frac{1}{\sqrt{16 - 8\cos\Omega + \cos^2\Omega + \sin^2\Omega}}$$

$$= \frac{1}{4} \frac{1}{\sqrt{17 - 8\cos\Omega}}$$

$$\measuredangle F(e^{i\Omega}) = -2\Omega - \operatorname{arctg} \frac{\sin\Omega}{4 - \cos\Omega}$$

b) $\Omega = \frac{\pi}{2}$

$$|F(e^{i\pi/2})| = \frac{1}{4} \frac{1}{\sqrt{17 - 8 \cdot 0}} = \frac{1}{4\sqrt{17}} = 0.061$$

$$\measuredangle F(e^{i\pi/2}) = -2\frac{\pi}{2} - \operatorname{arctg} \frac{1}{4} = -3.39 \text{ rad} = 2.89 \text{ rad}$$

c)

$$F(e^{i\Omega}) = \sum_{n=-\infty}^{\infty} f(n) e^{-i\Omega n}$$

$$F(e^{i(\Omega + 2\pi k)}) = \sum_{n=-\infty}^{\infty} f(n) e^{-i(\Omega + 2\pi k)n} = \sum_{n=-\infty}^{\infty} f(n) e^{-i\Omega n} e^{-i 2\pi k n}$$

$$= \sum_{n=-\infty}^{\infty} f(n) e^{-i\Omega n} \cdot (\underbrace{\cos 2\pi k n}_{1} - i \underbrace{\sin 2\pi k n}_{\emptyset})$$

$$= \sum_{n=-\infty}^{\infty} f(n) e^{-i\Omega n}$$

$$= F(e^{i\Omega})$$

B

5. $f_s: 1\,kHz$

$f(n)=\{\underline{-4},-2,2,4,-4,-2,2,4\}$

a) DFT u 8 točaka $\quad N=8$

$$F(k)=\sum_{n=0}^{7} f(n)\,e^{-j\frac{2\pi}{8}nk}=$$

$$=-4-2e^{-j\frac{2\pi}{8}k}+2e^{-j\frac{2\pi}{8}k\cdot 2}+4e^{-j\frac{2\pi}{8}k\cdot 3}-4e^{-j\frac{2\pi}{8}k\cdot 4}-2e^{-j\frac{2\pi}{8}k\cdot 5}+2e^{-j\frac{2\pi}{8}k\cdot 6}+4e^{-j\frac{2\pi}{8}k\cdot 7}$$

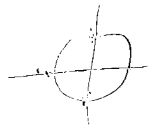
$$=-4\left(1+e^{-j\frac{2\pi}{8}k\cdot 4}\right)-2\left(e^{-j\frac{2\pi}{8}k}+e^{-j\frac{2\pi}{8}k\cdot 5}\right)+2\left(e^{-j\frac{2\pi}{8}k\cdot 2}+e^{-j\frac{2\pi}{8}k\cdot 6}\right)+4\left(e^{-j\frac{2\pi}{8}k\cdot 3}+e^{-j\frac{2\pi}{8}k\cdot 7}\right)$$

$$=-4e^{-j\frac{2\pi}{8}k\cdot 2}\left(e^{j\frac{2\pi}{8}k\cdot 2}+e^{-j\frac{2\pi}{8}k\cdot 2}\right)-2e^{-j\frac{2\pi}{8}k\cdot 3}\left(e^{j\frac{2\pi}{8}k\cdot 2}+e^{-j\frac{2\pi}{8}k\cdot 2}\right)$$

$$+2e^{-j\frac{2\pi}{8}k\cdot 4}\left(e^{j\frac{2\pi}{8}k\cdot 2}+e^{-j\frac{2\pi}{8}k\cdot 2}\right)+4e^{-j\frac{2\pi}{8}k\cdot 5}\left(e^{j\frac{2\pi}{8}k\cdot 2}+e^{-j\frac{2\pi}{8}k\cdot 2}\right)$$

$$=\left(e^{j\frac{2\pi}{8}k\cdot 2}+e^{-j\frac{2\pi}{8}k\cdot 2}\right)\cdot\left(-4e^{-j\frac{\pi}{2}k}-2e^{-j\frac{3\pi}{4}k}+2e^{-j\pi k}+4e^{-j\frac{5\pi}{4}k}\right)$$

$$=4\cos\frac{\pi}{2}k\cdot\left(-2e^{-j\frac{\pi}{2}k}-e^{-j\frac{3\pi}{4}k}+e^{-j\pi k}+2e^{-j\frac{5\pi}{4}k}\right)$$



$F(0)=4\cdot[-2-1+1+2]=0$

$F(1)=0$

$F(2)=4\cdot(-1)\cdot\left(-2e^{-j\pi}-e^{-j\frac{3\pi}{2}}+e^{-j2\pi}+2e^{-j\frac{5\pi}{2}}\right)=-4\cdot\left(+2-(-j\cdot(-1))+1+2\cdot(-j)\right)$

$\quad=-4(2-j+1-2j)=-12+12j$

$F(3)=0$

$F(4)=4\cdot\left(-2e^{-j2\pi}-e^{-j3\pi}+e^{-j4\pi}+2e^{-j5\pi}\right)=4(-2+1+1-2)=-8$

$F(5)=0$

$F(6)=-4\left(-2e^{-j3\pi}-e^{-j\frac{9\pi}{2}}+e^{-j6\pi}+2e^{-j\frac{15\pi}{2}}\right)=-4\left(-2\cdot(-1)-(-j)+1+2(-j\cdot(-1))\right)=$

$\quad=-4(2+j+1+2j)=-12-12j$

$F(7)=0$

$$F(k)=\{\underline{0},0,-12+12j,0,-8,0,-12-12j,0\}$$

b) za $k=2 \longrightarrow |F(k)|=\sqrt{12^2+12^2}=12\sqrt{2}$

$\Omega=\frac{2\pi}{N}\cdot k=\frac{2\pi}{8}\cdot 2=\frac{4\pi}{8}=\frac{\pi}{2}$ rad

c) $\Omega=\omega T$

$\omega=\frac{\Omega}{T}=\frac{\frac{\pi}{2}}{\frac{1}{f_s}}=\frac{\pi}{2}\cdot f_s=\frac{\pi}{2}\cdot 1000=500\pi$ rad/s