Diferencijalne jednadžbe (I) $\underset{a_0}{y''(t)} + \underset{a_1}{5y'(t)} + \underset{a_2}{6y(t)} = \underset{b_1}{u'(t)} + \underset{b_2}{u(t)}$, $u(t)=\mu(t)$, $y(0^-)=y'(0^-)=1$

(koef. b_0 bio bi uz $u''(t)$) poč. uvjeti

Homogeno rješenje $y_h(t) = Ce^{st}$ (uvrštavamo u poč. jednadžbu uz pobudu = 0)

$$\left.\begin{aligned} y_h(t) &= Ce^{st} \\ y_h'(t) &= sCe^{st} \\ y_h''(t) &= s^2Ce^{st} \end{aligned}\right\} \quad s^2Ce^{st} + 5sCe^{st} + 6Ce^{st} = 0$$

$$Ce^{st}(s^2+5s+6) = 0$$

$$s_{1,2} = \frac{-5 \pm \sqrt{25-24}}{2} = -3, -2$$

$$y_h(t) = C_1e^{-2t} + C_2e^{-3t}$$

Partikularno rješenje $y_p(t)$ ovisi o pobudi sustava (39. slajd 11. prezentacija)

Kad se pobuda sastoji od više dijelova, a sustav je LTI možemo tražiti odziv na svaku pobudu zasebno (15. tjedan auditorne vježbe Laplace transf. zad 1)

1) $u(t) = \mu(t) = 1$, $t \geq 0$ $\Rightarrow$ $y''(t) + 5y'(t) + 6y(t) = \mu(t)$

$$\left.\begin{aligned} y_{p1}(t) &= K, \quad t \geq 0 \\ y_{p1}'(t) &= 0 \\ y_{p1}''(t) &= 0 \end{aligned}\right\} \quad 0 + 5\cdot 0 + 6K = 1 \Rightarrow K = \frac{1}{6}$$

$$y_{p1}(t) = \frac{1}{6}\mu(t)$$

2) $u'(t) = \delta(t)$ $\Rightarrow$ $y''(t) + 5y'(t) + 6y(t) = \delta(t)$ $\Rightarrow$ Impulsni odziv

za $t=0$ $\quad h_1(t) = \delta(t)$ (jedino što se može pojaviti na izlazu je impuls)

za $t \geq 0^+$ $\quad \delta(t \geq 0^+) = 0$ $\Rightarrow$ $h''(t) + 5h'(t) + 6h(t) = 0$ (Homogena jednadžba uz poč. uvjete $h(0^-), h'(0^-) = 0$)

$h_2(t) = C_1e^{-2t} + C_2e^{-3t}$, $t \geq 0$

poč. uvjeti OBAVEZNO UVIJEK = 0

Impulsni odziv: $h(t) = h_1(t) + h_2(t) = \delta(t) + C_1e^{-2t} + C_2e^{-3t}$

(Komponentu $h_1(t)$ u daljnjem računu zanemarujemo ukoliko se eksplicitno ne traži izračunati impulsni odziv.)

$$a_0\left(h(0^+)-h(0^-)\right) = b_0\left(u(0^+)-u(0^-)\right)$$

PRERAČUNAVANJE UVJETA ZA IMP. ODZIV

$$1\cdot\left(h(0^+)-0\right) = 0\cdot\left(u(0^+)-u(0^-)\right) \Rightarrow \underline{h(0^+)=0}$$

$$a_0\left(h'(0^+)-h'(0^-)\right) + a_1\left(h(0^+)-h(0^-)\right) = b_0\left(u'(0^+)-u'(0^-)\right) + b_1\left(u(0^+)-u(0^-)\right)$$

$$1\cdot\left(h'(0^+)-0\right) + \underbrace{5\cdot 0}_{=0} = \underbrace{0\cdot\left(u'(0^+)-u'(0^-)\right)}_{=0} + 1\cdot\underbrace{\left(u(0^+)-u(0^-)\right)}_{=1} \Rightarrow \underline{h'(0^+)=1}$$

$$h(0^+) = C_1 e^{-2\cdot 0} + C_2 e^{-3\cdot 0} = C_1 + C_2 = 0 \Rightarrow C_1 = -C_2 = 1$$

$$h'(0^+) = -2C_1 e^{-2\cdot 0} - 3C_2 e^{-3\cdot 0} = -2C_1 - 3C_2 = 1 \Rightarrow C_2 = -1$$

IZRAČUN KONSTANTI

$$\underline{h_2(t) = e^{-2t} - e^{-3t}, \quad t \geq 0^+}$$

Partikularno rješenje $\underline{y_{p2}(t) = h_2(t) = e^{-2t} - e^{-3t}}$

Totalno rješenje dif. jedn. I

$$\underline{y(t) = y_h(t) + y_{p1}(t) + y_{p2}(t) = C_1 e^{-2t} + C_2 e^{-3t} + \frac{1}{6}\mu(t) + \left(e^{-2t} - e^{-3t}\right)\mu(t)}$$

$$\underbrace{a_0}_{=1}\left(y(0^+)-y(0^-)\right) = \underbrace{b_0}_{=0}\left(u(0^+)-u(0^-)\right) \Rightarrow \underline{y(0^+) = y(0^-) = 1}$$

PRERAČUN UVJETA TOTALNI ODZIV

$$\underbrace{a_0}_{=1}\left(y'(0^+)-\underbrace{y'(0^-)}_{=1}\right) + \underbrace{a_1}_{=5}\underbrace{\left(y(0^+)-y(0^-)\right)}_{=0} = \underbrace{b_0}_{0}\left(u'(0^+)-u'(0^-)\right) + \underbrace{b_1}_{1}\left(u(0^+)-u(0^-)\right)$$

$\mu(0^+)=1$

$$\Rightarrow y'(0^+) = u(0^+) + y'(0^-)$$

$$\underline{y'(0^+) = 2}$$

$$y(0^+) = C_1 e^{-2\cdot 0} + C_2 e^{-3\cdot 0} + \frac{1}{6}\mu(0) + 1\left(e^{-2\cdot 0} - e^{-3\cdot 0}\right)\mu(0) = C_1 + C_2 + \frac{1}{6} = 1 \quad /\cdot 2$$

$$y'(0^+) = -2C_1 - 3C_2 + \frac{1}{6}\delta(0) + (-2+3)\mu(0) + 0\cdot\delta(0) = -2C_1 - 3C_2 + \frac{1}{6} + 1 = 2$$

IZRAČUN KONSTANTI

$$2C_2 - 3C_2 + \frac{1}{3} + \frac{1}{6} = 3 \Rightarrow C_2 = -\frac{5}{2}$$

$$C_1 = 1 - \frac{1}{6} - C_2 = \frac{10}{3}$$

Tot. rješ. I $$\underline{y(t) = \underbrace{\frac{10}{3}e^{-2t} - \frac{5}{2}e^{-3t}}_{\text{PRIRODNI}} + \underbrace{\frac{1}{6}\mu(t) + \left(e^{-2t} - e^{-3t}\right)\mu(t)}_{\text{PRISILNI}}}$$

Diferencijalne jednadžbe (II) $\underset{a_0}{y''(t)} + \underset{a_1}{5y'(t)} + \underset{a_2}{6y(t)} = \underset{b_1}{u'(t)} + \underset{b_2}{u(t)}$, $u(t) = \mu(t)$, $y(0^-) = y'(0^-) = 1$

(b_0 je uz $u''(t)$, tj. $b_0 = 0$)

Totalno rješenje (II) $y(t) = y_{nepobuđeni} + y_{mirni}$

Nepobuđeni odziv $y_0(t) = y_h(t) = C_1 e^{-2t} + C_2 e^{-3t}$ (Konstante računamo iz poč. uvjeta, bez preračunavanja)

$$\left.\begin{aligned} y_0(0^-) &= C_1 e^{-2\cdot 0} + C_2 e^{-3\cdot 0} = C_1 + C_2 = 1 \quad /\cdot 2 \\ y_0'(0^-) &= -2C_1 e^{-2\cdot 0} - 3C_2 e^{-3\cdot 0} = -2C_1 - 3C_2 = 1 \end{aligned}\right\} \quad \begin{aligned} 2C_2 - 3C_2 = 3 &\Rightarrow C_2 = -3 \\ C_1 = 1 - C_2 &\Rightarrow C_1 = 4 \end{aligned}$$

$$y_0(t) = 4e^{-2t} - 3e^{-3t}$$

Mirni odziv $y_m(t) = y_h(t) + y_p(t)$ uz početne uvjete $y(0^-) = y'(0^-) = 0$ (potrebno preračunati!)

$$y_m(t) = C_1 e^{-2t} + C_2 e^{-3t} + \frac{1}{6}\mu(t) + \left(e^{-2t} - e^{-3t}\right)\mu(t) \quad \text{(iz Dif. jedn. (I))}$$

PREPAČUN UVJETA

$$\underbrace{a_0}_{=1}\left(y(0^+) - \underbrace{y(0^-)}_{=0}\right) = \underbrace{b_0}_{=0}\left(u(0^+) - u(0^-)\right) \Rightarrow y(0^+) = y(0^-) = 0$$

$$\underbrace{a_0}_{=1}\left(y'(0^+) - \underbrace{y'(0^-)}_{=0}\right) + \underbrace{a_1}_{=5}\underbrace{\left(y(0^+) - y(0^-)\right)}_{=0} = \underbrace{b_0}_{=0}\left(u'(0^+) - u'(0^-)\right) + \underbrace{b_1}_{=1}\left(u(0^+) - u(0^-)\right)$$

$\mu(0^+) = 1$

$$\Rightarrow y'(0^+) = 1$$

IZRAČUN KONSTANTI

$$y(0^+) = C_1 e^{-2\cdot 0} + C_2 e^{-3\cdot 0} + \frac{1}{6}\underbrace{\mu(0)}_{=1} + \underbrace{\left(e^{-2\cdot 0} - e^{-3\cdot 0}\right)}_{0}\mu(0) = C_1 + C_2 + \frac{1}{6} = 0 \quad /\cdot 2$$

$$y'(0^+) = -2C_1 e^{-2\cdot 0} - 3C_2 e^{-3\cdot 0} + \frac{1}{6}\underbrace{\delta(0)}_{=1} + (-2+3)\underbrace{\mu(0)}_{=1} + \underbrace{\left(e^{-2\cdot 0} - e^{-3\cdot 0}\right)}_{0}\delta(0) = -2C_1 - 3C_2 + \frac{1}{6} + 1 = 1$$

$$2C_2 - 3C_2 + \frac{1}{3} + \frac{1}{6} = 0$$

$$C_2 = +\frac{1}{2}$$

$$C_1 = -C_2 - \frac{1}{6} = -\frac{2}{3}$$

$$y_m(t) = -\frac{2}{3}e^{-2t} + \frac{1}{2}e^{-3t} + \frac{1}{6}\mu(t) + \left(e^{-2t} - e^{-3t}\right)\mu(t)$$

Totalno rješenje (II)

$$y(t) = \underbrace{4e^{-2t} - 3e^{-3t}}_{\text{NEPOBUĐENI}} \underbrace{- \frac{2}{3}e^{-2t} + \frac{1}{2}e^{-3t} + \frac{1}{6}\mu(t) + \left(e^{-2t} - e^{-3t}\right)\mu(t)}_{\text{MIRNI}}$$

$$= \frac{10}{3}e^{-2t} - \frac{5}{2}e^{-3t} + \frac{1}{6}\mu(t) + \left(e^{-2t} - e^{-3t}\right)\mu(t)$$

Jednadžbe diferencija (I) $y(n) - 5y(n-1) + 6y(n-2) = u(n) + u(n-1) + u(n-2)$, $u(n) = \mu(n)$

$y(-1) = 1$

1) Totalno rješenje (I) $y(n) = y_{homogeno} + y_{partikularno}$

$y(-2) = 0$

Homogeno rješenje $y_h(n) = Cq^n$ (uvrštavamo u poč. jedn. uz pobudu jednaku 0)

$$Cq^n - 5Cq^{n-1} + 6Cq^{n-2} = 0$$

$$Cq^{n-2}\left(q^2 - 5q + 6\right) = 0$$

$q = 0$ zanemarjemo $\quad q_{1,2} = \frac{5 \pm \sqrt{25-24}}{2} = 3, 2 \quad \Rightarrow \quad y_h(n) = C_1 3^n + C_2 2^n$

Partikularno rješenje $y_p(n)$ = ovisi isključivo o pobudi sustava $u(n)$

pobuda je konstantna

pobuda : $u(n) + u(n-1) + u(n-2) = \mu(n) + \mu(n-1) + \mu(n-2) = 3$, $n \geq 2 \Rightarrow y_p(n) = K$, $n \geq 2$

$$K - 5K + 6K = 1 + 1 + 1 \Rightarrow 2K = 3 \Rightarrow K = \frac{3}{2} \Rightarrow y_p(n) = \frac{3}{2} \;, n \geq 2$$

2) Tot. rj. (I) $y(n) = \left(C_1 3^n + C_2 2^n + \frac{3}{2}\right)\mu(n-2)$

Za izračunavanje konstanti obavezno preračunati uvjete (vrijedi samo za totalni odziv (I))

$n = 0$

$$y(0) - 5\underset{1}{y(-1)} + 6\underset{0}{y(-2)} = \underset{1}{u(0)} + \underset{0}{u(-1)} + \underset{0}{u(-2)} \Rightarrow y(0) = 6$$

$n = 1$

$$y(1) - 5\underset{6}{y(0)} + 6\underset{1}{y(-1)} = \underset{1}{u(1)} + \underset{1}{u(0)} + \underset{0}{u(-1)} \Rightarrow y(1) = 26$$

PRERAČUNAVANJE UVJETA

$$y(0) = C_1 3^0 + C_2 2^0 + \frac{3}{2} = C_1 + C_2 + \frac{3}{2} = 6 \quad / \cdot (-2)$$

$$y(1) = C_1 3^1 + C_2 2^1 + \frac{3}{2} = 3C_1 + 2C_2 + \frac{3}{2} = 26$$

IZRAČUN KONSTANTI

$$-2C_1 + 3C_1 - 3 + \frac{3}{2} = 14 \Rightarrow C_1 = \frac{31}{2}$$

$$C_2 = 6 - \frac{3}{2} - C_1 = -11$$

3) Totalno rješ. (I) $y(n) = \left(\underbrace{\frac{31}{2} 3^n - 11 2^n}_{\text{PRIRODNI ODZIV}} + \underbrace{\frac{3}{2}}_{\text{PRISILNI ODZIV}}\right)\mu(n-2)$

Jednadžbe diferencija (II) $y(n) - 5y(n-1) + 6y(n-2) = u(n) + u(n-1) + u(n-2)$

$u(n) = \mu(n)$

Totalno rješ. (II) $y(n) = y_{nepobuđeni} + y_{mirni}$ (Još se zove i odziv POBUĐENOG SUSTAVA) $y(-1) = 1$, $y(-2) = 0$

1) Nepobuđeni odziv $y_0(n) = y_h(n) = C_1 3^n + C_2 2^n$ (s tim da konstante računamo iz poč. uvjeta

$$y_0(-1) = C_1 3^{-1} + C_2 2^{-1} = \frac{C_1}{3} + \frac{C_2}{2} = 1 \Rightarrow -\frac{9}{12}C_2 + \frac{1}{2}C_2 = 1 \Rightarrow C_2 = -4$$

$$y_0(-2) = C_1 3^{-2} + C_2 2^{-2} = \frac{C_1}{9} + \frac{C_2}{4} = 0 \Rightarrow C_1 = -\frac{9}{4}C_2 = 9$$

$$y_0(n) = 9 \cdot 3^n - 4 \cdot 2^n$$

2) Mirni odziv $y_m(n) = y_h(n) + y_p(n) = C_1 \cdot 3^n + C_2 \cdot 2^n + \frac{3}{2}$ s tim da za računanje konstanti obavezno preračunavamo uvjete izjednačavajući ih s 0.

$y(-1) = y(-2) = 0$

$n = 0$

$$\underset{\uparrow}{y(0)} - 5\underset{\uparrow 0}{y(-1)} + 6\underset{\uparrow 0}{y(-2)} = \underset{\uparrow 1}{u(0)} + \underset{\uparrow 0}{u(-1)} + \underset{\uparrow 0}{u(-2)} \Rightarrow y(0) = 1$$

$n = 1$

$$y(1) - 5\underset{\uparrow 1}{y(0)} + 6\underset{\uparrow 0}{y(-1)} = \underset{\uparrow 1}{u(1)} + \underset{\uparrow 1}{u(0)} + \underset{\uparrow 0}{u(-1)} \Rightarrow y(1) = 7$$

$$y(0) = C_1 3^0 + C_2 2^0 + \frac{3}{2} = C_1 + C_2 + \frac{3}{2} = 1 \quad / \cdot (-2)$$

$$y(1) = C_1 3^1 + C_2 2^1 + \frac{3}{2} = 3C_1 + 2C_2 + \frac{3}{2} = 7$$

$$C_1 - 3 + \frac{3}{2} = 5 \Rightarrow C_1 = \frac{13}{2}$$

$$C_2 = 1 - \frac{3}{2} - C_1 \Rightarrow C_2 = -7$$

$$y_m(n) = \frac{13}{2}3^n - 7 2^n + \frac{3}{2}$$

3) Totalno rješenje (II) $y(n) = \underbrace{9 \cdot 3^n - 4 \cdot 2^n}_{NEPOBUĐENI} + \underbrace{\frac{13}{2}3^n - 7 2^n + \frac{3}{2}}_{MIRNI} = \frac{31}{2}3^n - 11 2^n + \frac{3}{2}$

Jednadžbe diferencija (III) $\quad y(n) - 5y(n-1) + 6y(n-2) = u(n) + u(n-1) + u(n-2)$

Impulsni odziv $h(n)$ određujemo za miran sustav $\left(y(-1) = y(-2) = 0\right)$ i za pobudu $u(n) = \delta(n)$

$$h(n) - 5h(n-1) + 6h(n-2) = \delta(n) + \delta(n-1) + \delta(n-2)$$

1) Vidimo da se za $n > 2$ jednadžba svodi na homogenu (jer su dirac-ovi $= 0$)

pa je $\quad h(n) = C_1 3^n + C_2 2^n \quad , n > 2$

2) Za računanje konstanti obavezno preračunavamo uvjete !

$n = 0$

$$h(0) - 5\overset{0}{\overset{\uparrow}{h(-1)}} + 6\overset{0}{\overset{\uparrow}{h(-2)}} = \overset{1}{\overset{\uparrow}{\delta(0)}} + \overset{0}{\overset{\uparrow}{\delta(-1)}} + \overset{0}{\overset{\uparrow}{\delta(-2)}} \Rightarrow h(0) = 1$$

$n = 1$

$$h(1) - 5\overset{1}{\overset{\uparrow}{h(0)}} + 6\overset{0}{\overset{\uparrow}{h(-1)}} = \overset{0}{\overset{\uparrow}{\delta(1)}} + \overset{1}{\overset{\uparrow}{\delta(0)}} + \overset{0}{\overset{\uparrow}{\delta(-1)}} \Rightarrow h(1) = 6$$

$n = 2$

$$h(2) - 5\overset{6}{\overset{\uparrow}{h(1)}} + 6\overset{1}{\overset{\uparrow}{h(0)}} = \overset{0}{\overset{\uparrow}{\delta(2)}} + \overset{0}{\overset{\uparrow}{\delta(1)}} + \overset{1}{\overset{\uparrow}{\delta(0)}} \Rightarrow h(2) = 25$$

PRERAČUNAVANJE UVJETA

$$h(0) = C_1 3^0 + C_2 2^0 = C_1 + C_2 = 1 \Rightarrow C_1 = 1 - C_2 \Rightarrow C_1 = 4$$

$$h(1) = C_1 3^1 + C_2 2^1 = 3C_1 + 2C_2 = 6 \Rightarrow 3 - 3C_2 + 2C_2 = 6 \Rightarrow C_2 = -3$$

RAČUNANJE KONSTANTI

3) $\underbrace{h(n) = \left(4 \cdot 3^n - 3 \cdot 2^n\right) \mu(n)}_{\text{IMPULSNI}}$ $\quad$ (za sve $n \geq 0$ zato jer smo uključili poč. uvjete $h(0), h(1), h(2)$)

Prijenosna funkcija (I) $y(n)+5y(n-1)+6y(n-2)=u(n)$

Računa se isključivo uz pobudu $u(n)=Uz^n$, pa je partikularno rješenje $y_p(n)=Yz^n$

$$Yz^n+5Yz^{n-1}+6Yz^{n-2}=Uz^n$$

$$Y\cancel{z^n}\left(1+5z^{-1}+6z^{-2}\right)=U\cancel{z^n}$$

$$Y=\frac{1}{1+5z^{-1}+6z^{-2}}U=\underbrace{\frac{z^2}{z^2+5z+6}}_{H(z)\,\Rightarrow\,\text{prijenosna funkcija}}U$$

$$y_p(n)=H(z)Uz^n$$

Frekvencijska karakteristika (II) $y(n)-y(n-1)=u(n)$

Računa se iz prijenosne funkcije $H(z)=\frac{1}{1-z^{-1}}=\frac{z}{z-1}$ uvrštavajući $z=e^{j\Omega}\Rightarrow H(e^{j\Omega})$ je

1) $H(e^{j\Omega})=\dfrac{e^{j\Omega}}{e^{j\Omega}-1}$ $\qquad y_p(n)=H(e^{j\Omega})Ue^{j\Omega n}$

2) $H(e^{-j\Omega})=\dfrac{e^{-j\Omega}}{e^{-j\Omega}-1}=\dfrac{1}{1-e^{j\Omega}}$ $\qquad y_p(n)=H(e^{-j\Omega})Ue^{-j\Omega n}$

3) pobuda: $u(n)=U\cos(\Omega n)=\dfrac{Ue^{j\Omega n}+Ue^{-j\Omega n}}{2}$

$$y_p(n)=\frac{H(e^{j\Omega})Ue^{j\Omega n}+H(e^{-j\Omega})Ue^{-j\Omega n}}{2}=2\,\mathrm{Re}\left(\frac{H(e^{j\Omega})Ue^{j\Omega n}}{2}\right)=\mathrm{Re}\left(|H(e^{j\Omega})|e^{j\angle H(e^{j\Omega})}Ue^{j\Omega n}\right)$$

$$y_p(n)=|H(e^{j\Omega})|\,U\cos\left(\Omega n+\angle H(e^{j\Omega})\right)$$ Formula (valja pamtiti)

Traženje partikularnog rješenja svodi se na traženje amplitudne $-|H(e^{j\Omega})|$ i fazne $\angle H(e^{j\Omega})$ frekvencijske karakteristike za slučaj (I) i (II) !!!

Prijenosna funkcija (I) $y''(t) - 5y'(t) + 6y(t) = u(t)$

Računa se isključivo za pobudu $\underline{u(t) = Ue^{st}}$ (kao pretpostavljeno homogeno rješenje), pa je partikularno rješenje $\underline{y_p(t) = Ye^{st}}$

$$\left.\begin{aligned} y_p(t) &= Ye^{st} \\ y_p'(t) &= sYe^{st} \\ y_p''(t) &= s^2Ye^{st} \end{aligned}\right\} \quad \begin{aligned} &s^2Ye^{st} - 5sYe^{st} + 6Ye^{st} = Ue^{st} \\ &Y\cancel{e^{st}}(s^2 - 5s + 6) = U\cancel{e^{st}} \end{aligned}$$

$$Y = \underbrace{\frac{1}{s^2 - 5s + 6}}_{} U = \frac{1}{(s-3)(s-2)} U$$

$$\underline{y_p(t) = H(s)Ue^{st}.}$$

$H(s) \Rightarrow$ prijenosna funkcija (pomoću nje lako određujemo partikularno rješenje)

Frekvencijska karakteristika (I) $y'(t) - 2y(t) = u(t)$

Računa se iz prijenosne funkcije $H(s) = \frac{1}{s-2}$ za $\underline{s = j\omega} \Rightarrow H(j\omega)$

1) pobuda : $u(t) = Ue^{j\omega t}$ $(s = j\omega)$

$$H(j\omega) = \frac{1}{j\omega - 2} \cdot \frac{j\omega + 2}{j\omega + 2} = \frac{2 + j\omega}{-(\omega^2 + 4)} = \overbrace{\frac{-2}{\omega^2+4}}^{Re} + j\overbrace{\frac{-\omega}{\omega^2+4}}^{Im}$$

$$\underline{y_p(t) = H(j\omega)Ue^{j\omega t}}$$

Kompleksno konjugirani !

2) pobuda : $u(t) = Ue^{-j\omega t}$ $(s = -j\omega)$

$$H(-j\omega) = \frac{1}{-j\omega - 2} \cdot \frac{j\omega - 2}{j\omega - 2} = -\frac{(j\omega - 2)}{-\omega^2 - 4} = \frac{j\omega - 2}{\omega^2 + 4} = \overbrace{-\frac{2}{\omega^2+4}}^{Re} + j\overbrace{\frac{\omega}{\omega^2+4}}^{Im}$$

$$\underline{y_p(t) = H(-j\omega)Ue^{-j\omega t}}$$

3) pobuda : $u(t) = U\cos(\omega t) = \frac{Ue^{j\omega t} + Ue^{-j\omega t}}{2}$

$$y_p(t) = \frac{H(j\omega)Ue^{j\omega t} + H(-j\omega)Ue^{-j\omega t}}{2} = 2Re\left(\frac{H(j\omega)Ue^{j\omega t}}{2}\right) = Re\left(|H(j\omega)|e^{j\angle H(j\omega)}Ue^{j\omega t}\right)$$

(valja zapamtiti) Formula : $\underline{y_p(t) = U|H(j\omega)|\cos(\omega t + \angle H(j\omega))}$

Laplaceova transformacija

Dvostrana : $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

$x(t) = e^{-at}\mu(t)$

→ uvjet $Re\{s+a\} > 0$ potreban je da bi dobili $e^{-\infty} = 0$

$$X(s) = \int_{-\infty}^{\infty} e^{-at}\mu(t) e^{-st} dt = \int_{0}^{\infty} e^{-t(a+s)} dt = -\frac{1}{s+a} e^{-t(s+a)}\Big|_0^{\infty} = -\frac{1}{s+a}\cdot(0-1) = \frac{1}{s+a},$$

$Re\{s+a\} > 0 \Rightarrow Re\{s\} > -a$

Jednostrana : $X(s) = \int_{0}^{\infty} x(t) e^{-st} dt$

za svaki signal $x(t)$ koji ne konvergira brže od eksponencijalnog signala ce^{at}, tj. $|x(t)| \leq ce^{at}$

$x(t) = \cos(\omega t)\mu(t)$

$$X(s) = \int_{0}^{\infty} \cos(\omega t) e^{-st}\mu(t) dt = \int_{0}^{\infty} \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) e^{-st} dt = \frac{1}{2}\int_{0}^{\infty} e^{-t(s-j\omega)} + e^{-t(s+j\omega)} dt$$

$$= \frac{1}{2}\left[-\frac{1}{s-j\omega} e^{-t(s-j\omega)}\Big|_0^{\infty} - \frac{1}{s+j\omega} e^{-t(s+j\omega)}\Big|_0^{\infty}\right] = \frac{1}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}s}{s^2+\omega^2}\right] = \frac{s}{s^2+\omega^2}, \quad \left.\begin{array}{l} Re\{s-j\omega\} > 0 \\ Re\{s+j\omega\} > 0 \end{array}\right\} Re\{s\} > 0$$

Inverz

$$X(s) = \frac{1}{s^2+5s+6} = \frac{1}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2} = \underbrace{\frac{-1}{s+3} + \frac{1}{s+2}}_{\text{TABLICA L. TRANSF. (ŠALABAHTER)}}$$

$A(s+2) + B(s+3) = 1$

$A + B = 0 \Rightarrow A = -B = -1$

$2A + 3B = 1 \Rightarrow B = 1$

$$\underline{x(s) = \left(-e^{-3t} + e^{-2t}\right)\mu(t)}$$

Primjena Laplaceove transf. u analizi lin. sustava (princip rješavanja)

$$y''(t) + 5y'(t) + 6y(t) = u(t) + 3u'(t) \qquad u(t)=\mu(t), \quad y(0^-)=1, \quad y'(0^-)=0$$

1) $$s^2Y(s) - sy(0^-) - y'(0^-) + 5\left[sY(s) - y(0^-)\right] + 6Y(s) = U(s) + 3\left[sU(s) - u(0^-)\right]$$

$$Y(s)\left[s^2+5s+6\right] - \underbrace{sy(0^-)}_{-s} - \underbrace{y'(0^-)}_{0} - \underbrace{5y(0^-)}_{-5} = U(s)\left[1+3s\right] - \underbrace{3u(0^-)}_{0}$$

$$Y(s)\left[s^2+5s+6\right] = U(s)\left[1+3s\right] + 5 + s$$

$$Y(s) = \frac{3s+1}{s^2+5s+6}U(s) + \frac{s+5}{s^2+5s+6}$$

2) $u(t)=\mu(t) \Rightarrow U(s)=\frac{1}{s}$

$$Y(s) = \frac{3s+1}{s(s^2+5s+6)} + \frac{s+5}{s^2+5s+6} = \frac{3s+1+s^2+5s}{s(s^2+5s+6)} = \frac{s^2+8s+1}{s\underbrace{(s^2+5s+6)}_{(s+2)(s+3)}} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$A(s^2+5s+6) + B(s^2+3s) + C(s^2+2s) = s^2+8s+1$$

$$\left.\begin{aligned} A+B+C &= 1 \quad / \cdot(-2) \\ 5A+3B+2C &= 8 \quad \swarrow + \end{aligned}\right\}$$

$$6A = 1 \Rightarrow A = \frac{1}{6}$$

$$3A + B = 6 \Rightarrow B = 6 - \frac{1}{2} = \frac{11}{2}$$

$$\Rightarrow C = 1 - A - B = -\frac{14}{3}$$

$$Y(s) = \frac{1}{6}\frac{1}{s} + \frac{11}{2}\frac{1}{s+2} - \frac{14}{3}\frac{1}{s+3}$$ (TABLICA TRANSFORMACIJA ŠALABAHTER)

3) $$y(t) = \underbrace{\frac{1}{6}\mu(t)}_{\text{PRISILNI}} + \underbrace{\frac{11}{2}e^{-2t}\mu(t) - \frac{14}{3}e^{-3t}\mu(t)}_{\text{PRIRODNI}} = \left(\frac{1}{6} + \frac{11}{2}e^{-2t} - \frac{14}{3}e^{-3t}\right)\mu(t) \Rightarrow \text{TOTALNI ODZIV}$$

Primjena z-transformacije u analizi lin. sustava (princip)

$$y(n)+5y(n-1)+6y(n-2)=u(n)+3u(n-1) \quad , \underbrace{u(n)=\mu(n)}_{\text{pobuda}} \quad , \underbrace{y(-1)=y(-2)=1}_{\text{poč. uvjeti}}$$

Traženje totalnog odziva $y(n)$

1) $$Y(z)+5\left[z^{-1}Y(z)-y(-1)\right]+6\left[z^{-2}Y(z)-z^{-1}y(-1)-y(-2)\right]=U(z)+3\left[z^{-1}U(z)-u(-1)\right]$$

$$Y(z)\left[1+5z^{-1}+6z^{-2}\right]\underbrace{-5y(-1)}_{-5}\underbrace{-6z^{-1}y(-1)}_{-6z^{-1}}\underbrace{-6y(-2)}_{-6}=U(z)\left[1+3z^{-1}\right]-3\underbrace{u(-1)}_{=0}$$

$$Y(z)\left[1+5z^{-1}+6z^{-2}\right]=U(z)\left[1+3z^{-1}\right]+6z+11$$

$$Y(z)=\frac{1+3z^{-1}}{1+5z^{-1}+6z^{-2}}U(z)+\frac{6z^{-1}+11}{1+5z^{-1}+6z^{-2}}=\frac{z^2+3z}{z^2+5z+6}U(z)+\frac{11z^2+6z}{z^2+5z+6}$$

2) $$u(n)=\mu(n) \Rightarrow U(z)=\frac{z}{z-1}$$

$$Y(z)=\frac{z^2+3z}{z^2+5z+6}\,\frac{z}{z-1}+\frac{11z^2+6z}{z^2+5z+6}=\frac{z^3+3z^2+11z^3+6z^2-11z^2-6z}{(z-1)(z+2)(z+3)}=\frac{12z^3-2z^2-6z}{(z-1)(z+2)(z+3)}$$

$$\frac{Y(z)}{z}=\frac{12z^2-2z-6}{(z-1)(z+2)(z+3)}=\frac{A}{z-1}+\frac{B}{z+2}+\frac{C}{z+3}$$

$$A(z^2+5z+6)+B(z^2+2z-3)+C(z^2+z-2)=12z^2-2z-6$$

$$\begin{aligned} A+B+C&=12 \\ 5A+2B+C&=-2 \\ 6A-3B-2C&=-6 \end{aligned} \qquad \begin{aligned} 12A&=4 \Rightarrow A=\frac{1}{3} \\ 4A+B&=-14 \Rightarrow B=-14-\frac{4}{3}=-\frac{46}{3} \\ C&=12-A-B=27 \end{aligned}$$

3) $$Y(z)=z\left[\frac{1}{3}\frac{1}{z-1}-\frac{46}{3}\frac{1}{z+2}+27\frac{1}{z+3}\right]=\frac{1}{3}\frac{z}{z-1}-\frac{46}{3}\frac{z}{z+2}+27\frac{z}{z+3}$$ (TABLICA TRANSF. ŠALABAHTER)

4) $$y(n)=\underbrace{\frac{1}{3}\mu(n)}_{\text{PRISILNI}}\underbrace{-\frac{46}{3}(-2)^n+27(-3)^n}_{\text{PRIRODNI}} \Rightarrow \text{Totalni odziv}$$

z - transformacija

Dvostrana : $X(z) = \sum_{-\infty}^{\infty} x(n) z^{-n}$

$x(n) = \alpha^n \mu(n)$

→ suma geometrijskog reda $\frac{1}{1-q}$, uz uvjet $|q| < 1$

$$X(z) = \sum_{-\infty}^{\infty} \alpha^n \mu(n) z^{-n} = \sum_{0}^{\infty} \alpha^n z^{-n} = \sum_{0}^{\infty} \left(\frac{\alpha}{z}\right)^n = \frac{1}{1-\frac{\alpha}{z}} = \frac{z}{z-\alpha}$$

uz uvjet : $\left|\frac{\alpha}{z}\right| < 1 \Rightarrow |\alpha| < |z|$ → Područje konvergencije (PK)

Jednostrana : $X(z) = \sum_{0}^{\infty} x(n) z^{-n}$

za svaki $x(n)$ koji ne raste brže od eksponencijalnog signala r^n : $|x(n)| \le r^n$

$x(n) = n\mu(n)$

$$X(z) = \sum_{0}^{\infty} n\mu(n) z^{-n} = \sum_{0}^{\infty} n z^{-n} \cdot \frac{z}{z} = -z \sum_{0}^{\infty} -n z^{-n-1} = -z\left(\sum_{0}^{\infty} z^{-n}\right)'$$

NAPOMENA: $\left|\frac{1}{z}\right| < 1$, $|z| > 1$

$$= -z\left(\frac{1}{1-\frac{1}{z}}\right)' = -z\left(\frac{z}{z-1}\right)' = -z\,\frac{1\cdot(z-1) - z\cdot 1}{(z-1)^2} = -z \cdot \frac{-1}{(z-1)^2} = \frac{z}{(z-1)^2}$$

Inverzna

$$X(z) = \frac{z^2}{z^2-5z+6} = \frac{z^2}{(z-3)(z-2)}$$

1) $\frac{X(z)}{z} = \frac{z}{(z-3)(z-2)} = \frac{A}{z-3} + \frac{B}{z-2} \quad / \cdot (z-3)(z-2)$

$A(z-2) + B(z-3) = z$

$$\left.\begin{aligned} A+B &= 1 \quad / \cdot 2 \\ -2A-3B &= 0 \end{aligned}\right\} \quad -B = 2 \Rightarrow B = -2$$

$A = 1 - B = 3$

2) $X(z) = z \cdot \left[\frac{3}{z-3} - \frac{2}{z-2}\right] = 3\underbrace{\frac{z}{z-3}} - 2\underbrace{\frac{z}{z-2}} = 3\cdot 3^n - 2\cdot 2^n = 3^{n+1} - 2^{n+1} = x(n)$

TABLICA Z-TRANSFORMACIJE (ŠALABAHTER)