Signali i sustavi – zadaci za aktivnost – tjedan 17.

Akademska školska godina 2006./2007.

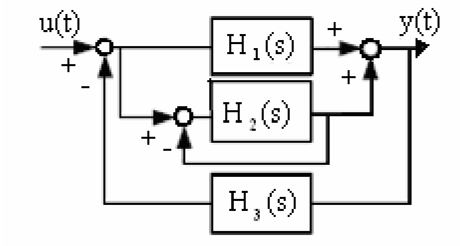
1. Matrice *A, B, C* i *D* diskretnog sustava su

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -\alpha \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix};\ C = \begin{bmatrix} 1 & -1 \end{bmatrix};\ D = \begin{bmatrix} 0 \end{bmatrix}.$$

Koliki je parametar α ako je odziv sustava na jediničnu stepenicu za $n \geq 0$

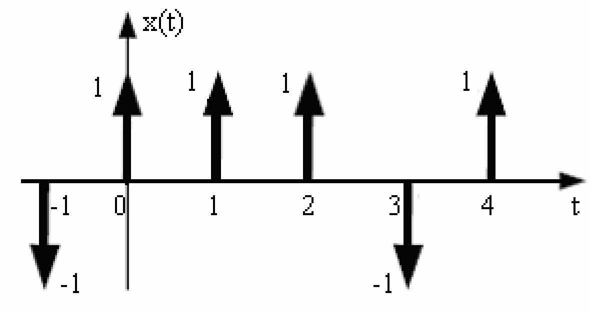
$$y(n) = \frac{3}{10}1^n - \frac{1}{2}(-1)^n + \frac{1}{5}(-4)^n\ ?$$

2. Kontinuirani sustav prikazan je pomoću blokovskog dijagrama (Slika 1.). Odredite ekvivalentnu prijenosnu funkciju cijelog sustava. Ako su dani podsustavi kauzalni, s prijenosnim funkcijama: $H_1 = \dfrac{1}{s+2}$; $H_2 = \dfrac{1}{s+1}$; $H_3 = \dfrac{1}{s+4}$, ispitati stabilnost cijelog sustava. Naći odziv sustava na jedinični skok (početni uvjeti su nula).



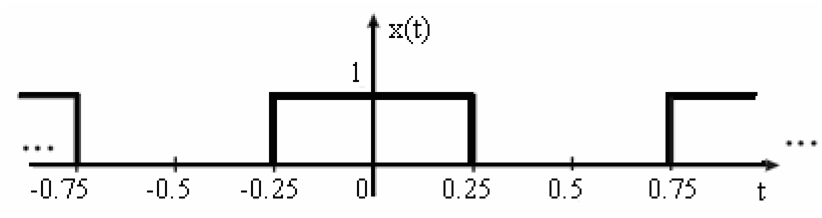
Slika 1.

3. Signal *x(t)* (Slika 2.) periodičan je s periodom *T=4 s*. Prikažite ovaj signal Fourierovim redom, te odredite koeficijente tog reda. Je li dobiveni red konvergentan i zašto?



Slika 2.

4. Slikom 3. dan je periodičan signal. Odredite srednju snagu ovog signala (u vremenskoj i u frekvencijskoj domeni), te aproksimirajte signal Fourierovim redom.



Slika 3.

5. Nađite Fourierove transformacije, te amplitudne, fazne, realne i imaginarne spektre sljedećih signala:

   a. $x(t) = e^{-t}\mu(t)$,
   b. $x(t) = e^{t}\mu(-t)$,
   c. $x(t) = e^{-|t|}$.

   Odredite energiju zadanih signala u vremenskoj i u frekvencijskoj domeni. U kakvom su odnosu ove dvije energije?

1.

$$A=\begin{bmatrix}-1 & 0\\ 0 & -\alpha\end{bmatrix} \qquad x(n+1)=Ax(n)+Bu(n)$$
$$B=\begin{bmatrix}1\\1\end{bmatrix} \qquad y(n)=Cx(n)+Du(n)$$
$$C=\begin{bmatrix}1 & -1\end{bmatrix}$$
$$D=[0]$$

$$\begin{bmatrix}x_1(n+1)\\x_2(n+1)\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -\alpha\end{bmatrix}\begin{bmatrix}x_1(n)\\x_2(n)\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}u(n)$$
$$y(n)=\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}x_1(n)\\x_2(n)\end{bmatrix}+0\cdot u(n)$$

$$H(z)=\left[C(zI-A)^{-1}B+D\right]$$

$$zI-A=\begin{bmatrix}z+1 & 0\\ 0 & z+\alpha\end{bmatrix} \longrightarrow (zI-A)^{-1}=\left[\begin{array}{cc|cc}z+1 & 0 & 1 & 0\\ 0 & z+\alpha & 0 & 1\end{array}\right] \sim \left[\begin{array}{cc|cc}1 & 0 & \frac{1}{z+1} & 0\\ 0 & 1 & 0 & \frac{1}{z+\alpha}\end{array}\right] = \begin{bmatrix}\frac{1}{z+1} & 0\\ 0 & \frac{1}{\alpha+z}\end{bmatrix}$$

$$H(z)=\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{z+1} & 0\\ 0 & \frac{1}{z+\alpha}\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}+0=\begin{bmatrix}\frac{1}{z+1} & -\frac{1}{z+\alpha}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\frac{1}{z+1}-\frac{1}{z+\alpha}=\boxed{\frac{\alpha-1}{(z+1)(z+\alpha)}}$$

ulaz $u(n)=\mu(n)$

$$\boxed{U(z)=\frac{z}{z-1}}$$

izlaz $y(n)=\frac{3}{10}\cdot 1^n-\frac{1}{2}(-1)^n+\frac{1}{5}(-4)^n$

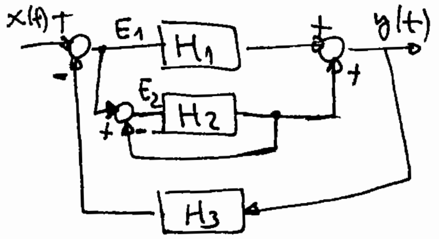
$$Y(z)=\frac{3}{10}\frac{z}{z-1}-\frac{1}{2}\frac{z}{z+1}+\frac{1}{5}\frac{z}{z+4}$$
$$=\frac{3z(z^2+5z+4)-5z(z^2+3z-4)+2z(z^2-1)}{10(z-1)(z+1)(z+4)}$$
$$=\frac{+0z^2+30z}{10(z-1)(z+1)(z+4)}=\boxed{\frac{3z}{(z-1)(z+1)(z+4)}}$$

$$H(z)=\frac{Y(z)}{U(z)}$$
$$=\frac{3\cancel{z}}{\cancel{(z-1)}(z+1)(z+4)}\cdot\frac{\cancel{z-1}}{\cancel{z}}=\boxed{\frac{3}{(z+1)(z+4)}}$$

$$\frac{\alpha-1}{(z+1)(z+\alpha)}=\frac{3}{(z+1)(z+4)}$$

$$\boxed{\alpha=4}$$
$$\alpha-1=3$$

2.



$$H_1 = \frac{1}{s+2}$$
$$H_2 = \frac{1}{s+1}$$
$$H_3 = \frac{1}{s+4}$$

$$E_1 H_1 + E_2 H_2 = y$$
$$E_1 = X - H_3 y$$
$$E_2 = E_1 - E_2 H_2$$
$$E_2 = \frac{E_1}{1+H_2}$$
$$E_2 = \frac{X - H_3 y}{1+H_2}$$

$$(X - H_3 y) H_1 + \frac{X - H_3 y}{1+H_2} H_2 = y$$
$$X H_1 - H_1 H_3 y + \frac{X H_2}{1+H_2} - \frac{H_3 H_2}{1+H_2} y = y$$
$$X\left[H_1(1+H_2) + H_2\right] = y\left[1 + H_2 + H_1 H_3 (1+H_2) + H_2 H_3\right]$$
$$\boxed{H = \frac{y}{X} = \frac{H_1 + H_1 H_2 + H_2}{1 + H_2 + H_1 H_3 + H_2 H_3 + H_1 H_2 H_3}}$$

$$H(s) = \frac{\frac{1}{s+2} + \frac{1}{(s+2)(s+1)} + \frac{1}{s+1}}{1 + \frac{1}{s+1} + \frac{1}{s+2}\cdot\frac{1}{s+4} + \frac{1}{s+1}\cdot\frac{1}{s+4} + \frac{1}{s+2}\,\frac{1}{s+1}\,\frac{1}{s+4}}$$

$$(s^2+3s+2)(s+4)$$
$$s^3 + 3s^2 + 2s$$
$$+ 4s^2 + 12s + 8$$

$$= \frac{\frac{s+1+1+s+2}{\cancel{(s+1)}(\cancel{s+2})}}{\frac{s^3+7s^2+14s+8+s^2+6s+8+s+1+s+2+1}{\cancel{(s+1)}(\cancel{s+2})(s+4)}}$$

$$= \frac{(2s+4)(s+4)}{s^3+8s^2+22s+20} = \frac{2(\cancel{s+2})(s+4)}{(\cancel{s+2})(s^2+6s+10)} = \boxed{\frac{2(s+4)}{s^2+6s+10}}$$

$$s^2 + 6s + 10 = 0$$
$$s_{1,2} = \frac{-6 \pm \sqrt{36-40}}{2} = \frac{-6 \pm 2i}{2} = -3 \pm i$$
$$\boxed{-3 < 0 \quad \text{STABILNO}}$$

ulaz $u(t) = \mu(t)$

$$\boxed{U(s) = \frac{1}{s}}$$

$$Y(s) = H(s)\cdot U(s) = \frac{2s+8}{s(s^2+6s+10)} = \frac{A}{s} + \frac{Bs+C}{s^2+6s+10} = \frac{\frac{4}{5}}{s} + \frac{\frac{4}{5}s - \frac{14}{5}}{s^2+6s+10}$$

$$A + B = 0$$
$$6A + C = 2 \rightarrow C = 2 - 6A = 2 - \frac{24}{5}$$
$$10A = 8$$
$$A = \frac{4}{5} \quad B = -\frac{4}{5} \quad C = \frac{-14}{5}$$

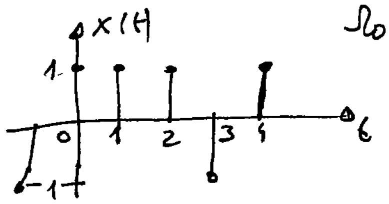
$$s^2 + 6s + 10 = (s+3)^2 + 1$$

$$= \frac{4}{5}\cdot\frac{1}{s} - \frac{4}{5}\,\frac{s+3}{(s+3)^2+1} \underbrace{- \frac{14}{5}\,\frac{1}{(s+3)^2+1} + \frac{\frac{12}{5}}{(s+3)^2+1}}_{\frac{-\frac{2}{5}}{(s+3)^2+1}}$$

$$\boxed{y(t) = \frac{4}{5}\mu(t) - \frac{4}{5}\cos t\, e^{-3t}\mu(t) - \frac{2}{5}\sin t\, e^{-3t}\mu(t)}$$

3.

$T_0 = 4s$



$\Omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\frac{\pi}{2}t}$$

$$X_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\frac{\pi}{2}t} dt$$

$$X_k = \frac{1}{4} \int_{0^-}^{4^-} x(t) e^{-jk\frac{\pi}{2}t} dt = \frac{1}{4}\Big[\int_{0^-}^{1^-} e^{-jk\frac{\pi}{2}t} \underbrace{\delta(t)}_{\delta(t-0)} dt + \int_{1^-}^{2^-} \underbrace{\delta(t)}_{\delta(t-1)} e^{-jk\frac{\pi}{2}t} dt + \int_{2^-}^{3^-} \delta(t-2) e^{-jk\frac{\pi}{2}t} dt + \int_{3^-}^{4^-} -\delta(t-3) e^{-jk\frac{\pi}{2}t} dt \Big]$$

$$= \frac{1}{4}\left[e^{-jk\frac{\pi}{2}\cdot 0} + e^{-jk\frac{\pi}{2}} + e^{-jk\frac{\pi}{2}\cdot 2} - e^{-jk\frac{\pi}{2}\cdot 3} \right]$$

$$= \frac{1}{4}\left[1 + \dots\right]$$

$\cos k\frac{\pi}{2} - j\sin k\frac{\pi}{2}$ → $1, -j, -1, j, 1, \dots$; $\left(e^{-j\frac{\pi}{2}}\right)^k$; $(-j)^k$

$\cos k\pi - j\sin k\pi$; $\left(e^{-\pi j}\right)^k$; $(-1)^k$

$\cos k\frac{3\pi}{2} - j\sin k\frac{3\pi}{2}$; $\left(e^{-j\frac{3\pi}{2}}\right)^k$; $(j)^k$

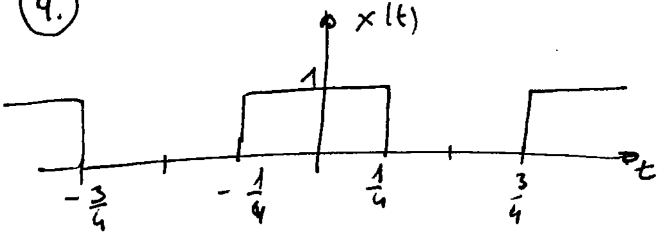
$$= \frac{1}{4}\left[1^k + (-j)^k + (-1)^k - j^k \right]$$

KONVERGENCIJA:

$$\int_{T_0} |x(t)|\, dt < \infty$$

konačan broj MIN, MAX, PREKIDA u 1 periodu

4.



$$x(t) = \sum_{k=-\infty}^{\infty} X_k \, e^{jk\Omega_0 t} \qquad T_0 = 1 \text{ s}$$

$$\Omega_0 = \frac{2\pi}{T_0} = 2\pi$$

$$X_k = \int_{-1/4}^{1/4} 1 \, e^{-jk\cdot 2\pi t} \, dt$$

$$= \frac{1}{-jk\cdot 2\pi} \, e^{-jk2\pi t} \Big|_{-1/4}^{1/4} = \frac{1}{jk\pi\cdot 2}\left[e^{-jk\cdot\frac{\pi}{2}} - e^{+jk\frac{\pi}{2}} \right]$$

$$\cancel{\cos k\tfrac{\pi}{2}} - j\sin\tfrac{k\pi}{2} - \cancel{\cos k\tfrac{\pi}{2}} - j\sin k\tfrac{\pi}{2}$$

$$= \frac{-\cancel{j}\cdot\cancel{2}\sin k\frac{\pi}{2}}{-\cancel{j}k\pi\cdot\cancel{2}} = +\frac{1}{2}\,\frac{\sin\frac{k\pi}{2}}{\frac{k\pi}{2}}$$

$$X_0 = \int_{-1/4}^{1/4} dt = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

SREDNJA SNAGA

$$P_x = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \int_{-1/4}^{1/4} dt = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P_x = \sum_{k=-\infty}^{\infty} |X_k|^2 = X_0^2 + \sum_{k=-\infty}^{0^-} |X_k|^2 + \sum_{k=0^+}^{\infty} |X_k|^2 = X_0^2 + 2\sum_{k=1}^{\infty} |X_k|^2$$

$$|X_0|^2 = \frac{1}{4}$$

$$|X_1|^2 = \left(\frac{\sin\frac{\pi}{2}}{\pi}\right)^2 = \left(\frac{1}{\pi}\right)^2 = 0.10132$$

$$|X_2|^2 = 0$$

$$|X_3|^2 = \left(\frac{-1}{3\pi}\right)^2 = 0.011258$$

$$|X_4|^2 = 0$$

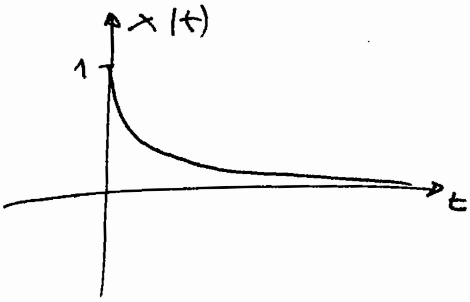
$$|X_5|^2 = \left(\frac{1}{5\pi}\right)^2 = 0.004052$$

$\vdots$

$$P_x = \frac{1}{4} + 2\cdot(0.101 + 0.011 + 0.004\ldots) \to 0.5$$

$$P_x = \frac{1}{4} + 2\cdot\left(\frac{1}{\pi}\right)^2 \underbrace{\sum_{k=0}^{\infty}\left(\frac{1}{2k+1}\right)^2}_{\frac{\pi^2}{8}} = \frac{1}{4} + \cancel{2}^{1}\,\frac{1}{\cancel{\pi^2}}\cdot\frac{\cancel{\pi^2}}{\cancel{8}_4} = \frac{2}{4} = \frac{1}{2}$$

5. a) $x(t) = e^{-t}\mu(t)$



$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\,dt$$
$$= \int_{-\infty}^{\infty} e^{-t}\mu(t)\, e^{-j\Omega t}\,dt$$
$$= \int_{0}^{\infty} e^{-(j\Omega+1)t}\,dt$$
$$= \frac{1}{-(1+j\Omega)} \cdot e^{-(j\Omega+1)t}\Big|_{0}^{\infty}$$
$$= \boxed{\frac{1}{1+j\Omega}} =$$
$$= \frac{1-j\Omega}{1+\Omega^2}$$

AMPLITUDNI SPEKTAR

$$|X(j\Omega)| = \sqrt{\frac{1+\Omega^2}{(1+\Omega^2)^2}} = \frac{1}{\sqrt{1+\Omega^2}}$$

FAZNI SPEKTAR

$$\measuredangle X(j\Omega) = \operatorname{arctg} \frac{\frac{-\Omega}{1+\Omega^2}}{\frac{1}{1+\Omega^2}} = \operatorname{arctg}(-\Omega)$$
$$= -\operatorname{arctg}\Omega$$

REALNI SPEKTAR

$$Re(X(j\Omega)) = \frac{1}{1+\Omega^2}$$

IMAGINARNI

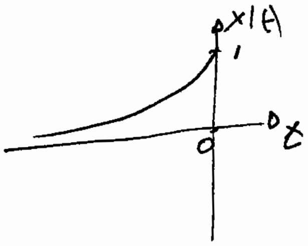
$$Im(X(j\Omega)) = -\frac{\Omega}{1+\Omega^2}$$

ENERGIJA

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |e^{-t}\mu(t)|^2 dt = \int_{0}^{\infty} e^{-2t}\,dt = \frac{1}{-2}\, e^{-2t}\Big|_{0}^{\infty} = \frac{1}{2}$$

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\Omega)|^2 d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{1}{1+j\Omega}\right)^2 d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+\Omega^2}\,d\Omega = \frac{1}{2\pi}\cdot\pi = \frac{1}{2}$$

b) $x(t) = e^{t}\mu(-t)$



$$X(j\Omega) = \int_{-\infty}^{\infty} e^{t}\mu(-t)\, e^{-j\Omega t}\,dt = \int_{-\infty}^{0} e^{(1-j\Omega)t}\,dt$$
$$= \frac{1}{1-j\Omega}\cdot e^{(1-j\Omega)t}\Big|_{-\infty}^{0} = \boxed{\frac{1}{1-j\Omega}} = \frac{1+j\Omega}{1+\Omega^2}$$

$$|X(j\Omega)| = \sqrt{\frac{1+\Omega^2}{(1+\Omega^2)^2}} = \frac{1}{\sqrt{1+\Omega^2}}$$

$$\measuredangle X(j\Omega) = \operatorname{arctg} \frac{\Omega}{1+\Omega^2} \Big/ \frac{1}{1+\Omega^2} = \operatorname{arctg}\Omega$$

$$Re(X(j\Omega)) = \frac{1}{1+\Omega^2}$$

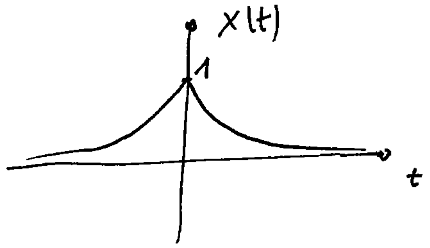
$$Im(X(j\Omega)) = \frac{\Omega}{1+\Omega^2}$$

ENERGIJA

$$E_x = \int_{-\infty}^{0} |e^{t}|^2\, dt = \int_{-\infty}^{0} e^{2t}\,dt = \frac{1}{2}\, e^{2t}\Big|_{-\infty}^{0} = \frac{1}{2}$$

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\Omega)|^2 d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{1}{1-j\Omega}\right|^2 d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+\Omega^2}\,d\Omega = \frac{1}{2\pi}\cdot\pi = \frac{1}{2}$$

c) $x(t) = e^{-|t|}$



$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

$$= \int_{-\infty}^{0} e^{t} e^{-j\Omega t} dt + \int_{0}^{\infty} e^{-t} e^{-j\Omega t} dt$$

$$= \int_{-\infty}^{0} e^{(1-j\Omega)t} dt + \int_{0}^{\infty} e^{-(1+j\Omega)t} dt$$

$$= \frac{1}{1-j\Omega} + \frac{1}{1+j\Omega} = \frac{1+j\Omega+1-j\Omega}{1+\Omega^2}$$

$$= \frac{2}{1+\Omega^2}$$

$$|X(j\Omega)| = \frac{2}{1+\Omega^2}$$

$$\measuredangle |X(j\Omega)| = \mathrm{arctg} \frac{0}{\frac{2}{1+\Omega^2}} = 0$$

$$\mathrm{Re}\,(X(j\Omega)) = \frac{2}{1+\Omega^2}$$

$$\mathrm{Im}\,(X(j\Omega)) = 0$$

ENERGIJA

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{0} e^{2t} dt + \int_{0}^{\infty} e^{-2t} dt = \frac{1}{2} e^{2t}\Big|_{-\infty}^{0} + \frac{1}{-2} e^{-2t}\Big|_{0}^{\infty}$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

$$E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2 d\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{2}{1+\Omega^2}\right)^2 d\Omega = \frac{1}{\cancel{2\pi}} \cdot \cancel{4} \cdot \frac{\cancel{\pi}}{\cancel{2}} = 1$$