Signali i sustavi

Međuispit (grupa A) – 26. travnja 2012.

1. (9 bodova) Zadan je vremenski kontinuiran signal $x(t) = \cos(100t) + \cos(200t)$.

a) **(4 boda)** Odredite razvoj signala $x(t)$ u vremenski kontinuirani Fourierov red (CTFS). Skicirajte amplitudni i fazni spektar signala.

b) **(3 boda)** Iz SPEKTRA izračunajte snagu signala.

c) **(2 boda)** Za koje frekvencije očitavanja je očitavanje signala $x(t)$ jednoznačno?

2. (9 bodova) Zadan je vremenski kontinuiran signal $x(t) = e^{-4t}\,\mu(t) + 5e^{5t}\,\mu(-t)$.

a) **(3 boda)** Odredite generaliziranu derivaciju zadanog signala.

b) **(3 boda)** Izračunajte vremenski kontinuiranu Fourierovu transformaciju (CTFT) zadanog signala.

c) **(3 boda)** Izračunajte energiju zadanog signala.

3. (9 bodova) Promatramo vremenski diskretan periodičan signal $x(n)$ perioda 6. Šest uzoraka jednog perioda počevši od koraka $n = 0$ su $\{\underline{-6}, 3, 0, 0, 3, 0\}$.

a) **(2 boda)** Odredite razvoj signala $x(n)$ u vremenski diskretan Fourierov red (DTFS).

b) **(2 boda)** Navedite svojstvo simetričnosti spektra X_k realnog signala $x(n)$. Pokažite da dobiveni X_k zadovoljava taj uvjet!

c) **(3 boda)** Izračunajte numeričke vrijednosti spektra X_k za $k \in \{0, 1, 2, 3, 4, 5\}$.

d) **(2 boda)** Skicirajte amplitudni i fazni spektar X_k

4. (9 bodova) Jedan period vremenski diskretne Fourierove transformacije (DTFT) nekog vremenski diskretnog signala $x(n)$ jest $X(e^{j\Omega}) = \Omega + 3\pi$, $-\pi < \Omega \leq \pi$.

a) **(4 boda)** Odredite vremenski diskretan signal $x(n)$.

b) **(3 boda)** Odredite energiju signala $x(n)$.

c) **(2 boda)** Odredite vremenski diskretnu Fourierovu transformaciju (DTFT) signala $y(n) = e^{j3\pi n}x(n)$.

5. (9 bodova) Zadan je vremenski kontinuiran signal $x(t) = \cos(100t) + \cos(200t)$.

a) **(1 bod)** Skicirajte amplitudni spektar vremenski kontinuirane Fourierove transformacije (CTFT) zadanog signala.

b) **(1 bod)** Ako signal očitamo s kružnom frekvencijom $\omega_S = 600$ skicirajte amplitudni spektar kontinuiranog očitanog signala $x(t)\,\mathrm{comb}_{T_S}(t)$.

c) **(2 boda)** Počevši od koraka $n = 0$ odredite prvih šest očitaka signala $x(t)$ uz $\omega_S = 600$. Iz tih očitaka izračunajte diskretnu Fourierovu transformaciju u šest točaka (DFT_6).

d) **(2 boda)** Kojim frekvencijama vremenski kontinuiranog signala odgovaraju članovi spektra $X(1)$ i $X(3)$ dobiveni pod c)?

e) **(1 bod)** Kolika je spektralna rezolucija ω_0 spektra pod c)?

f) **(2 boda)** Koliko treba biti trajanje signala za spektralnu rezoluciju $\omega_0 = 10$?

(A)

1. $x(t) = \cos 100t + \cos 200t$

a) period:

$100 T_1 = 2k\pi$ $\qquad$ $200 T_2 = 2k\pi$

$T_1 = \frac{2k\pi}{100}$ $\qquad$ $T_2 = \frac{2k\pi}{200} = \frac{k\pi}{100}$ $\qquad$ $\Big\}\ T = \frac{2\pi}{100}$

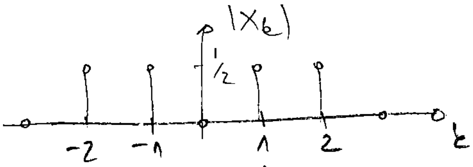
$T_1 = \frac{2\pi}{100}$ $\qquad$ $T_2 = \frac{\pi}{100}$

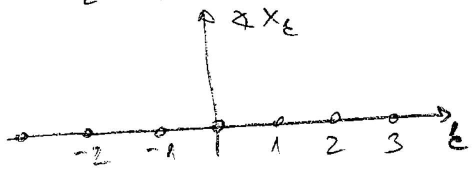
$$x(t) = \frac{1}{2}\left(e^{j100t} + e^{-j100t}\right) + \frac{1}{2}\left(e^{j200t} + e^{-j200t}\right)$$

$$= \frac{1}{2} e^{j\frac{2\pi}{\frac{2\pi}{100}}\cdot t} + \frac{1}{2} e^{-j\frac{2\pi}{\frac{2\pi}{100}}\cdot t} + \frac{1}{2} e^{j\frac{2\pi}{\frac{2\pi}{100}}\cdot t\cdot 2} + \frac{1}{2} e^{-j\frac{2\pi}{\frac{2\pi}{100}}\cdot t\cdot 2}$$

$X_1 = \frac{1}{2}$ $\qquad$ $X_2 = \frac{1}{2}$

$X_{-1} = \frac{1}{2}$ $\qquad$ $X_{-2} = \frac{1}{2}$





b) $P = \sum_{k=-\infty}^{\infty} |X_k|^2$

$P = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1$

c) $x(t) = \cos 100t + \cos 200t$ $\qquad$ $\omega = 2\pi f$

$\omega_1 = 100$ $\qquad$ $\omega_2 = 200$

$f_1 = \omega_1 / 2\pi$

$f_1 = \frac{100}{2\pi}$ Hz $\qquad$ $f_2 = \frac{200}{2\pi}$ Hz

Najveća frekvencija u signalu je $\frac{200}{2\pi}$ Hz

Očitavanje će biti jednoznačno za frekvencije veće od $\frac{400}{2\pi}$ Hz

ili $\omega > 400$ rad/s

2. $x(t) = e^{-4t}\mu(t) + 5e^{5t}\mu(-t)$

(A)

a) generalizirana derivacija

$$x'(t) = -4e^{-4t}\mu(t) + e^{-4t}\delta(t) + 25e^{5t}\mu(-t) + 5e^{5t}\delta(-t)\cdot(-1)$$

$$= -4e^{-4t}\mu(t) + 25e^{5t}\mu(-t) + \delta(t) - 5\delta(t)$$

$$= -4e^{-4t}\mu(t) + 25e^{5t}\mu(-t) - 4\delta(t)$$

b) CTFT

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-4t} e^{-j\omega t} dt + 5\int_{-\infty}^{0} e^{5t} e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-(4+j\omega)t} dt + 5\int_{-\infty}^{0} e^{(5-j\omega)t} dt$$

$$= \frac{e^{-(4+j\omega)t}}{-(4+j\omega)}\Big|_{0}^{\infty} + 5\,\frac{e^{(5-j\omega)t}}{5-j\omega}\Big|_{-\infty}^{0} = \frac{1}{4+j\omega} + \frac{5}{5-j\omega}$$

c) $$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(e^{j\omega})|^2 d\omega$$

$$= \int_{-\infty}^{\infty} \left(e^{-4t}\mu(t) + 5e^{5t}\mu(-t)\right)^2 dt = \int_{-\infty}^{\infty} \left(e^{-8t}\mu(t) + 25e^{10t}\mu(-t) + 10e^{t}\mu(t)\mu(-t)\right) dt$$

$$= \int_{0}^{\infty} e^{-8t} dt + \int_{-\infty}^{0} 25e^{10t} dt$$

$$= \frac{e^{-8t}}{-8}\Big|_{0}^{\infty} + 25\,\frac{e^{10t}}{10}\Big|_{-\infty}^{0} = \frac{1}{8} + \cancel{25}^{5}\cdot\frac{1}{\cancel{10}_{2}} = \frac{1+20}{8} = \frac{21}{8}$$

(A)

3. $x(n) = \{\underline{-6}, 3, 0, 0, 3, 0\}$

$N = 6$

a) DTFS

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$= \frac{1}{6}\sum_{n=0}^{5} x(n)\, e^{-j\frac{2\pi}{6}kn} = \frac{1}{6}\left(-6\cdot e^{-j\frac{\pi}{3}\cdot k\cdot 0} + 3e^{-j\frac{\pi}{3}\cdot k} + 3e^{-j\frac{\pi}{3}\cdot k\cdot 4}\right)$$

$$= \frac{1}{6}\left(-6 + 3e^{-j\frac{2\pi}{6}k} + 3e^{-j\frac{8\pi}{6}k}\right)$$

$$= -1 + \frac{1}{2}e^{-j\frac{5\pi}{6}\cdot k}\cdot\left(e^{j\frac{3\pi}{6}k} + e^{-j\frac{3\pi}{6}k}\right) = -1 + \frac{1}{2}e^{-j\frac{5\pi}{6}k}\cdot 2\cos\frac{3\pi}{6}k$$

$$= -1 + e^{-j\frac{5\pi}{6}k}\cdot\cos\frac{\pi}{2}k = -1 + \cos\frac{5\pi}{6}k\cos\frac{\pi}{2}k - j\sin\frac{5\pi}{6}k\cos\frac{\pi}{2}k$$

b) za realni signal vrijedi $X_k^* = X_{-k}$ → spektar je konjugirano simetričan

$$X_k^* = -1 + \cos\frac{5\pi}{6}k\cos\frac{\pi}{2}k + j\sin\frac{5\pi}{6}k\cos\frac{\pi}{2}k$$

$$X_{-k} = -1 + \cos\left(\frac{5\pi}{6}(-k)\right)\cos\left(\frac{\pi}{2}(-k)\right) - j\sin\left(\frac{5\pi}{6}(-k)\right)\cos\frac{\pi}{2}(-k)$$

$$= -1 + \cos\frac{5\pi}{6}k\cos\frac{\pi}{2}k + j\sin\frac{5\pi}{6}k\cos\frac{\pi}{2}k$$

$$X_k^* = X_{-k}$$

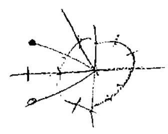
c) X_k za $k \in \{0, 1, 2, 3, 4, 5\}$

$$X_0 = -1 + e^{-j\cdot 0}\cos\frac{\pi}{2}\cdot 0 = 0$$

$$X_1 = -1 + e^{-j\frac{5\pi}{6}}\cos\frac{\pi}{2} = -1 = 1e^{j\pi}$$

$$X_2 = -1 + e^{-j\frac{5\pi}{6}\cdot 2}\cos\frac{\pi}{2}\cdot 2 = -1 + \left(\cos\frac{5\pi}{3} - j\sin\frac{5\pi}{3}\right)\cdot(-1)) = -1 - \frac{1}{2} + j\left(-\frac{\sqrt{3}}{2}\right)$$

$$= -\frac{3}{2} - \frac{\sqrt{3}}{2}j = \sqrt{\frac{12}{4}}\, e^{j\frac{7\pi}{6}} = \sqrt{3}\, e^{i\frac{7\pi}{6}} = \sqrt{3}\, e^{-i\frac{5\pi}{6}}$$



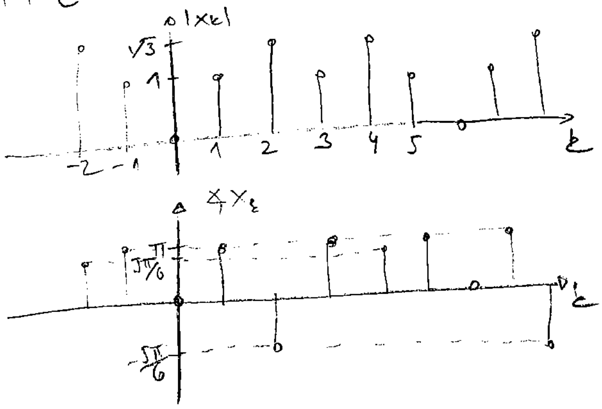
$$X_3 = -1 + e^{-j\frac{5\pi}{2}}\cos\left(\frac{\pi}{2}\cdot 3\right) = -1 = 1e^{i\pi}$$

$$X_4 = -1 + e^{-j\frac{10\pi}{3}}\cos(2\pi) = -1 + e^{-j\frac{10\pi}{3}} = -1 + \frac{\sqrt{3}}{2}j - \frac{1}{2} = -\frac{3}{2} + \frac{\sqrt{3}}{2}j$$

$$= \sqrt{3}\, e^{j\frac{5\pi}{6}}$$

$$X_5 = -1 + e^{-j\frac{25\pi}{6}}\cos\left(\frac{\pi}{2}\cdot 5\right) = -1 = 1e^{i\pi}$$

d)



b) $E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{i\Omega})|^2 d\Omega$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Omega + 3\pi|^2 d\Omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\Omega^2 + 6\pi\Omega + 9\pi^2) d\Omega$$

$$= \frac{4}{2\pi} \left(\frac{\Omega^3}{3} + 6\pi \frac{\Omega^2}{2} + 9\pi^2 \Omega \Big|_{-\pi}^{\pi} \right)$$

$$= \frac{4}{2\pi} \left(\frac{\pi^3}{3} + \frac{6\pi^3}{2} + 9\pi^3 - \left(\frac{-\pi^3}{3} + \frac{6\pi^3}{2} - 9\pi^3 \right) \right)$$

$$= \frac{4}{2\pi} \left(\frac{2\pi^3}{3} + 18\pi^3 \right) = \frac{28\,\cancel{56}\pi^{\cancel{3}}}{3 \cdot \cancel{2\pi}} = \frac{28\pi^2}{3}$$

c) $y(n) = e^{i3\pi n} x(n)$

$Y(e^{i\Omega}) = X(e^{i(\Omega - 3\pi)})$

$= (\Omega - 3\pi) + 3\pi = \Omega$

A

4. DTFT

$$X(e^{i\Omega}) = \Omega + 3\pi \quad , -\pi < \Omega \leq \pi$$

a) $$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{i\Omega}) e^{i\Omega n} \, d\Omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\Omega + 3\pi) e^{i\Omega n} \, d\Omega$$

$$= \left| \begin{array}{ll} \Omega + 3\pi = u & e^{i\Omega n} \, d\Omega = dv \\ d\Omega = du & \frac{e^{i\Omega n}}{jn} = v \end{array} \right|$$

$$= \frac{1}{2\pi} \left(\frac{e^{i\Omega n}}{jn} (\Omega + 3\pi) \right) \Bigg|_{-\pi}^{\pi} - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\Omega n}}{jn} \, d\Omega$$

$$= \frac{1}{2\pi} \left(\frac{1}{jn} e^{i\pi n} (\pi + 3\pi) - \frac{1}{jn} e^{-i\pi n} (-\pi + 3\pi) \right) - \frac{1}{2\pi} \frac{1}{jn} \frac{1}{jn} e^{i\Omega n} \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \left(\frac{4\pi}{jn} e^{i\pi n} - \frac{2\pi}{jn} e^{-i\pi n} \right) + \frac{1}{2\pi n^2} \left(e^{i\pi n} - e^{-i\pi n} \right)$$

$$= \frac{1}{jn} \left(2 e^{i\pi n} - e^{-i\pi n} \right) + \frac{1}{2\pi n^2} \cdot 2j \sin \pi n$$

$$= \frac{1}{jn} \cdot \left(2 \cos \pi n + 2j \sin \pi n - \cos \pi n + j \sin \pi n \right)$$

$$= \frac{1}{jn} \cdot \cos \pi n$$

$$= -j \frac{\cos \pi n}{n} = -j \frac{(-1)^n}{n} \qquad \text{za} \quad n \neq 0$$

za $n = 0$

$$x(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\Omega + 3\pi) \, d\Omega = \frac{1}{2\pi} \left(\frac{\Omega^2}{2} + 3\pi\Omega \right) \Bigg|_{-\pi}^{\pi}$$

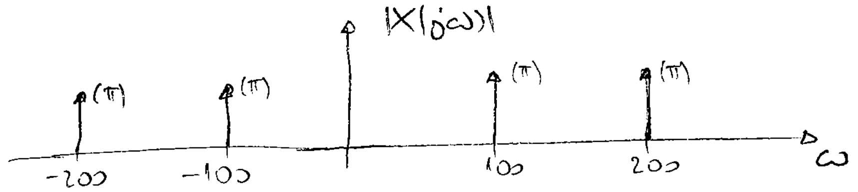
$$= \frac{1}{2\pi} \left(\frac{\pi^2}{2} + 3\pi^2 - \left(\frac{\pi^2}{2} - 3\pi^2 \right) \right)$$

$$= \frac{1}{2\pi} \left(6\pi^2 \right) = 3\pi$$

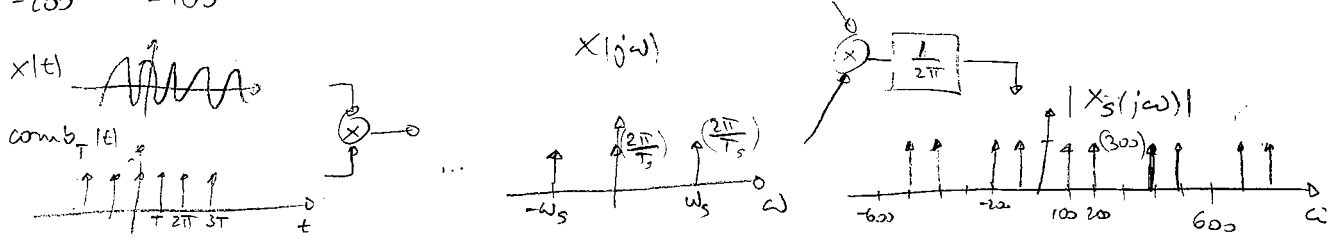
5. $x(t) = \cos 100t + \cos 200t$ (A)

a) CTFT $X(j\omega) = ?$

$$X(j\omega) = \pi\left(\delta(\omega-100) + \delta(\omega+100)\right) + \pi\left(\delta(\omega-200) + \delta(\omega+200)\right)$$
$$= \pi\,\delta(\omega-100) + \pi\,\delta(\omega+100) + \pi\,\delta(\omega-200) + \pi\,\delta(\omega+200)$$



b)



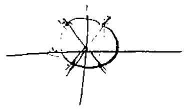
$\omega_s = \frac{2\pi}{T_s} = 600$

amplituda: $\frac{2\pi}{T_s} \cdot \pi \cdot \frac{1}{2\pi} = \frac{\pi}{T_s}$

$A = \frac{\pi}{T_s} = \frac{\pi}{\frac{2\pi}{\omega_s}} = \frac{\cancel{\pi} \cdot \omega_s}{2\cancel{\pi}} = 300$

c) $\omega_s = 600 \rightarrow T_s = \frac{2\pi}{600}$

$x(nT_s) = \cos 100 \frac{2\pi}{600} n + \cos 200 \cdot \frac{2\pi}{600} n = \cos \frac{2\pi}{6} n + \cos \frac{4\pi}{6} n$
$= \cos \frac{\pi}{3} n + \cos \frac{2\pi}{3} n$



$x(0) = 2$
$x(1) = \frac{1}{2} - \frac{1}{2} = 0$
$x(2) = -\frac{1}{2} - \frac{1}{2} = -1$
$x(3) = -1 + 1 = 0$
$x(4) = -\frac{1}{2} - \frac{1}{2} = -1$
$x(5) = \frac{1}{2} - \frac{1}{2} = 0$

$x(n) = \{\underline{2}, 0, -1, 0, -1, 0\}$

DFT_6

$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{5} x(n)\, e^{-j\frac{2\pi}{6}kn} = 2 - e^{-j\frac{2\pi}{6}\cdot 2k} - e^{-j\frac{2\pi}{6}\cdot 4k}$
$= 2 - e^{-j\frac{2\pi}{6}3k}\left(e^{j\frac{2\pi}{6}k} + e^{-j\frac{2\pi}{6}k}\right)$
$= 2 - e^{-j\pi k} \cdot 2\cos\frac{\pi}{3}k$
$= 2 - e^{-j\frac{2\pi}{3}k} - e^{-j\frac{4\pi}{3}k}$

$X(0) = 0$
$X(1) = 2 - 2\cdot e^{-j\pi} \cos\frac{\pi}{3} = 2 - \cancel{2}\cdot\frac{1}{\cancel{2}}\cdot(-1) = 3$
$X(2) = 2 - 2e^{-j2\pi} \cos\frac{2\pi}{3} = 2 - 2\cdot(-\frac{1}{2}) = 3$
$X(3) = 2 - 2e^{-j3\pi} \cos\pi = 2 - 2\cdot(-1)\cdot(-1) = 0$
$X(4) = 2 - 2e^{-j4\pi} \cos\frac{4\pi}{3} = 2 - 2(-\frac{1}{2}) = 3$
$X(5) = 2 - 2e^{-j5\pi} \cos\frac{5\pi}{3} = 2 - \cancel{2}\cdot(-1)\cdot\frac{1}{\cancel{2}} = 3$

$X(k) = \{\underline{0}, 3, 3, 0, 3, 3\}$

d) $\omega_s = 600$

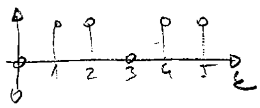
$\Omega = \frac{2\pi}{N} \cdot k$

$\Omega = \omega T_s = \omega \cdot \frac{2\pi}{\omega_s}$

$\omega = \frac{\omega_s}{2\pi} \cdot \Omega = \frac{\omega_s}{2\pi} \cdot \frac{2\pi}{N} \cdot k = \frac{\omega_s}{N} \cdot k$

$\omega_1 = \frac{600}{6} \cdot 1 = 100$ rad/s
$\omega_3 = \frac{600}{6} \cdot 3 = 300$ rad/s

e) ω_0



broj očitaka je $N=6$

frekv. očitavanja $\omega_s = 600 \rightarrow f_s = \frac{\omega_s}{2\pi} = \frac{600}{2\pi}$

postoji 5 razmaka između uzoraka.

ω_0 = razmak dva uzorka = rezolucija = $\frac{600}{5} = 120$ rad/s

ili $N=6$

$N_{\omega_0} \cdot \omega_0 = \omega_s \rightarrow \omega_0 = \frac{\omega_s}{N} = \frac{600}{6} = 100$ rad/s

f) ako je $\omega_0 = 10$

$\frac{600}{10} = 60$ razmaka među uzorcima

postoji 61 uzorak

$\omega_s = 600$ rad/s

$\omega_s = 2\pi f_s$

$f_s = \frac{600}{2\pi}$

trajanje signala

$$T_p = \frac{N}{f_s} = \frac{61}{\frac{600}{2\pi}} = \frac{61 \cdot 2\pi}{600} = 0.639 \text{ s}$$

ili

$$N = \frac{\omega_s}{\omega_0} = \frac{600}{10} = 60$$

$$T = \frac{N}{f_s} = \frac{N}{\frac{\omega_s}{2\pi}} = \frac{N \cdot 2\pi}{\omega_s} = \frac{6\cancel{0} \cdot 2\pi}{60\cancel{0}} = \frac{\pi}{5} \text{ s}$$

Signali i sustavi

Međuispit (grupa B) – 26. travnja 2012.

1. (9 bodova) Zadan je vremenski kontinuiran signal $x(t) = \cos(200t) + \cos(400t)$.

a) **(4 boda)** Odredite razvoj signala $x(t)$ u vremenski kontinuirani Fourierov red (CTFS). Skicirajte amplitudni i fazni spektar signala.

b) **(3 boda)** Iz SPEKTRA izračunajte snagu signala.

c) **(2 boda)** Za koje frekvencije očitavanja je očitavanje signala $x(t)$ jednoznačno?

2. (9 bodova) Zadan je vremenski kontinuiran signal $x(t) = e^{-3t}\,\mu(t) + 6e^{6t}\,\mu(-t)$.

a) **(3 boda)** Odredite generaliziranu derivaciju zadanog signala.

b) **(3 boda)** Izračunajte vremenski kontinuiranu Fourierovu transformaciju (CTFT) zadanog signala.

c) **(3 boda)** Izračunajte energiju zadanog signala.

3. (9 bodova) Promatramo vremenski diskretan periodičan signal $x(n)$ perioda 6. Šest uzoraka jednog perioda počevši od koraka $n = 0$ su $\{\underline{-6}, 0, 3, 0, 0, 3\}$.

a) **(2 boda)** Odredite razvoj signala $x(n)$ u vremenski diskretan Fourierov red (DTFS).

b) **(2 boda)** Navedite svojstvo simetričnosti spektra X_k realnog signala $x(n)$. Pokažite da dobiveni X_k zadovoljava taj uvjet!

c) **(3 boda)** Izračunajte numeričke vrijednosti spektra X_k za $k \in \{0, 1, 2, 3, 4, 5\}$.

d) **(2 boda)** Skicirajte amplitudni i fazni spektar X_k

4. (9 bodova) Jedan period vremenski diskretne Fourierove transformacije (DTFT) nekog vremenski diskretnog signala $x(n)$ jest $X(e^{j\Omega}) = \Omega + 2\pi$, $-\pi < \Omega \leq \pi$.

a) **(4 boda)** Odredite vremenski diskretan signal $x(n)$.

b) **(3 boda)** Odredite energiju signala $x(n)$.

c) **(2 boda)** Odredite vremenski diskretnu Fourierovu transformaciju (DTFT) signala $y(n) = e^{j3\pi n}x(n)$.

5. (9 bodova) Zadan je vremenski kontinuiran signal $x(t) = \cos(200t) + \cos(400t)$.

a) **(1 bod)** Skicirajte amplitudni spektar vremenski kontinuirane Fourierove transformacije (CTFT) zadanog signala.

b) **(1 bod)** Ako signal očitamo s kružnom frekvencijom $\omega_S = 1200$ skicirajte amplitudni spektar kontinuiranog očitanog signala $x(t)\,\mathrm{comb}_{T_S}(t)$.

c) **(2 boda)** Počevši od koraka $n = 0$ odredite prvih šest očitaka signala $x(t)$ uz $\omega_S = 1200$. Iz tih očitaka izračunajte diskretnu Fourierovu transformaciju u šest točaka (DFT_6).

d) **(2 boda)** Kojim frekvencijama vremenski kontinuiranog signala odgovaraju članovi spektra $X(1)$ i $X(3)$ dobiveni pod c)?

e) **(1 bod)** Kolika je spektralna rezolucija ω_0 spektra pod c)?

f) **(2 boda)** Koliko treba biti trajanje signala za spektralnu rezoluciju $\omega_0 = 10$?

B

1. $x(t) = \cos 200t + \cos 400t$

a) CTFS

period $\quad 200T_1 = 2k\pi \qquad 400T_2 = 2k\pi$

$$T_1 = \frac{2k\pi}{200} \qquad T_2 = \frac{2k\pi}{400}$$

$$T_1 = \frac{2\pi}{200} \qquad T_2 = \frac{2\pi}{400} = \frac{\pi}{200}$$

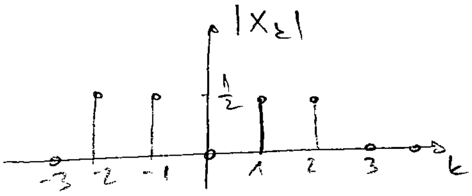
$$\Bigg\} \; T = \frac{2\pi}{200}$$

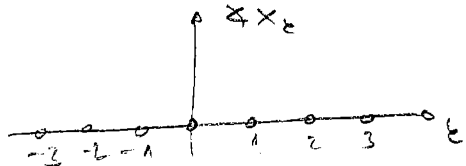
$$x(t) = \frac{1}{2}\left(e^{j200t} + e^{-j200t}\right) + \frac{1}{2}\left(e^{j400t} + e^{-j400t}\right)$$

$$= \frac{1}{2} e^{j\frac{2\pi}{\frac{2\pi}{200}} t \cdot 1} + \frac{1}{2} e^{-j\frac{2\pi}{\frac{2\pi}{200}} t \cdot 1} + \frac{1}{2} e^{j\frac{2\pi}{\frac{2\pi}{200}} \cdot t \cdot 2} + \frac{1}{2} e^{-j\frac{2\pi}{\frac{2\pi}{200}} \cdot t \cdot 2}$$

$X_1 = \frac{1}{2} \qquad X_2 = \frac{1}{2}$

$X_{-1} = \frac{1}{2} \qquad X_{-2} = \frac{1}{2}$





b) $P = \sum_{k=-\infty}^{\infty} |X_k|^2$

$$P = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1$$

---

c) $x(t) = \cos 200t + \cos 400t$

$\omega_1 = 200 \qquad \omega_2 = 400$

$f_1 = \frac{200}{2\pi}\,\text{Hz} \qquad f_2 = \frac{400}{2\pi}\,\text{Hz}$

Najveća frekvencija u signalu je $\frac{400}{2\pi}$ Hz

Očitavanje će biti jednoznačno za frekvenciju veću od $\frac{800}{2\pi}$ Hz

ili za $\omega > 800$ rad/s

2. $x(t) = e^{-3t}\mu(t) + 6e^{6t}\mu(-t)$

a) $x'(t) = -3e^{-3t}\mu(t) + e^{-3t}\delta(t) + 36e^{6t}\mu(-t) + 6e^{6t}\delta(-t)\cdot(-1)$

$= -3e^{-3t}\mu(t) + 36e^{6t}\mu(-t) + \delta(t) - 6\delta(t)$

$= -3e^{-3t}\mu(t) + 36e^{6t}\mu(-t) - 5\delta(t)$

b) CTFT

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$= \int_{0}^{\infty} e^{-3t}e^{-j\omega t}dt + \int_{-\infty}^{0} 6e^{6t}e^{-j\omega t}dt$$

$$= \int_{0}^{\infty} e^{-(3+j\omega)t}dt + 6\int_{-\infty}^{0} e^{(6-j\omega)t}dt$$

$$= \frac{e^{-(3+j\omega)t}}{-(3+j\omega)}\Bigg|_{0}^{\infty} + 6\,\frac{e^{(6-j\omega)t}}{6-j\omega}\Bigg|_{-\infty}^{0}$$

$$= \frac{1}{3+j\omega} + 6\,\frac{1}{6-j\omega}$$

c) $$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} |(e^{-3t}\mu(t) + 6e^{6t}\mu(-t)|^2 dt$$

$$= \int_{-\infty}^{\infty}\left(e^{-6t}\mu(t) + 36e^{12t}\mu(-t) + 6e^{3t}\mu(t)\mu(-t)\right) dt$$

$$= \int_{0}^{\infty} e^{-6t}dt + 36\int_{-\infty}^{0} e^{12t}dt$$

$$= \frac{e^{-6t}}{-6}\Bigg|_{0}^{\infty} + 36\,\frac{e^{12t}}{12}\Bigg|_{-\infty}^{0} = \frac{1}{6} + \frac{\cancel{36}^{18}}{\cancel{12}_{6}} = \frac{19}{6}$$

(B)

3. $x(n)=\{\underline{-6}, 0, 3, 0, 0, 3\}$ $N=6$

a) DTFS

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$= \frac{1}{6}\sum_{n=0}^{5} x(n)\, e^{-j\frac{2\pi}{6}kn} = \frac{1}{6}\left(-6e^{-j\frac{2\pi}{6}\cdot 0} + 3e^{-j\frac{2\pi}{6}\cdot 2k} + 3e^{-j\frac{2\pi}{6}\cdot 5k}\right)$$

$$= -1 + \frac{1}{2} e^{-j\frac{4\pi}{6}k} + \frac{1}{2} e^{-j\frac{10\pi}{6}k} = -1 + \frac{1}{2}\cos\frac{2\pi}{3}k + \frac{1}{2}\cos\frac{5\pi}{3}k + \frac{1}{2}j\left(-\sin\frac{2\pi}{3}k - \sin\frac{5\pi}{3}k\right)$$

b) za realni signal vrijedi $X_k^* = X_{-k}$ → spektar je konjugirano simetričan

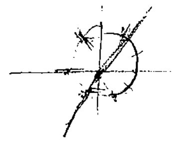
$$X_k^* = -1 + \frac{1}{2}\cos\frac{2\pi}{3}k + \frac{1}{2}\cos\frac{5\pi}{3}k - \frac{1}{2}j\left(-\sin\frac{2\pi}{3}k - \sin\frac{5\pi}{3}k\right)$$

$$X_{-k} = -1 + \frac{1}{2}\cos\left(\frac{2\pi}{3}(-k)\right) + \frac{1}{2}\cos\left(\frac{5\pi}{3}(-k)\right) + \frac{1}{2}j\left(-\sin\left(\frac{2\pi}{3}(-k)\right) - \sin\left(\frac{5\pi}{3}(-k)\right)\right)$$

$$= -1 + \frac{1}{2}\cos\frac{2\pi}{3}k + \frac{1}{2}\cos\frac{5\pi}{3}k + \frac{1}{2}j\left(\sin\frac{2\pi}{3}k + \sin\frac{5\pi}{3}k\right)$$

$$X_k^* = X_{-k}$$

c) X_k za $k=\{0,1,2,3,4,5\}$



$$X_0 = -1 + \frac{1}{2} + \frac{1}{2} = 0$$

$$X_1 = -1 + \frac{1}{2} e^{-j\frac{4\pi}{6}} + \frac{1}{2} e^{-j\frac{10\pi}{6}} = -1 + \frac{1}{2}(-1) + \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}j\left(-\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = -1 = e^{j\pi}$$

$$X_2 = -1 + \frac{1}{2}\left(-\frac{1}{2} - \frac{1}{2}\right) + \frac{1}{2}j\left(+\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = -\frac{3}{2} + \frac{\sqrt{3}}{2}j = \sqrt{3}\, e^{j\frac{5\pi}{6}}$$

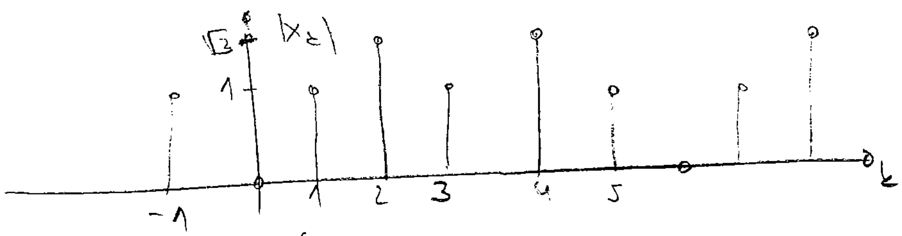
$$X_3 = -1 + \frac{1}{2}(1 - 1) + \frac{1}{2}j(0) = -1 = e^{j\pi}$$

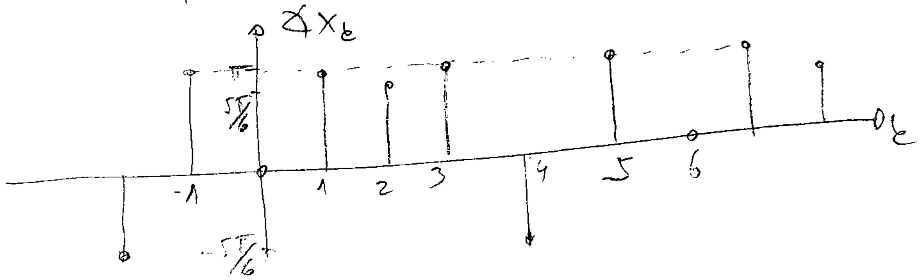
$$X_4 = -1 + \frac{1}{2}\left(\cos\frac{8\pi}{3} + \cos\frac{20\pi}{3}\right) + \frac{1}{2}j\left(-\sin\frac{8\pi}{3} - \sin\frac{20\pi}{3}\right) = -1 + \frac{1}{2}\left(-\frac{1}{2} - \frac{1}{2}\right) + \frac{1}{2}j\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)$$

$$= -\frac{3}{2} - \frac{\sqrt{3}}{2}j = \sqrt{3}\, e^{-j\frac{5\pi}{6}}$$

$$X_5 = -1 + \frac{1}{2}\left(\cos\frac{10\pi}{3} + \cos\frac{25\pi}{3}\right) + \frac{1}{2}j\left(-\sin\frac{10\pi}{3} - \sin\frac{25\pi}{3}\right) = -1 + \frac{1}{2}j\cdot 0 = -1 = e^{j\pi}$$

d)





(B)

4. $X(e^{j\Omega}) = \Omega + 2\pi \quad , -\pi < \Omega \le \pi$

a) $x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\Omega + 2\pi) e^{j\Omega n} d\Omega$

$= \left| \begin{array}{ll} \Omega + 2\pi = u & e^{j\Omega n} d\Omega = dv \\ d\Omega = du & \frac{e^{j\Omega n}}{jn} = v \end{array} \right|$

$= \frac{1}{2\pi} \left((\Omega + 2\pi) \frac{e^{j\Omega n}}{jn} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{e^{j\Omega n}}{jn} d\Omega \right)$

$= \frac{1}{2\pi} \left(\frac{\pi + 2\pi}{jn} e^{j\pi n} - \frac{(-\pi + 2\pi)}{jn} e^{-j\pi n} - \frac{e^{j\Omega n}}{-1n^2} \Big|_{-\pi}^{\pi} \right)$

$= \frac{1}{2\pi} \left(\frac{3\pi}{jn} e^{j\pi n} - \frac{\pi}{jn} e^{-j\pi n} + \frac{1}{n^2} (e^{j\pi n} - e^{-j\pi n}) \right)$

$= \frac{1}{2jn} (3e^{j\pi n} - 1e^{-j\pi n}) + \frac{1}{2\pi n^2} 2j \sin \pi n$

$= \frac{1}{2jn} (3 \cos \pi n + \cancel{3j \sin \pi n} - \cos \pi n + \cancel{j \sin \pi n})$

$= \frac{2 \cos \pi n}{2jn} = -j \frac{\cos \pi n}{n} = -\frac{j}{n} (-1)^n$

za $n = 0$

$x(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\Omega + 2\pi) d\Omega = \frac{1}{2\pi} \left(\frac{\Omega^2}{2} + 2\pi\Omega \right) \Big|_{-\pi}^{\pi}$

$= \frac{1}{2\pi} \left(\frac{\pi^2}{2} + 2\pi^2 - \left(\frac{\pi^2}{2} - 2\pi^2 \right) \right)$

$= \frac{1}{2\pi} (24\pi^2) = 2\pi$

b)

$$E = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\Omega})|^2 d\Omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\Omega + 2\pi)^2 d\Omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\Omega^2 + 4\pi\Omega + 4\pi^2) d\Omega$$

$$= \frac{1}{2\pi} \left(\frac{\Omega^3}{3} + 4\pi \frac{\Omega^2}{2} + 4\pi^2 \Omega \right) \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \left(\frac{\pi^3}{3} + \frac{4\pi^3}{2} + 4\pi^3 - \left(\frac{-\pi^3}{3} + \frac{4\pi^3}{2} - 4\pi^3 \right) \right)$$

$$= \frac{1}{2\pi} \left(\frac{2\pi^3}{3} + 8\pi^3 \right) = \frac{1}{2\pi} \frac{\overset{13}{26}\pi^3}{3} = \frac{13\pi^2}{3}$$

c)

$$y(n) = e^{j3\pi n} x(n)$$

DTFT

$$Y(e^{j\Omega}) = X(e^{j(\Omega - 3\pi)})$$

$$= (\Omega - 3\pi) + 2\pi$$

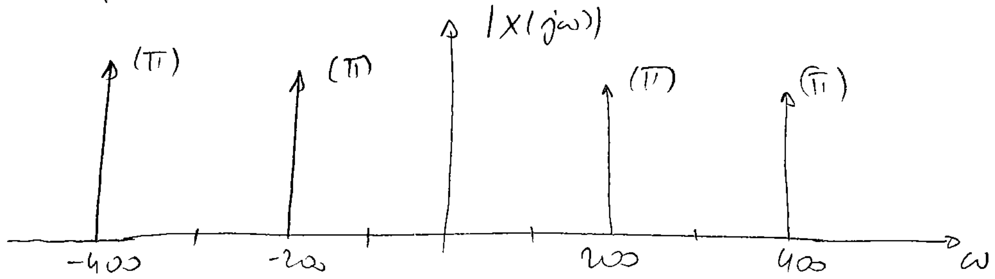
$$= \Omega - \pi$$

1B

5. $x(t) = \cos 200t + \cos 400t$

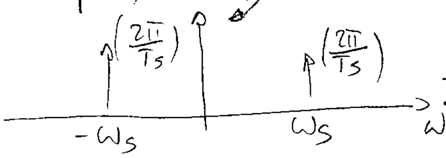
a) CTFT

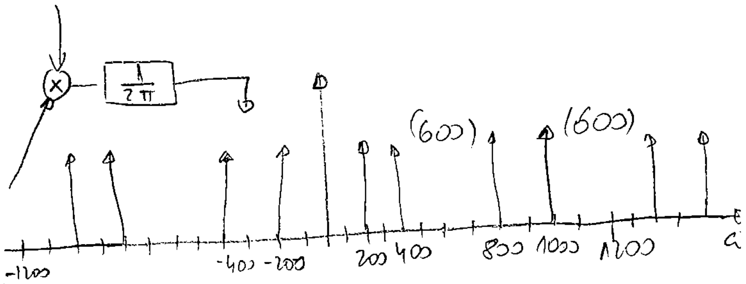
$X(j\omega) = \pi\left(\delta(\omega-200) + \delta(\omega+200)\right) + \pi\left(\delta(\omega-400) + \delta(\omega+400)\right)$



b) $\omega_s = 1200$

$x(t)\cdot \text{comb}_T(t) \longrightarrow$ spektar



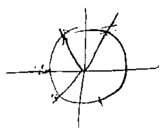


amplituda:

$A = \frac{2\pi}{T_s}\cdot\pi\cdot\frac{1}{2\pi} = \frac{\pi}{T_s} = \frac{\frac{\pi}{1}}{\frac{2\pi}{\omega_s}} = \frac{\pi\cdot\omega_s}{2\pi} = 600$

c) $T_s = \frac{2\pi}{1200}$

$x(nT_s) = \cos\left(200\frac{2\pi}{1200}n\right) + \cos\left(400\frac{2\pi}{1200}n\right) = \cos\left(\frac{\pi}{3}n\right) + \cos\left(\frac{2\pi}{3}n\right)$



$x(0) = 1 + 1 = 2$

$x(1) = \frac{1}{2} - \frac{1}{2} = 0$

$x(2) = -\frac{1}{2} - \frac{1}{2} = -1$ $\qquad x(n) = \{\underline{2}, 0, -1, 0, -1, 0\}$

$x(3) = -1 + 1 = 0$

$x(4) = -\frac{1}{2} - \frac{1}{2} = -1$

$x(5) = \frac{1}{2} - \frac{1}{2} = 0$

DFT_6

$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{5} x(n)e^{-j\frac{2\pi}{6}kn} = 2 - e^{-j\frac{2\pi}{6}2k} - e^{-j\frac{2\pi}{6}4k}$

$= 2 - e^{-j\frac{2\pi}{6}3k}\left(e^{j\frac{2\pi}{6}k} + e^{-j\frac{2\pi}{6}k}\right) = 2 - 2\cos\frac{\pi}{3}k\cdot(\cos\pi k - j\sin\pi k)$

$= 2 - 2\cos\frac{\pi}{3}k\cdot\cos\pi k$

$X(0) = 2 - 2 = 0$

$X(1) = 2 - 2\cos\frac{\pi}{3}\cos\pi = 2 - 2\cdot\frac{1}{2}\cdot(-1) = 3$

$X(2) = 2 - 2\cos\frac{2\pi}{3}\cos 2\pi = 2 + 2\cdot\frac{1}{2} = 3$ $\qquad X(k) = \{\underline{0}, 3, 3, 0, 3, 3\}$

$X(3) = 2 - 2\cos\pi\cos 3\pi = 0$

$X(4) = 2 - 2\cos\frac{4\pi}{3}\cos 4\pi = 2 - 2\cdot\left(-\frac{1}{2}\right) = 3$

$X(5) = 2 - 2\cos\frac{5\pi}{3}\cos 5\pi = 2 - 2\cdot\frac{1}{2}(-1) = 3$

d) $\omega_s = 600$

$\Omega = \frac{2\pi}{N} \cdot k$

$\Omega = \omega T_s = \omega \frac{2\pi}{\omega_s}$

$\omega = \frac{\omega_s}{2\pi} \cdot \Omega = \frac{\omega_s}{2\pi} \cdot \frac{2\pi}{N} k = \frac{\omega_s}{N} \cdot k$

$\omega_1 = \frac{1200}{6} \cdot 1 = 200$ rad/s

$\omega_3 = \frac{1200}{6} \cdot 3 = 600$ rad/s

e) spektralna rezolucija ω_0

6 uzoraka → 5 razmaka između njih

$\omega_s = 1200$ rad/s

$\omega_0 = \frac{1200}{5} = 240$ rad/s

ili

$N = 6$

$N_{\omega_s} \cdot \omega_0 = \omega_s \rightarrow \omega_0 = \frac{\omega_s}{N} = \frac{1200}{6} = 200$ rad/s

f) za $\omega_0 = 10$, $\omega_s = 1200$ $\omega_s = 2\pi f_s$

trajanje signala?

$f_s = \frac{\omega_s}{2\pi} = \frac{1200}{2\pi} = \frac{600}{\pi}$

$\omega_0 = \frac{1200}{N-1}$

$N - 1 = \frac{1200}{\omega_0} = 120$

$N = 121$ uzorak

trajanje signala $T_p = \frac{N}{f_s} = \frac{121}{\frac{600}{\pi}} = \frac{121\pi}{600} = 0{,}633$ s

ili

$N = \frac{\omega_s}{\omega_0} = \frac{1200}{10} = 120$

$T = \frac{N}{f_s} = \frac{N}{\frac{\omega_s}{2\pi}} = \frac{N \cdot 2\pi}{\omega_s} = \frac{120 \cdot 2\pi}{1200} = \frac{\pi}{5}$ rad/s