Signali i sustavi
Završni ispit – 24. lipnja 2008.

1. Zadan je kontinuirani sustav $y''(t) + 5y'(t) + 6y(t) = u(t)$. Pronađite:

a) odziv sustava u vremenskoj domeni ako je sustav pobuđen signalom $u(t) = (12t + 16)\,\mu(t)$ te ako su početni uvjeti $y(0^-) = 3$, $y'(0^-) = -8$. Rješavati bez korištenja Laplaceove transformacije,

b) amplitudnu i faznu karakteristiku sustava,

c) impulsni odziv sustava. Riješiti pomoću Laplaceove transformacije.

2. Zadan je diskretni sustav

$$y(n) - \frac{1}{2}y(n-1) = u(n).$$

Ako je sustav pobuđen signalom $u(n) = 2\sin\left(\frac{\pi}{3}n + \frac{\pi}{6}\right)$ te ako je početni uvjet $y(-1) = 1$, odredite:

a) prijenosnu funkciju sustava,

b) je li sustav stabilan i obrazložite zašto,

c) frekvencijsku karakteristiku sustava i pomoću nje prisilni odziv sustava,

d) totalni odziv sustava.

3. Diskretan kauzalan LTI sustav zadan je prijenosnom funkcijom

$$H(z) = \frac{z^{-1} + z^{-2} + z^{-3}}{(1 + 2z^{-1} + 3z^{-2})\,(1 - 2z^{-1})}.$$

Pronađite odziv mirnog sustava na kauzalnu pobudu $u(n) = \{\underline{1}, 2, 3, 1, 2, 3, \ldots\}$. Uočite da se $(1, 2, 3)$ ponavlja.

4. Vremenski kontinuiran sustav zadan je sustavom diferencijalnih jednadžbi:

$$\begin{aligned} y_1'(t) - 2y_2(t) &= u(t) \\ y_2'(t) + 3y_1(t) + 5y_2(t) &= u(t) \end{aligned}$$

Odredite:

a) matrice sustava **A, B, C, D** uz varijable stanja $x_1 = y_1, x_2 = y_2$,

b) prijenosnu matricu sustava,

c) impulsni odziv sustava,

d) odziv sustava na pobudu $u(t) = \delta(t) + 6e^{3t}\,\mu(t)$ uz početne uvjete jednake nuli.

5. Impulsni odziv diskretnog LTI sustava je $h(n) = \{\ldots, 0, 0, \underline{1}, 2, 3, 2, 1, 0, 0, \ldots\}$. Uočite da je niz različit od nule samo u pet točaka. Odredite:

a) vremenski diskretnu Fourierovu transformaciju (DTFT) impulsnog odziva $h(n)$,

b) diskretnu Fourierovu transformaciju (DFT) u 5 točaka za točke različite od nule,

c) energiju signala,

d) koja od izračunatih transformacija odgovara frekvencijskoj karakteristici zadanog sustava.

A

1. $y''(t) + 5y'(t) + 6y(t) = u(t)$

a) $u(t) = (12t + 16)\mu(t)$

$y(0^-) = 3 \rightarrow y(0^+) = 3$

$y'(0^-) = -8 \rightarrow y'(0^+) = -8$

$s^2 + 5s + 6 = 0 \qquad (s+2)(s+3) = 0$

$s_1 = -2$

$s_2 = -3$

$y_h(t) = C_1 e^{-2t} + C_2 e^{-3t}$

$$\left.\begin{aligned} y_p &= A + Bt \\ y_p'(t) &= B \\ y_p''(t) &= 0 \end{aligned}\right\} \quad 5B + 6A + 6Bt = 12t + 16$$

$6B = 12 \qquad 5B + 6A = 16$

$B = 2 \qquad 6A = 6$

$A = 1$

$y_p(t) = 1 + 2t$

$y_t(t) = C_1 e^{-2t} + C_2 e^{-3t} + 1 + 2t$

$y_t'(t) = -2C_1 e^{-2t} - 3C_2 e^{-3t} + 2$

$y_t(0^+) = C_1 + C_2 + 1 = 3 \qquad C_1 + C_2 = 2$

$y_t'(0^+) = -2C_1 - 3C_2 + 2 = -8 \qquad -2C_1 - 3C_2 = -10$

$-C_2 = -6$

$\boxed{y_t(t) = (-4e^{-2t} + 6e^{-3t} + 1 + 2t)\mu(t)} \qquad C_2 = 6 \qquad C_1 = -4$

b) $H(s) = \dfrac{Y(s)}{U(s)} = \dfrac{1}{s^2 + 5s + 6}$

$U(s) = 1$

$H(j\Omega) = \dfrac{1}{(j\Omega)^2 + 5j\Omega + 6} = \dfrac{1}{6 - \Omega^2 + j \cdot 5\Omega}$

$|H(j\Omega)| = \dfrac{1}{\sqrt{(6-\Omega^2)^2 + 25\Omega^2}} = \dfrac{1}{\sqrt{36 - 12\Omega^2 + \Omega^4 + 25\Omega^2}}$

$= \dfrac{1}{\sqrt{36 + 13\Omega^2 + \Omega^4}}$

$\measuredangle H(j\Omega) = -\operatorname{arctg} \dfrac{5\Omega}{6 - \Omega^2}$

c) $u(t) = \delta(t)$

$U(s) = 1$

$Y(s) = \dfrac{1}{s^2 + 5s + 6} = \dfrac{A}{s+2} + \dfrac{B}{s+3}$

$A + B = 0 \qquad A = -B$

$3A + 2B = 1 \rightarrow -3B + 2B = 1$

$B = -1$

$A = 1$

$= \dfrac{1}{s+2} - \dfrac{1}{s+3}$

$y(t) = (e^{-2t} - e^{-3t})\mu(t)$

A

2. $y(n) - \frac{1}{2} y(n-1) = u(n)$

a) $Y(z) - \frac{1}{2} z^{-1} Y(z) = U(z)$

$H(z) = \frac{Y(z)}{U(z)} = \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z-\frac{1}{2}}$

b) $z = \frac{1}{2}$ POL

$|z| < 1 \rightarrow$ STABILAN SUSTAV

c) $H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} = \frac{1}{1-\frac{1}{2}\cos\omega + \frac{1}{2} j \sin\omega}$

$|H(e^{j\omega})| = \frac{1}{\sqrt{(1-\frac{1}{2}\cos\omega)^2 + (\frac{1}{2}\sin\omega)^2}} = \frac{1}{\sqrt{1-\cos\omega + \underbrace{\frac{1}{4}\cos^2\omega + \frac{1}{4}\sin^2\omega}_{\frac{1}{4}}}}$

$= \frac{1}{\sqrt{\frac{5}{4} - \cos\omega}}$

$\sphericalangle H(e^{j\omega}) = -\operatorname{arctg} \frac{\frac{1}{2}\sin\omega}{1-\frac{1}{2}\cos\omega}$

pobuda:

$u(n) = 2 \sin(\frac{\pi}{3} n + \frac{\pi}{6})$

$y_p(n) = U |H(e^{j\omega})| \sin(\omega n + \varphi + \sphericalangle H(e^{j\omega}))$

$|H(e^{j\frac{\pi}{3}})| = \frac{1}{\sqrt{\frac{5}{4} - \cos\frac{\pi}{3}}} = \frac{1}{\sqrt{\frac{5}{4}-\frac{1}{2}}} = \frac{1}{\sqrt{\frac{3}{4}}} = \frac{2}{\sqrt{3}}$

$\sphericalangle H(e^{j\frac{\pi}{3}}) = -\operatorname{arctg} \frac{\frac{1}{2}\cdot\frac{\sqrt{3}}{2}}{1-\frac{1}{2}\cdot\frac{1}{2}} = -\operatorname{arctg} \frac{\frac{\sqrt{3}}{4}}{\frac{3}{4}} = -\operatorname{arctg} \frac{\sqrt{3}}{3} = -\frac{\pi}{6}$

$y_p(n) = 2 \cdot \frac{2}{\sqrt{3}} \sin(\frac{\pi}{3} n + \frac{\pi}{6} - \frac{\pi}{6})$

$= \frac{4}{\sqrt{3}} \sin(\frac{\pi}{3} n)$

d) $y_{TOT}(n) = y_h(n) + y_p(n)$

$y_h(n) = (\frac{1}{2})^n \cdot C_1$

$y_{TOT}(n) = C_1 (\frac{1}{2})^n + \frac{4}{\sqrt{3}} \sin(\frac{\pi}{3} n)$

$y_{TOT}(-1) = 1$

$y_{TOT}(-1) = 2C_1 + \frac{4}{\sqrt{3}} \sin(-\frac{\pi}{3}) = 2C_1 + \frac{\cancel{4}^{2}}{\cancel{\sqrt{3}}} \cdot \frac{-\cancel{\sqrt{3}}}{\cancel{2}} = 2C_1 - 2 = 1$

$2C_1 = 3$

$C_1 = \frac{3}{2}$

$y_{TOT}(n) = \frac{3}{2} (\frac{1}{2})^n + \frac{4}{\sqrt{3}} \sin(\frac{\pi}{3} n)$

3. $$H(z) = \frac{z^{-1}+z^{-2}+z^{-3}}{(1+2z^{-1}+3z^{-2})(1-2z^{-1})} = \frac{z^{-3}(z^2+z+1)}{z^{-2}\cancel{z^{-1}}(z^2+2z+3)(z-2)}$$

$$= \frac{z^2+z+1}{(z^2+2z+3)(z-2)}$$

$$u(n) = \{\underline{1}, 2, 3, 1, 2, 3, \dots\}$$

$$U(z) = z^0 + 2z^{-1} + 3z^{-2} + z^{-3} + 2z^{-4} + 3z^{-5} + \dots$$

$$= z^0(1+z^{-3}+z^{-6}+\dots) + 2z^{-1}(1+z^{-3}+z^{-6}) + 3z^{-2}(1+z^{-3}+z^{-6}+\dots)$$

$$= (1+z^{-3}+z^{-6}+\dots)(1+2z^{-1}+3z^{-2})$$

$$= \frac{z^2+2z+3}{z^2} \cdot \sum_{n=0}^{\infty} (z^{-3})^n =$$

$$= \frac{z^2+2z+3}{z^2} \quad \frac{1}{1-z^{-3}} = \frac{z^2+2z+3}{\cancel{z^2}} \cdot \frac{z^{\cancel{3}}}{z^3-1}$$

$$= \frac{(z^2+2z+3)z}{(z-1)(z^2+z+1)}$$

$$H(z) = \frac{Y(z)}{U(z)}$$

$$Y(z) = H(z) \cdot U(z)$$

$$= \frac{\cancel{z^2+z+1}}{(\cancel{z^2+2z+3})(z-2)} \cdot \frac{(\cancel{z^2+2z+3})z}{(z-1)(\cancel{z^2+z+1})} = \frac{z}{(z-2)(z-1)}$$

$$\frac{Y(z)}{z} = \frac{A}{z-2} + \frac{B}{z-1} = \frac{1}{(z-2)(z-1)}$$

z^1: $A + B = 0$

z^0: $-A - 2B = 1$

$-B = 1 \qquad A = 1$

$B = -1$

$$Y(z) = \frac{z}{z-2} - \frac{z}{z-1}$$

$$y(n) = 2^n \mu(n) - \mu(n)$$

A

4. $y_1' - 2y_2 = u$

$y_2' + 3y_1 + 5y_2 = u$

a) $x_1 = y_1$

$x_2 = y_2$

$\dot{x}_1 = u + 2y_2$

$= u + 2x_2$

$\dot{x}_2 = u - 3y_1 - 5y_2$

$= u - 3x_1 - 5x_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

b) $H(s) = C(sI - A)^{-1} B + D$

$$(sI - A)^{-1} = \begin{bmatrix} s & -2 \\ 3 & s+5 \end{bmatrix}^{-1} = \frac{1}{s^2+5s+6} \begin{bmatrix} s+5 & 2 \\ -3 & s \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s+5 & 2 \\ -3 & s \end{bmatrix} \frac{1}{s^2+5s+6} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+7}{s^2+5s+6} \\ \frac{s-3}{s^2+5s+6} \end{bmatrix}$$

c) $u(t) = \delta(t)$

$U(s) = 1$

$Y(s) = H(s)U(s)$

$$= \begin{bmatrix} \frac{s+7}{s^2+5s+6} \\ \frac{s-3}{s^2+5s+6} \end{bmatrix} = \begin{bmatrix} \frac{A_1}{s+2} + \frac{B_1}{s+3} \\ \frac{A_2}{s+2} + \frac{B_2}{s+3} \end{bmatrix}$$

$A_1 + B_1 = 1$

$3A_1 + 2B_1 = 7$

$-2A_1 - 2B_1 = -2$

$A_1 = 5$

$B_1 = -4$

$A_2 + B_2 = 1$

$3A_2 + 2B_2 = -3$

$-2A_2 - 2B_2 = -2$

$A_2 = -5$

$B_2 = 6$

$$= \begin{bmatrix} \frac{5}{s+2} - \frac{4}{s+3} \\ \frac{-5}{s+2} + \frac{6}{s+3} \end{bmatrix} \longrightarrow y(t) = \begin{matrix} (5e^{-2t} - 4e^{-3t})\mu(t) \\ (-5e^{-2t} + 6e^{-3t})\mu(t) \end{matrix}$$

d) $u(t) = \delta(t) + 6e^{3t}\mu(t)$

$U(s) = 1 + 6\frac{1}{s-3} = \frac{s+3}{s-3}$

$$Y(s) = \begin{bmatrix} \frac{s+7}{(s+2)\cancel{(s+3)}} \cdot \frac{\cancel{s+3}}{s-3} \\ \frac{\cancel{s-3}}{(s+2)\cancel{(s+3)}} \cdot \frac{\cancel{s+3}}{\cancel{s-3}} \end{bmatrix} = \begin{bmatrix} \frac{s+7}{(s+2)(s-3)} \\ \frac{1}{s+2} \end{bmatrix} = \begin{bmatrix} \frac{A}{s+2} + \frac{B}{s-3} \\ \frac{1}{s+2} \end{bmatrix}$$

$A + B = 1$

$-3A + 2B = 7$

$-2A - 2B = -2$

$-5A = 5$

$A = -1$

$B = 2$

$$= \begin{bmatrix} \frac{-1}{s+2} + \frac{2}{s-3} \\ \frac{1}{s+2} \end{bmatrix}$$

$$y(t) = \begin{bmatrix} (-e^{-2t} + 2e^{+3t})\mu(t) \\ (e^{-2t})\mu(t) \end{bmatrix}$$

5. $h(n) = \{\dots 0, \underline{1}, 2, 3, 2, 1, 0, \dots\}$ A

a) DTFT

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-i\omega n}$$

$$= 1e^{-i\omega\cdot 0} + 2e^{-i\omega 1} + 3e^{-i\omega\cdot 2} + 2e^{-i\omega\cdot 3} + e^{-i\omega\cdot 4}$$

$$= e^{-i\omega\cdot 2}\left[e^{j\omega 2} + e^{-i\omega\cdot 2} + 2\left(e^{+i\omega\cdot 1} + e^{-i\omega\cdot 1}\right) + 3e^{-i\omega\cdot 0}\right]$$

$$= e^{-j\omega\cdot 2}\left[2\cos 2\omega + 4\cos\omega + 3\right]$$

b) DFT

$$H(k) = \sum_{n=0}^{N-1} h(n)\, e^{-\frac{2\pi}{N}\cdot nk\cdot j}$$

$$= \sum_{n=0}^{4} h(n)\, e^{-j\cdot\frac{2\pi}{5}\cdot n\cdot k}$$

$$= e^{-i\frac{2\pi}{5}\cdot k\cdot 0} + 2e^{-i\frac{2\pi}{5}\cdot k\cdot 1} + 3e^{-i\frac{2\pi}{5}k\cdot 2} + 2e^{-i\frac{2\pi}{5}k\cdot 3} + e^{-i\frac{2\pi}{5}\cdot k\cdot 4} + \dots$$

$$= e^{-j\frac{4\pi}{5}k}\left(e^{j\frac{4\pi}{5}k} + 2\left(e^{j\frac{2\pi}{5}k} + e^{-i\frac{2\pi}{5}k}\right) + 3 + e^{-i\frac{4\pi}{5}k}\right]$$

$$= e^{-j\frac{4\pi}{5}k}\left[2\cos\frac{4\pi}{5}k + 4\cos\frac{2\pi}{5}k + 3\right]$$

c) $$E = \sum_{n=-\infty}^{\infty} |h(n)|^2$$

$$= 1^2 + 2^2 + 3^2 + 2^2 + 1^2 = 19$$

d) DTFT $H(e^{i\omega})$

$$H(z) = z^0 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4} + 0\dots$$

$$z = e^{i\omega}$$

$$H(e^{i\omega}) = e^{j\omega\cdot 0} + 2e^{-i\omega 1} + 3e^{-2i\omega} + 2e^{-i\omega\cdot 3} + e^{-i\omega\cdot 4}$$

$\downarrow$

frekvencijska karakteristika sustava
odgovara DTFT transformaciji.

Signali i sustavi

Završni ispit – 24. lipnja 2008.

1. Zadan je kontinuirani sustav $y''(t) + 6y'(t) + 8y(t) = u(t)$. Pronađite:

a) odziv sustava u vremenskoj domeni ako je sustav pobuđen signalom $u(t) = (8t + 22)\,\mu(t)$ te ako su početni uvjeti $y(0^-) = -5$, $y'(0^-) = 3$. Rješavati bez korištenja Laplaceove transformacije,

b) amplitudnu i faznu karakteristiku sustava,

c) impulsni odziv sustava. Riješiti pomoću Laplaceove transformacije.

2. Zadan je diskretni sustav

$$y(n) - \frac{1}{\sqrt{2}} y(n-1) = u(n).$$

Ako je sustav pobuđen signalom $u(n) = 2\sin\left(\frac{\pi}{4}n + \frac{\pi}{4}\right)$ te ako je početni uvjet $y(-1) = 2$, odredite:

a) prijenosnu funkciju sustava,

b) je li sustav stabilan i obrazložite zašto,

c) frekvencijsku karakteristiku sustava i pomoću nje prisilni odziv sustava,

d) totalni odziv sustava.

3. Diskretan kauzalan LTI sustav zadan je prijenosnom funkcijom

$$H(z) = \frac{z^{-1} + z^{-2} + z^{-3}}{(3 + 2z^{-1} + z^{-2})\,(1 - 4z^{-1})}.$$

Pronađite odziv mirnog sustava na kauzalnu pobudu $u(n) = \{\underline{3}, 2, 1, 3, 2, 1, \ldots\}$. Uočite da se $(3, 2, 1)$ ponavlja.

4. Vremenski kontinuiran sustav zadan je sustavom diferencijalnih jednadžbi:

$$\begin{aligned} y_1'(t) - 2y_2(t) &= u(t) \\ y_2'(t) + 4y_1(t) + 6y_2(t) &= u(t) \end{aligned}$$

Odredite:

a) matrice sustava **A, B, C, D** uz varijable stanja $x_1 = y_1, x_2 = y_2$,

b) prijenosnu matricu sustava,

c) impulsni odziv sustava,

d) odziv sustava na pobudu $u(t) = \delta(t) + 8e^{4t}\,\mu(t)$ uz početne uvjete jednake nuli.

5. Impulsni odziv diskretnog LTI sustava je $h(n) = \{\ldots, 0, 0, \underline{3}, 2, 1, 2, 3, 0, 0, \ldots\}$. Uočite da je niz različit od nule samo u pet točaka. Odredite:

a) vremenski diskretnu Fourierovu transformaciju (DTFT) impulsnog odziva $h(n)$,

b) diskretnu Fourierovu transformaciju (DFT) u 5 točaka za točke različite od nule,

c) energiju signala,

d) koja od izračunatih transformacija odgovara frekvencijskoj karakteristici zadanog sustava.

B

1. $y''(t) + 6y'(t) + 8y(t) = u(t)$

a) homogeno:

$s^2 + 6s + 8 = 0 \qquad (s+2)(s+4) = 0$

$s_1 = -2$

$s_2 = -4$

$y_h(t) = c_1 e^{-2t} + c_2 e^{-4t}$

partikularno

$u(t) = (8t + 22)\mu(t)$

$\left.\begin{array}{l} y_p(t) = A + Bt \\ y_p'(t) = B \\ y_p''(t) = 0 \end{array}\right\} \qquad 6B + 8A + 8Bt = 8t + 22$

$8B = 8 \qquad 6B + 8A = 22$

$B = 1 \qquad 8A = 22 - 6 = 16$

$A = 2$

$y_p(t) = 2 + t$

totalno

$y(t) = (c_1 e^{-2t} + c_2 e^{-4t} + 2 + t)\mu(t)$

konstante

$y(0^-) = -5 \rightarrow y(0^+) = -5$

$y'(0^-) = 3 \rightarrow y'(0^+) = 3$

$y'(t) = -2c_1 e^{-2t} + (-4)c_2 e^{-4t} + 1$

$y(0^+) = c_1 + c_2 + 2 = -5 \qquad c_1 + c_2 = -7$

$y'(0^+) = -2c_1 - 4c_2 + 1 = 3 \qquad -2c_1 - 4c_2 = 2$

$2c_1 = -26$

$c_1 = -13 \qquad c_2 = 6$

$\boxed{y(t) = (-13e^{-2t} + 6e^{-4t} + 2 + t)\mu(t)}$

b) $H(s) = \dfrac{Y(s)}{U(s)} = \dfrac{1}{s^2 + 6s + 8}$

$H(j\Omega) = \dfrac{1}{(j\Omega)^2 + 6j\Omega + 8} = \dfrac{1}{8 - \Omega^2 + j \cdot 6\Omega}$

$|H(j\Omega)| = \dfrac{1}{\sqrt{(8-\Omega^2)^2 + 36\Omega^2}} = \dfrac{1}{\sqrt{64 - 16\Omega^2 + \Omega^4 + 36\Omega^2}}$

$\boxed{|H(j\Omega)| = \dfrac{1}{\sqrt{64 + 20\Omega^2 + \Omega^4}} \qquad \measuredangle H(j\Omega) = -\operatorname{arctg}\dfrac{6\Omega}{8 - \Omega^2}}$

c) $U(s) = 1 \leftarrow u(t) = \delta(t)$

$Y(s) = \dfrac{1}{s^2 + 6s + 8} = \dfrac{1}{(s+2)(s+4)}$

$= \dfrac{A}{s+2} + \dfrac{B}{s+4} \longrightarrow (s+4)A + (s+2)B = 1$

$A + B = 0 \rightarrow A = -B$

$4A + 2B = 1 \rightarrow -4B + 2B = 1$

$-2B = 1 \rightarrow B = -\frac{1}{2}$

$A = \frac{1}{2}$

$= \frac{1}{2} \cdot \dfrac{1}{s+2} - \frac{1}{2} \dfrac{1}{s+4}$

$\boxed{y(t) = \left(\frac{1}{2} e^{-2t} - \frac{1}{2} e^{-4t}\right)\mu(t)}$

B

2. $y(n) - \frac{1}{\sqrt{2}} y(n-1) = u(n)$

a) Prijenosna funkcija

$$Y(z) - \frac{1}{\sqrt{2}} z^{-1} Y(z) = U(z)$$

$$H(z) = \frac{Y(z)}{U(z)} = \frac{1}{1 - \frac{1}{\sqrt{2}} z^{-1}} = \boxed{\frac{z}{z - \frac{1}{\sqrt{2}}}}$$

b) $z - \frac{1}{\sqrt{2}} = 0$

$z = \frac{1}{\sqrt{2}} \rightarrow$ POL

$|z| < 1 \rightarrow$ MANJI JE OD 1 PO APS. VRIJEDNOSTI $\rightarrow$ STABILAN SUSTAV

c) $$H(e^{j\omega}) = \frac{1}{1 - \frac{1}{\sqrt{2}} e^{-j\omega}} = \frac{1}{1 - \frac{1}{\sqrt{2}} \cos\omega + j \frac{1}{\sqrt{2}} \sin\omega}$$

$$|H(e^{j\omega})| = \frac{1}{\sqrt{(1 - \frac{1}{\sqrt{2}} \cos\omega)^2 + (\frac{1}{\sqrt{2}} \sin\omega)^2}} = \frac{1}{\sqrt{1 - \frac{2}{\sqrt{2}} \cos\omega + \underbrace{\frac{1}{2}\cos^2\omega + \frac{1}{2}\sin^2\omega}_{\frac{1}{2}}}}$$

$$\boxed{|H(e^{j\omega})| = \frac{1}{\sqrt{\frac{3}{2} - \frac{2}{\sqrt{2}} \cos\omega}}}$$

$$\boxed{\measuredangle H(e^{j\omega}) = -\text{arctg} \frac{\frac{1}{\sqrt{2}} \sin\omega}{1 - \frac{1}{\sqrt{2}} \cos\omega}}$$

pobuda $u(n) = 2 \sin(\frac{\pi}{4} n - \frac{\pi}{4})$

$\omega = \frac{\pi}{4}$

$$|H(e^{j\frac{\pi}{4}})| = \frac{1}{\sqrt{\frac{3}{2} - \frac{\cancel{2}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\cancel{2}}}} = \frac{1}{\sqrt{\frac{1}{2}}} = \sqrt{2}$$

$$\measuredangle H(e^{j\frac{\pi}{4}}) = -\text{arctg} \frac{\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{2}}{1 - \frac{1}{\sqrt{2}} \frac{\sqrt{2}}{2}} = -\text{arctg} \frac{\frac{1}{2}}{\frac{1}{2}} = -\text{arctg}\, 1 = -\frac{\pi}{4}$$

$$y_P(n) = U \cdot |H(e^{j\omega})| \sin(\omega n + \varphi + \measuredangle H(e^{j\omega}))$$

$$= 2 \cdot \sqrt{2} \sin(\frac{\pi}{4} n + \frac{\pi}{4} + (-\frac{\pi}{4}))$$

$$\boxed{y_P(n) = 2\sqrt{2} \sin(\frac{\pi}{4} n)}$$

d) $y_h(n) = C_1 g^n$

$= C_1 (\frac{1}{\sqrt{2}})^n$

$y_{TOT}(n) = C_1 (\frac{1}{\sqrt{2}})^n + 2\sqrt{2} \sin \frac{\pi}{4} n$

$y_{TOT}(-1) = 2$

$$y_{TOT}(-1) = C_1 \cdot \sqrt{2} + 2\sqrt{2} \sin(-\frac{\pi}{4}) = \sqrt{2} C_1 + \cancel{2}\sqrt{2} \cdot \frac{-\sqrt{2}}{\cancel{2}} = \sqrt{2} C_1 - 2 = 2$$

$$C_1 = \frac{4\sqrt{2}}{\sqrt{2}\sqrt{2}} = 2\sqrt{2}$$

$$\boxed{y_{TOT}(n) = 2\sqrt{2} (\frac{1}{\sqrt{2}})^n + 2\sqrt{2} \sin \frac{\pi}{4} n}$$

$$3.\quad H(z) = \frac{z^{-1} + z^{-2} + z^{-3}}{(3 + 2z^{-1} + z^{-2})(1 - 4z^{-1})}$$ B

$$= \frac{\cancel{z^{-3}}(z^2 + z + 1)}{z^{-2}\cdot\cancel{z^{-1}}(3z^2 + 2z + 1)(z-4)} = \frac{z^2 + z + 1}{(3z^2 + 2z + 1)(z-4)}$$

$$u(n) = \{\underline{3}, 2, 1, 3, 2, 1, \dots\}$$

$$U(z) = 3z^0 + 2z^{-1} + z^{-2} + 3z^{-3} + 2z^{-4} + z^{-5} + \dots$$

$$= 3z^0(1 + z^{-3} + z^{-6} + \dots) + 2z^{-1}(1 + z^{-3} + z^{-6} + \dots) + z^{-2}(1 + z^{-3} + z^{-6} + \dots)$$

$$= (3z^0 + 2z^{-1} + z^{-2})(1 + z^{-3} + z^{-6} + \dots)$$

$$= \frac{3z^2 + 2z + 1}{z^2} \cdot \sum_{n=0}^{\infty} (z^{-3})^n$$

$$= \frac{3z^2 + 2z + 1}{z^2} \cdot \frac{1}{1 - z^{-3}}$$

$$= \frac{3z^2 + 2z + 1}{\cancel{z^2}} \cdot \frac{z^{\cancel{3}}}{z^3 - 1}$$

$$= \frac{(3z^2 + 2z + 1)\cdot z}{(z-1)(z^2 + z + 1)}$$

$$H(z) = \frac{Y(z)}{U(z)}$$

$$Y(z) = H(z)\cdot U(z)$$

$$= \frac{\cancel{(3z^2 + 2z + 1)}z}{(z-1)\cancel{(z^2 + z + 1)}} \cdot \frac{\cancel{z^2 + z + 1}}{(z-4)\cancel{(3z^2 + 2z + 1)}}$$

$$= \frac{z}{(z-1)(z-4)}$$

$$\frac{Y(z)}{z} = \frac{1}{(z-1)(z-4)} = \frac{A}{z-1} + \frac{B}{z-4}$$

$$z^1: \quad A + B = 0$$

$$z^0: \quad -4A - B = 1$$

$$-3A = 1$$

$$A = -\frac{1}{3} \qquad B = \frac{1}{3}$$

$$Y(z) = -\frac{1}{3}\,\frac{z}{z-1} + \frac{1}{3}\,\frac{z}{z-4}$$

$$y(n) = -\frac{1}{3}\mu(n) + \frac{1}{3}(4)^n\mu(n)$$

B

4. $y_1' - 2y_2 = u$

$y_2' + 4y_1 + 6y_2 = u$

a) $x_1 = y_1$

$x_2 = y_2$

$\dot{x}_1 = u + 2y_2$

$= u + 2x_2$

$\dot{x}_2 = u - 4y_1 - 6y_2$

$= u - 4x_1 - 6x_2$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 2\\ -4 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix}u$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 0\end{bmatrix}u$$

b) $\dot{x} = Ax + Bu$

$\underline{y = Cx + Du}$

$(sI - A)X = BU$

$X = (sI - A)^{-1}BU$

$y = (C(sI-A)^{-1}B + D)U$

$$sI - A = \begin{bmatrix}s & -2\\ 4 & s+6\end{bmatrix}$$

$$(sI-A)^{-1} = \frac{1}{s^2+6s+8}\cdot\begin{bmatrix}s+6 & 2\\ -4 & s\end{bmatrix}$$

$$H(s) = \frac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D$$

$$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{s+6}{s^2+6s+8} & \frac{2}{s^2+6s+8}\\ \frac{-4}{s^2+6s+8} & \frac{s}{s^2+6s+8}\end{bmatrix}\cdot\begin{bmatrix}1\\ 1\end{bmatrix} + \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$= \begin{bmatrix}\frac{s+8}{s^2+6s+8}\\ \frac{s-4}{s^2+6s+8}\end{bmatrix}$$

c) Impulsni odziv $\to u(t) = \delta(t)$

$U(s) = 1$

$$H(s) = \begin{bmatrix}\frac{A_1}{(s+2)} + \frac{B_1}{(s+4)}\\ \frac{A_2}{s+2} + \frac{B_2}{s+4}\end{bmatrix}$$

$A_1 + B_1 = 1$

$\underline{4A_1 + 2B_1 = 8}$

$\underline{-4A_1 - 4B_1 = -4}$

$-2B_1 = 4$

$B_1 = -2$

$A_1 = 3$

$A_2 + B_2 = 1$

$\underline{4A_2 + 2B_2 = -4}$

$\underline{-4A_2 - 4B_2 = -4}$

$-2B_2 = -8$

$B_2 = 4$

$A_2 = -3$

$$= \begin{bmatrix}\frac{3}{s+2} - \frac{2}{s+4}\\ \frac{-3}{s+2} + \frac{4}{s+4}\end{bmatrix} \to h(t) = \begin{bmatrix}(3e^{-2t} - 2e^{-4t})\mu(t)\\ (-3e^{-2t} + 4e^{-4t})\mu(t)\end{bmatrix}$$

d) $u(t) = \delta(t) + 8e^{4t}\mu(t)$

$U(s) = 1 + 8\,\frac{1}{s-4} = \frac{s+4}{s-4}$

$$y(s) = H(s)\cdot U(s) = \begin{bmatrix}\frac{(s+8)\cancel{(s+4)}}{(s+2)\cancel{(s+4)}(s-4)}\\ \frac{\cancel{(s-4)}\cancel{(s+4)}}{(s+2)\cancel{(s+4)}\cancel{(s-4)}}\end{bmatrix} = \begin{bmatrix}\frac{s+8}{(s+2)(s-4)}\\ \frac{1}{s+2}\end{bmatrix} = \begin{bmatrix}\frac{A_3}{s+2} + \frac{B_3}{s-4}\\ \frac{1}{s+2}\end{bmatrix}$$

$A_3 + B_3 = 1$

$\to -4A_3 + 2B_3 = 8$

$\underline{-2A_3 - 2B_3 = -2}$

$-6A_3 = 6$

$A_3 = -1$

$B_3 = 2$

$$y(s) = \begin{bmatrix}\frac{-1}{s+2} + \frac{2}{s-4}\\ \frac{1}{s+2}\end{bmatrix} \to y(t) = \begin{bmatrix}(-e^{-2t} + 2e^{4t})\mu(t)\\ e^{-2t}\mu(t)\end{bmatrix}$$

B

5. $h(n) = \{\ldots 0, \underline{3}, 2, 1, 2, 3, 0, \ldots\}$

a) DTFT

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

$$= 3e^{-j\omega \cdot 0} + 2e^{-j\omega \cdot 1} + 1e^{-j\omega \cdot 2} + 2e^{-j\omega \cdot 3} + 3e^{-j\omega \cdot 4} + 0 \ldots$$

$$= e^{-j\omega 2}(3 \cdot 2\cos 2\omega + 2 \cdot 2\cos\omega + 1)$$

$$= e^{-j\omega \cdot 2}(6\cos 2\omega + 4\cos\omega + 1)$$

b) DFT

$$H(k) = \sum_{n=0}^{N-1} h(n) e^{-\frac{2\pi}{N} nk \cdot j}$$

$$= 3e^{-\frac{2\pi}{5} 0 k j} + 2e^{-\frac{2\pi}{5} \cdot 1 k \cdot j} + e^{-\frac{2\pi}{5} \cdot 2 k j} + 2e^{-\frac{2\pi}{5} \cdot 3 k j} + 3e^{-\frac{2\pi}{5} \cdot 4 k j}$$

$$= e^{-j\frac{4\pi}{5} \cdot k}\left[3 \cdot 2\cos\left(\frac{4\pi}{5}k\right) + 2 \cdot 2\cos\left(\frac{2\pi}{5}k\right) + 1\right]$$

$$= e^{-j\frac{4\pi}{5}k}\left[6\cos\frac{4\pi}{5}k + 4\cos\frac{2\pi}{5}k + 1\right)$$

c) $E = \sum_{n=-\infty}^{\infty} |h(n)|^2$

$$= 3^2 + 2^2 + 1^2 + 2^2 + 3^2$$

$$= 9 + 5 + 4 + 9 = 27$$

d)

$$H(z) = 3z^{-0} + 2z^{-1} + 1z^{-2} + 2z^{-3} + 3z^{-4} + \ldots$$

$$z = e^{+j\omega}$$

$$H(e^{j\omega}) = 3 + 3e^{-j\omega} + e^{-2j\omega} + 2e^{-j\omega \cdot 3} + 3e^{-j\omega \cdot 4}$$

što odgovara DTFT transformaciji