MASS. 3ciklus

2.

$$y(n) - \frac{1}{2} y(n-2) = u(n)$$

a) PRIJENOSNA FUNKCIJA

- omjer izlaza i ulaza uz nulte poč. uvjete $y(-1) = y(-2) = 0$

$y(n) \circ\!\!-\!\!\bullet\ Y(z)$

$y(n-2) \circ\!\!-\!\!\bullet\ z^{-2} Y(z) - \cancel{z^{-1} y(-1) - y(-2)}$

$u(n) \circ\!\!-\!\!\bullet\ U(z)$

$$Y(z) - \frac{1}{4} z^{-2} Y(z) = U(z)$$

$$H(z) = \frac{Y(z)}{U(z)} = \frac{1}{1 - \frac{1}{4} z^{-2}} = \frac{z^2}{z^2 - \frac{1}{4}}$$

b) A-F karakteristika

$H(j\omega) = H(s)$ CTFT

$H(e^{j\omega}) = H(z)$ DTFT

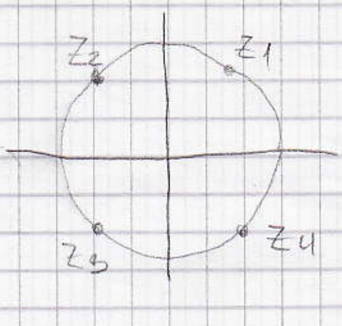
NESTABILNI SUSTAVI nemaju frekv. karakteristiku

$$H(e^{j\omega}) = \frac{e^{2j\omega}}{e^{2j\omega} - \frac{1}{4}} = \frac{\cos(2\omega) + j\sin(2\omega)}{\cos(2\omega) + j\sin(2\omega) - \frac{1}{4}}$$

$$A(\omega) = |H(e^{j\omega})| = \frac{\sqrt{\cos^2(2\omega) + \sin^2(2\omega)}}{\sqrt{\left(\cos(2\omega) - \frac{1}{4}\right)^2 + \left(\sin(2\omega)\right)^2}} = \frac{1}{\sqrt{\frac{17}{16} - \frac{1}{2}\cos(2\omega)}} \;//$$

$$\varphi(\omega) = \operatorname{arctg}\frac{\sin(2\omega)}{\cos(2\omega)} - \operatorname{arctg}\frac{\sin(2\omega)}{\cos(2\omega) - \frac{1}{4}} = 2\omega - \operatorname{arctg}\frac{\sin(2\omega)}{\cos(2\omega) - \frac{1}{4}} \;//$$

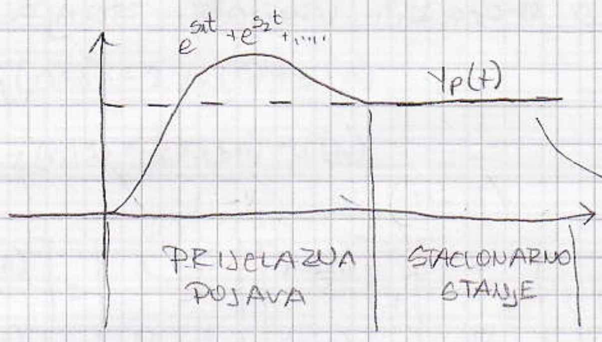
$z_1 = 1 + j$	$\measuredangle z_1 = 45^\circ$
$z_2 = -1 + j$	$\measuredangle z_2 = -45^\circ + 180^\circ = 135^\circ$
$z_3 = -1 - j$	$\measuredangle z_3 = 45^\circ + 180^\circ = 225^\circ$
$z_4 = 1 - j$	$\measuredangle z_4 = -45^\circ$



PRIJENOSNA FUNKCIJA SUSTAVA $H(s)$ je laplaceova transformacija impulsnog odziva $h(t)$

c) $u(n) = \sin\left(\frac{\pi}{2}n + \frac{\pi}{2}\right)$

STACIONARNO STANJE



Ako je sustav nestabilan staci. stanje ne postoji

Stacionarno stanje jednako je partikularnom rješenju (ali samo za stab. sustave)

$$y_p(n) = K\sin\left(\frac{\pi}{2}n + \psi\right)$$

$$K = 1 \cdot A(\omega)\Big|_{\omega=\frac{\pi}{2}} = 1 \cdot \frac{4}{5} = \frac{4}{5}$$

$$\psi = \frac{\pi}{2} + \varphi(\omega)\Big|_{\omega=\frac{\pi}{2}} = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

$\pi - \pi = 0$

$$y_p(n) = \frac{4}{5}\sin\left(\frac{\pi}{2}n + \psi\right) = \frac{4}{5}\sin\left(\frac{\pi}{2}n + \frac{\pi}{2}\right)$$

u kompleksnom načinu rada

shift + (−)

$$H(e^{j\omega})\Big|_{\omega=\frac{\pi}{2}} = \frac{e^{j\pi}}{e^{j\pi} - \frac{1}{4}} = \frac{1 \angle 180^\circ}{1\angle 180^\circ - \frac{1}{4}} = \frac{4}{5}\angle 0^\circ$$

$$u = Ue^{j\alpha} \longrightarrow \boxed{\text{SUSTAV}} \xrightarrow{y = Ye^{j\psi}}$$

$$H(e^{j\omega}) = A(\omega)e^{j\varphi(\omega)}$$

$$\underset{n\times 1}{\dot{x}(t)} = \underset{n\times n}{A}\,\underset{n\times 1}{x(t)} + \underset{n\times m}{B}\,\underset{m\times 1}{u(t)}$$

$$\underset{p\times 1}{Y(t)} = \underset{p\times n}{C}\,\underset{n\times 1}{x(t)} + \underset{p\times m}{D}\,\underset{m\times 1}{u(t)}$$

Koliki je red sustava toliko mora biti varijabli stanja

$X \in \mathbb{R}^{n\times 1}$ $\quad u \in \mathbb{R}^{m\times 1}$ $\quad Y \in \mathbb{R}^{p\times 1}$

$x(t) \circ\!\!-\!\!\bullet\ X(s)$

$\dot{x}(t) \circ\!\!-\!\!\bullet\ sX(s) - x(0^-)$

$u(t) \circ\!\!-\!\!\bullet\ U(s)$

$sX(s) - x(0^-) = AX(s) + BU(s)$

$sX(s) - AX(s) = BU(s) + x(0^-)$

$(sI - A)\,X(s) = BU(s) + x(0^-)$ / inverz matrice tj. $(sI-A)^{-1}$

mora biti s lijeve strane jer su s i A s lijeve strane

$$\boxed{X(s) = (sI-A)^{-1}BU(s) + (sI-A)^{-1}x(0^-)}$$

Odziv varijabli stanja sustava

$$\boxed{\phi(z) = z(zI-A)^{-1}} \quad \boxed{\phi(s) = (sI-A)^{-1}}$$ → matrica karakterističnih frekvencija

$$Y(s) = C(sI-A)^{-1}BU(s) + C(sI-A)^{-1}x(0^-) + DU(s)$$

$$\boxed{Y(s) = \underbrace{\overbrace{[C(sI-A)^{-1}B + D]}^{\text{MIRNI ODZIV SUSTAVA}}}_{\text{PRIJENOSNA MATRICA}}U(s) + \overbrace{C(sI-A)^{-1}x(0^-)}^{\text{NEPOBUĐENI ODZIV SUSTAVA}}}$$ → Odziv sustava

PRIJENOSNA MATRICA → mora imati dimenzije matrice D

$$\boxed{H(s) = C(sI-A)^{-1}B + D}$$

4.

$$y_1(n+1) - 2y_2(n) = u(n)$$

$$y_2(n+1) - 2y_1(n) = u(n)$$

$$x_1(n) = y_1(n), \quad x_2(n) = y_2(n)$$

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} u(n)$$

$$x_1(n+1) = 2x_2(n) + u(n)$$

$$x_2(n+1) = 2x_1(n) + u(n)$$

$$\begin{bmatrix} y_1(n) \\ y_2(n) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{D} u(n)$$

b)

$$(zI - A) = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} z & -2 \\ -2 & z \end{bmatrix}$$

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad M^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$(zI - A)^{-1} = \frac{1}{z^2-4} \begin{bmatrix} z & 2 \\ 2 & z \end{bmatrix}$$

$$\Phi(z) = z(zI - A)^{-1} = \frac{z}{z^2-4} \begin{bmatrix} z & 2 \\ 2 & z \end{bmatrix}$$

$$\Phi_{1,1}(z) = \frac{z^2}{z^2-4}, \quad \frac{\Phi_{1,1}(z)}{z} = \frac{z}{z^2-4} = \frac{z}{(z-2)(z+2)} = \frac{\frac{1}{2}}{z-2} + \frac{\frac{1}{2}}{z+2}$$

$$\Phi_{1,1}(z) = \frac{1}{2} \cdot \frac{z}{z-2} + \frac{1}{2} \frac{z}{z+2} \multimap \frac{1}{2}(2)^n + \frac{1}{2}(-2)^n = \frac{1}{2}\left[(2)^n + (-2)^n\right] \mu(n)$$

$$\Phi_{1,2}(z) = \frac{2z}{z^2-4}, \quad \frac{\Phi_{1,2}(z)}{z} = \frac{2}{z^2-4} = \frac{2}{(z-2)(z+2)} = \frac{\frac{1}{2}}{z-2} + \frac{-\frac{1}{2}}{z+2}$$

$$\Phi_{1,2}(z) = \frac{1}{2} \frac{z}{z-2} - \frac{1}{2} \frac{z}{z+2} \multimap \frac{1}{2}(2)^n - \frac{1}{2}(-2)^n = \frac{1}{2}\left((2)^n - (-2)^n\right) \mu(n)$$

$$\Phi(z) \bullet\!\!-\!\!\circ\ A^n = \frac{1}{2}\begin{bmatrix} 2^n+(-2)^n & 2^n-(-2)^n \\ 2^n-(-2)^n & 2^n+(-2)^n \end{bmatrix}\mu(n) \Rightarrow$$ fundamentalna matrica sustava

$$H(z) = \underbrace{C}_{\text{jedinična matrica}}(zI-A)^{-1}B + \underbrace{D}_{\text{nul matrica}} = (zI-A)^{-1}B = \begin{bmatrix} \frac{1}{z-2} \\ \frac{1}{z-2} \end{bmatrix} \Rightarrow$$ prijenosna matrica sustava

IMPULSNI ODZIV → $H(z)$ u vrem. domenu

$$H_{1,1}(z) = \frac{1}{z-2}, \quad \frac{H_{1,1}(z)}{z} = \frac{1}{z(z-2)} = \frac{-\frac{1}{2}}{z} + \frac{\frac{1}{2}}{z-2}$$

ne stoji kod $\delta(n)$

$$H_{1,1}(z) = -\frac{1}{2} + \frac{1}{2}\frac{z}{z-2} \bullet\!\!-\!\!\circ\ -\frac{1}{2}\delta(n) + \frac{1}{2}(2)^n \mu(n)$$

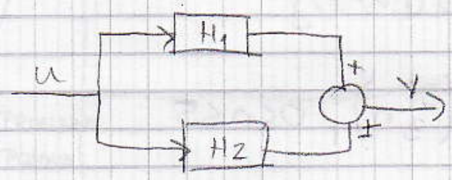
$$h(n) = \begin{bmatrix} -\frac{1}{2}\delta(n) + \frac{1}{2}2^n\mu(n) \\ -\frac{1}{2}\delta(n) + \frac{1}{2}2^n\mu(n) \end{bmatrix}$$

SLOŽENI SUSTAVI

(1.) KASKADA

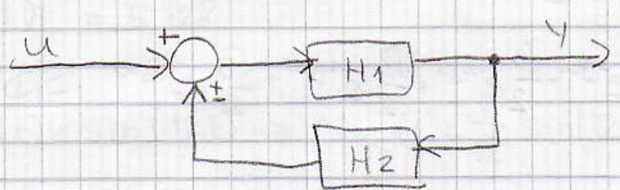
$$H = \frac{Y}{U} = H_1 \cdot H_2$$

(2.) PARALELA



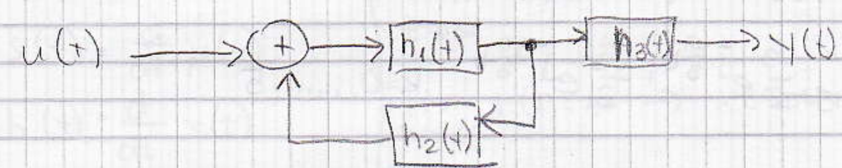
$$H = \frac{Y}{U} = H_1 \pm H_2$$

(3.) POVRATNA VEZA



$$H = \frac{Y}{U} = \frac{H_1}{1 \mp H_1 H_2}$$

(3.)



$$h_1(t) = \cos(2t)\,\mu(t) \circ\!\!-\!\!\bullet\, H_1(s) = \frac{s}{s^2+4}$$

$$h_2(t) = 13\,\mu(t) \circ\!\!-\!\!\bullet\, H_2(s) = \frac{13}{s}$$

$$h_3(t) = -48 e^{-5t} \mu(t) \circ\!\!-\!\!\bullet\, H_3(s) = \frac{-48}{s+5}$$

a)

$$H_{1,2}(s) = \frac{H_1(s)}{1 - H_1(s) H_2(s)} = \frac{\frac{s}{s^2+4}}{1 - \frac{\cancel{s}}{s^2+4} \cdot \frac{13}{\cancel{s}}} = \frac{s}{s^2-9}$$

$$H(s) = H_{1,2}(s) \cdot H_3(s) = \frac{s}{s^2-9} \cdot \frac{-48}{s+5} = \frac{-48s}{(s-3)(s+3)(s+5)}$$

b) $s_1 = 3$ NESTAB, $s_2 = -3$ STAB, $s_3 = -5$ STAB $\rightarrow$ NESTABILAN

$Re\{s\} < 0 \rightarrow$ STAB

c) $u(t) = \mu(t) \circ\!\!-\!\!\bullet\, U(s) = \frac{1}{s}$

$$Y_m(s) = H(s)\,U(s) = \frac{-48\cancel{s}}{\cancel{s}(s+3)(s+5)(s-3)} = \frac{4}{(s+3)} + \frac{-3}{s+5} + \frac{-1}{s-3} \bullet\!\!-\!\!\circ$$

$$Y_m(t) = \left(4e^{-3t} - 3e^{-5t} - e^{3t}\right)\mu(t)$$

① $x(t)=\sin\left(\frac{5\pi}{3}t\right)\left(\mu(t)-\mu(t-6)\right)$

a) $F_S = 1\,Hz$

$\Omega_S = 2\pi < \frac{10\pi}{3}$ → ima aliasinga

$$x(n)=x(t)\Big|_{t=nT_S} = \sin\left(\frac{5\pi}{3}\cdot n\right)\left[\mu(nT_S)-\mu(nT_S-6)\right]=$$

$$=\sin\left(\frac{5\pi}{3}n\right)\left(\mu(n)-\mu(n-6)\right)=\sin\left(\frac{5\pi}{3}n\right),\ 0\leq n\leq 5$$

b) $x(n)=\left\{\underline{0}, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right\}$

$$X(e^{j\omega})=\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = -\frac{\sqrt{3}}{2}e^{-j\omega}-\frac{\sqrt{3}}{2}e^{-2j\omega}+\frac{\sqrt{3}}{2}e^{-4j\omega}+\frac{\sqrt{3}}{2}e^{-5j\omega} \rightarrow DTFT$$

c) $$X(k)=\sum_{n=0}^{N-1}x(n)e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{5}x(n)e^{-j\frac{\pi}{3}kn}=$$

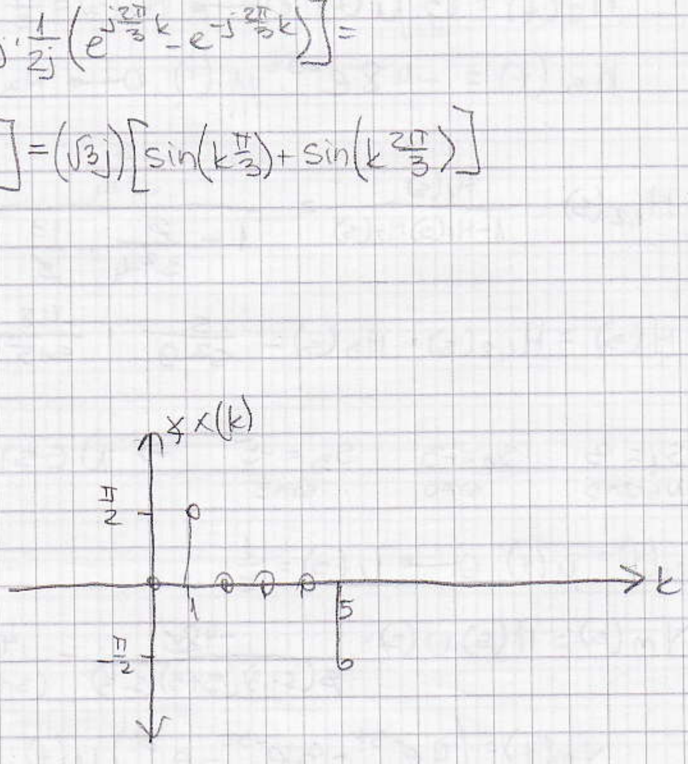
$$=-\frac{\sqrt{3}}{2}e^{-j\frac{\pi}{3}k}-\frac{\sqrt{3}}{2}e^{-j\frac{2\pi}{3}k}+\frac{\sqrt{3}}{2}e^{-j\frac{4\pi}{3}k}+\frac{\sqrt{3}}{2}e^{-j\frac{5\pi}{3}k},\quad k=0,\ldots,5$$

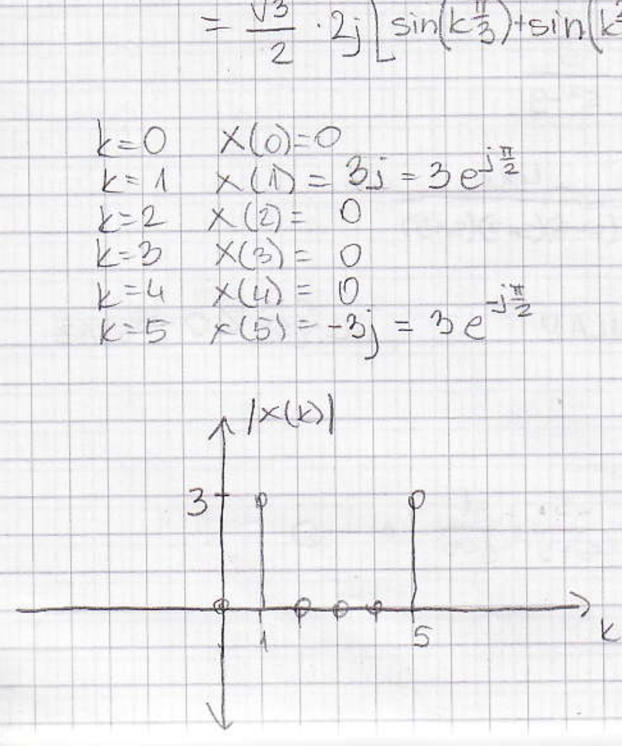
$$X(k)=-\frac{\sqrt{3}}{2}\left(e^{-j\frac{\pi}{3}k}+e^{-j\frac{2\pi}{3}k}-e^{+j\frac{2\pi}{3}k}-e^{+j\frac{\pi}{3}k}\right)=$$

$$=\frac{\sqrt{3}}{2}\left[2j\cdot\frac{1}{2j}\left(e^{j\frac{\pi}{3}k}-e^{-j\frac{\pi}{3}k}\right)+2j\cdot\frac{1}{2j}\left(e^{j\frac{2\pi}{3}k}-e^{-j\frac{2\pi}{3}k}\right)\right]=$$

$$=\frac{\sqrt{3}}{2}\cdot 2j\left[\sin\left(k\frac{\pi}{3}\right)+\sin\left(k\frac{2\pi}{3}\right)\right]=\left(\sqrt{3}j\right)\left[\sin\left(k\frac{\pi}{3}\right)+\sin\left(k\frac{2\pi}{3}\right)\right]$$

$k=0 \quad X(0)=0$
$k=1 \quad X(1)=3j=3e^{j\frac{\pi}{2}}$
$k=2 \quad X(2)=0$
$k=3 \quad X(3)=0$
$k=4 \quad X(4)=0$
$k=5 \quad X(5)=-3j=3e^{-j\frac{\pi}{2}}$

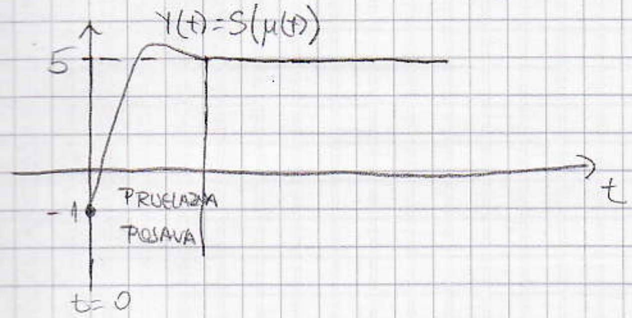




5. LTI sustav

$$H(s)=\frac{b_2}{s^2+a_1 s+a_2}$$

$$y''(t)+a_1 y'(t)+a_2 y(t)=b_2 u(t)$$



$$y(t)\Big|_{t=1}=5,128$$

$$u(t)=\sin(12t)\Rightarrow y_{ss}(t)=\frac{1}{\sqrt{2}}\sin\left(12t-\frac{3\pi}{4}\right)$$

a)

$$h(t)=S(\delta(t))$$

$$\delta(t)=\frac{d}{dt}\mu(t)$$

$$h(t)=\frac{d}{dt}y(t)$$

$\lim_{t\to\infty} h(t)=0 \Rightarrow$ STABILAN

$\lim_{t\to\infty} h(t)=\infty \rightarrow$ NESTABILAN

SVE DRUGO $\rightarrow$ granično stabilan

b)

$$u(t)=\mu(t)\Rightarrow y_p(t)=5\mu(t)$$

$$u(t)=\sin(12t)\Rightarrow y_p(t)=\frac{1}{\sqrt{2}}\sin\left(12t-\frac{3\pi}{4}\right)$$

$$y_p(t)=K,\quad K=5$$

1) $(K)''+a_1(K)'+a_2K=b_2\cdot 1$

$$\boxed{b_2=5a_2}$$

2)

$$y_p''+a_1y_p'+a_2y_p=5a_2\,u(t)$$

$$-\frac{1}{\sqrt{2}}(12)^2\sin\left(12t-\frac{3\pi}{4}\right)+\frac{a_1}{\sqrt{2}}12\cos\left(12t-\frac{3\pi}{4}\right)+\frac{a_2}{\sqrt{2}}\sin\left(12t-\frac{3\pi}{4}\right)=5a_2\sin(12)t \quad /\cdot\sqrt{2}$$

$$(-144+a_2)\sin\left(12t-\frac{3\pi}{4}\right)+(12a_1)\cos\left(12t-\frac{3\pi}{4}\right)=5\sqrt{2}\,a_2\sin(12t)$$

$$\sin\left[\left(12t-\frac{3\pi}{4}\right)+\frac{3\pi}{4}\right]=\sin\left(12t-\frac{3\pi}{4}\right)\underbrace{\cos\left(\frac{3\pi}{4}\right)}_{-\frac{\sqrt{2}}{2}}+\cos\left(12t-\frac{3\pi}{4}\right)\underbrace{\sin\left(\frac{3\pi}{4}\right)}_{\frac{\sqrt{2}}{2}}$$

$$(a_2-144)\sin\left(12t-\frac{3\pi}{4}\right)+(12a_1)\cos\left(12t-\frac{3\pi}{4}\right)=(-5a_2)\sin\left(12t-\frac{3\pi}{4}\right)+(5a_2)\cos\left(12t-\frac{3\pi}{4}\right)$$

$a_2-144=-5a_2 \qquad 12a_1=5a_2$

$a_2=24$
$a_1=10$
$b_2=120$

$$H(s)=\frac{120}{s^2+10s+24}$$

c) $y''(t)+10y'(t)+24y(t)=120\,u(t)$

d) $y(0^+)=-1=y(0^-)$

$y'(0^+)=y'(0^-)=?$

$s_1=-4 \quad s_2=-6$

$y_{TOT}(t)=c_1e^{-4t}+c_2e^{-6t}+5$

$y_{TOT}'(t)=-4c_1e^{-4t}-6c_2e^{-6t}$

$y_{TOT}'(0)=-4c_1-6c_2$

$y_{TOT}(0)=c_1+c_2+5=-1 \rightarrow c_1+c_2=-6$

$y(t)\big|_{t=1}=5{,}128$

$y_{TOT}(1)=c_1e^{-4}+c_2e^{-6}+5=5{,}128 \rightarrow c_1e^{-4}+c_2e^{-6}=0{,}128$

$c_1=$ $\qquad$ $c_2=$