3. svibanj 2009
8:07

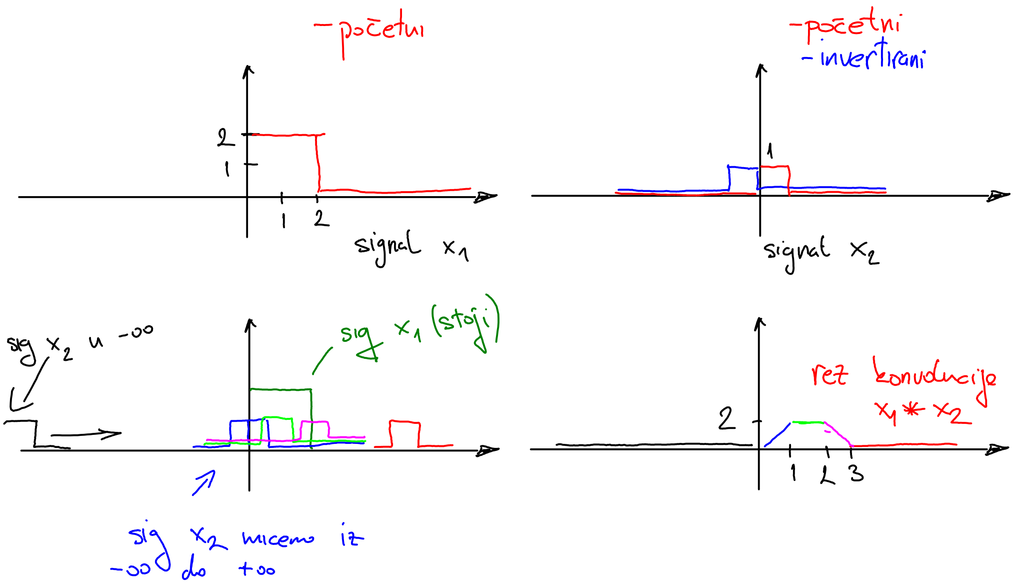
Konvolucija

$$y(n) = u(n) * h(n) = \sum_{m=-\infty}^{\infty} u(m)\, h(n-m)$$

$$y(t) = u(t) * h(t) = \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau$$

Svojstva konvolucije

- Komutativnost $(x_1 * x_2)(t) = (x_2 * x_1)(t)$
- Distributivnost $(x_1 * (x_2 + x_3))(t) = (x_1 * x_2 + x_1 * x_3)(t)$
- Asocijativnost $(x_1 * (x_2 * x_3))(t) = ((x_1 * x_2) * x_3)(t)$
- Pomak $y(t) = x_1(t) * x_2(t)\ ;\ x_1(t-T_1) * x_2(t-T_2) = y(t-T_1-T_2)$
- Konvolucija s impulsom $x(t) * \delta(t) = x(t)$



---

(16) $$\big(x(t) + y(t) * \delta(t+2)\big) * \delta(t-1) =$$

$$y(t) * \delta(t) = y(t)$$

$$y(t) * \delta(t+2) = y(t+2) \Rightarrow \big(x(t) + y(t+2)\big) * \delta(t-1) =$$

$$= x(t) * \delta(t-1) + y(t+2) * \delta(t-1) =$$

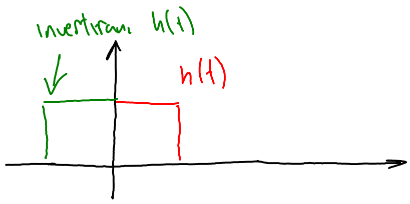
$$= x(t-1) + y(t+1)$$

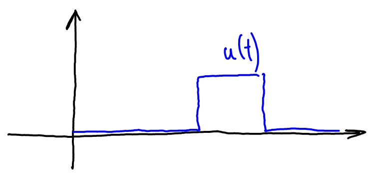
15

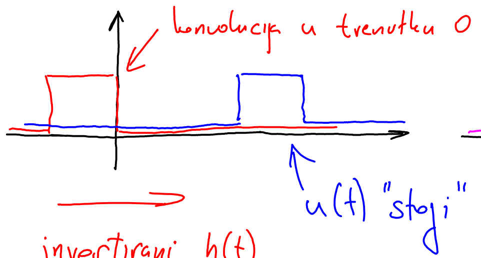
$$h(t) = \begin{cases} 1 & , 0 < t < 1 \\ 0 & , \text{inače} \end{cases}$$ ← impulsni odziv

$$u(t) = \begin{cases} 1, & 2 < t < 3 \\ 0, & \text{inače} \end{cases}$$

$$y(t) = h(t) * u(t)$$



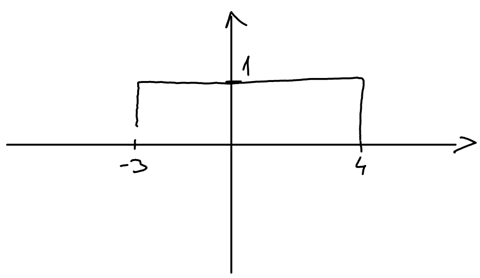


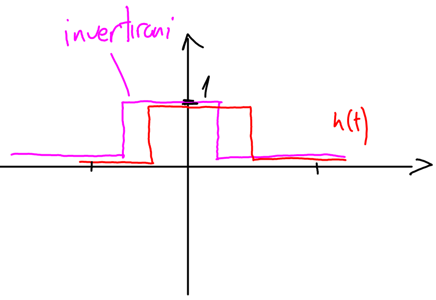


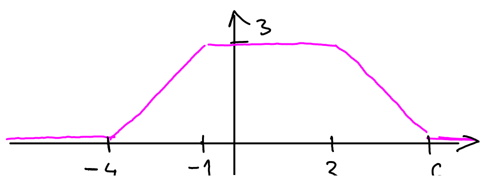
invertirani h(t) "putuje" iz $-\infty$ do $+\infty$ i rezultat konvolucije je površina presjeka s u(t)

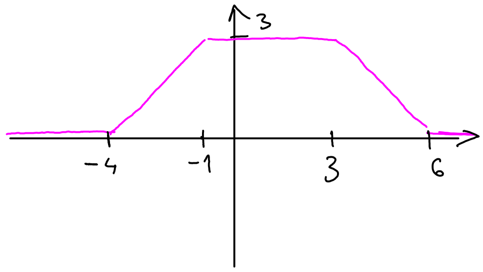
$$h(t) = \mu(t+1) - \mu(t-2)$$

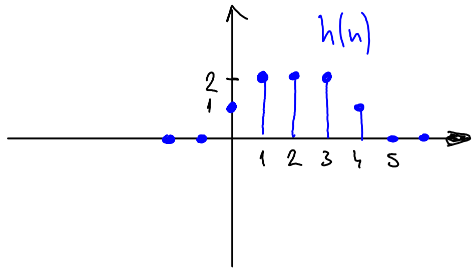
$$u(t) = \mu(t+3) - \mu(t-4)$$

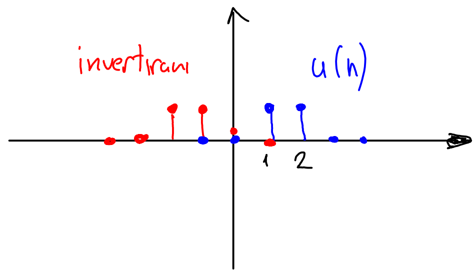






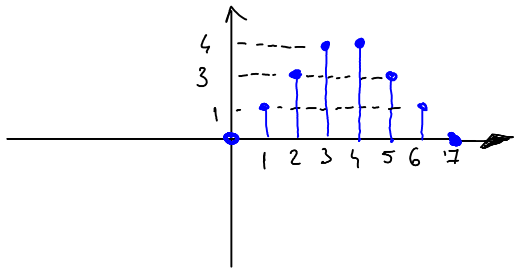




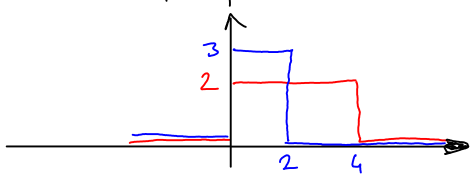


n_1 = trajanje prvog
n_2 = trajanje drugog
$n = n_1 + n_2 - 1$

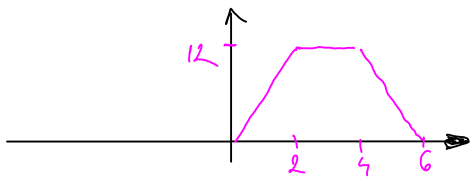
trajanje konvolucije



Brzi primjer



– 2 signala



– rez konvolucije

3. svibanj 2009
9:56

Jednadžbe diferencija

$y(n)+2y(n-1)+y(n-2)=u(n)$ – diff jed. 2 reda

$y(n-1)-y(n-3)\approx y(n)-y(n-2)$

$y(n+1)-y(n-3)=0$ – diff jed 4 reda

$3-(-1)=4$

Primer

$y(n)+2y(n-1)+y(n-2)=u(n)$

$u(n)=4\mu(n)$

$y(-1)=2$

$y(-2)=2$

uber zajeb

$y(n)=u(n)-2y(n-1)-y(n-2)$

$y(0)=u(0)-2y(-1)-y(-2)=-2$

$y(1)=u(1)-2y(0)-y(-1)=6$

$y(2)=u(2)-2y(1)-y(0)=-6$

$\vdots$

Homogena jednadžba

$y_h(0)=Cq^n$

$y_h(n)+2y_h(n-1)+y_h(n-2)=0$

$Cq^n+2Cq^{n-1}+Cq^{n-2}=0 \Rightarrow Cq^n(1+2q^{-1}+q^{-2})=0$

$Cq^nq^{-2}(q^2+2q+1)=0$

$\underbrace{Cq^{n-2}}_{\neq 0}\underbrace{(q^2+2q+1)}_{=0}=0$

$$q^2+2q+1=0$$

$$(q+1)^2=0$$

$$q_1=q_2=-1$$

q – karakteristična frekv. sustava

Opće homogeno rješenje

$$y_h(n)=(-1)^n(C_1+C_2 n)$$

Partikularno rješenje

$$y_p(n)=K$$

$$y_p(n)+2y_p(n-1)+y_p(n-2)=u(n)$$

$$K+2K+K=4$$

$$K=1 \Rightarrow y_p(n)=1 \quad ; \quad n\geq 0$$

Totalni odziv

$$y(n)=y_h(n)+y_p(n)$$

$$y(n)=(-1)^n(C_1+C_2 n)+1 \quad , \quad n\geq 0$$

jer jednadžba je definirana za $n\geq 0$, početni uvjeti moraju biti za trenutak veći ili jednak 0

$$y(0)=u(0)-2y(-1)-y(-2)=-2$$

$$y(1)=u(1)-2y(0)-y(-1)=6$$

$$y(0)=C_1+1=-2 \Rightarrow C_1=-3$$

$$y(1)=-C_1-C_2+1=6 \Rightarrow C_2=-2$$

Homogena jednadžba

Homogena jednadžba

Pretpostavljen oblik jednadžbe

$$y_h(n) = C q^n \Rightarrow \text{ iz ovog dobijemo } q\text{-ove}$$

1) Jednostruke karakteristične frekvencije

$$q_1 = 2 \quad q_2 = 4 \quad q_3 = 7 \quad q_4 = -1$$

$$y_h(n) = C_1 q_1^n + C_2 q_2^n + C_3 q_3^n + C_4 q_4^n$$

$$y_h(n) = C_1 2^n + C_2 4^n + C_3 7^n + C_4 (-1)^n$$

2) Višestruke karakteristične frekvencije

$$q_1 = 2 \quad q_2 = q_3 = q_4 = 5 \quad q_5 = q_6 = q_7 = q_8 = q_{10} = -2$$

$$y_h(n) = C_1 2^n + (C_2 + C_3 n + C_4 n^2) 5^n + (C_5 + C_6 n + C_7 n^2 + C_8 n^3 + C_9 n^4)(-2)^n$$

3) Kompleksne karakteristične freq.

$$q_1 = a + bj \quad q_2 = a - bj$$

$$y_h(n) = C_1 (a+bj)^n + C_2 (a-bj)^n$$ — nije praktičan zapis

$$q_{1,2} = a \pm bj \; ; \; q_{1,2} = |q| e^{\pm j\phi}$$

$$y_h(n) = C_1 (|q| e^{j\phi})^n + C_2 (|q| e^{-j\phi})^n = C_1 |q|^n e^{jn\phi} + C_2 |q|^n e^{-jn\phi}$$

$$y_h(n) = |q|^n (C_1 e^{jn\phi} + C_2 e^{-jn\phi}) = |q|^n (C_1 \cos(n\phi) + j C_1 \sin(n\phi) + C_2 \cos(n\phi) - j C_2 \sin(n\phi)) =$$

$$y_h(n) = |q|^n \Big(\underbrace{(C_1 + C_2)}_{A} \cos(n\phi) + \underbrace{j(C_1 - C_2)}_{B} \sin(n\phi) \Big)$$

$$y_h(n) = |q|^n (A \cos(n\phi) + B \sin(n\phi))$$

Partikularna jednadžba - ovisi o pobudi u karakt. [illegible]

Partikularna jednadžba

- ovisi o pobudi u karakt. freq

Pobuda	Partikularni oblik
1. A	K
2. Ar^n, $r \neq q$	Kr
3. Ar^n $r = q$	$Kr^n n$
4. Ar^M	$K_0 + k_1 n + \dots + k_M n^n$
5. $r^n n^M$	$(K_0 + k_1 n + \dots K_M n^M) r^n$

① • $q_1 = 2$ $q_2 = 3$ $u(n) = 5\mu(n)$

pobuda je konst. => partikularno rješenje je konst. => (1)

$y_p(n) = k$

② • $q_1 = 2$ $q_2 = 3$ $u(n) = 5 \cdot 4^n \mu(n)$

pobuda je konst. puta nešto na n, a nešto $\neq q_i$

$y_p(n) = k \cdot 4^n$

③ • $q_1 = 2$ $q_2 = 3$ $u(n) = 5\, 3^n \mu(n)$

pobuda je konst puta nešto na n s time da je to nešto karakteristična freq. sustava

$y_p(n) = k \cdot 3^n \cdot n$

• $q_1 = 2$ $q_2 = q_3 = 3$ $u(n) = 5 \cdot 3 \mu(n)$

$y_p(n) = k \cdot 3^n \cdot n^2$

② + ③ $u(n) = Ar^n \Rightarrow y_p(n) = k \cdot r^n \cdot n^z$

z – koliko puta se r pojavljuje kao frekvencija sustava

④ • $q_1 = 2 \qquad q_2 = 3 \qquad u(n) = (n^3+1)\mu(n)$

$$y_p(n) = K_0 + K_1 n + K_2 n^2 + K_3 n^3$$

Pobuda je polinom M-tog stupnja pa je i partikularni oblik isto polinom M-tog stupnja ali s nepoznatim koeficijentima

• $q_1 = 2 \qquad q_2 = q_3 = q_4 = 3 \qquad u(n) = (n^2+n+1)5^n\mu(n)$

⑤ $$y_p(n) = (K_0 + K_1 n + K_2 n^2)5^n$$

• $q_1 = 2 \qquad q_2 = q_3 = q_4 = 3 \qquad u(n) = n^2 3^n \mu(n)$

$$y_p(n) = (K_0 + K_1 n + K_2 n^2)3^n \cdot n^3$$

⑤ $\Rightarrow u(n) = n^M \cdot r^n$

$$y_p(n) = (K_0 + K_1 n + \dots + K_M n^M) r^n n^z$$

CAUTION

• $q_1 = 3 \qquad q_2 = q_3 = q_4 \qquad u(n) = 6\mu(n)$

$\cancel{y_p(n) = K}$ $\qquad u(n) = 6(1)^n \mu(n)$

$$y_p(n) = K(1)^n n^2 = K n^2$$

• $q_1 = q_2 = q_3 = 1 \qquad q_4 = 2 \qquad u(n) = n\mu(n)$

$\cancel{y_p(n) = K_0 + K_1 n}$

$$u(n) = n \cdot (1)^n \mu(n) \Rightarrow y_p(n) = (K_0 + K_1 n)n^3$$

3. svibanj 2009
11:40

1. Mirni odziv

$$y_m(n) = y_h(n) + y_p(n)$$

$\cancel{y(-1) = \ldots}$ $\cancel{y(-2) = \ldots}$

$y(-1) = 0$ $y(-2) = 0$ - mirni odziv, sustav je mirovao prije početka pobude

2. Nepobuđeni odziv

$y_n(n) = y_h(n)$ - nema pobude
$\Rightarrow$ nema partik. rješenja

$y(-1) = \ldots$
$y(-2) = \ldots$

3. Prisilni odziv

$$y_{Prisilni}(n) = y_p(n)$$

4. Prirodni odziv

Onaj dio totalnog koji pripada homogenom djelu

5. Totalni odziv

(1) način

$$y(n) = y_h(n) + y_p(n)$$

$y(-1) = \ldots$ $y(-2) = \ldots$ $\Rightarrow$ obično $y(0)$ i $y(1)$

(2) način

$$y(n) = y_M(n) + y_n(n)$$

Zadatak:

① $y(n)-2y(n-1)-y(n-2)=u(n)$

$y(-1)=1 \quad y(-2)=2 \qquad u(n)=4\mu(n)$

Homogena jed.

$$Cq^{n-2}\left(q^2-2q+1\right)=0$$

$$(q-1)^2=0$$

$$q_1=q_2=1$$

poseban oprez za jedinicu

$$y_h(n)=(1)^n(C_1+C_2 n)$$

$$y_h(n)=C_1+C_2 n$$

Partikularna jedn.

$$y_p(n)=K\cdot n^2$$

$$y_p(n)-2y_p(n-1)-y_p(n-2)=u(n)$$

$$Kn^2-2K(n-1)^2-K(n-2)^2=4$$

$$Kn^2+2Kn^2-4Kn+2K-Kn^2+2Kn+4K=4$$

$$n^2(\underbrace{K-2K+K}_{0})+n(\underbrace{4K-4K}_{0})+(-2K+4)=4$$

$$2K=4$$

$$K=2$$

uvijek mora biti jedinstven K

Mirni odziv

$$y_m(n)=C_1+C_2 n+2n^2 \quad , n\geq 0$$

$$\cancel{y(-1)=1} \qquad \cancel{y(-2)=2}$$

$$y(-1)=y(-2)=0$$

$$y(n)=u(n)+2y(n-1)-y(n-2)$$

$$y(0)=u(0)+2y(-1)-y(-2)=4$$

$$y(1)=u(1)+2y(0)-y(-1)=12$$

$$y(1) = u(1) + 2y(0) - y(-1) = 12$$

$$y_M(0) = C_1 = 4$$

$$y_M(1) = C_1 + C_2 + 2 = 12 \Rightarrow C_2 = 6$$

$$\boxed{y_M(n) = 4 + 6n + 2n^2}, \quad n \geq 0$$

Nepobuđeni odziv

$$y_n(n) = y_h(n) = C_1 + C_2 n \qquad \Leftarrow \text{prije početka pobude}$$

$$y(-1) = 1 \quad y(-2) = 2$$

$$\left.\begin{aligned} y_n(-1) &= C_1 - C_2 n = 1 \\ y_n(-2) &= C_1 - C_2 n = 2 \end{aligned}\right\} \quad \begin{aligned} C_1 &= 0 \\ C_2 &= -1 \end{aligned}$$

$$\boxed{y_n(n) = -n}$$

Totalni odziv (1)

$$y(n) = C_1 + C_2 n + 2n^2, \quad n \geq 0$$

$$y(-1) = 1 \quad y(-2) = 2$$

$$y(n) = u(n) + 2y(n-1) - y(n-2)$$

$$y(0) = u(0) + 2y(-1) - y(-2) = 4$$

$$y(1) = u(1) + 2y(0) - y(-1) = 11$$

$$y(0) = C_1 = 4$$

$$y(1) = C_1 + C_2 + 2 = 11 \Rightarrow C_2 = 5$$

$$\boxed{y(n) = 4 + 5n + 2n^2; \quad n \geq 0}$$

$$\| \checkmark$$

(2) način

$$y(n) = y_m(n) + y_n(n) \Rightarrow \boxed{y(n) = 4 + 5n + 2n^2, \quad n \geq 0}$$

prirodni (pripada homogenom)

prisilni (pripada partikularnom)

3. svibanj 2009
12:35

$$y(n) - 7y(n-1) + 12y(n-2) = u(n)$$
$$u(n) = (6n+7)\mu(n)$$
$$y(-1)=0 \qquad y(-2)=1$$

Homogena jednadžba

$$y_h(n) = Cq^n$$
$$\underbrace{Cq^{n-2}}_{\neq 0}\underbrace{(q^2 - 7q + 12)}_{0} = 0$$
$$q^2 - 7q + 12 = 0$$
$$q_1 = 3 \quad q_2 = 4$$
$$\boxed{y_h(n) = C_1 3^n + C_2 4^n}$$

Partikularna jednadžba

$$y_p(n) = K_0 + K_1 n$$
$$y_p(n) - 7y_p(n-1) + 12y_p(n-2) = u(n)$$
$$\vdots$$
$$(6K_1)n + (6K_0 - 17K_1) = 6n + 7$$
$$6K_1 = 6 \Rightarrow K_1 = 1 \Rightarrow K_0 = 4$$
$$y_p(n) = 4 + n \quad , n \geq 0$$

Mirni odziv

$$y_m(n) = y_h(n) + y_p(n)$$
$$y_m(n) = C_1 3^n + C_2 4^n + 4 + n \quad ; n \geq 0$$
$$\cancel{y(-1)=0} \quad \cancel{y(-2)=1}$$
$$y(-1) = y(-2) = 0$$

$$y(n) = u(n) + 7y(n-1) - 12y(n-2)$$
$$y(0) = u(0) + 7y(-1) - 12y(-2) = 7$$
$$y(1) = u(1) + 7y(0) - 12y(-1) = 62$$

$$\left.\begin{aligned} y_m(0) &= C_1 + C_2 + 4 = 7 \\ y_m(1) &= 3C_1 + 4C_2 + 4 = 62 \end{aligned}\right\} \begin{aligned} C_1 &= -45 \\ C_2 &= 48 \end{aligned}$$

$$\boxed{y_m(n) = -45 \cdot 3^n + 48 \cdot 4^n + 4 + n \quad , n \geq 0}$$

$$\boxed{y_m(n) = -45\cdot 3^n + 48\cdot 4^n + 4 + n \quad , n \geq 0}$$

Nepobuđeni odziv

$$y_n(n) = C_1 3^n + C_2 4^n$$

$$y(-1) = 0 \qquad y(-2) = 1$$

$$\left.\begin{array}{l} y_n(-1) = C_1 3^{-1} + C_2 4^{-1} = 0 \\ y_n(-2) = C_1 3^{-2} + C_2 4^{-2} = 1 \end{array}\right\} \begin{array}{l} C_1 = 36 \\ C_2 = -48 \end{array}$$

$$\boxed{y_n(n) = 36\cdot 3^n - 48\cdot 4^n}$$

Totalni odziv

$$y(n) = y_n(n) + y_p(n)$$

$$y(n) = C_1 3^n + C_2 4^n + 4 + n \;; \; n \geq 0$$

$$y(-1) = 0 \qquad y(-2) = 1$$

$$y(0) = u(0) + 7y(-1) - 12y(-2) = -5$$

$$y(1) = u(0) + 7y(0) - 12y(-1) = -22$$

$$\left.\begin{array}{l} y(0) = C_1 + C_2 + 4 = -5 \\ y(1) = 3C_1 + 4C_2 + 4 + 1 = -22 \end{array}\right\} \begin{array}{l} C_1 = -9 \\ C_2 = 0 \end{array}$$

$$\boxed{y(n) = -9\cdot 3^n + 4 + n \;; \; n \geq 0}$$

prirodni prisilni

2-način

$$y(n) = y_m(n) + y_n(n)$$

$$y(n) = -9\cdot 3^n + 4 + n \;, \; n \geq 0$$

3. svibanj 2009
12:55

$$y(n) - 2y(n-1) + 2y(n-2) = u(n)$$

$$u(n) = 5 \cdot 3^n \mu(n)$$

$$y(-1) = 0 \qquad y(-2) = 1$$

Homogeno

$$\underbrace{c g^{n-2}}_{\neq 0} \left(\underbrace{g^2 - 2g + 2}_{0} \right) = 0$$

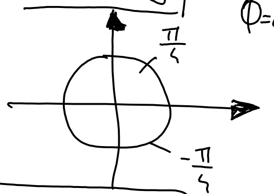
$$|g| = \sqrt{2}$$

$$g_{1,2} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2j}{2} = 1 \pm j$$

$$\phi = \operatorname{arctg} \frac{Im}{Re}$$

$$g_1 = 1 + j = |\sqrt{2}| e^{j\frac{\pi}{4}}$$

$$g_2 = 1 - j = |\sqrt{2}| e^{-j\frac{\pi}{4}}$$



$$\boxed{y_h(n) = \sqrt{2}^n \left(A\cos\left(\frac{\pi}{4} n\right) + B\sin\left(n\frac{\pi}{4}\right) \right)}$$

- Partikularno

$$y_p(n) = K 3^n$$

$$y_p(n) - 2y_p(n-1) + 2y(n-2) = u(n)$$

$$K3^n - 2K3^{n-1} + 2K3^{n-2} = 5 \cdot 3^n$$

$$3^n \left(\underbrace{K - 2K \cdot 3^{-1} + 2K \cdot 3^{-2}}_{= 5 \Rightarrow K = 9} \right) = 5 \cdot 3^n$$

$$y_p(n) = 9 \cdot 3^n, \quad n \geq 0$$

- Mirni odziv

$$y_m(n) = \sqrt{2}^n \left(A\cos\left(\frac{\pi}{4} n\right) + B\sin\left(\frac{\pi}{4} n\right) \right) + 9 \cdot 3^n$$

$$\cancel{y(-2) = -1} \qquad \cancel{y(-1) = 0}$$

$$y(-2) = y(-1) = 0$$

$$y(0) = u(0) + 2y(-1) - 2y(-2) = 5$$
$$y(1) = u(1) + 2y(0) - 2y(-1) = 25$$

$$y_m(0) = A + 9 = 5 \Rightarrow A = -4$$
$$y_m(1) = \cancel{\sqrt{2}}\left(A\frac{\sqrt{2}}{\cancel{2}} + B\frac{\cancel{\sqrt{2}}}{\cancel{2}}\right) + 27 = 25$$

$$A + B = -2$$
$$B = 2$$

$$\boxed{y_m(n) = \sqrt{2}^n\left(-4\cos\left(n\frac{\pi}{4}\right) + 2\sin\left(n\frac{\pi}{4}\right)\right) + 9\cdot 3^n}$$

∘ Nepobuđeni

$$y_n(n) = \sqrt{2}^n(\ldots\ldots)$$

$$y(-1) = 0 \qquad y(-2) = -1$$

$$y_n(-1) = \frac{1}{\cancel{\sqrt{2}}}\left(A\frac{\cancel{\sqrt{2}}}{2} - B\frac{\cancel{\sqrt{2}}}{2}\right) = \frac{1}{2}A - \frac{1}{2}B = 0$$

$$y_n(-2) = \frac{1}{2}(A\cdot 0 - B) = -1 \qquad A = B = 2$$

$$\boxed{y_n(n) = \sqrt{2}^n\left(2\cos\left(n\frac{\pi}{4}\right) + 2\sin\left(n\frac{\pi}{4}\right)\right)}$$

Totalni odziv (1)

$$y(n) = \sqrt{2}^n\left(A\cos\left(n\frac{\pi}{4}\right) + B\sin\left(n\frac{\pi}{4}\right)\right) + 9\cdot 3^n \;;\; n \geq 0$$

$$y(-1) = 0 \qquad y(-2) = -1 \Rightarrow y(0) = 7$$
$$y(1) = 29$$

$$y(1) = -2 + B + 27 = 29 \Rightarrow B = 4$$

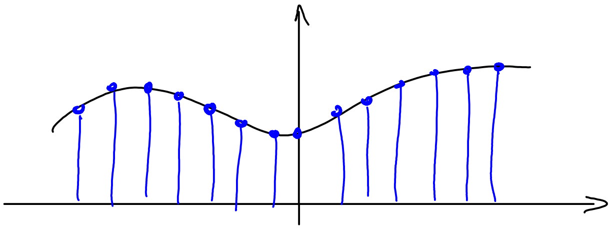
$$y(n) = \sqrt{2}^{\,n}\left(-2\cos\left(n\frac{\pi}{4}\right) + 4\sin\left(n\frac{\pi}{4}\right)\right) + 9\cdot 3^n,\ n \geq 0$$

Totalni odziv (2)

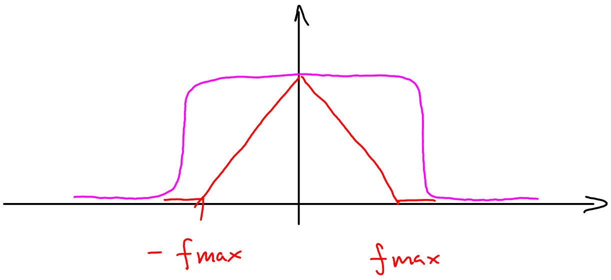
$$y(n) = y_m(n) + y_n(n) \Rightarrow y(n) =$$

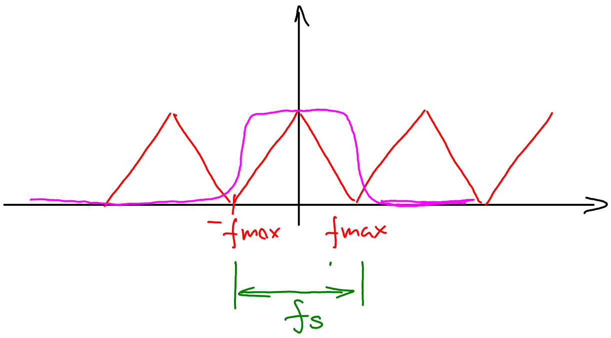
3. svibanj 2009
13:36

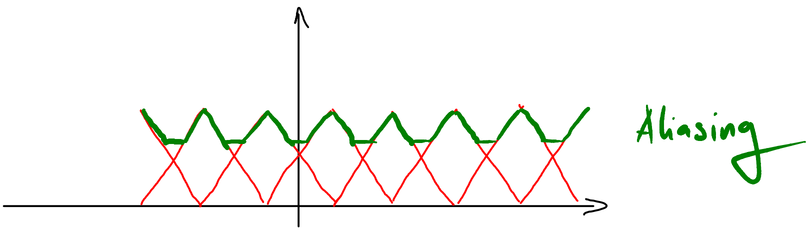
Otipkavanje



$f_s > 2 f_{max}$

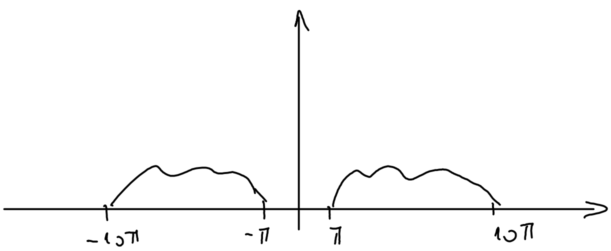






1) $w \in \langle -10\pi, -\pi \rangle$

$w \in \langle \pi, 10\pi \rangle$

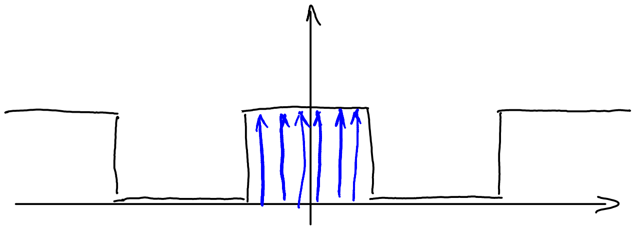


$w_s > 20\pi$

$$2\pi f_s > 20\pi$$

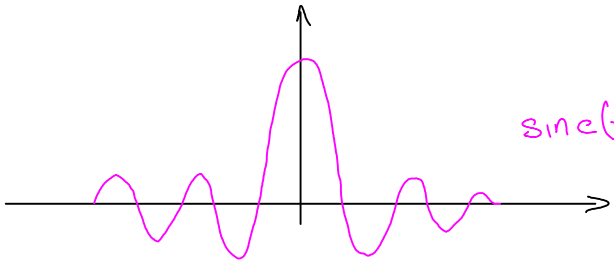
$$f_s > 10$$

② T_p T_i



Period ⇒ Diskretan

Kont ⇒ Aperiod



$$\mathrm{sinc}(x) = \frac{\sin(x)}{x}$$

3. svibanj 2009
13:50

DFT

$$X(k)=\sum_{n=0}^{N-1} x(n)e^{-2\pi j k \frac{n}{N}}$$

$$x(n)=\frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{2\pi j k \frac{n}{N}}$$

5 $x(n)=\{\underline{0},1,0,0\}$

$$X(k)=\sum_{n=0}^{3} x(n)e^{-2\pi j k \frac{n}{4}}=\underbrace{\cancel{x(0)}}_{0}+\underbrace{\cancel{x(1)e^{-jk\frac{\pi}{2}}}}_{0}+x(2)e^{-jk\pi}+\underbrace{\cancel{x(3)e^{-jk\frac{3\pi}{2}}}}_{0}$$

$$=e^{-jk\frac{\pi}{2}}$$

$$X(3)=e^{-j\frac{3\pi}{2}}=e^{j\frac{\pi}{2}}=\cos\left(\frac{\pi}{2}\right)+j\sin\left(\frac{\pi}{2}\right)$$