Signali i sustavi

Pismeni ispit – 24. travnja 2014.

1. **(9 bodova)** Zadan je vremenski kontinuiran signal $f(t) = t^2\big(\mu(t+5) - \mu(t-5)\big)$.

   a) **(2 boda)** Izračunajte energiju signala.

   b) **(2 boda)** Izračunajte i skicirajte prvu derivaciju signala.

   c) **(2 boda)** Očitajte signal i njegovu prvu derivaciju s periodom očitavanja $T_s = 2$.

   d) **(3 boda)** Iz očitaka signala izračunajte prvu derivaciju signala pomoću aproksimacije derivacije silaznom diferencijom.

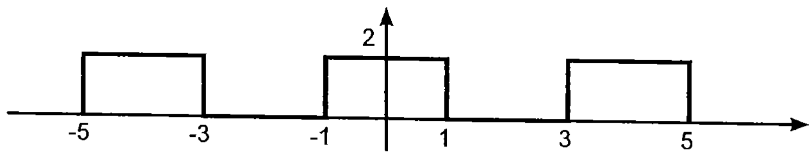
2. **(9 bodova)** Vremenski kontinuiran periodičan signal zadan je slikom.

   a) **(5 bodova)** Odredite i skicirajte amplitudni i fazni spektar signala za $k \in \{-3, -2, -1, 0, 1, 2, 3\}$.

   b) **(2 boda)** Objasnite Gibbsovu pojavu. Navedite primjer signala kod kojeg se javlja i primjer signala kod kojeg se ne javlja Gibbsova pojava.

   c) **(2 boda)** Pokažite da za vremenski kontinuirane realne signale $f(t)$ za koje postoji CTFS vrijedi

$$f(t) = \sum_{k=-\infty}^{\infty} F_k e^{jk\omega_0 t} = F_0 + \sum_{k=1}^{\infty} 2|F_k| \cos\left(k\omega_0 t + \angle F_k\right).$$



3. **(9 bodova)** Spektar vremenski kontinuiranog signala $f(t)$ je $F(j\omega) = e^{-2|\omega|}$.

   a) **(3 boda)** Odredite signal $f(t)$.

   b) **(3 boda)** Izračunajte energiju signala.

   c) **(3 boda)** Signal $f(t)$ očitali smo s periodom očitavanja $T_s = 1\,\text{ms}$. Koliko točaka očitanog signala moramo uzeti ako želimo numerički odrediti spektar s rezolucijom od $f_0 = 5\,\text{Hz}$?

4. **(9 bodova)** Zadan je vremenski diskretan signal $f(n) = \begin{cases} 3^{-n}, & \text{za } n > 1 \\ 0, & \text{inače} \end{cases}$.

   a) **(4 boda)** Odredite amplitudni i fazni spektar signala (nije potrebno skicirati).

   b) **(2 boda)** Izračunajte vrijednost amplitudnog i faznog spektra za $\Omega = \frac{\pi}{2}$.

   c) **(3 boda)** Pokažite da je spektar vremenski diskretnog aperiodičnog signala periodičan s osnovnim periodom 2π.

5. **(9 bodova)** Vremenski kontinuiran signal $f(t)$ očitan je u osam točaka s frekvencijom očitavanja $f_s = 1\,\text{kHz}$, te je dobiven vremenski diskretan signal $f(n) = \{\underline{-3}, -1, 1, 3, -3, -1, 1, 3\}$.

   a) **(5 bodova)** Izračunajte DFT u osam točaka vremenski diskretnog signala $f(n)$.

   b) **(2 boda)** Odredite frekvenciju Ω na kojoj amplitudni spektar DFT-a vremenski diskretnog signala $f(n)$ poprima maksimum.

   c) **(2 boda)** Odredite dominantnu spektralnu komponentu vremenski kontinuiranog signala $f(t)$.

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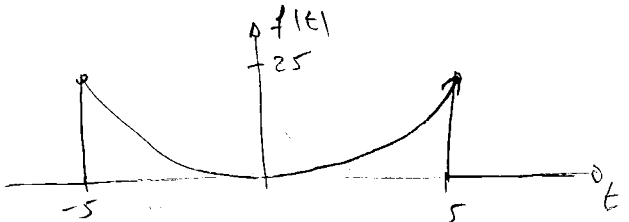
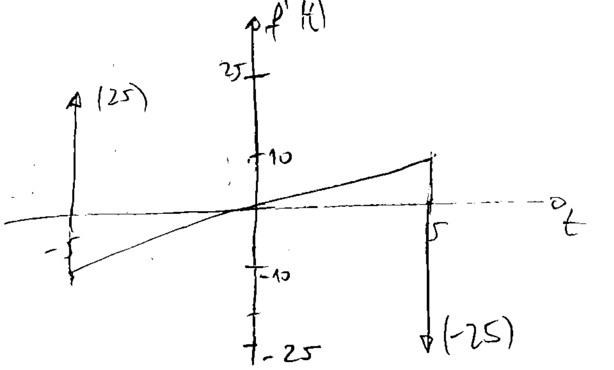
1. $f(t) = t^2 \left(\mu(t+5) - \mu(t-5)\right)$

a) $E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-5}^{5} t^4 dt = \frac{t^5}{5} \Big|_{-5}^{5} = \frac{5^5 - (-5)^5}{5} = \frac{2 \cdot 5^5}{5} = 2 \cdot 5^4 = 1250$

b)



$f'(t) = 2t \left(\mu(t+5) - \mu(t-5)\right)$
$\quad + t^2 \left(\delta(t+5) - \delta(t-5)\right)$
$= 2t \left(\mu(t+5) - \mu(t-5)\right)$
$\quad + 25 \left(\delta(t+5) - \delta(t-5)\right)$



c) $T_s = 2$

$f(n) = (2n)^2 \left(\mu(2n+5) - \mu(2n-5)\right) = \begin{cases} 4n^2 & \text{za } n = -2, -1, 0, 1, 2 \\ 0 & \text{inače} \end{cases}$

$= \{\dots 0, 16, 4, \underline{0}, 4, 16, 0, \dots\}$

$f_1(n) = 2 \cdot 2n = \begin{cases} 4n & \text{za } n = -2, -1, 0, 1, 2 \\ 0 & \text{inače} \end{cases}$

$= \{\dots, 0, -8, -4, \underline{0}, 4, 8, 0, \dots\}$

d)

$f_1(n) = \frac{1}{T_s} \{f(n) - f(n-1)\} = \frac{1}{2} \{4n^2 - 4(n-1)^2\}$

$= \frac{1}{2} \{4n^2 - 4(n^2 - 2n + 1)\}$

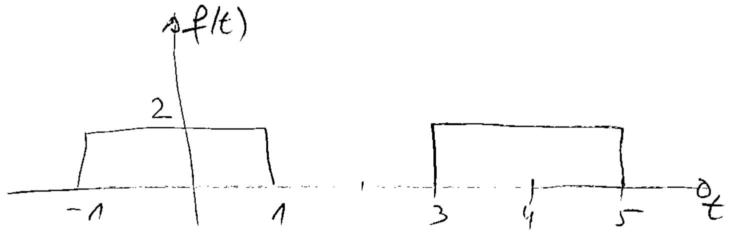
$= \frac{1}{2} (\cancel{4n^2} - \cancel{4n^2} + 8n - 4)$

$= 4n - 2 \qquad \text{za } n = -2, -1, 0, 1, 2$

$= \{\dots 0, -10, -6, \underline{-2}, 2, 6, 0, \dots\}$

2.

A



a) CTFS

$$X_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-j\frac{2\pi}{T_0}kt}dt$$

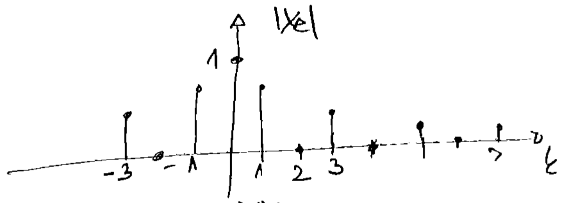
$$= \frac{1}{4}\int_{-1}^{1} 2e^{-j\frac{2\pi}{4}kt}dt = \frac{1}{2}\left.\frac{e^{-j\frac{2\pi}{4}kt}}{-j\frac{2\pi}{4}k}\right|_{-1}^{1}$$

$$= \frac{1}{-2j\frac{2\pi}{4}k}\left(e^{-j\frac{\pi}{2}k} - e^{+j\frac{\pi}{2}k}\right) = \frac{1}{+j\pi k}\left(e^{+j\frac{\pi}{2}k} - e^{-j\frac{\pi}{2}k}\right)$$

$$= \frac{1}{j\pi k}\cdot 2j\sin\frac{\pi}{2}k = \frac{2\sin\frac{\pi}{2}k}{\pi k} = \frac{\sin\frac{\pi}{2}k}{\frac{\pi}{2}k}$$

$$|X_k| = \left|\frac{\sin\frac{\pi}{2}k}{\frac{\pi}{2}k}\right|$$

$$\measuredangle X_k = \measuredangle\frac{\sin\frac{\pi}{2}k}{\frac{\pi}{2}k} = \pm\pi \quad \text{kad je} \quad \frac{\sin\frac{\pi}{2}k}{\frac{\pi}{2}k} < 0$$

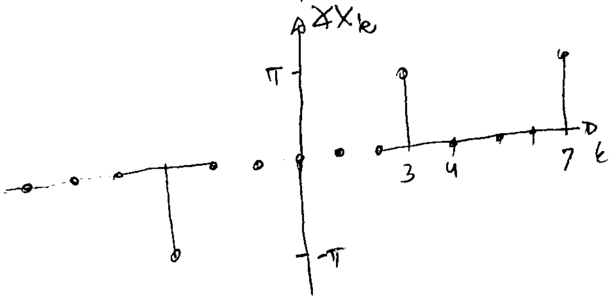


$$X_0 = \frac{1}{4}\int_{-1}^{1} 2\,dt = \frac{2}{4}\left.t\right|_{-1}^{1} = \frac{1}{2}(1+1) = 1$$

$$X_{-3} = \frac{1}{-\frac{\pi}{2}\cdot 3} = -\frac{2}{3\pi} \qquad X_3 = \frac{-1}{\frac{\pi}{2}\cdot 3} = -\frac{2}{3\pi}$$

$$X_{-2} = 0 \qquad X_2 = 0$$

$$X_{-1} = \frac{-1}{-\frac{\pi}{2}} = \frac{2}{\pi} \qquad X_1 = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$$



b) Gibbsova pojava se javlja kod nastanka kontinuiranih signala s diskontinuitetom u CTFS u obliku valovitosti na mjestima diskontinuiteta. Energija greške teži u nulu, ali Fourierova reprezentacija signala u točkama diskontinuiteta ne teži prema vrijednosti signala. Javlja se kod pravokutnika koji se ponavlja, a ne javlja se kod sinusa.

A

2. c)

$$f(t) = \sum_{k=-\infty}^{\infty} F_k e^{jk\omega_0 t} = F_0 + \sum_{k=1}^{\infty} 2|F_k| \cos(k\omega_0 t + \measuredangle F_k)$$

$$\sum_{k=-\infty}^{\infty} F_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{-1} F_k e^{jk\omega_0 t} + F_0 + \sum_{k=1}^{\infty} F_k e^{jk\omega_0 t}$$

$$= F_0 + \sum_{k=+1}^{+\infty} F_{-k} e^{-jk\omega_0 t} + \sum_{k=1}^{\infty} F_k e^{jk\omega_0 t}$$

$$\left.\begin{aligned} F_k &= |F_k| e^{j\measuredangle F_k} \\ F_{-k} &= |F_k| e^{-j\measuredangle F_k} \end{aligned}\right\} \text{ signal je realan} \rightarrow \text{koeficijenti su konjugirano simetrični}$$

$$f(t) = F_0 + \sum_{k=1}^{\infty} |F_k| e^{-j(k\omega_0 t + \measuredangle F_k)} + \sum_{k=1}^{\infty} |F_k| e^{j(k\omega_0 t + \measuredangle F_k)}$$

$$= F_0 + \sum_{k=1}^{\infty} |F_k| \left[\cos(k\omega_0 t + \measuredangle F_k) - j \cancel{\sin(k\omega_0 t + \measuredangle F_k)}\right]$$

$$+ \sum_{k=1}^{\infty} |F_k| \cdot \left[\cos(k\omega_0 + \measuredangle F_k) + j \cancel{\sin(k\omega_0 t + \measuredangle F_k)}\right]$$

$$= F_0 + \sum_{k=1}^{\infty} |F_k| \cdot 2 \cos(k\omega_0 + \measuredangle F_k)$$

A

3. $F(j\omega) = e^{-2|\omega|}$

a) $$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-2|\omega|} e^{j\omega t}\, d\omega$$

$$e^{(2+jt)\cdot\omega} = e^{2\omega}\cdot e^{j\omega t} = e^{-2\infty}\cdot e^{j\omega t} = 0$$
$$e^{(-2+jt)\omega} = e^{-2\infty}\cdot e^{jt\cdot\infty} = 0$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{2\omega} e^{j\omega t}\, d\omega + \int_{0}^{\infty} e^{-2\omega} e^{j\omega t}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{(2+jt)\omega}}{2+jt}\Bigg|_{-\infty}^{0} + \frac{e^{(-2+jt)\omega}}{-2+jt}\Bigg|_{0}^{\infty}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{2+jt} - \frac{1}{-2+jt}\right] = \frac{1}{2\pi}\cdot\frac{-2+jt-2-jt}{-4-t^2} = \frac{-4}{-2\pi(4+t^2)} = \frac{2}{\pi(4+t^2)}$$

b) $$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-4|\omega|}\, d\omega =$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{4\omega}\, d\omega + \int_{0}^{\infty} e^{-4\omega}\, d\omega\right] =$$

$$= \frac{1}{2\pi}\left[\frac{e^{4\omega}}{4}\Bigg|_{-\infty}^{0} + \frac{e^{-4\omega}}{-4}\Bigg|_{0}^{\infty}\right] = \frac{1}{2\pi}\left[\frac{1}{4} - \frac{1}{-4}\right] = \frac{1}{2\pi}\cdot\frac{1}{2} = \frac{1}{4\pi}$$

c) $T_s = 1\,ms$ očitamo u vremenskoj domeni $\quad \omega_s = \frac{2\pi}{T_s} = 2\pi \rightarrow f_s = \frac{1}{T_s} = 10^3\,Hz$

$f_0 = 5\,Hz$ rezolucija u spektru $\quad T_p = \frac{1}{f_0} = \frac{1}{5}\,s$

broj točaka $\quad N_{\omega_0} = N_T = T_p \cdot f_s = \frac{1}{5}\cdot 1000 = 200$

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4. $f(n) = \begin{cases} 3^{-n}, & \text{za } n > 1 \\ 0, & \text{inače} \end{cases} = \{0, 0, 3^{-2}, 3^{-3}, 3^{-4}, \dots\}$
$= \{0, 0, (\frac{1}{3})^2, (\frac{1}{3})^3, (\frac{1}{3})^4, \dots\}$

a) DTFT

$$F(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} f(n)\, e^{-j\Omega n}$$
$$= \sum_{n=2}^{\infty} f(n)\, e^{-j\Omega n}$$
$$= 3^{-2} e^{-j\Omega \cdot 2} + 3^{-3} \cdot e^{-j\Omega 3} + 3^{-4} e^{-j\Omega 4} + \dots$$
$$= 3^{-2} e^{-j\Omega 2} (1 + 3^{-1} e^{-j\Omega} + 3^{-2} e^{-j\Omega 2} + \dots)$$
$$= \frac{1}{9} e^{-j\Omega \cdot 2} \sum_{n=0}^{\infty} (3^{-1} e^{-j\Omega})^n$$
$$= \frac{1}{9} e^{-j\Omega \cdot 2} \cdot \frac{1}{1 - 3^{-1} e^{-j\Omega}} = \frac{1}{9} e^{-j\Omega 2} \cdot \frac{1}{\frac{3e^{j\Omega} - 1}{e^{j\Omega} \cdot 3}}$$
$$= \frac{1}{9} e^{-j\Omega \cdot 2} \cdot \frac{3 e^{j\Omega}}{3 e^{j\Omega} - 1} = \frac{1}{3} e^{-j\Omega} \frac{1}{3e^{j\Omega} - 1}$$
$$= \frac{\frac{1}{3}(\cos\Omega - j\sin\Omega)}{3\cos\Omega + 3j\sin\Omega - 1}$$

$$|F(e^{j\Omega})| = \frac{\frac{1}{3}}{\sqrt{(3\cos\Omega - 1)^2 + (3\sin\Omega)^2}} = \frac{1}{3} \cdot \frac{1}{\sqrt{9\cos^2\Omega - 6\cos\Omega + 1 + 9\sin^2\Omega}}$$
$$= \frac{1}{3} \cdot \frac{1}{\sqrt{10 - 6\cos\Omega}}$$

$$\sphericalangle F(e^{j\Omega}) = -\Omega - \operatorname{arctg} \frac{3\sin\Omega}{3\cos\Omega - 1}$$

b) $\Omega = \frac{\pi}{2}$

$$|F(e^{j\Omega})| = \frac{1}{3} \frac{1}{\sqrt{10 - 6 \cdot \cos\frac{\pi}{2}}} = \frac{1}{3\sqrt{10}}$$

$$\sphericalangle F(e^{j\Omega}) = -\frac{\pi}{2} - \operatorname{arctg} \frac{3\sin\frac{\pi}{2}}{3\cos\frac{\pi}{2} - 1} = -\frac{\pi}{2} - \operatorname{arctg}\frac{3}{-1} = -\frac{\pi}{2} - 1{,}89 = -3{,}46$$
$$= 2{,}82 \text{ rad}$$

c) $F(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} f(n)\, e^{-j\Omega n}$

$$F(e^{j(\Omega + 2\pi k)}) = \sum_{n=-\infty}^{\infty} f(n)\, e^{-j(\Omega + 2\pi k)n} = \sum_{n=-\infty}^{\infty} f(n) e^{-j\Omega n} e^{-j2\pi k n}$$
$$= \sum_{n=-\infty}^{\infty} f(n) e^{-j\Omega n} \cdot (\underbrace{\cos 2\pi k n}_{1} - \underbrace{j\sin 2\pi k n}_{0})$$
$$= \sum_{n=-\infty}^{\infty} f(n) e^{-j\Omega n} = F(e^{j\Omega})$$

A

5.

$f(t)$

$T_s = 1\,kHz$

$f(n) = \{\underline{-3}, -1, 1, 3, -3, -1, 1, 3\}$

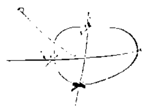
a) DFT

$$F(k) = \sum_{n=0}^{7} f(n)\, e^{-j\frac{2\pi}{8}nk} = -3\left(1+e^{-j\frac{2\pi}{8}4k}\right) - 1\left(e^{-j\frac{2\pi}{8}k} + e^{-j\frac{2\pi}{8}5k}\right) + \left(e^{-j\frac{2\pi}{8}2k} + e^{-j\frac{2\pi}{8}6k}\right)$$
$$+3\left(e^{-j\frac{2\pi}{8}3k} + e^{-j\frac{2\pi}{8}7k}\right)$$

$$= -3\, e^{-j\frac{2\pi}{8}2k}\left(e^{j\frac{2\pi}{8}2k} + e^{-j\frac{2\pi}{8}2k}\right) + (-1)\, e^{-j\frac{2\pi}{8}3k}\left(e^{j\frac{2\pi}{8}2k} + e^{-j\frac{2\pi}{8}2k}\right)$$
$$+3\, e^{-j\frac{2\pi}{8}5k}\left(e^{j\frac{2\pi}{8}2k} + e^{-j\frac{2\pi}{8}2k}\right) + e^{-j\frac{2\pi}{8}4k}\left(e^{j\frac{2\pi}{8}2k} + e^{-j\frac{2\pi}{8}2k}\right)$$

$$= -3\, e^{-j\frac{\pi}{2}k}\left(2\cos\tfrac{4\pi}{8}k\right) - e^{-j\frac{3\pi}{4}k}\cdot 2\cos\tfrac{4\pi}{8}k + e^{-j\frac{4\pi}{4}k}\cdot 2\cos\tfrac{4\pi}{8}k + 3\, e^{-j\frac{5\pi}{4}k}\cdot 2\cos\tfrac{4\pi}{8}k$$

$$= 2\cos\tfrac{\pi}{2}k\left[-3e^{-j\frac{\pi}{2}k} - e^{-j\frac{3\pi}{4}k} + e^{-j\pi k} + 3e^{-j\frac{5\pi}{4}k}\right]$$



$F(0) = 2\cdot[-3 - 1 + 1 + 3] = 0$

$F(1) = 0$

$F(2) = -2\cdot\left[-3e^{-j\pi} - e^{-j\frac{3\pi}{2}} + e^{-j2\pi} + 3e^{-j\frac{5\pi}{2}}\right] = -2\left[-3\cdot(-1) - (+j) + 1 + 3(-j)\right] =$

$= -2[3 - j + 1 - 3j] = -8 + 8j = 8\sqrt{2}\, e^{j\frac{3\pi}{4}} \quad = -$

$F(3) = 0$

$F(4) = 2\cdot\left[-3e^{-j2\pi} - e^{-j3\pi} + e^{-j4\pi} + 3e^{-j5\pi}\right] = 2\left[-3 - (-1) + 1 + 3(-1)\right] =$

$= 2[-3 + 1 + 1 - 3] = -8 = 8e^{j\pi}$

$F(5) = 0$

$F(6) = -2\left[-3e^{-j3\pi} - e^{-j\frac{9\pi}{2}} + e^{-j6\pi} + 3e^{-j\frac{15\pi}{2}}\right] = -2\left[-3(-1) - (-j) + 1 + 3j\right]$

$= -2[3 + j + 1 + 3j] = -8 - 8j = 8\sqrt{2}\, e^{-j\frac{3\pi}{4}}$

$F(7) = 0$

$F(k) = \{\underline{0}, 0, -8+8j, 0, -8, 0, -8-8j\}$

A

5. b)

ze $k=2 \rightarrow |F(k)| = 8\sqrt{2}$

$\Omega = \frac{2\pi}{N} \cdot k$

$= \frac{2\pi}{8} \cdot 2$

$\Omega = \frac{4\pi}{8}$

$\Omega = \frac{\pi}{2}$ rad

c) $\Omega = \omega T$ $\qquad f_s = 1\,kHz = 10^3\,Hz \rightarrow T = 10^{-3}\,s$

$\omega = \frac{\Omega}{T} = \frac{\frac{\pi}{2}}{\frac{10^3}{1}}$

$= \frac{\pi}{2 \cdot 10^{-3}} = \frac{\pi}{2} \cdot 1000 = 500\pi$ rad/s