Signali i sustavi

Auditorne vježbe 11.

LS&S
FER – ZESOI

Razlaganje sustava na jednostavnije podsustave i izbor varijabli stanja

DIREKTNA METODA
(NORMALNE VARIJABLE STANJA)

Zadatak 1.

- Koristeći *direktnu metodu* naći model linearnog sustava opisanog diferencijalnom jednadžbom tj. ekvivalentnom prijenosnom funkcijom.

$$y''' + 2y'' + 5y' + 6y = u.$$

Rješenje :

- $y''' + 2y'' + 5y' + 6y = u.$
- Primjenimo Laplaceovu transformaciju:
  $s^3Y(s) + 2s^2Y(s) + 5sY(s) + 6\,Y(s) = U(s)$ (1)
- (početni uvjeti neka su nula).
- Nakon izlučivanja $Y(s)$ imamo:
  $Y(s) \cdot [\, s^3 + 2s^2 + 5s + 6 \,] = U(s)$,
- i konačno:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^3 + 2s^2 + 5s + 6}.$$

- To je prijenosna funkcija sustava.

Nastavak ...

- Izbor varijabli stanja:

  $x_1(t) = y(t)$,

  $x_2(t) = y'(t) = x_1'$,

  $x_3(t) = y''(t) = x_2'$.
- To uvrstimo u diferencijalnu jednadžbu

  $x_3' + 2x_3 + 5x_2 + 6x_1 = u$.

Rješenje nastavak ...

- Jednadžbe stanja:

  $\mathbf{x}' = \mathbf{A}\,\mathbf{x} + \mathbf{B} \cdot \mathbf{u}$.
- U našem slučaju:

  $x_1' = x_2$,

  $x_2' = x_3$,

  $x_3' = -6x_1 - 5x_2 - 2x_3 + u$.

Matrični oblik jed. stanja ...

- U matričnom obliku, to izgleda ovako:

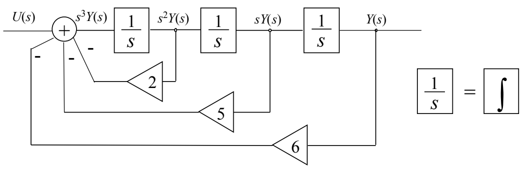
$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -5 & -2 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{B}} u.$$

Izlazna jednadžba?

$$\mathbf{y} = \mathbf{C}\cdot\mathbf{x} + \mathbf{D}\cdot\mathbf{u},$$
$$y = x_1.$$

$$y = \underbrace{\begin{bmatrix}1 & 0 & 0\end{bmatrix}}_{\mathbf{C}}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \underbrace{\begin{bmatrix}0\end{bmatrix}}_{\mathbf{D}}\cdot u.$$

Simulacijski blok dijagram?



- iz (1) $\Rightarrow$ $s^3Y(s) = U(s) - 6Y(s) - 5sY(s) - 2s^2Y(s)$.

Zadatak 2.

- Koristeći *direktnu metodu* naći model linearnog sustava opisanog diferencijalnom jednadžbom tj. ekvivalentnom prijenosnom funkcijom:

$$H(s) = \frac{s^3 + 2s^2 + 3s + 4}{s^3 + 2s^2 + 5s + 6}$$

- Rješenje:

$$H(s) = \frac{B(s)}{A(s)} \qquad Y(s) = H(s)\cdot U(s) = B(s)\underbrace{\frac{U(s)}{A(s)}}_{Z(s)} = B(s)\cdot Z(s).$$

Zadatak 2.

- Najprije realiziramo $Z(s)$:

$$Z(s) = \frac{1}{s^3 + 2s^2 + 5s + 6} \cdot U(s).$$

$z''' + 2z'' + 5z' + 6z = u,$ (1)

- ovo je isti slučaj kao i u prethodnom zadatku!

Što ćemo s brojnikom B(s)?

$Y(s) = (s^3 + 2s^2 + 3s + 4) \cdot Z(s),$

$y(t) = z''' + 2z'' + 3z' + 4z,$ (2)

$z''' + 2z'' + 5z' + 6z = u.$ (1)

- Jednadžbe stanja (iz (1)) su :

$x_1 = z,$

$x_2 = z'$ $\quad x_1' = x_2,$

$x_3 = z''$ $\quad x_2' = x_3,$

$x_3' = -6x_1 - 5x_2 - 2x_3 + u.$

Rješenje nastavak ...

- Izlazna jednadžba

$y = z''' + 2z'' + 3z' + 4z,$ (2)

$= (-6x_1 - 5x_2 - 2x_3 + u) + 2x_3 + 3x_2 + 4x_1,$

$= -2x_1 - 2x_2 + u.$

Rješenje nastavak ...

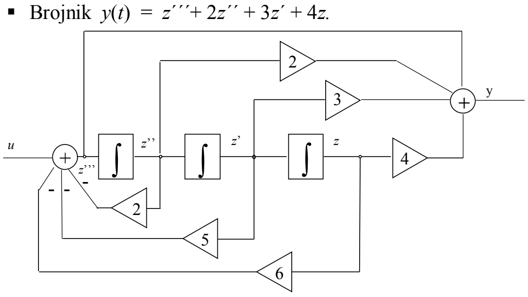
- Matrični oblik:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} -2 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 1 \cdot u.$$

Simulacijski blok dijagram?

- Nazivnik → $z''' + 2z'' + 5z' + 6z = u$.
  - Ovo znamo realizirati (prethodni zadatak).
- Brojnik $y(t) = z''' + 2z'' + 3z' + 4z$.



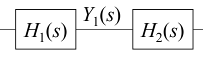
Kaskadna realizacija (Iterativne varijable stanja)

Zadatak 3.

Za sustav zadan prijenosnom funkcijom nacrtati model i napisati jednadžbe stanja *kaskadnom metodom*.

$$H(s) = \frac{(s+2)(s+1)}{(s+3)(s+4)}.$$

- Rješenje:



Kaskadna realizacija

- Faktoriziramo razlomak

$$H(s)=\frac{s+2}{s+3}\cdot\frac{s+1}{s+4}=H_1(s)\cdot H_2(s),$$

$$H_1(s)=\frac{s+2}{s+3},$$

$$Y_1(s)=H_1(s)\cdot U(s), \qquad X_1(s)$$

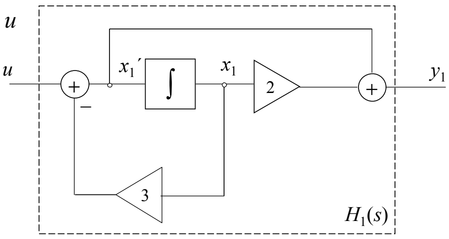
$$=\frac{s+2}{s+3}U(s),$$

$$X_1(s)=\frac{U(s)}{s+3}\Rightarrow x_1^{'}=-3x_1+u, \qquad (1)$$

$$Y_1(s)=(s+2)\cdot X_1(s)\Rightarrow y_1=x_1^{'}+2x_1=-x_1+u.$$

Kaskadna realizacija

$x_1´=-3x_1+u$

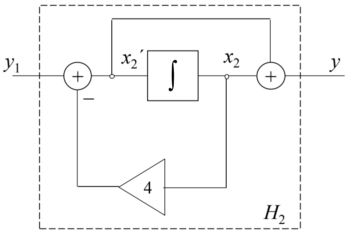


$$Y(s)=H_2(s)\cdot Y_1(s)=\frac{s+1}{s+4}\cdot Y_1(s), \qquad X_2(s)$$

$$X_2(s)=\frac{Y_1(s)}{s+4}\Rightarrow x_2^{'}=-4x_2+y_1=-x_1-4x_2+u. \qquad (2)$$

Kaskadna realizacija

$$Y(s)=(s+1)X_2(s)\Rightarrow y=x_2^{'}+x_2=-x_1-3x_2+u.$$



Kaskadna realizacija

- Jednadžbe stanja:

$x_1' = -3x_1 + u,$

$x_2' = -x_1 - 4x_2 + u.$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot u,$$

- Izlazna jednadžba

$y = -x_1 - 3x_2 + u.$

$$y = \begin{bmatrix} -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 1 \cdot u.$$

Tipična donja trokutasta matrica (kod kaskadne realizacije)

Kaskadna realizacija

Zadatak 4.

Za sustav zadan prijenosnom funkcijom nacrtati model i napisati jednadžbe stanja kaskadnom metodom.

$$H(s) = \frac{(s+1)(s+2)s}{(s+3)(s+4)(s^2+2s+2)} = \underbrace{\frac{s+1}{s+3}}_{H_1} \cdot \underbrace{\frac{s}{s+4}}_{H_2} \cdot \underbrace{\frac{s+2}{s^2+2s+2}}_{H_3}.$$

ne razbija se dalje, da ne dobijemo imaginarne koeficijente (–1±j)

$$U(s) \longrightarrow \boxed{H_1(s)} \xrightarrow{Y_1(s)} \boxed{H_2(s)} \xrightarrow{Y_2(s)} \boxed{H_3(s)} \longrightarrow Y(s)$$

Kaskadna realizacija

- $\boldsymbol{H_1}$

$$Y_1(s) = H_1(s) \cdot U(s),$$

$$= \frac{s+1}{s+3} U(s), \quad X_1(s)$$

$$X_1(s) = \frac{U(s)}{s+3} \Rightarrow x_1' = -3x_1 + u,$$

$$Y_1(s) = (s+1) \cdot X_1(s) \Rightarrow y_1 = x_1' + x_1 = -2x_1 + u.$$

Kaskadna realizacija

- ***H*₂**

$$Y_2(s) = H_2(s)\cdot Y_1(s),$$

$$= \underbrace{\frac{s}{s+4}Y_1(s)}_{X_2(s)},$$

$$X_2(s) = \frac{Y_1(s)}{s+4} \Rightarrow x_2' = -4x_2 + y_1 = -2x_1 - 4x_2 + u,$$

$$Y_2(s) = s\cdot X_2(s) \Rightarrow y_2 = x_2' = -2x_1 - 4x_2 + u.$$

Kaskadna realizacija

- ***H*₃**

$$Y(s) = H_3(s)\cdot Y_2(s),$$

$$= \underbrace{\frac{s+2}{s^2+2s+2}Y_2(s)}_{Z(s)},$$

$$Z(s) = \frac{Y_2(s)}{s^2+2s+2} \Rightarrow z'' + 2z' + 2z = y_2,$$

realizirati direktnom metodom

$$x_3 = z,$$

$$x_4 = z' = x_3',$$

$$x_4' = z'' = -2x_3 - 2x_4 + y_2,$$

$$= -2x_1 - 4x_2 - 2x_3 - 2x_4 + u.$$

Kaskadna realizacija

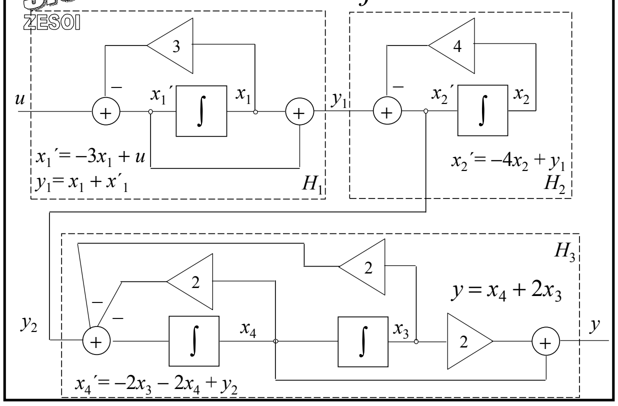
$$Y(s) = (s+2)\cdot Z(s) \Rightarrow y = z' + 2z = x_4 + 2x_3.$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 & 0 \\ -2 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -4 & -2 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot u,$$

$$y = \begin{bmatrix} 0 & 0 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + 0\cdot u.$$

Trokutasti oblik "pokvaren" zbog direktne realizacije sekcije II reda

Kaskadna realizacija



Paralelna realizacija
(Kanonske varijable stanja)

$$H(s)=\frac{\mathrm{b}_m s^m+\ldots+\mathrm{b}_0}{\mathrm{a}_n s^n+\ldots+\mathrm{a}_0}=\mathrm{d}_0+\frac{\mathrm{c}_1}{s-s_1}+\ldots+\frac{\mathrm{c}_n}{s-s_n}.$$

Rastav na parcijalne razlomke.

s_{i}, $i = 1,...,n$ - jednostruki realni polovi.

$$\mathrm{d}_0=\lim_{s\to\infty}H(s) \qquad \mathrm{c}_k=(s-s_k)H(s)\big|_{s=s_k},$$

$X_k(s)$

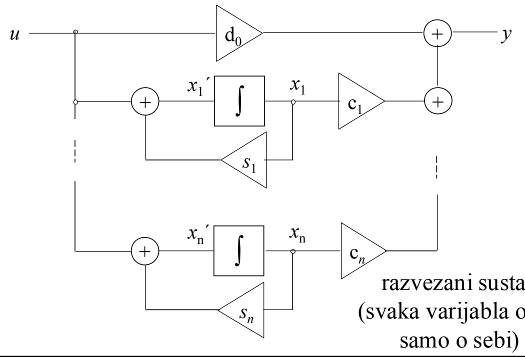
$$Y(s)=H(s)\cdot U(s)=\mathrm{d}_0\cdot U(s)+\sum_{k=1}^{n}\mathrm{c}_k\frac{U(s)}{s-s_k}.$$

Paralelna realizacija

$$X_k(s)=\frac{\mathrm{U(s)}}{s-s_k}\Rightarrow x_k'=s_k\cdot x_k+u,$$

$$Y(s)=\mathrm{d}_0U(s)+\sum_{k=1}^{n}\mathrm{c}_kX_k(s)\Rightarrow y=\mathrm{d}_0u+\sum_{k=1}^{n}\mathrm{c}_kx_k.$$

Paralelna realizacija



razvezani sustav (svaka varijabla ovisi samo o sebi)

Paralelna realizacija

Zadatak 5

Nacrtati simulacijski dijagram i napisati stanja pomoću kanonskih varijabli (paralelna realizacija)

$$H(s)=\frac{s^2+7s+12}{s(s+1)^2(s+2)}.$$

Rješenje:

$$H(s)=\mathrm{d}_0+\underbrace{\frac{\mathrm{c}_{11}}{(s+1)^2}+\frac{\mathrm{c}_{12}}{(s+1)}}+\frac{\mathrm{c}_{21}}{s}+\frac{\mathrm{c}_{31}}{(s+2)},$$

uočiti rastav višestrukog pola!!!

Paralelna realizacija

- Višestruki polovi

višestrukost pola s_i

$$\mathrm{c}_{ij}=\frac{1}{(j-1)!}\cdot\frac{d^{j-1}}{ds^{j-1}}\left[(s-s_i)^n H(s)\right]_{s=s_i},$$

$$\mathrm{c}_{11}=(s+1)^2\cdot H(s)\Big|_{s=-1}=(s+1)^2\cdot\frac{s^2+7s+12}{s(s+1)^2(s+2)}\Bigg|_{s=-1}=\ldots=-6,$$

$$\mathrm{c}_{12}=1\cdot\frac{d}{ds}\left[\frac{s^2+7s+12}{s(s+2)}\right]_{s=-1}=$$

$$=\frac{(2s+7)s(s+2)-(s^2+7s+12)(2s+2)}{[s(s+2)]^2}\Bigg|_{s=-1}=\ldots=-5,$$

$$\mathrm{c}_{21}=sH(s)\big|_{s=0}=6,\qquad \mathrm{c}_{31}=(s+2)H(s)\big|_{s=-2}=-1.$$

Paralelna realizacija, izbor varijabli stanja

$$Y(s) = -6\cdot\underbrace{\frac{1}{(s+1)^2}U(s)}_{X_1} - 5\cdot\underbrace{\frac{1}{s+1}U(s)}_{X_2} + 6\cdot\underbrace{\frac{1}{s}U(s)}_{X_3} - \underbrace{\frac{1}{s+2}U(s)}_{X_4},$$

$$X_1(s) = \frac{U(s)}{(s+1)^2} = \frac{X_2(s)}{s+1} \quad \Rightarrow x_1' = -x_1 + x_2,$$

$$X_2(s) = \frac{U(s)}{s+1} \quad \Rightarrow x_2' = -x_2 + u,$$

$$X_3(s) = \frac{U(s)}{s} \quad \Rightarrow x_3' = u,$$

$$X_4(s) = \frac{U(s)}{s+2} \quad \Rightarrow x_4' = -2x_4 + u.$$

Paralelna realizacija

- Jednadžbe stanja: Jordanov blok, −1 višestruki korjen

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot u.$$

Paralelna realizacija

Izlazna jednadžba:

$$y = \begin{bmatrix} -6 & -5 & 6 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + 0 \cdot u.$$

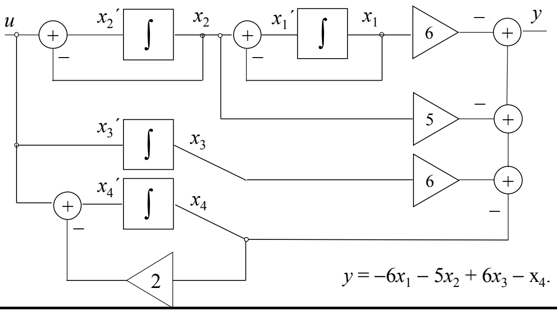
- Za jednostruke polove u matrici A ostaju samo dijagonalni elementi.
- Za n-struke polove p javlja se Jordanov blok (n > 1).

Jordanov blok u općem slučaju:

$$\begin{bmatrix} p & 1 & 0 & 0 & 0 \\ 0 & p & 1 & 0 & 0 \\ 0 & 0 & p & 1 & 0 \\ 0 & 0 & 0 & p & 1 \\ 0 & 0 & 0 & 0 & p \end{bmatrix} n = 5$$

Blok dijagram (realizacija):

$x_1' = -x_1 + x_2,\quad x_2' = -x_2 + u,\quad x_3' = u,\quad x_4' = -2x_4 + u.$



$y = -6x_1 - 5x_2 + 6x_3 - x_4.$

Zadatak 6., paralelna realizacija

$$H(s) = \frac{(s+2)\cdot(s+1)}{(s+3)\cdot\underbrace{(s^2+4s+5)}_{}},$$

$s_{1,2} = -2 \pm j \Rightarrow$ konjugirano kompleksna rješenja

$$H(s) = d_0 + \frac{C_1}{s+3} + \frac{C_2 \cdot s + C_3}{s^2+4s+5},$$

$$d_0 = \lim_{s\to\infty} H(s) = 0,$$

$$C_1 = (s+3)H(s)\big|_{s=-3} = 1.$$

nastavak

- C_2 , C_3 = ?
- Metoda jednakih koeficijenata.

$$H(s)=\frac{(s+2)(s+1)}{(s+3)(s^2+4s+5)}=\frac{1}{s+3}+\frac{C_2\cdot s+C_3}{s^2+4s+5},$$

$$\frac{s^2+3s+2}{(s+3)(s^2+4s+5)}$$

nastavak

- Izjednačimo brojnike

$$s^2+3s+2=s^2(1+C_2)+s(4+3C_2+C_3)+(5+3C_3),$$

$$1+C_2=1 \quad \Rightarrow C_2=0,$$

$$5+3C_3=2 \quad \Rightarrow C_3=-1,$$

$$4+3C_2+C_3=3,$$

$$4+3\cdot 0-1=3,$$

$$3=3.$$

nastavak

$$H(s)=\frac{1}{s+3}-\frac{1}{s^2+4s+5},$$

$$Y(s)=H(s)\cdot U(s)=\frac{1}{s+3}U(s)-\frac{1}{s^2+4s+5}U(s),$$

$$X_1(s)=\frac{U(s)}{s+3} \quad \Rightarrow x_1'=-3x_1+u,$$

$$X_2(s)=\frac{U(s)}{s^2+4s+5} \quad \Rightarrow x_2''+4x_2'+5x_2=u,$$

$$x_2'=x_3,$$

$$x_3'+4x_3+5x_2=u,$$

$$x_3'=-5x_2-4x_3+u.$$

nastavak

$$Y(s) = x_1(s) - x_2(s),$$

$$y = x_1 - x_2\,.$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & -4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot u.$$

Matrica A nije dijagonalna i nema samo Jordanove blokove.

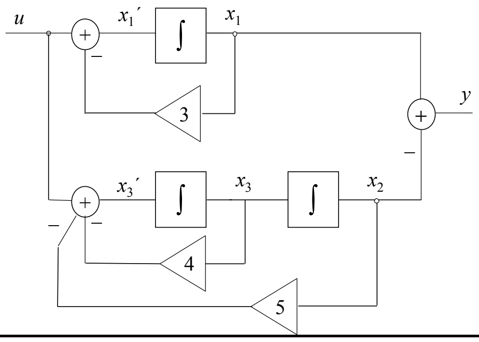
$$y = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0 \cdot u.$$

nastavak

$x_1' = -3x_1 + u,$

$x_2' = x_3,$ $\qquad y = x_1 - x_2.$

$x_3' = -5x_2 - 4x_3 + u,$



Zadatak 7., paralelna realizacija

$$H(s) = \frac{s(s+1)}{(s+2)(s+1)}$$

- s + 1 se ne smije pokratiti u slučaju kada se traže varijable stanja

- par pol/nula postoji u sustavu, utječe na njegovo vladanje, ali je nevidljiv s ulazno-izlaznih stezaljki

Zadatak 7., paralelna realizacija

$$H(s)=\frac{s(s+1)}{(s+2)(s+1)}$$

Rješenje:

$$H(s)=d_0+\frac{C_1}{s+2}+\frac{C_2}{s+1}$$

$$\left.\begin{aligned} d_0 &= 1 \\ C_1 &= -2 \\ C_2 &= 0 \end{aligned}\right\} H(s)=1-2\cdot\frac{1}{s+2}+0\cdot\frac{1}{s+1}$$

nastavak

$$Y(s)=U(s)-2\cdot\underbrace{\frac{U(s)}{s+2}}_{\mathrm{X}_1(\mathrm{s})}+0\cdot\underbrace{\frac{U(s)}{s+1}}_{\mathrm{X}_2(\mathrm{s})}$$

$$x_1^{'}=-2x_1+u$$

$$x_2^{'}=-x_2+u$$

$$y=-2x_1+0\cdot x_2+u$$

nastavak

Matrični oblik:

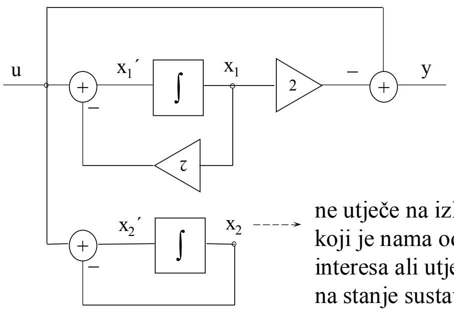
$$\begin{bmatrix} x_1^{'} \\ x_2^{'} \end{bmatrix}=\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}+\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cdot u$$

$$y=\begin{bmatrix} -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}+1\cdot u$$

$$Y(s)=U(s)-2X_1(s)+0\cdot X_2(s)$$

nastavak

$y = -2x_1 + 0 \cdot x_2 + u$



ne utječe na izlaz koji je nama od interesa ali utječe na stanje sustava