Signali i sustavi

Pismeni ispit – 22. travnja 2015.

1. (8 bodova) Zadan je vremenski kontinuiran signal $f(t) = \begin{cases} 1, & \text{za } t < 0, \\ e^{-2t}, & \text{za } t \geq 0. \end{cases}$

a) **(2 boda)** Odredite vremenski diskretan signal $f(n)$ koji dobijemo očitavanjem vremenski kontinuiranog signala $f(t)$ s periodom očitavanja T_s.

b) **(2 boda)** Odredite vremenski diskretan signal $f_{ad}(n)$ koji dobijemo aproksimacijom derivacije metodom silazne diferencije, ako je period očitavanja T_s.

c) **(2 boda)** Odredite vremenski diskretan signal $f_d(n)$ koji dobijemo očitavanjem generalizirane derivacije vremenski kontinuiranog signala $f(t)$ s periodom očitavanja T_s.

d) **(2 boda)** Izračunajte energiju greške $E_g = \sum_{n=-\infty}^{\infty} |f_d(n) - f_{ad}(n)|^2$ aproksimacije derivacije metodom silazne diferencije.

2. (8 bodova) Zadan je vremenski diskretan signal $f(n) = \cos\left(\frac{\pi}{2}n\right)\left(\mu(n) - \mu(n-4)\right)$.

a) **(3 boda)** Odredite vremenski diskretnu Fourierovu transformaciju zadanog signala (DTFT).

b) **(2 boda)** Je li dobiveni spektar periodičan ili aperiodičan? Ako je periodičan, koliki mu je osnovni period?

c) **(3 boda)** Odredite koeficijente vremenski diskretnog Fourierovog reda (DTFS) vremenski diskretnog signala $g(n) = \cos\left(\frac{\pi}{2}n\right)$.

3. (8 bodova) Zadan je spektar vremenski kontinuiranog signala $F(j\omega) = j\left[-\mu(\omega + 2\pi) + 2\,\mu(\omega) - \mu(\omega - 2\pi)\right]$

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c) **(3 boda)** Izvedite Parsevalovu relaciju za vremenski kontinuiranu Fourierovu transformaciju (CTFT).

4. (8 bodova) Zadan je spektar vremenski kontinuiranog signala $F(j\omega) = (\omega+\pi)(\mu(\omega+\pi)-\mu(\omega))+(-\omega+\pi)(\mu(\omega)-\mu(\omega-\pi))$.

a) **(2 boda)** Možemo li očitati odgovarajući signal u vremenskoj domeni tako da ne dođe do aliasinga u frekvencijskoj domeni? Ako da, objasnite zašto da i odredite minimalnu frekvenciju očitavanja tako da ne dođe do aliasinga, a ako ne objasnite zašto ne.

b) **(2 boda)** Ako signal $f(t)$ očitamo frekvencijom $\omega_s = \frac{3\pi}{2}$, skicirajte amplitudni spektar očitanog kontinuiranog signala.

c) **(2 boda)** Ako signal $f(t)$ očitamo frekvencijom $\omega_s = 3\pi$, skicirajte amplitudni spetar očitanog kontinuiranog signala.

d) **(2 boda)** Objasnite postupak rekonstrukcije kontinuiranog signala iz očitanog kontinuiranog signala u vremenskoj i frekvencijskoj domeni.

5. (8 bodova) Zadani su vremenski kontinuirani signali $f(t)$ i $g(t)$ za koje vrijedi $g(t) = f(at)$, $a > 0$.

a) **(2 boda)** Ako je energija signala $f(t)$ konačna i iznosi E_f, odredite energiju signala $g(t) = f(at)$, E_g (izvedite izraz).

b) **(3 boda)** Odredite energiju vremenski kontinuiranog signala $h(t) = \frac{\sin(10t)}{10t}$.

c) **(3 boda)** Ako je signal $f(t)$ periodičan s osnovnim periodom T_0 te ako je njegova snaga konačna i iznosi P_f, odredite snagu signala $g(t) = f(at)$, P_g (izvedite izraz).

Signali i sustavi

Pismeni ispit – 22. travnja 2015.

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c) **(3 boda)** Izvedite Parsevalovu relaciju za vremenski kontinuiranu Fourierovu transformaciju (CTFT).

4. (8 bodova) Zadan je spektar vremenski kontinuiranog signala $F(j\omega) = (\omega+2\pi)(\mu(\omega+2\pi) - \mu(\omega)) + (-\omega+2\pi)(\mu(\omega) - \mu(\omega-2\pi))$.

a) **(2 boda)** Možemo li očitati odgovarajući signal u vremenskoj domeni tako da ne dođe do aliasinga u frekvencijskoj domeni? Ako da, objasnite zašto da i odredite minimalnu frekvenciju očitavanja tako da ne dođe do aliasinga, a ako ne objasnite zašto ne.

b) **(2 boda)** Ako signal $f(t)$ očitamo frekvencijom $\omega_s = 3\pi$, skicirajte amplitudni spektar očitanog kontinuiranog signala.

c) **(2 boda)** Ako signal $f(t)$ očitamo frekvencijom $\omega_s = 6\pi$, skicirajte amplitudni spetar očitanog kontinuiranog signala.

d) **(2 boda)** Objasnite postupak rekonstrukcije kontinuiranog signala iz očitanog kontinuiranog signala u vremenskoj i frekvencijskoj domeni.

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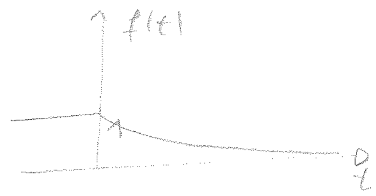
A

1. $$f(t)=\begin{cases}1 & \text{za } t<0\\ e^{-2t} & \text{za } t\geq 0\end{cases}$$

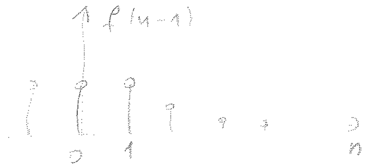


a) $f(n)=?$

T_s period očitavanja

$$f(n)=\begin{cases}1, & \text{za } n<0\\ e^{-2T_s\cdot n}, & \text{za } n\geq 0\end{cases}$$

b) $f_{ad}(n)=?$



$$f_{ad}(n)=\frac{1}{T_s}\left\{f(n)-f(n-1)\right\}$$

$$f_{ad}(n)=\frac{1}{T_s}\left\{1-1\right\}=0 \quad \text{za } n\leq 0$$

$$f_{ad}(n)=\frac{1}{T_s}\left\{e^{-2T_sn}-e^{-2T_s(n-1)}\right\}=\frac{1}{T_s}e^{-2T_sn}\left(1-e^{2T_s}\right) \quad \text{za } n>0$$

c) $f_d(n)=?$

$$f'(t)=\begin{cases}0, & t<0\\ -2e^{-2t}, & t\geq 0\end{cases}$$

$$f_d(n)=\begin{cases}0, & n<0\\ -2e^{-2T_sn}, & n\geq 0\end{cases}$$

d) $$E_g=\sum_{n=-\infty}^{\infty}\left|f_d(n)-f_{ad}(n)\right|^2$$

$$=\sum_{n=-\infty}^{-1}|0-0|^2+\sum_{n=1}^{\infty}\left|\frac{1}{T_s}\left(e^{-2T_sn}-e^{-2T_sn}\cdot e^{2T_s}\right)-\left(-2e^{-2T_sn}\right)\right|^2+\left(-2e^{-2T_s\cdot 0}-0\right)^2$$

$$=\sum_{n=1}^{\infty}\left|\left(\frac{1}{T_s}-\frac{1}{T_s}e^{2T_s}+2\right)e^{-2T_sn}\right|^2+4$$

$$=\left(\frac{1}{T_s}-\frac{1}{T_s}e^{2T_s}+2\right)^2\cdot\sum_{n=1}^{\infty}\left(e^{-4T_s}\right)^n+4$$

$$=\left(\frac{1}{T_s}-\frac{1}{T_s}e^{2T_s}+2\right)^2\cdot\sum_{n=0}^{\infty}\left(e^{-4T_s}\right)^n-\left(\frac{1}{T_s}-\frac{1}{T_s}e^{2T_s}+2\right)^2\cdot 1+4$$

$$=\left(\frac{1-e^{2T_s}+2T_s}{T_s}\right)^2\cdot\frac{1}{1-e^{-4T_s}}-\frac{\left(1-e^{2T_s}+2T_s\right)^2}{T_s^2}+4$$

$$=4+\frac{\left(1-e^{2T_s}+2T_s\right)^2\left(1-1+e^{-4T_s}\right)}{T_s^2\left(1-e^{-4T_s}\right)}$$

$$=4+\frac{\left(1-e^{2T_s}+2T_s\right)^2\cdot e^{-4T_s}}{T_s^2\left(1-e^{-4T_s}\right)}$$

2. $f(n) = \cos\frac{\pi}{2}n \cdot (\mu(n) - \mu(n-4)) = \{\underline{1}, 0, -1, 0\}$ $\boxed{A}$

a) DTFT

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$= e^{-j\Omega \cdot 0} - e^{-j\Omega \cdot 2}$$

$$= e^{-j\Omega}\left(e^{j\Omega} - e^{-j\Omega}\right)$$

$$= e^{-j\Omega}\left(2j \sin\Omega\right)$$

$$= 2j \sin\Omega\, e^{-j\Omega}$$

$$= 2 \sin\Omega\, e^{j\frac{\pi}{2}} e^{-j\Omega}$$

$$= 2 \sin\Omega\, e^{j(-\Omega + \frac{\pi}{2})}$$

b) Spektar je periodičan sa 2π

$$X(e^{j(\Omega + 2k\pi)}) = 2\sin(\Omega + 2k\pi) \cdot e^{j(-(\Omega + 2k\pi) + \frac{\pi}{2})}$$

$$= 2\sin(\Omega + 2k\pi) \cdot e^{j(-\Omega + \frac{\pi}{2})} \cdot \underbrace{e^{-j \cdot 2k\pi}}_{\underbrace{\cos 2k\pi}_{1} - j \underbrace{\sin 2k\pi}_{\emptyset}}$$

$$= 2\sin\Omega \cdot e^{j(-\Omega + \frac{\pi}{2})}$$

c) DTFS

$$g(n) = \cos\frac{\pi}{2}n$$

$$= \frac{1}{2} e^{j\frac{\pi}{2}n} + \frac{1}{2} e^{-j\frac{\pi}{2}n}$$

Period $g(n)$:

$$\frac{\pi}{2}N = 2k\pi$$

$$N = 4k$$

$$N_0 = 4$$

$$g(n) = \frac{1}{2} e^{j\frac{2\pi}{4} \cdot n \cdot 1} + \frac{1}{2} e^{j\frac{2\pi}{4} \cdot n \cdot (-1)}$$

$$= \frac{1}{2} e^{j\frac{2\pi}{4} \cdot n \cdot 1} + \frac{1}{2} e^{j\frac{2\pi}{4} \cdot n \cdot 3}$$

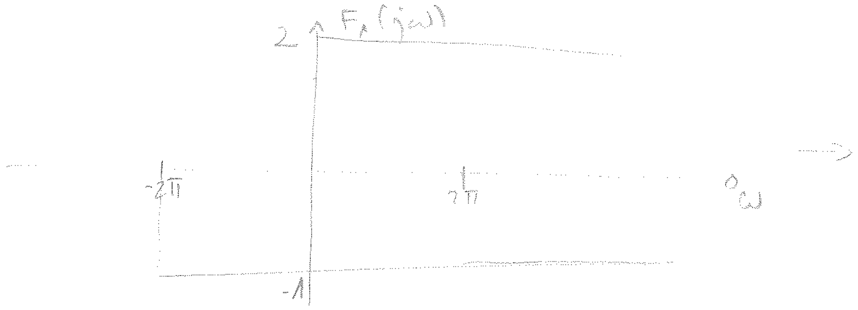
$$\boxed{G_1 = \frac{1}{2} \qquad G_3 = \frac{1}{2}}$$

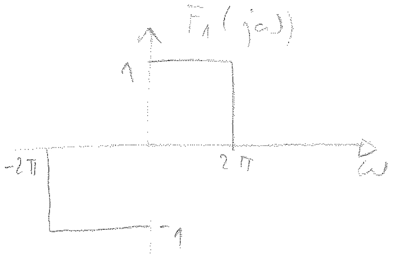
A

3

$$F(j\omega) = j\left[-\mu(\omega+2\pi) + 2\mu(\omega) - \mu(\omega-2\pi)\right] = j \cdot F_1(j\omega)$$

a) $$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$





$$f(t) = \frac{1}{2\pi}\int_{-2\pi}^{0} (-j)\, e^{j\omega t}\, d\omega + \frac{1}{2\pi}\int_{0}^{2\pi} j\, e^{j\omega t}\, d\omega$$

$$= \frac{1(-j)}{2\pi} \cdot \left.\frac{e^{j\omega t}}{jt}\right|_{-2\pi}^{0} + \frac{1}{2\pi} \cdot \cancel{j} \left.\frac{e^{j\omega t}}{\cancel{j}t}\right|_{0}^{2\pi}$$

$$= \frac{-1}{2\pi t}\left(1 - e^{-2\pi j t}\right) + \frac{1}{2\pi t}\left(e^{j \cdot 2\pi \cdot t} - e^{j \cdot 0}\right)$$

$$= -\frac{1}{2\pi t}(1+1) + \frac{1}{2\pi t}\left(e^{-j2\pi t} + e^{j2\pi t}\right)$$

$$= -\frac{\cancel{2}}{\cancel{2}\pi t} + \frac{1}{2\pi t} \cdot 2\cos 2\pi t$$

$$= 2 \cdot \frac{\cos 2\pi t - 1}{2\pi t}$$

$$f(0) = \frac{1}{2\pi}\left[\int_{-2\pi}^{0} (-j)\, d\omega + \int_{0}^{2\pi} j\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[-j\omega\Big|_{-2\pi}^{0} + j\omega\Big|_{0}^{2\pi}\right]$$

b) $$g(t) = f(t-4)$$

$$G(j\omega) = F(j\omega)\, e^{-j\omega 4}$$

$$= j e^{-j\omega \cdot 4}\left[-\mu(\omega+2\pi) + 2\mu(\omega) - \mu(\omega-2\pi)\right]$$

3. c) Parsevalova relacija za CTFT

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega$$

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} f(t) \cdot f^*(t) dt =$$

$$= \int_{-\infty}^{\infty} f(t) \cdot \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(j\omega) e^{-j\omega t} d\omega \right] dt$$

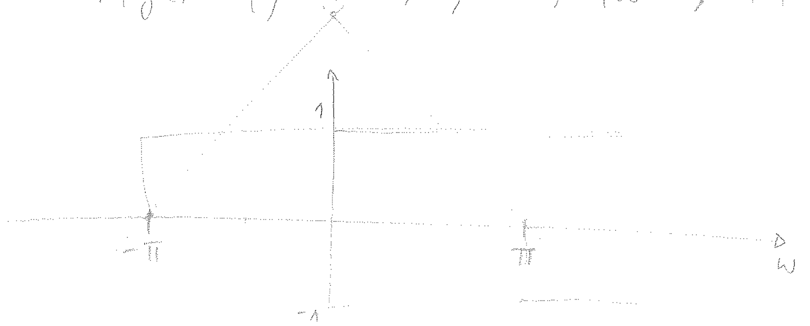
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(j\omega) \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right] d\omega$$

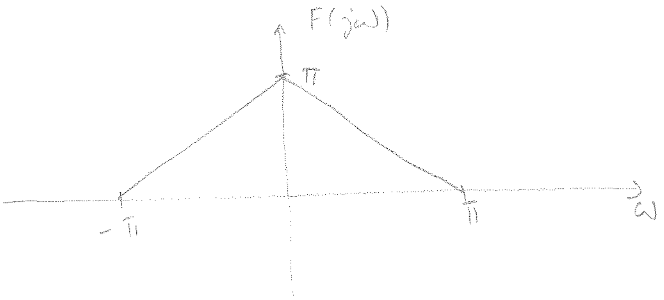
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(j\omega) \quad F(j\omega) \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega$$

A

4. $F(j\omega) = \left(\mu(\omega+\pi) - \mu(\omega)\right)\cdot(\omega+\pi) + (-\omega+\pi)\left(\mu(\omega) - \mu(\omega-\pi)\right)$



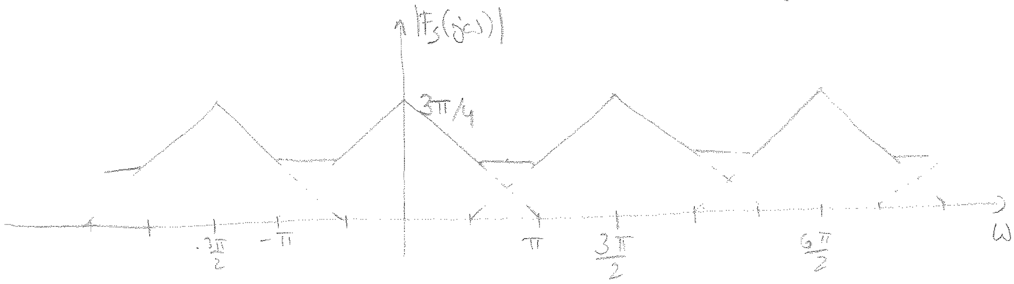


a) Maksimalna frekvencija u signalu je π.
Kako postoji maksimalna frekvencija koja je konačan broj — ovaj signal se može jednoznačno očitati u vremenskoj domeni.

Minimalna frekvencija očitavanja je $2\cdot\omega_{max} = 2\pi$

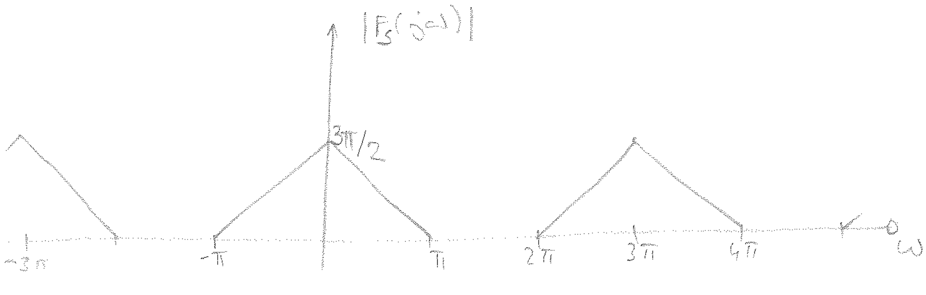
b) $\omega_s = \frac{3\pi}{2} \longrightarrow T_s = \frac{2\pi}{\omega_s} = \frac{2\pi}{\frac{3\pi}{2}} = \frac{4\pi}{3\pi} = \frac{4}{3}$

$\omega_s < 2\pi$ dolazi do aliasinga



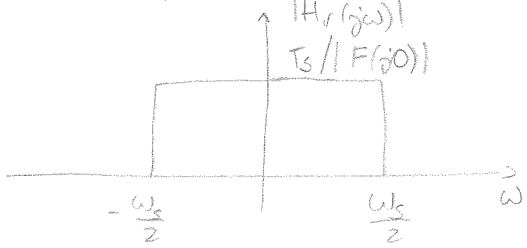
c) $\omega_s = 3\pi \longrightarrow T_s = \frac{2\pi}{\omega_s} = \frac{2\pi}{3\pi} = \frac{2}{3}$

$\omega_s > 2\pi$ neće doći do aliasinga



4 d) Rekonstrukcija kontinuiranog signala iz očitanog

- u frekvencijskoj domeni množenjem idealnim filtrom



$X(j\omega) = X_s(j\omega) \cdot H_r(j\omega)$

- u vremenskoj domeni je to konvolucija očitanog kontinuiranog signala s $ICTFT\{H_r(j\omega)\}$

$$h_r(t) = ICTFT\{H_r(j\omega)\} = \frac{1}{2\pi}\int_{-\omega_s/2}^{\omega_s/2} \frac{T_s}{|F(j0)|} e^{j\omega t} d\omega$$

$$= \frac{T_s}{2\pi|F(j0)|} \cdot \left.\frac{e^{j\omega t}}{jt}\right|_{-\omega_s/2}^{\omega_s/2}$$

$$= \frac{T_s}{2\pi|F(j0)|jt}\left(e^{j\frac{\omega_s}{2}t} - e^{-j\frac{\omega_s}{2}t}\right)$$

$$= \frac{T_s \cdot 2j\sin\frac{\omega_s}{2}t}{2j\pi t\cdot|F(j0)|}$$

$$= \frac{\cancel{2\pi}\cdot 2\sin\frac{\omega_s}{2}t}{\cancel{2\pi}\, t\cdot|F(j0)|\cdot\omega_s}$$

$$= \frac{1}{|F(j0)|}\cdot\frac{\sin\frac{\omega_s}{2}t}{\frac{\omega_s}{2}t}$$

$x(t) = x_s(t) * h_r(t)$

$\boxed{A}$

5. $f(t)$, $g(t)$

$g(t) = f(at)$, $a > 0$

a) $E_f = E(f(t)) < \infty$

$E_g = ?$

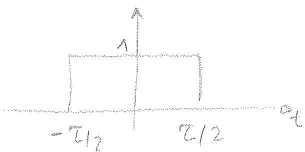
$$E_f = \int_{-\infty}^{\infty} |f(t)|^2 \, dt$$

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 \, dt = \int_{-\infty}^{\infty} |f(at)|^2 \, dt = \left| \begin{array}{ll} at = \tau & \\ a\,dt = d\tau & \\ t = -\infty & \tau = -\infty \\ t = \infty & \tau = \infty \end{array} \right|$$

$$= \int_{-\infty}^{\infty} |f(\tau)|^2 \frac{1}{a} \, d\tau = \frac{1}{a} \int_{-\infty}^{\infty} |f(\tau)|^2 \, d\tau$$

$$= \frac{1}{a} E_f$$

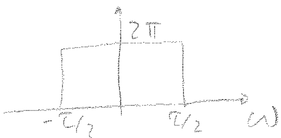
b) $h(t) = \dfrac{\sin 10t}{10t}$



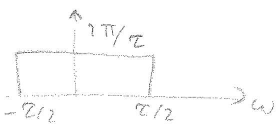
$\xleftrightarrow{CTFT}$ $\tau \cdot \dfrac{\sin \frac{\tau}{2}\omega}{\frac{\tau}{2}\omega}$

SVOJSTVO DUALNOSTI

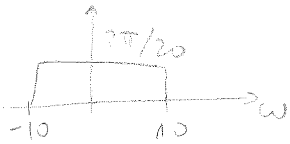
$\tau \dfrac{\sin \frac{\tau}{2} t}{\frac{\tau}{2} t}$ $\xleftrightarrow{CTFT}$



$\dfrac{\sin \frac{\tau}{2} t}{\frac{\tau}{2} t}$ $\xleftrightarrow{CTFT}$



$\dfrac{\sin 10t}{10t}$ $\xleftrightarrow{CTFT}$



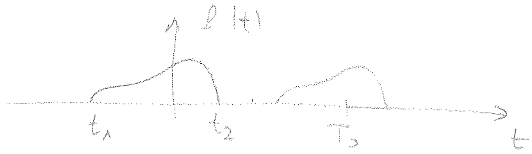
$$E_h = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(j\omega)|^2 \, d\omega$$

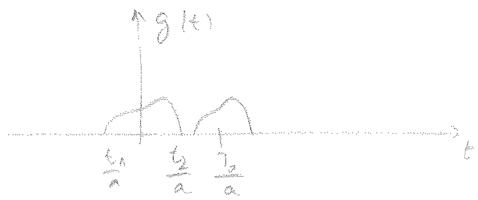
$$= \frac{1}{2\pi} \int_{-10}^{10} \left| \frac{2\pi}{\cancel{20}\,10} \right|^2 d\omega$$

$$= \frac{1}{2\pi} \frac{\pi^2}{10^2} \, \omega \Big|_{-10}^{10} = \frac{\pi}{2 \cdot 100} (10 + 10) = \frac{20\pi}{200} = \frac{\pi}{10}$$

c) $f(t)$ periodičen s T_0

$g(t) = f(at)$





$$P_f = \frac{1}{T_0}\int_{T_0} |f(t)|^2 dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} |f(t)|^2 dt$$

$$P_g = \frac{1}{\frac{T_0}{a}} \int_{T_0/a} |f(at)|^2 dt$$

$$= \frac{a}{T_0}\int_{-T_0/2a}^{T_0/2a} |f(at)|^2 dt = \left| \begin{array}{ll} at = \tau & t = -\frac{T_0}{2a} \to \tau = -\frac{T_0}{2a}\cdot a = -\frac{T_0}{2} \\ a\,dt = d\tau & \\ dt = \frac{1}{a}d\tau & t = \frac{T_0}{2a} \to \tau = \frac{T_0}{2a}\cdot a = \frac{T_0}{2} \end{array} \right|$$

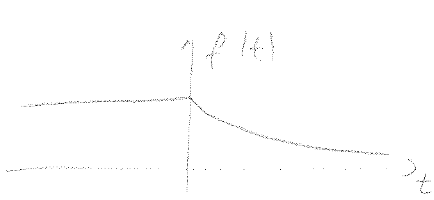
$$= \frac{a}{T_0}\int_{-T_0/2}^{T_0/2} |f(\tau)|^2 \frac{1}{a} d\tau = \frac{a}{T_0}\cdot\frac{1}{a}\int_{-T_0/2}^{T_0/2} |f(\tau)|^2 d\tau$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} |f(\tau)|^2 d\tau$$

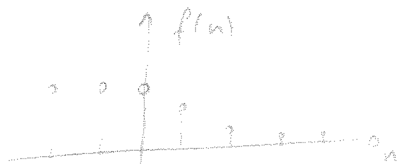
$$= P_f$$

B

1. $$f(t)=\begin{cases}1, & t<0\\ e^{-4t}, & t\geq 0\end{cases}$$



a) $$f(n)=\begin{cases}1 & n<0\\ e^{-4T_s\cdot n}, & n\geq 0\end{cases}$$



b) $$f_{ad}(n)=\frac{1}{T_s}\left(f(n)-f(n-1)\right)$$

$$f_{ad}(n)=\frac{1}{T_s}\left(1-1\right)=0\quad,\quad n\leq 0$$

$$f_{ad}(n)=\frac{1}{T_s}\left(e^{-4T_s\cdot n}-e^{-4T_s(n-1)}\right)=$$

$$=\frac{1}{T_s}e^{-4T_s n}\left(1-e^{4T_s}\right)\quad,\quad n>0$$

c) $$f'(t)=\begin{cases}0 & ; t<0\\ -4e^{-4t} & ; t\geq 0\end{cases}$$

$$f_d(n)=\begin{cases}0 & n<0\\ -4e^{-4T_s\cdot n}, & n\geq 0\end{cases}$$

d) $$E_g=\sum_{n=-\infty}^{\infty}\left|f_d(n)-f_{ad}(n)\right|^2$$

$$=\sum_{n=-\infty}^{-1}|0-0|^2+\left|-4e^{-4T_s\cdot 0}-0\right|^2+\sum_{n=1}^{\infty}\left|-4e^{-4T_s\cdot n}-\frac{1}{T_s}e^{-4T_s n}\left(1-e^{4T_s}\right)\right|^2$$

$$=16+\sum_{n=1}^{\infty}\left|e^{-4T_s\cdot n}\left(-4-\frac{1}{T_s}\left(1-e^{4T_s}\right)\right)\right|^2$$

$$=16+\sum_{n=0}^{\infty}\left|e^{-4T_s\cdot n}\right|^2\cdot\left|-4-\frac{1}{T_s}\left(1-e^{4T_s}\right)\right|^2-\left(-4-\frac{1}{T_s}\left(1-e^{4T_s}\right)\right)^2$$

$$=16-\left(4+\frac{1-e^{4T_s}}{T_s}\right)^2+\left(4+\frac{1-e^{4T_s}}{T_s}\right)^2\cdot\sum_{n=0}^{\infty}\left(e^{-8T_s}\right)^n$$

$$=16-\frac{\left(4T_s+1-e^{4T_s}\right)^2}{T_s^2}+\frac{\left(4T_s+1-e^{4T_s}\right)^2}{T_s^2}\cdot\frac{1}{1-e^{-8T_s}}$$

$$=\frac{16T_s^2\left(1-e^{-8T_s}\right)-\left(1-e^{-8T_s}\right)\left(4T_s+1-e^{4T_s}\right)^2+\left(4T_s+1-e^{4T_s}\right)^2}{T_s^2\left(1-e^{-8T_s}\right)}$$

$$=\frac{16T_s^2\left(1-e^{-8T_s}\right)+\left(4T_s+1-e^{4T_s}\right)^2e^{-8T_s}}{T_s^2\left(1-e^{-8T_s}\right)}$$

(B)

2.

$f(n) = \cos\left(\frac{3\pi}{2}n\right)\left(\mu(n) - \mu(n-4)\right) = \{\underline{1}, 0, -1, 0\}$

a) DTFT

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$= 1 \cdot e^{-j\Omega 0} - 1\, e^{-j\Omega 2}$$

$$= e^{-j\Omega}\left(e^{j\Omega} - e^{-j\Omega}\right)$$

$$= 2j \sin\Omega\, e^{-j\Omega}$$

$$= 2 \sin\Omega\, e^{j\left(-\Omega + \frac{\pi}{2}\right)}$$

b) Spektar je periodičan sa 2π.

Dokaz:

$$X(e^{j(\Omega + 2k\pi)}) = 2\sin(\Omega + 2k\pi)\, e^{j\left(-(\Omega + 2k\pi) + \frac{\pi}{2}\right)}$$

$$= 2\sin(\Omega + 2k\pi) \cdot e^{j\left(-\Omega + \frac{\pi}{2}\right)} \cdot \underbrace{e^{-j2k\pi}}_{\underbrace{\cos 2k\pi}_{1} - j\underbrace{\sin 2k\pi}_{0}}$$

$$= 2\sin\Omega \cdot e^{j\left(-\Omega + \frac{\pi}{2}\right)}$$

c) DTFS

$g(n) = \cos\frac{3\pi}{2}n$

Period $\quad \frac{3\pi}{2}N = 2k\pi$

$N = \frac{4k}{3}$

$N_0 = 4$

$$g(n) = \frac{1}{2}\, e^{j\frac{3\pi}{2}n} + \frac{1}{2}\, e^{-j\frac{3\pi}{2}n}$$

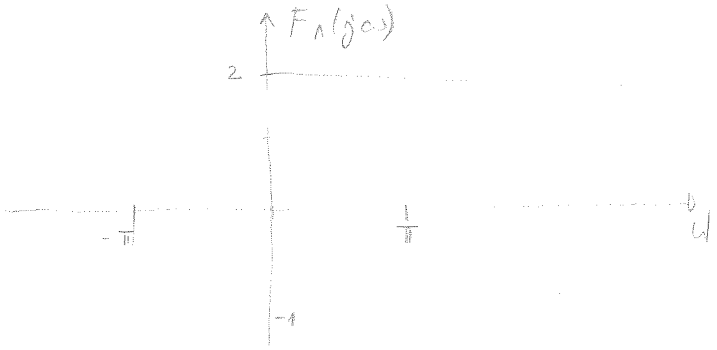
$$= \frac{1}{2}\, e^{j\frac{2\pi}{4}\cdot n \cdot 3} + \frac{1}{2}\, e^{j\frac{2\pi}{4}\cdot n \cdot (-3)}$$

$$= \frac{1}{2}\, e^{j\frac{2\pi}{4}\cdot n \cdot 3} + \frac{1}{2}\, e^{j\frac{2\pi}{4}\cdot n \cdot 1}$$

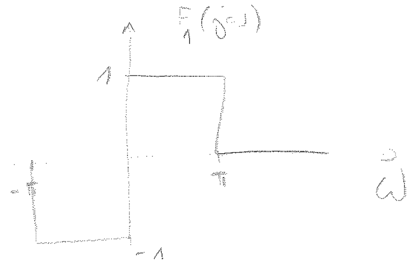
$\boxed{G_1 = \frac{1}{2}}$ $\quad$ $\boxed{G_3 = \frac{1}{2}}$

3. $F(j\omega) = j\left[-\mu(\omega+\pi) + 2\mu(\omega) - \mu(\omega-\pi)\right] = j \cdot F_1(j\omega)$ $\boxed{B}$

a) $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$



$\longrightarrow$



$$f(t) = \frac{1}{2\pi}\int_{-\pi}^{0}(-j)\, e^{j\omega t}\, d\omega + \frac{1}{2\pi}\int_{0}^{\pi} j\, e^{j\omega t}\, d\omega$$

$$= \frac{1\cdot(-j)}{2\pi} \cdot \frac{e^{j\omega t}}{jt}\Bigg|_{-\pi}^{0} + \frac{j}{2\pi}\,\frac{e^{j\omega t}}{jt}\Bigg|_{0}^{\pi}$$

$$= \frac{-j}{2\pi jt}\left(e^{0} - e^{-jt\pi}\right) + \frac{j}{2\pi jt}\left(e^{j\pi t} - e^{0}\right)$$

$$= \frac{1}{2\pi t}\left(e^{-j\pi t} + e^{j\pi t}\right) - \frac{1}{2\pi t} - \frac{1}{2\pi t}$$

$$= \frac{2\cos \pi t}{2\pi t} - \frac{2}{2\pi t}$$

$$= \frac{\cos \pi t}{\pi t} - \frac{1}{\pi t}, \quad t \neq 0$$

$$f(0) = \frac{1}{2\pi}\int_{-\pi}^{0}(-j)\,d\omega + \frac{1}{2\pi}\int_{0}^{\pi} j\,d\omega$$

$$= \frac{-j}{2\pi}\,\omega\Big|_{-\pi}^{0} + \frac{j}{2\pi}\,\omega\Big|_{0}^{\pi}$$

$$= \frac{-j}{2\pi}(0+\pi) + \frac{j}{2\pi}(\pi - 0)$$

$f(0) = 0$

b) $G(j\omega) = ?$ $\qquad$ in tablici: $x(t-t_0) \xleftrightarrow{CTFT} X(j\omega)\, e^{-j\omega t_0}$

$g(t) = f(t-5)$

$$G(j\omega) = F(j\omega)\, e^{-j\omega \cdot 5}$$

$$= j \cdot e^{-j\omega 5} \cdot \left[-\mu(\omega+\pi) + 2\mu(\omega) - \mu(\omega-\pi)\right]$$

3. c) Parsevalova relacija za CTFT

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega$$

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} f(t) \cdot f^*(t) dt =$$

$$= \int_{-\infty}^{\infty} f(t) \cdot \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(j\omega) e^{-j\omega t} d\omega \right] dt$$

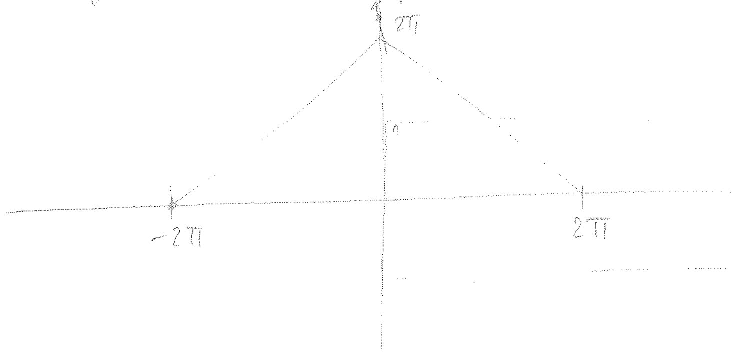
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(j\omega) \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right] d\omega$$

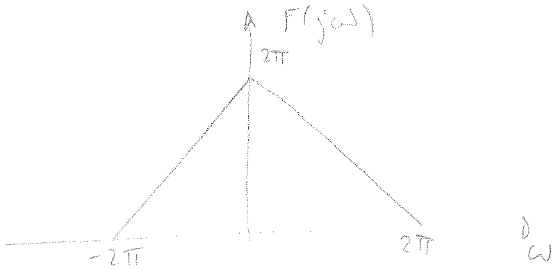
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(j\omega) \; F(j\omega) \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega$$

$\boxed{B}$

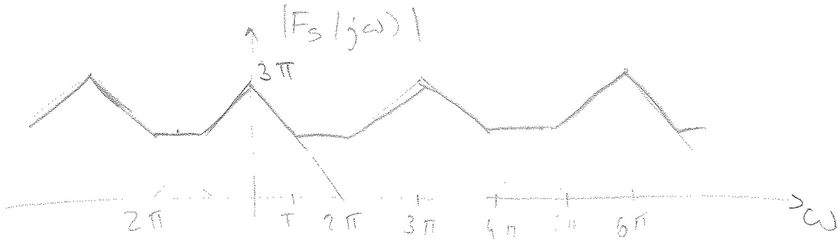
4. $F(j\omega) = (\omega + 2\pi)\left(\mu(\omega+2\pi) - \mu(\omega)\right) + (-\omega + 2\pi)\left(\mu(\omega) - \mu(\omega - 2\pi)\right)$





a) Maksimalna frekvencija u signalu je 2π.
Kako maksimalna frekvencija postoji (konačan broj) - ovaj signal se može jednoznačno očitati u vremenskoj domeni.
Minimalna frekvencija očitavanja je $2 \cdot \omega_{max} = 4\pi$

b) $\omega_s = 3\pi$



$T_s = \frac{2\pi}{\omega_s} = \frac{2\pi}{3\pi} = \frac{2}{3}$

$\omega_s < 2\pi$ dolazi do aliasinga

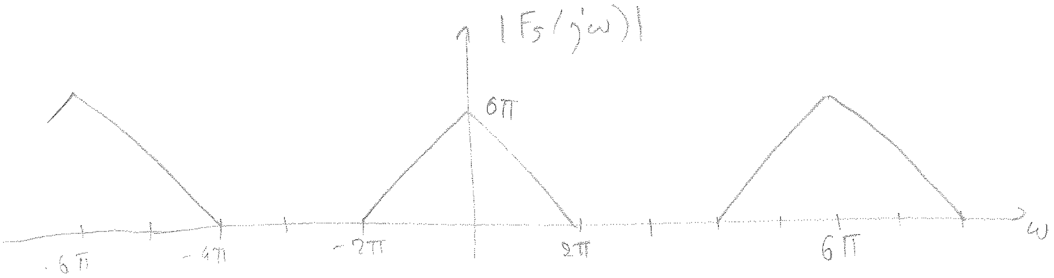
amplituda u $\omega = 0$:
$2\pi \frac{1}{T_s} = 2\pi \cdot \frac{3}{2} = 3\pi$

c) $\omega_s = 6\pi$

$\omega_s > 2\pi$ neće doći do aliasinga

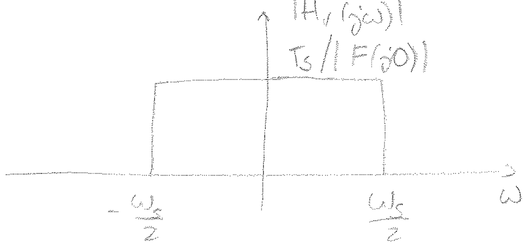
$T_s = \frac{2\pi}{6\pi} = \frac{1}{3}$

amplituda $\frac{2\pi}{T_s} = 2\pi \cdot 3 = 6\pi$



4 d) Rekonstrukcija kontinuiranog signala iz očitanog

- u frekvencijskoj domeni množenjem idealnim filtrom



$X(j\omega) = X_s(j\omega)\cdot H_r(j\omega)$

- u vremenskoj domeni je to konvolucija očitanog kontinuiranog signala s $ICTFT\{H_r(j\omega)\}$

$$h_r(t) = ICTFT\{H_r(j\omega)\} = \frac{1}{2\pi}\int_{-\omega_s/2}^{\omega_s/2} \frac{T_s}{|F(j0)|} e^{j\omega t} d\omega$$

$$= \frac{T_s}{2\pi|F(j0)|}\cdot \left.\frac{e^{j\omega t}}{jt}\right|_{-\omega_s/2}^{\omega_s/2}$$

$$= \frac{T_s}{2\pi|F(j0)|jt}\left(e^{j\frac{\omega_s}{2}t} - e^{-j\frac{\omega_s}{2}t}\right)$$

$$= \frac{T_s\cdot 2j\sin\frac{\omega_s}{2}t}{2j\pi t\cdot|F(j0)|}$$

$$= \frac{\cancel{2\pi}\cdot 2\sin\frac{\omega_s}{2}t}{\cancel{2\pi}\, t\cdot|F(j0)|\cdot\omega_s}$$

$$= \frac{1}{|F(j0)|}\cdot\frac{\sin\frac{\omega_s}{2}t}{\frac{\omega_s}{2}t}$$

$x(t) = x_s(t) * h_r(t)$

5. $g(t) = f(at), \quad a > 0$

B

a) $E_f = E(f(t)) < \infty$

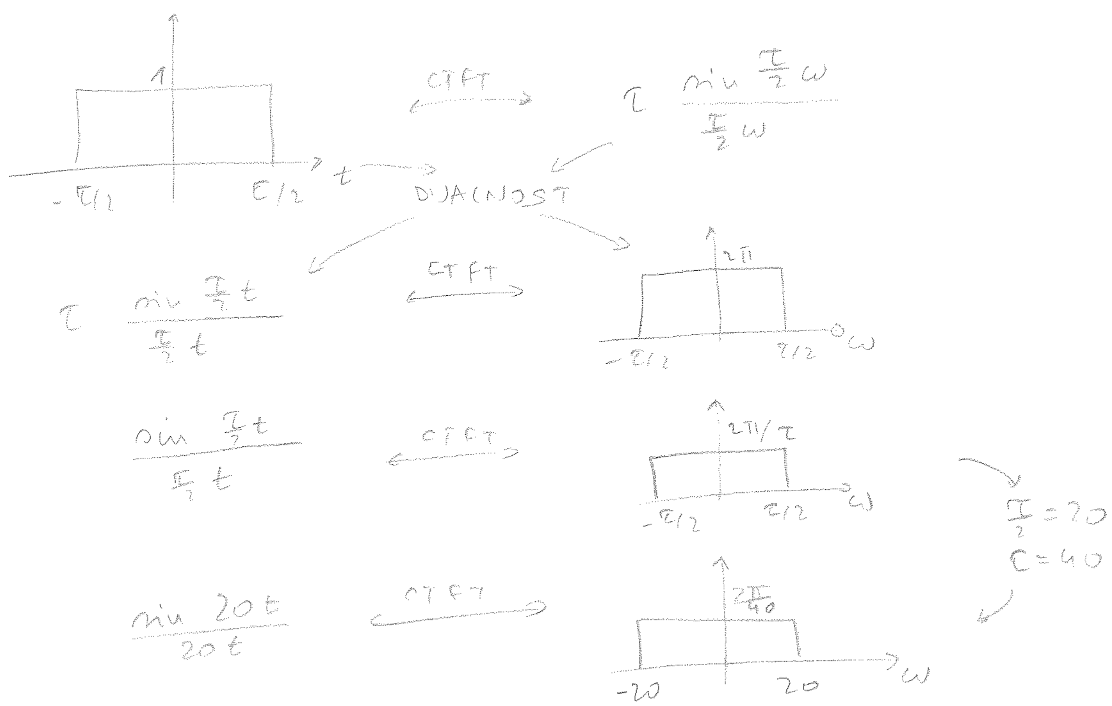
$$E_f = \int_{-\infty}^{\infty} |f(t)|^2 \, dt$$

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 \, dt = \int_{-\infty}^{\infty} |f(at)|^2 \, dt = \left| \begin{array}{ll} at = \tau & \\ a\,dt = d\tau & \\ t = -\infty & \tau = -\infty \\ t = \infty & \tau = \infty \end{array} \right|$$

$$= \int_{-\infty}^{\infty} \frac{1}{a} |f(\tau)|^2 \, d\tau = \frac{1}{a} \int_{-\infty}^{\infty} |f(\tau)|^2 \, d\tau$$

$$= \frac{1}{a} E_f$$

b) $h(t) = \dfrac{\sin 20t}{20t}$



$$E_h = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(j\omega)|^2 \, d\omega$$

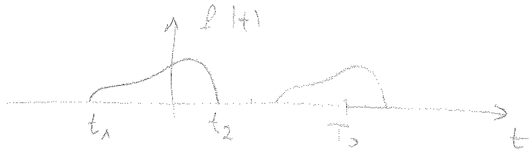
$$= \frac{1}{2\pi} \int_{-20}^{20} \left| \frac{2\pi}{\cancel{40}\,20} \right|^2 d\omega$$

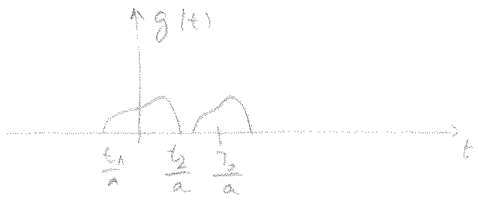
$$= \frac{1}{2\pi} \cdot \frac{\pi^2}{20 \cdot 20} \, \omega \Big|_{-20}^{20} = \frac{\pi}{2 \cdot 20 \cdot 20} (20 + 20) = \frac{2 \cdot \cancel{20} \cdot \pi}{2 \cdot \cancel{20} \cdot 20} = \frac{\pi}{20}$$

ili $E_h = \int_{-\infty}^{\infty} \left(\frac{\sin 20t}{20t} \right)^2 dt = \left| \begin{array}{l} 20t = \tau \\ 20\,dt = d\tau \end{array} \right| = \int_{-\infty}^{\infty} \left(\frac{\sin \tau}{\tau} \right)^2 \frac{1}{20} d\tau = \frac{1}{20} \cdot 2 \cdot \underbrace{\int_{0}^{\infty} \left(\frac{\sin \tau}{\tau} \right)^2 d\tau}_{\text{TABLICA } \frac{\pi}{2}} = \frac{1}{20} \cdot 2 \cdot \frac{\pi}{2} = \frac{\pi}{20}$

c) $f(t)$ periodična sa T_0

$g(t) = f(at)$





$$P_f = \frac{1}{T_0} \int_{T_0} |f(t)|^2 dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |f(t)|^2 dt$$

$$P_g = \frac{1}{\frac{T_0}{a}} \int_{T_0/a} |f(at)|^2 dt$$

$$= \frac{a}{T_0} \int_{-T_0/2a}^{T_0/2a} |f(at)|^2 dt = \left| \begin{array}{ll} at = \tau & t = -\frac{T_0}{2a} \to \tau = -\frac{T_0}{2a} \cdot a = -\frac{T_0}{2} \\ a\,dt = d\tau & \\ dt = \frac{1}{a} d\tau & t = \frac{T_0}{2a} \to \tau = \frac{T_0}{2a} \cdot a = \frac{T_0}{2} \end{array} \right|$$

$$= \frac{a}{T_0} \int_{-T_0/2}^{T_0/2} |f(\tau)|^2 \frac{1}{a} d\tau = \frac{a}{T_0} \cdot \frac{1}{a} \int_{-T_0/2}^{T_0/2} |f(\tau)|^2 d\tau$$

$$= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |f(\tau)|^2 d\tau$$

$$= P_f$$