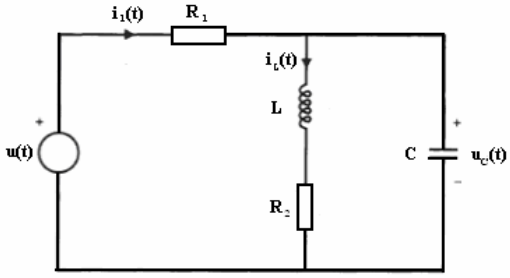
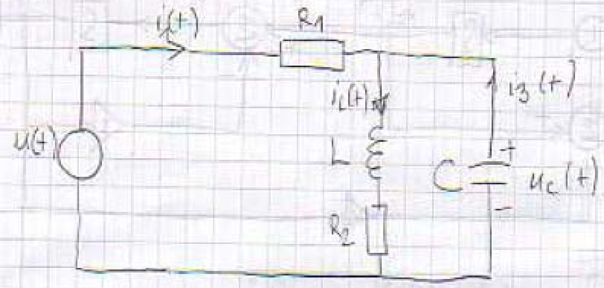
1. Na slici 1. je dana RLC mreža. Nađite model s varijablama stanja za ovu mrežu (matrice *A, B, C* i *D*), ako je *u(t)* ulaz, $x_1(t)=i_L(t)$ i $x_2(t)=u_C(t)$ su varijable stanja, a izlazi su $y_1(t)=i_1(t)$ i $y_2(t)=u_C(t)$. Zadane su vrijednosti elemenata mreže: $R_1=R_2=1\Omega$, $L=1H$ i $C=1F$.



Slika 1.

1) $u(t)$
$x_1(t)=i_L(t)$
$x_2(t)=u_C(t)$
$y_1(t)=i_1(t)$
$y_2(t)=u_C(t)$
$R_1=R_2=1\Omega$
$L=1H$
$C=1F$



$$C\frac{du_C(t)}{dt}=-i_3(t)$$
$$i_3(t)=-C\frac{du_C(t)}{dt}$$
$$i_L(t)=i_1(t)+i_3(t)$$
$$i_1(t)=x_1(t)+C\frac{dx_2(t)}{dt}$$

$$u_C(t)=L\frac{di_L(t)}{dt}+R_2 i_L(t)$$
$$x_2(t)=L\frac{dx_1(t)}{dt}+R_2 x_1(t)$$
$$\frac{dx_1(t)}{dt}=-\frac{R_2}{L}x_1(t)+\frac{1}{L}x_2(t)$$

$$u(t)-i_1(t)R_1=u_C(t)$$
$$u(t)-\left(x_1(t)+C\frac{dx_2(t)}{dt}\right)R_1=x_2(t)$$
$$u(t)-R_1x_1(t)-R_1C\frac{dx_2(t)}{dt}=x_2(t)$$
$$\frac{dx_2(t)}{dt}=-\frac{1}{C}x_1(t)-\frac{1}{R_1C}x_2(t)+\frac{1}{R_1C}u(t)$$

$$\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}=\begin{bmatrix}-\frac{R_2}{L} & \frac{1}{L}\\-\frac{1}{C} & -\frac{1}{R_1C}\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}+\begin{bmatrix}0\\\frac{1}{R_1C}\end{bmatrix}u(t)$$
$$=\begin{bmatrix}-1 & 1\\-1 & -1\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}u(t)$$

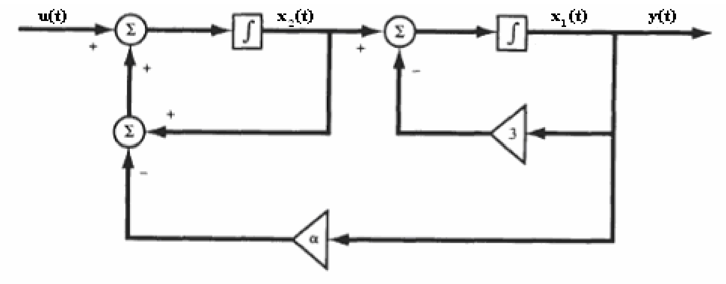
$$y_1(t)=i_1(t)=x_1(t)+C\frac{dx_2(t)}{dt}=-\frac{1}{R_1}x_2(t)+\frac{1}{R_1}u(t)$$
$$y_2(t)=u_C(t)=x_2(t)$$

$$\begin{bmatrix}y_1(t)\\y_2(t)\end{bmatrix}=\begin{bmatrix}0 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}u(t)$$

$$A=\begin{bmatrix}-1 & 1\\-1 & 1\end{bmatrix}\qquad B=\begin{bmatrix}0\\1\end{bmatrix}$$

$$C=\begin{bmatrix}0 & -1\\0 & 1\end{bmatrix}\qquad D=\begin{bmatrix}1\\0\end{bmatrix}$$

2. Vremenski kontinuirani LTI sustav dan je Slikom 2. Nađite model s varijablama stanja $x_1(t)$ i $x_2(t)$ kako su odabrane na slici (matrice A, B, C i D).



Slika 2.

$$x_2(t) = \int \left[x_2(t) + u(t) - \alpha x_1(t) \right] dt \quad \Big/ \frac{d}{dt}$$

$$x_1(t) = \int \left[x_2(t) - 3 x_1(t) \right] dt \quad \Big/ \frac{d}{dt}$$

$$\dot{x}_2(t) = x_2(t) + u(t) - \alpha x_1(t)$$

$$\dot{x}_1(t) = x_2(t) - 3 x_1(t)$$

$$y(t) = x_1(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -\alpha & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} + D u(t)$$

$$A = \begin{bmatrix} -3 & 1 \\ -\alpha & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

3. Audio oscilator je sustav koji proizvodi sinusoidalni signal dane frekvencije ω. Ovaj sustav je moguće prikazati pomoću modela s varijablama stanja:

$$A = \begin{bmatrix} \cos(\omega) & -\sin(\omega) \\ \sin(\omega) & \cos(\omega) \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D=0.$$

a. Matematičkom indukcijom dokažite: $A^n = \begin{bmatrix} \cos n\omega & -\sin n\omega \\ \sin n\omega & \cos n\omega \end{bmatrix}$.

b. Nađite odziv stanja nepobuđenog sustava, te odziv nepobuđenog sustava, ako je početno stanje $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

c. Nađite impulsni odziv mirnog sustava.

(3) $A = \begin{bmatrix} \cos(w) & -\sin(w) \\ \sin(w) & \cos(w) \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ $D = 0$

(A) 1. $n=1$

$$A = \begin{bmatrix} \cos(w) & -\sin(w) \\ \sin(w) & \cos(w) \end{bmatrix}$$

2. $n=k$

$$A = \begin{bmatrix} \cos(kw) & -\sin(kw) \\ \sin(kw) & \cos(kw) \end{bmatrix}$$

3. $n=k+1$

$$A^n = A^k \cdot A = \begin{bmatrix} \cos(kw) & -\sin(kw) \\ \sin(kw) & \cos(kw) \end{bmatrix} \begin{bmatrix} \cos(w) & -\sin(w) \\ \sin(w) & \cos(w) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(kw)\cos(w) - \sin(kw)\sin(w) & -\cos(kw)\sin(w) - \sin(kw)\cos(w) \\ \sin(kw)\cos(w) + \cos(kw)\sin(w) & -\sin(kw)\sin(w) + \cos(kw)\cos(w) \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} \cos[(k+1)w] + \cancel{\cos[(k-1)w]} - \cancel{\cos[(k-1)w]} + \cos[(k+1)w] & -\sin[(k+1)w] + \cancel{\sin[(k-1)w]} - \sin[(k+1)w] - \cancel{\sin[(k-1)w]} \\ \sin[(k+1)w] - \cancel{\sin[(1-k)w]} + \sin[(k+1)w] - \cancel{\sin[(k-1)w]} & -\cancel{\cos[(k-1)w]} + \cos[(k+1)w] + \cos[(k+1)w] + \cancel{\cos[(k-1)w]} \end{bmatrix}$$

$$= \begin{bmatrix} \cos[(k+1)w] & -\sin[(k+1)w] \\ \sin[(k+1)w] & \cos[(k+1)w] \end{bmatrix} = \begin{bmatrix} \cos(nw) & -\sin(nw) \\ \sin(nw) & \cos(nw) \end{bmatrix}$$

(B) odziv stanja nepobuđenog sustava:

$x(n) = A^n x(0)$, $n \geq 0$

$$x(n) = \begin{bmatrix} \cos(nw) & -\sin(nw) \\ \sin(nw) & \cos(nw) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x(n) = \begin{bmatrix} -\sin(nw) \\ \cos(nw) \end{bmatrix}$$

odziv nepobuđenog sustava:

$$y(n) = \begin{cases} Cx(0) & n=0 \\ CA^n x(0) & n>0 \end{cases}$$

$$y(n) = \begin{cases} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & n=0 \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(nw) & -\sin(nw) \\ \sin(nw) & \cos(nw) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & n>0 \end{cases}$$

$$y(n) = \begin{cases} 0 & n=0 \\ -\sin(nw) & n>0 \end{cases}$$

(C) impulsni odziv.

$$y(n) = h(n) = \begin{cases} 0 & n<0 \\ D & n=0 \\ CA^{n-1}B & n>0 \end{cases} = \begin{cases} 0 & n<0 \\ 0 & n=0 \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \cos[(n-1)w] & -\sin[(n-1)w] \\ \sin[(n-1)w] & \cos[(n-1)w] \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & n>0 \end{cases}$$

$$y(n) = \begin{cases} 0 & n<0 \\ 0 & n=0 \\ -\sin[(n-1)w] & n>0 \end{cases}$$

4. Dana je matrica A vremenski diskretnog SISO LTI sustava $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, te vektor $B=\begin{bmatrix}0\\ 1\end{bmatrix}$. Pretpostavite da je $x(0)=\begin{bmatrix}0\\ 0\end{bmatrix}$. Nađite ulaznu sekvencu *u(0), u(1)* takve da je stanje u drugom koraku $x(2)=\begin{bmatrix}1\\ 2\end{bmatrix}$.

4. $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ $B=\begin{bmatrix}0\\ 1\end{bmatrix}$ $x(0)=\begin{bmatrix}0\\ 0\end{bmatrix}$ $x(2)=\begin{bmatrix}1\\ 2\end{bmatrix}$

$$x(n+1)=Ax(n)+Bu(n)$$
$$x(2)=Ax(1)+Bu(1)$$
$$x(1)=Ax(0)+Bu(0)$$
$$x(2)=A(Ax(0)+Bu(0))+Bu(1)=$$
$$=A^2\cdot x(0)+ABu(0)+Bu(1)$$
$$\begin{bmatrix}1\\ 2\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}0\\ 0\end{bmatrix}+\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}u(0)+\begin{bmatrix}0\\ 1\end{bmatrix}u(1)$$
$$\begin{bmatrix}1\\ 2\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}u(0)+\begin{bmatrix}0\\ 1\end{bmatrix}u(1)$$
$$u(0)=1 \qquad u(1)=2$$

5. Zadan je vremenski diskretan LTI sustav impulsnim odzivom:

$$h(n)=\begin{cases}1, & n=0,1\\ 0, & \text{inace}\end{cases}$$

Nađite ulazno – izlaznu relaciju (jednadžbu diferencija) za ovaj sustav.

$$h(n)=\begin{cases}1, & n=0,1\\ 0, & \text{INAČE}\end{cases}$$

$$y(n)=\sum_{m=-\infty}^{+\infty} h(n-m)\cdot x(m), \qquad h(n-m)=\begin{cases}1, & m=n,\ n-1\\ 0, & \text{INAČE}\end{cases}$$

$$y(n)=\sum_{m=n-1}^{n} h(n-m)\cdot x(m)$$

$$y(n)=h(1)\cdot x(n-1)+h(0)\cdot x(n)$$

$$y(n)=x(n-1)+x(n)$$

6. Nađite odziv diskretnog sustava na pobudu $u(n)=\alpha^n\mu(n)$, ako je poznat impulsni odziv sustava $h(n)=\beta^n\mu(n)$.

6. $u(n)=\alpha^n\mu(n) \qquad h(n)=\beta^n\mu(n)$

$$y(n)=\sum_{k=-\infty}^{\infty} h(k)\,u(n-k)$$

$$=\sum_{k=-\infty}^{\infty} \beta^k\mu(k)\cdot\alpha^{n-k}\mu(n-k)$$

$$\left(\mu(k)=\begin{cases}0, & k<0\\ 1, & k\geq 0\end{cases}\right)$$

$$=\sum_{k=0}^{\infty}\beta^k\alpha^{n-k}\mu(n-k) \quad \left[\mu(n-k)=\begin{cases}0 & k>n\\ 1 & k\leq n\end{cases}\right]$$

$$=\sum_{k=0}^{n}\beta^k\alpha^{n-k}=\sum_{k=0}^{n}\left(\frac{\beta}{\alpha}\right)^k\alpha^n=\alpha^n\sum_{k=0}^{n}\left(\frac{\beta}{\alpha}\right)^k$$

IMAMO GEOMETRIJSKI NIZ S $q=\frac{\beta}{\alpha}$

$$\Rightarrow\quad y(n)=\alpha^n\frac{1-\left(\frac{\beta}{\alpha}\right)^{n+1}}{1-\frac{\beta}{\alpha}}$$

Za $\alpha=\beta$ → $y(n)=\alpha^n(n+1)$

7. Dokažite svojstva konvolucije vremenski kontinuiranog sustava:

   a. $u(t) * \delta(t) = u(t)$
   b. $u(t) * \delta(t - t_0) = u(t - t_0)$
   c. $u(t) * \mu(t) = \int_{-\infty}^{t} u(\tau) d\tau$
   d. $u(t) * \mu(t - t_0) = \int_{-\infty}^{t-t_0} u(\tau) d\tau$

a) $u(t) * \delta(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)\,d\tau \quad , \; t \neq \tau \; , \; \delta(t-\tau) = 0$

$$\int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)\,d\tau = u(t) \cdot \int_{-\infty}^{\infty} \delta(t-\tau)\,d\tau = u(t) \cdot 1 = \underline{\underline{u(t)}}$$

b) $u(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} u(\tau)\delta(t-t_0-\tau)\,d\tau$

za $\tau \neq t-t_0 \; , \; \delta(t-t_0-\tau) = 0$

$$\int_{-\infty}^{\infty} u(\tau)\delta(t-t_0-\tau)\,d\tau = u(t-t_0)\int_{-\infty}^{\infty} \delta(t-t_0-\tau)\,d\tau =$$

$$= u(t-t_0) \cdot 1 = \underline{\underline{u(t-t_0)}}$$

c) $u(t) * \mu(t) = \int_{-\infty}^{\infty} u(\tau)\mu(t-\tau)\,d\tau = \left| \mu(t-\tau) = \begin{cases} 0, & \tau > t \\ 1, & \tau \leq t \end{cases} \right|$

$$= \int_{-\infty}^{t} u(\tau) \cdot 1\,d\tau + 0 = \underline{\underline{\int_{-\infty}^{t} u(\tau)\,d\tau}}$$

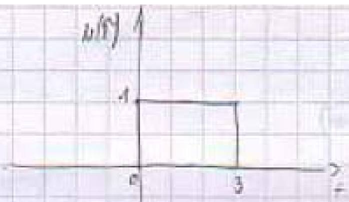
d) $u(t) \cdot \mu(t-t_0) = \int_{-\infty}^{\infty} u(\tau)\mu(t-t_0-\tau)\,d\tau = \left| \mu(t-t_0-\tau) = \begin{cases} 0, & \tau > t-t_0 \\ 1, & \tau \leq t-t_0 \end{cases} \right|$

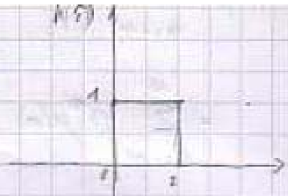
$$= \int_{-\infty}^{t-t_0} u(\tau) \cdot 1\,d\tau + 0 = \underline{\underline{\int_{-\infty}^{t-t_0} u(\tau)\,d\tau}}$$

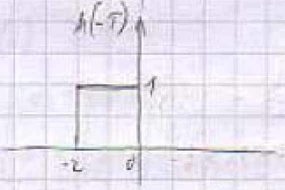
8. Nađite odziv kontinuiranog sustava na pobudu $u(t) = \begin{cases} 1, & 0 < t \leq 3 \\ 0, & \text{inace} \end{cases}$, ako je impulsni odziv $h(t) = \begin{cases} 1, & 0 < \text{t} \leq 2 \\ 0, & \text{inace} \end{cases}$.

8) $u(t) = \begin{cases} 1, & 0 < t \leq 3 \\ 0, & \text{inače} \end{cases}$

$h(t) = \begin{cases} 1, & 0 < t \leq 2 \\ 0, & \text{inače} \end{cases}$







$u(\tau)$ i $h(t-\tau)$ se preklapaju u 3 intervala

- $t \in [0,2]$ djelomično $y(t) = \int_0^t u(\tau)h(t-\tau)\,d\tau = \int_0^t d\tau = t$
- $t \in [2,3]$ potpuno $y(t) = \int_{t-2}^{t} u(\tau)h(t-\tau)\,d\tau = \int_{t-2}^{t} d\tau = 2$
- $t \in [3,5]$ djelomično $y(t) = \int_{t-2}^{3} u(\tau)h(t-\tau)\,d\tau = \int_{t-2}^{3} d\tau = 5-t$

$$y(t) = \begin{cases} 0 & , t \leq 0 \\ t & , 0 \leq t \leq 2 \\ 2 & , 2 \leq t \leq 3 \\ 5-t & , 3 < t \leq 5 \\ 0 & , t \geq 5 \end{cases}$$