Mass drugi dio [voditelj Malbert]

25. ožujak 2009
18:07

8. cl. ne ulaze u 1. MI (npr. DFS)
Moguće da konvolucija ne ulazi

Fourierov red

- kontinuirani, periodični
- radi spektralnu analizu signala, signal iz vremenskog u freq.
- oznaka: CTFS

Analiza

$$X_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt$$

Sinteza

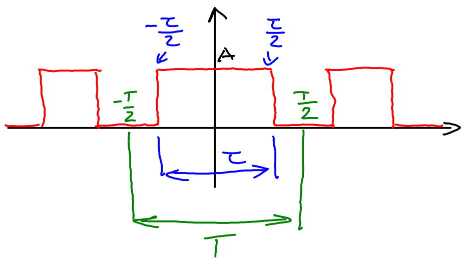
$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}, \quad X_k \in \mathbb{C}$$

$$X_k = |X_k| e^{j\measuredangle X_k}$$

Snaga po Pars.

$$P = \sum_{k=-\infty}^{\infty} |X_k|^2$$

Primjer 1:



$$x(t) = \begin{cases} A, & -\frac{\tau}{2} \le t \le \frac{\tau}{2} \\ 0, & \frac{\tau}{2} < |t| < \frac{T}{2} \end{cases}$$

$$X_k = \frac{1}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A e^{-jk\omega_0 t}dt$$

$$X_k = A\,\frac{1}{T}\cdot\frac{1}{k\omega_0}\cdot\frac{(-1)}{j}\,e^{-jk\omega_0 t}\Big|_{-\frac{\tau}{2}} = A\,\frac{1}{T}\,\frac{1}{k\omega_0}\,\frac{(-1)}{j}\left(e^{-jk\omega_0\frac{\tau}{2}} - e^{jk\omega_0\frac{\tau}{2}}\right) =$$

$$= A\,\frac{1}{T}\,\frac{1}{k\omega_0}\,\underbrace{\frac{2}{2j}\left(e^{jk\omega_0\frac{\tau}{2}} - e^{-jk\omega\frac{\tau}{2}}\right)}_{\sin\left(k\omega_0\frac{\tau}{2}\right)} =$$

$$\sin x = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right)$$

$$\cos x = \frac{1}{2}\left(e^{jx} + e^{-jx}\right)$$

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x}$$

$$\mathrm{sinc}(0) = 1$$

$$= A\,\frac{1}{T}\,\frac{1}{k\omega_0}\cdot 2\sin\left(k\omega_0\frac{\tau}{2}\right) =$$

$$= A\,\frac{1}{T}\,\frac{\tau}{\cancel{2}}\,\underbrace{\frac{\cancel{2}}{k\omega_0\frac{\tau}{2}}\sin\left(k\omega_0\frac{\tau}{2}\right)}_{\mathrm{sinc}\left(k\omega_0\frac{\tau}{2}\right)}$$

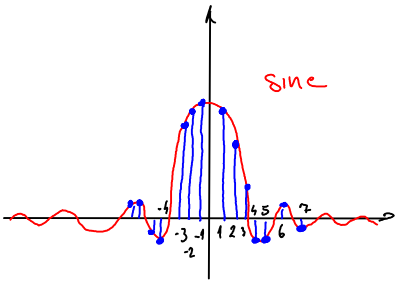
Dodali da stvorimo sinc(x)

$$= A\,\frac{\tau}{T}\,\mathrm{sinc}\left(k\omega_0\frac{\tau}{2}\right) =$$

$$\omega_0 = \frac{2\pi}{T}$$

$$= A\,\frac{\tau}{T}\,\mathrm{sinc}\left(k\pi\frac{\tau}{T}\right)$$

$$X_0 = A\,\frac{\tau}{T} \qquad k=0$$



Traženje nultočke

$$m\cancel{\pi} = k\cancel{\pi}\,\frac{\tau}{T}$$

$$k = m\,\frac{T}{\tau}$$

$$k, m \in \mathbb{Z}$$

Formule vrijede za svaki pravokutni signal simetričan na ishodište

Primjer zadatka:

Prva nultočka se pojavljuje,

$k=6$, $A=10$, $T=12$

$\tau = ?$

$$6 = 1\cdot\frac{12}{\tau} \Rightarrow \tau = 2s$$

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Primjer 2:

$$x(t) = \cos\left(50\pi t + \frac{\pi}{4}\right) + 5\sin\left(120\pi t + \frac{\pi}{3}\right)$$

$\omega_1 = 50\pi$ $\qquad$ $\omega_2 = 120\pi$

ω_0 je najveći zajednički djelitelj ω_1, ω_2

ω_0

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} = \dots \; x_{-5} e^{j(-5)\omega_0 t} \dots + x_{12} e^{j12\omega_0 t} \dots$$

$$x(t) = \frac{1}{2}\left(e^{j\left(50\pi t + \frac{\pi}{4}\right)} + e^{-j\left(50\pi t + \frac{\pi}{4}\right)}\right) + \frac{5}{2j}\left(e^{j\left(120\pi t + \frac{\pi}{3}\right)} - e^{-j\left(120\pi t + \frac{\pi}{3}\right)}\right)$$

$$= \frac{1}{2}\left(e^{j50\pi t} \cdot e^{j\frac{\pi}{4}} + e^{-j\frac{\pi}{4}} e^{-j50\pi t}\right) + \frac{5}{2j}\left(e^{j120\pi t} e^{j\frac{\pi}{3}} - e^{-j120\pi t} e^{-j\frac{\pi}{3}}\right) =$$

$$= \underbrace{\frac{1}{2} e^{j\frac{\pi}{4}}}_{X_5} e^{j(5)\overset{\omega_0}{(10\pi)}t} + \underbrace{\frac{1}{2} e^{-j\frac{\pi}{4}}}_{X_{-5}} e^{j(-5)10\pi t} + \underbrace{\frac{5}{2j} e^{j\frac{\pi}{3}}}_{X_{12}} e^{j(12)10\pi t} - \underbrace{\frac{5}{2j} e^{-j\frac{\pi}{3}}}_{X_{-12}} e^{j(-12)10\pi t} =$$

$x_5 = \frac{1}{2} e^{j\frac{\pi}{4}}$, $\measuredangle \frac{\pi}{4}$

$x_{-5} = \frac{1}{2} e^{-j\frac{\pi}{4}}$, $\measuredangle \frac{\pi}{4}$

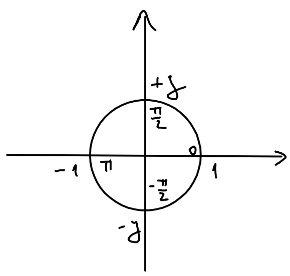
$X_k = |X_k| e^{j\measuredangle X_k}$

$x_{12} = \frac{5}{2j} e^{j\frac{\pi}{3}} = \frac{5}{2j} \cdot \frac{j}{j} e^{j\frac{\pi}{3}} = \frac{5}{2}(-j) e^{j\frac{\pi}{3}} = \frac{5}{2} e^{j\left(\frac{-\pi}{2}\right)} e^{j\frac{\pi}{3}} = \frac{5}{2} e^{j\left(-\frac{\pi}{6}\right)}$, $\measuredangle\left(-\frac{\pi}{6}\right)$

$x_{-12} = \frac{5}{2j} e^{-j\frac{\pi}{3}} = \frac{5}{2}(-j) e^{j\frac{\pi}{3}}$

$x_n = -3e^{j\frac{\pi}{4}}$

$x_n = 3e^{j\frac{\pi}{4}} e^{j\pi}$



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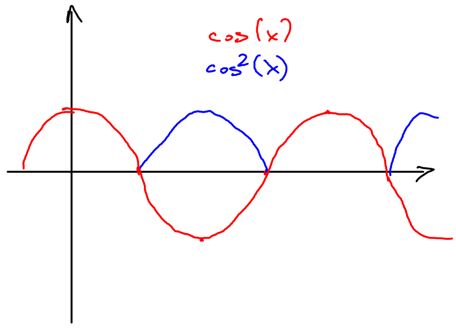
12 $x(t) = \cos^2\left(2t + \frac{\pi}{6}\right)$

$\omega_0 = 4$

$X_0 = ?$



$\cos^2 x = \frac{1}{2}\left(1 + \cos 2x\right)$

$\sin^2 x = \frac{1}{2}\left(1 - \cos 2x\right)$

$x(t) = \frac{1}{2} + \frac{1}{2}\cos\left(4t + \frac{\pi}{3}\right)$

$x(t) = \frac{1}{2} + \frac{1}{4} e^{4t} e^{\frac{\pi}{3}} + \frac{1}{4} e^{-4t} e^{-j\frac{\pi}{3}} =$

$= \frac{1}{2} e^{j(0)t} + \frac{1}{4} e^{4t} e^{\frac{\pi}{3}} + \frac{1}{4} e^{-4t} e^{-j\frac{\pi}{3}}$

$k = 0$

$x_0 = \frac{1}{2}$

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Fourierova transformacija

CTFT

F. analiza

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

ω nije kružna freq.

$$F(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \quad \text{– Laplace}$$

$$s = \sigma + j\omega$$

F. sinteza

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Parsevalova energija

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega$$

"Energija u freq. i vremenskoj domeni mora biti jednaka"

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⑥ $x(t) = e^{-5t}\mu(t-5)$

$$X(j\omega) = \int_5^\infty e^{-5t} e^{-j\omega t}\, d\omega = \int_5^\infty e^{-t(5+j\omega)}\, d\omega =$$

$$= \frac{(-1)}{5+j\omega}\, e^{-t(5+j\omega)} \Big|_5^\infty = \frac{-1}{5+j\omega}\left(\underbrace{\cancel{e^{-\infty}}}_{0} - e^{-5(5+j\omega)}\right) =$$

$$= \frac{1}{5+j\omega}\, e^{-25-5j\omega}$$

Traže se Im, Re, ∢

$$X(j\omega) = \frac{1}{3+j\omega}$$

$$X(j\omega) = \frac{1}{3+j\omega} \cdot \frac{3-j\omega}{3-j\omega} = \underbrace{\frac{3}{9+\omega}}_{Re} + j\underbrace{\frac{(-\omega)}{9+\omega}}_{Im}$$

$$\measuredangle = \operatorname{arctg}\frac{Im}{Re} = \frac{\frac{-\omega}{\cancel{9+\omega}}}{\frac{3}{\cancel{9+\omega}}} = -\frac{\omega}{3}$$

$$|X(j\omega)| = \sqrt{Re^2 + Im^2}$$

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$$x(t) = \begin{cases} t & t \in \langle -\pi, \pi \rangle \\ 0 & \text{inače} \end{cases}$$

$$\Omega = 1$$

$$X(j\omega) = \int_{-\pi}^{\pi} t e^{-j\omega t} dt = \left| \begin{array}{l} u = t \\ du = dt \\ dv = e^{-j\omega t} dt \\ v = \frac{(-1)}{j\omega} e^{-j\omega t} \end{array} \right| = uv - \int v\,du =$$

$$= t \cdot \frac{(-1)}{j\omega} e^{-j\omega t} - \int \frac{(-1)}{j\omega} e^{-j\omega t} dt = t \frac{(-1)}{j\omega} e^{-j\omega t} + \frac{1}{j\omega} \frac{(-1)}{j\omega} e^{-j\omega t} =$$

$$= \frac{(-1)}{j\omega} e^{-j\omega t} \left(t + \frac{1}{j\omega} \right) \Bigg|_{-\pi}^{\pi} = \left(\frac{-1}{j\omega} \right) e^{-j\omega\pi} \left(\pi + \frac{1}{j\omega} \right) - \frac{(-1)}{j\omega} e^{j\omega\pi} \left(-\pi + \frac{1}{j\omega} \right)$$

$$\omega = 1$$

$$X(1) = \frac{(-1)}{j} \left(\pi + \frac{1}{j} \right) + j(-\pi - j) = -j\pi \cancel{+1} - j\pi \cancel{-1} = \underline{-2j\pi}$$

$$e^{-j\pi} = \underset{1}{\underset{\shortparallel}{\cos\pi}} - j \underset{0}{\underset{\shortparallel}{\sin\pi}}$$

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Svojstva

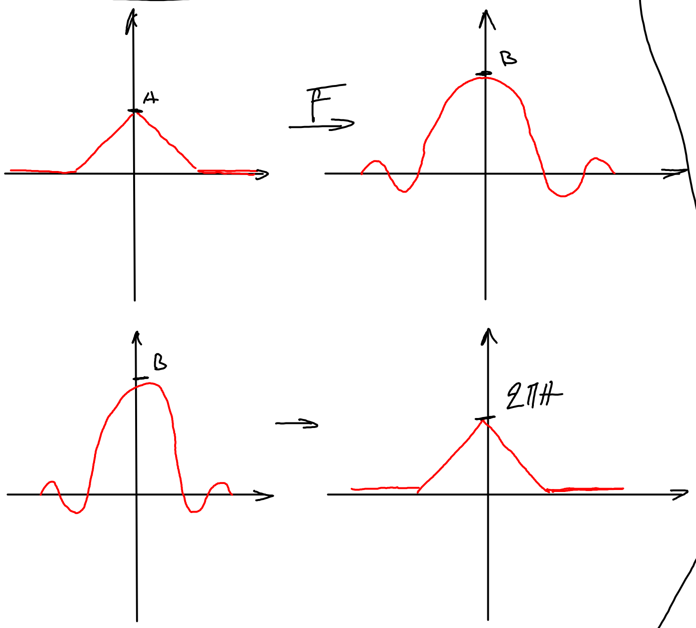
$x(t) \circ\!\!- X(j\omega)$

$x(t-t_0) \circ\!\!- X(j\omega)e^{-j\omega t_0}$

$x(at) \circ\!\!- \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$

$e^{j\omega t} x(t) \circ\!\!- X\left(j(\omega-\omega_0)\right)$

Dualnost



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(8) $\delta(t+t_0)$, $t_0>0$

~~a) $\varphi(\omega)=0$, $\omega<0$~~

~~b) $\varphi(\omega)=0$, $\omega>0$~~

~~c) $\varphi(\omega)<0$, $\omega>0$~~

d) $\varphi(\omega)>0$, $\omega>0$ ✓

~~e) Ništa~~

$x(t)=\delta(t)$

$$X(j\omega)=\int_{-\infty}^{\infty}\delta(t-0)e^{-j\omega t}\,dt=e^{-j\omega 0}=1$$

$$\boxed{\int_{-\infty}^{\infty}\delta(t-\tau_0)f(t)\,dt=f(t_0)}$$

$$\delta(t)\overset{\mathcal{F}}{\longleftrightarrow}1$$

$$\delta(t-t_0)\longleftrightarrow 1e^{j\omega t_0}$$

$$\varphi(\omega)=\omega t_0$$

$$t_0>0$$

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(9)

$$x(t)\circ\!\!-\!\bullet\, X(j\omega)$$

$$x(-t)\circ\!\!-\!\bullet\, X\left(j\frac{\omega}{-1}\right)=X(-j\omega)$$

$$\circ\!\!-\!\bullet\, X(-j\omega)\,e^{-j\omega 10}$$

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(10)

$$x(t)\circ\!\!-\!\bullet\, X(j\omega)$$

$$g(t)\circ\!\!-\!\bullet\, G(j\omega)\quad,\quad g(t)=x(t+7)$$

$$|X(j\omega)|-|G(j\omega)|=?$$

$$g(t)\circ\!\!-\!\bullet\,|G(j\omega)|=|X(j\omega)|\,\underbrace{|e^{7j\omega}|}_{1}$$

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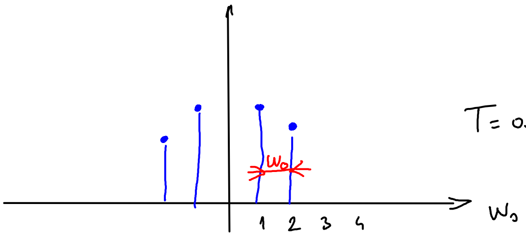

$T = \infty \Rightarrow w = 0$

Zadatak: Periodican signal ima diskretan spektar
Aperiodican -II- ima kontinuirani spektar

vrijedi li obrat? Vrijedi! ✓

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20 Zadatak

$u(n) = \cos(2n)\mu(n)$ $f: \mathbb{Z} \to \mathbb{R}$

aperiodican

diskretan

pa je spektar periodican i kontinuiran

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Fourierova transformacija

DTFT

Analiza

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jwn}$$

Sinteza

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw}) e^{jwn} \, dw$$

Parseval

$$e^{jw} = e^{j(w+2\pi)} = e^{jw} \underbrace{e^{j2\pi}}_{1}$$

$$E = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| X(e^{jw}) \right|^2 dw$$

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$$u(n) = \begin{cases} n, & |n| \leq 4 \\ 0, & \text{inače} \end{cases}$$

sin (4w)

$$-4 \frac{2j}{2j} \left(e^{j4w} - e^{-j4w} \right)$$

$$X(e^{jw}) = \sum_{n=-4}^{4} x(n) e^{-jwn} = -4e^{jw4} - 3e^{jw3} - 2e^{jw2} - e^{jw} + e^{-jw} + 2e^{-jw2} + 3e^{-jw3} + 4e^{-jw4} =$$

$$= -8j\sin(4w) - 6j\sin(3w) - 4j\sin(2w) - 2j\sin(w)$$

$w = \frac{\pi}{2}$

$$= 6j - 2j = 4j$$

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$$X(e^{j\omega}) = \begin{cases} 2, & |\omega| \leq a \\ 0, & a < |\omega| < \pi \end{cases}$$

$$x(n) = \frac{1}{2\pi}\int_{-a}^{a} 2e^{j\omega n}d\omega = \frac{1}{\pi}\frac{1}{j\pi}e^{j\omega n}\Big|_{-a}^{a} =$$

$$= \frac{1}{\pi n}\frac{2}{2j}\left(e^{jan} - e^{-jan}\right) = \frac{2}{\pi n}\sin(an)$$

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$$X(e^{j\omega}) = \begin{cases} 2\pi, & |\omega| \leq a \\ 0, & a < |\omega| < \pi \end{cases}$$

Parseval

$$E = \frac{1}{2\pi}\int_{-a}^{a} 4\pi^2 d\omega = 2\pi\,\omega\Big|_{-a}^{a} = 4a\pi$$

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Svojstva DTFT

$$x(n) \circ\!\!-\!\!\bullet X(e^{j\omega})$$

$$x(n-n_0) \circ\!\!-\!\!\bullet X(e^{j\omega})e^{-j\omega n_0}$$

$$e^{j\omega_0 n}x(n) \circ\!\!-\!\!\bullet X\left(e^{j(\omega-\omega_0)}\right)$$

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(19) $$x(n) \circ\!\!-\!\!\bullet X(e^{j\omega})$$

$$x(n)-x(n-1) \circ\!\!-\!\!\bullet X(e^{j\omega})-X(e^{j\omega})e^{-j\omega}=$$

$$=\left(1-e^{-j\omega}\right)X(e^{j\omega})$$

(18) $$x(n) \circ\!\!-\!\!\bullet X(e^{j\omega})$$

$$x(n)\cos(\omega_0 n-\alpha)=\frac{1}{2}x(n)e^{j\omega_0 n}e^{-j\alpha}+\frac{1}{2}x(n)e^{-j\omega_0 n}e^{j\alpha}$$

$$\circ\!\!-\!\!\bullet$$

$$\frac{1}{2}e^{-j\alpha}X\left(e^{j(\omega-\omega_0)}\right)+\frac{1}{2}e^{j\alpha}X\left(e^{j(\omega+\omega_0)}\right)$$

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Generalizirana transformacija

$$x(t) = e^{jt}\mu(-t)$$

$$x(t) = e^{t}\mu(-t) = \int_{-\infty}^{0} e^{t} e^{-j\omega t}\,dt =$$

$$e^{j\omega_0 t}x(t) \circ\!\!-\!\bullet\ X\big(j(\omega-\omega_0)\big)$$

$$\mu(t) \circ\!\!-\!\bullet\ \pi\delta(\Omega) + \frac{1}{j\Omega}$$

$$\mu(-t) \circ\!\!-\!\bullet\ \pi\delta(-\Omega) - \frac{1}{j\Omega}$$

$$e^{jt}\mu(-t) \circ\!\!-\!\bullet\ \overbrace{\pi\delta\big(-(\Omega-1)\big)}^{Re} - \overbrace{\frac{1}{j(\Omega-1)}}^{Im}$$

$$= \frac{1}{\Omega-1} = \frac{-1}{1-\Omega}$$