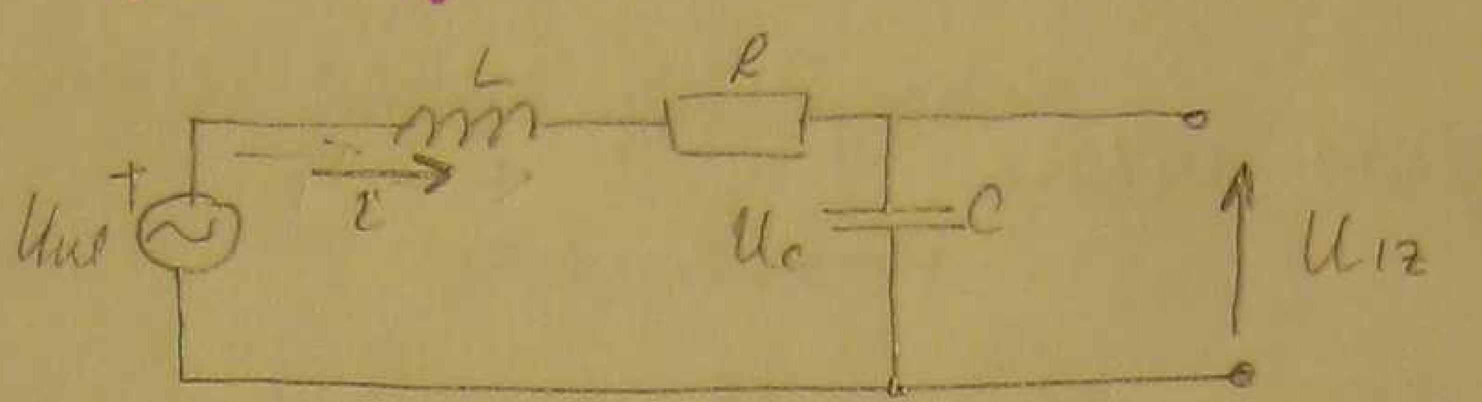
MASS – SIS – 3. ciklus

KONTINUIRANI SUSTAVI

- Diferencijalne jednadžbe



$C = L = 1$
$R = 2$

$$u_{ul} = u_L + u_R + u_C$$

$$u_{ul} = L\frac{di}{dt} + iR + u_{iz}$$

$$u_{iz} = u_C = \frac{1}{C}\int i\,d\tau \Rightarrow \int i\,d\tau = C \cdot u_{iz}$$

$$\Rightarrow i = C \cdot \dot{u}_{iz}$$

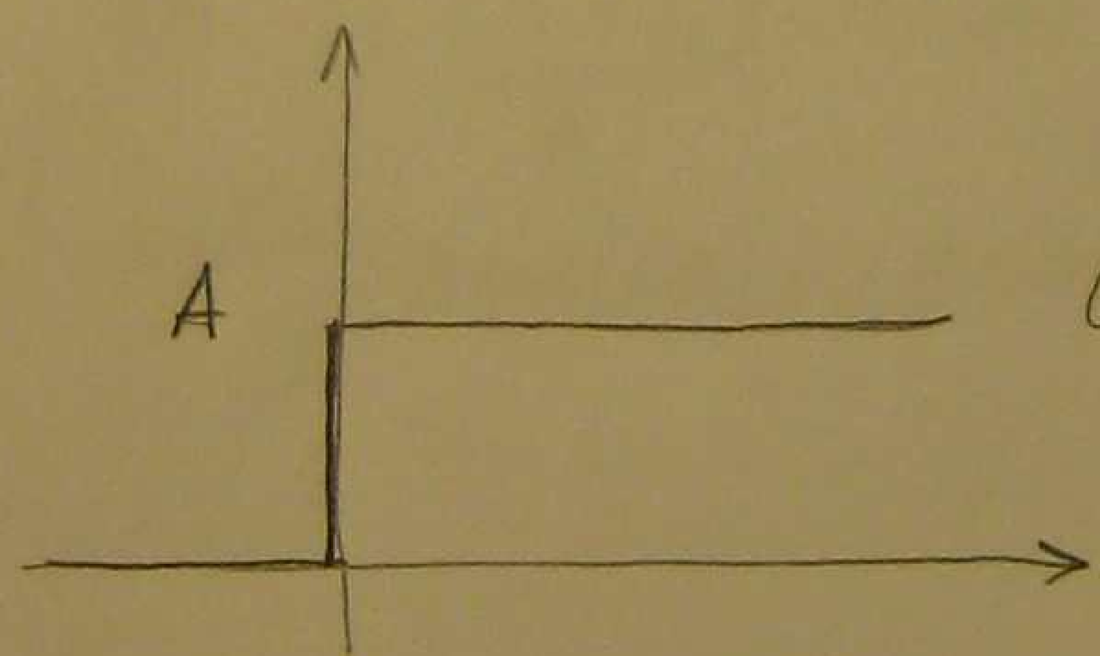
$$i' = C \cdot \ddot{u}_{iz}$$

$$\Rightarrow u_{ul} = L \cdot C \ddot{u}_{iz} + RC\dot{u}_{iz} + u_{iz}$$

$u := u_{ul}$
$y := u_{iz}$

$\underbrace{LC\ddot{y} + RC\dot{y} + y = u}$ → dif. jedn. sustava

$u = 4\mu(t)$
$i_L(0^-) = 0$
$u_C(0^-) = 1$



$0^- \to 0$
$0^+ \to A$

$$\Rightarrow y(0^-) = u_C(0^-) = u_{iz}(0^-) = 1$$

$$y'(0^-) = \frac{1}{C} i(0^-) = 0$$

$$\Rightarrow LC\ddot{y} + RC\dot{y} + y = u$$

$u = 4\mu(t)$
$i_L(0^-) = 0$
$u_C(0^-) = 1$
$y(0^-) = 1$
$\dot{y}(0^-) = 0$

$y'' + 2y' + y = u$
$u = 4\mu(t)$.
$y(0^-)=1,\ y'(0^-)=0$

$\rightarrow y_h(t) = Ce^{st} \quad , \quad y_h'(t) = Cse^{st} \ , \quad y_h'' = Cs^2e^{st}$

$y_h''(t) + 2y_h'(t) + y_h(t) = 0$

$\underbrace{Ce^{st}}_{\neq 0}\underbrace{(s^2+2s+1)}_{=0} = 0 \rightarrow$ karakteristična jedn. sustava

$s^2+2s+1=0 \Rightarrow s_{1,2}=-1$

$$\Rightarrow \underline{\underline{y_h(t) = (C_1+C_2t)e^{-t}}}$$

$\rightarrow Y_p(t) = k \ , \ Y_p'(t) = 0 \ , \ Y_p''(t) = 0$

$Y_p''(t) + 2Y_p'(t) + Y_p(t) = 4$

$k=4$

$$\Rightarrow \underline{\underline{Y_p(t) = 4, \quad t \geq 0}}$$

$\rightarrow$ totalni odziv $\rightarrow$ uvjeti nakon početka djelovanja pobude

$y(0^+) = y(0^-) = 1$
$y'(0^+) = y'(0^-) = 0$

$\{ \rightarrow$ ako nema derivacije pobude $\Rightarrow (0^+ = 0^-) \}$

$y(t) = (C_1+C_2t)e^{-t} + 4 \quad = Y_h(t) + Y_p(t)$

$y'(t) = (C_2 - C_1 - C_2t)e^{-t}$

$y(t) \rightarrow y(0^+) = C_1 + 4 = 1 \quad \Rightarrow C_1 = -3$

$y'(t) \rightarrow y'(0) = C_2 - C_1 = 0 \Rightarrow C_2 = -3$

$$\Rightarrow \underline{\underline{y(t) = (-3-3t)e^{-t} + 4, \ t \geq 0}}$$

Homogena jednadžba

$$y_h(t) = Ce^{st}$$

$$y''(t) - 2y'(t) + y(t) = u(t)$$

$$s^2 - 2s + 1 = 0 \rightarrow s_1 = s_2 = 1$$

① jednostruke karakt. frekvencije

$$s_1 = 2,\ s_2 = 4,\ s_3 = 6,\ s_4 = -4$$

$$y_h(t) = C_1 e^{2t} + C_2 e^{4t} + C_3 e^{6t} + C_4 e^{-4t} \qquad \left\{ y_h(t) = \sum_{i=1}^{m} C_i e^{s_i t} \right\}$$

② višestruke karakt. frekvencije

$$s_1 = 2,\ s_2 = s_3 = 4,\ s_4 = s_5 = s_6 = s_7 = 5$$

$$y_h(t) = C_1 e^{2t} + (C_2 + C_3 t)e^{4t} + (C_4 + C_5 t + C_6 t^2 + C_7 t^3)e^{5t}$$

③ kompleksno konjugirane karakt. frekvencije

$$s_1 = \sigma + j\omega,\ s_2 = \sigma - j\omega \quad \rightarrow \text{uvijek u parovima}$$

$$y_h(t) = C_1 e^{(\sigma + j\omega)t} + C_2 e^{(\sigma - j\omega)t}$$

$$= e^{\sigma t}\left(C_1 e^{j\omega t} + C_2 e^{-j\omega t}\right)$$

$$= e^{\sigma t}\left(C_1\cos(\omega t) + jC_1\sin(\omega t) + C_2\cos(\omega t) - jC_2\sin(\omega t)\right)$$

$$= e^{\sigma t}\left(\underbrace{(C_1 + C_2)}_{A}\cos(\omega t) + j\underbrace{(C_1 - C_2)}_{B}\sin(\omega t)\right)$$

$$= e^{\sigma t}\left(A\cos(\omega t) + jB\sin(\omega t)\right) \text{//}$$

Partikularno rješenje

- ovisi o pobudi i karakt. frekvencijama sustava

→ TABLICA ⇒ pogledat u skripti

Primjeri

- $u(t) = 4\mu(t)$, $s_1 = 2$, $s_2 = 3$

  $y_p(t) = k$, $t \geq 0$

- $u(t) = 2e^{-3t}\mu(t)$, $s_1 = 2$, $s_2 = 3$ ⇒ $\xi = -3$

  $y_p(t) = ke^{-3t} \cdot t^0 = ke^{-3t}$

- $u(t) = 2e^{-3t}\mu(t)$, $s_1 = s_2 = -3$, $s_3 = 2$ ⇒ $\xi = 2$

  $y_p(t) = ke^{-3t} \cdot t^2$

- $u(t) = (t^3 + 1)\mu(t)$, $s_1 = 2$, $s_2 = 3$

  $y_p(t) = (k_0 + k_1 t + k_2 t^2 + k_3 t^3$

- $u(t) = t^2 e^{4t}$, $s_1 = s_2 = s_3 = 3$, $s_4 = 5$ ⇒ $\xi = 4$

  $y_p(t) = (k_0 + k_1 t + k_2 t^2) e^{4t} \cdot t^0$

- $u(t) = t^2 e^{4t}$, $s_1 = s_2 = s_3 = 4$, $s_4 = 5$ ⇒ $\xi = 4$

  $y_p(t) = (k_0 + k_1 t + k_2 t^2) e^{4t} \cdot t^3$

- $u(t) = 3\mu(t)$, $s_1 = s_2 = 0$, $s_3 = 2$

  $\cancel{y_p(t) = k}$ jer pobuda može pisati kao $u(t) \, 3 \cdot e^{0t}\mu(t)$

  ⇒ $y_p(t) = ke^{0t} \cdot t^2 = kt^2$

- $u(t) = (t^2 + 1)\mu(t)$, $s_1 = 0$, $s_2 = 3$

  ⇒ $u(t) = (t^2 + 1)e^{0t}\mu(t)$ → $y_p(t) = (k_0 + k_1 t)e^{0t} \cdot t^1$

Početni uvjeti sustava

→ prvo tražimo diferencijalnu jednadžbu:

$$y'(t)+a_1y(t)=b_0u'(t)+b_1u(t)$$

$$y''(t)+a_1y'(t)+a_2y(t)=b_0u''(t)+b_1u'(t)+b_2u(t)$$

$$y'''(t)+a_1y''(t)+a_2y'(t)+a_3y(t)=b_0u'''(t)+b_1u''(t)+b_2u'(t)+b_3u(t)$$

$\Delta y = y(0^+)-y(0^-)$

↓ tražimo ↓ zadano → uglavnom!

$\Delta y^{(i)}$ = i-ta derivacija od y

$\Delta y^{(i)} = y^{(i)}(0^+)-y^{(i)}(0^-)$

Ako sustav nema derivaciju pobude → početni uvjeti u 0^+ i 0^- jednaki!

$\Delta y = \underset{\to 0}{b_0}u(0^+) \Rightarrow y(0^+)=y(0^-)$

$y'(0^+)-y'(0^-)+0,\cdot 0=0 \Rightarrow y'(0^+)=y'(0^-)$

– primjeri –

- $2y'''(t)-y(t)=u''(t)+3u(t)$ $/:2$ ⇒ prvo se njega 2

  $y(0^-)=1,\ y'(0^-)=2,\ y''(0^-)=3,\ u(t)=4\mu(t)$

  $y'''(t)+a_1y''(t)+a_2y'(t)+a_3y(t)=b_0u'''(t)+b_1u''(t)+b_2u'(t)+b_3u(t)$

  $\Rightarrow a_1=0,\ a_2=0,\ a_3=-1,\ b_0=0,\ b_1=1,\ b_2=0,\ b_3=3$

  slažemo jednadžbe

  $\Delta y = y(0^+)-y(0^-)=\underset{=0}{b_0}u(0^+)=0 \Rightarrow y(0^+)-1=0 \Rightarrow y(0^+)=1$ //

  $y'(0^+)-y'(0^-)+a_1\Delta y=\underset{=0}{b_0}u'(0^+)+\underset{=1}{b_1}u(0^+)=0+4\cdot 1 \Rightarrow y'(0^+)=4+y'(0^-)=6$ //

- $y''(t)-y(t)=u'(t)+2u(t)$

  $y(0^-)=1,\ y'(0^-)=3,\ u(t)=3\mu(t)$

  $y''(t)+a_1y'(t)+a_2y(t)=b_0u''(t)+b_1u'(t)+b_2u(t)$

  $\Rightarrow a_1=0;\ a_2=-1;\ b_0=0;\ b_1=1;\ b_2=2$

  $\Delta y = y(0^+)-y(0^-)=b_0u(0^+)$

  $y(0^+)-1=0\cdot 3$

  $\Rightarrow y(0^+)=1$

$$y'(0^+) - y'(0^-) + a_1(y(0^+) - y(0^-)) = b_0 u'(0^+) + b_1 u(0^+) \quad \{u'(t) = 0 + 3\delta(t)\}$$

$$y'(0^+) - 3 + 0(1-1) = 0 \cdot 3\delta(t) + 1 \cdot 3$$

$$y'(0^+) = 3 + 3$$

$$\underline{\underline{y'(0^+) = 6}}$$

Postupci određivanja odziva sustava

1. Mirni odziv

$$y_m(t) = y_h(t) + y_p(t)$$

$$y^{(n-1)}(0^-) = \dots = y^{(1)}(0^-) = y(0^-) = 0 \quad , \text{ sustav } n\text{-tog reda}$$

→ početne uvjete iz $0^- \to 0^+$

→ dobivene uvjete uvrštavamo u jednu → koeficijente

2. Nepobuđeni odziv

$$y_n(t) = y_h(t)$$

- iz uvjeta ~~prije~~ prije pobude ⇒ $y(0^-)$!

→ uvjete u homogenu! ⇒ koeficijente

3. Totalni odziv

$$y(t) = y_h(t) + y_p(t) = y_m(t) + y_n(t)$$

→ uzimamo zadane uvjete od (0^-) i takve uvrštavamo u jednu → 0^+ ⇒ koeficijente

4. Prirodni odziv

$$y_{pr}(t) = y(t) - y_p(t)$$ → homogeni dio totalnog

5. Prisilni odziv

$$y_{pri}(t) = y_p(t)$$ 4

6. Impulsni odziv

$$h_A(t) = y_h(t)$$ ⇒ međukorak

$$h_A(0^+) = h_A^{(1)}(0^+) = \dots = h_A^{(n-2)}(0^+) = 0 \;,\quad h_A^{(n-1)}(0^+) = 1$$ ⇒ samo najveća derivacija $= 1$

- sustav prvog reda → poč. uvjet u $h(0^+) = 1$

$$h(t) = \begin{cases} \sum_{m=0}^{M} (b_{N-m} D^m) \cdot h_A(t) \,, & t \geq 0,\ N > M \\ b_0 \delta(t) + \sum_{m=0}^{M} (b_{N-m} D^m) h_A(t) \,, & t \geq 0,\ N = M \end{cases} \quad D^7 h_A = h_A^{(7)}$$

N = red sustava , M = stupanj najveće derivacije

Primjer (Problem 2)

LTI = linearan vrem. nepromjenjiv

$y''(t) + 3y'(t) + 2y(t) = u(t)$

$y(0^-) = 2, \; y'(0^-) = 1$

$u(t) = 4\mu(t)$

$y(t), \; y_m(t), \; y_n(t) = ?$

$\xrightarrow{h}$ $s^2 + 3s + 2 = 0 \rightarrow s_1 = -1, \; s_2 = -2$

$y_h(t) = C_1 e^{-t} + C_2 e^{-2t}$

$\xrightarrow{p}$ $y_p(t) = k, \; y_p'(t) = y_p''(t) = k$

$2k = 4 \Rightarrow k = 2$

$y_p(t) = 2 \quad , \; t \geq 0$

$\xrightarrow{t.o}$ $y(t) = y_h(t) + y_p(t)$

$y(t) = C_1 e^{-t} + C_2 e^{-2t} + 2$

$y(0^+) = y(0^-) = 2 \rightarrow y(t) = C_1 e^{-t} + C_2 e^{-2t} + 2 = C_1 + C_2 = 2$

$y'(0^+) = y'(0^-) = 1 \rightarrow y'(t) = -C_1 e^{-t} - 2C_2 e^{-2t} = -C_1 - 2C_2 = 1$ $\Big\}$ $C_1 = 1$, $C_2 = -1$

$\Rightarrow y(t) = e^{-t} - e^{-2t} + 2, \; t \geq 0$

$\xrightarrow{m.o.}$ $y_m(t) = y_h(t) + y_p(t)$

$y(t) = C_1 e^{-t} + C_2 e^{-2t} + 2$

$y'(t) = -C_1 e^{-t} - 2C_2 e^{-2t}$

$y(0^-) = y'(0^-) = 0 \rightarrow y(0^+) = y'(0^+) = 0$

MIRNI ODZIV
$\Rightarrow$ NEMA POČETNIH UVJETA !
$\neq$ nepobuđeni !

$y(0^+) = C_1 + C_2 + 2 = 0$
$y'(0^+) = -C_1 - 2C_2 = 0$ $\Big\}$ $C_1 = -4$, $C_2 = 2$

$y_m(t) = -4e^{-t} + 2e^{-2t} + 2, \; t \geq 0$

$\xrightarrow{n.o.}$ $y_n(t) = y_h(t)$

$y_n(t) = C_1 e^{-t} + C_2 e^{-2t}$

$y(0^-) = 2, \; y'(0^-) = 1$

$y_n'(t) = -C_1 e^{-t} - 2C_2 e^{-2t}$

$\Rightarrow C_1 = 5, \; C_2 = -3$

$\Rightarrow y_n(t) = 5e^{-t} - 3e^{-2t}$

Primjer (Problem 3.)

$y'''(t) - y'(t) = 2u(t)$

$y(0^-) = 4$; $y'(0^-) = 1$; $y''(0^-) = 3$

$u(t) = 3\mu(t)$ $\Rightarrow u(t) = 3$, (t)

$y(t), y_m(t), y_n(t) = ?$

$s^3 - s = 0$

$s(s^2 - 1) = 0$

$s_1 = 0 \quad s_2 = -1, s_3 = -1$

$h \rightarrow$ $y_h(t) = C_1 e^{0t} + C_2 e^{t} + C_3 e^{-t} = C_1 + C_2 e^{t} + C_3 e^{-t}$

$p \rightarrow$ $y_p(t) = k e^{0t} \cdot t = k t \mu(t)$

$y'_p(t) = k$ $\Rightarrow$ $\underline{\underline{y_p(t) = -6t, \ t \geq 0}}$

$y''_p(t) = y'''_p(t) = 0$

$0 - k = 3 \cdot 2$

$\underline{\underline{k = -6}}$

$t.o. \rightarrow$ $y(t) = y_h(t) + y_p(t)$

$y(t) = (C_1 + C_2 e^{t} + C_3 e^{-t} - 6t)\mu(t)$; $y'(t) = C_2 e^{t} - C_3 e^{-t} - 6$; $y''(t) = C_2 e^{t} + C_3 e^{-t}$

$y(0^+) = y(0^-) = 4$

$y'(0^+) = y'(0^-) = 1$

$y''(0^+) = y''(0^-) = 3$

$C_2 - C_3 + 6 = 1$

$C_2 + C_3 = 3$

$C_1 + C_2 + C_3 = 4$

$C_1 + 3 = 4$

$\underline{\underline{C_1 = 1}}$

$C_2 - C_3 = 1 + 6$

$C_2 - C_3 = 7$

$C_2 + C_3 = 3$

$2C_2 = 10$

$\underline{\underline{C_2 = 5}}$

$\underline{\underline{C_3 = -2}}$

$\Rightarrow y(t) = \underline{\underline{(1 + 5e^{t} - 2e^{-t} - 6t)}}\mu(t)$

m.e. →

$y_m(t) = y_h(t) + y_p(t)$

$y_m(t) = C_1 + C_2 e^{t} + C_3 e^{-t} - 6t$

$y'_m(t) = C_2 e^{t} - C_3 e^{-t} - 6$

$y''_m(t) = C_2 e^{t} + C_3 e^{-t}$

$(y(0^-) = y'(0^-) = y''(0^-) = 0 \rightarrow y(0^+) = y'(0^+) = y''(0^+) = 0) \rightarrow$ samo tako što nema $u', u''\ldots$

↓ ne direktno

$y(0^-) = 0 \rightarrow y(0^+) = 0$
$y'(0) = 0 - \ldots$
} formule ! $\Delta y \ldots$

$0 = C_1 + C_2 + C_3$
$0 = C_2 - C_3 - 6$
$0 = C_2 + C_3$

$\underline{\underline{C_1 = 0}}$

$C_2 + C_3 = 0$
$C_2 - C_3 = 6$

$2C_2 = 6$

$\underline{\underline{C_2 = 3}}$

$\underline{\underline{C_3 = -3}}$

$\underline{\underline{y_m(t) = (3e^{t} - 3e^{-t} - 6t)u(t)}}$

n.o. →

$y_n(t) = y_h(t)$

$y_n(t) = C_1 + C_2 e^{t} + C_3 e^{-t}$

$y'_n(t) = C_2 e^{t} - C_3 e^{-t}$

$y''_n(t) = C_2 e^{t} + C_3 e^{-t}$

$4 = C_1 + C_2 + C_3$
$1 = C_2 - C_3$
$3 = C_2 + C_3$

$4 = 2C_2$

$\underline{\underline{C_2 = 2}}$

$4 = C_1 + 3$

$\underline{\underline{C_1 = 1}}$

$\underline{\underline{C_3 = 1}}$

$\Rightarrow \underline{\underline{y_n(t) = 1 + 2e^{t} + e^{-t}}}$

Primjer (Problem 4)

$$y''(t) - y(t) = u_1(t) - u_2(t) = u_1(t) + (-u_2(t))$$

$y(0^-) = 2$, $y'(0) = 1$

$u_1(t) = 3e^{-2t}\mu(t)$

$u_2(t) = 5\mu(t)$

$y(t) = ?$

I.) $y''(t) - y(t) = u_1(t)$

$y(0^-) = 2$, $y'(0^-) = 1$

$u_1(t) = 3e^{-2t}\mu(t)$

$y_h(t) \ldots$

$y_p(t) \ldots$

$y_1(t) = \ldots$

II.) $y''(t) - y(t) = -u_2(t)$

$y(0^-) = 2$, $y'(0^-) = 1$

$u_2(t) = 5\mu(t)$

$y_h(t) \ldots$

$y_p(t) \ldots$

$y_2(t) \ldots$

$$\Rightarrow y(t) = y_1(t) + y_2(t)$$

Impulsni odziv

Primjer (Problem 5)

$$y''(t) + 2y'(t) + y(t) = 2u''(t) + u'(t) + u(t) \quad ; h_A = y_h$$

$h \rightarrow s^2 + 2s + 1 = 0$

$s_1 = s_2 = -1 \quad \Rightarrow \quad y_h(t) = (C_1 + C_2 t)e^{-t}$

$h_A(t) = (C_1 + C_2 t)e^{-t}$

$h_A(0^+) = 0 \rightarrow$ sve nula

$h_A'(0^+) = 1 \rightarrow$ najveća derivacija $= 1$

$h_A'(t) = (C_2 - C_1 - C_2 t)e^{-t}$

$0 = C_1$

$C_2 - C_1 = 1$

$C_1 = 0$

$C_2 = 1$

$$\Rightarrow h_A(t) = te^{-t}$$

$$y'' + a_1 y' + a_2 y = b_0 u'' + b_1 u' + b_2 u$$

$a_1 = 2$; $a_2 = 1$; $b_0 = 2$; $b_1 = b_2 = 1$; $N = 2$; $M = 2$

$$\Rightarrow N = M \rightarrow \quad h(t) = b_0 \delta(t) + \sum_{m=0}^{M} (b_{N-m} D^m) h_A(t)$$

$$h(t) = 2\delta(t) + \sum_{m=0}^{2} (b_{2-m} D^m) h_A(t)$$

$$h(t) = 2\delta(t) + b_2 D^0 h_A(t) + b_1 D^1 h_A(t) + b_0 D^2 h_A(t)$$

$$h(t) = 2\delta(t) + 1 \cdot h_A(t) + 1 \cdot h_A'(t) + 2 h_A''(t) .$$

$h_A'(t) = \ldots$

$h_A''(t) = \ldots$

$h(t) = \ldots$

• Laplaceova transformacija

$\rightsquigarrow X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$ = dvostrana $\mathcal{L}$-transformacija

• $x(t) = e^{-\alpha t}\mu(t) \circ\!\!-\!\!\bullet\ ?$

$$X(s) = \int_{-\infty}^{\infty} e^{-\alpha t}\mu(t) e^{-st} dt = \int_{0}^{\infty} e^{-\alpha t} e^{-st} dt = \int_{0}^{\infty} e^{-t(s+\alpha)} dt =$$

$$= \frac{-1}{s+\alpha} e^{-t(s+\alpha)} \Big|_{0}^{\infty} = \frac{(-1)}{s+\alpha}\left(\lim_{t\to\infty} \underbrace{e^{-t(s+\alpha)}}_{\to 0} - \lim_{t\to 0} \underbrace{e^{-t(s+\alpha)}}_{\to 1}\right) =$$

$$= \frac{1}{s+\alpha}$$

$\boxed{e^{-\alpha t}\mu(t) \circ\!\!-\!\!\bullet \frac{1}{s+\alpha}}$, $s+\alpha > 0 \Rightarrow \boxed{s > -\alpha}$

• $x(t) = -e^{-\alpha t}\mu(-t) \circ\!\!-\!\!\bullet\ ?$

$$X(s) = \int_{-\infty}^{\infty} -e^{-\alpha t}\mu(-t) e^{-st} dt = -\int_{-\infty}^{0} e^{t(-s-\alpha)} dt =$$

$$= \frac{-1}{-s-\alpha} e^{t(-s-\alpha)} \Big|_{-\infty}^{0} = \frac{1}{s+\alpha}\left(\lim_{t\to 0} \underbrace{e^{t(-s-\alpha)}}_{\to 1} - \lim_{t\to -\infty} \underbrace{e^{t(-s-\alpha)}}_{\to 0}\right) =$$

$$= \frac{1}{s+\alpha}$$

$\boxed{-e^{-\alpha t}\mu(-t) \circ\!\!-\!\!\bullet \frac{1}{s+\alpha}}$, $-\alpha - s > 0 \Rightarrow \boxed{s < -\alpha}$

• $X(s) = \frac{1}{s+\alpha} \bullet\!\!-\!\!\circ\ ???$

$\rightsquigarrow X(s) = \int_{0}^{\infty} x(t) e^{-st} dt$ = jednostrana $\mathcal{L}$-transformacija

Svojstva $\mathcal{L}$-transformacije

1 Linearnost

$x(t) \circ\!\!-\!\!\bullet X(s)$

$y(t) \circ\!\!-\!\!\bullet Y(s)$

$\Big\}\ \underline{a x(t) + b y(t) \circ\!\!-\!\!\bullet a X(s) + b Y(s)}$

2 Pomak u vremenu

$x(t) \circ\!\!-\!\!\bullet X(s)$

$\underline{x(t-t_0) \circ\!\!-\!\!\bullet X(s) e^{-st_0}}$

npr.

$x(t) = \mu(t)$

$x(t-5) \circ\!\!-\!\!\bullet ?$

$x(t) \circ\!\!-\!\!\bullet X(s)$

$x(t-5) \circ\!\!-\!\!\bullet X(s) \cdot e^{-5s}$

$\mu(t-5) \circ\!\!-\!\!\bullet \frac{1}{s} e^{-5s}$ //

3 Frekvencijski pomak

$x(t) \circ\!\!-\!\!\bullet X(s)$

$\underline{e^{at} x(t) \circ\!\!-\!\!\bullet X(s-a)}$

npr.

a) $x(t) = e^{4t} \cdot 3\mu(t)$

$3\mu(t) \circ\!\!-\!\!\bullet \frac{3}{s}$

$e^{4t} \cdot 3\mu(t) \circ\!\!-\!\!\bullet \frac{3}{s-4}$ //

b) $x(t) = e^{4t} \cdot \cos(3t)\mu(t)$

$\cos(3t)\mu(t) \circ\!\!-\!\!\bullet \frac{s}{s^2+9}$

$e^{4t} \cdot \cos(3t)\mu(t) \circ\!\!-\!\!\bullet \frac{s-4}{(s-4)^2+9}$

4 Vremenska kompresija

$x(t) \circ\!\!-\!\!\bullet X(s)$

$\underline{x(at) \circ\!\!-\!\!\bullet \frac{1}{a} X\left(\frac{s}{a}\right)}$

npr.

$x(t) = 3\mu(t) \circ\!\!-\!\!\bullet \frac{3}{s}$

$x\left(\frac{t}{2}\right) = 3\mu\left(\frac{t}{2}\right) \circ\!\!-\!\!\bullet 2 \cdot \frac{3 \cdot 2}{a \cdot 2} = \frac{3}{a}$ //

5. Konvolucija u vremenu

$$\left.\begin{array}{l} x(t) \circ\!-\!\bullet X(s) \\ y(t) \circ\!-\!\bullet Y(s) \end{array}\right\} \underline{x(t) * y(t) \circ\!-\!\bullet X(s) \cdot Y(s)}$$

$$\underline{x(t) \cdot y(t) \circ\!-\!\bullet X(s) * Y(s)}$$

6. Derivacija u vremenu

$$x(t) \circ\!-\!\bullet X(s)$$

$$\left.\begin{array}{l} x'(t) \circ\!-\!\bullet sX(s) - x(0^-) \\ x''(t) \circ\!-\!\bullet s(sX(s) - x(0^-)) - x'(0^-) \\ \qquad \circ\!-\!\bullet s^2X(s) - sx(0^-) - x'(0^-) \end{array}\right\} \; !$$

$$\vdots$$

7. Integracija u vremenu

$$x(t) \circ\!-\!\bullet X(s)$$

$$\int_{0^-}^{t} x(t)\,dt \circ\!-\!\bullet \frac{1}{s} X(s)$$

$$\iiiint\!\!\int_{0^-}^{t} x(t)\,dt\,dt\,dt\,dt\,dt \circ\!-\!\bullet \frac{1}{s^5} X(s)$$

• Inverzna Laplaceova transformacija

• $X(s) = \frac{s+5}{s^2+2s+1} = \frac{s+5}{(s+1)^2}$

1. $\frac{5}{(s+1)(s+2)} = \frac{C_{11}}{s+1} + \frac{C_{21}}{s+2}$ na kojoj nul-točki

2. $\frac{5}{(s+1)^3(s+2)} = \frac{C_{11}}{(s+1)^1} + \frac{C_{12}}{(s+1)^2} + \frac{C_{13}}{(s+1)^3} + \frac{C_{21}}{(s+2)^1}$ potencija nazivnika

1. $C_{11} = \left\{ \text{nul-točka nazivnika: } -1 \Rightarrow \frac{5}{s+2} = \frac{5}{-1+2} = \frac{5}{1} = 5 \right\} =$

$$\boxed{C_{i1} = \lim_{s \to s_i} \left(X(s) \cdot (s - s_i) \right)} \to \text{metoda prikrivanja}$$

$$C_{11} = \lim_{s \to -1} \frac{5}{\cancel{(s+1)}(s+2)} \cdot \cancel{(s+1)} = 5$$

$$C_{21} = \frac{5}{s+1} = \frac{5}{-2+1} = -5$$

$$\Rightarrow \frac{5}{(s+1)(s+2)} = \frac{5}{s+1} - \frac{5}{s+2} \circ\!\!-\!\!\!-\!\!\circ (5e^{-t} - 5e^{-2t})\mu(t)$$

2. $C_{21} = \frac{5}{(s+1)^3} \Big|_{s=-2} = \frac{5}{(-1)^3} = -5$

$$\boxed{C_{ij} = \frac{1}{(r_i - j)!} \cdot \lim_{s \to s_i} \left\{ \frac{d^{(r_i-j)}}{ds^{(r_i-j)}} \left[X(s) \cdot (s - s_i)^r \right] \right\}}$$

r_i = višestrukost korijena

$$C_{11} = \frac{1}{(3-1)!} \lim_{s \to -1} \left\{ \frac{d^{(3-1)}}{ds^{(3-1)}} \left[\frac{5}{\cancel{(s+1)^3}(s+2)} \cdot \cancel{(s+1)^3} \right] \right\} =$$

$$= \frac{1}{2!} \lim_{s \to -1} \left\{ \frac{d^2}{ds^2} \left(\frac{5}{s+2} \right) \right\} =$$

$$= \frac{1}{2} \lim_{s \to -1} \left\{ \frac{d}{ds} \left(\frac{-5}{(s+2)^2} \right) \right\} =$$

$$= \frac{1}{2} \lim_{s \to -1} \left(\frac{\cancel{0 \cdot (s+2)^2} + 5 \cdot 2(s+2)}{(s+2)^4} \right) =$$

$$= \frac{1}{2} \lim_{s\to -1} \frac{10(s+2)}{(s+2)^4} = \frac{1}{2}\cdot 10 = 5$$ //

$$a_{12} = \frac{1}{(3-2)!} \lim_{s\to -1} \left\{ \frac{d^{(3-2)}}{ds^{(3-2)}} \left[X(s)\cdot(s-s_i) \right] \right\} =$$

$$= 1 \lim_{s\to -1} \frac{d}{ds}\left(\frac{5}{s+2}\right) =$$

$$= \lim_{s\to -1} \left(-\frac{5}{(s+2)^2}\right) = -5$$ //

$$a_{13} = \frac{1}{0!} \lim_{s\to -1} \left[\frac{d^0}{ds^0} \frac{5}{s+2} \right]$$

$$= 1\cdot \lim_{s\to -1} 5$$

$$= 5$$ //

$$\Rightarrow \frac{5}{(s+1)^3(s+2)} = \frac{5}{s+1} - \frac{5}{(s+1)^2} + \frac{5}{(s+1)^3} - \frac{5}{s+2}$$ //

3. $\frac{s^3+s^2+1}{s^2-4} = (s+1) + \frac{4s+5}{s^2-4}$; $s \circ\!\!-\!\!\circ\ \delta'(t)$, $1 \circ\!\!-\!\!\circ\ \delta(t)$

$$(s^3+s^2+1) : (s^2-4) = s+1$$

$$-(s^3 \quad -4s)$$

$$0 + s^2 - 4s + 1$$

$$-\,s^2 \quad -4$$

$$4s+5$$

Primjer (Problem 2)

- u frekvencijskoj domeni

$y'' + 3y' + 2y = u$

$y(0^-) = 2 \; ; \; y'(0^-) = 1$

$u = 4\mu(t)$

$y(t) \circ\!\!-\!\!\bullet\, Y(s)$

$y'(t) \circ\!\!-\!\!\bullet\, sY(s) - y(0^-) = sY(s) - 2$

$y''(t) \circ\!\!-\!\!\bullet\, s^2Y(s) - sy(0^-) - y'(0^-) = s^2Y(s) - 2s - 1$

$u(t) \circ\!\!-\!\!\bullet\, U(s) = \frac{4}{s}$

$s^2Y(s) - 2s - 1 + 3(sY(s) - 2) + 2Y(s) = U(s)$

$s^2Y(s) - 2s - 1 + 3sY(s) - 6 + 2Y(s) = U(s)$

$Y(s)(s^2 + 3s + 2) = U(s) + 7 + 2s$

$$Y(s) = \frac{U(s) + 7 + s}{s^2 + 3s + 2} = \underbrace{\frac{U(s)}{s^2+3s+2}}_{\text{MIRNI ODZIV}} + \underbrace{\frac{2s+7}{s^2+3s+2}}_{\text{NEPOBUĐENI ODZIV}}$$

posljedica poč. uvjeta

$$\frac{4}{s(s+2)(s+1)} = \frac{C_{11}}{s} + \frac{C_{21}}{s+2} + \frac{C_{31}}{s+1} = \frac{2}{s} + \frac{2}{s+2} - \frac{4}{s+1}$$

$\bullet\!\!-\!\!\circ\; (2 - 4e^{-t} + 2e^{-2t})\mu(t) = y_m(t)$

$C_{11} = \frac{4}{2} = 2$

$C_{21} = \frac{4}{2} = 2$

$C_{31} = \frac{4}{-1} = -4$

$$\frac{2s+7}{(s+1)(s+2)} = \frac{C_{11}}{s+1} + \frac{C_{21}}{s+2} = \frac{5}{s+1} - \frac{3}{s+2}$$

$\bullet\!\!-\!\!\circ\; (5e^{-t} - 3e^{-2t})\mu(t) = y_n(t)$

$C_{11} = 5$

$C_{21} = -3$

Primjer (u frekvencijskoj domeni)

$y'' - y = u' + u \Rightarrow s^2 - 1$

$y(0^-) = 2$, $y'(0) = -1$

$u = \mu(t)$

$y'' - y = \underbrace{f(t)}_{u_1} + \underbrace{\mu(t)}_{u_2}$

$\Rightarrow$ posebno odziv na u_1 i na u_2

$\Rightarrow$ zbrojiti

vrem. domena

$y(t) \circ\!\!-\!\!\bullet Y(s)$

$y'(t) \circ\!\!-\!\!\bullet sY(s) - y(0^-) = sY(s) - 2$

$y''(t) \circ\!\!-\!\!\bullet s^2Y(s) - 2s + 1$

$u(t) \circ\!\!-\!\!\bullet U(s)$

$u'(t) \circ\!\!-\!\!\bullet sU(s)$

$s^2Y(s) - 2s + 1 - Y(s) = sU(s) + U(s)$

$Y(s)(s^2 - 1) = sU(s) + U(s) + 2s - 1$

$$Y(s) = \frac{sU(s) + U(s)}{s^2 - 1} + \frac{2s - 1}{s^2 - 1}$$

$$Y(s) = \frac{U(s)(s+1)}{s^2 - 1} + \frac{2s - 1}{s^2 - 1}$$ } nule } polovi

$$Y(s) = \frac{U(s)\cancel{(s+1)}}{(s-1)\cancel{(s+1)}} + \frac{2s - 1}{(s-1)(s+1)} = \underbrace{\frac{1}{s(s-1)}}_{\text{MIRNI}} + \underbrace{\frac{2s - 1}{(s-1)(s+1)}}_{\text{NEPOBUĐENI}}$$

$$\frac{1}{s(s-1)} = \frac{C_{11}}{s} + \frac{C_{21}}{s-1} = \frac{-1}{s} + \frac{1}{s-1}$$

$C_{11} = -1$

$C_{21} = 1$