LAPLACEOVA TRANSFORMACIJA

1. $y''(t) - 5y'(t) + 6y(t) = u(t) + 3u'(t)$

$y(0^-) = 3, \quad y'(0^-) = 0$

$u(t) = 2\mu(t)$

a) $y_h(t) = Ce^{st}$

$s^2 - 5s + 6 = 0$

$s_{1,2} = \frac{5 \pm \sqrt{25-24}}{2} = \frac{5 \pm 1}{2}$

$s_1 = 3 \quad s_2 = 2$

$y_h(t) = C_1 e^{3t} + C_2 e^{2t}$

$y_p(t) = K$
$y'_p(t) = 0$

$6K = 2$

$K = \frac{1}{3}$

$y_p(t) = \frac{1}{3}, \quad t \geq 0$

$y(t) = C_1 e^{3t} + C_2 e^{2t} + \frac{1}{3}, \quad t \geq 0$

poč. uvjeti: $y(0^+) - y(0^-) = b_0 u(0^+)$

$y(0^+) = y(0^-) = 3$

$y'(0^+) - y'(0^-) + a_1(y(0^+) - y(0^-)) = b_1 u'(0^+) + b_2 u(0^+) \qquad b_1 = 1, \ b_2 = 1$

$y'(0^+) = 6$

$y'(t) = 3C_1 e^{st} + 2C_2 e^{st} \qquad y(t) = C_1 e^{3t} + C_2 e^{2t} + \frac{1}{3}$

$y'(0) = 3C_1 + 2C_2 = 6 \qquad y(0) = C_1 + C_2 + \frac{1}{3} = 3$

$\left.\begin{array}{l} 3C_1 + 2C_2 = 6 \\ C_1 + C_2 = \frac{8}{3} \end{array}\right\} C_1 = \frac{2}{3}, \ C_2 = 2$

$y(t) = \frac{2}{3} e^{3t} + 2e^{2t} + \frac{1}{3}, \quad t \geq 0$

b) $x'' = s^2X(s) - sx(0^-) - x'(0^-)$

$x' = sX(s) - x(0^-)$

$y''(t) - 5y'(t) + 6y(t) = u(t) + 3u'(t)$

$s^2Y(s) - 3s - 5(sY(s) - 3) + 6Y(s) = \frac{2}{s} + 6$

$s^2Y(s) - 5sY(s) + 6Y(s) = \frac{2}{s} + 6 + 3s - 15$

$Y(s)(s^2 - 5s + 6) = \frac{3s^2 - 9s + 2}{s}$

$Y(s) = \frac{3s^2 - 9s + 2}{s(s^2 - 5s + 6)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s-3}$

$A(s-2)(s-3) + Bs(s-3) + Cs(s-2) = As^2 - 5As + 6A + Bs^2 - 3Bs + Cs^2 - 2Cs$

$s^2(A+B+C) + s(-5A - 3B - 2C) + 6A = 3s^2 - 9s + 2$

$A + B + C = 3$
$-5A - 3B - 2C = -9$
$6A = 2 \Rightarrow A = \frac{1}{3}$

$B + C = \frac{8}{3}$ $\quad B = 2$

$3B + 2C = 9 - \frac{5}{3}$

$8 - 3C + 2C = \frac{22}{3}$

$C = \frac{2}{3}$

$Y(s) = \frac{1}{3s} + \frac{2}{s-2} + \frac{2}{3} \cdot \frac{1}{s-3}$

$y(t) = \left[\frac{1}{3} + 2e^{2t} + \frac{2}{3}e^{3t}\right]\mu(t)$

② $y''(t) + 5y'(t) + 6y(t) = u(t)$ $\quad u(t) = \delta(t) \to U(s) = 1$

a) $s^2Y(s) + 5sY(s) + 6Y(s) = U(s)$

$Y(s)(s^2 + 5s + 6) = U(s)$

$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2+5s+6} = \frac{1}{(s+2)(s+3)} = \frac{1}{s+2} - \frac{1}{s+3}$

$= \frac{C_{11}}{(s+2)} + \frac{C_{21}}{(s+3)}$

$h(t) = (e^{-2t} - e^{-3t})\mu(t)$

b) $y''(t) + 5y'(t) + 6y(t) = u(t)$

$u(t) = (12t + 16)\mu(t)$

$y(0^-) = 3, \quad y'(0^-) = -8$

(1) bez Laplaceove transf.

$y_h(t) = Ce^{st}$

$Ce^{st}(s^2 + 5s + 6) = 0$

$s^2 + 5s + 6 = 0$

$s_1 = -3 \quad s_2 = -2$

$y_h(t) = C_1e^{-3t} + C_2e^{-2t}$

$y_p(t) = K_0 + K_1 t$
$y_p'(t) = K_1$
$y_p''(t) = 0$

$5K_1 + 6K_0 + 6K_1 t = 12t + 6$

$$\left.\begin{array}{r} 5K_1 + 6K_0 = 6 \\ 6K_1 = 12 \end{array}\right\} K_1 = 2, \quad K_0 = -\frac{2}{3}$$

$y_p(t) = -\frac{2}{3} + 2t$

$y(t) = C_1e^{-3t} + C_2e^{-2t} + 2t - \frac{2}{3} \quad \Rightarrow \quad y'(t) = -3C_1e^{3t} - 2C_2e^{2t} + 2$

$y(0^+) = b_0 u(0^+) + y(0^-) = y(0^-) = 3$

$y'(0^+) = y'(0^-) = -8$

$y(0) = C_1 + C_2 - \frac{2}{3} = 3$

$y'(0) = -3C_1 - 2C_2 + 2 = -8$

$$\begin{array}{r} 2C_1 + 2C_2 = \frac{22}{3} \\ -3C_1 - 2C_2 = -10 \\ \hline C_1 = \frac{8}{3}, \quad C_2 = 1 \end{array}$$

$y(t) = \frac{8}{3}e^{-3t} + e^{-2t} + 2t - \frac{2}{3}$

(2) s Laplaceovom transf.

$$y''(t) + 5y'(t) + 6y(t) = u(t)$$

$$u(t) = (12t+6)\mu(t)$$

$$y(0^-) = 3, \quad y'(0^-) = -8$$

$$s^2Y(s) - sy(0^-) - y'(0^-) + 5(sY(s) - y(0^-)) + 6Y(s) = \frac{12}{s^2} + \frac{6}{s}$$

$$s^2Y(s) - 3s + 8 + 5sY(s) - 15 + 6Y(s) = \frac{12}{s^2} + \frac{6}{s}$$

$$Y(s)(s^2 + 5s + 6) = \frac{12}{s^2} + \frac{6}{s} + 7 + 3s \quad / \cdot \frac{s^2}{s^2}$$

$$Y(s)(s^2 + 5s + 6) = \frac{12 + 6s + 7s^2 + 3s^3}{s^2}$$

$$Y(s) = \frac{3s^3 + 7s^2 + 6s + 12}{s^2(s+3)(s+2)} = \frac{c_{11}}{s} + \frac{c_{12}}{s^2} + \frac{c_{21}}{s+3} + \frac{c_{31}}{s+2}$$

$$c_{11} = \frac{1}{(2-1)!} \lim_{s\to 0} \left\{ \frac{d^{2-1}}{ds^{2-1}} \left[\frac{3s^3 + 7s^2 + 6s + 12}{(s+3)(s+2)} \right] \right\} =$$

$$= \frac{(9s^2 + 14s + 6)(s^2 + 5s + 6) - (3s^3 + 7s^2 + 6s + 12)(2s+5)}{(s^2 + 5s + 6)^2} = \frac{36 - 60}{36} = -\frac{2}{3}$$

$$c_{12} = \frac{1}{(2-2)!} \lim_{s\to 0} \left\{ \frac{d^{2-2}}{ds^{2-2}} \left[\frac{3s^3 + 7s^2 + 6s + 12}{(s+3)(s+2)} \right] \right\} = \frac{12}{6} = 2$$

$$c_{21} = \lim_{s\to -3} \frac{3s^3 + 7s^2 + 6s + 12}{s^2(s+2)} = \frac{-81 + 63 - 18 + 12}{9 \cdot (-1)} = \frac{24}{9} = \frac{8}{3}$$

$$c_{31} = \lim_{s\to -2} \frac{3s^3 + 7s^2 + 6s + 12}{s^2(s+3)} = \frac{-24 + 28 - 12 + 12}{4 \cdot 1} = 1$$

$$Y(s) = -\frac{2}{3} \cdot \frac{1}{s} + 2 \cdot \frac{1}{s^2} + \frac{8}{3} \frac{1}{s+3} + \frac{1}{s+2}$$

$$y(t) = \left[-\frac{2}{3} + 2t + \frac{8}{3} e^{-3t} + e^{-2t} \right] \mu(t)$$

③ $h(t)=2te^{-t}\mu(t)$

a) $H(s)=2\,\dfrac{1}{(s+1)^2}$

b) $u(t)=2\mu(t)$

$y(0^-)=2, \quad y'(0^-)=0$

$\dfrac{Y(s)}{U(s)}=\dfrac{2}{(s+1)^2}$

$(s+1)^2\,Y(s)=2U(s)$

$y''(t)+2y'(t)+y(t)=2u(t)$

hom... $s^2+2s+1=0$

$s_{1,2}=-1$

$y_h(t)=(C_1+C_2t)e^{-t}$

part... $y''(t)+2y'(t)+y(t)=4\mu(t)$

$y_p(t)=K$

$K=4$

$y_p(t)=4, \quad t>0$

$y(0^-)=y(0^+)=2$

$y'(0^-)=y'(0^+)=0$

$y(t)=(C_1+C_2t)e^{-t}+4$

$y'(t)=-C_1e^{-t}-C_2te^{-t}+C_2e^{-t}$

$$\left.\begin{aligned} y(0^+)&=C_1+4=2\\ y'(0^+)&=-C_1+C_2=0\end{aligned}\right\}\quad C_1=-2,\ C_2=-2$$

$y(t)=\left[(-2-2t)e^{-t}+4\right]\mu(t)$

(4) $y'(t) + 4y(t) = u(t) + 2u'(t)$

$u(t) = \mu(t), \quad y(0^-) = 2$

a) $y(0^+) = b_0 u(0^+) + y(0^-)$ $\quad b_0 = 2, \; b_1 = 1, \; a_1 = 4$

$y(0^+) = 4$

b) $y_h(t) = Ce^{st}$

$s + 4 = 0$

$s = -4 \rightarrow y_h(t) = C_1 e^{-4t}$

$y_p(t) = K$

$y_p'(t) + 4y_p(t) = 1$

$4K = 1 \rightarrow K = \frac{1}{4}$

$y_p(t) = \frac{1}{4}\mu(t)$

$y(t) = y_h(t) + y_p(t)$

$y(t) = C_1 e^{-4t} + \frac{1}{4}$

$y(0^+) = C_1 + \frac{1}{4} = 4$

$C_1 = \frac{15}{4}$

$y(t) = \left(\frac{15}{4} e^{-4t} + \frac{1}{4}\right)\mu(t)$

c) $y'(t) + 4y(t) = u(t) + 2u'(t)$

$sY(s) - y(0^-) + 4Y(s) = U(s) + 2sU(s) - 2u(0^-)$

$(s+4)\,Y(s) = (2s+1)\,U(s) + y(0^-)$

$Y(s) = \frac{2s+1}{s+4} U(s) + \frac{2}{s+4}$

$Y(s) = \underbrace{H(s) \cdot U(s)}_{Y_m(s)} + Y_0(s)$

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$Y(s) = \frac{2s+1}{s+4} \cdot \frac{1}{s} + \frac{2}{s+4} = \frac{2s+1+2s}{s(s+4)} = \frac{4s+1}{s(s+4)} = \frac{C_1}{s} + \frac{C_2}{s+4}$

$= \frac{1}{4} \cdot \frac{1}{s} - \frac{15}{4} \cdot \frac{1}{s+4}$

$y(s) = \left(\frac{1}{4} + \frac{15}{4} e^{-4t}\right)\mu(t), \; t \geq 0$

1) $H(s) = \frac{2s+1}{4}$

pošto je sustav kauzalan područje stabilnosti je lijeva poluravnina

$\sigma < 0$