Orbits & Simulation

Part 2

In Part 1 we looked at the basic equations of motion. In Part 2 we will examine the effects of gravity and try some simple simulations. I will no longer expand every expression in term of $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ but remember each non-italic value is a 3 dimensional vector.

Gravity

Newton tells us that the gravitational force between two bodies is proportional to the product of their masses and inversely proportional to the square of the distance between them. If there are two bodies, $\boldsymbol{p}$ and $\boldsymbol{q}$, located at $\mathbf{r}_p$ and $\mathbf{r}_q$ in our coordinate system then the distance between them is $r_{pq} = |\mathrm{r}_q - \mathrm{r}_p|$.

If the two bodies have masses $\boldsymbol{m_p}$ and $\boldsymbol{m_q}$ the force between them will be:

2.1 $$f_{pq} = \frac{Gm_p m_q}{r_{pq}^2}$$

$\boldsymbol{G}$ is the gravitational constant which we will discuss later.

We need the direction of the force as well as its magnitude, so we have to split its value into $\boldsymbol{x}$,$\boldsymbol{y}$, $\boldsymbol{z}$ components to generate a force vector $\mathbf{f}$.

2.2 $$\mathbf{f}_{pq} = \frac{Gm_p m_q}{r_{pq}^2}\hat{\mathbf{r}}_{pq}$$

Where $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} \equiv \left\{\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right\}$ is the "unit vector" between the objects.

And this can simplify to:

2.3 $$\mathbf{f}_{pq} = \frac{Gm_p m_q}{r_{pq}^3}\mathbf{r}_{pq}$$

Newton tells us that when a force $\boldsymbol{f}$ acts on a body of mass $\boldsymbol{m}$ the body will accelerate at a rate $\boldsymbol{a}$ of:

2.4 $$a = \frac{f}{m} \quad \text{or, expressed as a vector} \quad \mathbf{a} = \frac{\mathbf{f}}{m}$$

Newton finally tells us that the force acts with equal magnitude and opposite direction on the bodies such that they are attracted towards each other.

Putting it all together we get:

2.5 $$\mathbf{a}_p = \frac{Gm_q}{r_{pq}^3}\mathbf{r}_{pq} \qquad \mathbf{a}_q = -\frac{Gm_p}{r_{pq}^3}\mathbf{r}_{pq}$$

The n-body problem

When there are 2 bodies in a gravitational system the equations of motion can be integrated to show that they move in orbit around their common centre of gravity and the shape of the orbit is a conic section. Normally we are concerned with objects in elliptical orbits and it would be quite easy to draw the orbits using basic equations of an ellipse.

When there are more than 2 bodies in a gravitational system the equations of motion cannot be integrated mathematically. That is to say there is no straight-forward equation that specifies the position and velocity of each body into the future.

In a multi-body system the forces acting on any one body are the vector sum of the gravitational attractions from all the other bodies in the system so for any body $\boldsymbol{p}$:

2.6
$$\mathbf{a}_p = G \sum_{\substack{q=1, \\ q \neq p}}^{q=n} \frac{m_q}{r_{pq}^3} \mathbf{r}_{pq}$$

When we say "calculate accelerations" we mean evaluate the above for every body in the system.

Numerical Integration

We can estimate the movement of all the bodies by Numerical Integration. The basic approach is:

1. Start with the state vectors of each body.
2. Calculate the gravitational accelerations for each body.
3. Calculate new positions and velocities for a short time step $\boldsymbol{\delta t}$.
4. Repeat 2-3 for as long as you like.

Remember the basic equations of motion (1.9, 1.11):

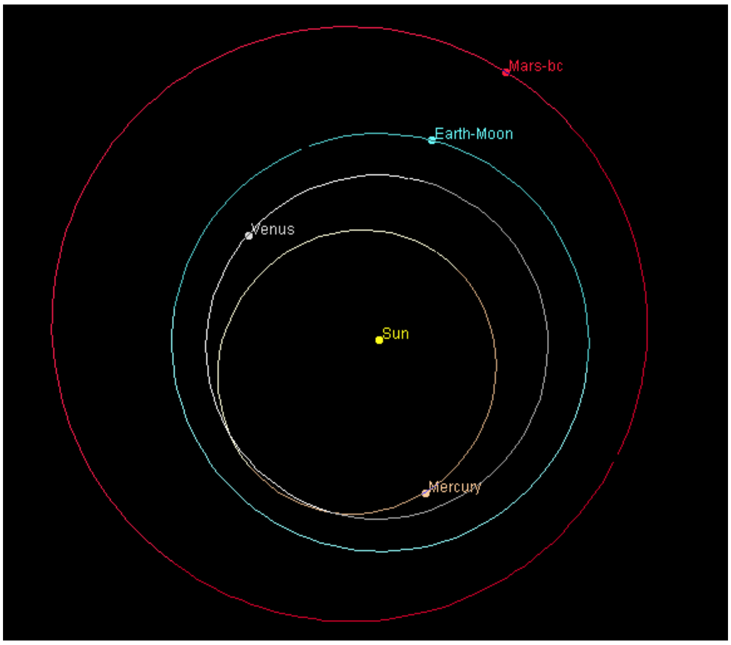
$$\mathrm{r}' = \mathrm{r} + \mathrm{v}\delta t$$
$$\mathrm{v}' = \mathrm{v} + \mathrm{a}\delta t$$

We repeat these for each iteration $\boldsymbol{i}$ such that:

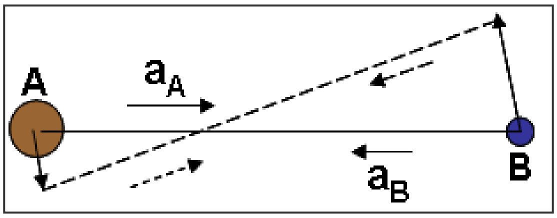
$$\mathrm{r}_{i+1} = \mathrm{r}_i + \mathrm{v}_i\delta t$$
$$\mathrm{v}_{i+1} = \mathrm{v}_i + \mathrm{a}_i\delta t$$

This is the **Euler** method of numerical integration. We can run this integration with a 1-day timestep and see that the orbits quickly appear to go wrong. The image below shows the results after about 1 year of simulation. An animated version is shown attached to the Discussion post.

Euler integration for one year using 1-day timesteps.



The orbits visibly depart from their proper paths immediately. Reducing the timestep can slow down the appearance of visible errors, but never removes them. Why?



Consider two bodies A and B. At the start of the time step acceleration is along the solid line but by the end of the time-step it is along the dotted line. Both velocity and acceleration will change during the time step but the Euler method assumes that they remain constant. If the time step is small the error may be small but a similar error will occur in each step and we will need many steps to integrate a given period of time.

The integration methods we will look at adopt increasingly sophisticated approaches to reducing the errors caused by the fact that position, velocity and acceleration are constantly changing.

---to be continued…