Orbits & Simulation

Part 3

In Part 2 we looked at the effects of gravity and attempted to solve the n-body problem using numerical integration and the method of **Euler**. It did not work very well, so let's look at a slightly better solution.

Leapfrog (Verlet) method

Euler assumed velocity and acceleration did not change during the timestep but of course they do change. With Leapfrog we try to reduce the error by estimating the velocity $\mathbf{v}'$ half way through the time step…

8.1 $$\mathbf{v}' = \mathbf{v}_i + \frac{1}{2}\mathbf{a}_i\boldsymbol{\delta t}$$

…and using that as the average velocity to calculate an improved final position:

8.2 $$\mathbf{r}_{i+1} = \mathbf{r}_i + \mathbf{v}'\boldsymbol{\delta t}$$

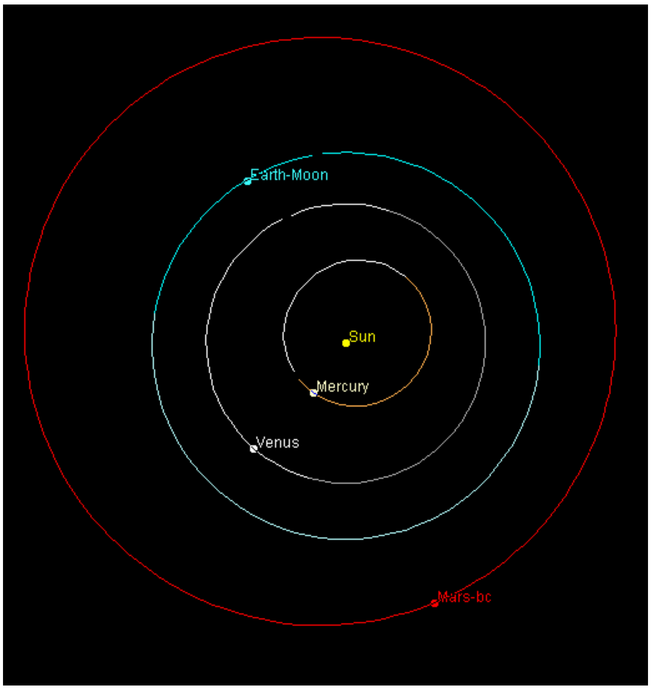
Then we re-calculate the accelerations $\mathbf{a}_{i+1}$ at the new position and use this to get an improved final velocity. The first half of the step uses the initial acceleration and the second half of the step uses the final acceleration.

8.3 $$\mathbf{v}_{i+1} = \mathbf{v}' + \frac{1}{2}\mathbf{a}_{i+1}\boldsymbol{\delta t}$$

In the above, $\mathbf{v}'$ is an intermediate result representing the velocity half way through the time step (hence the "Leapfrog" name).

The "improved" new positions and velocities still have an errors because the acceleration will not be linear nor will the rate of change of acceleration be linear. But they are better than using the initial velocity and acceleration for the whole time step as Euler did.

100 year integration with Leapfrog method and 1-day timestep.



Testing this method we see that, with a time step of 1 day, the orbits look pretty good (see gif attached to the forum post and image above). Although the orbits look good, a detailed examination of the state vectors will show that the planets gradually become a bit out of position. Also, Mercury's orbit is precessing (the direction of the perihelion is changing) much faster than it should do.

Before we move on to yet more sophisticated integration methods we need to talk about some general matters.

Order

When integrating orbits it is clear that smaller time steps δt can lead to smaller errors.

Methods of integration can be classified by the amount of error they introduce, expressed as a power of δt . For example, in a **first order** method the error will be proportional to δt while in a **second order** method the error will be proportional to δt^2. If you reduce the time step of a second order method by a factor of 10 then the error should be reduced by a factor of about 100. Euler is a 1st order method and, as we have seen, not very good. Leapfrog is a second order method and much improved, yet not accurate enough for serious orbit calculations. We will examine integrators up to 8th order.

Conserving Energy

The energy in a gravitational system is the sum of kinetic and potential energy. A good integration method will conserve energy and the degree to which energy is conserved can be a measure of the quality of its results.

The kinetic energy ***KE*** of body $\boldsymbol{p}$ is due to its mass $\boldsymbol{m}$ and velocity $\boldsymbol{v}$. It reflects the energy that would be expended in accelerating the body from rest to the given velocity and is given by:

3.1 $$KE_p = \frac{1}{2} m_p v_p^2$$

The potential energy ***PE*** between two bodies $\boldsymbol{p}$ and $\boldsymbol{q}$ is due to their gravitational interaction. It is the energy required to separate them against the force of gravity. By convention we say that two bodies that are an infinite distance apart have zero potential energy. Potential energy is lost by allowing the objects to fall towards each other so ***PE*** is always negative.

3.2 $$PE_{pq} = -G \frac{m_p m_q}{r_{pq}}$$

The ***PE*** of a system of ***n*** bodies is the sum of the ***PE'***s for each combination of two bodies:

3.3 $$PE_{total} = -G \sum_{p=1}^{p=n-1} \sum_{q=p+1}^{q=n} \frac{m_p m_q}{r_{pq}}$$

The total energy of the system at any time is the sum of all the kinetic and potential energies of all the bodies. The total energy of the system should remain unchanged during the integration and calculating its value at regular intervals is one way of checking how much error is being introduced.

In our Euler demonstration there was a 1% energy error after only one year. In our Leapfrog demonstration the error was about 3 parts in 10^{11} after 100 years. Nevertheless, we have seen that this Leapfrog example shows a wrong rate of precession for Mercury's orbit so energy conservation alone is not the only measure of how well an integrator is working.

Rounding

While smaller time steps reduce the individual errors we must guard against very small time steps that introduce rounding errors. For an example with 9 significant figures:

$\boldsymbol{v} = 1.23456789 \times 10^{-1}$	AU per day
$\boldsymbol{a} = 3.45678912 \times 10^{-6}$	AU per day per time step
$\boldsymbol{v'} = \boldsymbol{a} + \boldsymbol{v} = 1.23460246 \times 10^{-1}$	AU per day

In effect we have used only 4 significant digits from the $\boldsymbol{a}$ figure and "lost" nearly 1/10000[th] of the acceleration. This may not seem a lot but if we integrate for a million time steps the error will become significant.

This problem exists even with double precision floating point numbers where the loss of a large number of “insignificant” figures can add up to a significant error. We are therefore looking for integration methods that can produce good results with reasonable sized timesteps. This is also good for keeping the processing time down.

Barycentre

We have previously discussed using a coordinate system centred on the Sun. However, the Sun is not static and we may want to examine its movements relative to the planets. Also, if you are going to display the positions of the bodies in an animated display then the centre of the system may drift away from the centre of the screen.

A suitable alternative coordinate system is one based on the centre of mass (COM) or “barycentre” of the entire system. If we start with the coordinates $\mathbf{r}_{1\,\text{-}\,n}$ and velocities $\mathbf{v}_{1\,\text{-}\,n}$ of $\boldsymbol{n}$ bodies expressed in any Cartesian coordinate system then the coordinates of the system barycentre (in that coordinate system) is given by:

3.4 $$\mathbf{r}_{\mathbf{com}} = \frac{1}{M}\sum_{p=1}^{n} m_p \mathbf{r}_p$$

(where $\boldsymbol{M}$ is the total mass of all the bodies)

The barycentre may be moving (i.e. the whole system is moving) and its velocity is given by:

3.5 $$\mathbf{v}_{\mathbf{com}} = \frac{1}{M}\sum_{p=1}^{n} m_p \mathbf{v}_p$$

(Remember $\mathbf{r}$ and $\mathbf{v}$ are vectors so you have to do the calculation for $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$.)

To convert all the positions and velocities from their current coordinate system to one with the barycentre as its origin, subtract $\mathbf{r}_{\text{com}}$ from the position of each body and $\mathbf{v}_{\text{com}}$ from the velocity of each body. Do this at the start of the integration so that the barycentre becomes the point {0,0,0} in your coordinate system. This point will not move and any movement of the central mass (Sun) will be visible.

---to be continued…