Orbits & Simulation

Part 1

Introduction

My objective was to obtain an understanding of celestial mechanics while developing an orbit simulator. The original simulator was developed as a Java Applet but, as Applets were deemed insecure, it was converted to a desktop application. A subset of its functions were implemented in JavaScript for delivery in webpages. This version is much slower and less accurate, but can still demonstrate some interesting orbital mechanics. It can be seen here.

My intention is to explain the basic maths used in creating the simulator and to examine some interesting orbital behaviours such as Trojans and Horseshoe Orbits.

Due to the limitations of the discussions board, I will deliver these notes in a set of PDFs hosted on Github. Eventually it may be possible to migrate these to Slooh.

Please feel free to comment and find the deliberate mistakes!.

Coordinate System

Position.

Positions of bodies are described using 3 dimensional Cartesian coordinates. Each body has a position given by its $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ coordinates.

Origin {0,0,0}.

As observers we often use Earth as the origin of our coordinate systems (geocentric) or the position of an individual observer (topocentric). Working with the Solar System it is better to adopt the Sun as the origin (heliocentric). This is normally the case when we talk about orbits and orbital elements and for the time being I will adopt the Sun as our origin.

x-axis

Following the tradition of centuries, the positive direction of the $\boldsymbol{x}$ axis points to the 1st Point of Aries. This is the point in the sky where the the ecliptic crosses the equator and is the position of the Sun at the Vernal Equinox. The $\boldsymbol{x}$ axis is a line through the Sun with positive direction towards the 1st Point of Aries.

y-axis

This is perpendicular to the the $\boldsymbol{x}$ axis and set in the plane of the ecliptic (the plane of Earth's orbit round the Sun). The positive direction is 90 degrees east (anticlockwise) from the positive direction of the x axis when looking down on the system from the north.

z-axis

This is perpendicular to the other two axes with positive towards north.

Epoch

The Sun is not static, the Earth's orbit "wobbles" and the 1st Point of Aries changes with recession. So we have to specify that our coordinates use the Sun and the plane of the ecliptic as at a certain point in time. That point of time is 2000-01-01 12:00 UTC which is called the J2000 Epoch.

These coordinates are called Heliocentric Ecliptic J2000 coordinates.

State Vectors

Position

A body's position is expressed in terms of its radius vector $\mathbf{r}$ from the origin:

1.1 $\mathrm{r} \equiv \{x, y, z\}$
(normal font is vector; italic font is scalar).

Pythagoras tells us the body's distance from the origin is:

1.2 $r = |\mathrm{r}| = \sqrt{x^2 + y^2 + z^2}$
($|\mathrm{r}|$ means magnitude of $\mathbf{r}$)

If we have two bodies $\boldsymbol{p}$ and $\boldsymbol{q}$ then their positions will be at

1.3 $\{x_p, y_p, z_p\}$ and $\{x_q, y_q, z_q\}$.

The line drawn between the two positions is a vector that represents both the distance and direction between the two bodies:

1.4 $\mathrm{r}_{pq} = \mathrm{r}_q - \mathrm{r}_p = \{(x_q - x_p), (y_q - y_p), (z_q - z_p)\}$

The distance between the bodies is:

1.5 $r_{pq} = |\mathrm{r}_{pq}| = \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2 + (z_q - z_p)^2}$

Velocity

Celestial bodies are always moving. If it takes a body time $\boldsymbol{\delta t}$ to go from position 1 to positions 2, then it has an average velocity of:

1.6 $\bar{\mathrm{v}} = \frac{(\mathrm{r}_2 - \mathrm{r}_1)}{\delta t} = \left\{\frac{x_2 - x_1}{\delta t}, \frac{y_2 - y_1}{\delta t}, \frac{z_2 - z_1}{\delta t}\right\}$
(a line over a value mean average)

A dot over a value represents its rate of change or 'differential' with respect to time. So $\dot{x}$ means the rate of change of the x coordinate of the body. Velocity is the rate of change of position.

1.7 $\mathrm{v} = \dot{\mathrm{r}} = \{\dot{x}, \dot{y}, \dot{z}\}$

The straight line average velocity (speed) of the object is the magnitude of v:

1.8 $$v = |\mathrm{v}| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

The values of the radius vector $\mathbf{r}$ and the velocity vector $\mathbf{v}$ together represent the **state vectors** of an object at a specific point in time called the **epoch**. You can obtain the state vectors of Solar System objects at any chosen epoch from services like JPL Horizons or using software like Find_Orb.

Acceleration

Newton told us that a body will continue at the same speed in a straight line unless acted upon by a force. So if a body is at position $\mathbf{r}$ and moving with velocity $\mathbf{v}$ then after time $\boldsymbol{\delta t}$ it will be at a new position $\mathbf{r}'$:

1.9 $$\mathrm{r}' = \mathrm{r} + \mathrm{v}\delta t = \{x, y, z\} + \{\dot{x}\delta t, \dot{y}\delta t, \dot{z}\delta t\}$$

For most objects, speed and direction will change continuously. The rate of change of speed is called acceleration $\mathbf{a}$. Two dots over a value represents the rate of change of the rate of change (otherwise called the second differential with respect to time). Velocity is the rate of change of position and acceleration is the rate of change of velocity. So we get:

1.10 $$\mathrm{a} \equiv \dot{\mathrm{v}} \equiv \ddot{\mathrm{r}} \equiv \{\ddot{x}, \ddot{y}, \ddot{z}\}$$

If a body is moving with velocity $\mathbf{v}$ and experiences acceleration $\mathbf{a}$ then, after time $\boldsymbol{\delta t}$, it will have a new velocity $\mathbf{v}'$**:**

1.11 $$\mathrm{v}' = \mathrm{v} + \mathrm{a}\delta t = \{\dot{x}, \dot{y}, \dot{z}\} + \{\ddot{x}\delta t, \ddot{y}\delta t, \ddot{z}\delta t\}$$

The average velocity during time $\boldsymbol{\delta t}$ is:

1.12 $$\bar{\mathrm{v}} = \frac{1}{2}(\mathrm{v} + \mathrm{v}')$$

So the position $\mathbf{r}'$ at end of time $\boldsymbol{\delta t}$ is:

1.13 $$\mathrm{r}' = \mathrm{r} + \bar{\mathrm{v}}\delta t = \mathrm{r} + \mathrm{v}\delta t + \frac{1}{2}\mathrm{a}\delta t^2$$

At the initial epoch plus $\boldsymbol{\delta t}$ the new state vectors are $\{\mathbf{r}', \mathbf{v}'\}$.

Note that the above is only true if the acceleration does not change during the period $\boldsymbol{\delta t}$.

---to be continued…