Orbits & Simulation

Part 5

We are continuing to look for integration methods that are fast and accurate. One of the most popular ones is Runge-Kutta.

Runge-Kutta method

The Mathematicians C. Runge and M.W. Kutta developed a procedure that can take account of any desired number of terms in the Taylor series by successive use of the technique of employing $(x_{i+1} - x_i)/\delta t$ to estimate $\dot{x}$.

In general, the method can be likened to a multi-step Leapfrog where the acceleration is recalculated at several provisional positions before the final result is achieved. The advantage is we can reduce errors per time step but the disadvantage is we have to recalculate the accelerations several times for each time step.

Here we look at 4th and 5th Order versions of the RK formula.

RK4

In this 4th Order method, acceleration forces have to be recalculated three times at different "predicted" positions during the step. The accelerations are then combined according to the Runge-Kutta formula to yield an accurate new position.

The full derivation is beyond the scope of these pages but a long explanation can be found in Hut & Makino 2003-7

In the following, $\mathbf{a_0}$, $\mathbf{a_1}$, and $\mathbf{a_2}$ contain the accelerations at the start and the two intermediate positions. The expression $\boldsymbol{A}(\boldsymbol{g})$ means calculate the accelerations at the position denoted by the expression $\boldsymbol{g}$.

Calculate the first set of accelerations at the starting positions.

$$\mathbf{a_0} = \boldsymbol{A}(\mathbf{r}_{\boldsymbol{i}})$$

Calculate the accelerations at two other intermediate positions

$$\mathbf{a}_1 = \boldsymbol{A}\left(\mathbf{r}_i + \frac{1}{2}\mathbf{v}_i\delta t + \frac{1}{8}\mathbf{a_0}\delta t^2\right)$$

$$\mathbf{a}_2 = \boldsymbol{A}\left(\boldsymbol{r}_i + \mathbf{v}_i\delta t + \tfrac{1}{2}\mathbf{a_1}\delta t^2\right)$$

Assemble the final position and velocity:

$$\mathbf{r}_{i+1} = \mathbf{r}_i + \mathbf{v}_i\delta t + \frac{1}{6}(\mathbf{a}_0 + 2\mathbf{a}_1)\delta t^2$$

$$\mathbf{v}_{i+1} = \mathbf{v}_i + \frac{1}{6}(\mathbf{a}_0 + 4\mathbf{a}_1 + \mathbf{a}_2)\delta t$$

Each step should be carried out for all the bodies in the system so that the accelerations calculated at the intermediate positions are produced when all the bodies are at their intermediate positions.

RK5

The principle is the same as the 4th order RK method but it takes 6 acceleration evaluations to produce a 5th order method. It is questionable whether the extra processing is worth the extra order of accuracy compared to using the RK4 method with a shorter time step. The structure of the procedure is as for the RK4 method but with the extra stages.

The various accelerations are calculated according to:

$$\begin{aligned}
\boldsymbol{a_0} &= A(\boldsymbol{r}_i) \\
\boldsymbol{a_1} &= A\left(\boldsymbol{r_i} + \frac{1}{4}\boldsymbol{v}_i\delta t\right) \\
\boldsymbol{a_2} &= A\left(\boldsymbol{r}_i + \frac{1}{4}\boldsymbol{v_i}\delta t + \frac{1}{32}\boldsymbol{a_0}\delta t^2\right) \\
\boldsymbol{a_3} &= A\left(\boldsymbol{r_i} + \frac{1}{2}\boldsymbol{v_i}\delta t + \frac{1}{8}\boldsymbol{a_1}\delta t^2\right) \\
\boldsymbol{a_4} &= A\left(\boldsymbol{r_i} + \frac{3}{4}\boldsymbol{v_i}\delta t + \frac{9}{16}\boldsymbol{a_2}\delta t^2 - \frac{9}{32}\boldsymbol{a_1}\delta t^2\right) \\
\boldsymbol{a_5} &= A\left(\boldsymbol{r_i} + \boldsymbol{v_i}\delta t + \frac{\delta t^2}{14}(7\boldsymbol{a_0} + 15\boldsymbol{a_1} - 24\boldsymbol{a_2} + 9\boldsymbol{a_3})\right)
\end{aligned}$$

The expression ***A*(g)** means calculate accelerations based on positions **g**.

The accelerations are then combined to produce the new positions and velocities:

$$\mathbf{r}_{i+1} = \mathbf{r}_i + \mathbf{v}\delta\mathbf{t} + \frac{\mathbf{1}}{\mathbf{90}}\left(\mathbf{7a}_0 + \mathbf{24a}_2 + \mathbf{6a}_3 + \mathbf{8a}_4\right)\boldsymbol{\delta t}^2$$

$$\mathbf{v}_{i+1} = \mathbf{v}_i + \frac{\mathbf{1}}{\mathbf{90}}\left(\mathbf{7a}_0 + \mathbf{32a}_2 + \mathbf{12a}_3 + \mathbf{32a}_4 + \mathbf{7a}_5\right)\boldsymbol{\delta t}$$

The various numerical coefficients have been derived from Boulet (1991) and their derivation is available in a separate document.

The RK5 integrator will run with an energy error of the order of 1×10^{-7} in 100 years using a 1 day time step and 10^{-12} using a 0.1 day time step. With the 0.1 day time step, the integration will run at 4 years per second and the precession of Mercury is correct to within a fraction of an arcsecond per century (when relativity is taken into consideration, which we will address in future notes).

The Runge-Kutta integration methods have been around for a long time and are very popular. However, they are not necessarily the fastest integrators for computers and several alternative approaches have been researched in recent decades.

….. to be continued.