ELEC 5360 Fall 2014
Homework 1

Due on Sept. 25, 2014, before class

1. **Deriving bounds for Q function**.
(a) Use the Chernoff bound to show that

$$Q(x) \leq e^{-x^2/2}. \tag{1}$$

(b) Integrate by parts to derive the upper and lower bounds

$$Q(x) < \frac{1}{\sqrt{2\pi x^2}} e^{-x^2/2}, \tag{2}$$

and

$$Q(x) > (1 - \frac{1}{x^2}) \frac{1}{\sqrt{2\pi x^2}} e^{-x^2/2}, \tag{3}$$

respectively.

(c) Here is another way to establish these tight upper and lower bounds. By using a simple change of variables, firstly show that

$$Q(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \int_0^\infty \exp(-\frac{y^2}{2} - xy) dy. \tag{4}$$

Then show that

$$1 - \frac{y^2}{2} \leq \exp(-\frac{y^2}{2}) \leq 1. \tag{5}$$

Putting these together, derive the bounds of part (b).

2. **Outage probability calculation**.
(a) The capacity of a wireless channel with Rayleigh fading is given by

$$C = \log_2(1 + |h|^2\gamma), \tag{6}$$

where $\gamma > 0$ is the signal-to-noise ratio (SNR) and $h \sim \mathcal{CN}(0,1)$ represents the channel coefficient. Calculate the following outage probability

$$p_{\text{out}}(R) = \Pr\{\log_2(1 + |h|^2\gamma) < R\}, \tag{7}$$

where $R > 0$ is a predefined threshold. The outage probability is the probability that the capacity of the channel is below the threshold.

(b) The capacity of a parallel fading channel is given by

$$C = \sum_{l=1}^{L} \log_2(1 + |h_l|^2\gamma), \tag{8}$$

where the random variables h_l's are mutually independent with $h_l \sim \mathcal{CN}(0,1), \forall l$. Calculate a lower bound for the following outage probability

$$p_{\text{out}}(R) = \Pr\left\{\frac{1}{L}\sum_{l=1}^{L} \log_2(1 + |h_l|^2\gamma) \leq R\right\}. \tag{9}$$

via Jensen inequality. [Hint: If f is convex, $x_1, \ldots, x_k$ in its domain, and $\theta_1, \ldots, \theta_k \geq 0$ with $\theta_1 + \cdots + \theta_k = 1$, then the Jensen's inequality can be stated as:

$$f(\theta_1 x_1 + \cdots + \theta_k x_k) \leq \theta_1 f(x_1) + \cdots + \theta_k f(x_k). \tag{10}$$

In particular, the function $-\log(x)$ is convex.]

3. **Stochastic process**

Let $\{N(t), t \geq 0\}$ be a Poisson processes with rate $\lambda > 0$, i.e.,

$$\Pr\{N(t) = k\} = \frac{(\lambda t)^k}{k!} \exp(-\lambda t). \tag{11}$$

The telegraph signal $X(t)$ is given by the following stochastic process

$$X(t) = X_0 \cdot (-1)^{N(t)}, \tag{12}$$

where X_0 equals ± 1 with equal probability and is independent with $N(t)$.

a) Calculate the autocorrelation function of the stochastic process $X(t)$, i.e., $R_X(t, s) = \mathbb{E}[X(t)X(s)]$ with $t \geq s$.

b) Calculate the power spectral density of the stochastic process $X(t)$.

4. **Huffman coding**

The optimum four-level nonuniform quantizer for a Gaussian-distributed signal amplitude results in the four levels a_1, a_2, a_3 and a_4, with corresponding probabilities of occurrence $p_1 = p_2 = 0.3365$ and $p_3 = p_4 = 0.1635$.

a) Design a Huffman code that encodes a signal level at a time.

b) Determine the average number of binary digits per source level.

c) Determine the entropy $H(X)$ of the source.

5 . **Lempel-Ziv algorithm** [Section 6.3 of Proakis, Page 346-348]

Find the Lempel-Ziv source code for the binary source sequence

0001 001 00000011 00001 00000001 0000001 01 00001 0000001 101 00000001 100

Recover the original sequence back from the Lempel-Ziv source code. [Hint: You require two passes of the binary sequence to decide on the size of the dictionary.]