ELEC 5360 Homework 5

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1. (a)

First recall that for a matrix A,

$$\det(A) = \prod_{i=1}^{n} \lambda_i$$

where λ_i's are the eigenvalues of A.

This is because, when obtaining λ_i's, we have $det(A - \lambda I) = 0$. Written in *characteristic polynomial* form, we have $det(A - \lambda I) = (\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_n)$. Set $\lambda = 0$, we have $\det(A) = \lambda_1 \lambda_2 ... \lambda_n$.

Now,

$$\det(HH^*) = \prod_{i=1}^{n_{min}} \lambda_i$$

where λ_i's are eigenvalues of HH^* and $n_{min} = \min(N_t, N_r)$.

Therefore

$$\det(\frac{\gamma}{N_t} HH^*) = \prod_{i=1}^{n_{min}} \frac{\gamma}{N_t} \lambda_i$$

And

$$\det(I_{N_r} + \frac{\gamma}{N_t} HH^*) = \prod_{i=1}^{n_{min}} 1 + \frac{\gamma}{N_t} \lambda_i$$

this is because, for a matrix A, and eigenvalue λ_i, eigenvector $\vec{v_i}$, $A\vec{v_i} = \lambda_i \vec{v_i}$, we have $(A+I)\vec{v_i} = (\lambda_i + 1)\vec{v_i}$ Now applying *logarithmic operation* on both sides, we have

$$log_2 \det(\frac{\gamma}{N_t} HH^*) = \sum_{i=1}^{n_{min}} log_2(1 + \frac{\gamma}{N_t} \lambda_i)$$

(b)

We have

$$\begin{aligned} C &= \sum_{i=1}^{n_{min}} \mathbb{E}[log_2(1+\frac{\gamma}{N_t}\lambda_i)] \\ &= \sum_{i=1}^{n_{min}} \mathbb{E}[\frac{log(1+\frac{\gamma}{N_t}\lambda_i)}{log2}] \\ &= \sum_{i=1}^{n_{min}} \mathbb{E}[log(1+\frac{\gamma}{N_t}\lambda_i)]\, log_2 e \end{aligned}$$

Notice for small x, we have $log(1+x) \approx x$. When $\gamma \to 0$, we have

$$\begin{aligned} C &\approx \sum_{i=1}^{n_{min}} \frac{\gamma}{N_t}\mathbb{E}[\lambda_i]\, log_2 e \\ &= \frac{\gamma}{N_t}\mathbb{E}[tr(HH^*)]\, log_2 e \\ &= \frac{\gamma}{N_t}\mathbb{E}[\sum_{i,j} |h_{i,j}|^2]\, log_2 e \end{aligned}$$

because $tr[A] = \sum_i \lambda_i(A)$, and in the matrix HH^*, the i'th diagonal element is $\sum_j h_{ij}h_{ji}$. Now notice that $\mathbb{E}[\sum_{i,j} |h_{i,j}|^2] = N_t N_r$, assuming the normalization of power.

So we get

$$C \approx N_r \gamma log_2 e$$

2. (a)

Notice that $\vec{y} - H\vec{x} = \vec{n} \sim \mathcal{N}(0, N_0)$, therefore we get

$$\begin{aligned} p(\vec{p}|H,\vec{x}) &= \frac{1}{(\pi N_0)^{N_r}} exp(\frac{-||\vec{y}-H\vec{x}||^2}{N_0}) \\ &= \frac{1}{(\pi N_0)^{N_r}} exp(-\frac{\sum_{m=1}^{N_r} |y_m - \sum_{n=1}^{N_t} h_{mn}x_n|^2}{N_0}) \end{aligned}$$

(b)

Estimation of signal $\hat{x}$ of an *ML receiver* is

$$\hat{x} = \text{argmin}_{x\in\{s\}}||\vec{y} - H\vec{x}||^2$$

Using pseduoinverse, we will have

$$\hat{x} = H^*(HH^*)^{-1}\vec{y}$$

(c)

$$\begin{aligned} p(\vec{x_k} \to \vec{x_l}|H) &= Q(\frac{||H(\vec{x_k} - \vec{x_l})||}{\sqrt{2N_0}}) \\ &= Q(\sqrt{\frac{E_s||H(\vec{s_k} - \vec{s_l})||^2}{2N_0N_r}}) \\ &= Q(\sqrt{\frac{E_b||H(\vec{s_k} - \vec{s_l})||^2}{2N_0}}) \end{aligned}$$

(d)

Written the row vector of matrix H as $h_1, h_2, ..., h_{N_r}$, then

$$\begin{aligned} ||H(\vec{s_k} - \vec{s_l})||^2 &= ||[h_1^T, h_2^T, ..., h_{N_r}^T]^T(\vec{s_k} - \vec{s_l})||^2 \\ &= \begin{pmatrix} h_1^T(\vec{s_k} - \vec{s_l}) \\ h_2^T(\vec{s_k} - \vec{s_l}) \\ ... \\ h_{N_r}^T(\vec{s_k} - \vec{s_l}) \end{pmatrix}^2 \end{aligned}$$

Notice that each row follows Complex Gaussian Distribution as $\mathcal{CN}(0, ||\vec{s_k} - \vec{s_l}||^2)$.

Hence, by the relation between *Gaussian Distribution* and *Chi-Square Distribution*,

$$||H(\vec{s_k} - \vec{s_l})||^2 \sim \frac{||(\vec{s_k} - \vec{s_l})||}{2}\chi^2(2N_r)$$

(e)

By *Chernoff Bound* of Q-function, we know for average error probability

$$
\begin{aligned}
p(\vec{x_k} \to \vec{x_l}|H) &= \mathbb{E}(Q(\sqrt{\frac{E_b||H(\vec{s_k}-\vec{s_l})||^2}{2N_0}})) \\
&\leq \frac{1}{2}\mathbb{E}[exp(\frac{E_b||H(\vec{s_k}-\vec{s_l})||^2}{4N_0})]
\end{aligned}
$$

Now, as we know $||H(\vec{s_k}-\vec{s_l})||^2$ follows χ^2 distribution, from the moment generating function, for χ^2 distributed x, $\mathbb{E}[e^{tx}] = \frac{1}{(1-2t)^{k/2}}$.

Then we have

$$
p(\vec{x_k} \to \vec{x_l}|H) = \frac{1}{2(1-\frac{E_b||H(\vec{s_k}-\vec{s_l})||^2}{4N_0})^{N_r}}
$$

3. (a)

$$
H = \begin{bmatrix} 1.0 & 0.2 \\ -0.1 & 1.5 \end{bmatrix}
$$

We have

$$
HH^* = \begin{bmatrix} 1.0 & 0.2 \\ -0.1 & 1.5 \end{bmatrix} \begin{bmatrix} 1.0 & -0.1 \\ 0.2 & 1.5 \end{bmatrix} = \begin{bmatrix} 1.04 & 0.2 \\ 0.2 & 2.26 \end{bmatrix}
$$

Now we determine the eigenvalues and eigenvectors, let

$$
det \begin{pmatrix} 1.04-\lambda & 0.2 \\ 0.2 & 2.26-\lambda \end{pmatrix} = 0
$$

Solving the *characteristic equation*, we have

$$
\lambda_1 = 1.008, \lambda_2 = 2.292
$$

with corresponding eigenvectors,

$$\vec{v_1} = [0.158, 0.988]^T, \vec{v_2} = [0.988, -0.158]^T$$

Therefore, we get the singular value,

$$\sqrt{\lambda_1} = 1.004, \sqrt{\lambda_2} = 1.514$$

Likewise we can do similar process for H^*H, and we get the *SVD decomposition* as follows

$$H = \begin{bmatrix} 0.158 & 0.988 \\ 0.988 & -0.158 \end{bmatrix} \begin{bmatrix} 1.514 & 0 \\ 0 & 1.004 \end{bmatrix} \begin{bmatrix} 0.039 & 0.999 \\ 0.999 & -0.033 \end{bmatrix}$$

(b)

For equal power allocation, we have

$$\begin{aligned} C &= log_2 \det(I_{N_r} + \frac{E_s}{N_t N_0} HH^*) \\ &= \sum_{i=0}^{r} log_2\,(1 + \frac{E_s}{N_t N_0} \lambda_i(HH^*)) \\ &= log_2(1 + \frac{1}{2 \times 0.1} \times 1.008) + log_2(1 + \frac{1}{2 \times 0.1} \times 2.292) \\ &= 6.233 \text{ bps/Hz} \end{aligned}$$

(c)

For water filling, we have

$$C = \max_{\sum_{k=1}^{r} \gamma_k = N_r} \sum_{k=1}^{r} log_2\,(1 + \frac{E_s \lambda_k}{N_t N_0} \gamma_k)$$

By *Lagrange Multiplier*, we will have optimal γ_k^* satisfies

$$\gamma_k^* = (\mu - \frac{N_t N_0}{E_s \lambda_k})^+$$
$$\sum_{k=1}^{r} \gamma_i^* = N_r$$

where $(x)^+ = x$ if $x > 0$, and $= 0$, otherwise.

Now, assuming both terms in summation are not zero, we have $\gamma_1 + \gamma_2 = 2$, which expands to

$$(\mu - \frac{2 \times 0.1}{1 \times 1.008}) + (\mu - \frac{2 \times 0.1}{1 \times 2.292}) = 2$$

This leads to $\mu = 1.143$.

Substitute μ back, we have

$$\gamma_1 = 0.944, \gamma_2 = 1.056$$

Furthermore,

$$C = log_2\,(1 + \frac{1 \times 1.008}{2 \times 0.1} \times 0.944) + log_2\,(1 + \frac{1 \times 2.292}{2 \times 0.1} \times 1.056) = 6.237 \text{ bps/Hz}$$

So *water-filling* obtains a higher capacity than *equal power* does, which is within expectation, as high quality channel is utilized better.

4. (a)

For the general *BER* expression, the average error probability can be calculated as follows

$$
\begin{aligned}
&\int_0^\infty \alpha Q(\sqrt{\beta x})\frac{1}{\bar{\gamma}}e^{-\frac{x}{\bar{\gamma}}}dx \\
&= \int_0^\infty \alpha \int_{\sqrt{\beta x}}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy\frac{1}{\bar{\gamma}}e^{-\frac{x}{\bar{\gamma}}}dx \\
&= \int_0^\infty \alpha \int_0^{\frac{y^2}{\beta}} \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\frac{1}{\bar{\gamma}}e^{-\frac{x}{\bar{\gamma}}}dxdy \\
&= \int_0^\infty \alpha\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}(-e^{-\frac{x}{\bar{\gamma}}}\Big|_0^{\frac{y^2}{\beta}})dy \\
&= \frac{\alpha}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{y^2}{2}}(1-e^{-\frac{y^2}{\beta\bar{\gamma}}})dy \\
&= \frac{\alpha}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{y^2}{2}}dy - \frac{\alpha}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{y^2}{2}-\frac{y^2}{\beta\bar{\gamma}}}dy \\
&= \frac{\alpha}{\sqrt{2\pi}}(\frac{\sqrt{2\pi}}{2} - \frac{\sqrt{2\pi}}{2}\sqrt{\frac{\bar{\gamma}\beta/2}{\bar{\gamma}\beta/2+1}}) \\
&= \frac{\alpha}{2}(1-\sqrt{\frac{\bar{\gamma}\beta/2}{\bar{\gamma}\beta/2+1}})
\end{aligned}
$$

(b)

For *BPSK* $\alpha = 1, \beta = 2$, then

$$\bar{p_b} = \frac{1}{2}(1-\sqrt{\frac{\bar{\gamma}}{\bar{\gamma}+1}}) = \frac{1}{2}(1-\sqrt{\frac{1}{1+\frac{1}{\bar{\gamma}}}})$$

For high *SNR* (large $\bar{\gamma}$), we denote $\epsilon = \frac{1}{\bar{\gamma}}$, which is small

$$
\begin{aligned}
\frac{1}{\sqrt{1+\epsilon}} &= 1 + (\frac{1}{\sqrt{1+\epsilon}})'\Big|_{\epsilon=0}\epsilon + o(\epsilon^2) \\
&= 1 - \frac{1}{2}(1+\epsilon)^{-\frac{3}{2}}\Big|_{\epsilon=0}\epsilon + o(\epsilon^2) \\
&\approx 1 - \frac{1}{2}\epsilon
\end{aligned}
$$

Substitute this result back, we obtain for high SNR,

$$\bar{p_b} \approx \frac{1}{2}[1 - 1 + \frac{1}{2\bar{\gamma}}] = \frac{1}{4}\gamma^{-1}$$

(c)

$$\begin{aligned}
p_b &= \int_0^\infty Q(\sqrt{2\gamma_b})p(\gamma_b)d\gamma_b \\
&= \int_0^\infty \int_{\sqrt{2\gamma_b}}^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \frac{1}{(L-1)!\bar{\gamma}_c^L} \gamma_b^{L-1} e^{-\gamma_b/\gamma_c} d\gamma_b \\
&= \frac{1}{(L-1)!\bar{\gamma}_c^L} \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \int_0^{y^2/2} \gamma_b^{L-1} e^{-\gamma_b/\gamma_c} d\gamma_b \, dy \\
&= \frac{1}{(L-1)!\bar{\gamma}_c^L} \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \gamma_c^L (L! - (L-1)!\Gamma[L, \frac{y^2}{2\gamma_c}]) dy \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} (L - \sum_{k=0} \frac{y^{2k}}{(2\gamma_c)^k k!}) dy \\
&= \frac{L}{2} - \sum_{k=0} \int_0^\infty \frac{e^{-y^2/2} y^{2k}}{(2\gamma_c)^k k!} dy \\
&= \frac{L}{2} - \sum_{k=0} L - 1 \frac{1}{(\gamma_c)^k k!} 2^{-1/2+k} \Gamma(\frac{1}{2} + k)
\end{aligned}$$

which is numerically equivalent to

$$[\frac{1}{2}(1-\mu)]^L \sum_{k=0}^{L-1} \begin{pmatrix} L-1+k \\ k \end{pmatrix} [\frac{1}{2}(1+\mu)]^k$$

where $\mu = \sqrt{\frac{\bar{\gamma}_c}{1+\bar{\gamma}_c}}$