**Q function**: $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{t^2/2} dt$

Transform a Gaussian Distribution to Normal Distribution:

$F_X \sim \mathcal{N}(m, \sigma^2)$

$F_X(x) = \Phi(\frac{x-m}{\sigma})$, where $\Phi \sim \mathcal{N}(0,1)$

$Tail(x) = Q(\frac{x-m}{\sigma})$

Some bounds: $Q(x) \le e^{-x^2/2}$

$Q(x) < \frac{1}{\sqrt{2\pi x^2}} e^{-x^2/2}$

$Q(x) > (1 - \frac{1}{x^2}) \frac{1}{\sqrt{2\pi x^2}} e^{-x^2/2}$

$Q(x) \le \frac{1}{2} e^{-x^2/2}$

Chebyshv: $Q(x) \le \frac{1}{2x^2}$

**Markov Inequality**: $P(x \le a) \le \frac{E(x)}{a}$

**Chebyshv Inequality**: $P(|x-m| \ge a) \le \frac{\sigma^2}{a^2}$

**Chernoff Bound**: $P(X \le a) \le e^{-\widetilde{v}a} E(e^{\widetilde{v}X})$

Chernoff Bound holds for $\widetilde{v} > 0$, the *tightest* bound achieves when

$\frac{d}{d\widetilde{v}}(E(e^{\widetilde{v}X})) = E(Xe^{\widetilde{v}X}) = aE(e^{\widetilde{v}X})$

**Poisson Process**: $Pr(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$

$N(t)$ stands for number of occurrence, and this is a *memoryless* process.

**Real Gaussian Vector**

$\vec{w} \sim \mathcal{N}(0, I)$

Definition: $P(\vec{w}) = \frac{1}{(\sqrt{2\pi})^n} e^{-(\frac{||w||^2}{2})}$

Each definition of $\vec{w}$ follows $\mathcal{N}(0,1)$

For $\vec{x} = A\vec{w} + \vec{b}$

$p(\vec{x}) = \frac{1}{(\sqrt{2\pi})^n \sqrt{det(A^T A)}} e^{\frac{1}{2}(\vec{x}-\vec{b})^T (A^T A)^{-1} (\vec{x}-\vec{b})}$

Moreover $\vec{x} \sim \mathcal{N}(\vec{b}, AA^T)$

And $\forall \vec{c} \in \mathcal{R}^n$, $\vec{c}^T\vec{x} \sim \mathcal{N}(\vec{c}^T\vec{b}, \vec{c}^T AA^T \vec{c})$

For 1D case, if $X \sim \mathcal{N}(0, \sigma^2)$, then $aX + b \sim \mathcal{N}(b, a^2\sigma^2)$

**Complex Random Variable**

$x \sim C\mathcal{N}(0,1)$ (or $x \sim C\mathcal{N}(0, \sigma^2)$)

*Real* and *Imaginary* components yields

$Re, Im \sim \mathcal{N}(0, 1/2)$ (or $Re, Im \sim \mathcal{N}(0, \sigma^2/2)$)

The *angle* is uniformly distributed on $[0, 2\pi)$

$|x|$ follows *Rayleigh* Distribution

$P(r) = 2re^{-r^2}$ for $C\mathcal{N}(0,1)$

(or $P(r) = \frac{r}{\sigma_0^2} e^{\frac{-r^2}{2\sigma_0^2}}$ for $C\mathcal{N}(0, \sigma^2)$,

$\sigma_0^2 = \sigma^2/2$ is for $Re$ and $Im$ component)

**Relations between different distributions**

For $X_1 \sim \mathcal{N}(m_1, \sigma^2)$, $X_2 \sim \mathcal{N}(m_2, \sigma^2)$

If $m_1 = m_2 = 0$, $X = \sqrt{X_1^2 + X_2^2}$ follows *Rayleigh* Distribution, $p(x) = \frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}}, x > 0$

If $m_1, m_2 > 0$, $X$ follows *Ricean* Distribution, $p(x) = \frac{x}{\sigma^2} I_0(\frac{sx}{\sigma^2}) e^{-\frac{x^2+s^2}{2\sigma^2}}$, where $s = \sqrt{m_1^2 + m_2^2}$, $I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{xcos\theta} d\theta$

For $Z_i \sim \mathcal{N}(0,1)$, then $\sum_{i=1}^k Z_i^2$ follows *Chi-Square* distribution with *degree of freedom* k.

$p(x,k) = \frac{x^{(k/2-1)} e^{-x/2}}{2^{k/2}\Gamma(k/2)}, x > 0.$

$(p(x,1) = \frac{e^{-x/2}}{\sqrt{2x\pi}}$,

$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dt, \Gamma(t) = t!$ if $t$ is integer)

*Nakagami* Random Variable is typical for Fading Channel.

$p(x) = \frac{2}{\Gamma(m)} (\frac{m}{\Omega})^m x^{2m-1} e^{-mx^2/\Omega}, x > 0$, where

$\Omega = E(x^2)$, and $m = \frac{\Omega^2}{E[(x^2-\Omega)^2]}, m \leq 1/2$, called fading figure.

**Lloyd-Max Algorithm**

Notice $b_0 = 0, b_n = \infty$, update algorithm: *representation* point $a_i = \frac{\int_{b_{i-1}}^{b_i} x f_X(x)dx}{\int_{b_{i-1}}^{b_i} f_X(x)dx}$, which is basically the distribution 'center' and the *endpoint* $b_i = \frac{a_{i-1}+a_i}{2}$

**Nyquist Criterion**

$g(t) = p(t) * h(t) * q(t)$ satisfies no *inter-symbol interference* property

$$g(jT - kT) = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{otherwise} \end{cases}$$

in frequency domain,

$\sum_{m\in\mathbb{Z}} \hat{g}(f - m/T) rect(fT) = T\, rect(fT)$

which can be simplified to

$\frac{1}{T}\sum_{m\in\mathbb{Z}} \hat{g}(f - m/T) = 1$

**Gaussian Process**

Definition: $\{Z(t)\}$, for $i$ in a finite set of $\{n\}$, $\{Z(t_i)\}$ are jointly Gaussian set of r.v.

Basically *i.i.d. Gaussian r.v.* on each time instance.

Let $\{Z(t)\}$ be White Gaussian Noise Process. $V = \int g(t)Z(t)dt$ will be Gaussian r.v. with zero mean.

Notice from *Linear Functional of WSS process* $E[V^2] = \int_{-\infty}^{\infty} |\hat{g}(f)|^2 S_Z(f)df$,

where $S_Z(f)$ is spectral density.

Now note that $\sigma^2 = E[V^2] - E[V]^2$

Typically, $E[V] = E[\int g(t)Z(t)dt] = 0$,

because Z is zero-mean

$E[V^2] = E[\int g(t)Z(t)dt \int g(\tau)Z(\tau)d\tau]$

$= \int\int E[Z(t)Z(\tau)]g(t)g(\tau)dtd\tau$

$= \int\int \frac{N_0}{2}\delta(t-\tau)g(t)g(\tau)dtd\tau$

(This is because the gaussian r.v. on different time are independent, and $E(Z^2) - E(Z)^2 = \sigma^2 = N_0/2$, assuming spectral density $\frac{N_0}{2}$)

**Error Probability**

*bit error probability*: $Pr(e) = Q(\frac{d(a_0,a_1)/2}{\sigma}) = Q(\frac{d(a_0,a_1)}{\sqrt{2N_0}})$, assuming WGN spectral density $N_0/2$

*Symbol distance & Symbol Energy*

In general, the energy for a point in constellation graph is its $norm^2$, take mean of them for the average symbol energy

Further, bit energy $E_b = E_S/log_2 M$

*FSK*: $d = \sqrt{2E_s}$ (holds for every two symbols)

*M-PSK*: $d = 2\sqrt{E_s}sin(\pi/M)$

*M-PAM/M-ASK*: $d = \sqrt{\frac{12E_{savg}}{M^2-1}}$

*M-QAM*: just derive from stretch, but for M = 4, 16, 64, 256 $d = \sqrt{\frac{6E_{savg}}{M-1}}$

$P_c(M-QAM) = P_c^2(\sqrt{M}-PAM) = (1 - P_e(\sqrt{M}-PAM))^2$

$P_e(M-QAM) = 1 - (1 - P_e(\sqrt{M}-PAM))^2$