ELEC 5360 Fall 2014
Homework 5

Due on Nov. 27, 2014, before class

1. **MIMO Capacity**

(a) Please verify the following equality for the MIMO capacity on Slide 32 of Lecture Note 8:

$$\log_2 \det\left(\mathbf{I}_{N_r} + \frac{\gamma}{N_t}\mathbf{H}\mathbf{H}^{\mathsf{H}}\right) = \sum_{i=1}^{n_{\min}} \log_2\left(1 + \frac{\gamma}{N_t}\lambda_i\right),$$

where λ_i is the i-th largest eigenvalue of $\mathbf{H}\mathbf{H}^{\mathsf{H}}$ and $n_{\min} = \min\{N_t, N_r\}$.

(b) On Slide 33 of Lecture Note 8, we discussed the capacity gain of MIMO. At low SNR, when $\gamma \to 0$, we have the average capacity

$$\begin{aligned} C &\approx \sum_{i=1}^{n_{\min}} \frac{\gamma}{N_t}\mathbb{E}[\lambda_i]\log_2 e & (1)\\ &= \frac{\gamma}{N_t}\mathbb{E}\left[\mathrm{tr}(\mathbf{H}\mathbf{H}^{\mathsf{H}})\right]\log_2 e & (2)\\ &= \frac{\gamma}{N_t}\mathbb{E}\left[\sum_{i,j}|h_{ij}|^2\right]\log_2 e & (3)\\ &= N_r\gamma\log_2 e, & (4) \end{aligned}$$

Please justify each step, i.e., from Step (1) to Step (4).

2. **MIMO Receiver**

In this problem, we will investigate MIMO maximum likelihood detection (MLD). The received signal of a MIMO channel can be written as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}, \text{ with } \mathbf{x} = \sqrt{\frac{E_s}{N_t}}\mathbf{s}, \tag{5}$$

where E_s is the symbol energy, N_t is the number of transmit antennas. $\mathbf{H} \in \mathbb{C}^{N_r \times N_t}$ is the channel matrix for the flat fading channel, whose elements are independent zero-mean complex Gaussian random variables with unit variance. $\mathbf{s} \in \mathbb{C}^{N_t \times 1}$ is the transmit signal vector with $\mathbb{E}[|s_2|^2] = 1$, and $\mathbf{n}$ is the additive noise vector with zero mean and variance N_0.

(a) Write the expression of $p(\mathbf{y}|\mathbf{x}, \mathbf{H})$. [Hint: the distribution of $p(\mathbf{y}|\mathbf{x}, \mathbf{H})$] is jointly Gaussian.]

(b) Write down the estimate signal $\hat{\mathbf{x}}$ of an ML receiver.

(c) Derive an expression for the pairwise error probability (PEP) of the ML detector $P(\mathbf{x}_k \to \mathbf{x}_l|\mathbf{H})$. [Hint: first find the Euclidean distance between two vectors $\mathbf{x}_k$ and $\mathbf{x}_l$. Express the PEP in a form of Q-function, w.r.t., E_b/N_0.]

(d) Show the random variable $\|\mathbf{H}(\mathbf{s}_k - \mathbf{s}_l)\|^2$ with $\mathbf{s}_k$ and $\mathbf{s}_l$ fixed distributed as $\frac{\|\mathbf{s}_k - \mathbf{s}_l\|^2}{2}\chi^2(2N_r)$.

(e) Derive an upper bound of the average pairwise error probability of the ML detector by applying the Chernoff bound. [Hint: The Chernoff bound of the Q-function is $Q(x) \le \frac{1}{2}e^{-\frac{x^2}{2}}, x > 0$; the moment generating function of a random variable X, which follows a χ^2 distribution with k degrees of freedom, is given by $M_X(t) = \mathbb{E}[e^{tX}] = \frac{1}{(1-2t)^{k/2}}, t \in \mathbb{R}$.]

3. **Water-filling**

For a 2×2 MIMO system with AWGN, the channel matrix is

$$\mathbf{H} = \begin{bmatrix} 1.0 & 0.2 \\ -0.1 & 1.5 \end{bmatrix}. \tag{6}$$

The received signal vector is written as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}. \tag{7}$$

The total transmit power is $1W$ and the noise power at each receive branch is $0.1W$.

(a) Calculate the singular values of $\mathbf{H}$ and determine the SVD of $\mathbf{H}$.

(b) Determine the capacity of this MIMO system with equal power allocation.

(c) What is the capacity with water filling power allocation? Compared it with the previous question.

4. **Error Probability for M-ary Modulations**

In this problem, we will derive the average error probability for general M-ary modulations in the fading channel. The error probability of different modulations in the AWGN channel can be written in the general form

$$P_b \approx \alpha Q(\sqrt{\beta x}), \tag{8}$$

where x denotes the received SNR. For example, $\alpha = 1, \beta = 2$ for BPSK. Now we want to calculate the average error probability

$$\bar{P}_b = \mathbb{E}\left[\alpha Q\left(\sqrt{\beta x}\right)\right], \tag{9}$$

where for the Rayleigh fading channel the pdf of the received SNR is given as

$$p_X(x) = \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}}, x \geq 0, \tag{10}$$

where $\bar{\gamma}$ is the average SNR.

(a) From the general BER expression, show that the average error probability in the Rayleigh fading channel is given by

$$\bar{P}_b = \frac{\alpha}{2}\left[1 - \sqrt{\frac{\bar{\gamma}\beta/2}{\bar{\gamma}\beta/2 + 1}}\right], \tag{11}$$

based on the following alternative representation of the Q-function:

$$Q(z) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left[\frac{-z^2}{2\sin^2\theta}\right] d\theta. \tag{12}$$

[Hint: $\int \frac{1}{\alpha + \sin^2 x} dx = \frac{1}{\sqrt{\alpha(\alpha+1)}} \arctan\left(\sqrt{\frac{\alpha+1}{\alpha}} \tan x\right), \alpha > 0.$]

(b) Show that for BPSK, at high SNR, the average error probability can be approximated by

$$P_b \approx \frac{1}{4}\bar{\gamma}^{-1}, \tag{13}$$

i.e., the error probability decays inversely proportional to the SNR.

(c) Derive the BER expression of BPSK with MRC. Assume there are L diversity channels. The received signals from these independent channels can be written in a vector form

$$\mathbf{y} = \mathbf{h}x + \mathbf{n}, \mathbf{h} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_L \end{bmatrix}, h_k \sim \mathcal{CN}(0,1), \mathbf{n} \sim \mathcal{CN}(\mathbf{0}, N_0\mathbf{I}). \tag{14}$$

[Hint: the combined SNR $\gamma_b = \frac{E_b}{N_0}\sum_{k=1}^{L} |h_k|^2 \triangleq \sum_{k=1}^{L} \gamma_k$, and the SNR follows a chi-square distribution $p(\gamma_b)$ with $2L$ degrees of freedom. The BER of a BPSK signal can be calculated as $p_b = \int_0^\infty Q(\sqrt{2\gamma_b}) p(\gamma_b) d\gamma_b$.]