ELEC 5360 Homework 1

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1. (a)

Note that the Q function is given by:

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dx$$

By Chernott bound:

$$Q(x) \le E(e^{\widetilde{v}(X-x)}))$$

where $\widetilde{v}$ satisfies

$$E(Xe^{\widetilde{v}X}) = xE(e^{\widetilde{v}X})$$

Now

$$E(e^{\widetilde{v}X}) = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} e^{\widetilde{v}t} dt = e^{\widetilde{v}^2/2}$$

While

$$E(Xe^{\widetilde{v}^2/2}) = \frac{d}{d\widetilde{v}} E(e^{\widetilde{v}X}) = \widetilde{v}e^{\widetilde{v}^2/2}$$

Thus

$$\widetilde{v}e^{\widetilde{v}^2/2} = xe^{\widetilde{v}^2/2}$$

$$\widetilde{v} = x$$

Now

$$\begin{aligned} Q(x) &\le E(e^{x(X-x)}) \\ &= e^{-x^2} E(e^{xX}) \\ &= e^{-x^2} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} e^{xt} dt \\ &= e^{-x^2} e^{x^2/2} \\ &= e^{-x^2/2} \end{aligned}$$

(b)

Integration by part gives

$$\begin{aligned}
Q(x) &= \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \\
&= \int_x^\infty t \frac{1}{t} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \\
&= -\frac{1}{\sqrt{2\pi}} \frac{1}{t} e^{-t^2/2} \Big|_x^\infty - \int_x^\infty -\frac{1}{\sqrt{2\pi}} (-\frac{1}{t^2}) e^{-t^2/2} dt \\
&= \frac{1}{\sqrt{2\pi x^2}} e^{-x^2/2} - \int_x^\infty \frac{1}{t^2} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt
\end{aligned}$$

Drop the second term, which is a positive number, we get

$$Q(x) < \frac{1}{\sqrt{2\pi x^2}} e^{-x^2/2}$$

Instead, if we apply integration by part on the second term

$$\begin{aligned}
\int_x^\infty \frac{1}{t^2} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt &= \int_x^\infty t \frac{1}{t^3} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \\
&= -\frac{1}{\sqrt{2\pi}} \frac{1}{t^3} e^{-t^2/2} \Big|_x^\infty - \int_x^\infty -\frac{1}{\sqrt{2\pi}} (-\frac{3}{t^4}) e^{-t^2/2} dt \\
&= \frac{1}{x^2\sqrt{2\pi x^2}} e^{-x^2/2} - \int_x^\infty \frac{3}{t^4} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt
\end{aligned}$$

Substitute back to the result obtained before

$$Q(x) = (1 - \frac{1}{x^2}) \frac{1}{\sqrt{2\pi x^2}} e^{-x^2/2} + \int_x^\infty \frac{3}{t^4} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

Drop the last term, which is a positive number, we get

$$Q(x) > (1 - \frac{1}{x^2}) \frac{1}{\sqrt{2\pi x^2}} e^{-x^2/2}$$

(c)

Substitute $t = y + x$ in the Q function,

$$\begin{aligned} Q(x) &= \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \\ &= -\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(y+x)^2/2} dy \\ &= -\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \int_0^\infty e^{-y^2/2-xy} dy \end{aligned}$$

Now to show the upper and lower bound of Q function in (b), we only need to show

$$\frac{1}{x} - \frac{1}{x^3} < \int_0^\infty e^{-y^2/2-xy} dy < \frac{1}{x}$$

Notice that since $-y^2/2 \leq 0$

$$e^{-y^2/2} \leq 1$$

Also by Taylor series, for $x > 0$

$$e^{-x} = 1 - x + x^2 + o(x^3) \geq 1 - x$$

Thus

$$e^{-y^2/2} \geq 1 - y^2/2$$

Hence

$$\int_0^\infty (1 - y^2/2) e^{-xy} dy \leq \int_0^\infty e^{-y^2/2-xy} dy \leq \int_0^\infty e^{-xy} dy$$

At this point, notice that

$$\int_0^\infty e^{-xy} dy = -\frac{1}{x} e^{-xy} \Big|_0^\infty = \frac{1}{x}$$

Also from integration by part

$$\begin{aligned}
\int_0^\infty \frac{y^2}{2} e^{-xy} dy &= -\frac{1}{x}\frac{y^2}{2} e^{-xy}\Big|_0^\infty - \int_0^\infty -\frac{1}{x} y e^{-xy} dy \\
&= \int_0^\infty -\frac{y}{x} e^{-xy} dy \\
&= -\frac{y}{x^2} e^{-xy}\Big|_0^\infty - \int_0^\infty -\frac{1}{x^2} e^{-xy} dy \\
&= \frac{1}{x^2}\int_0^\infty e^{-xy} dy \\
&= \frac{1}{x^3}
\end{aligned}$$

Hence this leads to

$$\frac{1}{x} - \frac{1}{x^3} < \int_0^\infty e^{-y^2/2-xy} dy < \frac{1}{x}$$

which completes the proof.

2. (a)

Notice

$$log_2(1 + |h|^2\gamma) < R$$

is equivalent to

$$-\sqrt{\frac{2^R-1}{\gamma}} < |h| < \sqrt{\frac{2^R-1}{\gamma}}$$

Since $h \sim C\mathcal{N}(0,1)$, $|h|$ follows *Rayleigh distribution* as

$$Pr(|h|) = |h| e^{-|h|^2/2}$$

Therefore (note $|h| \geq 0$)

$$\begin{aligned}
P_{out}(R) &= \int_0^{\sqrt{\frac{2^R-1}{\gamma}}} r e^{-r^2/2} dr \\
&= -e^{-r^2/2}\Big|_0^{\sqrt{\frac{2^R-1}{\gamma}}} \\
&= 1 - e^{-\frac{2^R-1}{\gamma}/2}
\end{aligned}$$

(b)
Notice that

$$\frac{1}{L}\sum_{l=1}^{L} log_2(1+|h_l|^2\gamma) \leq R$$

is equivalent to

$$\frac{1}{L}\sum_{l=1}^{L} -log_2(1+|h_l|^2\gamma) \geq R$$

By Jensen inequality, as $-log_2(x)$ is convex,

$$\begin{aligned}
\frac{1}{L}\sum_{l=1}^{L} log_2(1+|h_l|^2\gamma) &\geq -log_2(\sum_{l=1}^{L}\frac{1}{L}(1+|h_l|^2\gamma)) \\
&= -log_2(1+\frac{\gamma}{L}\sum_{l=1}^{L}|h_l|^2) \\
&\geq -log_2(1+\gamma(\sum_{l=1}^{L}\frac{1}{L}|h_l|)^2) \\
&= -log_2(1+\frac{\gamma}{L^2}(\sum_{l=1}^{L}|h_l|)^2)
\end{aligned}$$

Now

$$Pr\{-log_2(1+\frac{\gamma}{L^2}(\sum_{l=1}^{L}|h_l|)^2) \leq R\}$$

will give a lower bound for the original outage probability $P_{out}(R)$

However, the analytic solution for sum of Rayleigh distribution does not exist. An approximation is adopted from Hu et al., **Accurate Simple Closed-Form Approximations to Rayleigh Sum Distributions and Densities**, *IEEE Communication Letter, Vol 9, No. 2, Feb 2005.*

The pdf of sum of Rayleigh distribution is approximately given by

$$f_{sum} = \frac{t^{2L-1}e^{-t^2/2b}}{2^{L-1}b^L(L-1)!}$$

where

$$b = \frac{1}{L}((2L-1)!!)^{1/2}$$

Hence, the lower bound calculated based on this method will be

$$\int_0^{\sqrt{\frac{L^2(2^R-1)}{\gamma}}} f_{sum}dt$$

which is

$$-\frac{2^(L+1)b^L}{(L-1)!}\Gamma(2L,\frac{t}{2b})\Big|_0^{\sqrt{\frac{L^2(2^R-1)}{\gamma}}}$$

3. (a)
Notice that

$$\begin{aligned}
Pr\{(-1)^{N(t)}=1\} &= Pr\{N(t)\ is\ even\}\\
&= e^{-\lambda t}(1+\frac{(\lambda t)^2}{2!}+\frac{(\lambda t)^4}{4!}+...)\\
&= e^{-\lambda t}(\frac{\sum\limits_{m=0}^{\infty}\frac{(\lambda t)^m}{m!}+\sum\limits_{m=0}^{\infty}\frac{(-\lambda t)^m}{m!}}{2})\\
&= e^{-\lambda t}(\frac{e^{\lambda t}+e^{-\lambda t}}{2})\\
&= e^{-\lambda t}cosh(\lambda t)
\end{aligned}$$

Likewise,

$$\begin{aligned}
Pr\{(-1)^{N(t)}=-1\} &= Pr\{N(t)\ is\ odd\}\\
&= e^{-\lambda t}(\frac{(\lambda t)^1}{1!}+\frac{(\lambda t)^3}{3!}+...)\\
&= e^{-\lambda t}(\frac{e^{\lambda t}-e^{-\lambda t}}{2})\\
&= e^{-\lambda t}sinh(\lambda t)
\end{aligned}$$

Also note that $R(t,s)=E(X(t),X(s))$ is independent with sign of X_0, as $X_0^2=1$ anyway.

Now, utilizing the *memoryless property* of *Poisson Process*, the conditional probability depends *only* on whether the sign is flipped or not, then we will have

$$\begin{aligned}
Pr\{X(t)X(s)=1\} &= Pr\{X(s)=1\}Pr\{X(t)=1|X(s)=1\}\\
&\quad + Pr\{X(s)=-1\}Pr\{X(t)=-1|X(s)=-1\}\\
\\
&= e^{-\lambda s}cosh(\lambda s)e^{-\lambda(t-s)}cosh(\lambda(t-s))\\
&\quad + e^{-\lambda s}sinh(\lambda s)e^{-\lambda(t-s)}cosh(\lambda(t-s))
\end{aligned}$$

Likewise,

$$\begin{aligned}
Pr\{X(t)X(s) = -1\} &= Pr\{X(s) = 1\}Pr\{X(t) = -1|X(s) = 1\} \\
&\quad + Pr\{X(s) = -1\}Pr\{X(t) = 1|X(s) = -1\} \\
\\
&= e^{-\lambda s}cosh(\lambda s)e^{-\lambda(t-s)}sinh(\lambda(t-s)) \\
&\quad + e^{-\lambda s}sinh(\lambda s)e^{-\lambda(t-s)}sinh(\lambda(t-s))
\end{aligned}$$

These are the only two probabilities of the value of $X(t)X(s)$, hence the pdf of $X(t)X(s)$ is *delta function*.

Therefore

$$R(s,t) = E(X(s)X(t)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ProbFunc\ (\delta(x_s \pm 1)\delta(x_t \pm 1))dx_s dx_t$$

where the $ProbFunc$ equals to the expression discussed above when x_s and x_t equal to corresponding values, otherwise, 0.

Hence

$$\begin{aligned}
R(s,t) &= e^{-\lambda s}cosh(\lambda s)e^{-\lambda(t-s)}cosh(\lambda(t-s)) \\
&\quad + e^{-\lambda s}sinh(\lambda s)e^{-\lambda(t-s)}cosh(\lambda(t-s)) \\
&\quad - e^{-\lambda s}cosh(\lambda s)e^{-\lambda(t-s)}sinh(\lambda(t-s)) \\
&\quad - e^{-\lambda s}sinh(\lambda s)e^{-\lambda(t-s)}sinh(\lambda(t-s)) \\
\\
&= e^{-2\lambda(s-t)}
\end{aligned}$$

(b)
Notice that the above random process yields wide-sense stationary, the power spectral density will be (we denote $|s-t|$ to be τ)

$$\begin{aligned}
S_X(f) &= \mathcal{F}(R(\tau)) \\
&= \int_{-\infty}^{\infty} e^{-2\lambda|\tau|}e^{-if\tau}d\tau \\
&= \int_{-\infty}^{0} e^{2\lambda\tau}e^{-if\tau}d\tau + \int_{0}^{\infty} e^{-2\lambda\tau}e^{-if\tau}d\tau \\
&= \frac{1}{2\lambda - if} + \frac{1}{2\lambda + if} \\
&= \frac{4\lambda}{4\lambda^2 + f^2}
\end{aligned}$$

4. (a)
step 1

P1	0.3365	
P2	0.3365	
P3	0.1635	0
P4	0.1635	1

combined probability: 0.1635+0.1635 = 0.327

step 2 (notice $0.327 < 0.3365$)

P1	0.3365	
P2	0.3365	1
P3	0.1635	00
P4	0.1635	10

combined probability: 0.327+0.3365 = 0.6635

step 3

P1	0.3365	1
P2	0.3365	10
P3	0.1635	000
P4	0.1635	100

combined probability: 0.6635+0.3365 = 1

Hence, a possible Huffman code is

Sympble	*codeword*
P1	1
P2	01
P3	000
P4	001

(b)
Average number of binary digit per source level is

$$1 \times 0.3365 + 2 \times 0.3365 + 3 \times 0.327 = 1.9905$$

(c)
The entropy $H(X)$ is as follows

$$\sum_{i=1}^{4} -p_i log_2(p_i) = -2 \times 0.3365 \; log_2(0.3365) - 2 \times 0.1635 \; log_2(0.1635)$$
$$= 1.9118$$

5.
By Lempel-Ziv Algorithm, the original binary sequence will be separated as follows,

0, 00, 1, 001, 000, 0001, 10, 00010, 0000, 0010, 00000, 101, 00001, 000000, 11, 01, 0000000, 110, 0

There are *19* phrases in total, therefore, we need $\lceil log_2(19) \rceil = 5$ bits for position representation. Hence the dictionary construction is as follows,

Dictionary Location	Location in binary	Original content	LZ codeword
1	00001	0	000000
2	00010	00	000010
3	00011	1	000001
4	00100	001	000101
5	00101	000	000100
6	00110	0001	001011
7	00111	10	000110
8	01000	00010	001100
9	01001	0000	001010
10	01010	0010	001000
11	01011	00000	010010
12	01100	101	001111
13	01101	00001	010011
14	01110	000000	010110
15	01111	11	000111
16	10000	01	000011
17	10001	000000	011100
18	10010	110	011110
19	10011	0	00000

The LZ source code is the concatenation of each of the LZ codeword in the tabular above.

For decode LZ code back to original code, the decoder will separate the incoming source code into blocks with size 6 (except the last one). The dictionary can be built up as the source code coming and the original content can be decoded accordingly.