ELEC 5360 Homework 2

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1.

The initial value of a_i determines b_i as follows:

$$b_1 = \frac{a_1 + a_2}{2} = 0.4$$
$$b_2 = \frac{a_2 + a_3}{2} = 0.65$$
$$b_3 = \frac{a_3 + a_4}{2} = 0.8$$

According to *Lloyd-Max Algorithm*,

$$a_1 = \frac{\int_{b_0}^{b_1} x f_X(x)dx}{\int_{b_0}^{b_1} f_X(x)dx} = \frac{\int_0^{0.4} x(-2x+2)dx}{\int_0^{0.4} -2x+2\ dx} = \frac{-2/3x^3 + x^2\Big|_0^{0.4}}{-x^2+2x\Big|_0^{0.4}} = \frac{11}{60}$$

$$a_2 = \frac{\int_{b_1}^{b_2} x f_X(x)dx}{\int_{b_1}^{b_2} f_X(x)dx} = \frac{\int_{0.4}^{0.65} x(-2x+2)dx}{\int_{0.4}^{0.65} -2x+2\ dx} = \frac{-2/3x^3 + x^2\Big|_{0.4}^{0.65}}{-x^2+2x\Big|_{0.4}^{0.65}} = \frac{293}{570}$$

$$a_3 = \frac{\int_{b_2}^{b_3} x f_X(x)dx}{\int_{b_2}^{b_3} f_X(x)dx} = \frac{\int_{0.65}^{0.8} x(-2x+2)dx}{\int_{0.65}^{0.8} -2x+2\ dx} = \frac{-2/3x^3 + x^2\Big|_{0.65}^{0.8}}{-x^2+2x\Big|_{0.65}^{0.8}} = \frac{79}{110}$$

$$a_4 = \frac{\int_{b_3}^{b_4} x f_X(x)dx}{\int_{b_3}^{b_4} f_X(x)dx} = \frac{\int_{0.8}^{1} x(-2x+2)dx}{\int_{0.85}^{1} -2x+2\ dx} = \frac{-2/3x^3 + x^2\Big|_{0.8}^{1}}{-x^2+2x\Big|_{0.8}^{1}} = \frac{13}{15}$$

New b_i can be then calculated accordingly,

$$b_1 = \frac{a_1 + a_2}{2} = \frac{53}{152}$$
$$b_2 = \frac{a_2 + a_3}{2} = \frac{3863}{6270}$$
$$b_3 = \frac{a_3 + a_4}{2} = \frac{523}{600}$$

2. (a)

To check the orthogonality, we look into

$$\int_0^4 f_1(t)f_2^*(t)dt = \int_0^2 \frac{1}{2}\frac{1}{2}dt + \int_2^4 \frac{1}{2}(-\frac{1}{2})dt = -\frac{2}{4} + \frac{2}{4} = 0$$
$$\int_0^4 f_1(t)f_3^*(t)dt = \int_0^1 \frac{1}{2}\frac{1}{2}dt + \int_1^2 \frac{1}{2}(-\frac{1}{2})dt + \int_2^3 \frac{1}{2}\frac{1}{2}dt + \int_3^4 \frac{1}{2}(-\frac{1}{2})dt = \frac{1}{4} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4} = 0$$
$$\int_0^4 f_1(t)f_3^*(t)dt = \int_0^1 \frac{1}{2}\frac{1}{2}dt + \int_1^2 \frac{1}{2}(-\frac{1}{2})dt + \int_2^3 (-\frac{1}{2})\frac{1}{2}dt + \int_3^4 (-\frac{1}{2})(-\frac{1}{2})dt = \frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4} = 0$$

To check the normality, we look into

$$\int_0^4 f_1(t)f_1^*(t)dt = \int_0^4 \frac{1}{2}\frac{1}{2}dt = \frac{4}{4} = 1$$
$$\int_0^4 f_2(t)f_2^*(t)dt = \int_0^2 \frac{1}{2}\frac{1}{2}dt + \int_2^4 (-\frac{1}{2})(-\frac{1}{2})dt = \frac{2}{4} + \frac{2}{4} = 1$$
$$\int_0^4 f_3(t)f_3^*(t)dt = \int_0^1 \frac{1}{2}\frac{1}{2}dt + \int_1^2 (-\frac{1}{2})(-\frac{1}{2})dt + \int_2^3 \frac{1}{2}\frac{1}{2}dt + \int_3^4 (-\frac{1}{2})(-\frac{1}{2})dt = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

Hence the bases are orthonormal to each other.

(b)

Assume $x(t)$ can be decomposed into the set of bases defined above, then

$$x(t) = a_1 f_1(t) + a_2 f_2(t) + a_3 f_3(t)$$

Now

$$a_1 = \int_0^4 x(t)f_1^*(t)dt = \int_0^1 (-1)\frac{1}{2}dt + \int_1^3 1\frac{1}{2}dt + \int_3^4 (-1)\frac{1}{2}dt = -\frac{1}{2} + \frac{2}{2} - \frac{1}{2} = 0$$
$$a_2 = \int_0^4 x(t)f_2^*(t)dt = \int_0^1 (-1)\frac{1}{2}dt + \int_1^2 1\frac{1}{2}dt + \int_2^3 1(-\frac{1}{2})dt + \int_3^4 (-1)(-\frac{1}{2})dt = -\frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} = 0$$
$$a_3 = \int_0^4 x(t)f_3^*(t)dt = \int_0^1 (-1)\frac{1}{2}dt + \int_1^2 1(-\frac{1}{2})dt + \int_2^3 1\frac{1}{2}dt + \int_3^4 (-1)(-\frac{1}{2})dt = -\frac{1}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 0$$

All the coefficients are 0, which means the signal doesn't live in the space span by the bases.

3. (a)

According to *Gram-Schemidt method*, denote m_i to be the waveform after removing components of constructed bases, and x_i to be the normalized waveform basis.

$$
\begin{aligned}
m_1(t) &= s_1(t) \\
x_1(t) &= m_1(t) * \frac{1}{\int_0^T m_1(t)dt} = m_1(t) * \frac{1}{\int_0^{T/3} 1\ dt} = \frac{3}{T}s_1(t) \\
m_2(t) &= s_2(t) - [\int_0^T s_2(t)x_1^*(t)]x_1(t) = s_2(t) - \frac{T}{3}x_1(t) \\
x_2(t) &= m_2(t) * \frac{1}{\int_0^T m_2(t)dt} = m_1(t) * \frac{1}{\int_{T/3}^{2T/3} 1\ dt} = \frac{3}{T}(s_2(t) - s_1(t)) \\
m_3(t) &= s_3(t) - [\int_0^T s_3(t)x_1^*(t)]x_1(t) - [\int_0^T s_3(t)x_2^*(t)]x_2(t) = s_3(t) - \frac{T}{3}x_2(t) \\
x_3(t) &= m_3(t) * \frac{1}{\int_0^T m_3(t)dt} = m_3(t) * \frac{1}{\int_{2T/3}^{T} 1\ dt} = \frac{3}{T}(s_3(t) - s_2(t) + s_1(t)) \\
m_4(t) &= s_4(t) - [\int_0^T s_4(t)x_1^*(t)]x_1(t) - [\int_0^T s_4(t)x_2^*(t)]x_2(t) - [\int_0^T s_4(t)x_3^*(t)]x_3(t) \\
&= s_4(t) - \frac{T}{3}x_1(t) - \frac{T}{3}x_2(t) - \frac{T}{3}x_3(t) = 0
\end{aligned}
$$

So the there are 3 orthonormal bases for these 4 signals.

$$
x_1(t) = \begin{cases} \frac{3}{T}, & \text{if } 0 \le t \le \frac{T}{3} \\ 0, & \text{otherwise} \end{cases}
$$

$$
x_2(t) = \begin{cases} \frac{3}{T}, & \text{if } \frac{T}{3} \le t \le \frac{2T}{3} \\ 0, & \text{otherwise} \end{cases}
$$

$$
x_3(t) = \begin{cases} \frac{3}{T}, & \text{if } \frac{2T}{3} \le t \le T \\ 0, & \text{otherwise} \end{cases}
$$

(b)

In general, for $i = 1, 2, 3, 4$

$$s_i(t) = \sum_{j=1}^{3} [\int_0^T s_i(t) x_j^*(t)] x_j(t)$$

Hence,

$$\begin{aligned}
s_1(t) &= \frac{T}{3} x_1(t) \\
s_2(t) &= \frac{T}{3} x_1(t) + \frac{T}{3} x_2(t) \\
s_3(t) &= \frac{T}{3} x_2(t) + \frac{T}{3} x_3(t) \\
s_4(t) &= \frac{T}{3} x_1(t) + \frac{T}{3} x_2(t) + \frac{T}{3} x_3(t)
\end{aligned}$$

4. (a)

$g(t)$ must satisfy *No inter-symobol interference* property, namely,

$$g(jT - kT) = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{otherwise} \end{cases}$$

Nyquist criteria, in formal, states that

$$\sum_{m \in \mathbb{Z}} \hat{g}(f - m/T) rect(fT) = T\, rect(fT)$$

In this case, it can be simplified to

$$\frac{1}{T} \sum_{m \in \mathbb{Z}} \hat{g}(f - m/T) = 1$$

(b)

According to *Nyquist criteria*, and $T = 1/2$

$$\sum_m \hat{g}(f - 2m) = \frac{1}{2}$$

Now we look into $|f| \leq 2$, since

$$\hat{g}(f) = \hat{p}(f)\hat{h}(f)\hat{q}(f)$$

When $|f| \leq 0.6$, we will have

$$\hat{q}(f) = \frac{1}{2\hat{p}(f)\hat{h}(f)}$$

Hence

$$\hat{q}(f) = \begin{cases} 1, & \text{if } |f| \leq 0.5 \\ \frac{1}{3-2|f|}, & 0.5 \leq |f| \leq 0.6 \end{cases}$$

Notice that the values of $\hat{q}(f)$ in region $0.6 \leq |f| \leq 2$ can be any value because $\hat{h}(f) = 0$ there anyway.

(c)

If $\hat{h}(f)\hat{q}(f) = 0$ then $\hat{g}(f) = 0$, which means the *Nyquist criteria* does not hold.

Hence, in order to avoid inter-symbol interference,

for each$0 \leq f \leq 1/(2T)$,

$$\exists m \in \mathbb{Z}, \text{such that } \hat{h}(f - m/T)\hat{p}(f - m/T) \neq 0$$

5. (a)

Notice that $V_i = \int \phi_i(t)Z(t)dt$ is a Gaussian random variable.

Also, with Spectral density $S_Z(f) = N_0/2$, we have

$$
\begin{aligned}
E(V_i^2) &= E[\int_{-\infty}^{\infty} \phi_i(t)Z(t)dt \int_{-\infty}^{\infty} \phi_i(\tau)Z(\tau)d\tau] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_i(t)\phi_i(\tau)E[Z(t)Z(\tau)]dtd\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_i(t)\phi_i(\tau)\frac{N_0}{2}\delta(t-\tau)dtd\tau
\end{aligned}
$$

(This is because the gaussian r.v. on different time are independent, and $E(Z^2) - E(Z)^2 = \sigma^2 = N_0/2$)

Since $\{\phi_i(t)\}$ is set of orthonormal functions, we have

$$
\int_{-\infty}^{\infty} |\phi_i(t)|^2 dt = \int_{-\infty}^{\infty} \phi_i(t)\phi_i^*(t)dt = 1
$$

Therefore, we have $E(V_i^2) = N_0/2$. Then we have

$$
\sigma_i^2 = E(V_i^2) - E(V_i)^2 = N_0/2
$$

Hence, we have the distribution for V_i

$$
V_i \sim \mathcal{N}(0, \frac{N_0}{2})
$$

(b)

Notice that

$$
Y = \int_0^T Z(t)dt = \int_{-\infty}^{\infty} g(t)Z(t)dt
$$

where

$$
g(t) = \begin{cases} 1, & \text{if } 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}
$$

and Y is a Gaussian random variable.

Now, notice that

$$
\begin{aligned}
E(Y^2) &= E[\int_{-\infty}^{\infty} g(t)Z(t)dt \int_{-\infty}^{\infty} g(\tau)Z(\tau)d\tau] = \int_0^T S_Z(f)dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(t)g(\tau)E[Z(t)Z(\tau)]dtd\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(t)g(\tau)\frac{N_0}{2}\delta(t-\tau)dtd\tau = \frac{N_0}{2}T
\end{aligned}
$$

Then we have

$$\sigma_Y^2 = E(Y^2) - E(Y)^2 = \frac{N_0}{2}T$$

$$Y \sim \mathcal{N}(0, \frac{N_0}{2}T)$$