263F - Homework 2

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Task 1: Implicit Simulation of Beam Deflection

Problem setup

A simply supported aluminum hollow tube beam of length $L = 1$ m is modeled using $N = 50$ nodes. The outer and inner radii of the circular cross section are $R = 0.013$ m and $r = 0.011$ m, respectively. The modulus of elasticity is $E = 70$ GPa and the density is $\rho = 2700$ kg/m^3. The beam is subjected to a concentrated load $P = 2000$ N acting vertically downward at $x = 0.75$ m from the left end. The first node is fixed in both x and y directions, while the last node is constrained only in y direction. The time integration is performed implicitly using the backward Euler method with a time step $\Delta t = 10^{-2}$ s, for the time range $0 \leq t \leq 1$ s.

Numerical method

The beam is represented as a discrete mass–spring system consisting of N nodes connected by stretching and bending elements. At each time step, the dynamic equilibrium equation is solved implicitly using the Newton–Raphson iteration:

$$\mathbf{f}(\mathbf{q}^{k+1}) = \frac{\mathbf{M}}{\Delta t^2}(\mathbf{q}^{k+1} - \mathbf{q}^k) - \frac{\mathbf{M}}{\Delta t}\mathbf{u}^k - \mathbf{F}_{\text{int}}(\mathbf{q}^{k+1}) - \frac{\mathbf{C}}{\Delta t}(\mathbf{q}^{k+1} - \mathbf{q}^k) - \mathbf{W} = \mathbf{0},$$

where $\mathbf{M}$ is the lumped mass matrix, $\mathbf{C}$ is the damping matrix (here set to zero), $\mathbf{F}_{\text{int}}$ is the internal elastic force vector, and $\mathbf{W}$ is the external load vector. The internal force and tangent stiffness are derived from the discrete stretching and bending energies:

$$E_s = \frac{1}{2}EA\sum_{i=1}^{N-1}\varepsilon_i^2\Delta L, \qquad E_b = \frac{1}{2}\frac{EI}{\Delta L}\sum_{i=2}^{N-1}\kappa_i^2.$$

Results

Figure 1 shows the maximum vertical deflection $y_{\max}$ of the beam as a function of time. The deflection rapidly increases in the first few time steps and then reaches a steady value as the system approaches static equilibrium.

The steady-state maximum deflection predicted by the simulation is approximately

$$y_{\max}^{\text{sim}} \approx 0.038 \text{ m}.$$

For comparison, the analytical deflection according to the Euler–Bernoulli beam theory is given by

$$y_{\max}^{\text{EB}} = \frac{P\,c\,(L^2 - c^2)^{1.5}}{9\sqrt{3}\,E\,I\,L}, \qquad c = \min(d,\, L - d), \tag{1}$$

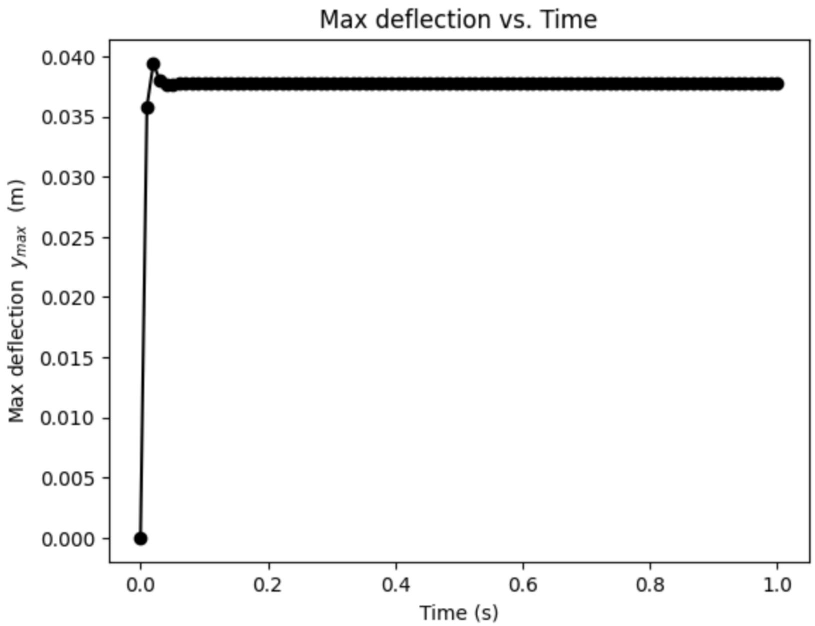

Figure 1: Maximum vertical deflection $y_{\max}$ as a function of time under a concentrated load $P = 2000$ N.

where $d = 0.75$ m is the distance of the load from the left support. Substituting $E = 70 \times 10^9$ Pa, $I = \frac{\pi}{4}(R^4 - r^4)$, $L = 1$ m, and $c = 0.25$ m yields

$$y_{\max}^{\mathrm{EB}} = 0.0380 \text{ m}.$$

The simulated steady-state deflection agrees almost exactly with the theoretical value from Eq. (1), showing that the implicit Euler solver accurately reproduces the static deformation predicted by classical Euler–Bernoulli beam theory. The small transient oscillations observed in the first few steps result from the instantaneous application of the load and are quickly damped out by the inherent numerical damping of the implicit integration scheme.

Discussion

The results confirm that the implicit solver is stable and capable of reaching static equilibrium within 1 s of simulation. The numerical model successfully reproduces the theoretical deflection for small deformations. In the next task (Task 2), the same framework will be used to analyze the deviation between the nonlinear discrete simulation and the linear Euler–Bernoulli prediction under large loads.

Task 2: Load–Deflection Relationship and Large-Deformation Effects

Objective

The goal of Task 2 is to investigate the nonlinear behavior of the beam under increasing concentrated loads and to compare the simulated deflections with the theoretical prediction from the Euler–Bernoulli (E–B) beam theory.

Procedure

The same beam model and implicit solver as in Task 1 are used, with $N = 50$, $\Delta t = 10^{-2}\,\mathrm{s}$, and $T = 1\,\mathrm{s}$. The load is applied at $x = 0.75\,\mathrm{m}$ and increased gradually from $P = 20$ N to $P = 20\,000$ N in twenty increments. For each load case, the beam deformation is computed until steady state is reached, and the maximum vertical deflection $y_{\max}$ is recorded. The analytical deflection is calculated from the Euler–Bernoulli theory as

$$y_{\max}^{\mathrm{EB}} = \frac{P\,c\,(L^2 - c^2)^{1.5}}{9\sqrt{3}\,E\,I\,L}, \qquad c = \min(d, L - d), \tag{2}$$

where $d = 0.75\,\mathrm{m}$ and $L = 1\,\mathrm{m}$.

Results

Figure 2 presents the load–deflection curve obtained from the implicit simulation (black circles) together with the linear prediction from Euler–Bernoulli theory (red dashed line).

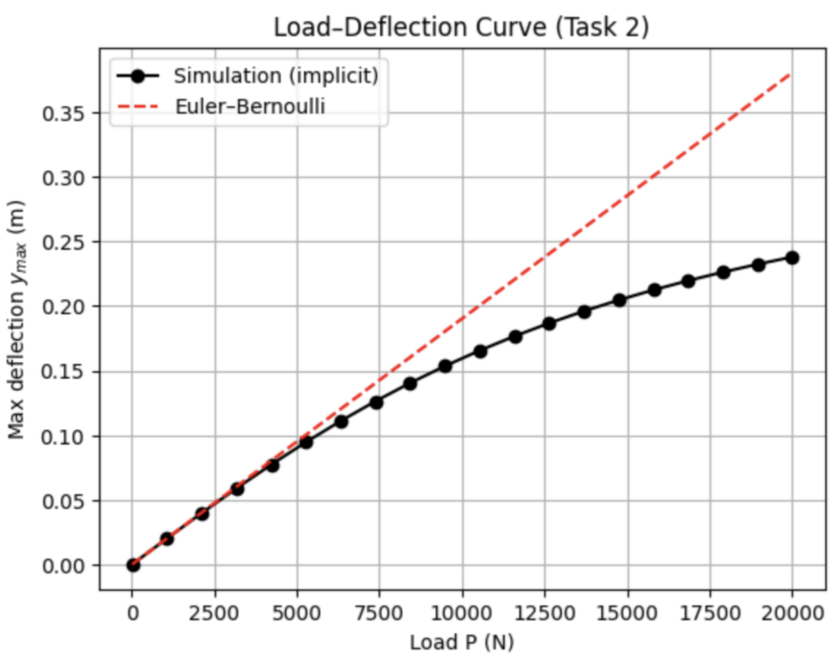

Figure 2: Load–deflection curve of the simply supported aluminum tube beam ($L = 1\,\mathrm{m}$, $R = 0.013\,\mathrm{m}$, $r = 0.011\,\mathrm{m}$, $E = 70\,\mathrm{GPa}$).

For small loads ($P \leq 4000$ N), the simulation results coincide with the Euler–Bernoulli prediction, indicating that the beam deformation remains in the linear (small-deflection) regime. As the load increases beyond approximately $P^* \approx 7000$ N, the simulated deflection becomes noticeably smaller than the linear prediction. This deviation indicates the onset of **geometric nonlinearity**: as the beam deflects significantly, axial stretching introduces additional stiffness, causing the actual deflection to be smaller than that predicted by linear theory.

Discussion

The nonlinear implicit solver successfully captures both the small- and large-deflection responses:

- In the small-load region ($P < 4000$ N), $y_{\max}^{\text{sim}} \approx y_{\max}^{\text{EB}}$, confirming the accuracy of the model.
- Around $P^* \approx 7000$ N, the relative difference $\varepsilon = |y_{\max}^{\text{sim}} - y_{\max}^{\text{EB}}|/y_{\max}^{\text{EB}}$ exceeds roughly 5%, marking the transition to nonlinear behavior.
- For $P > 10\,000$ N, the simulation predicts substantially smaller deflections than Euler–Bernoulli theory, due to geometric stiffening effects ignored by the linear model.

The implicit integration scheme remains stable throughout the entire load range, demonstrating its robustness for nonlinear static equilibrium analysis. Overall, the simulation accurately reproduces the expected physical trend: linear response at small loads and geometric stiffening at large deflections.