Massachusetts Institute of Technology

Problems in higher dimensions

Some examples on how things generalize beyond 1d

18.303 Linear Partial Differential Equations: Analysis and Numerics

Poisson equation in 2d

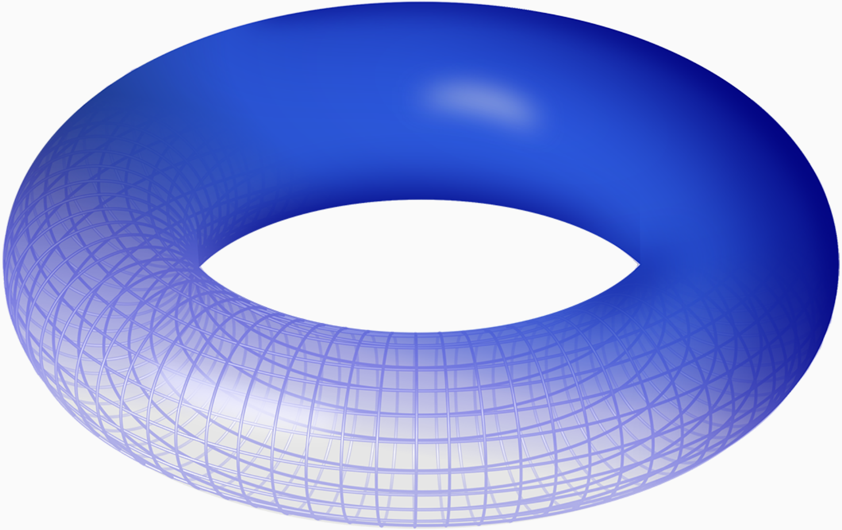
Consider a box of size (L_x, L_y) with periodic boundaries.

Poisson's equation with periodic boundaries

$$\Delta u(x, y) = \frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = f(x, y);$$

$$u(0, y) = u(L_x, y),\ u(x, 0) = u(x, L_y)$$

We require that $(x, y) \in \Omega = [0, L_x] \times [0, L_y]$.



How Ω looks like (topologically).

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(Here $u_x = \partial_x u$ and $u_y = \partial_y u$).

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How do we solve this?

As we did before, let's try finding the eigenfunctions in order to invert the Laplacian. This gives the eigenvalue problem

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Eigenproblem for the Laplacian with periodic boundaries

$$\frac{\partial^2 \phi^{(\mathsf{n})}(x,y)}{\partial x^2} + \frac{\partial^2 \phi^{(\mathsf{n})}(x,y)}{\partial y^2} = -\lambda_{\mathsf{n}}^2 \phi^{(\mathsf{n})}(x,y);$$
$$\phi^{(\mathsf{n})}(0,y) = \phi^{(\mathsf{n})}(L_x,y),\ \phi^{(\mathsf{n})}(x,0) = \phi^{(\mathsf{n})}(x,L_y);$$
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We can again try out a solution that is separable. Let $\phi^{(\mathrm{n})}(x,y) = X_n(x)Y_m(y)$. This gives

$$Y_m(y)X_n''(x) + X_n(x)Y_m''(y) = -\lambda_{n,m}^2 X_n(x)Y_m(y).$$

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Dividing by XY gives

$$\frac{X_n''(x)}{X_n(x)} + \frac{Y_m''(y)}{Y_m(y)} = -\lambda_{n,m}^2.$$

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Before proceeding with the calculation, let's talk about the boundary conditions. We have

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Substituting $\phi^{(\mathrm{n})} = X_n Y_m$ gives

$$X_n(0) = X_n(L_x),\ Y_m(0) = Y_m(L_y);$$
$$X_n'(0) = X_n'(L_x),\ Y_m'(0) = Y_m'(L_y)$$

i.e. we have periodic boundary conditions for both X_n and Y_m.

We see that the problem separates nicely. We have two unrelated eigenvalue problems for X_n and Y_m. Let's solve one of these explicitly since we haven't done that in class.

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We know that $X_n(x) = \exp(i\mu_n x)$ and we require

$$X_n(0) = X_n(L_x) \Leftrightarrow 1 = \exp(i\mu_n L_x).$$

This gives

$$\mu_n = \frac{2\pi n}{L_x},$$

where $n \in \mathbb{Z}$.

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Similarly, we get

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where $\nu_m = 2\pi m/L_y$. One should mention that we have $\lambda_{n,m}^2 = \mu_n^2 + \nu_m^2$.

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Now, the solution can be expressed as

$$u(x,y) = \sum_{n=-\infty}^{\infty} \hat{X}_n e^{2\pi i n x/L_x} \sum_{m=-\infty}^{\infty} \hat{Y}_m e^{2\pi i m x/L_x} = \sum_{n,m=-\infty}^{\infty} \hat{X}_n \hat{Y}_m e^{2\pi i (nx/L_x + my/L_y)}.$$

We get new Fourier coefficients that depend on two indices. We can write u now as

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Taking the differential operator of u gives

$$\begin{aligned} \sum_{n,m=-\infty}^{\infty} \hat{u}_{n,m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) e^{2\pi i (nx/L_x + my/L_y)} = \\ - \sum_{n,m=-\infty}^{\infty} \hat{u}_{n,m} \left[\left(\frac{2\pi n}{L_x} \right)^2 + \left(\frac{2\pi m}{L_y} \right)^2 \right] e^{2\pi i (nx/L_x + my/L_y)}. \end{aligned}$$

This gives us

$$-\sum_{n,m=-\infty}^{\infty} \hat{u}_{n,m} \left[\left(\frac{2\pi n}{L_x} \right)^2 + \left(\frac{2\pi m}{L_y} \right)^2 \right] e^{2\pi i (nx/L_x + my/L_y)} = f(x, y). \tag{2}$$

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$$\langle \phi^{(n,m)}, \phi^{(j,k)} \rangle = \int_0^{L_x} \int_0^{L_y} e^{2\pi i ((j-n)x/L_x + (k-m)y/L_y)} \mathrm{d}x \mathrm{d}y = \delta_{j,n} \delta_{k,m} L_x L_y$$

i.e. it gives $L_x L_y$ if $j = n$ and $k = m$ and zero otherwise.

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Exercise: Show this.

Now we get the coefficients

$$\hat{u}_{n,m} = -\frac{\langle \phi^{(n,m)}, f \rangle}{L_x L_y \lambda_{n,m}^2} = -\frac{\hat{f}_{n,m}}{\lambda_{n,m}^2},$$

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What if $\lambda_{n,m} = 0$? This actually happens for $n, m = 0$. This corresponds to the eigenfunction $\exp(0) = 1$ i.e. the constant part of the function. For periodic domains constant solutions are always in the null space of the Laplacian operator. Some extra information is needed to get this.

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Question: Why do we have the latter equality?

Fourier coefficients

We get the coefficients as

$$\hat{f}_{\mathbf{n}} = \frac{1}{\mu(\Omega)} \int_{\Omega} \exp\left(-i\mathbf{k}_{\mathbf{n}} \cdot \mathbf{x}\right) f(\mathbf{x}) \mathrm{d}\mathbf{x}, \tag{3}$$

where $\mathbf{n} = (n_x, n_y)$ and $\mathbf{k}_{\mathbf{n}} = 2\pi(n_x/L_x, n_y/L_y)$. The function $\mu(\Omega) = L_x L_y$ gives the "volume" of the space.

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The inverse transform gives back f i.e. $\mathcal{F}^{-1}(\hat{f})(x) = f(x)$. This is also given by

$$f(x) = \sum_{\mathbf{n} \in \mathbb{Z}^2} \hat{f}_{\mathbf{n}} \exp(i\mathbf{k}_{\mathbf{n}} \cdot \mathbf{x}).$$

In literature $\mathcal{F}$ is sometimes called the finite Fourier transform. We will talk about the actual Fourier transform later on.

What if f is real?

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For the (finite) Fourier transform it doesn't really matter but we get a nice relation. Because for real functions we have $f(x)^* = f(x)$, the Fourier expansion gives

$$f(x)^* = f(x) \Leftrightarrow \sum_{\mathsf{n}\in\mathbb{Z}^2} \hat{f}_{\mathsf{n}}^* \exp(-i\mathsf{k}_{\mathsf{n}} \cdot \mathsf{x}) = \sum_{\mathsf{n}\in\mathbb{Z}^2} \hat{f}_{\mathsf{n}} \exp(i\mathsf{k}_{\mathsf{n}} \cdot \mathsf{x}) = \sum_{\mathsf{n}\in\mathbb{Z}^2} \hat{f}_{-\mathsf{n}} \exp(-i\mathsf{k}_{\mathsf{n}} \cdot \mathsf{x}).$$

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Because the basis is orthogonal, the coefficients have to be the same. This gives

$$\hat{f}_{-\mathbf{n}} = \hat{f}_{\mathbf{n}}^*$$

i.e. we only need to calculate, say, the coefficients with $n_x \geq 0$ and the ones corresponding to the negative indices are given by taking the complex conjugate of the coefficients corresponding to positive indices. The numerical algorithms using this fact are called *real Fourier transforms* (`rfft`).

Exercise 1

Assume that we have 1d Fourier expansion of a real function

$$f(x) = \sum_{n=0}^{\infty} \alpha_n \sin(k_n x) + \beta_n \cos(k_n x) = \sum_{n=-\infty}^{\infty} \hat{f}_n \exp(ik_n x).$$

Whats the relation between $\hat{f}_n$ and α_n and β_n?

Heat equation

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We have

$$\frac{\partial}{\partial t}u(t, \mathsf{x}) = \Delta u(t, \mathsf{x})$$

and the same spatial boundary conditions for all t and in addition we are given

$$u(0, \mathsf{x}) = u^{(0)}(\mathsf{x}).$$

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We can actually solve this the same way we did before using the separation $u(t, \mathsf{x}) = T_{\mathsf{n}}(t)X_{\mathsf{n}}(\mathsf{x})$ but now the spatial part depends on two coordinates.

As before we get

$$\begin{aligned} T_{\mathsf{n}}'(t) &= -\lambda_{\mathsf{n}}^2 T_{\mathsf{n}}(t), \\ \Delta X_{\mathsf{n}}(\mathsf{x}) &= -\lambda_{\mathsf{n}}^2 X_{\mathsf{n}}(\mathsf{x}). \end{aligned}$$

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We know that

$$T_n(t) = T_n^{(0)} \exp(-\lambda_n^2 t)$$

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\end{aligned}
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We know that

$$T_{\mathsf{n}}(t) = T_{\mathsf{n}}^{(0)} \exp(-\lambda_{\mathsf{n}}^2 t)$$

and that

$$X_{\mathsf{n}} = \exp(i\mathbf{k}_{\mathsf{n}} \cdot \mathbf{x}).$$

(Also, in this notation $\lambda_{\mathsf{n}}^2 = |\mathbf{k}_{\mathsf{n}}|^2$.)

The solution is given by

$$u(t, \mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^2} T_{\mathbf{n}}^{(0)} X_{\mathbf{n}} \exp(-k_{\mathbf{n}}^2 t),$$

where the coefficients $T_{\mathbf{n}}^{(0)}$ are the same as the Fourier coefficients of the initial condition

$$u^{(0)}(x) = \sum_{\mathbf{n} \in \mathbb{Z}^2} \hat{u}_{\mathbf{n}}^{(0)} X_{\mathbf{n}}.$$

Remark: remember that $T_0^{(0)}$ is related to the average value of u. This doesn't evolve in time, which tells us that the heat equation preserves the average value of u.

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Conclusion

- We saw how we can use separation of variables to solve problems in many dimensions
- **Warning:** this only works for highly symmetric cases as the box we considered today
- The heat equation example showed what I pointed out previously, namely that if you can solve the eigenproblem for the spatial part, solving for the time evolution is easy
- We briefly talked about finite Fourier transforms and how they can be used both for real and imaginary functions