MA3227 Numerical Analysis II

Tutorial 6: Explicit Runge-Kutta methods

1 Trajectory of a cannonball

Consider a cannonball which leaves the barrel at time $t = 0$ and location $x(0) = 0 \in \mathbb{R}^2$ with an initial velocity $\dot{x}(0) = v_0 \in \mathbb{R}^2$. Let us assume the trajectory $x : [0, T] \to \mathbb{R}^2$ of this cannonball satisfies the ODE

$$\ddot{x} = -D \, \|\dot{x}\|_2 \, \dot{x} - g \, e_2, \qquad x(0) = 0, \qquad \dot{x}(0) = v_0$$

where the term $-D \, \|\dot{x}\|_2 \, \dot{x}$ with $D > 0$ describes drag and the term $-g \, e_2$ with $g > 0$ and $e_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$ describes gravity.

1. Rewrite the above ODE in the form $\dot{y} = f(y)$, $y(0) = y_0$.
2. Implement the resulting $f(y)$ in `cannonball_f(y,D,g)`.
3. Run `cannonball_trajectory()` and verify that you get a physically plausible trajectory.

2 Butcher tableaus

Complete the functions `midpoint_step()` and `ssprk3_step()` such that they perform single Runge-Kutta steps according to the following Butcher tableaus:

$$\text{Midpoint:} \quad \begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array} \qquad \qquad \text{SSPRK3:} \quad \begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ \hline & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{array}$$

The midpoint method is second-order convergent and SSPRK3 is third-order convergent. Verify that your implementation achieves these convergence orders by running `convergence()`.

3 Generic two-stage Runge-Kutta method

A generic two-stage Runge-Kutta method is of the form

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \theta & \theta & 0 \\ \hline & w_1 & w_2 \end{array} .$$

Show that this method is third-order consistent if and only if

$$w_1 = 1 - \tfrac{1}{2\theta}, \qquad w_2 = \tfrac{1}{2\theta}.$$