MA3227 Numerical Analysis II

Solutions to Tutorial 5

GMRES convergence

- Matrix A has only 4 distinct eigenvalues; hence we must have $r_3 = 0$. The only plot satisfying this criterion is plot 3.
- We have the bound

$$\min_{q\in\mathcal{P}_n} \max_{x\in[c,d]} \frac{|q(x)|}{|q(0)|} \;\leq\; 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^n \qquad \text{where} \qquad \kappa = \tfrac{d}{c},$$

  which implies that GMRES converges faster for smaller κ. For matrix B we have $\kappa = \frac{10}{1} = 10$ while for matrix C we have $\kappa = \frac{40}{10} = 4$; hence the faster converging plot 2 must correspond to C and the more slowly converging plot 1 must correspond to B.

Krylov convergence plots

- Plot 1 must correspond to conjugate gradients because MinRes and restarted GMRES reduce the residual monotonically.
- Plot 2 must correspond to restarted GMRES because MinRes and conjugate gradients converge after at most $\mathrm{length}(b) = 40$ iterations.
- Plot 3 must therefore correspond to MinRes because MinRes is the only method left at this point.

MinRes vs LU factorisation

We know from Lecture 5 that the MinRes residuals satisfy

$$\min_{p\in\mathcal{P}_n} \|b_N - A_N\, p(A_N)\, b_N\|_2 \quad\leq\quad \|b\|_2 \min_{q\in\mathcal{P}_{n+1}} \max_{k} \tfrac{|q(\lambda_k)|}{|q(0)|}.$$

The right-hand side in this bound is independent of N; hence we can choose the polynomial degree / number of MinRes iterations n independent of N, and the MinRes runtime is given by

$$O\big(n\,\mathrm{nnz}(A) + n\,\mathrm{length}(b)\big) = O\big(N^2 + N\big) = O\big(N^2\big).$$

By contrast, LU factorisation requires $O(N^3)$ operations because A is a dense matrix.