MA3227 Numerical Analysis II

Solutions to Assignment 2

Simon Etter, 2019/2020
Deadline: 1 April 2020, 12.00 (noon)
Total marks: 10

1 Explicit trapezoidal method [5 marks]

1.

$$\left(\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

2. See code.
3. We obtain using $f(y) = y$ that

$$\begin{aligned}
\tilde{y}(t) &= y_0 + \tfrac{1}{2}\, f(y_0)\, t + \tfrac{1}{2}\, f\big(y_0 + f(y_0)\, t\big)\, t, \\
\dot{\tilde{y}}(t) &= \tfrac{1}{2}\, f(y_0) + \tfrac{1}{2}\, f\big(y_0 + f(y_0)\, t\big) + \tfrac{1}{2}\, f'\big(y_0 + f(y_0)\, t\big)\, f(y_0)\, t, \\
\ddot{\tilde{y}}(t) &= f'\big(y_0 + f(y_0)\, t\big)\, f(y_0) + \tfrac{1}{2}\, f''\big(y_0 + f(y_0)\, t\big)\, f(y_0)^2\, t
\end{aligned}$$

   and observe

$$\tilde{y}(0) = y_0 = y(0), \qquad \dot{\tilde{y}}(0) = f(y_0) = \dot{y}(0), \qquad \ddot{\tilde{y}}(0) = f'(y_0)\, f(y_0) = \ddot{y}(0).$$

   Since the derivatives up to second order of $\tilde{y}(t)$ and $y(t)$ match at $t = 0$, we conclude that the method is at least third-order consistent.

4. We compute using $f(y) = y$

$$\tilde{y}(t) = y_0 + \tfrac{1}{2}\, y_0\, t + \tfrac{1}{2}\, (y_0 + y_0\, t)\, t = \big(1 + t + \tfrac{t^2}{2}\big)\, t.$$

   Hence the stability function is $R(z) = 1 + z + \frac{z^2}{2}$.

5. See code.

2 Implicit trapezoidal method [5 marks]

6.

$$\left(\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 0 & 1 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

7. See code.
8. Using $f(y) = y$ we obtain

$$f_2 = y_0 + f_2\, t \qquad \Longleftrightarrow \qquad f_2 = \tfrac{y_0}{1-t}.$$

Inserting this result into the formula for $\tilde{y}(t)$ yields

$$\tilde{y}(t) = y_0 + \tfrac{1}{2}\, y_0\, t + \tfrac{1}{2}\, \tfrac{y_0}{1-t}\, t = \left(1 + \tfrac{t}{2} + \tfrac{t}{2-2t}\right) y_0 = \tfrac{2-2t+t\,(1-t)+t}{2-2t}\, y_0 = \tfrac{2-t^2}{2-2t}\, y_0;$$

hence the stability function is $R(z) = \frac{2-z^2}{2-2z}$

9. Yes, we do have a step-size constraint because $|R(z)| \to \infty$ for $|z| \to \infty$, which implies that the stability domain is bounded.

10. See code.