MA3227 Numerical Analysis II

Lab Session 6

Simon Etter, 2019/2020

1 Trajectory of a cannonball (continued)

Recall from Lab Session 5 the ODE

$$\ddot{\vec{x}} = -D\,\|\dot{\vec{x}}\|_2\,\dot{\vec{x}} - g\,\vec{e}_2, \qquad \vec{x}(0) = \vec{0}, \qquad \dot{\vec{x}}(0) = \vec{v}_0 \tag{1}$$

which describes the trajectory of a cannonball of mass $m = 1$ subject to drag with drag coefficient $D > 0$ and gravity with gravitational constant $g > 0$. Equivalently, the ODE (1) can be split into the two coupled ODEs

$$\dot{\vec{v}} = -D\,\|\vec{v}\|_2\,\vec{v} - g\,\vec{e}_2 \quad \text{with} \quad \vec{v}(0) = \vec{v}_0 \qquad \text{and} \qquad \dot{\vec{x}} = \vec{v} \quad \text{with} \quad \vec{x}(0) = \vec{0}$$

which is more convenient for some of the following tasks.

1. We observe that the ODE $\dot{\vec{v}} = \vec{f}(\vec{v})$ for $\vec{v}$ has a fixed point $\vec{f}(\vec{v}_F) = \vec{0}$ for $\vec{v}_F = -\sqrt{\frac{g}{D}}\,\vec{e}_2$, which corresponds to the cannonball falling straight down with a velocity such that drag exactly balances the acceleration due to gravity. Verify the following computations for determining $\nabla\vec{f}(\vec{v}_F)$.

$$\begin{aligned}
\vec{f}(\vec{v}) &= \begin{pmatrix} -D\,\sqrt{v_1^2 + v_2^2}\,v_1 \\ -D\,\sqrt{v_1^2 + v_2^2}\,v_2 - g \end{pmatrix}, \\
\nabla\vec{f}(\vec{v}) &= -D\begin{pmatrix} \frac{v_1^2}{\sqrt{v_1^2+v_2^2}} + \sqrt{v_1^2+v_2^2} & \frac{v_1\,v_2}{\sqrt{v_1^2+v_2^2}} \\ \frac{v_1\,v_2}{\sqrt{v_1^2+v_2^2}} & \frac{v_2^2}{\sqrt{v_1^2+v_2^2}} + \sqrt{v_1^2+v_2^2} \end{pmatrix}, \\
\nabla\vec{f}(\vec{v}_F) &= \begin{pmatrix} -\sqrt{gD} & 0 \\ 0 & -2\sqrt{gD} \end{pmatrix}.
\end{aligned}$$

2. Let us denote by $\lambda = -2\sqrt{gD}$ the more negative of the two eigenvalues. We have seen in class that under these circumstances, explicit Runge-Kutta methods are stable only if the time step `dt` is chosen such that $|R(\lambda\,\mathtt{dt})| \le 1$, where $R(z) = 1 + z$ for Euler's method and $R(z) = 1 + z + \frac{z^2}{2}$ for the midpoint method. For both methods, determine $\mathtt{dt} > 0$ such that $|R(\lambda\,\mathtt{dt})| = 1$. Test your answer by replacing the placeholder `TODO` in `stability()` with the determined values of `dt`. If your answer is correct, you will find that the distance $d = |\vec{v}_2 + \sqrt{g/D}|$ between $\vec{v}_2$ and its fixed-point value $(\vec{v}_F)_2 = -\sqrt{g/D}$ is approximately constant rather than exponentially decaying.

   Hint. You will find that d decays slightly for the midpoint method. This is due to the nonlinearity of the ODE, which is not captured by our linearised analysis around the fixed point.

3. The time-step constraints derived in the previous two tasks can be avoided by switching to an implicit Runge-Kutta scheme, but doing so would require us to solve a system of

nonlinear equations which we would like to avoid. Instead, we consider a semi-implicit Euler method given by

$$\tilde{\vec{v}}(t) = \vec{v}(0) - D\,\|\vec{v}(0)\|_2\,\tilde{\vec{v}}(t)\,t - g\,\vec{e}_2\,t, \qquad \tilde{\vec{x}}(t) = \vec{x}(0) + \vec{v}(0)\,t.$$

These are almost the equations of the explicit Runge-Kutta method except that we have a single factor of $\tilde{\vec{v}}(t)$ appearing on the right-hand side of the equation for $\tilde{\vec{v}}(t)$. The advantage of these equations is that we can write down an explicit formula for $\tilde{\vec{v}}(t)$, namely

$$\tilde{\vec{v}}(t) = \frac{1}{1 + D\,\|\vec{v}(0)\|_2\,t}\,\big(\vec{v}(0) - g\,\vec{e}_2\,t\big).$$

Implement this scheme in the function `semi_implicit_euler_step()`. You can test your code using the provided function `convergence()`. If your code is correct, you will find that the error e_n decays as $e_n = \mathcal{O}\big(n^{-1}\big)$ for this semi-implicit Euler method.

4. Uncomment the line for the semi-implicit Euler method in `stability()`. Note how even with a time step `dt` $= 10^3$, the distance $d = |\vec{v}_2 + \sqrt{g/D}|$ still decays (albeit slowly) for the semi-implicit Euler method.

   The practical implications of this are as follows: Once the velocity $\vec{v}(t)$ approaches its steady state $\vec{v}_F$, we should be able to take arbitrarily large time steps `dt` since the two ODEs for position and velocity simplify to $\dot{\vec{v}} = 0$, $\dot{\vec{x}} = \vec{v}$ and these equations can be solved *exactly* with a single step of any of the methods considered above. However, it is not possible to increase the time step `dt` beyond some `max_dt` $< \infty$ for the explicit methods due to the stability constraint. The semi-implicit method has no such constraint; hence for large enough time intervals $[0, T]$ the semi-implicit method can be arbitrarily much faster than the explicit methods.