MA3227 Numerical Analysis II

Lab Session 7

Simon Etter, 2019/2020

1 Computing π using the Monte Carlo method

Assume $(x_k)_{k=1}^N$ are samples of a random variable $X \sim \text{Uniform}[0,1]^2$, i.e. X is uniformly distributed in the unit square $[0,1]^2$. Since

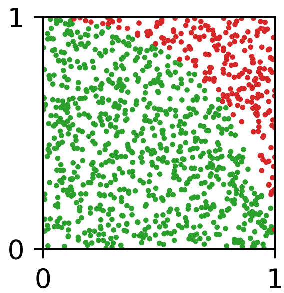
$$\frac{\text{area}\left\{x \in [0,1]^2 \mid \|x\|_2 \le 1\|\right\}}{\text{area}\,[0,1]^2} = \frac{\pi}{4},$$

we expect that

$$\pi \approx \frac{4}{N}\sum_{k=1}^{N} \chi(x) \qquad \text{where} \qquad \chi(x) = \begin{cases} 1 & \text{if } \|x\|_2 \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Implement this Monte Carlo scheme and verify that the error asymptotically decays as $\mathcal{O}(N^{-1/2})$.

Hint. You can generate samples of X using `rand(2)`.



2 Shooting method for nonlinear boundary value problems

Consider the boundary value problem

$$\text{Find } u : [0,1] \to \mathbb{R} \text{ such that } \begin{cases} -u''(x) = f(x) - u^2(x) & \text{for all } x \in (0,1), \\ u(0) = u(1) = 0 & \text{otherwise.} \end{cases} \tag{1}$$

where $f : [0,1] \to \mathbb{R}$ denotes some given function. We can interpret this boundary value problem as an initial value problem

$$\text{Find } u : [0,1] \to \mathbb{R} \text{ such that } \begin{cases} u''(x) = -f(x) + u^2(x) & \text{for all } x \in (0,1), \\ u(0) = 0, \qquad u'(0) = d_0. \end{cases} \tag{2}$$

where $d_0 = u'(0)$ is determined by the right-hand boundary condition $u(1) = u(1, d_0) = 0$.

The file `lab07_shooting.jl` contains a function `main()` which uses Newton's method to determine d_0 such that the solution to the initial value problem (2) satisfies the boundary value problem (1). Fill in the blanks in this function.

Hint. The idea of this exercise is to practice the $\frac{\partial}{\partial t}\frac{\partial y}{\partial \theta} = \nabla f(y)\,\frac{\partial y}{\partial \theta}$ technique introduced in Lecture 20.

Remark. The above technique for solving a boundary value problem is called the shooting method because we can interpret it as "shooting" from $x = 0$ at various angles d_0 until we hit the target $u(1) = 0$.