MA3227 Numerical Analysis II

Solutions to Assignment 1

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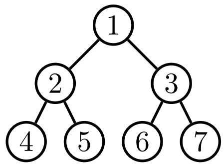
Total marks: 10

1 Fill path theorem [4 marks]

1. The graph is a tree:



2. According to the fill-path theorem, we have $(L+U)[i,j] \neq 0$ in the structural sense if and only if there is a fill-path from j to i. Using this result, we obtain the following sparsity pattern for the LU factorisation.

$$A = \left(\begin{array}{c|c|c|c|c|c|c}
1 & \bullet & \bullet & & & & \\ \hline
\bullet & 2 & \mathtt{x} & \bullet & \bullet & & \\ \hline
\bullet & \mathtt{x} & 3 & \mathtt{x} & \mathtt{x} & \bullet & \bullet \\ \hline
 & \bullet & \mathtt{x} & 4 & \mathtt{x} & \mathtt{x} & \mathtt{x} \\ \hline
 & \bullet & \mathtt{x} & \mathtt{x} & 5 & \mathtt{x} & \mathtt{x} \\ \hline
 & & \bullet & \mathtt{x} & \mathtt{x} & 6 & \mathtt{x} \\ \hline
 & & \bullet & \mathtt{x} & \mathtt{x} & \mathtt{x} & 7
\end{array}\right)$$

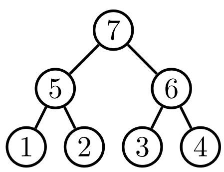
In particular:

- $L[3,2] \neq 0$ because we have a fill path $2 \to 1 \to 3$.
- $L[6,2] = 0$ because the only path connecting these two vertices is $2 \to 1 \to 3 \to 6$ and this is not a fill path because $3 > 2$.
- $L[4,3] \neq 0$ because we have a fill path $3 \to 1 \to 2 \to 4$.

3. The trick is to enumerate the vertices from bottom to top, e.g. using the permutation

i	1	2	3	4	5	6	7
$\pi(i)$	4	5	6	7	2	3	1

which yields the enumeration



We now verify that this new enumeration indeed does not permit fill paths of length > 1. (Note that only fill paths of length > 1 produce fill-in. For fill paths $j \to i$ of length one, we have $(L+U)[i,j] \neq 0$ but also $A[i,j] \neq 0$ and hence this is not a fill-in entry.)

- No fill path of length > 1 starts in the vertices 1, 2, 3 or 4.

All paths of length > 1 starting in vertex 1 are of the form $1 \to 5 \to \ldots$. If we add vertices after 5, this will no longer be a fill path because $1 < 5$; hence there are no fill paths of length > 1 starting in vertex 1. The argument for the other vertices is analogous.

- No fill path of length > 1 starts in the vertices 5 or 6.

  All paths of length > 1 starting in vertex 5 are of the form $5 \to 7 \to \ldots$. No such path can be a fill path because $7 > 5$. The argument for vertex 6 is analogous.

- No fill path of length > 1 starts in vertex 7.

  All paths of length > 1 starting in vertex 7 are of the form $7 \to 5 \to \ldots$ or $7 \to 6 \to \ldots$. All possible vertices following 5 or 6 are numbered lower than 5 or 6, so these paths cannot be fill paths.

2 Convergence of GMRES [2 marks]

- Matrix C has only 5 distinct eigenvalues and hence we must have $\|Cx_k - b\|_2 = 0$ for $k = 5$. The only plot satisfying this criterion is plot 3.
- We have the bound
$$\min_{q_k \in \mathcal{P}_k} \max_{x \in [1,\kappa]} \frac{|q_k(x)|}{|q_k(0)|} \leq 2 \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^k$$
  which implies that GMRES converges faster for smaller κ. For matrix A we have $\kappa = 10$ while for matrix B we have $\kappa = 40$; hence the faster converging plot 1 must correspond to matrix A and the more slowly converging plot 2 must correspond to matrix B.

3 Conjugate gradients [4 marks]

See `sheet1_solutions.jl`.