MA3227 Numerical Analysis II

Solutions to Assignment 2

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Total marks: 10

1 Multigrid in two dimensions [4 marks]

The error recursion formula for the relaxed Jacobi iteration is

$$x_{k+1} - x = R\,(x_k - x) \qquad \text{where} \qquad R = I - \theta\, D^{-1} A.$$

Using $D = 4\,(n+1)^2\, I$, we find that the eigenvalues $\hat\lambda^{(2)}_{k_1,k_2}$ of R can be expressed in terms of the eigenvalues $\lambda^{(2)}_{k_1,k_2}$ of $A = -\Delta^{(2)}_n$ as

$$\hat\lambda^{(2)}_{k_1,k_2} = 1 - \theta\,\frac{\lambda^{(2)}_{k_1,k_2}}{4\,(n+1)^2}.$$

Using $\lambda^{(2)}_{k_1,k_2} = \lambda^{(1)}_{k_1} + \lambda^{(1)}_{k_2}$ and the bounds on $\lambda^{(1)}_k$ mentioned in the hint, we find that the $\hat\lambda^{(2)}_{k_1,k_2}$ of largest absolute value are

$$\hat\lambda^{(2)}_{\frac{n}{2},1} = \hat\lambda^{(2)}_{1,\frac{n}{2}} \approx 1 - \theta\,\frac{2\,(n+1)^2}{4\,(n+1)^2} = 1 - \frac{\theta}{2}$$

and

$$\hat\lambda^{(2)}_{n,n} \approx 1 - \theta\,\frac{8\,(n+1)^2}{4\,(n+1)^2} = 1 - 2\theta.$$

Thus, $|\hat\lambda_{\text{best}}| = \max\big\{|1 - \frac{\theta}{2}|, |1-2\theta|\big\}$ is minimised if

$$1 - \tfrac{\theta_{\text{best}}}{2} = -(1 - 2\,\theta_{\text{best}}) \qquad \Longleftrightarrow \qquad \theta_{\text{best}} = \tfrac{4}{5}$$

and we have

$$|\hat\lambda_{\text{best}}| = |1 - 2\,\theta_{\text{best}}| = \tfrac{3}{5}.$$

See also `sheet2_multigrid_solution.jl`.

2 Gradient descent for classification [6 marks]

See `sheet2_classification_solution.jl`.