MA3227 Numerical Analysis II

Lab Session 9

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1 Sampling theorems

Consider the distributions $\mathcal{F}$ and $\mathcal{G}$ on $[0, 1]$ whose probability density functions are given by, respectively,

$$f(x) = 6\,x\,(1-x) \qquad \text{and} \qquad g(x) = \tfrac{\pi}{2}\,\sin(\pi x).$$

1. Verify that the cumulative distribution functions of $\mathcal{F}$ and $\mathcal{G}$ are given by, respectively,

   $$\hat{F}(x) = 3x^2 - 2x^3, \qquad \hat{G}(x) = \tfrac{1}{2}\big(1 - \cos(\pi x)\big).$$

   Observe that there is a simple formula for $\hat{G}^{-1}(u)$ but not for $\hat{F}^{-1}(u)$. We will therefore use transformation sampling for simulating $G \sim \mathcal{G}$, but rejection sampling with $\mathcal{G}$ as proposal distribution for simulating $F \sim \mathcal{F}$.

2. Complete the function `rand_sin()` such that it simulates $G \sim \mathcal{G}$ using transformation sampling.

3. Complete the function `rand_quad()` such that it simulates $F \sim \mathcal{F}$ using rejection sampling with $\mathcal{G}$ as proposal distribution.

   Hint. You will need to compute $M = \sup_{x\in[0,1]} \frac{f(x)}{g(x)}$. A plot of $\frac{f(x)}{g(x)}$ reveals that $M = \lim_{x\to 0} \frac{f(x)}{g(x)}$, and this limit can be computed using L'Hôpital's rule.

4. Test your answers to Tasks 2 and 3 using the provided `histogram()`.

5. Given `rand_sin()` and `rand_quad()`, we can estimate $\mathbb{E}[F]$ either through direct Monte Carlo or through importance sampling Monte Carlo with $\mathcal{G}$ as proposal distribution. Determine which of the two is more efficient based on the output of `importance_sampling()`. In particular, explain why the `Var[F]*M` line is needed to make a fair comparison.