MA3227 Numerical Analysis II

Lab Session 3

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1 Jacobi preconditioning for Poisson's equation

Consider the linear system $Ax = b$ and denote by D the diagonal of A. Lecture 10 introduced the Jacobi iteration

$$x_{k+1} = D^{-1}\big(b - (A - D)\, x_k\big).$$

which can also be written in the form

$$x_{k+1} = x_k + D^{-1}\left(b - A\, x_k\right). \tag{1}$$

Equation (1) suggests that Jacobi implicitly treats D^{-1} as an approximation to A^{-1}, for if we replace D^{-1} with A^{-1} in this formula, we obtain the iteration $x_{k+1} = x_k + A^{-1}\,(b - Ax_k)$ which reaches the exact solution in a single step. For this reason, the matrix $D = \operatorname{diag}(A)$ is called the *Jacobi preconditioner*.

Let us now consider the finite-difference-discretised Poisson equation $A = -\Delta_n^{(d)}$. Show that Jacobi preconditioning

$$Ax = b \qquad \longrightarrow \qquad D^{-1}Ax = D^{-1}b$$

does not change the sequence of approximations $x_k \approx x$ computed by Krylov subspace methods.

2 Gauss-Seidel preconditioning for Poisson's equation

Consider the linear system $Ax = b$ and denote by D the diagonal, by L the lower-triangular and by U the upper-triangular part of A. The Gauss-Seidel iteration can then be written as

$$x_{k+1} = (D + L)^{-1}\big(b - Ux_k\big).$$