MA3227 Numerical Analysis II

Tutorial 6: Explicit Runge-Kutta methods

3 Generic two-stage Runge-Kutta method

We have

$$\begin{aligned}\tilde{y}(t) &= y_0 \;+\; f(y_0)\,w_1\,t \;+\; f\big(y_0 + f(y_0)\,\theta\,t\big)\,w_2\,t \\ &= y_0 \;+\; f(y_0)\,w_1\,t \;+\; f(y_0)\,w_2\,t \;+\; f'(y_0)\,f(y_0)\,\theta\,w_2\,t^2 \;+\; O(t^3) \\ &= y(0) \;+\; (w_1 + w_2)\,\dot{y}(0)\,t \;+\; \theta\,w_2\,\ddot{y}(0)\,t^2 \;+\; O(t^3),\end{aligned}$$

where the identity on the first line is derived from the Butcher tableau on the tutorial sheet, on the second line I do a Taylor expansion of $t \mapsto f\big(y_0 + f(y_0)\,\theta\,t\big)$ around $t = 0$, and on the third line I use that $y(0) = y_0$, $\dot{y}(0) = f(y_0)$ and $\ddot{y}(0) = f'(y_0)\,f(y_0)$.

Matching the coefficients of the above series with the Taylor series of $y(t)$, we obtain

$$w_1 + w_2 = 1, \qquad \theta\,w_2 = \tfrac{1}{2}$$

which yields

$$w_1 = 1 - \tfrac{1}{2\theta}, \qquad w_2 = \tfrac{1}{2\theta}$$

as claimed.