MA3227 Numerical Analysis II

Assignment 3

Simon Etter, 2019/2020
Deadline: 1 April 2020, 12.00 (noon)
Total marks: 10

1 Explicit trapezoidal method [5 marks]

The lectures presented the midpoint method as an example of a two-stage, second-order-convergent Runge-Kutta method. In this assignment, we will study another such method known as the trapezoidal method whose one-step equation is given by

$$\tilde{y}(t) = y_0 + \tfrac{1}{2}\, f_1\, t + \tfrac{1}{2}\, f_2\, t \qquad \text{where} \qquad f_1 = f(y_0), \qquad f_2 = f\big(y_0 + f_1\, t\big) \approx f\big(y(t)\big). \tag{1}$$

1. Write down the Butcher tableau for the trapezoidal method.
2. Complete the function `trapezoidal_step()`.
3. Show that the trapezoidal method is at least third-order consistent. You do not have to show that that the trapezoidal method is not fourth-order consistent.

   Hint. You can check your answers to Tasks 2 and 3 using `convergence()`. Recall that third-order consistency implies second-order convergence.
4. Determine the stability function $R(z)$ of the trapezoidal method.
5. Check your answer to Task 4 by completing `R_trapezoidal()` and running `stability()`. If your answer is correct, the dashed and solid blue lines should be parallel.

2 Implicit trapezoidal method [5 marks]

The above (explicit) trapezoidal method can be turned into an implicit trapezoidal method by replacing the equation for f_2 with

$$f_2 = f\big(y_0 + f_2\, t\big).$$

Assume in the following that the equations for $\tilde{y}(t)$ and f_1 are as in (1).

6. Write down the Butcher tableau for the implicit trapezoidal method.
7. Complete the function `implicit_trapezoidal_step()`.

   You may solve the nonlinear equation for f_2 using the `find_zero(f,x0) -> x` function provided by the `Roots` package, which returns `x` such that `f(x) = 0` using `x0` as an initial guess for `x`. Use $\hat{f}_2 = f(y_0 + f_1\, t)$ as initial guess for f_2.

   The implicit trapezoidal method is second-order convergent. You can therefore check that your implementation of `implicit_trapezoidal_rule()` is correct using `convergence()`.
8. Determine the stability function $R(z)$ of the implicit trapezoidal method.
9. Does the implicit trapezoidal method exhibit spurious exponential blow-up when applied the ODE $\dot{y} = -y$ with a very large time step? Motivate your answer by referring to the stability function $R(z)$ determined in Task 8.
10. Check your answer to Task 8 by completing `R_trapezoidal_implicit()` and running `stability()`. If your answer is correct, the dashed and solid orange lines should be parallel.