MA3227 Numerical Analysis II

Tutorial 5: Krylov subspace methods

GMRES convergence

Match the following matrices with their GMRES convergence histories.

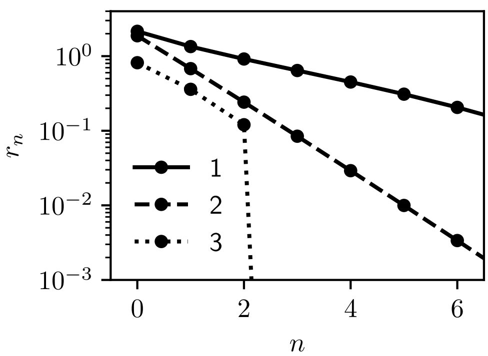
- Matrices:

$$A = \mathrm{diag}(1, \ldots, 4), \qquad B = \mathrm{diag}(1, \ldots, 10), \qquad C = \mathrm{diag}(10, \ldots, 40)$$

  where

$$\mathrm{diag}(v_1, \ldots, v_n)[i, j] = \begin{cases} v_k & \text{if } i = j = k, \\ 0 & \text{otherwise.} \end{cases}$$

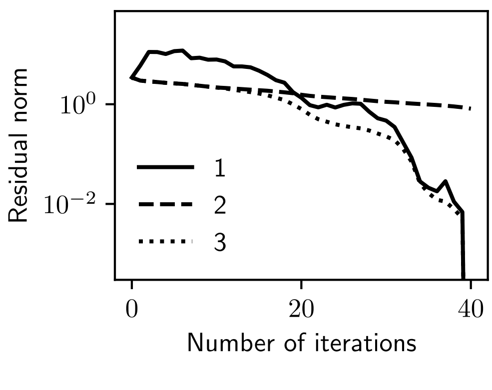
- Convergence histories:



where $\qquad r_n = \min_{p \in \mathcal{P}_n} \|b - A\, p(A)\, b\|_2.$

Krylov convergence plots

The methods on the right applied to a symmetric and positive definite linear system $Ax = b$ of size 40×40 results in the convergence plot on the left.



(A) Restarted GMRES with $n_{\text{inner}} = 10$
(B) MinRes
(C) Conjugate gradients

Match the methods and convergence plots.

For restarted GMRES, the convergence plot shows the residual after every inner iteration.

MinRes vs LU factorisation

Let $A_N \in \mathbb{R}^{N\times N}$ be a sequence of dense and symmetric matrices such that the eigenvalues of all these matrices is contained in some set $S \subset \mathbb{R} \setminus \{0\}$ independent of N, and assume we want to determine approximate solutions $\tilde{x}_N$ to $A_N\, x_N = b_N$ such that

$$\|b_N - A_N\, \tilde{x}_N\|_2 \;\leq\; \tau\, \|b_N\|_2$$

for some given $\tau > 0$ and $b_N \in \mathbb{R}^N$.

Show that MinRes can solve this problem in $O(N^2)$ runtime while dense LU factorisation requires $O(N^3)$ runtime.