MA3227 Numerical Analysis II

Solutions to Assignment 4

Simon Etter, 2019/2020
Deadline: 17 April 2020, 18.00
Total marks: 10

1 Uniformly distributed points on the disk [5 marks]

1. The probability density functions for Uniform $[-1,1]^2$ and Uniform D are given by

$$p_{[-1,1]^2}(x) = \begin{cases} \frac{1}{4} & \text{if } x \in [-1,1]^2, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad p_D(x) = \begin{cases} \frac{1}{\pi} & \text{if } x \in D, \\ 0 & \text{otherwise,} \end{cases}$$

respectively; hence we have $p_D(x) \leq M\, p_{[-1,1]^2}(x)$ for $M = \frac{4}{\pi}$ and the acceptance probability for a given $G \sim$ Uniform $[-1,1]^2$ is given by

$$\frac{p_D(G)}{M\, p_{[-1,1]^2}(G)} = \begin{cases} \frac{\frac{1}{\pi}}{\frac{4}{\pi}\frac{1}{4}} = 1 & \text{if } G \in D, \\ \frac{0}{\frac{4}{\pi}\frac{1}{4}} = 0 & \text{otherwise.} \end{cases}$$

In this case, rejection sampling thus amounts to generating $G \sim$ Uniform $[-1,1]^2$ until $G \in D$ and then setting $X = G$. See code.

2. We observed in Task 1 that $M = \frac{4}{\pi}$. According to Lecture 23, this means that we on average need to generate $M = \frac{4}{\pi} \approx 1.27$ samples of Uniform $[-1,1]^2$ to generate a sample of Uniform D.

   A more direct way to see this is to observe that according to Task 1, the number K of samples of Uniform $[-1,1]^2$ per sample of Uniform D is geometrically distributed with success probability $p = P(G \in D) = \frac{\pi}{4}$, and we have $\mathbb{E}[K] = p^{-1} = P(G \in D)^{-1} = \frac{4}{\pi} = M$.

3. The task implicitly asks you to show that $P\big(X(R,\Phi) \in A\big) = \frac{\text{area}(A)}{\text{area}(D)} = \frac{1}{\pi}\,\text{area}(A)$ for all $A \subset D$. To this end, let us introduce the indicator function

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

and let us denote the probability densities of $\mathcal{R}$ and Uniform $[0, 2\pi]$ by

$$p(r) = 2r \qquad \text{and} \qquad q(\phi) = \frac{1}{2\pi}, \qquad \text{respectively.}$$

Since R and Φ are independent, we then have that

$$\begin{aligned} P\big(X(R,\Phi) \in A\big) &= \int_0^1 \int_0^{2\pi} \chi_A\big(X(r,\phi)\big)\, p(r)\, q(\phi)\, d\phi\, dr \\ &= \int_0^1 \int_0^{2\pi} \chi_A\big(X(r,\phi)\big)\, \tfrac{2r}{2\pi}\, d\phi\, dr \\ &= \tfrac{1}{\pi} \int_D \chi_A(x)\, dx = \tfrac{1}{\pi}\,\text{area}(A) \end{aligned}$$

as required.

4. $\Phi \sim \text{Uniform}[0, 2\pi]$ can be sampled using $\Phi(U) = 2\pi U$ where $U \sim \text{Uniform}[0, 1]$, since then we have for all $[a, b] \subset [0, 2\pi]$ that

$$P\big(\Phi(U) \in [a, b]\big) = P\big(U \in \tfrac{1}{2\pi}\,[a, b]\big) = \tfrac{b-a}{2\pi}$$

as required.

$R \sim \mathcal{R}$ can be sampled using transformation sampling: the cumulative distribution function of $\mathcal{R}$ is given by

$$F(r) = \int_0^r 2t\,dt = r^2;$$

hence we have for $U \sim \text{Uniform}[0, 1]$ that

$$R = F^{-1}(U) = \sqrt{U} \sim \mathcal{R}.$$

See code.

2 Importance sampling for highly concentrated integrals [5 marks]

See code.