MA3227 Numerical Analysis II

Lab Session 5

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1 Trajectory of a cannonball

Consider a cannonball which leaves the barrel at time $t = 0$ and location $\vec{x}(0) = \vec{0}$ with an initial velocity $\dot{\vec{x}}(0) = \vec{v}_0$. Let us assume the trajectory $\vec{x} : [0, T] \to \mathbb{R}^2$ of this cannonball satisfies the ODE

$$\ddot{\vec{x}} = -D \, \|\dot{\vec{x}}\|_2 \, \dot{\vec{x}} - g \, \vec{e}_2, \qquad \vec{x}(0) = \vec{0}, \qquad \dot{\vec{x}}(0) = \vec{v}_0$$

where the term $-D \, \|\dot{\vec{x}}\|_2 \, \dot{\vec{x}}$ with $D > 0$ describes drag and the term $-g \, \vec{e}_2$ with $g > 0$ and $e_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$ describes gravity.

Rewrite the above ODE in the form $\dot{y} = f(y)$, $y(0) = y_0$. Implement the resulting $f(y)$ in `cannonball_f(y,D,g)`, and run `plot_trajectory()` to verify that you get a physically plausible trajectory.

2 Butcher tableaus

Complete the functions `midpoint_step()` and `ssprk3_step()` such that they perform single Runge-Kutta steps according to the following Butcher tableaus:

$$\text{Midpoint:} \quad \begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array} \qquad\qquad \text{SSPRK3:} \quad \begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ \hline & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{array}$$

The midpoint method is second-order convergent and SSPRK3 is third-order convergent. Verify that your implementation achieves these convergence orders by running `convergence()`.

3 Generic two-stage Runge-Kutta method

A generic two-stage Runge-Kutta method is of the form

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \theta & \theta & 0 \\ \hline & w_1 & w_2 \end{array} \,. \tag{1}$$

Show that this method is third-order consistent if and only if

$$w_1 = 1 - \tfrac{1}{2\theta}, \qquad w_2 = \tfrac{1}{2\theta}.$$

Hint. The one-step formula corresponding to (1) is given by

$$\tilde{y}(t) = y_0 + f(y_0) \, w_1 \, t + f\big(y_0 + f(y_0) \, \theta\big) \, w_2 \, t.$$

Taylor-expand this formula around $t = 0$ and make sure the first three terms match the Taylor expansion of the exact solution.