MA3227 Numerical Analysis II

Lecture 10: Jacobi Method

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Jacobi Method

Introduction

Recap: our overarching goal is to find an $\mathcal{O}(N)$ solver for $-\Delta_n^{(d)} u_n = f$.

So far, we have the following:

	LU	Krylov
$d = 1$	$\mathcal{O}\big(N\big)$	$\mathcal{O}(N^2)$
$d = 2$	$\mathcal{O}\big(N^{3/2}\big)$	$\mathcal{O}(N^{3/2})$
$d = 3$	$\mathcal{O}\big(N^2\big)$	$\mathcal{O}(N^{4/3})$

Jacobi's method is yet another method for the same problem which scales even worse than LU or Krylov methods.

The reason why we discuss Jacobi is because it provides the foundation for an algorithm which finally achieves our $\mathcal{O}(N)$ goal.

Jacobi Method

Jacobi iteration for Poisson equation

Assume we have an initial guess x_0 for the linear system

$$(3+1)^2 \begin{pmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{pmatrix} \begin{pmatrix} x[1] \\ x[2] \\ x[3] \end{pmatrix} = \begin{pmatrix} b[1] \\ b[2] \\ b[3] \end{pmatrix}.$$

Solving each equation for the “diagonal” unknown yields

$$\begin{aligned} x_1[1] &= \tfrac{1}{2}\left(\tfrac{b[1]}{(3+1)^2} + x_0[2]\right), \\ x_1[2] &= \tfrac{1}{2}\left(\tfrac{b[2]}{(3+1)^2} + x_0[1] + x_0[3]\right), \\ x_1[3] &= \tfrac{1}{2}\left(\tfrac{b[3]}{(3+1)^2} + x_0[2]\right). \end{aligned}$$

The resulting x_1 is not the exact solution in general, but we may hope that it is a better approximation to x than x_0.

Idea: iterate the map $x_0 \mapsto x_1$ until convergence.
See `jacobi_step()`.

Jacobi Method

Gauss-Seidel iteration

Minor modification of Jacobi:

on each line, we use the most recent version of $x[i]$ currently available.

$$x_1[1] = \tfrac{1}{2}\left(\tfrac{b[1]}{(3+1)^2} + x_0[1]\right),$$
$$x_1[2] = \tfrac{1}{2}\left(\tfrac{b[2]}{(3+1)^2} + x_1[1] + x_0[3]\right),$$
$$x_1[3] = \tfrac{1}{2}\left(\tfrac{b[3]}{(3+1)^2} + x_1[2]\right).$$

See `gauss_seidel_step()`.

Gauss-Seidel is a well-known algorithm that you should have heard of.

It has some minor advantages and disadvantages compared to Jacobi, but overall the two methods are very similar. If you understand Jacobi, you will have no trouble understanding Gauss-Seidel.

I will only discuss Jacobi in the following for simplicity.

Jacobi Method

Discussion of Jacobi-type methods

- Good: single iterations run very fast.
- Bad: many iterations are needed to reach a reasonable accuracy.

See `plot_convergence()`.

Next steps

As usual, we want to quantify the rate of convergence through an estimate of the form $\|x_k - x\| = \mathcal{O}\big(f(k)\big)$.

A first step towards such an estimate is to have a matrix formula for the Jacobi iteration. This is provided on the next slide.

Jacobi Method

Jacobi iteration (abstract definition)

Let A be an invertible matrix with nonzero diagonal D.
The Jacobi iteration can then be written as follows.

Algorithm 1 Jacobi iteration

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for $k = 1, 2, \ldots$ **do**

$\quad x_k = D^{-1}\Big(b - (A - D)\, x_{k-1}\Big)$

end for

The next slide demonstrates that the general definition reduces to the concrete Jacobi iteration when applied to the Poisson matrix.

Jacobi Method

Example: Jacobi iteration for Poisson matrix

Consider the matrix

$$A = (3+1)^2 \begin{pmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{pmatrix}$$

The Jacobi iteration takes the form

$$\begin{aligned} x_1 &= D^{-1}\left(b - (A-D)\,x_0\right) \\ &= \begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix}^{-1} \left(\frac{1}{(3+1)^2} \begin{pmatrix} b[1] \\ b[2] \\ b[3] \end{pmatrix} - \begin{pmatrix} & -1 & \\ -1 & & -1 \\ & -1 & \end{pmatrix} \begin{pmatrix} x_0[1] \\ x_0[2] \\ x_0[3] \end{pmatrix} \right) \\ &= \begin{pmatrix} \frac{1}{2}\left(\frac{b[1]}{(3+1)^2} + x_0[2]\right), \\ \frac{1}{2}\left(\frac{b[2]}{(3+1)^2} + x_0[1] + x_0[3]\right), \\ \frac{1}{2}\left(\frac{b[3]}{(3+1)^2} + x_0[2]\right). \end{pmatrix}. \end{aligned}$$

This is precisely the iteration that we had before.

Jacobi Method

Towards a convergence estimate for Jacobi iteration

Using the Jacobi iteration formula and $b = Ax$, we obtain

$$\begin{aligned} x_k - x &= D^{-1}\Big(b - (A-D)\,x_{k-1}\Big) - x \\ &= D^{-1}\Big(Ax - Dx - (A-D)\,x_{k-1}\Big) \\ &= -D^{-1}(A-D)\big(x_{k-1} - x\big). \end{aligned}$$

Applying this formula repeatedly yields

$$x_k - x = R^k\big(x_0 - x\big) \qquad \text{where} \qquad R = -D^{-1}(A-D).$$

Let us expand the initial error in terms of the eigenvectors u_ℓ of R,

$$x_0 - x = \sum_{\ell=1}^{N} c_\ell\, u_\ell.$$

Denoting the associated eigenvalues by $\hat{\lambda}_\ell$, we obtain

$$x_k - x = R^k\big(x_0 - x\big) = \sum_{\ell=1}^{N} c_\ell\, R^k\, u_\ell = \sum_{\ell=1}^{N} c_\ell\, \hat{\lambda}_\ell^k\, u_\ell.$$

Jacobi Method

Towards a convergence estimate for Jacobi iteration

From previous slide:

$$x_k - x = \sum_{\ell=1}^{N} c_\ell \, \hat{\lambda}_\ell^k \, u_\ell.$$

Assume $\|u_\ell\| = 1$ and introduce $|\hat{\lambda}_{\mathsf{max}}| = \arg\max_\ell |\hat{\lambda}_\ell|$. Then,

$$\|x_k - x\| \le \sum_{\ell=1}^{N} |c_\ell| \, |\hat{\lambda}_\ell|^k \le \left(\sum_{\ell=1}^{N} |c_\ell| \right) |\hat{\lambda}_{\mathsf{max}}|^k.$$

Conclusion

Jacobi iterates x_k satisfy

$$\|x_k - x\| \le C \, |\hat{\lambda}_{\mathsf{max}}|^k$$

where $\hat{\lambda}_{\mathsf{max}}$ is the eigenvalue of largest absolute value of

$$R = -D^{-1}(A - D).$$

Jacobi Method

Jacobi's method applied to Poisson's equation

$$A = -\Delta_n^{(1)} = (n+1)^2 \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \quad \Longrightarrow \quad D = (n+1)^2 \begin{pmatrix} 2 & & \\ & \ddots & \\ & & 2 \end{pmatrix}.$$

Let us introduce the following notation:

- $\lambda_\ell = (n+1)^2 \left(2 - 2\cos\left(\pi \frac{\ell}{n+1}\right)\right)$: eigenvalues of $-\Delta_n^{(1)}$.
- $\hat{\lambda}_\ell$: eigenvalues of $R = -D^{-1}(A - D)$.

Since $D \propto I$, we obtain

$$\hat{\lambda}_\ell = -\frac{1}{2\,(n+1)^2}\left(\lambda_\ell - 2\,(n+1)^2\right) = \cos\left(\pi \frac{\ell}{n+1}\right).$$

Largest absolute value is achieved for $\ell = 1$ and $\ell = n$ for which we have

$$\hat{\lambda}_1 = -\hat{\lambda}_n = \cos\left(\frac{\pi}{n+1}\right) = 1 - \mathcal{O}(n^{-2}).$$

The same result holds for $\Delta_n^{(d)}$ because we have $\lambda_{\max}^{(d)} = d\,\lambda_{\max}^{(1)}$ but also $D^{(d)} = 2d\,(n+1)^2\, I$ and thus the d cancel.

Jacobi Method

Jacobi's method applied to Poisson's equation (continued)

It follows from the above that the Jacobi iterates satisfy

$$\|x_k - x\| \le C \cos\left(\tfrac{\pi}{n+1}\right)^k = C\left(1 - \mathcal{O}(n^{-2})\right)^k$$

In particular, to guarantee $\|x_k - x\| \le \varepsilon$ we must choose

$$k = \frac{\log(\varepsilon/C)}{\log\left(\cos\left(\tfrac{\pi}{n+1}\right)\right)} = \mathcal{O}(n^2).$$

This yields the following runtimes.

	LU	Krylov	Jacobi
$d = 1$	$\mathcal{O}(N)$	$\mathcal{O}(N^2)$	$\mathcal{O}(N^3)$
$d = 2$	$\mathcal{O}(N^{3/2})$	$\mathcal{O}(N^{3/2})$	$\mathcal{O}(N^2)$
$d = 3$	$\mathcal{O}(N^2)$	$\mathcal{O}(N^{4/3})$	$\mathcal{O}(N^{5/3})$

Jacobi Method

Summary

- Jacobi's method is

$$x_k = D^{-1}\left(b - (A - D)\,x_{k-1}\right)$$

  with $D = \mathrm{diag}(A)$.
- Jacobi iterates satisfy

$$\|x_k - x\| \leq C\,|\hat{\lambda}_{\max}|^k$$

  where $\hat{\lambda}_{\max}$ is the eigenvalue of largest absolute value of $R = -D^{-1}\,(A - D)$.
- For $A = -\Delta_n^{(d)}$, we have $|\lambda_{\max}| = 1 - \mathcal{O}(n^2)$.