MA3227 Numerical Analysis II

Lab Session 3

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1 Jacobi and Gauss-Seidel methods

We have seen in Lecture 10 that the Jacobi and Gauss-Seidel methods solve a linear system $Ax = b$ by constructing a sequence x_k such that $x_k \to x$ for $k \to \infty$. These sequences are defined by the following recurrence relations where D denotes the diagonal of A, and L and U denote the strictly lower- and upper-triangular parts of A, respectively.

$$\begin{aligned} \text{Jacobi:} \quad & x_{k+1} = D^{-1}\big(b - (A-D)\,x_k\big), \\ \text{Gauss-Seidel:} \quad & x_{k+1} = (D+U)^{-1}\big(b - Lx_k\big). \end{aligned}$$

1. We have seen that the Jacobi iterates satisfy the error recursion

$$x_k - x = R^k\,(x_0 - x) \qquad \text{where} \qquad R = -D^{-1}\,(A-D).$$

   Derive the analogous error recursion for the Gauss-Seidel iteration.

2. Construct a matrix A such that the Jacobi iteration diverges. Verify numerically that $\|Ax_k - b\|$ diverges with the expected rate.

   Hint for finding A. Construct a matrix R with at least one eigenvalue larger than 1 and then find the matrix A which corresponds to this R.

   Hints for implementation.

   - Have a look at the functions `Diagonal()`, `triu()` and `tril()` from the `LinearAlgebra` package.
   - The next task will ask for similar numerical experiments. Try to structure your code such that it contains as little repetition as possible.

3. Consider the two matrices

$$A_1 = \begin{pmatrix} 1 & \frac{3}{4} & -\frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ 1 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & \frac{3}{4} & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ 1 & 0 & 1 \end{pmatrix}.$$

   Verify numerically that Jacobi converges but Gauss-Seidel diverges for A_1, but the converse is true for A_2.