MA3227 Numerical Analysis II

Lecture 9: Krylov Methods Applied to Poisson's Equation

Simon Etter

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Krylov Methods Applied to Poisson's Equation

Introduction

Our overarching goal in the first half of this module is to develop solvers for the finite-difference-discretised Poisson equation $-\Delta_n^{(d)} u_n = f$ which achieve optimal $\mathcal{O}(N) = \mathcal{O}(n^d)$ runtime and memory.

We have seen that LU factorisation achieves this goal only for $d = 1$.

This lecture addresses the following question:

Can Krylov methods achieve $\mathcal{O}(N)$ runtime and memory for any d?

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Preparing for the discussion

- $-\Delta_n^{(d)}$ is spd; hence we can use the conjugate gradient algorithm.
- Runtime of conjugate gradients is $\mathcal{O}(Nk)$. Memory is $\mathcal{O}(N)$.
- The eigenvalues of $-\Delta_n^{(d)}$ are asymptotically contained in

$$\mathcal{E} = \left[d\,\pi^2, 4\,d\,(n+1)^2\right].$$

We have seen in Lecture 2 that the eigenvalues of $\Delta_n^{(1)}$ are given by

$$\lambda_k = (n+1)^2\left(2\cos\left(\pi\tfrac{k}{n+1}\right) - 2\right) \in [-4\,(n+1)^2, -\pi^2 + \mathcal{O}(n^{-2})].$$

$\mathcal{E}$ for $\Delta_n^{(d)}$ follows from the Kronecker-product argument on page 8 of Lecture 3.

- If all eigenvalues of A are contained in $[1, \kappa]$, the conjugate gradient iterates x_k satisfy the bound

$$\|Ax_k - b\| \le C\,\rho^k \qquad \text{where} \qquad \rho = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}.$$

- We are usually interested in achieving a fixed error $\|Ax_k - b\| \le \varepsilon$; hence we should choose $k = \frac{\log(\varepsilon/C)}{\log(\rho)}$

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Putting things together

Combining the last three points yields

$$\kappa = \frac{\lambda_{\max}}{\lambda_{\min}} = \frac{4\,(n+1)^2}{\pi^2} = \mathcal{O}(n^2),$$
$$\rho = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1} = \frac{1-\kappa^{-1/2}}{1+\kappa^{-1/2}} = 1-\mathcal{O}(\kappa^{-1/2}) = 1-\mathcal{O}(n^{-1}),$$
$$k = \frac{\log(\varepsilon/C)}{\log(\rho)} = \mathcal{O}\Big(\log(\varepsilon)\,(1-\rho)^{-1}\Big) = \mathcal{O}\Big(\log(\varepsilon)\,n\Big).$$

Conclusion

To achieve a fixed error ε independent of n, we have to choose $k \propto n$.

Thus, the runtime of conjugate gradients is

$$\mathcal{O}(Nk) = \mathcal{O}(n^{d+1}) = \mathcal{O}(N^{(d+1)/d}).$$

This is not the desired $\mathcal{O}(N)$ scaling. How does it compare to LU?

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Discussion: LU vs. conjugate gradients

Conjugate gradients has optimal $\mathcal{O}(N)$ memory; hence it is as good as LU for $d = 1$, and better than LU for $d > 1$.

The situation is more complicated for the runtime:

	LU runtime	CG runtime
$d = 1$	$\mathcal{O}(N)$	$\mathcal{O}(N^2)$
$d = 2$	$\mathcal{O}(N^{3/2})$	$\mathcal{O}(N^{3/2})$
$d = 3$	$\mathcal{O}(N^2)$	$\mathcal{O}(N^{4/3})$

Outlook

The following slides demonstrate the convergence of Krylov methods when applied to the Poisson equation.

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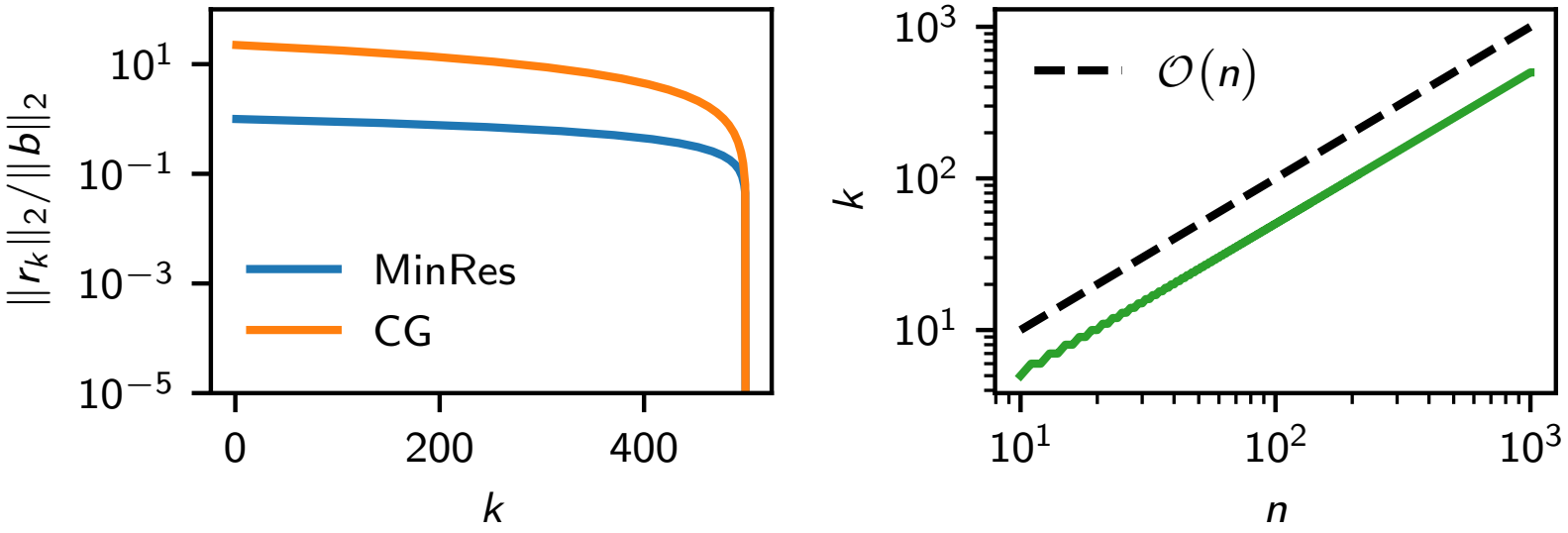
One-dimensional Poisson equation ($n = 1000$)



Left: convergence history of MinRes and conjugate gradients for $n = 1000$.
Right: number of CG iterations to achieve $\|Ax_k - b\|_2 \leq 10^{-8} \|b\|_2$.

As predicted, we get hardly any convergence for $k \ll n$ (left plot), and the number of iterations k required to achieve a fixed error is $\mathcal{O}(N)$ (right plot).

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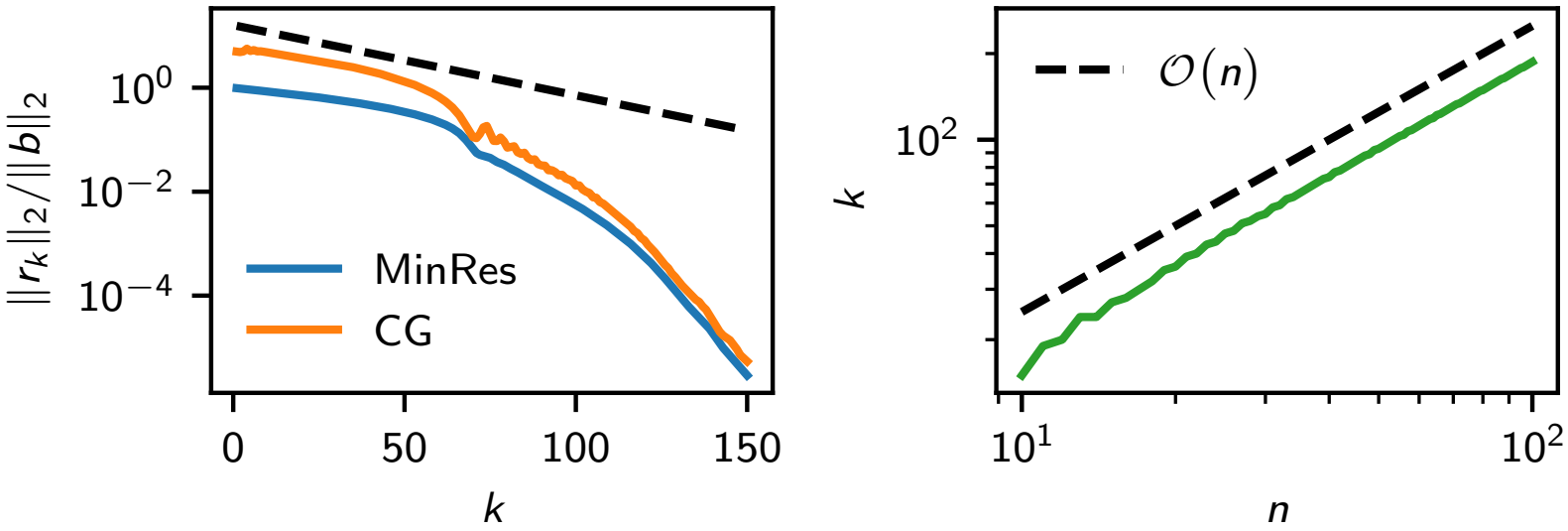
Two-dimensional Poisson equation



Left: convergence history of MinRes and conjugate gradients for $n = 100$.
Right: number of CG iterations to achieve $\|Ax_k - b\|_2 \leq 10^{-8} \|b\|_2$.

Again, convergence kicks in only for $k \approx n$ (left plot), and we observe $k = \mathcal{O}(n)$ (right plot).

Additionally, we observe that $\|Ax_k - b\|_2$ does not decay monotonically for conjugate gradients. This is expected since conjugate gradients minimises $\|x_k - x\|_A$ rather than $\|Ax_k - b\|_2$.

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Discussion

The above seems to suggest that Krylov methods are not very helpful.

This is partly because so far we have not made use of preconditioning: if $Ax = b$ is too hard to solve, then solve $P^{-1}Ax = P^{-1}b$ instead where P is some matrix which makes the problem easier.

The next slide demonstrates the effect of the ILU(τ) preconditioner introduced in Lecture 7. This preconditioner is given by $P = \tilde{L}\tilde{U}$ where $\tilde{L}, \tilde{U}$ are computed by applying LU factorisation to A but ignoring entries which are smaller than $\texttt{tol} \times 4\,(n+1)^2$.

$4\,(n+1)^2$ is the magnitude of the largest entry in $\Delta_n^{(2)}$.

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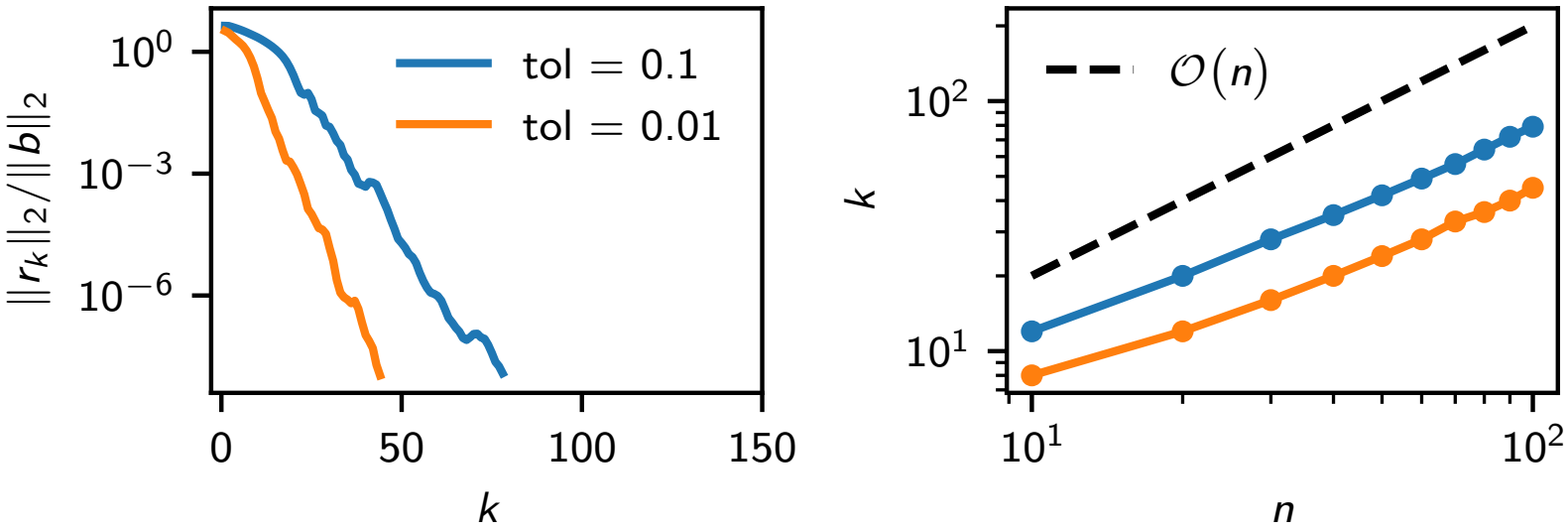
Two-dimensional Poisson equation with ILU preconditioning



Left: convergence history of conjugate gradients for $n = 100$.
Right: number of CG iterations to achieve $\|Ax_k - b\|_2 \leq 10^{-8} \|b\|_2$.

Unsurprisingly, the convergence improves for smaller `tol`. However, smaller `tol` also means that the preconditioner becomes more expensive to compute and apply.

Overall, we conclude that ILU preconditioning reduces the prefactors but does not change the $k = \mathcal{O}(n)$ scaling.

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Summary / Discussion

- Krylov methods almost always require a preconditioner to be effective.
- ILU preconditioning is a simple "black-box" preconditioner which may help reduce the prefactors, but typically it does not lead to asymptotically better runtimes.