

### Direct Methods for the solution of Linear Systems.

- 1 Given a matrix  $A \in \mathbb{R}^{n \times n}$  and the vector  $x_{true} = (1, 1, \dots, 1)^T \in \mathbb{R}^n$ , write a script that:
- Computes the right-hand side of the linear system  $b = Ax_{true}$ .
  - Computes the condition number in 2-norm of the matrix  $A$ . It is ill-conditioned? What if we use the  $\infty$ -norm instead of the 2-norm?
  - Solves the linear system  $Ax = b$  with the function `np.linalg.solve()`.
  - Computes the relative error between the solution computed before and the true solution  $x_{true}$ . Remember that the relative error between  $x_{true}$  and  $x$  in  $\mathbb{R}^n$  can be computed as

$$E(x_{true}, x) = \frac{\|x - x_{true}\|_2}{\|x_{true}\|_2}$$

- Plot a graph (using `matplotlib.pyplot`) with the relative errors as a function of  $n$  and (in a new window) the condition number in 2-norm  $K_2(A)$  and in  $\infty$ -norm, as a function of  $n$ .
- 2 Test the program above with the following choices of  $A \in \mathbb{R}^{n \times n}$ :
- A random matrix (created with the function `np.random.rand()`) with size varying with  $n = \{10, 20, 30, \dots, 100\}$ .
  - The Vandermonde matrix (`np.vander`) of dimension  $n = \{5, 10, 15, 20, 25, 30\}$  with respect to the vector  $x = \{1, 2, 3, \dots, n\}$ .
  - The Hilbert matrix (`scipy.linalg.hilbert`) of dimension  $n = \{4, 5, 6, \dots, 12\}$ .

### Floating Point Arithmetic.

- 1 The Machine epsilon  $\epsilon$  is the distance between 1 and the next floating point number. Compute  $\epsilon$ , which is defined as the smallest floating point number such that it holds:

$$fl(1 + \epsilon) > 1$$

*Tips:* use a `while` structure.

- 2 Let's consider the sequence  $a_n = (1 + \frac{1}{n})^n$ . It is well known that:

$$\lim_{n \rightarrow \infty} a_n = e$$

where  $e$  is the Euler constant. Choose different values for  $n$ , compute  $a_n$  and compare it to the real value of the Euler constant. What happens if you choose a large value of  $n$ ? Guess the reason.

3. Let's consider the matrices:

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}, B = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$$

Compute the rank of A and B and their eigenvalues. Are A and B full-rank matrices? Can you infer some relationship between the values of the eigenvalues and the full-rank condition? Please, corroborate your deduction with other examples.

*Tips:* Please, have a look at `np.linalg`.