1.
Oppenheim 课本，2.0 节 - 2.5 节

二.必做题（共1 题,33.3分）

1.
Oppenheim 课本，第二章课后习题，2.2，2.3，2.7，2.10，2.14，2.15，2.18

三.选做题（共1 题,33.4分）

1.
下列题目中任选至少 3 道完成
Oppenheim 课本，第二章课后习题，2.26，2.27，2.28，2.43，2.48，2.63

2.2. (a) The impulse response $h[n]$ of an LTI system is known to be zero, except in the interval $N_0 \le n \le N_1$. The input $x[n]$ is known to be zero, except in the interval $N_2 \le n \le N_3$. As a result, the output is constrained to be zero, except in some interval $N_4 \le n \le N_5$. Determine N_4 and N_5 in terms of N_0, N_1, N_2, and N_3.

(b) If $x[n]$ is zero, except for N consecutive points, and $h[n]$ is zero, except for M consecutive points, what is the maximum number of consecutive points for which $y[n]$ can be nonzero?

[解答]:

(a).

$h[n]$是LTI系统的冲激响应($Impulse\ Response$)，不妨记输出为$y[n]$，从而会有：

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

依据下面给出的图 1，我们可以得到N_4 N_5:

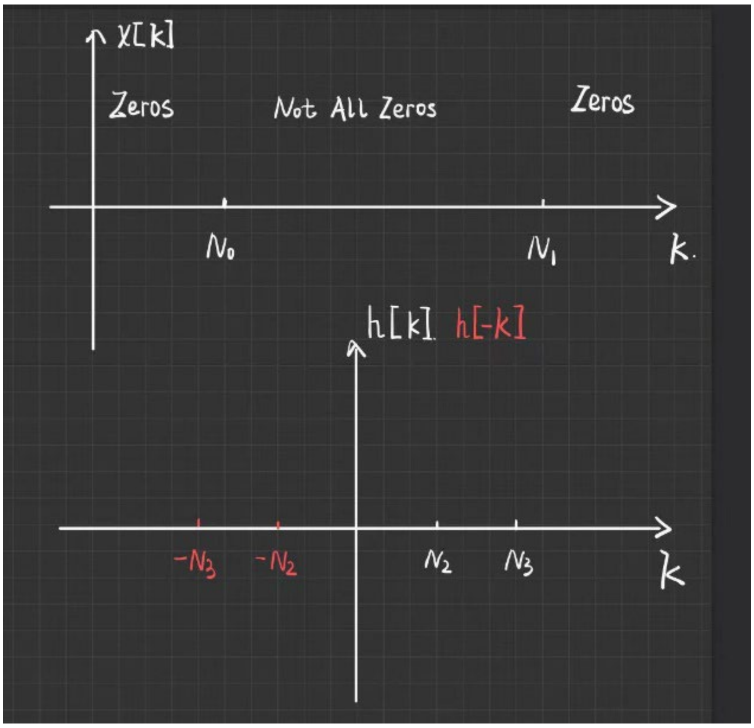

图 1 卷积示意图

$$N_4 = N_0 + N_2$$
$$N_5 = N_1 + N_3$$

(b).

$x[n]$是N点连续全非 0 序列，$h[n]$是M点连续全非 0 序列，我们可以利用(a)中的结论：

$$Length = N_5 - N_4 + 1 = (N_1 - N_0) + (N_3 - N_2) + 1$$
$$M = (N_1 - N_0) + 1$$
$$N = (N_3 - N_2) + 1$$
$$Length = (N_1 - N_0) + (N_3 - N_2) + 1 = (M-1) + (N-1) + 1 = M + N - 1$$

我们能得到的结果，最多最多有$(M + N - 1)$点是非 0 的

2.3. By direct evaluation of the convolution sum, determine the unit step response ($x[n] = u[n]$) of an LTI system whose impulse response is

$$h[n] = a^{-n}u[-n], \qquad 0 < a < 1.$$

[解答]:

已知目标系统为LTI系统，并且已知LTI系统本身的冲激响应($Impulse\ Response$)为:

$$h[n] = a^{-n}u[-n] \qquad 0 < a < 1$$

我们给入的输入为:

$$x[n] = u[n]$$

记$y[n]$为输出，从而会有:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

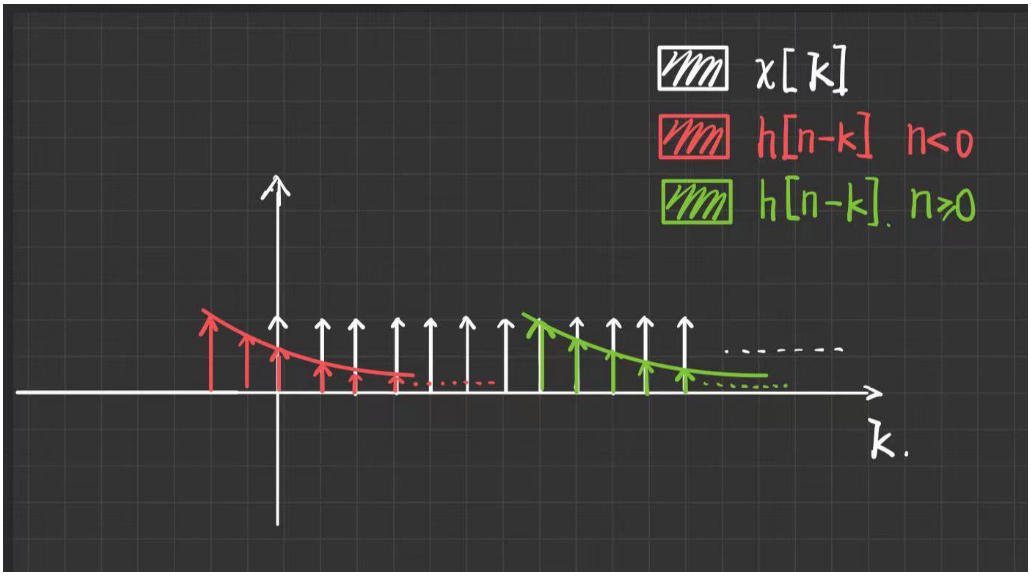

图 2 卷积示意图

根据图 2，我们可以判断出，这里需要分类讨论。因为卷积本质上是算两个函数之间的重合部分在离散轴上的累加(Sum)或者连续轴上的积分($Intergrate$)。红色情况$n < 0$很明显是有一部分缺失掉了，绿色是没有缺失的。所以红色部分会随着n变化而变化，绿色明显就不会。

$n \geq 0$ (绿色情况):

$$y[n] = x[n] * h[n] = \sum_{s=0}^{+\infty} a^s = \frac{1}{1-a}$$

$n < 0$ (红色情况):

$$y[n] = x[n] * h[n] = a^{-n}\sum_{k=0}^{+\infty} a^k = \frac{a^{-n}}{1-a}$$

2.7. Determine whether each of the following signals is periodic. If the signal is periodic, state its period.

(a) $x[n] = e^{j(\pi n/6)}$
(b) $x[n] = e^{j(3\pi n/4)}$
(c) $x[n] = [\sin(\pi n/5)]/(\pi n)$
(d) $x[n] = e^{j\pi n/\sqrt{2}}$.

[解答]:

(a) (b)是周期性信号，(c) (d)不是周期性信号

对于(a) $x[n] = e^{j\left(\frac{\pi n}{6}\right)}$来说有：

$$T_a = 12$$

对于(b) $x[n] = e^{j\left(\frac{3\pi n}{4}\right)}$来说有：

$$T_b = 8$$

2.10. Determine the output of an LTI system if the impulse response $h[n]$ and the input $x[n]$ are as follows:

(a) $x[n] = u[n]$ and $h[n] = a^n u[-n-1]$, with $a > 1$.
(b) $x[n] = u[n-4]$ and $h[n] = 2^n u[-n-1]$.
(c) $x[n] = u[n]$ and $h[n] = (0.5)2^n u[-n]$.
(d) $h[n] = 2^n u[-n-1]$ and $x[n] = u[n] - u[n-10]$.

Use your knowledge of linearity and time invariance to minimize the work in parts (b)–(d).

[解答]:

已知所有的系统全都是$LTI(Linear\ Time\ Invariant)$，记输出为$y[n]$

$$y[n] = x[n] * h[n]$$

题目的本意是，我们求出来(a)的结果，然后使用(a)的结果对(b) (c) (d)进行推导

(a)

$$x[n] = u[n]$$
$$h[n] = a^n u[-n-1] \qquad a > 1$$
$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} x[n-k]h[k]$$

经过画图得知我们需要分两段讨论：

$n \geq -1$时有：

$$y[n] = \sum_{k=-\infty}^{+\infty} x[n-k]h[k] = \sum_{k=-\infty}^{-1} a^k = \sum_{k=1}^{+\infty} \left(\frac{1}{a}\right)^k = \frac{\frac{1}{a}}{1-\frac{1}{a}} = \frac{1}{a-1}$$

$n < -1$时有：

$$y[n] = \sum_{k=-\infty}^{+\infty} x[n-k]h[k] = \sum_{k=-\infty}^{n} a^k = \sum_{k=-n}^{+\infty} \left(\frac{1}{a}\right)^k = a^n \frac{1}{1-\frac{1}{a}} = \frac{a^{n+1}}{a-1}$$

(b)

令$a=2$，并且输入为$x'[n]=x[n-4]=u[n-4]$

$$y'[n]=x'[n]*h[n]=\sum_{k=-\infty}^{+\infty}x'[k]h[n-k]$$

$$=\sum_{k=-\infty}^{+\infty}x[k-4]h[n-k]\overset{k'=k-4}{=}\sum_{k'=-\infty}^{+\infty}x[k']h[n-k'-4]$$

$$y'[n]=y[n-4]$$

$$y[n]|_{a=4}=\begin{cases}1 & n\geq -1\\ 2^{n+1} & n<-1\end{cases}$$

$$y'[n]|_{a=4}=\begin{cases}1 & n\geq 3\\ 2^{n-3} & n<3\end{cases}$$

(c)

这里令$a=2$

$$h'^{[n]}=(0.5)2^n u[-n]=2^{n-1}u[-n]=2^{n-1}u[-(n-1)-1]=h[n-1]$$

$$y''[n]=y[n-1]$$

$$y''[n]=\begin{cases}1 & n\geq 0\\ 2^n & n<0\end{cases}$$

(d)

令$a=2$

$$x'''[n]=u[n]-u[n-10]=x[n]-x[n-10]$$

$$h[n]=a^n u[-n-1]=2^n u[-n-1]$$

$$y'''[n]=h[n]*x'''[n]=y[n]-y[n-10]$$

$$y'''[n]=\begin{cases}2^{n+1}-2^{n-9} & n\leq -2\\ 1-2^{n-9} & -1\leq n\leq 8\\ 0 & n\geq 9\end{cases}$$

2.14. A single input–output relationship is given for each of the following three systems:

(a) System A: $x[n] = (1/3)^n$, $y[n] = 2(1/3)^n$.

(b) System B: $x[n] = (1/2)^n$, $y[n] = (1/4)^n$.

(c) System C: $x[n] = (2/3)^n u[n]$, $y[n] = 4(2/3)^n u[n] - 3(1/2)^n u[n]$.

Based on this information, pick the strongest possible conclusion that you can make about each system from the following list of statements:

(i) The system cannot possibly be LTI.

(ii) The system must be LTI.

(iii) The system can be LTI, and there is only one LTI system that satisfies this input–output constraint.

(iv) The system can be LTI, but cannot be uniquely determined from the information in this input–output constraint.

If you chose option (iii) from this list, specify either the impulse response $h[n]$ or the frequency response $H(e^{j\omega})$ for the LTI system.

[解答]:

这道题是很经典的题目，它涉及LTI系统的一个重要知识点：系统的本征函数$(Eigenfunction)$

我们考虑现在有$x[n] = t^n$这样的输入，那么根据LTI的性质有：

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[n-k]h[k] = \sum_{k=-\infty}^{+\infty} t^{n-k}h[k] = t^n \sum_{k=-\infty}^{+\infty} t^{-k}h[k]$$

对于一个固定的输入$x[n] = t^n$，和一个我们研究的特定、已知单位冲激响应$(Impulse\ Response)$的LTI系统有：

$$\sum_{k=-\infty}^{+\infty} t^{-k}h[k] = Constant = C$$

请注意，上面我们默认是可以收敛的，如果不能收敛的话，这里没什么研究的意义和必要性了。

那么会有：

$$y[n] = Ct^n$$

这本质上其实是LTI系统对于指数函数输入的输出特性。

(a)

$$x[n] = \left(\frac{1}{3}\right)^n \qquad y[n] = 2\left(\frac{1}{3}\right)^n$$

这里我们有很多手段去取$h[n]$，使得有：

$$\sum_{k=-\infty}^{+\infty} t^{-k}h[k] = Constant = C = 2$$

从而这个情况应该是(IV)，并且值得注意的是，这个系统不一定会是LTI的，我可以很轻松做到这一点，对任何输入都输出$y[n] = 2\left(\frac{1}{3}\right)^n$，这显然$LTI$，所以(Ⅱ)不会选。

最后补充的是，非线性情况占我们实际遇到的系统的主要，所以(Ⅱ)这个选项就不太可能。

(b)

显然这不可能是LTI系统，从而这个情况对应(Ⅰ)

(c)

$$x[n] = \left(\frac{2}{3}\right)^n u[n] \qquad y[n] = 4\left(\frac{2}{3}\right)^n u[n] - 3\left(\frac{1}{2}\right)^n u[n]$$

这里请大家注意了！我们上面推导的针对的是$x[n] = t^n$这个情况，这里$x[n]$形如$t^n u[n]$并不在我们推导的讨论的范畴内，所以$y[n]$中产生$\left(\frac{1}{2}\right)^n u[n]$这样的结果不奇怪。

我们能做的：假定这是个LTI，尝试是否可以推导得出$h[n]$

$$y[n] = h[n] * x[n]$$

直接在离散时域上做不好做，我们考虑频域上做了反变换回来：

$$DTFT\{y[n]\} = DTFT\{h[n] * x[n]\} = DTFT\{h[n]\}DTFT\{x[n]\}$$

对给定信号$x[n]$有：

$$DTFT\{x[n]\} = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} = X\left(e^{j\omega}\right)$$

所以有：

$$H\left(e^{j\omega}\right) = \frac{Y\left(e^{j\omega}\right)}{X(e^{j\omega})} = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

这是经典的反变换，对于$x[n] = a^n u[n] \qquad 0 < a < 1$

$$DTFT\{x[n]\} = \sum_{n=-\infty}^{+\infty} a^n u[n]e^{-j\omega n} = \sum_{n=0}^{+\infty} a^n e^{-j\omega n} = \sum_{n=0}^{+\infty}\left(ae^{-j\omega}\right)^n = \frac{1}{1 - ae^{-j\omega}}$$

所以有：

$$H\left(e^{j\omega}\right) = \left(\frac{1}{2}\right)^n u[n]$$

所以选(Ⅲ)如果是LTI那么可以唯一确定

2.15. Consider the system illustrated in Figure P2.15. The output of an LTI system with an impulse response $h[n]=\left(\frac{1}{4}\right)^n u[n+10]$ is multiplied by a unit step function $u[n]$ to yield the output of the overall system. Answer each of the following questions, and briefly justify your answers:

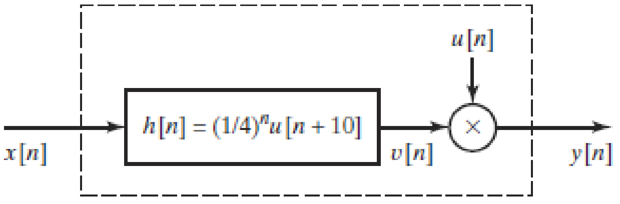

Figure P2.15

(a) Is the overall system LTI?
(b) Is the overall system causal?
(c) Is the overall system stable in the BIBO sense?

[解答]:

(a).

验证LTI，其实比较直接简单的方法是：1.验证线性性质($Linear$)　2.验证时不变性质($Time\ Invariant$)

考虑输入为$x[n]$

1.先验证线性性质

纯量乘法：

$$v[n]=x[n]*h[n]$$
$$y[n]=u[n]v[n]$$
$$v'[n]=\{\alpha x[n]\}*h[n]=\alpha v[n]$$
$$y'[n]=v'[n]u[n]=\alpha v[n]u[n]=\alpha y[n]$$

线性叠加：

$$v_1[n]=x_1[n]*h[n]\qquad v_2[n]=x_2[n]*h[n]$$
$$y_1[n]=v_1[n]u[n]\qquad y_2[n]=v_2[n]u[n]$$
$$v'[n]=\{x_1[n]+x_2[n]\}*h[n]=v_1[n]+v_2[n]$$
$$y'[n]=v'[n]u[n]=(v_1[n]+v_2[n])u[n]=y_1[n]+y_2[n]$$

这确实是一个线性系统

2.再验证时不变性质

如果一个系统是时不变的那么会有：

$$y[n]=System\{x[n]\}$$
$$y'[n]=System\{x[n-n_d]\}=y[n-n_d]$$

考虑输入为$x[n]$

$$y[n]=v[n]u[n]$$

输入经过延迟为$x[n-n_d]$

$$y'[n]=v'[n]u[n]=v[n-n_d]u[n]\neq y[n-n_d]$$

本质上是$u[n]$处只是乘法运算，所以$u[n]$中没有延迟项，导致不一定结果延迟。

这一定不是个LTI系统。

(b)

考虑我们输入为$x[n] = \delta[n]$，延时 1 点，有$x[n] = \delta[n-1]$

$$v[n] = h[n-1] = \left(\frac{1}{4}\right)^{n-1} u[n+9]$$

$$y[n] = v[n]u[n] = \left(\frac{1}{4}\right)^{n-1} u[n]$$

$n < 1$时$(n = 0)$会有$x[n] = 0 \quad y[n] \neq 0$

(c)

考虑一个有界输入$x[n]$

$$|x[n]| \leq M_x \qquad \forall n \in N$$

$$v[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

$$|v[n]| = \left|\sum_{k=-\infty}^{+\infty} h[k]x[n-k]\right| \leq \sum_{k=-\infty}^{+\infty} |h[k]x[n-k]| \leq \sum_{k=-\infty}^{+\infty} |h[k]||x[n-k]| \leq M_x \sum_{k=-\infty}^{+\infty} |h[k]|$$

经过判断，我们可以得知，$\sum_{k=-\infty}^{+\infty} |h[k]|$本身有界，也就是说$h[n] = \left(\frac{1}{4}\right)^{n} u[n+10]$本身是一个$BIBO$稳定系统。

又有：

$$y[n] = v[n]u[n]$$

$$|y[n]| = |v[n]u[n]| \leq |v[n]| \leq M_v = M_x M_h$$

所以这个系统一定$BIBO$稳定

2.18. For each of the following impulse responses of LTI systems, indicate whether or not the system is causal:

(a) $h[n] = (1/2)^n u[n]$
(b) $h[n] = (1/2)^n u[n-1]$
(c) $h[n] = (1/2)^{|n|}$
(d) $h[n] = u[n+2] - u[n-2]$
(e) $h[n] = (1/3)^n u[n] + 3^n u[-n-1]$.

[解答]:
如果我们要判断一个LTI系统是不是因果的:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

$y[n]$只能和$x[n]$ $x[n-1]$ $x[n-2]$ ……………有关
从而会有:

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k] = \sum_{k=0}^{+\infty} h[k]x[n-k]$$

所以有:

$$h[n] = 0 \qquad \forall n < 0$$

也就是:

$$h[n] = h[n]u[n]$$

(a)是　(b)是　(c)不是　(d)不是　(e)不是

2.26. For each of the systems in Figure P2.26, pick the strongest valid conclusion that you can make about each system from the following list of statements:

(i) The system must be LTI and is uniquely specified by the information given.
(ii) The system must be LTI, but cannot be uniquely determined from the information given.
(iii) The system could be LTI and if it is, the information given uniquely specifies the system.
(iv) The system could be LTI, but cannot be uniquely determined from the information given.
(v) The system could not possibly be LTI.

For each system for which you choose option (i) or (iii), give the impulse response $h[n]$ for the uniquely specified LTI system. One example of an input and its corresponding output are shown for each system.

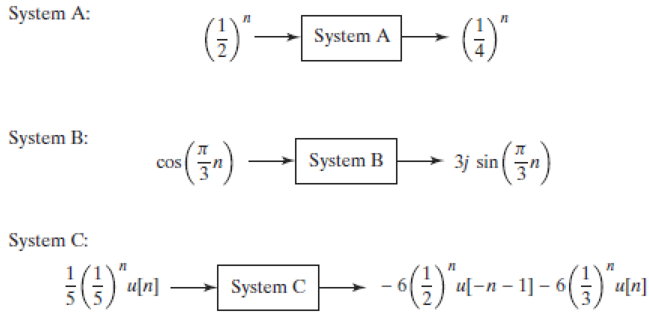

Figure P2.26

[解答]:

细心的读者看了我 2.14 的解答，其实就能会心一笑，这个题 System A 和那个题的前两问没有区别，一定不是LTI；System C 和那个题的最后一问没有区别，我们无法直接判断，出现什么样的结果都不奇怪，因为引入了$u[n]$；那么这个题现在主要问题会在 System B 上面，下面我们推导一下LTI针对复指数信号(形如$\exp(j\omega)$)的作用：

考虑现在我们对于一个单位冲激响应为$h[n]$的LTI系统给入的输入为$x[n]=e^{j\omega_{in}n}$

$$y[n]=h[n]*x[n]=\sum_{k=-\infty}^{+\infty}h[k]e^{j\omega_i(n-k)}=e^{j\omega_i n}\sum_{k=-\infty}^{+\infty}h[k]e^{-j\omega_i k}=e^{j\omega_i n}H\left(e^{j\omega}\right)\big|_{\omega=\omega_i}$$
$$=e^{j\omega_i n}H\left(e^{j\omega_i}\right)$$

所以很显然，如果我们给一个复指数型的输入$x[n]=e^{j\omega_i n}$进入LTI系统，输出形式为:

$$y[n]=x[n]H\left(e^{j\omega_i}\right)$$

只是在输入的基础上做纯量乘法。

欧拉公式告诉我们，三角和复指数没有本质区别:

$$e^{ix}=\cos(x)+i\sin(x)$$

所以直接回到这题的 System B:

$$x[n]=\cos\left(\frac{\pi}{3}n\right)=\frac{e^{j\frac{\pi}{3}n}+e^{-j\frac{\pi}{3}n}}{2}$$
$$y[n]=3j\sin\left(\frac{\pi}{3}n\right)=\frac{3j}{2j}\left(e^{j\frac{\pi}{3}n}-e^{-j\frac{\pi}{3}n}\right)=\frac{3}{2}\left(e^{j\frac{\pi}{3}n}-e^{-j\frac{\pi}{3}n}\right)$$

也就是$H\left(e^{j\frac{\pi}{3}}\right)=3$　　$H\left(e^{j\left(-\frac{\pi}{3}\right)}\right)=-3$，$\left|H\left(e^{j\frac{\pi}{3}}\right)\right|=\left|H\left(e^{j\left(-\frac{\pi}{3}\right)}\right)\right|$，这合情合理

但是我们就知道两个点是不够的，所以这个系统不唯一确定。

最后 3 个系统结果请大家自行判断，我已经将推论给出了，大家要做必要的练习。

2.27. For each of the systems in Figure P2.27, pick the strongest valid conclusion that you can make about each system from the following list of statements:

(i) The system must be LTI and is uniquely specified by the information given.
(ii) The system must be LTI, but cannot be uniquely determined from the information given.
(iii) The system could be LTI, and if it is, the information given uniquely specifies the system.
(iv) The system could be LTI, but cannot be uniquely determined from the information given.
(v) The system could not possibly be LTI.

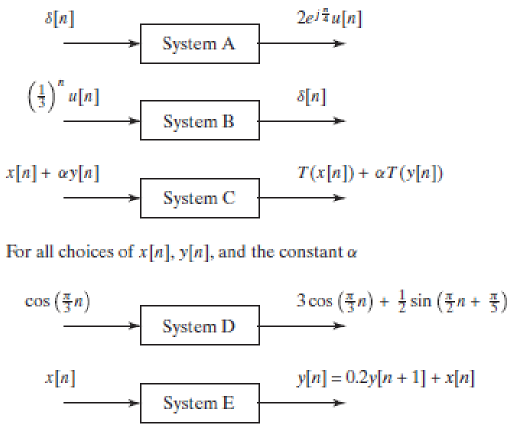

Figure P2.27

[解答]:

System A：可能是，并且它把单位冲激响应都给我们了，(III)

System B：输入为$x[n] = \left(\frac{1}{3}\right)^n u[n]$所以输出不必一定为形如$y[n] = C\left(\frac{1}{3}\right)^n$.

如果 System B 是LTI系统：

$$y[n] = h[n] * x[n]$$

$$DTFT\{y[n]\} = DTFT\{h[n] * x[n]\} = DTFT\{h[n]\}DTFT\{x[n]\}$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = 1$$

$$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{3}e^{-j\omega}}$$

$$H(e^{j\omega}) = 1 - \frac{1}{3}e^{-j\omega}$$

$$h[n] = \delta[n] - \frac{1}{3}\delta[n-1]$$

所以可能是，并且单位冲激响应唯一确定，(III)

System C：(III)

System D：显然不会是，因为出现新的频率了，(V)

System E：(IV) 请大家看 Oppenheimer 书 P27 对于使用差分方程刻画LTI系统的描述，我们需要给出一个初始松弛条件。

并且 0 输入情况下，需要系统 0 输出，也就是ZIR（$Zero\ Input\ Response$）为 0，否则不满足线性。

2.28. Four input–output pairs of a particular system S are specified in Figure P2.28-1:

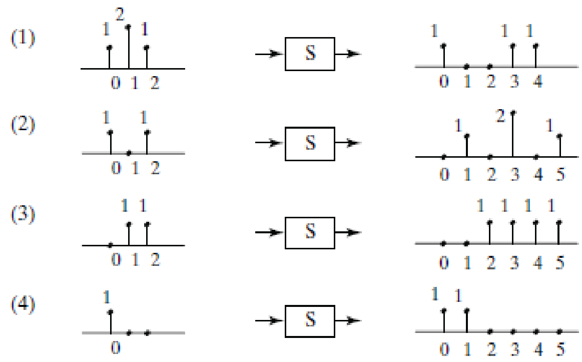

Figure P2.28–1

(a) Can system S be time-invariant? Explain.
(b) Can system S be linear? Explain.
(c) Suppose (2) and (3) are input–output pairs of a particular system S_2, and the system is known to be LTI. What is $h[n]$, the impulse response of the system?
(d) Suppose (1) is the input–output pair of an LTI system S_3. What is the output of this system for the input in Figure P2.28-2:

Figure P2.28–2

[解答]:

(a)

显然时变，结合(3)　(4)看即可

（3）的$n = 1$处不应该是 0

(b)

先验证是否线性，如果线性就需要满足保纯量乘法和加法:

$$2(x_3[n] + x_4[n]) = x_1[n] + x_2[n]$$
$$2(y_3[n] + y_4[n]) \neq y_1[n] + y_2[n]$$

非线性系统

(c)

不妨设$\delta[n]$的输出为$y[n]$

那么会有:

$$y_2[n] = y[n] + y[n-2]$$
$$y_3[n] = y[n-1] + y[n-2]$$
$$y[n] = h[n] = \delta[n-1] + \delta[n-3]$$

(d)

2.28.2 输入为$x_5[n] = x_1[n] + x[n-2]$

$$y_5[n] = y_1[n] + y_1[n-2] = \delta[n] + \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5] + \delta[n-6]$$