二.必做题（共1 题,33.3分）

1.
Oppenheim 课本，第二章课后习题，2.17，2.33，2.36，2.38，2.45，2.46
Oppenheim 课本，第三章课后习题，3.3，3.6，3.9，3.13，3.21，3.25

答案：

答案解析：

难度： 易

知识点：

三.选做题（共1 题,33.4分）

1.
下列题目中任选至少 3 道完成
Oppenheim 课本，第二章课后习题，2.56，2.60，2.65，2.84，2.85，2.86
下列题目中任选至少 3 道完成
Oppenheim 课本，第三章课后习题，3.22，3.42，3.49，3.52，3.56，3.57

2.17. **(a)** Determine the Fourier transform of the sequence

$$r[n] = \begin{cases} 1, & 0 \le n \le M, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Consider the sequence

$$w[n] = \begin{cases} \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{M}\right)\right], & 0 \le n \le M, \\ 0, & \text{otherwise.} \end{cases}$$

Sketch $w[n]$ and express $W(e^{j\omega})$, the Fourier transform of $w[n]$, in terms of $R(e^{j\omega})$, the Fourier transform of $r[n]$. (*Hint*: First express $w[n]$ in terms of $r[n]$ and the complex exponentials $e^{j(2\pi n/M)}$ and $e^{-j(2\pi n/M)}$.)

(c) Sketch the magnitude of $R(e^{j\omega})$ and $W(e^{j\omega})$ for the case when $M = 4$.

[解]

(a).

$$r[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & otherwise \end{cases}$$

$$DTFT\{r[n]\} = \sum_{n=-\infty}^{+\infty} r[n]e^{-j\omega n} = \sum_{n=0}^{M} e^{-j\omega n} = \frac{1 - e^{-j\omega(M+1)}}{1 - e^{-j\omega}}$$

$$= \frac{\left(e^{+j\frac{\omega}{2}(M+1)} - e^{-j\frac{\omega}{2}(M+1)}\right)e^{-j\frac{\omega}{2}(M+1)}}{\left(e^{+j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)e^{-j\frac{\omega}{2}}} = e^{-j\frac{\omega}{2}M}\frac{\sin\left(\frac{\omega}{2}(M+1)\right)}{\sin\left(\frac{\omega}{2}\right)}$$

(b).

根据题目给出的 Hint，我们了解需要从(a)中找寻思路：

$$w[n] = \begin{cases} \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{M}\right)\right] & 0 \le n \le M \\ 0 & otherwise \end{cases}$$

$$DTFT\{w[n]\} = \frac{1}{2}DTFT\{r[n]\} - \frac{1}{2}\sum_{n=-\infty}^{+\infty}\cos\left(\frac{2\pi n}{M}\right)e^{-j\omega n}$$

$$\cos\left(\frac{2\pi n}{M}\right) = \frac{1}{2}\left(e^{+j\frac{2\pi}{M}n} + e^{-j\frac{2\pi}{M}n}\right)$$

这里记$R(e^{j\omega}) = DTFT\{r[n]\}$　　$DTFT\{w[n]\} = W(e^{j\omega})$

$$\frac{1}{2}\sum_{n=-\infty}^{+\infty}\cos\left(\frac{2\pi n}{M}\right)e^{-j\omega n} = \frac{1}{4}R\left(e^{j\left(\omega-\frac{2\pi}{M}\right)}\right) + \frac{1}{4}R\left(e^{j\left(\omega+\frac{2\pi}{M}\right)}\right)$$

$$W(e^{j\omega}) = \frac{1}{2}R(e^{j\omega}) - \frac{1}{4}R\left(e^{j\left(\omega-\frac{2\pi}{M}\right)}\right) - \frac{1}{4}R\left(e^{j\left(\omega+\frac{2\pi}{M}\right)}\right)$$

其中$R(e^{j\omega}) = e^{-j\frac{\omega}{2}M}\frac{\sin\left(\frac{\omega}{2}(M+1)\right)}{\sin\left(\frac{\omega}{2}\right)}$

无论是考试还是作业，表达到这个形式就很够了，不需要化简。

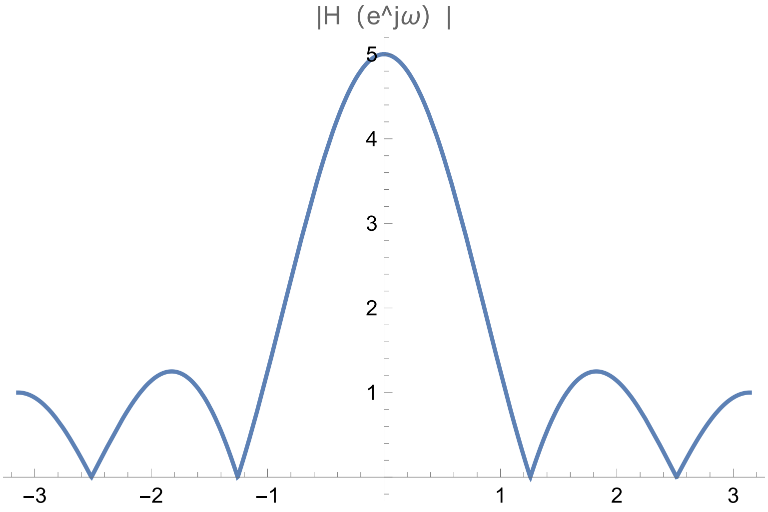
(c).

$$M = 4$$

我们只需要做幅频特性的图即可(Magnitude)

$$R\left(e^{j\omega}\right)=e^{-j\frac{\omega}{2}M}\frac{\sin\left(\frac{\omega}{2}(M+1)\right)}{\sin\left(\frac{\omega}{2}\right)}=\left|R\left(e^{j\omega}\right)\right|e^{+j\angle R\left(e^{j\omega}\right)}=\left|\frac{\sin\left(\frac{\omega}{2}(M+1)\right)}{\sin\left(\frac{\omega}{2}\right)}\right|e^{+j\left(-\frac{\omega}{2}M\right)}$$

$$R\left(e^{j\omega}\right)=\left|\frac{\sin\left(\frac{5\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}\right|e^{+j(-2\omega)}$$



$$W\left(e^{j\omega}\right)=W\left(e^{j\omega}\right)=\frac{1}{2}R\left(e^{j\omega}\right)-\frac{1}{4}R\left(e^{j\left(\omega-\frac{\pi}{2}\right)}\right)-\frac{1}{4}R\left(e^{j\left(\omega+\frac{\pi}{2}\right)}\right)$$

频域左右平移$\frac{\pi}{2}$后取幅频特性即可，图省略。

2.33. Consider an LTI system defined by the difference equation

$$y[n] = -2x[n] + 4x[n-1] - 2x[n-2].$$

(a) Determine the impulse response of this system.

(b) Determine the frequency response of this system. Express your answer in the form

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\omega n_d},$$

where $A(e^{j\omega})$ is a real function of ω. Explicitly specify $A(e^{j\omega})$ and the delay n_d of this system.

(c) Sketch a plot of the magnitude $|H(e^{j\omega})|$ and a plot of the phase $\angle H(e^{j\omega})$.

(d) Suppose that the input to the system is

$$x_1[n] = 1 + e^{j0.5\pi n} \qquad -\infty < n < \infty.$$

Use the frequency response function to determine the corresponding output $y_1[n]$.

(e) Now suppose that the input to the system is

$$x_2[n] = (1 + e^{j0.5\pi n})u[n] \qquad -\infty < n < \infty.$$

Use the defining difference equation or discrete convolution to determine the corresponding output $y_2[n]$ for $-\infty < n < \infty$. Compare $y_1[n]$ and $y_2[n]$. They should be equal for certain values of n. Over what range of values of n are they equal?

[解]:

(a).

这里我们需要求取所谓的"impulse response"，根据定义我们只需要令：

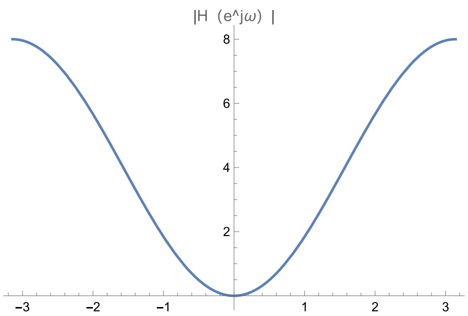
$$x[n] = \delta[n]$$
$$y[n] = -2\delta[n] + 4\delta[n-1] - 2\delta[n-2]$$
$$h[n] = y[n]|_{x[n]=\delta[n]} = -2\delta[n] + 4\delta[n-1] - 2\delta[n-2]$$

(b).

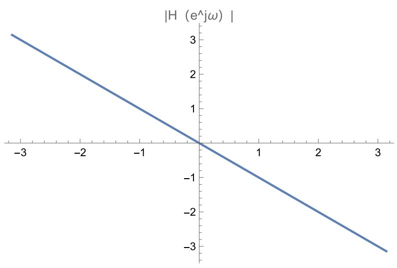
$$H(e^{j\omega}) = DTFT\{-2\delta[n] + 4\delta[n-1] - 2\delta[n-2]\} = -2 + 4e^{-j\omega} - 2e^{-j2\omega}$$
$$= 2e^{-j\omega}[2 - (e^{+j\omega} + e^{-j\omega})] = 2e^{-j\omega}(2 - 2cos\omega) = 4e^{-j\omega}(1 - cos\omega)$$
$$H(e^{j\omega}) = 4(1 - cos\omega)e^{-j\omega n_d} = 4(1 - cos\omega)e^{-j\omega}$$

(c).

$$|H(e^{j\omega})| = |4(1 - cos\omega)| = 4(1 - cos\omega)$$



$$\angle H(e^{j\omega}) = -\omega$$



(d).

$$x_1[n] = 1 + e^{+j\frac{\pi}{2}n}$$

$$DTFT\{x_1[n]\} = X_1(e^{j\omega}) = 2\pi\delta(\omega) + 2\pi\delta\left(\omega - \frac{\pi}{2}\right)$$

$$Y_1\left(e^{j\omega}\right) = X_1\left(e^{j\omega}\right)H\left(e^{j\omega}\right) = 4(1 - cos\omega)e^{-j\omega}2\pi\delta(\omega) + 4(1 - cos\omega)e^{-j\omega}2\pi\delta\left(\omega - \frac{\pi}{2}\right)$$

$$Y_1(j\omega) = 0 + 2\pi * 4\left(1 - \cos\left(\frac{\pi}{2}\right)\right)e^{-j\frac{\pi}{2}}\delta\left(\omega - \frac{\pi}{2}\right) = 2\pi * \left(4e^{-j\frac{\pi}{2}}\right)\delta\left(\omega - \frac{\pi}{2}\right)$$

$$y_1[n] = -4j\ e^{+j\frac{\pi}{2}n}$$

(e).

$$x_2[n] = \left(1 + e^{+j\frac{\pi}{2}n}\right)u[n]$$

$$y_2[n] = -2x_2[n] + 4x_2[n-1] - 2x_2[n-2]$$

$$x_2[n-1] = \left(1 + e^{+j\frac{\pi}{2}(n-1)}\right)u[n-1] = \left(1 - je^{+j\frac{\pi}{2}n}\right)u[n-1]$$

$$x_@[n-2] = \left(1 + e^{+j\frac{\pi}{2}(n-2)}\right)u[n-2] = \left(1 - e^{+j\frac{\pi}{2}n}\right)u[n-2]$$

$$y_2[n] = -2\left(1 + e^{+j\frac{\pi}{2}n}\right)u[n] + 4\left(1 - je^{+j\frac{\pi}{2}n}\right)u[n-1] - 2\left(1 - e^{+j\frac{\pi}{2}n}\right)u[n-2]$$

$n \geq 2$时有

$$y_2[n] = y_1[n]$$

2.36. An LTI discrete-time system has frequency response given by

$$H(e^{j\omega}) = \frac{(1 - je^{-j\omega})(1 + je^{-j\omega})}{1 - 0.8e^{-j\omega}} = \frac{1 + e^{-j2\omega}}{1 - 0.8e^{-j\omega}} = \frac{1}{1 - 0.8e^{-j\omega}} + \frac{e^{-j2\omega}}{1 - 0.8e^{-j\omega}}.$$

(a) Use one of the above forms of the frequency response to obtain an equation for the impulse response $h[n]$ of the system.

(b) From the frequency response, determine the difference equation that is satisfied by the input $x[n]$ and the output $y[n]$ of the system.

(c) If the input to this system is

$$x[n] = 4 + 2\cos(\omega_0 n) \quad \text{for} - \infty < n < \infty,$$

for what value of ω_0 will the output be of the form

$$y[n] = A = \text{constant}$$

for $-\infty < n < \infty$? What is the constant A?

[解]:

我们考虑经典变换，形如$x[n] = a^n u[n] \quad |a| < 1$

$$DTFT\{x[n]\} = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{+\infty} a^n e^{-j\omega n} = \sum_{n=0}^{+\infty} (ae^{-j\omega})^n = \lim_{N\to+\infty} \frac{1 - (ae^{-j\omega})^{N+1}}{1 - ae^{-j\omega}}$$

$$= \frac{1}{1 - ae^{-j\omega}}$$

(a).

显然，这里$a = 0.8$

$$IDTFT\left\{\frac{1}{1 - 0.8e^{-j\omega}}\right\} = h_1[n] = (0.8)^n u[n]$$

$$IDTFT\left\{\frac{e^{-j2\omega}}{1 - 0.8e^{-j\omega}}\right\} = h_1[n-2] = h_2[n] = (0.8)^{n-2} u[n-2]$$

$$h[n] = h_1[n] + h_2[n] = (0.8)^n u[n] + (0.8)^{n-2} u[n-2]$$

(b).

$$h[n] = (0.8)^n u[n] + (0.8)^{n-2} u[n-2]$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 + e^{-j2\omega}}{1 - 0.8e^{-j\omega}}$$

$$X(e^{j\omega})(1 + e^{-j2\omega}) = Y(e^{j\omega})(1 - 0.8e^{-j\omega})$$

形如$X(e^{j\omega})e^{-j2\omega}$,我们可以作$IDTFT$:

$$IDTFT\{X(e^{j\omega})e^{-j2\omega}\} = x[n-2]$$

所以有差分方程为:

$$x[n] + x[n-2] = y[n] - 0.8y[n-1]$$

保险起见，最后移项:

$$y[n] = 0.8y[n-1] + x[n] + x[n-2]$$

(c).

$$x[n] = 4 + 2\cos(\omega_0 n) = 4\cos(\omega n)|_{\omega=0} + 2\cos(\omega_0 n)$$

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{+j\angle H(e^{j\omega})}$$

问题其实就是形如$\cos(\omega_0 n)$输入$LTI\ System$，输出会是如何

$$\cos(\omega_0 n)=\frac{1}{2}\left(e^{+j\omega_0 n}+e^{-j\omega_0 n}\right)$$

先给出我们之前推到过的结论:

考虑现在我们对于一个单位冲激响应为$h[n]$的LTI系统给入的输入为$x[n]=e^{j\omega_{in}n}$

$$y[n]=h[n]*x[n]=\sum_{k=-\infty}^{+\infty}h[k]e^{j\omega_i(n-k)}=e^{j\omega_i n}\sum_{k=-\infty}^{+\infty}h[k]e^{-j\omega_i k}=e^{j\omega_i n}H\left(e^{j\omega}\right)\big|_{\omega=\omega_i}$$
$$=e^{j\omega_i n}H\left(e^{j\omega_i}\right)$$

$$x[n]=\cos(\omega_0 n)=\frac{1}{2}\left(e^{+j\omega_0 n}+e^{-j\omega_0 n}\right)$$

$$y[n]=\frac{1}{2}e^{+j\omega_0 n}H\left(e^{+j\omega_0}\right)+\frac{1}{2}e^{-j\omega_0 n}H\left(e^{-j\omega_0}\right)$$

考虑这是一个实系统:

$$H\left(e^{j\omega}\right)=\left|H\left(e^{j\omega}\right)\right|e^{+j\angle H\left(e^{j\omega}\right)}$$

那么会有（我们习题课讲过这个推论）:

$$\left|H\left(e^{j\omega}\right)\right|=\left|H\left(e^{j(-\omega)}\right)\right|$$

$$\angle H\left(e^{j\omega}\right)=-\angle H\left(e^{j(-\omega)}\right)$$

所以会有:

$$y[n]=\frac{1}{2}\left|H\left(e^{+j\omega_0}\right)\right|e^{+j\left[\omega_0 n\angle H\left(e^{j\omega_0}\right)\right]}+\frac{1}{2}\left|H\left(e^{+j\omega_0}\right)\right|e^{-j\left[\omega_0 n+\angle H\left(e^{j\omega_0}\right)\right]}$$

$$=\left|H\left(e^{+j\omega_0}\right)\right|\cos\left(\omega_0 n+\angle H\left(e^{j\omega}\right)\right)$$

所以本题结果为:

$$y[n]=4\left|H\left(e^{+j0}\right)\right|+2\left|H\left(e^{+j\omega_0}\right)\right|\cos\left(\omega_0 n+\angle H\left(e^{j\omega}\right)\right)=constant=A$$

$$\left|H\left(e^{+j\omega_0}\right)\right|=0$$

$$\left|H\left(e^{+j\omega_0}\right)\right|=\left|\frac{1+e^{-j2\omega_0}}{1-0.8e^{-j\omega_0}}\right|=0$$

$$1+e^{-j2\omega_0}=0$$

$$\boldsymbol{\omega_0=\frac{\pi}{2}+2k\pi\qquad k\in N}$$

$$\boldsymbol{A=40}$$

2.38. Consider the cascade of two LTI systems shown in Figure P2.38.

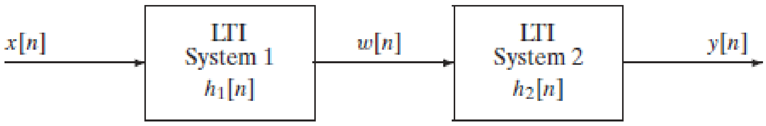


Figure P2.38

The impulse responses of the two systems are:

$$h_1[n] = u[n-5] \qquad \text{and} \qquad h_2[n] = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & \text{otherwise.} \end{cases}$$

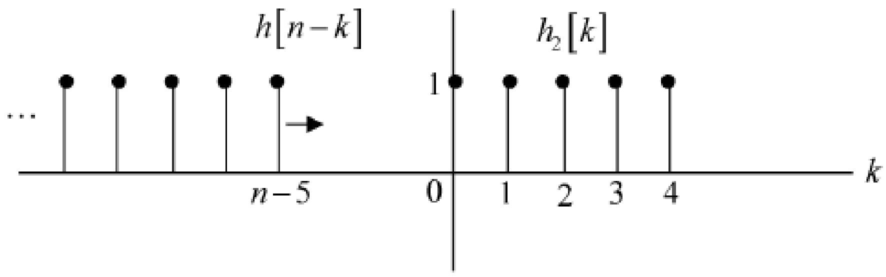
(a) Make a sketch showing both $h_2[k]$ and $h_1[n-k]$ (for some arbitrary $n < 0$) as functions of k.

(b) Determine $h[n] = h_1[n] * h_2[n]$, the impulse response of the overall system. Give your answer as an equation (or set of equations) that define $h[n]$ for $-\infty < n < \infty$ or as a carefully labelled plot of $h[n]$ over a range sufficient to define it completely.

[解]:

本题中两个系统产生级联($Cascade$)关系

A.

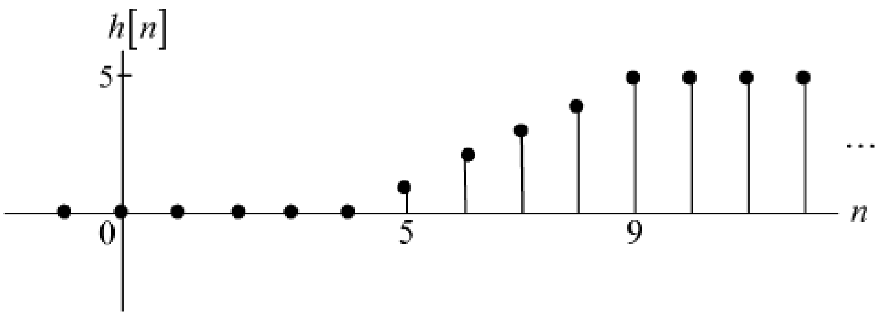


B. Clearly, $h[n]=0$ for $n-5<0$ or $n<5$.

Then $h[n]$ increases linearly until $n-5=4$ or $n=9$.

After $n=9$ the output is constant at $h[n]=5$.

$$h[n] = \begin{cases} 0, & n<5 \\ n-4, & 5 \le n < 9 \\ 5, & n>9. \end{cases}$$



2.45. Consider the cascade of LTI discrete-time systems shown in Figure P2.45.

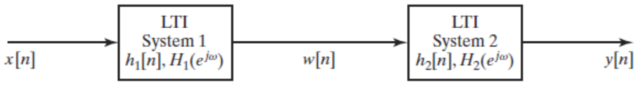


Figure P2.45

System 1 is described by the difference equation

$$w[n] = x[n] - x[n-1],$$

and System 2 is described by

$$h_2[n] = \frac{\sin(0.5\pi n)}{\pi n} \Longleftrightarrow H_2(e^{j\omega}) = \begin{cases} 1 & |\omega| < 0.5\pi \\ 0 & 0.5\pi < |\omega| < \pi. \end{cases}$$

The input $x[n]$ is

$$x[n] = \cos(0.4\pi n) + \sin(.6\pi n) + 5\delta[n-2] + 2u[n].$$

Determine the overall output $y[n]$.

(*With careful thought, you will be able to use the properties of LTI systems to write down the answer by inspection.*)

[解]:

先对输入予以更正

$$x[n] = cos(0.4\pi n) + sin(0.6\pi n) + 5\delta[n-2] + 2u[n]$$

这个里面其实本质上只有$u[n]$非常不好处理，因为我们求过$u[n]$的$DTFT$，频域上直接变换不现实，那样会相当地复杂。

$$w[n] = \cos(0.4\pi n) - \cos[0.4\pi(n-1)] + \sin(0.6\pi n) - \sin[0.6\pi(n-1)] + 5\delta[n-2] - 5\delta[n-3] + 2u[n] - 2u[n-1]$$

第一个LTI系统这么设计是有用意的，这样就会消去$u[n]$

$$2u[n] - 2u[n-1] = 2\delta[n]$$

$$w[n] = \cos(0.4\pi n) - \cos[0.4\pi(n-1)] + \sin(0.6\pi n) - \sin[0.6\pi(n-1)] + 5\delta[n-2] - 5\delta[n-3] + 2\delta[n]$$

所以说，本质上我们只需要探究$\cos(\omega_0 n)$与$\sin(\omega_0 n)$通过LTI系统后结果如何

其实我们在 2.36 题就已经深入、细致讨论过这个事情，当时我们讨论的是$\cos(\omega_0 n)$的情况，所以这里我们有：

$$\sin(\omega_0 n) = \cos\left[\omega_0\left(n - \frac{\frac{\pi}{2}}{\omega_0}\right)\right] = \cos\left[\omega_0\left(n - \frac{\pi}{2\omega_0}\right)\right]$$

本质上sin相对于cos也只是做了一个延迟，他们信号的本质性质都没有区别

这里我们使用 2.36 的结论

所以会有：

$$y[n] = \frac{1}{2}\left|H\left(e^{+j\omega_0}\right)\right|e^{+j[\omega_0 n \angle H(e^{j\omega_0})]} + \frac{1}{2}\left|H\left(e^{+j\omega_0}\right)\right|e^{-j[\omega_0 n + \angle H(e^{j\omega_0})]}$$

$$= \left|H\left(e^{+j\omega_0}\right)\right|\cos\left(\omega_0 n + \angle H\left(e^{j\omega}\right)\right)$$

$$H\left(e^{j\omega}\right) = \begin{cases} 1 & |\omega| < \frac{\pi}{2} \\ 0 & otherwise \end{cases}$$

对于$x_1[n]=\cos(0.4\pi n)-\cos[0.4\pi(n-1)]$有：

$$\left|H\left(e^{j(0.4\pi)}\right)\right|=1 \qquad \angle H\left(e^{j(0.4\pi)}\right)=0$$

输出为：

$$y_1[n]=\cos(0.4\pi n)-\cos\left(0.4\pi(n-1)\right)$$

有同学大概会对这个延迟有疑惑，这里其实本质上是这个样子的：

$$DTFT\{x[n-1]\}=e^{-j\omega}X\left(e^{j\omega}\right)$$
$$Y\left(e^{j\omega}\right)=X\left(e^{j\omega}\right)H\left(e^{j\omega}\right)=DTFT\{x[n]*h[n]\}$$
$$Y'\left(e^{j\omega}\right)=e^{-j\omega}X\left(e^{j\omega}\right)H\left(e^{j\omega}\right)=e^{-j\omega}Y\left(e^{j\omega}\right)=DTFT\{x[n-1]*h[n]\}$$
$$y'[n]=IDTFT\left\{Y'\left(e^{j\omega}\right)\right\}=y[n-1]$$

其实就是卷积中有一个有延迟，结果也延迟，这个是需要烂熟于心的

对于$x_2[n]=\sin(0.6\pi n)-\sin[0.6\pi(n-1)]$有：

$$\left|H\left(e^{j(0.4\pi)}\right)\right|=0 \qquad \angle H\left(e^{j(0.4\pi)}\right)=0$$

$$y_2[n]=0$$

对于$x_3[n]=5\delta[n-2]-5\delta[n-3]+2\delta[n]$有：

$$y_3[n]=x_3[n]*h[n]=5\frac{\sin\left(0.5\pi(n-2)\right)}{\pi(n-2)}-5\frac{\sin\left(0.5\pi(n-3)\right)}{\pi(n-3)}+5\frac{\sin(0.5\pi n)}{\pi n}$$

最后总结果$y[n]=y_1[n]+y_2[n]+y_3[n]$

$$y[n]=\cos(0.4\pi n)-\cos\left(0.4\pi(n-1)\right)+5\frac{\sin\left(0.5\pi(n-2)\right)}{\pi(n-2)}-5\frac{\sin\left(0.5\pi(n-3)\right)}{\pi(n-3)}$$
$$+5\frac{\sin(0.5\pi n)}{\pi n}$$

2.46. The DTFT pair

$$a^n u[n] \Longleftrightarrow \frac{1}{1-ae^{-j\omega}} \qquad |a|<1 \qquad \text{(P2.46-1)}$$

is given.

(a) Using Eq. (P2.46-1), determine the DTFT, $X(e^{j\omega})$, of the sequence

$$x[n] = -b^n u[-n-1] = \begin{cases} -b^n & n \le -1 \\ 0 & n \ge 0. \end{cases}$$

What restriction on b is necessary for the DTFT of $x[n]$ to exist?

(b) Determine the sequence $y[n]$ whose DTFT is

$$Y(e^{j\omega}) = \frac{2e^{-j\omega}}{1+2e^{-j\omega}}.$$

[解]:

(a).

$$DTFT\{x[n]\} = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{-1} (-b^n)e^{-j\omega n} = -\sum_{n=-\infty}^{-1} b^n e^{-j\omega n} = -\sum_{n'=1}^{+\infty} \left(b^{-1}e^{+j\omega}\right)^{n'}$$

上式中我们做了一步换元:

$$n' = -n$$

$$DTFT\{x[n]\} == -\sum_{n'=1}^{+\infty} \left(b^{-1}e^{+j\omega}\right)^{n'} = -\left\{\lim_{N\to+\infty} \frac{1-\left(b^{-1}e^{+j\omega}\right)^{N+1}}{1-b^{-1}e^{+j\omega}} - 1\right\} = 1 - \frac{1}{1-b^{-1}e^{+j\omega}}$$

上面式子的收敛需要:

$$\left|b^{-1}e^{+j\omega}\right| < 1$$
$$|b| > 1$$

(b)

$$DTFT\{y[n]\} = Y\left(e^{j\omega}\right) = \frac{2e^{-j\omega}}{1+2e^{-j\omega}} = \frac{2}{e^{j\omega}+2} = \frac{1}{1+\frac{1}{2}e^{j\omega}}$$

带入(a)即可

$$y[n] = \begin{cases} -2^n & n \le 0 \\ 0 & otherwise \end{cases}$$

这里因为少了一个 1，所以需要凑一下上下限，没有难度，大家仔细想一下即可。

选做部分

2.60. Consider a discrete-time LTI system with frequency response $H(e^{j\omega})$ and corresponding impulse response $h[n]$.

(a) We are first given the following three clues about the system:

(i) The system is causal.

(ii) $H(e^{j\omega}) = H^*(e^{-j\omega})$.

(iii) The DTFT of the sequence $h[n+1]$ is real.

Using these three clues, show that the system has an impulse response of finite duration.

(b) In addition to the preceding three clues, we are now given two more clues:

(iv) $\frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})d\omega = 2$.

(v) $H(e^{j\pi}) = 0$.

Is there enough information to identify the system uniquely? If so, determine the impulse response $h[n]$. If not, specify as much as you can about the sequence $h[n]$.

[解]:

(a).

这里需要我们根据三个线索，证明符合这个线索的LTI系统会是有限冲激响应的，也就是到一定点数之后就全是 0 了，有截断($Truncation$)

(i)系统是因果的(Causal):

$$h[n] = h[n]u[n]$$

这个条件阐释了时域负半轴部分

(ii)系统正负频率所对应的频域函数是共轭关系的:

$$H(e^{j\omega}) = H^*(e^{j(-\omega)})$$

我们在习题课里面其实讲过，这个系统很显然就是一个实系统，也就是$h[n] \in R$

(iii)系统时移后的$DTFT$是实的:

$$DTFT\{h[n+1]\} = \sum_{n=-\infty}^{+\infty} h[n+1]e^{-j\omega n} = \sum_{n=-1}^{+\infty} h[n+1]e^{-j\omega n}$$

第三个条件说明了，系统只有三点，并且 $h[0] = h[2]$，从而会有:

$$DTFT\{h[n+1]\} = h[0]e^{+j\omega} + h[1] + h[2]e^{-j\omega} = h[0]\{2\cos(\omega)\} + h[1]$$

所以确实是有限长度的。

(b).

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H(e^{j\omega})e^{+j\omega n}d\omega$$

$$h[0] = \left\{\frac{1}{2\pi}\int_{-\pi}^{+\pi} H(e^{j\omega})e^{+j\omega n}d\omega\right\}|_{n=0} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H(e^{j\omega})d\omega = 2$$

$$h[2] = h[0] = 2$$

$$H(e^{j\omega}) = h[0]\{2\cos(\omega)\} + h[1]$$

$\omega = \pi$情况下有:

$$H(e^{j\pi}) = -2h[0] + h[1] = 0$$

$$h[1] = 4$$

所以这五个条件可以明确地确定这个系统的唯一性$h[0] = 2 \quad h[1] = 4 \quad h[2] = 2$

2.65. The LTI system

$$H(e^{j\omega}) = \begin{cases} -j, & 0 < \omega < \pi, \\ j, & -\pi < \omega < 0, \end{cases}$$

is referred to as a 90° phase shifter and is used to generate what is referred to as an analytic signal $w[n]$ as shown in Figure P2.65-1. Specifically, the analytic signal $w[n]$ is a complex-valued signal for which

$$\mathcal{R}e\{w[n]\} = x[n],$$

$$\mathcal{I}m\{w[n]\} = y[n].$$

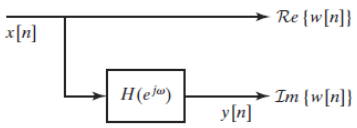


Figure P2.65–1

If $\mathcal{R}e\{X(e^{j\omega})\}$ is as shown in Figure P2.65-2 and $\mathcal{I}m\{X(e^{j\omega})\} = 0$, determine and sketch $W(e^{j\omega})$, the Fourier transform of the analytic signal $w[n] = x[n] + jy[n]$.

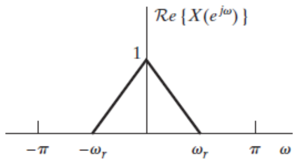


Figure P2.65–2

[解]:

这个题目的背景本质上就是$Hilbert$变换，我们在后续会学到。

$$Re\{w[n]\} = x[n]$$

$$Im\{w[n]\} = y[n]$$

$$w[n] = x[n] + jy[n]$$

$$DTFT\{w[n]\} = DTFT\{x[n] + jy[n]\}$$

由于$DTFT$变换是线性变换，自然有：

$$DTFT\{w[n]\} = W(e^{j\omega}) = DTFT\{x[n]\} + jDTFT\{y[n]\} = X(e^{j\omega}) + jY(e^{j\omega})$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \begin{cases} -jX(e^{j\omega}) & 0 < \omega < \pi \\ +jX(e^{j\omega}) & -\pi < \omega < 0 \end{cases}$$

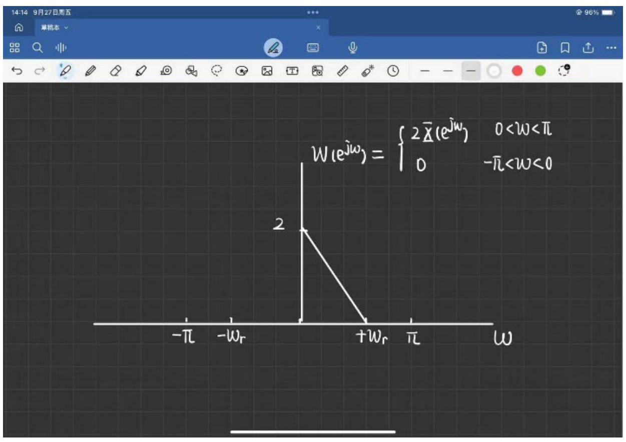
$$W(e^{j\omega}) = X(e^{j\omega}) + jY(e^{j\omega}) = \begin{cases} 2X(e^{j\omega}) & 0 < \omega < \pi \\ 0 & -\pi < \omega < 0 \end{cases}$$

已知$Re\{X(e^{j\omega})\}$如图，并且会有$Im\{X(e^{j\omega})\} = 0$

所以会有：

$$X(e^{j\omega}) = Re\{X(e^{j\omega})\} + jIm\{X(e^{j\omega})\} = Re\{X(e^{j\omega})\}$$

从而会有$W(e^{j\omega})$的图：



2.56. For the system in Figure P2.56, determine the output $y[n]$ when the input $x[n]$ is $\delta[n]$ and $H(e^{j\omega})$ is an ideal lowpass filter as indicated, i.e.,

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \pi/2, \\ 0, & \pi/2 < |\omega| \le \pi. \end{cases}$$

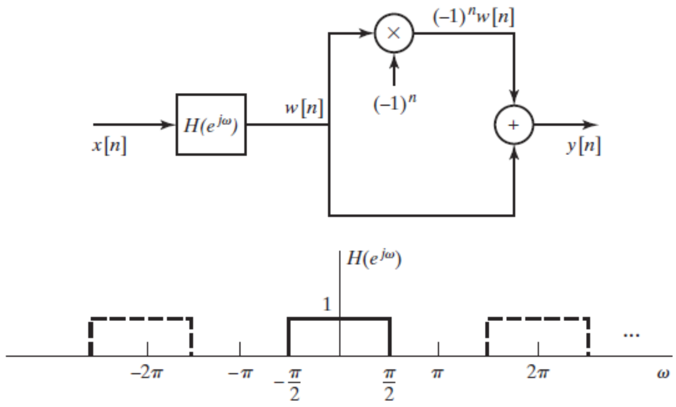


Figure P2.56

[解]:

$$x[n] = \delta[n]$$

$$W(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \begin{cases} 1 & |\omega| < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\omega| < \pi \end{cases}$$

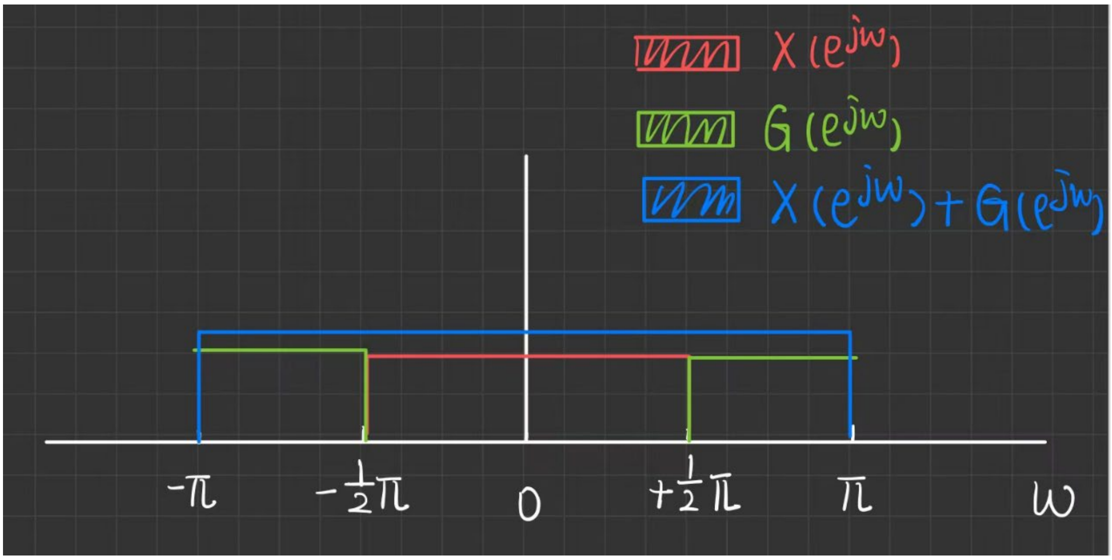
$$y[n] = x[n] + (-1)^n x[n]$$

$$g[n] = (-1)^n x[n]$$

$$G(e^{j\omega}) = DTFT\{g[n]\} = \sum_{n=-\infty}^{+\infty} g[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} (-1)^n x[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x[n]e^{-j(\omega+\pi)n}$$

$$= X(e^{j(\omega+\pi)})$$

如下图:



红绿蓝三色信号其实是一样高，为了明显才画的不一样高

$$y[n] = \delta[n]$$

3.3. Determine the z-transform of each of the following sequences. Include with your answer the ROC in the z-plane and a sketch of the pole–zero plot. Express all sums in closed form; α can be complex.

(a) $x_a[n] = \alpha^{|n|}, \qquad 0 < |\alpha| < 1.$

(b) $x_b[n] = \begin{cases} 1, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$

(c) $x_c[n] = \begin{cases} n+1, & 0 \le n \le N-1, \\ 2N-1-n, & N \le n \le 2(N-1), \\ 0, & \text{otherwise.} \end{cases}$

Hint: Note that $x_b[n]$ is a rectangular sequence and $x_c[n]$ is a triangular sequence. First, express $x_c[n]$ in terms of $x_b[n]$.

[解]:

(a)

$$x_a[n] = \alpha^{|n|} \qquad 0 < |\alpha| < 1$$

$$\begin{aligned} X_a(z) &= \sum_{n=-\infty}^{-1} \alpha^{-n} z^{-n} + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\ &= \sum_{n=1}^{\infty} \alpha^n z^n + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\ &= \frac{\alpha z}{1-\alpha z} + \frac{1}{1-\alpha z^{-1}} = \frac{z(1-\alpha^2)}{(1-\alpha z)(z-\alpha)}, \qquad |\alpha| < |z| < \frac{1}{|\alpha|} \end{aligned}$$

(b)

$$x_b = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & N \le n \\ 0, & n < 0 \end{cases} \Rightarrow X_b(z) = \sum_{n=0}^{N-1} z^{-n} = \frac{1-z^{-N}}{1-z^{-1}} = \frac{z^N - 1}{z^{N-1}(z-1)} \qquad z \ne 0$$

(c)

$$x_c[n] = x_b[n-1] * x_b[n] \Leftrightarrow X_c(z) = z^{-1} X_b(z) \cdot X_b(z)$$

$$X_c(z) = z^{-1} \left(\frac{z^N - 1}{z^{N-1}(z-1)} \right)^2 = \frac{1}{z^{2N-1}} \left(\frac{z^N - 1}{z-1} \right)^2 \qquad z \ne 0, 1$$

图略

3.6. Following are several z-transforms. For each, determine the inverse z-transform using both methods—partial fraction expansion and power series expansion—discussed in Section 3.3. In addition, indicate in each case whether the Fourier transform exists.

(a) $X(z) = \dfrac{1}{1+\frac{1}{2}z^{-1}}, \qquad |z| > \dfrac{1}{2}$

(b) $X(z) = \dfrac{1}{1+\frac{1}{2}z^{-1}}, \qquad |z| < \dfrac{1}{2}$

(c) $X(z) = \dfrac{1-\frac{1}{2}z^{-1}}{1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}, \qquad |z| > \dfrac{1}{2}$

(d) $X(z) = \dfrac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{4}z^{-2}}, \qquad |z| > \dfrac{1}{2}$

(e) $X(z) = \dfrac{1-az^{-1}}{z^{-1}-a}, \qquad |z| > |1/a|$

[解]:

先给出两个最经典的Z变换，所有的Z变换都是在它们的基础上进行变换:

$$x[n] = a^n u[n]$$

$$Z\{x[n]\} = \sum_{n=-\infty}^{+\infty} x[n]z^{-n} = \sum_{n=0}^{+\infty} a^n z^{-n} = \frac{1}{1-az^{-1}}$$

$$ROC: |az^{-1}| < 1 \qquad |a| < |z|$$

$$x[n] = -a^n u[-n-1]$$

$$Z\{x[n]\} = \sum_{n=-\infty}^{+\infty} x[n]z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\left(\sum_{n=-\infty}^{0} a^n z^{-n} - 1\right) = 1 - \sum_{n=-\infty}^{0} a^n z^{-n}$$

$$= 1 - \sum_{n'=0}^{+\infty} (a^{-1}z)^{n'} = 1 - \frac{1}{1-a^{-1}z} = \frac{-a^{-1}z}{1-a^{-1}z} = \frac{1}{1-az^{-1}}$$

$$ROC: |a^{-1}z| < 1 \qquad |z| < |a|$$

(a).

$$X(z) = \frac{1}{1+\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$

$$x[n] = \left(-\frac{1}{2}\right)^n u[n]$$

(b)

$$X(z) = \frac{1}{1+\frac{1}{2}z^{-1}} \qquad |z| < \frac{1}{2}$$

$$x[n] = -\left(-\frac{1}{2}\right)^n u[-n-1]$$

(c)

$$X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{4}{1 + \frac{1}{2}z^{-1}} + \frac{-3}{1 + \frac{1}{4}z^{-1}} \qquad |z| > \frac{1}{2}$$

$$x[n] = 4\left(-\frac{1}{2}\right)^n u[n] - 3\left(-\frac{1}{4}\right)^n u[n]$$

(d)

$$X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}} = \frac{1 - \frac{1}{2}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)} \qquad |z| > \frac{1}{2}$$

$$X(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$

$$x[n] = \left(-\frac{1}{2}\right)^n u[n]$$

(e).

$$X(z) = \frac{1 - az^{-1}}{z^{-1} - a} = \frac{1 - az^{-1}}{a(a^{-1}z^{-1} - 1)} = -\frac{1}{a}\frac{1 - az^{-1}}{1 - a^{-1}z^{-1}}$$

$$= -\frac{1}{a}\left(\frac{1}{1 - a^{-1}z^{-1}} - a\frac{z^{-1}}{1 - a^{-1}z^{-1}}\right) \qquad |z| > |a^{-1}|$$

$$x[n] = -\frac{1}{a}\{(a^{-1})^n u[n] - a * (a^{-1})^{n-1} u[n-1]\}$$

$$x[n] = -a^{-n-1}u[n] + a^{-n+1}u[n-1]$$

3.9. A causal LTI system has impulse response $h[n]$, for which the z-transform is

$$H(z) = \frac{1+z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}.$$

(a) What is the ROC of $H(z)$?
(b) Is the system stable? Explain.
(c) Find the z-transform $X(z)$ of an input $x[n]$ that will produce the output

$$y[n] = -\frac{1}{3}\left(-\frac{1}{4}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1].$$

(d) Find the impulse response $h[n]$ of the system.

[解]:

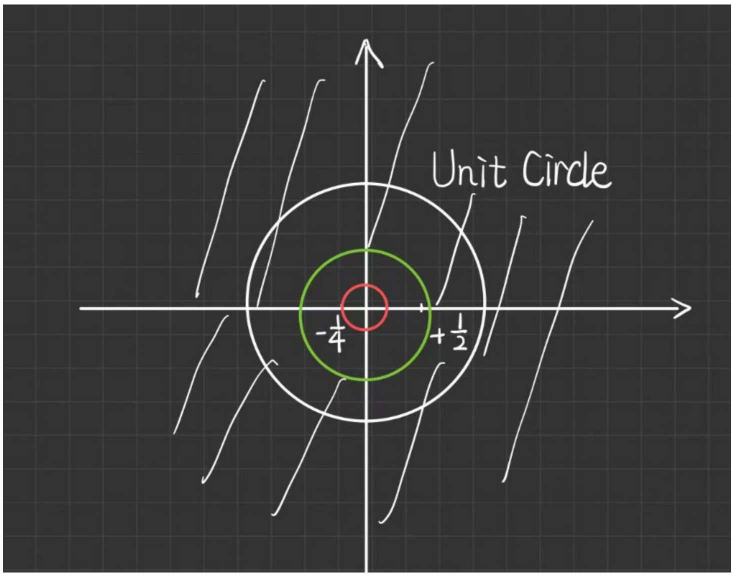
(a).

因为系统是因果的:

$$h[n] = h[n]u[n]$$

ROC形如$|z| > |a|$

如下图，必须有交



$$ROC: |z| > \frac{1}{2}$$

(b).

我们考虑一个系统如果是因果的，那么会有:

$$\exists M > 0 \quad such\ that \quad \sum_{n=-\infty}^{+\infty} |h[n]| < M$$

$$\left|\sum_{n=-\infty}^{+\infty} h[n]\right| < \sum_{n=-\infty}^{+\infty} |h[n]| < M$$

一个必要条件是$|z| = 1$所表征的$Unit\ Circle$需要在ROC之内，当然张老师上课降到了这个点，这只是个必要条件，是否一定如此判决还有待商榷。但是根据 Oppenheimer 书上所述，这里确实是属于$Stable$的范畴。

(c)

$$Y(z)=\sum_{n=-\infty}^{+\infty}y[n]z^{-n}=-\frac{1}{3}\frac{1}{1-\left(-\frac{1}{4}\right)z^{-1}}+(-1)\left(-\frac{4}{3}\right)\frac{1}{1-2z^{-1}}$$

$$Y(z)=-\frac{1}{3}\frac{1}{1+\frac{1}{4}z^{-1}}+\frac{4}{3}\frac{1}{1-2z^{-1}}\qquad ROC{:}\frac{1}{4}<|z|<2$$

$$Y(z)=\frac{1+z^{-1}}{\left(1+\frac{1}{4}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}\qquad ROC{:}\frac{1}{4}<|z|<2$$

$$X(z)=\frac{1-\frac{1}{2}z^{-1}}{1-2z^{-1}}\qquad |z|<2$$

$$\begin{aligned}x[n]&=-(2)^n u[-n-1]+\left(-\frac{1}{2}\right)(-1)(2)^{n-1}u[-(n-1)-1]\\&=-2^n u[-n-1]+2^{n-2}u[-n]\end{aligned}$$

(d)

$$h[n]=\left(\frac{1}{2}\right)^{n-1}u[n]-\left(-\frac{1}{4}\right)^{n}u[n]$$

3.13. A causal sequence $g[n]$ has the z-transform

$$G(z) = \sin(z^{-1})(1 + 3z^{-2} + 2z^{-4}).$$

Find $g[11]$.

[解]:

$$G(z) = \sin(z^{-1})\,(1 + 3z^{-2} + 2z^{-4})$$

我们对$\sin(z^{-1})$进行$Taylor$展开，或者更精确地来说，这里是关于 0 点的$Taylor$展开，也就是退化为了$Maclaurin$展开

$$\sin(z) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

这里我们替换z为z^{-1}

$$\sin(z^{-1}) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} z^{-(2n+1)}$$

$$G(z) = \sum_{n=-\infty}^{+\infty} g[n] z^{-n}$$

令$n = 11$

$$g[11] \to z^{-11}$$

于是会有:

$$-(2n_0 + 1) + 0 = -11$$
$$n_0 = 5$$

$$-(2n_1 + 1) - 2 = -11$$
$$n_1 = 4$$

$$-(2n_2 + 1) - 4 = -11$$
$$n_2 = 3$$

对比系数可以有:

$$g[11] = \frac{(-1)^5}{11!} + \frac{(-1)^4}{9!} * 3 + \frac{(-1)^3}{7!} * 2 = -\frac{1}{11!} + \frac{3}{9!} - \frac{2}{7!}$$

3.21. A causal LTI system has the following system function:

$$H(z) = \frac{4 + 0.25z^{-1} - 0.5z^{-2}}{(1 - 0.25z^{-1})(1 + 0.5z^{-1})}$$

(a) What is the ROC for $H(z)$?

(b) Determine if the system is stable or not.
(c) Determine the difference equation that is satisfied by the input $x[n]$ and the output $y[n]$.
(d) Use a partial fraction expansion to determine the impulse response $h[n]$.
(e) Find $Y(z)$, the z-transform of the output, when the input is $x[n] = u[-n-1]$. Be sure to specify the ROC for $Y(z)$.
(f) Find the output sequence $y[n]$ when the input is $x[n] = u[-n-1]$.

[解]:

(a)

此LTI系统是因果($Causal$)系统，所以会有ROC形如：$|z| > |a|$

$$ROC: |z| > 0.5$$

(b)

$|z| = 1$包含在ROC之中，所以是稳定的

(c)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{4 + 0.25z^{-1} - 0.5z^{-2}}{1 + 0.25z^{-1} - 0.125z^{-2}}$$

差分方程为:

$$y[n] + 0.25y[n-1] - 0.125y[n-2] = 4x[n] + 0.25x[n-1] - 0.5x[n-2]$$

保险起见，我建议大家都移项变成标准形式:

$$y[n] = 4x[n] + 0.25x[n-1] - 0.5x[n-2] + 0.125y[n-2] - 0.25y[n-1]$$

(e)

$$x = u[-n-1]$$

$$X(z) = \frac{-1}{1 - z^{-1}} \qquad |z| < 1$$

$$Y(z) = H(z)X(z) \qquad ROC: ROC_X \cap ROC_H$$

$$Y(z) = \frac{-4 - 0.25z^{-1} + 0.5z^{-2}}{(1 - 0.25z^{-1})(1 + 0.5z^{-1})(1 - z^{-1})} \qquad ROC: 0.5 < |z| < 1$$

(f)

$$Y(z) = \frac{-4 - 0.25z^{-1} + 0.5z^{-2}}{(1 - 0.25z^{-1})(1 + 0.5z^{-1})(1 - z^{-1})} = \frac{-\frac{1}{3}}{(1 - 0.25z^{-1})} + \frac{-\frac{1}{3}}{(1 + 0.5z^{-1})} + \frac{-\frac{10}{3}}{(1 - z^{-1})}$$

$$y[n] = -\frac{1}{3}(+0.25)^n u[n] - \frac{1}{3}(-0.5)^n u[n] + \frac{10}{3}u[-n-1]$$

3.25. Sketch each of the following sequences and determine their z-transforms, including the ROC:

(a) $\sum_{k=-\infty}^{\infty} \delta[n-4k]$

(b) $\frac{1}{2}\left[e^{j\pi n} + \cos\left(\frac{\pi}{2}n\right) + \sin\left(\frac{\pi}{2} + 2\pi n\right)\right] u[n]$

[解]:

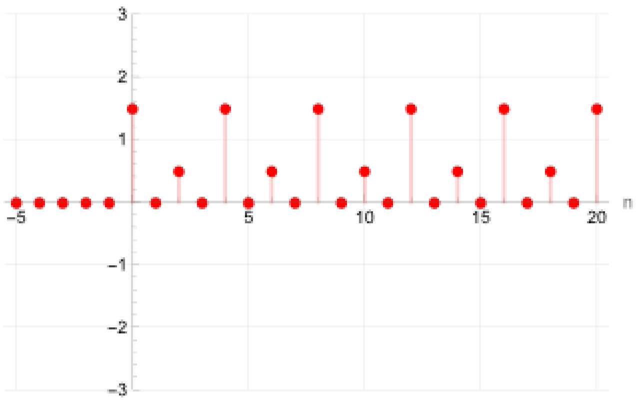
(a)

$$g[n] = \sum_{k=-\infty}^{+\infty} \delta[n-4k]$$

$$Z\left\{\sum_{k=-\infty}^{+\infty} \delta[n-4k]\right\} = \sum_{n=-\infty}^{+\infty} \left\{\sum_{k=-\infty}^{+\infty} \delta[n-4k]\right\} z^{-n} = G(z)$$

$$G(z) = \sum_{k=-\infty}^{+\infty} z^{-4k}$$

(b)



我们画图可以看出:

$$m[n] = \frac{1}{2} u[n] \left\{e^{j\pi n} + \cos\left(\frac{\pi}{2}n\right) + \sin\left(\frac{\pi}{2} + 2\pi n\right)\right\}$$

这个函数本质上是两个周期信号的叠加，幅值一个是$\frac{3}{2}$,一个是$\frac{1}{2}$，周期都是 4

$$M(z) = \sum_{k=0}^{+\infty} \frac{3}{2} z^{-4k} + \sum_{k=0}^{+\infty} \frac{1}{2} z^{-4k-2} = \frac{\frac{3}{2}}{1-z^{-4}} + \frac{\frac{1}{2}z^{-2}}{1-z^{-4}} \quad ROC: |z| > 1$$

选做部分

3.22. A causal LTI system has system function

$$H(z) = \frac{1-4z^{-2}}{1+0.5z^{-1}}.$$

The input to this system is

$$x[n] = u[n] + 2\cos\left(\frac{\pi}{2}n\right) \qquad -\infty < n < \infty,$$

Determine the output $y[n]$ for large positive n; i.e., find an expression for $y[n]$ that is asymptotically correct as n gets large. (*Of course, one approach is to find an expression for $y[n]$ that is valid for all n, but you should see an easier way.*)

[解]:

这个题本质上和 2.45 一个道理，无非是$DTFT$变为了Z变换，本质上也是一个道理。

这里只是$u[n]$会难以处理一些

$$x_{input}[n] = u[n]$$

$$y_{output}[n] = x_{input}[n] * h[n] = \sum_{k=-\infty}^{+\infty} u[k]h[n-k] = \sum_{k=0}^{+\infty} h[n-k] = \sum_{n'=-\infty}^{n} h[n'] \quad (n' = n-k)$$

n足够大的时候会有：

$$y_{output}[n] = \sum_{n'=-\infty}^{n} h[n'] = \sum_{n'=-\infty}^{+\infty} h[n'] = \sum_{n'=-\infty}^{+\infty} h[n']z^{-n'}|_{n'=0} = H(z)|_{z=1} = -2$$

3.42. In Figure P3.42, $H(z)$ is the system function of a causal LTI system.

(a) Using z-transforms of the signals shown in the figure, obtain an expression for $W(z)$ in the form

$$W(z) = H_1(z)X(z) + H_2(z)E(z),$$

where both $H_1(z)$ and $H_2(z)$ are expressed in terms of $H(z)$.

(b) For the special case $H(z) = z^{-1}/(1 - z^{-1})$, determine $H_1(z)$ and $H_2(z)$.

(c) Is the system $H(z)$ stable? Are the systems $H_1(z)$ and $H_2(z)$ stable?

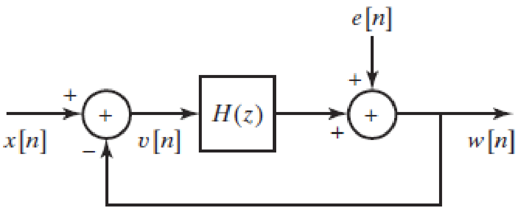


Figure P3.42

[解]:

(a)

$$v[n] = x[n] - w[n]$$
$$q[n] = v[n] * h[n]$$
$$w[n] = e[n] + q[n]$$
$$W(z) = E(z) + Q(z)$$
$$Q(z) = V(z)H(z) = \big(X(z) - W(z)\big)H(z) = X(z)H(z) - W(z)H(z)$$
$$W(z) = E(z) + X(z)H(z) - W(z)H(z)$$
$$H_1(z) = \frac{H(z)}{1 + H(z)} \qquad H_2(z) = \frac{1}{1 + H(z)}$$

(b)

$$H(z) = \frac{z^{-1}}{1 - z^{-1}}$$
$$H_1(z) = z^{-1} \qquad H_2(z) = 1 - z^{-1}$$

(c)

$H(z)$的极点是$z = 1$，所以是非稳定状态

$H(z)$要因果且结果收敛就有:

$$|z| > 1$$

完全不包含$Unit\ Circle$

对于$H_1(z)$　$H_2(z)$　$|z| = 1$是收敛的，所以稳定

3.56. A real finite-duration sequence whose z-transform has no zeros at conjugate reciprocal pair locations and no zeros on the unit circle is uniquely specified to within a positive scale factor by its Fourier transform phase (Hayes et al., 1980).

An example of zeros at conjugate reciprocal pair locations is $z = a$ and $(a^*)^{-1}$. Even though we can generate sequences that do not satisfy the preceding set of conditions, almost any sequence of practical interest satisfies the conditions and therefore is uniquely specified to within a positive scale factor by the phase of its Fourier transform.

Consider a sequence $x[n]$ that is real, that is zero outside $0 \le n \le N-1$, and whose z-transform has no zeros at conjugate reciprocal pair locations and no zeros on the unit circle. We wish to develop an algorithm that reconstructs $cx[n]$ from $\angle X(e^{j\omega})$, the Fourier transform phase of $x[n]$, where c is a positive scale factor.

(a) Specify a set of $(N-1)$ linear equations, the solution to which will provide the recovery of $x[n]$ to within a positive or negative scale factor from $\tan\{\angle X(e^{j\omega})\}$. You do not have to prove that the set of $(N-1)$ linear equations has a unique solution. Further, show that if we know $\angle X(e^{j\omega})$ rather than just $\tan\{\angle X(e^{j\omega})\}$, the sign of the scale factor can also be determined.

(b) Suppose

$$x[n] = \begin{cases} 0, & n < 0, \\ 1, & n = 0, \\ 2, & n = 1, \\ 3, & n = 2, \\ 0, & n \ge 3. \end{cases}$$

Using the approach developed in part (a), demonstrate that $cx[n]$ can be determined from $\angle X(e^{j\omega})$, where c is a positive scale factor.

[解]:

这题本身的描述是比较复杂的，为了方便大家理解，我们先读题：

“一个有限长、实的序列，它的 Z 变换在共轭倒数和单位圆上是不存在零点的。这个有限长实序列可以被唯一地由傅里叶变换的相位确定（这里唯一确定过程中我们忽略掉了可能存在影响的正幅度加权因子）”——Hayes 1980

题目还描述：尽管我们其实是可以创造一些不满足上述条件的序列（也就是它的 Z 变换不满足所谓“在共轭倒数和单位圆上不存在零点”），但是实际上有意思的（这里可能指的是值得咱们研究的）序列几乎全都满足上面 Hayes 等人在 1980 年提出的那个条件。所以几乎上，所有我们感兴趣的、有研究价值的序列，都可以在不考虑所谓的正幅度加权因子基础上，使用傅里叶变换根据它的相位唯一确定。

接下来就是对问题的考察了。考虑现阶段我们有一个实序列$x[n]$，这个序列满足在$[0, N-1]$之外全部都是 0，并且它的 Z 变换对应的零点不在单位圆上也不共轭对称成对出现。我们需要试着去使用$\tan\{\angle X(e^{j\omega})\}$构造线性方程恢复$x[n]$。更进一步要探究陈述，如果我们只得到不是$\tan\{\angle X(e^{j\omega})\}$而是更加本质的$\angle X(e^{j\omega})$，我们可以确认正幅度加权因子。

这题难度很大，大家不会做很正常，毕竟是 1980 年代的 Trans 文章，说明是那个年代的巅峰之作，这是语音信号的顶会 ICASSP，是 CCF-B 会议。

我们考虑直接构造出所谓的$\angle X(e^{j\omega})$

$$X(e^{j\omega})=\sum_{n=-\infty}^{+\infty}x[n]e^{-j\omega n}=\sum_{n=0}^{N-1}x[n]e^{-j\omega n}=Re\{X(e^{j\omega})\}+Im\{X(e^{j\omega})\}$$

$$=|X(e^{j\omega})|e^{+j\angle X(e^{j\omega})}$$

这里因为我们研究的是实序列，所以会有：

$$\sum_{n=0}^{N-1}x[n]e^{-j\omega n}=\sum_{n=0}^{N-1}x[n]\cos(\omega n)-j\sum_{n=0}^{N-1}x[n]\sin(\omega n)$$

$$=\sum_{n=0}^{N-1}x[n]\cos(\omega n)+j\left\{-\sum_{n=0}^{N-1}x[n]\sin(\omega n)\right\}$$

事实上我们会有：

$$|X(e^{j\omega})|=\sqrt{\left(\sum_{n=0}^{N-1}x[n]\cos(\omega n)\right)^2+\left(\sum_{n=0}^{N-1}x[n]\sin(\omega n)\right)^2}$$

$$\tan\angle X(e^{j\omega})=\frac{Im\left(X(e^{j\omega})\right)}{Re\left(X(e^{j\omega})\right)}=-\frac{\sum_{n=0}^{N-1}x[n]\sin(\omega n)}{\sum_{n=0}^{N-1}x[n]\cos(\omega n)}$$

就是这一步，我们需要直接构造出来题目中最重要的量$X(e^{j\omega})$或者比它更弱一些的量$\tan\angle X(e^{j\omega})$，这里我们得到的信息只允许我们构造出$\tan\angle X(e^{j\omega})$

我们需要使用$\tan\angle X(e^{j\omega})$来重构$x[n]$，解上述等式即可：

$$\tan\angle X(e^{j\omega})\left[\sum_{n=0}^{N-1}x[n]\cos(\omega n)\right]=-\sum_{n=0}^{N-1}x[n]\sin(\omega n)$$

这里大家需要仔细想一下乘法分配律到底可不可以做：

$$\sum_{n=0}^{N-1}x[n]\left[\cos(\omega n)\tan\angle X(e^{j\omega})\right]=-\sum_{n=0}^{N-1}x[n]\sin(\omega n)$$

所以其实有：

$$\sum_{n=0}^{N-1}x[n]\left[\sin(\omega n)+\cos(\omega n)\tan\angle X(e^{j\omega})\right]=0$$

需要注意的是，这里$n=0$的情况本质上是退化的：

$$\left\{x[n]\left[\sin(\omega n)+\cos(\omega n)\tan\angle X(e^{j\omega})\right]\right\}|_{n=0}=x[0]\tan\angle X(e^{j\omega})$$

所以其实原式等价于下式，这就能给出所谓的线性方程：

$$x[0]\tan\angle X(e^{j\omega})+\sum_{n=1}^{N-1}x[n]\left[\sin(\omega n)+\cos(\omega n)\tan\angle X(e^{j\omega})\right]=0$$

这个地方 Hayes 认为，我们本质上有这个等式（也就是辅助角后的结果）：

$$\sum_{n=1}^{N-1}\left\{\left[\sqrt{\left(tan\angle X(e^{j\omega})\right)^2+1}\,x[n]\right]\sin\left(\omega n+\angle X(e^{j\omega})\right)\right\}=-\left[x[0]\sqrt{\left(tan\angle X(e^{j\omega})\right)^2+1}\right]\sin\left(\angle X(e^{j\omega})\right)$$

从而会有：

$$\sum_{n=1}^{N-1}x[n]\sin\left(\omega n+\angle X\left(e^{j\omega}\right)\right)=-x[0]\sin\left(\angle X\left(e^{j\omega}\right)\right)$$

所以只需要采样$(N-1)$个ω对应的$\angle X\left(e^{j\omega}\right)$值即可

这也就是题目中所叙说的$(N-1)$个等式，本质上也就是$(N-1)$个ω对应的等式，也就是$\omega_k \quad k\in[1,N-1]$

大家可能注意到了我们并没有使用所谓的“Z 变换在共轭倒数和单位圆上是不存在零点的”这个条件。这个地方也是助教百思不得其解的地方，直到我看到了 Hayes 的原文，Hayes 用了一种相当精妙的手法去证明这件事。

这个条件就是证明一件事：如果信号$x[n]$与信号$y[n]$，我们得到它们的角度一致$\angle X\left(e^{j\omega}\right)=\angle Y\left(e^{j\omega}\right)\quad\forall\,\omega\in[-\pi,\pi)$那么事实上$x[n]$与$y[n]$仅仅相差一个幅度调制系数$\beta$：

$$\beta x[n]=y[n]$$