PROOFS FOR TOPOLOGY

TONY DENG

Proposition 2.4. If $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ and $g : (Y, \mathcal{T}_Y) \to (Z, \mathcal{T}_Z)$ are continuous, then so is $g \circ f : (X, \mathcal{T}_X) \to (Z, \mathcal{T}_Z)$.

Version: 1
Comments / Collaborators: no comments at this point; no collaboration allowed for this version.

Proof. Pick an arbitrary open set U from Z, we know that $g^{-1}(U)$ is open in Y by the continuity of g. Denote $g^{-1}(U)$ as W. Since W is open in Y, we know that $f^{-1}(W)$ is open X by the continuity of f. Denote $f^{-1}(W)$ as V. So for any open sets in Z we know that $(g \circ f)^{-1}(U)$ is open in X. Thus, $g \circ f$ is continuous as well. $\square$

Date: September 20, 2021.

Example 2.5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -1 & x < 0, \\ 1 & x \geq 0. \end{cases}$$

Determine which (if any) of the following maps between topological spaces are continuous.

(1) $f : (\mathbb{R}, \mathcal{T}_l) \to (\mathbb{R}, \mathcal{T}_l)$
(2) $f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T}_l)$
(3) $f : (\mathbb{R}, \mathcal{T}_l) \to (\mathbb{R}, \mathcal{T})$

Version: 1
Comments / Collaborators: no comments at this point; no collaboration allowed for this version.

Proof. (1) is not a continuous map.

Choose $[1, 2)$ from $\mathcal{T}_l$, $f^{-1}([1, 2)) = \{1\}$ which is not an open set in $\mathcal{T}_l$.

(2) is not a continuous map.

Choose $[1, 2)$ from $\mathcal{T}_l$, $f^{-1}([1, 2)) = \{1\}$ which is not an open set in $\mathcal{T}$.

(3) is not a continuous map.

Choose $(1, 2)$ from $\mathcal{T}$, $f^{-1}([1, 2)) = \{1\}$ which is not an open set in $\mathcal{T}_l$.

$\square$

Lemma 2.6. (1) Let $f : X \to Y$ be a continuous map. Suppose that $A \subset X$. The restriction of f to A (denoted $f|_A$, and defined by $f|_A(x) = f(x)$ if $x \in A$) is continuous as a function $f|_A : A \to Y$ with respect to the subspace topology on the domain

(2) Let $f : X \to B$ be a map between topological spaces with $B \subset Y$ having the subspace topology. Let $i : B \to Y$ be the inclusion map (i.e., $i(b) = b$ for all $b \in B$). The map f is continuous if and only if $i \circ f$ is continuous.

Version: 1
Comments / Collaborators: no comments at this point; no collaboration allowed for this version.

Proof. (1)

Want to show that $f|_A$ is continuous.

Since f is continuous, we know that for any open sets, V, in $\mathcal{T}_Y$, $f^{-1}(V)$ is in $\mathcal{T}_X$. Denote $f^{-1}(V)$ as $\tilde{U}$. Let $U = A \cap \tilde{U}$, we know that $U \in \mathcal{T}_A$. So we know that $f|_A^{-1}(V)$ sends to some $\tilde{U}$ in $\mathcal{T}_X$, which we can take intersection with A and get an open set U in $\mathcal{T}_A$. Thus, $f|_A$ is continuous on $(A, \mathcal{T}_A)$ to $(Y, \mathcal{T}_Y)$.

(2)

Forward direction.

Suppose f is continuous.

Pick an arbitrary open set U in $\mathcal{T}_Y$. If $U \cap B = \emptyset$, then we know that $(i \circ f)^{-1}(U) = \emptyset$ which is contained in $\mathcal{T}_X$. If $U \cap B = \tilde{U}$, we know that $\tilde{U}$ is in $\mathcal{T}_B$ by construction of subspace topology. So we have

$$(i \circ f)^{-1}(U) = (i \circ f)^{-1}(\tilde{U}) = f^{-1}(\tilde{U}) \in \mathcal{T}_X \tag{1}$$

Thus $i \circ f$ is continuous.

Backward direction.

Suppose $i \circ f$ is continuous. Pick an arbitrary $\tilde{U}$ from $\mathcal{T}_B$. We know that for all $U \in \mathcal{T}_Y$, $i^{-1}(U) = \tilde{U} : \tilde{U} = U \cap B$. So we have

$$(i \circ f)^{-1}(U) = f^{-1}(\tilde{U}) \in \mathcal{T}_X \tag{2}$$

Thus f is continuous. $\square$