PROOFS FOR TOPOLOGY

TONY DENG

1. Rewrite of previous weeks' problems

Example 5.15. If $E \subset \mathbb{R}^2$ is countable, then $\mathbb{R}^2/E$ is path connected.

Version: 2
Comments / Collaborators:

Proof. Suppose $\mathbb{R}^2/E$ is not path connected, i.e., for every pair of points in $\mathbb{R}^2/E$, we can not find a path between them in $\mathbb{R}^2/E$.

Since we know that $\mathbb{R}^2$ is connected, so we can find paths between this pair in $\mathbb{R}^2$. In fact, we can find uncountably many paths between this pair in $\mathbb{R}^2$.

By assumption, non of these path is connected in $\mathbb{R}^2/E$. This implies that at least one point on each of the path is in E. Since there are uncountably many paths, there are at least uncountably many point in E. However, since E is countable, we've reached a contradiction. $\square$

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2. This Week's Problem

1. Let D denote the closed unit disk. Define an equivalence relation on D as follows: $x \sim y$ if and only if either $x = y$ or x, y are both distance 1 from the origin. Prove that $D/\sim$ is homeomorphic to the 2-sphere.

Version: 1
Comments / Collaborators: lalala

Proof. Define a map $f : D/\sim \to S^2$, where $f(r, \theta) = (1, \theta, \pi r - \pi/2)$.

Now we want to show that this map is a homeomorphism.

First, show that the map is a bijection. For all points on S^2, it can be written as $(1, \theta, \phi)$ where θ is the angle in the horizontal plane of the sphere and ϕ is the angle vertically. Then we know that we can always find a point in D where that point is $((\phi + \pi/2)/\pi, \theta)$ such that it's the preimage of the point on the sphere.

For all points on S^2, if a random pair of points differ by θ, the preimages are definitely different; if a random pair differ by ϕ, since ϕ is between $-\pi/2$ and $\pi/2$ so there can be no two ϕ such that $(\phi_1 + \pi/2)/\pi = (\phi_2 + \pi/2)/\pi$. Hence different preimages.

Thus, the function is a bijection.

Now, we want to show that the map is continuous. Pick a random open set that doesn't contain the north point of the sphere. Since the function is a linear transformation, we know that it is continuous.

Pick a random open set that contains the north point of the sphere. The preimage would be an open set that contains the boundary of the disk, which is again open in the quotient topology.

Thus, the function is continuous.

Lastly, show that the inverse function is continuous as well. Pick a random open set that doesn't contain the boundary. Since the transformation is again, linear, it is continuous.

Pick a random open set that contains the boundary. The preimage is an open set around the north point on the sphere, which again is open in the subspace topology of S^2.

Thus, the inverse function is continuous.

Therefore, the function is a homeomorphism.

$\square$

2. Prove that if $f : X \to Y$ is a continuous surjection, X is compact, and Y is Hausdorff, then f is a quotient map.

Version: 1
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Proof. Suppose U is a set in Y, and $f^{-1}(U)$ is an open set in X.

We want to show that U is open in Y.

U is open in Y if and only if U^C is closed in Y.

Since $f^{-1}(U)$ is open in X, we know that $f^{-1}(U)^C$ is closed in X. Then we also know that $f^{-1}(U^C)$ is closed in X.

Since X is compact, f is a continuous surjection, by prop 4.12, Y is compact as well.

Since X is compact, and $f^{-1}(U^C)$ is closed in X, we know that $f^{-1}(U^C)$ is compact in X as well.

Pick an arbitrary open cover for U^C in Y. Send this open cover back to X, this will be an open cover for $f^{-1}(U^C)$. Since $f^{-1}(U^C)$ is compact in X, we know that there exists finite subcover. Pick the corresponding finite subcover in Y for U^C. Since the open cover we picked was an arbitrary one, so for any open cover of U^C, there exists a finite subcover. Thus, U^C is compact. By prop 4.7, we know that U^C is closed, thus U is open.

□

3. (Proof of the "maps on quotients" lemma) Suppose $f : X \to Z$ is a continuous map and that $\pi : X \to Y$ is a quotient map. Suppose also that $f(x) = f(x')$ whenever $\pi(x) = \pi(x')$. Prove that there exists a continuous map $\bar{f}$ so that $\bar{f} \circ \pi = f$.

Version: 1
Comments / Collaborators:

Proof. Want to show that $\bar{f}$ is continuous.

Pick an arbitrary open set in Z. Want to show that $\bar{f}^{-1}(Z)$ is open in Y.

Since f is continuous, we know that $f^{-1}(Z)$ is open in X. Since π is a quotient map, we know that $\pi(f^{-1}(Z))$ is open in Y.

Since $\bar{f} \circ \pi = f$, $\bar{f} = f \circ \pi^{-1}$, and $\bar{f}^{-1} = \pi \circ f^{-1}$.

Thus, we see that $\bar{f}^{-1}(Z) = \pi(f^{-1}(Z))$ which is open in Y. $\square$