PROOFS FOR TOPOLOGY

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1. Rewrite of previous weeks' problems

Lemma 3.5 (Pasting Lemma). Let $X = A \cup B$, where A and B are closed sets. Let $f : A \to Y$ and $g : B \to Y$ be continuous functions that agree on $A \cap B$. Then there exists a continuous function $h : X \to Y$ such that

$$h(x) = \begin{cases} f(x) & x \in A, \\ g(x) & x \in B. \end{cases}$$

Version: 3
Comments / Collaborators:

Proof. Pick an arbitrary closed set U from Y. We want to show that $h^{-1}(U)$ is closed in X.

There are several possible cases:

(i) $h^{-1}(U) = f^{-1}(U)$. In this case, since f is continuous, $f^{-1}(U)$ is closed in A. By the construction of subspace topology, there exists closed set $\tilde{U} \in X$ such that $\tilde{U} \cap A = f^{-1}(U)$. Since both A and $\tilde{U}$ are closed in X, $f^{-1}(U)$ is closed in X. So $h^{-1}(U)$ is closed in X.

(ii) $h^{-1}(U) = g^{-1}(U)$. Similar to the above case, since g is continuous, $g^{-1}(U)$ is closed in X, so $h^{-1}(U)$ is closed in X.

(ii) $h^{-1}(U) = f^{-1}(U) \cup g^{-1}(U)$. Since f and g are continuous functions, $f^{-1}(U)$ and $g^{-1}(U)$ are closed sets in X. The union of two closed sets are still closed, so $h^{-1}(U)$ is closed in X.

Thus h is continuous. □

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Proposition 3.22. All metric spaces are normal.

Version: 2
Comments / Collaborators: the hint helped a lot.

Proof. Suppose X is a metric space. So there exists a metric d on X. Suppose A and B are arbitrary disjoint closed sets in X.

Want to show there exists open sets U containing A and V containing B such that $U \cap V = \emptyset$.

Let $d(a, B) = \inf d(a, b) : \forall b \in B$. Then for each $a \in A$ we can have a $d(a, B)$. Pick $d = \inf d(a, B)_{a \in A}$

Construct $U_a = B(a, d/3)$ for each $a \in A$, take $U = \bigcup_{a \in A} U_a$. Since U_a are open, we know that U is open. We also know that there exists no element of B in U since the radius of the open balls are one third the smallest distance between A and B.

Construct V in the same fashion. $V = \bigcup_{b \in B} V_b$.

We know that there can be no element in both U and V since no element can be in the one third of the shortest distance of A and B at the same time. $\square$

2. This week's problems

Proposition 4.7. Let X be a Hausdorff topological space and let A be a subset. Prove that if A is compact in the subspace topology, then A is closed in X.

Version: 1
Comments / Collaborators: The diagram you drew on class helped a lot.

Proof. Pick an x from X not in A. Pick an a from A. Since X is Hausdorff, we know that there exists open sets U and V such that $a \in U$ and $x \in V$ and $U \cap V = \emptyset$. Find open sets U for all $a \in A$, then we know that $A \subset \bigcup_{a \in A} U_a$. Since A is compact, we know that there exists finitely many U_a's in $\{U_a\}_{a \in A}$ such that $A \subset \bigcup_n U_{a_n}$. Find the corresponding V_n's for x. Since there are finitely many V_n's, we know that the intersection of these V_n's is still open and contains x.

We can do that to all $x \in X$ that is not in A. Then union the intersection of the V_n's, $\bigcup_{x \in X} \bigcap_n V_n$,we find the A compliment, which should be open since it's the union of open sets.

Thus, A is closed. $\square$

Tarik's Corner. Show that a sequentially compact metric space is second countable. Hint: for a given natural number n, consider the ball of radius $1/n$ about a given point, x. If there is a point outside of this ball, consider the ball of radius $1/n$ about that point as well. Continue choosing new points until or unless there are no more points outside of the balls you've already formed. Show that this process has to terminate after a finite number of steps. Then, repeat for bigger and bigger n to get the desired countable basis.

Version: 1
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Proof. Suppose (X, d) is sequentially compact metric space.

Pick a $N \in \mathbb{N}$, pick $x_1 \in X$. Construct open ball $B(x_1, 1/N)$.

Pick $x_2 \notin B(x_1, 1/N)$. Construct open ball $B(x_2, 1/N)$.

Pick $x_3 \notin \bigcup_{i=1}^{2} B(x_i, 1/N)$. Construct open ball $B(x_3, 1/N)$.

Claim: we can only do this for finite number of steps.

Proof. Proof by contradiction, suppose we can do this forever.

Then we know that $x_1, x_2, \ldots, x_n \ldots$ forms a sequence. Since X is sequentially compact, there exists a convergent subsequence, x_{n_i}, such that it converges to some $x \in X$. And we know that $x \notin \bigcup_{j=n_1}^{n_i} B(x_j, 1/N)$.

Since we know that x is the limit of x_{n_i}, we know that any open sets that contains x, contains some element of x_{n_i}. Construct the open ball around x, $B(x, 1/2N)$, then we know that this open ball has to contains some x_{n_i}. However, this contradicts the fact that $x \notin B(x_{n_i}, 1/N)$ for all possible n_i. Thus, there can only be finite many x_n's. $\square$

Since for each x_n, there is a countably many open balls, and since finite union of countable sets is still countable. We have our countable basis. $\square$

Assume now that we have an open cover **U** *(***U** *is a collection of open sets) of a sequentially compact metric space, and let* **B** *be a countable basis of balls in* X *(which, by (2), we know exists).*
Tarik's Corner. Let $\mathbf{B}'$ be the subcollection of balls in $\mathbf{B}$ consisting of those balls that are contained in at least one set of $\mathbf{U}$. Show that if x is any point in X, then x is contained in at least one ball in $\mathbf{B}'$.

Version: 1
Comments / Collaborators:

Proof. Pick an arbitrary $x \in X$, Since $\mathbf{U}$ is an open cover of X, there exists some $U \subset \mathbf{U}$, such that $x \in U$. Since $\mathbf{B}$ is a basis of X, so there exist some $B \in \mathbf{B}$ such that $x \in B \subset U$. Since $B \subset U$, we know that $B \in \mathbf{B}'$. So we know that for any $x \in X$, there exists some $B \in \mathbf{B}'$ such that $x \in B$. □

Tarik's Corner. For each ball B in $\mathbf{B}'$, choose a set U_B in $\mathbf{U}$ containing B. Show that $\{U_B\}_{B \in \mathbf{B}'}$ is an open cover of X and therefore deduce that our open cover has a countable sub-cover.

Version: 1
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Proof. For any $x \in X$, there exists some $B \in \mathbf{B}'$ such that $x \in B$. Hence, there exists $U_B \in \{U_B\}_{B \in \mathbf{B}'}$ such that $x \in U_B$. Thus, $\{U_B\}_{B \in \mathbf{B}'}$ is an open cover of X. Since $\mathbf{B}$ is countable, $\mathbf{B}'$ has to be countable, thus $\{U_B\}_{B \in \mathbf{B}'}$ is a countable open cover. □

Let $\mathbf{U}'$ *denote our countable sub-cover, and choose an enumeration of it (i.e., an injection from* $\mathbf{U}'$ *to the natural numbers) so that we can list the sets as* $U_1, U_2, U_3, \ldots$*. Suppose, by way of contradiction, there exists no finite subcover.*
Tarik's Corner. Choose a point x_1 not in U_1, then another point x_2 not in the union of U_1 and U_2, then another point x_3 not in the union of the first three U's, etc. etc. Use sequential compactness to reach the desired contradiction from here.

Version: 1
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Proof. Suppose there exists no finite subcover. That means we always have an x_n that is not in the first n U's. We can have a sequence with the x_n's. Since X is sequentially compact, there exists a convergent subsequence x_{n_i} in x_n such that x_{n_i} converges to some x in X but not

$\bigcup_{n_i} U_{n_i}$. Since x is the limit of x_{n_i}, we know that for all open sets, U, containing x, U should contain infinite amount of x_{n_i}. Since $\mathbf{U}'$ are the open sets that contains the open balls, we can reach the same contradiction as in problem (2).

Thus, there must exists finite many sub-covers. $\square$

Tarik's Corner. Show that if X, Y are compact and Hausdorff and if $f : X \to Y$ is a continuous bijection, then f has to be a homeomorphism. Hint: We need to show that if U is open in X, then $f(U)$ is open in Y (think about why). It suffices to prove that the complement of $f(U)$ is compact (think about why). Take an open cover of the complement of $f(U)$ and think about what happens to it when you take the preimage under f.

Version: 1
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Proof. Since we know that f is continuous and bijective, all that's left is to show that f^{-1} is continuous as well.

Let g denote f^{-1}, g is continuous if and only if for all open set $U \in X$, $g^{-1}(U) = f(U)$ is open in Y.

$f(U)$ is open if and only if the complement of $f(U)$ is closed.

Since Y is compact and Hausdorff, $(f(U))^C$ being compact implies $(f(U))^C$ being closed.

Hence, we want to show that for all open cover of $(f(U))^C$ in Y, there exists a finite subcover.

Suppose $\bigcup^{\infty} V$ is an open cover for $(f(U))^C$. Then we know that $(f(U))^C \subset \bigcup^{\infty} V$. Take the preimage on both side, $U^C \subset f^{-1}(\bigcup^{\infty} V)$. Since U^C is closed in X, it is compact and thus there exists finite subcover, say $f^{-1}(\bigcup_1^n V_n)$ that contains U^C.

Take the image on both side, we have $(f(U))^C \subset \bigcup_1^n V_n$.

Thus, there exists finite subcover for $(f(U))^C$.

Therefore f^{-1} is continuous, f is homeomorphic.

$\square$