335 HW

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1. Rewrite

Proposition 1.6: A function $f : \mathbb{R} \to \mathbb{R}$ is $\epsilon - \delta$ continuous if and only if $f^{-1}(U)$ is open in $\mathbb{R}$ for any open set $U \subset \mathbb{R}$.

Proof. Forward direction.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is $\epsilon - \delta$ continuous, and U is an open set in $\mathbb{R}$.

Then we know that given $\epsilon > 0$, for any x, there exist $\delta > 0$, such that whenever $|x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$.

For any element $f(x) \in U$, we can find an open ball $B(f(x), \epsilon)$. By the continuity, we can find corresponding open ball $B(x, \delta)$ as a subset of the pre-image of $B(f(x), \epsilon)$. Since we know that $B(f(x), \epsilon) \subset U$, by continuity, $f^{-1}(B(f(x), \epsilon)) \subset f^{-1}(U)$. So $B(x, \delta) \subset f^{-1}(U)$. Therefore $f^{-1}(U)$ is open.

Backward direction.

Suppose there exist $U \subset \mathbb{R}$ an open set, we know that $f^{-1}(U)$ is an open set as well. Without lost of any generality, let U be an open ball $B(f(x), \epsilon)$. Since $f^{-1}(U)$ is open, we know that there exists some open ball $B(x, \delta) \subset f^{-1}(U)$. Since for all $|x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$, we have $\epsilon - \delta$ continuity for f. □

2. This week's problems

Example 1.15: (3) $\mathcal{B}_H = \mathcal{B} \cup \{(a, b) \backslash H : a < b\}$, where $H = \{\frac{1}{n}\}_{n \in \mathbb{Z}}$, is a basis for the harmonic topology $\mathcal{T}_H$.

Proof. Pick an arbitrary $x \in \mathbb{R}$.

Case 1: If $x = 0$, there exist $\epsilon > 0$ such that $0 \in B(0, \epsilon)$. Suppose $x \in B_1$ and $x \in B_2$.

Sub-case 1: $B_1, B_2 \in \mathcal{B}$, let $B_3 = (0, \min(\epsilon_1, \epsilon_2))$, where ϵ_i is the radius of the open ball B_i.

Sub-case 2: Without the loss of generality, suppose $B_1 \in \mathcal{B}$ and $B_2 = (c, d) \in \{(a, b) \backslash H : a < b\}$. If B_1 is an open ball that is a subset of B_2, it can be our wanted B_3 and satisfies condition B2. If B_1 is not a subset of B_2. Let $\epsilon_3 = \min(|c|, |d|)$ Since we know that there exist no element of H in (c, d), so for every element in $B(0, \epsilon_3)$, they don't exist in H. We know that ϵ_3 will be smaller than ϵ_1 since B_1 is not a subset of B_2.

(Personally, I think this sub-case is impossible since H is oscillating around 0, so it should be impossible to find such B_2.)

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Sub-case 3: $B_1 = (a, b)$ and $B_2 = (c, d)$ for some $a < b$ and $c < d$, and B_1, B_2 doesn't have any element in H.

Let $e = \max(a, c)$ and $f = \min(b, d)$. Let $B_3 = (e, f)$, we know that 0 is in B_3 and, there's no element of H in B_3, which is a subset of the intersection of B_1 and B_2 by observation.

Case 2: If $x \notin H$ and $x \neq 0$, by the formation of $\mathcal{B}_H$, we can find such $a, b \in \mathbb{R}$ such that $x \in (a, b)\backslash H$. If such $x \in (a, b)\backslash H$ and in $(c, d)\backslash H$, pick new interval $(e, f) = (\max(a, c), \min(b, d))$. We know that $x \in (e, f)$ for sure and $(e, f) \subset (a, b) \cap (c, d)$.

Case 3: If $x \in H$, find open ball $B(x, \epsilon)$ with $\epsilon > 0$. we know that $x \in B(x, \epsilon)$. If such x is in $B(x, \epsilon_1)$ and $B(x, \epsilon_2)$ for $\epsilon_1, \epsilon_2 \in \mathbb{R}$ and greater than 0, we know that $x \in B(x, \min(\epsilon_1, \epsilon_2)$ and $B(x, \min(\epsilon_1, \epsilon_2) \subset B(x, \epsilon_1) \cap B(x, \epsilon_2)$. $\square$

Example 1.18: On $\mathbb{R}$, compare $\mathcal{T}, \mathcal{T}_l$, and $\mathcal{T}_{l,\mathbb{Q}}$.

Claim: $\mathcal{T}_l$ is strictly finer than $\mathcal{T}$, and $\mathcal{T}_l$ is finer than $\mathcal{T}_{l,\mathbb{Q}}$.

Proof. Without the loss of generality, pick an arbitrary open set $(a, b) \in \mathbb{R}$. Let $x = \frac{a+b}{2}$, which is the in the middle of (a, b).

Construct a union of open sets in $\mathcal{T}_l$ such that the lower limit of the open sets goes to a yet never touches it; the upper limit of the open sets can b. By T2, we know that the union of the sets in $\mathcal{T}_l$ has to be in $\mathcal{T}_l$ as well, so we know that (a, b) is in $\mathcal{T}_l$ as well.

Thus, we know $\mathcal{T} \subset \mathcal{T}_l$.

Pick an arbitrary open set in $\mathcal{T}_l$, say, $[a, b)$.

Proof by contradiction. Suppose $[a, b)$ is in $\mathcal{T}$. Then there exist an open ball $B(a, \epsilon), \epsilon > 0$ such that it equals to $[a, b)$. However, since it's an open ball, there exists elements that is smaller than a which is not in $[a, b)$. Which contradicts with our assumption. Thus $\mathcal{T}_l$ is strictly finer than $\mathcal{T}$.

$\square$

We want to show that $\mathcal{T}_l$ is finer than $\mathcal{T}_{l,\mathbb{Q}}$.

Proof. For every open set in $\mathcal{T}_{l,\mathbb{Q}}$, we know that it's in $\mathcal{T}_l$ since every rational number is a real number. So for any $[a, b)$ in $\mathcal{T}_{l,\mathbb{Q}}$, we know that it has to be in $\mathcal{T}_l$. $\square$

Example 1.22: There is a metric on $\mathbb{R}$ whose associated topology is not the same as the standard topology.

Proof. Let our metric d be a function such that $d(x, y) = |x^3 - y^3|$. Since it's in the absolute value sign, we know that $d(x, y) \geq 0$.

Want to show that $d(x, y) = 0$ if and only if $x = y$.

Suppose $d(x, y) = 0$. So we know that $|x^3 - y^3| = 0$. Without the loss of generality, suppose $x^3 \geq y^3$. Then we have

$$x^3 - y^3 = 0$$
$$x^3 = y^3$$

Take the cube root, since cube doesn't change signs, so we don't need to change signs.

$$x = y$$

Suppose $x = y$. Then $d(x, y) = |x^3 - y^3| = |x^3 - x^3| = 0$.

So $d(x, y) = 0$ if and only if $x = y$.

Want to show that $d(x, y) = d(y, x)$. Without loss of generality, suppose $x \geq y$. Then we have

$$\begin{aligned} d(x, y) &= x^3 - y^3 \\ &= -(y^3 - x^3) \\ &= |y^3 - x^3| = d(y, x) \end{aligned}$$

Want to show that $d(x, y) + d(y, z) \geq d(x, z)$.

$$d(x, y) + d(y, z) = |x^3 - y^3| + |y^3 - z^3|$$

By the triangle inequality, we have

$$|x^3 - y^3| + |y^3 - z^3| \geq |x^3 - y^3 + y^3 - z^3| = d(x, z)$$

Construct the metric topology $\mathcal{T}_d$ on the metric space $(\mathbb{Q}, d)$, defined by the basis

$$\mathcal{B}_d = \{B(x, \epsilon) = \{y \in \mathbb{Q} : d(x, y) < \epsilon\} \text{for all } x \in \mathbb{Q}, \epsilon > 0\}$$

Want to show that $\mathcal{T}_d$ is not the standard topology.

For each open balls in $\mathcal{T}_d$, when hold the ϵ the same, the open balls in $\mathcal{T}$ will always be bigger and $\mathcal{T}_d$ is not the same as $\mathcal{T}$. $\square$