PROOFS FOR TOPOLOGY

TONY DENG

1. Rewrite of previous weeks' problems

Proposition 4.7. Let X be a Hausdorff topological space and let A be a subset. Prove that if A is compact in the subspace topology, then A is closed in X.

Version: 2
Comments / Collaborators: The diagram you drew on class helped a lot.

Proof. Pick an x from X not in A. Pick an a from A. Since X is Hausdorff, we know that there exists open sets U and V such that $a \in U$ and $x \in V$ and $U \cap V = \emptyset$. Find open sets U for all $a \in A$, then we know that $A \subset \bigcup_{a \in A} U_a$. Since A is compact, we know that there exists finitely many U_a's in $\{U_a\}_{a \in A}$ such that $A \subset \bigcup_n U_{a_n}$. Find the corresponding V_n's for x. Since there are finitely many V_n's, we know that the intersection of these V_n's is still open and contains x.

Since these V_n's are disjoint from corresponding U_n's, then the intersection of these V_n's are definitly disjoint from the union of the U_n's. Thus, the intersection of these V_n's are disjoint from A.

We can do that to all $x \in X$ that is not in A. Then union the intersection of the V_n's, $\bigcup_{x \in X} \bigcap_n V_n$,we find the A compliment, which should be open since it's the union of open sets.

Thus, A is closed. $\square$

Date: November 9, 2021.

2. This week's problem

In class yesterday, I misspoke when I said that the theorem we ended with could technically be proved without the invariance of domain theorem. The point of the first exercise is to run through a correct proof of that theorem.

We want to show that if M is a connected manifold, then there exists a natural number n so that for every point $x \in M$, there is an open neighborhood of x that is homeomorphic toR^n. And recall that the invariance of domain theorem states that R^n is homeomorphic to R^m if and only if $m = n$ (the best way to prove this theorem uses algebraic topology...we'll cover some basic algebraic topology at the end of the semester but we won't get to the machinery needed for this theorem. But in case you're interested, the machinery is called homology theory).

In Exercise (2), we will in fact use the slightly stronger sounding version of the invariance of domain theorem (it's actually equivalent but it sounds a little stronger): if n is strictly less than m, then no subspace of R^n is homeomorphic to R^m.

Tarik's Corner: exercise (1). (a) Consider the function $f : M \to$ (Natural numbers, subspace topology with respect to the standard topology on R) which for each $x \in M$, outputs the minimum n such that there is a neighborhood about x which is homeomorphic to R^n. Use the invariance of domain theorem to prove that f is continuous.

(b) Use the fact that M is connected to show that f must be a constant function.

So, in a connected manifold M, there is indeed one n such that every point has a neighborhood homeomorphic to R^n.

Version: 1
Comments / Collaborators: I assumed that any open ball in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ without proving it...

Proof. (a): We want to show that f is continuous. Showing that for all open sets, U, from $(\mathbb{N}, \mathcal{T})$, $f^{-1}(U)$ is open in M would achieve our goal.

By the definition of subspace topology, we would have singleton sets $\{n\}$, where $n \in N$, open in $\mathbb{N}$.

Take the preimage of $\{n\}$, we get

(1)
$$V = \{x \in M : \exists U \text{ such that } x \in U, \text{ and } U \text{ is homeomorphic to } \mathbb{R}^n, \\ \text{where } n \text{ is the minimum possible integer.}\}$$

Claim: $V = \bigcup_x U_x$.

Proof. Since for all $x \in V$, x is in some U_x, so V is a subset of $\bigcup_x U_x$.

Want to show that there exists no y in U_x such that there exists U_y containing y homeomorphic to some $\mathbb{R}^m$ where $m < n$.

Let $U = U_x \cap U_y$. Since U_x and U_y are both open sets U must be open as well. Since U_x is homeomorphic to $\mathbb{R}^n$, and $U \subset U_x$, then we know that U is homeomorphic to $\mathbb{R}^n$. By the same logic U is homeomorphic to $\mathbb{R}^m$ as well. However, this contradicts with the invariance domain theorem. Thus, there exists no $y \in U_x$ such that y is contained in another open set that is homeomorphic to some $\mathbb{R}^m$ where $m < n$. $\square$

Since U_x's are open, V is open.

Thus f is continuous.

(b) Proof by contrapositive.

Suppose that f is not a constant function, i.e., there exists $n, m \in \mathbb{N}$ such that $f(U) \to n$ or $f(V) \to m$, where U, V are open in M.

Let $U, V \neq \emptyset$ by the virtue of construction.

We know that $U \cap V = \emptyset$ by the same argument around the invariance domain theorem in the last proof.

We also know that $U \cap V = M$, since otherwise, there would exists some $w \in \mathbb{N}$, such that $f(M - U - V) = w$, which is false by construction.

Thus, we have a separation for M.

Thus, if M is connected, then f has to be a constant function. $\square$

Tarik's Corner: exercise (2). Use the invariance of domain theorem again to show that in a connected manifold M, there is exactly one n such that every point has a neighborhood homeomorphic to R^n.

Exercises (1)+(2) imply that it makes sense to a refer to a connected manifold as an $n-$dimensional manifold (or a $n-$manifold for short).

Version: 1
Comments / Collaborators:

Proof. In part (1), we've shown that for all $x \in M$, the smallest n for x to have a neighborhood that is homeomorphic to $\mathbb{R}^n$ is the same for a connected manifold. Thus, we now need to show that there is no $m \in \mathbb{N}$ such that $m > n$, and all points in M has a neighborhood that is homeomorphic to $\mathbb{R}^m$.

Use proof by contradiction, we suppose that there exists some $m > n$ such that for all $x \in M$, we can find a neighborhood of x, say U_m such that U_m is homeomorphic to $\mathbb{R}^m$.

Since we also know that for all $x \in M$, there also exists U_n such that U_n is homeomorphic to $\mathbb{R}^n$, we can take the intersection of U_n and U_m and still have an open set.

Let $U = U_m \cap U_n$, we know that U is still open in M.

Since $U \subset U_m$, we know that U is homeomorphic to $\mathbb{R}^m$. Since $U \subset U_n$, we know that U is homeomorphic to $\mathbb{R}^n$.

So U is homeomorphic to both $\mathbb{R}^n$ and $\mathbb{R}^m$, which contradicts the invariance domain theorem.

Thus, there is only one n. $\square$

(3). Prove that the unit sphere in R^3 is a connected 2-manifold.

Version: 1
Comments / Collaborators: I used that an open ball in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ again...

Proof. Construct an open cover of the unit sphere. Let the open sets in our cover be the intersection between open balls with radius 1/3 and the unit sphere.

We know that the shape of our open sets will be a curved circle with radius less than 1/3.

Construct the homeomorphism by taking the normal vector of our open sets, align it with the direction where it is perpendicular to the $x - y$ plane, project our open set on $\mathbb{R}^2$.

We can observe that the projection is an open ball in $\mathbb{R}^2$ which is homeomorphic to $\mathbb{R}^2$.

So S^2 is a 2-manifold.

Now we need to show that the unit sphere is a connected space. If the unit sphere is path connected, then it is connected. Given any two points, label them using spherical coordinates, say (R, ϕ_1, θ_1) and (R, ϕ_2, θ_2). Construct the path between these two points by first moving in the ϕ direction ,then in θ direction.

□

(4). The real projective plane is a topological space defined as follows: as a set, it is the set of all lines that go through the origin in R^3. We put a topology on this by saying that a set of lines is open if and only if the set of all points (except for the origin!) in R^3 contained on any of those lines is an open subset of R^3-{origin} (in the subspace topology with respect to the standard topology on R^3). Prove that the real projective plane is compact. (Hint: try to find a continuous onto function from a space that you can easily prove is compact by using the Heine-Borel theorem.)

Version: 1
Comments / Collaborators: i'm lost...

Proof. Pick the unit sphere in $\mathbb{R}^4$. Since we know that the unit sphere in $\mathbb{R}^4$ is closed and bounded, we know that S^4 is a sequentially compact by the Heine-Borel theorem.

Construct a function $\phi : S^2 \to$ the real projective plane, such that

$$\phi((x_1, x_2, x_3, x_4)) = \begin{cases} (x_1, x_2, x_3) & \text{if } x_4 = 0, \\ (x_1/x_4, x_2/x_4, x_3/x_4) & \text{else.} \end{cases}$$

This function is onto since for all possible (x, y, z) in $\mathbb{R}^3$, if (x, y, z) is on the unit sphere, then it can be hit by $(x, y, z, 0)$; if not on the unit sphere, we can always find the corresponding (x_1, x_2, x_3, x_4) by solving the system of equations $x = x_1/x_4, y = x_2/x_4, z = x_3/x_4, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$.

Now, we want to show that this function is continuous.

Pick an arbitrary open set from the projective plane, take the preimage of this open set, (I don't know how to continue from here...I'm not sure if the function I found is continuous... $\square$