PROOFS FOR TOPOLOGY

TONY DENG

BEFORE PRESENTATION

Lemma 3.4. A function $f : X \to Y$ is continuous if and only if for every closed set $C \subset Y$, $f^{-1}(C)$ is closed in X.

Version: 1
Comments / Collaborators: very intuitive.

Proof. Suppose $f : X \to Y$ is a continuous function. For every open set U in Y, we know that $f^{-1}(U)$ is an open set in X. Let $C = Y/U$, we know that C is closed. $f^{-1}(C) = f^{Y/U} = X/f^{-1}(U)$. Since $f^{-1}(U)$ is open in X, $f^{-1}(C)$ is closed in X.

Suppose for all closed sets, C, in Y, $f^{-1}(C)$ is closed in X. Pick an arbitrary U open in Y. We know that there exists closed set C such that $U = Y/C$. $f^{-1}(U) = f^{-1}(Y/C) = X/f^{-1}(C)$. Since $f^{-1}(C)$ is closed in X, $f^{-1}(U)$ is open in X. Thus f is continuous. $\square$

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Lemma 3.5 (Pasting Lemma). Let $X = A \cup B$, where A and B are closed sets. Let $f : A \to Y$ and $g : B \to Y$ be continuous functions that agree on $A \cap B$. Then there exists a continuous function $h : X \to Y$ such that

$$h(x) = \begin{cases} f(x) & x \in A, \\ g(x) & x \in B. \end{cases}$$

Version: 1
Comments / Collaborators: Wonder if it would work when A, B are both open, one open one closed...

Proof. Pick an arbitrary closed set U from Y. We want to show that $h^{-1}(U)$ is closed in X.

There are several possible cases:

(i) $h^{-1}(U) = f^{-1}(U)$. In this case, since f is continuous, $f^{-1}(U)$ is closed, so $h^{-1}(U)$ is closed.

(ii) $h^{-1}(U) = g^{-1}(U)$. Similar to the above case, since g is continuous, $g^{-1}(U)$ is closed, so $h^{-1}(U)$ is closed.

(ii) $h^{-1}(U) = f^{-1}(U) \cup g^{-1}(U)$. Since f and g are continuous functions, $f^{-1}(U)$ and $g^{-1}(U)$ are closed sets. The union of two closed sets are still closed, so $h^{-1}(U)$ is closed.

Thus h is continuous. $\square$

Proposition ?? 3.7. If A and B are subsets of a topological space X, then:

(1) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(2) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

Version: 1

Comments / Collaborators: Used some knowledge from Analysis.

Proof. (1) proof by double containment.

For all elements in $\overline{A \cup B}$, we know that this element is either in A or in B or serves as a limit point of either A or B. So, we can express this as $A \cup B \cup L_A \cup L_B$, where L_A is the set of limit points of A and L_B is the set of limit points of B.

Since A and L_A is equal to $\overline{A}$ by definition of the closure, and B and L_B is equal to $\overline{B}$ by the same reasoning, we can see that all possible element from $\overline{A \cup B}$ is in $\overline{A} \cup \overline{B}$.

The backward direction is the same since all relationships above are in equal sign.

(2) This claim is false.

Let $X = \mathbb{R}$ with the standard topology. Let $A = (0, 1), B = (1, 2)$. Then $\overline{A} = [0, 1], \overline{B} = [1, 2]$, and $\overline{A \cup B} = \emptyset$. Obviously, $\overline{A} \cup \overline{B} = [0, 2] \neq \emptyset = \overline{A \cup B}$. ∎