Topology Lecture Notes

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All errors in these notes are my responsability, and not of the presenters. On the other way around, all nice things from these notes are the presenters' responsability, not mine.

Proposition (Proposition 1.27)

The collection $\mathcal{B}_\times$ defined above is a basis for a topology.

Proof (by Jess). Let $\mathcal{B}_\times = \{U \times V : U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y\}$. Note that the elements of $\mathcal{B}_\times$ are sets whose elements are ordered pairs (u, v), in which $u \in U$ and $v \in V$.

For example, if we consider $(U, \mathcal{T}_X) = (V, \mathcal{T}_Y) = (\mathbb{R}, \mathcal{T})$, then $\{(x, y) : 0 < x < 1, 0 < y < 1\} \in \mathcal{B}_\times$.

We want to show that $\mathcal{B}_\times$ is a basis, so we need to prove the two defining conditions.

(B1) Given $a = (x, y) \in X \times Y$, there exists $B \in \mathcal{B}_\times$, such that $a \in B$.

Because $\mathcal{T}_X$ is a topology on X, there exists $U \in T_X$ so that $x \in U$. Because $\mathcal{T}_Y$ is a topology on Y, there exists $V \in T_Y$ so that $y \in V$.

Then, $U \times V \in \mathcal{B}_X$ and $a = (x, y) \in U \times V$.

(B2) If $x \in B_1 \cap B_2$, with $B_1, B_2 \in \mathcal{B}_\times$, then there exists $B_3 \in \mathcal{B}_\times$ so that $x \in B_3 \subset B_1 \cap B_2$.

We'll prove a stronger condition. We'll prove that, given B_1, B_2 in the basis, then their intersection is also in the basis.

Note that

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2), \tag{1}$$

and $U_1 \cap U_2 \in \mathcal{T}_X$ and $V_1 \cap V_2 \in \mathcal{T}_Y$.

Therefore, $\mathcal{B}_\times$ is, in fact, a basis. ∎

Questions from the Audience

Q. I don't understand equation 1.

A. If $(x, y) \in U_1 \times V_1$, then $x \in U_1$ and $y \in V_1$. If $(x, y) \in U_2 \times V_2$, then $x \in U_2$ and $y \in V_2$. Therefore, $x \in U_1 \cap U_2$ and $y \in V_1 \cap V_2$. Thus, $(x, y) \in (U_1 \cap U_2) \times (V_1 \cap V_2)$.

Tarik's Corner

Exercise (Exercise 7). Draw a picture in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ to show that if we define the product topology to consist of products of open sets, then axiom T2 of a topology does not hold.

Proof. Suppose $\mathcal{T}$ is a topology. Consider the open squares $(0,1) \times (0,1)$ and $(2,3) \times (2,3)$. Axiom T2 implies that their union should be open, but their union is not in $\mathcal{T}_\times$, as it is defined. ∎

Question. What about the product topology of infinitely many sets?

Definition (Product Topology)

If $\{X_i\}_{i \in I}$, where $card(I)$ is inifinite, then the *product topology* on $\prod_{i \in I} X_i$ is given by a basis

$$\mathcal{B}_\times = \left\{ \prod_{i \in I} U_i : U_i \text{ is open in } X_i \text{ and } U_i = X_i \text{ for all but finitely many } i \right\}$$

Definition (Box Topology)

If $\{X_i\}_{i \in I}$, where $card(I)$ is inifinite, then the *box topology* on $\prod_{i \in I} X_i$ is given by a basis

$$\mathcal{B}_\times = \left\{ \prod_{i \in I} U_i : U_i \text{ is open in } X_i \right\}$$

Theorem (Theorem we'll prove in the future)

The ∞-dimensional cube $\prod_{i=1}^{\infty} [0,1]$ is compact in the product topology, but not compact on the box topology.

Proposition (Example 1.28)

The product topology $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the same as the standard topology on $\mathbb{R}^2$, given by the Euclidean distance.

Proof (by Young). Let $\mathcal{T}_\times$ be the product topology, $\mathcal{T}_d$ the standard topology, and consider their bases

$$\begin{aligned} \mathcal{B}_\times &= \{U \times V : U, V \text{ open in } \mathbb{R}\}, \\ \mathcal{B}_d &= \{B((x,y),\epsilon) : (x,y) \in \mathbb{R}^2 \text{ and } \epsilon > 0\}. \end{aligned}$$

Proposition 1.17 implies that it is enough to show that $\mathcal{B}_\times \subset \mathcal{T}_d$ and $\mathcal{B}_d \subset \mathcal{T}_\times$.

First, we'll show that $\mathcal{B}_\times \subset \mathcal{T}_d$.

Let $(x,y) \in B = U \times V \in \mathcal{B}_\times$. Therefore, $x \in U$ and $y \in V$. Since U and V are open in $\mathbb{R}$, there are open intervals centered on x, and y that are contained in U, and V, respectively. Let's call those intervals $(x - \epsilon_1, x + \epsilon_1) \subset U$ and $(y - \epsilon_2, y + \epsilon) \subset V$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then,

$$B(\alpha, \epsilon) \subset (x - \epsilon_1, x + \epsilon_1) \times (y - \epsilon_2, y + \epsilon_2) \subset U \times V,$$

therefore, every $B \in \mathcal{B}_\times$ is also in $\mathcal{T}_d$.

Now, we'll show that $\mathcal{B}_d \subset \mathcal{T}_\times$.

Let $(x,y) \in B \in \mathcal{T}_d$. Since B is open in the standard topology in $\mathbb{R}$, there exists a ball centered at (x,y) contained inside B, namely $B((x,y),\delta)$.

Then $I = (x - \delta/2, x + \delta/2) \times (y - \delta/2, y + \delta/2)$ is inside $B((x,y),\delta)$; therefore, every $B \in \mathcal{B}_d$ is also in $\mathcal{T}_\times$. ∎

Definition (Definition 2.1)

A function $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is *continuous* if $f^{-1}(U)$ is open in X for every open set $U \subset Y$.

Lemma (Lemma 2.2)

Suppose $\mathcal{B}$ is a basis for $\mathcal{T}_Y$. Then a function f is continuous if $f^{-1}(B)$ is open for all $B \in \mathcal{B}$.

Proof (by Nico). Let $U \in \mathcal{T}_Y$. Recall that we can write $U = \bigcup_{\alpha \in I}$ for a collection $\{B_\alpha\}_{\alpha \in I} \subset \mathcal{B}$. However,

$$\begin{aligned} f^{-1}(U) &= f^{-1}\left(\bigcup_{\alpha \in I} B_\alpha\right) \\ &= \{x \in X : f(x) \in \bigcup_{\alpha \in I} B_\alpha\} \\ &= \{x \in X : f(x) \in B_\alpha \text{ for some } \alpha \in I\} \\ &= \{x \in X : x \in f^{-1}(B_\alpha) \text{ for some } \alpha \in I\} \\ &= \bigcup_{\alpha \in I} f^{-1}(B_\alpha), \end{aligned}$$

which is a union of open sets, thus $f^{-1}(U)$ is open. ∎