TOPOLOGY COURSE NOTES (FALL 2021)

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PROOFS BY STUDENTS OF MATH 335

Contents

Preface

The goal of this course is to introduce you to the fundamental examples, problems, and machinery of topology. The central concepts in topology are those of *nearness* of points in a set and *continuity* of functions between sets for which a concept of nearness has been defined. The goal of topology is to study properties of sets that are preserved under continuous functions, as well as properties of the continuous functions themselves. The result is a general and powerful theory that can make deep qualitative statements in a variety of situations, from the classic "rubber sheet geometry" formulation of topology, to vector calculus, dynamical systems, and even to data analysis!

In particular, I hope to get to the following theorems: the Brouwer Fixed Point Theorem (2D version), Invariance of Domain (2D version), the reason

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your keys don't fall off your keychain, and how to keep your eye upon the doughnut and not upon the hole.

You have already encountered some topological ideas in your real analysis course: there, a norm gave a notion of nearness, and you studied such topological properties as continuity, compactness, and connectedness. After pinning down more abstract notions of nearness and continuity, we will spend the first part of the course revisiting these — and other — properties in a more general setting. In the second part of the course, we will develop methods for constructing examples — especially for bootstrapping from simple to more interesting ones. In the final part of the course, we will use algebraic tools to analyze topological problems once they have been adequately described; this will lead us into the realm of "higher connectedness" and algebraic topology.

The Moore Method. These notes are designed to be used in a course taught using some version of the Moore Method. That is, the students reading these notes are expected to create to proofs of the propositions and examples, guided by the professor. Here are several typographical features of these notes to keep in mind:

(1) A statement labeled "Proposition??" is really a true-or-false question: you should either prove the statement or give a counterexample.
(2) A statement labeled "Lemma…" or "Proposition…" is really a fill-in-the-blank question: you should complete the statement of the lemma or proposition and then prove it.
(3) A statement with a $\star$ after its label requires a more difficult proof, for which you can collect up to two bonus points (see the syllabus!).
(4) No written proofs are required (or accepted) for statements labeled "Exercise", but these statements are useful to think about.

Since you will be bereft of two of the main sources you used to prove things in your previous courses — i.e. lecture notes and a textbook — it will behoove you to remember the following proof-creation strategies when — *not* if — you get stuck:

- Do *something*! Don't stare at a blank sheet of paper / chalkboard / spot on the ceiling!
- Remember that the logical form of the proof is dictated by the logical form of the theorem.
- Draw a picture, whether it be an accurate representation of an example or a cartoon picture of an abstract theorem (a "cartoon picture" in point-set topology is frequently a picture in $\mathbb{R}^2$ with its usual Euclidean topology).
- Work out a special case. That is, start by working out an example that satisfies the hypotheses, or add additional hypotheses to make the theorem easier.

- Work backwards: think of a statement that would imply the conclusion, and then try to prove that statement. This is also known as using "wishful thinking".
- Do you understand where the hypotheses come from? Can you prove the theorem without one or more of the hypotheses, or can you construct a counterexample?

Analysis I is a prerequisite for this course, and these notes take advantage of that fact. You may assume that any proposition or theorem labeled "from Analysis" has already been proven. Note that there will be a few propositions that you saw in analysis, but that you will need to prove again — you might recognize these propositions, but they will not be labeled "from Analysis".

Sources. These notes have mostly grown out of the lecture notes and homework I have used for previous, lecture-based incarnations of Math 335. I was also directly inspired by and borrowed from Starbird's Moore Method course notes [4] and Su's Moore Method course notes [5]. My old lecture notes themselves came out of courses based on Munkres' *Topology* [3], McCleary's *A First Course in Topology* [2], and Hatcher's *Algebraic Topology* [1].

Part 1. Point-Set Topology

1. Topological Spaces

1.1. Motivation from Analysis.

Let us begin our study of topology with a familiar definition from analysis (and hence with familiar intuition and a familiar store of examples). We let $|\cdot|$ denote the usual absolute value function on $\mathbb{R}$; the key idea is that we can use the absolute value to define the distance between x and y to be $|x-y|$. We can then define continuity of a function at a point.

Definition 1.1. A function $f : \mathbb{R} \to \mathbb{R}$ is $\epsilon - \delta$ **continuous at** $a \in \mathbb{R}$ if for all $\epsilon > 0$, there exists $\delta > 0$ so that if $|x-a| < \delta$, then $|f(x) - f(a)| < \epsilon$. We say that f is $\epsilon - \delta$ **continuous** if it is $\epsilon - \delta$ continuous at all points in its domain.

As you did in analysis, we would like to rephrase this definition in terms of sets rather than inequalities. To do so, we begin by defining the **open ball** of radius ϵ about a point $a \in \mathbb{R}$ to be:

$$B(a, \epsilon) = \{x \in \mathbb{R} \ : \ |x-a| < \epsilon\}.$$

Exercise 1. Use the definitions of subsets and of preimages to convince yourself that the implication in Definition 1.1 can be rewritten as $B(a, \delta) \subset f^{-1}(B(f(a), \epsilon)$.

Generalizing the definition of an open ball further, we obtain:

Definition 1.2. A set $U \subset \mathbb{R}$ is **open** if for all $x \in U$, there exists a $\delta_x > 0$ so that $B(x, \delta_x) \subset U$.

Let us pause to investigate some basic properties of open sets in $\mathbb{R}$.

Proposition 1.3. *Both $\mathbb{R}$ and the empty set are open.*

Exercise 2. Convince yourself that, in fact, any open interval (a, b) is open, but that the set $[0, 1)$, say, is not open.

Proposition 1.4. *If each set in the collection $\{U_\alpha\}_{\alpha \in I}$ of subsets of $\mathbb{R}$ is open, then so is $\bigcup_\alpha U_\alpha$ (i.e. arbitrary unions of open sets are open).*

Proposition?? 1.5. *If $\{U_\alpha\}_{\alpha \in I}$ is a collection of open sets, then $\bigcap_\alpha U_\alpha$ is also open.*

We now return to the question of how to use open sets to think about continuity. Here is the key proposition:

Proposition 1.6. *A function $f : \mathbb{R} \to \mathbb{R}$ is $\epsilon - \delta$ continuous if and only if $f^{-1}(U)$ is open in $\mathbb{R}$ for any open set $U \subset \mathbb{R}$.*

1.2. **The Definition of a Topology.** In order to extend the ideas of nearness and continuity to a wider range of situations, we take the properties of open sets in $\mathbb{R}$ in the propositions above to be the axioms for a topology.

Definition 1.7. A **topological space** $(X, \mathcal{T})$ is a set X together with a collection $\mathcal{T}$ (called a **topology**) of subsets of X that satisfies:

- **T1:** $\emptyset, X \in \mathcal{T}$,
- **T2:** If $U_\alpha \in \mathcal{T}$ for all $\alpha \in I$, then $\bigcup_\alpha U_\alpha \in \mathcal{T}$, and
- **T3:** If $U_0, U_1 \in \mathcal{T}$, then $U_0 \cap U_1 \in \mathcal{T}$.

We say that the sets in $\mathcal{T}$ are **open**, or more precisely **open with respect to the topology $\mathcal{T}$**.

You should keep in mind that open sets are *elements* of $\mathcal{T}$ and *subsets* of X. It is not a bad idea to read the definition above out loud to yourself, saying "is open" in place of "$\in \mathcal{T}$". We will eventually drop the word "topological" and refer to a topological space as simply a "space".

Note that Propositions 1.3, 1.4, and the corrected Proposition 1.5 prove that the open sets of Definition 1.2 form a topology (called the **standard topology $\mathcal{T}$**) on $\mathbb{R}$.

Proposition 1.8. *Let $(X, \mathcal{T})$ be a topological space, and let $A \subset X$. Suppose that for each $x \in A$, there is an open set U containing x such that $U \subset A$. The set A is open (i.e. that $A \in \mathcal{T}$).*

Example 1.9. *The following are some toy examples (as well as future counterexamples for you to use!) to help get you used to the definition of a topology. We will gradually build up more sophisticated examples over the next few weeks.*

(1) *The **discrete topology** on a set X is given by:*

$$\mathcal{T}_{disc} = \mathcal{P}(X).$$

(2) *The **indiscrete topology** on a set X is given by:*

$$\mathcal{T}_{indisc} = \{\emptyset, X\}.$$

(3) *The **finite complement topology** on an infinite set X is given by:*

$$\mathcal{T}_{FC} = \{U \subset X \ : \ X \setminus U \text{ is finite or all of } X\}.$$

1.3. **Basis for a Topology.** In the examples above, we directly specified *all* of the open sets in the topology. In general, however, axiom T2 will necessitate that open sets become extremely complicated. Can we find a more efficient way of specifying a topology? By analogy with your study of linear algebra, can we specify *some* of the open sets and let them somehow generate all of the others in the same way that we can use linear combinations of a basis for a vector space to specify all vectors? The model we want to pursue is the motivating example from analysis: we specified the open sets of $\mathbb{R}$ by specifying open balls (a simple collection of sets!) and then defined open sets using open balls.

Definition 1.10. A collection $\mathcal{B}$ of subsets of X is a **basis for a topology** if it satisfies:

B1: For all $x \in X$, there exists $B \in \mathcal{B}$ so that $x \in B$, and
B2: If $x \in B_1 \cap B_2$ for $B_i \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Exercise 3. Draw a "cartoon picture" of condition B2.

Definition 1.11. If $\mathcal{B}$ is a basis on a set X, then

$$\mathcal{T}_{\mathcal{B}} = \{U \subset X \ : \ \forall x \in U, \exists B_x \in \mathcal{B} \text{ s.t. } x \in B_x \subset U\}$$

is the **topology generated by** $\mathcal{B}$; we also say that $\mathcal{B}$ is a basis for $\mathcal{T}_{\mathcal{B}}$.

Proposition 1.12. *$(X, \mathcal{T}_{\mathcal{B}})$ is a topological space.*

The following proposition describes an alternative way to characterize the open sets in the topology generated by a basis.

Proposition 1.13. *Given a set X and a basis $\mathcal{B}$ let*

$$\mathcal{T}'_{\mathcal{B}} = \Big\{U \subset X \ : \ U = \bigcup_{\alpha \in I} B_\alpha \text{ for some collection } \{B_\alpha\}_{\alpha \in I} \subset \mathcal{B}\Big\}.$$

Then $\mathcal{T}_{\mathcal{B}} = \mathcal{T}'_{\mathcal{B}}$.

Example 1.14. *In $\mathbb{R}$, the collection of open balls $\mathcal{B} = \{B(x, \epsilon) \ : \ x \in \mathbb{R}, \epsilon > 0\}$ is a basis for a topology, and the topology it generates is the standard topology on $\mathbb{R}$.*

Example 1.15. *In $\mathbb{R}$, the following collections of sets are bases for a topology on $\mathbb{R}$:*

(1) $\mathcal{B}_\ell = \{[a,b) \;:\; a < b\}$ *is a basis for the* ***lower limit topology*** $\mathcal{T}_\ell$.
(2) $\mathcal{B}_{\ell,\mathbb{Q}} = \{[a,b) \;:\; a < b$ *with* $a, b \in \mathbb{Q}\}$ *is a basis for the* ***rational lower limit topology*** $\mathcal{T}_{\ell,\mathbb{Q}}$.
(3) $\mathcal{B}_H = \mathcal{B} \cup \{(a,b) \setminus H \;:\; a < b\}$, *where* $H = \{\frac{1}{n}\}_{n\in\mathbb{Z}}$, *is a basis for the* ***harmonic topology*** $\mathcal{T}_H$.

We now have several different topologies on $\mathbb{R}$: $\mathcal{T}$, $\mathcal{T}_\ell$, $\mathcal{T}_{\ell,\mathbb{Q}}$, $\mathcal{T}_{disc}$, $\mathcal{T}_{indisc}$, and $\mathcal{T}_{FC}$. How are they related? The language we need to properly ask this question is:

Definition 1.16. Given two topologies $\mathcal{T}$ and $\mathcal{T}'$ on a set X, we say that $\mathcal{T}$ is **finer** than $\mathcal{T}'$ if $\mathcal{T}' \subset \mathcal{T}$ (that is, $\mathcal{T}$ has more open sets than $\mathcal{T}'$). We say that $\mathcal{T}$ is **strictly finer** than $\mathcal{T}'$ if $\mathcal{T}' \subsetneq \mathcal{T}$.

Exercise 4. What can you say about comparisons to $\mathcal{T}_{disc}$? To $\mathcal{T}_{indisc}$?

A useful criterion for showing that one topology is finer than another uses the idea of a basis:

Proposition 1.17. *Suppose* $\mathcal{B}$ *and* $\mathcal{B}'$ *are bases on a set* X. *Then* $\mathcal{T}_\mathcal{B}$ *is finer than* $\mathcal{T}_{\mathcal{B}'}$ *if and only if for all* $B' \in \mathcal{B}'$, *there exists a collection of sets* $\{B_\alpha\}_{\alpha\in I}$ *in* $\mathcal{B}$ *such that* $B' = \bigcup_\alpha B_\alpha$ *(i.e.,* $\mathcal{B}' \subset \mathcal{T}_\mathcal{B}$*).*

Example 1.18. *On* $\mathbb{R}$, *compare* $\mathcal{T}$, $\mathcal{T}_\ell$, *and* $\mathcal{T}_{\ell,\mathbb{Q}}$.

Proposition 1.19 $(\star)$**.** *On* $\mathbb{R}$, *there exist non-comparable topologies, i.e. topologies* $\mathcal{T}$ *and* $\mathcal{T}'$ *such that* $\mathcal{T}$ *is not finer than* $\mathcal{T}'$ *and* $\mathcal{T}'$ *is not finer than* $\mathcal{T}$.

1.4. **More Examples of Topological Spaces.** In this section, we discuss how to further generalize the usual distance in $\mathbb{R}$ and how to create new topological spaces out of old ones.

1.4.1. *The Metric Topology.* In analysis, you generalized the distance on $\mathbb{R}$ to a *norm* on a vector space. A metric is a further generalization that allows us to discuss distance on sets that are not necessarily vector spaces.

Definition 1.20. A **metric space** (X, d) is a set X together with a **metric** d, i.e. a function $d : X \times X \to \mathbb{R}$ such that:
(1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$, and
(3) $d(x, y) + d(y, z) \geq d(x, z)$.

Exercise 5. If $\|\cdot\|$ is a norm on $\mathbb{R}^n$, then $d(x, y) = \|x - y\|$ is a metric.

Just as the absolute value may be used to define the standard topology on $\mathbb{R}$, so a metric generates a topology on (X, d):

Definition 1.21. The **metric topology** $\mathcal{T}_d$ on a metric space (X, d) is defined by the basis

$$\mathcal{B}_d = \{B(x,\epsilon) = \{y \in X \;:\; d(x,y) < \epsilon\} \text{ for all } x \in X, \epsilon > 0\}.$$

Exercise 6. Is $\mathcal{B}_d$ a basis for a topology?

Example 1.22. *There is a metric on* $\mathbb{R}$ *whose associated topology is not the same as the standard topology.*

The following two examples are here more for your edification than for you to supply a proof.

Example 1.23. *On the unit sphere* S^2 *in* $\mathbb{R}^3$*, define a metric by letting* $d(x, y)$ *be the shortest length of any curve in* S^2 *that connects* x *to* y*.*

Example 1.24. *Let* $C^0[0, 1]$ *be the set of continuous functions* $f : [0, 1] \to \mathbb{R}$*. Define a metric* d *on* $C^0[0, 1]$ *by*

$$d(f, g) = \sup\{|f(x) - g(x)| \,:\, x \in [0, 1]\}.$$

This metric is known as the ***sup*** *or* ***infinity*** *metric.*

The metric topology is intuitively appealing: the notion of nearness is built into it, and continuity has the nice interpretation of preserving this easy-to-understand nearness. Thus, it is important to understand when a topology comes from a metric; we say that a space $(X, \mathcal{T})$ is **metrizable** if there exists a metric d on X so that $\mathcal{T} = \mathcal{T}_d$.

Proposition?? 1.25 ($\star$)**.** *There exists a topology on* $\mathbb{R}$ *that is not metrizable.*

1.4.2. *The Product Topology.* We can form new topological spaces out of a pair of old ones $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ using the Cartesian product $X \times Y$. An obvious idea would be to define $\mathcal{T}_{X\times Y}$ to consist of products of open sets in $\mathcal{T}_X$ and $\mathcal{T}_Y$. This does not quite work, however:

Exercise 7. Draw a picture in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ to show that if we define the product topology to consist of products of open sets, then axiom T2 of a topology does not hold.

Instead, we will define the product topology via a basis.

Definition 1.26. Given topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, the **product topology** on $X \times Y$ is generated by the basis

$$\mathcal{B}_\times = \{U \times V \,:\, U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}.$$

Proposition 1.27. *The collection* $\mathcal{B}_\times$ *defined above is a basis for a topology.*

Example 1.28. *The product topology on* $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ *is the same as the topology induced from the Euclidean metric (i.e.* $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$*).*

Exercise 8. Think about how to generalize all of the material in this section to any finite product $X_1 \times \cdots \times X_n$. If you have to think too hard, then you're not making the proper generalizations!

1.4.3. *The Subspace Topology.* Given a subset $A \subset (X, \mathcal{T})$, how can we put a topology on A?

Definition 1.29. The **subspace topology** on $A \subset (X, \mathcal{T})$ is given by:

$$\mathcal{T}_A = \{U \subset A \ : \ \exists \tilde{U} \in \mathcal{T} \text{ s.t. } U = A \cap \tilde{U}\}.$$

Exercise 9. Consider $[0, 2] \subset (\mathbb{R}, \mathcal{T})$. Convince yourself that the set $[0, 1)$ is open in the subspace topology. Further, convince yourself that the set $\{2\}$ is open in the subspace topology of $[0, 2] \subset (\mathbb{R}, \mathcal{T}_\ell)$.

Proposition 1.30. *The subspace topology on $A \subset (X, \mathcal{T})$ is a topology.*

The subspace topology allows us to define many important examples that we will return to throughout the course. Here are a few (note that there is nothing to prove in these exmaples):

Example 1.31. *The unit n-disk $D^n \subset (\mathbb{R}^{n+1}, \mathcal{T})$ is defined to be the set*

$$\{x \in \mathbb{R}^{n+1} \ : \ d_2(x, 0) \leq 1\}.$$

Example 1.32. *The n-sphere $S^n \subset (\mathbb{R}^{n+1}, \mathcal{T})$ is defined to be the set*

$$\{x \in \mathbb{R}^{n+1} \ : \ d_2(x, 0) = 1\}.$$

Of particular note are S^1, the unit circle in $\mathbb{R}^2$, and S^2, the unit sphere in $\mathbb{R}^3$.

Example 1.33. *The 2-torus $T^2 \subset \mathbb{R}^3$ is the set obtained by revolving the unit circle in the xz plane centered at $(2, 0, 0)$ around the z axis.*

Example 1.34. *The general linear group $GL(n, \mathbb{R}) \subset M_{n\times n} = \mathbb{R}^{n^2}$, which consists of all $n \times n$ invertible matrices, may be given the subspace topology. Similarly, we can put the subspace topology on $SL(n, \mathbb{R})$ (matrices with determinant 1), $O(n)$ (orthogonal matrices), etc.*

> **Aside:** The general linear group, special linear group, orthogonal group, and their complex counterparts are the canonical examples of **topological groups**, i.e. topological spaces G such that the multiplication map $\mu : G \times G \to G$ and the inverse map $\iota : G \to G$ are both continuous (we'll examine the topology on $G \times G$ in the next section).

While we are here ...

> **Aside:** The group $SO(3)$ of 3×3 orthogonal matrices is the key to understanding the Belt Trick: think of the belt as a continuous map $b : [0, 1] \to SO(3)$ with $b(0) = b(1)$. Can this path be deformed to the constant path? Hopefully, we will get to talk about this in Part 3.

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2. Continuity

We saw in Section 1.1 that continuity of functions $f : \mathbb{R} \to \mathbb{R}$ really just depended on open sets. Now that we have generalized the notion of an open set from the standard topology on $\mathbb{R}$ to a topological space, we should also generalize the notion of a continuous mapping to topological spaces. That is, now that we have described a mathematical structure, we want to try to understand the functions that respect that structure.

> **Aside:** What I'm hinting at here is the world of category theory. Loosely speaking, a category C is a collection $\mathcal{O}_C$ of *objects* together with collections $\hom_C(x, y)$ of *morphisms* between any pair (x, y) of objects in $\mathcal{O}_C$. In group theory, the objects are groups and the morphisms are group homomorphisms; in linear algebra, the objects are vector spaces (over a fixed field, say) and the morphisms are linear maps; in topology, the objects are topological spaces and the morphisms are continuous maps.

2.1. The Definition of Continuity.

Since we are working in topological spaces, open sets are the only things we have left, so Proposition 1.6 becomes our *definition* of continuity.

Definition 2.1. A function $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is **continuous** if for every open set $U \subset Y$, $f^{-1}(U)$ is open in X.

Note that we can rephrase this definition in terms of bases:

Lemma... 2.2. *In Definition 2.1, suppose that $\mathcal{B}$ is a basis for $\mathcal{T}_Y$. Then f is continuous if ....*

Remember that Proposition 1.6 lets you import much of your old intuition — and examples! — for continuity from your analysis course. We will not reprove any of the facts about continuous functions between Euclidean spaces that you learned in analysis; you may use them without proof. Those facts are listed in the following proposition.

Proposition 2.3 (from Analysis)**.** *The following functions $f : A \subset \mathbb{R} \to \mathbb{R}$ are continuous on their domains (with A given the subspace topology if $A \neq \mathbb{R}$):*

- *Polynomials, exponential functions, trig functions, log functions, root functions;*
- *Sums and scalar multiples of continuous functions; and*
- *Compositions of continuous functions (we will prove this in more generality below).*

One way to see the beauty and power of the Definition 2.1 is to see how easy it is to prove basic properties. Here is one example.

Proposition 2.4. *If $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ and $g : (Y, \mathcal{T}_Y) \to (Z, \mathcal{T}_Z)$ are continuous, then so is $g \circ f : (X, \mathcal{T}_X) \to (Z, \mathcal{T}_Z)$.*

You do need to be careful about the topologies you are using when discussing continuity of functions $f : \mathbb{R} \to \mathbb{R}$, however, as the following examples and lemmas show.

Example 2.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}.$$

Determine which (if any) of the following maps between topological spaces are continuous.

(1) $f : (\mathbb{R}, \mathcal{T}_\ell) \to (\mathbb{R}, \mathcal{T}_\ell)$
(2) $f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T}_\ell)$
(3) $f : (\mathbb{R}, \mathcal{T}_\ell) \to (\mathbb{R}, \mathcal{T})$

Here is an example of how continuity interacts with the subspace topology:

Lemma 2.6.
(1) *Let $f : X \to Y$ be a continuous map.*[1] *Suppose that $A \subset X$. The restriction of f to A (denoted $f|_A$, and defined by $f|_A(x) = f(x)$ if $x \in A$) is continuous as a function $f|_A : A \to Y$ with respect to the subspace topology on the domain.*
(2) *Let $f : X \to B$ be a map between topological spaces with $B \subset Y$ having the subspace topology. Let $i : B \to Y$ be the inclusion map (i.e. $i(b) = b$ for all $b \in B$). The map f is continuous if and only if $i \circ f$ is continuous.*

To understand interactions with the product topology, we introduce the **projection map** $\pi_X : X \times Y \to X$, defined by $\pi_X(x, y) = x$. The projection map π_Y is defined similarly.

Lemma 2.7. *The projection map π_X is continuous.*

Proposition 2.8. *A map $f : Z \to X \times Y$ is continuous if and only if $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous. (What does this proposition mean "practically"?)*

2.2. **Homeomorphisms.** How does continuity help us formulate topological problems? The answer lies in the notion of a homeomorphism, which is akin to an isomorphism in group theory or an invertible linear map in linear algebra. That is, a homeomorphism tells us when two topological spaces are equivalent. Topology is really the study of topological spaces *up to homeomorphism.*[2]

[1] For typographical efficiency, I will frequently omit explicit notation for the topologies on the domain and codomain. You can add them back in if this makes you uncomfortable.

[2] This is a bit of a lie, as there are other interesting and important equivalence relations in topology, such as homotopy equivalence, which we shall discuss in the last part of the course.

Definition 2.9. A function $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a **homeomorphism** if it is bijective, continuous, and has a continuous inverse. We then say that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are **homeomorphic**.

Let us look at a few examples to gain an intuitive and technical understanding of the concept.

Example 2.10. *The identity function $f : (\mathbb{R}, \mathcal{T}_\ell) \to (\mathbb{R}, \mathcal{T})$ given by $f(x) = x$ is continuous and invertible, but it is not a homeomorphism. Note that this example does not prove that* no *homeomorphism exists, just that f isn't one.*

Example 2.11. *The function $f : (-\pi/2, \pi/2) \to \mathbb{R}$ (with the standard topologies) defined by $f(x) = \tan x$ is a homeomorphism. In particular, a topologist cannot tell the difference between $(-\pi/2, \pi/2)$ (or any open interval, for that matter) and all of $\mathbb{R}$.*

This is an easy example to use for intuition: spaces that differ by a continuous stretching are homeomorphic. This leads to the old saw that a topologist cannot tell the difference between a doughnut and a coffee cup.

Example 2.12. *The n-sphere S^n minus the north pole $N = (0, \dots, 0, 1)$ is homeomorphic to $\mathbb{R}^n$ via the **stereographic projection** map $\pi : S^n \setminus \{N\} \to \mathbb{R}^n$, which is defined as follows: for a point $q \in S^n \setminus \{N\}$, consider the line l that contains N and q. Let $\pi(q) \in \mathbb{R}^n$ be the point where l intersects the xy-plane.*

Example 2.13. *The topological group $SO(2)$ is homeomorphic to S^1.*

Example 2.14 ($\star$)**.** *The 2-torus described in Example 1.33 is homeomorphic to the product $S^1 \times S^1$.*

As hinted by the previous examples, we can develop an intuition for when two spaces are homeomorphic — and even explicitly prove it at times — but it is more difficult to tell when two spaces are *not* homeomorphic. For this, we introduce the idea of a "topological property" or a "topological invariant". We say that a property of a topological space is "topological" if whenever $(X, \mathcal{T}_X)$ has the property, then so does any space homeomorphic to it. Example 2.11 shows that "bounded with respect to the Euclidean norm" is not a topological property.

We will study a large number of topological properties in the coming sections, including generalizations of compactness and connectedness. Before moving on, however, let us look at two somewhat subtle properties. For the first, we say that a topological space X has the **fixed point property** if for any continuous function $f : X \to X$, there exists an $x \in X$ such that $f(x) = x$.

Proposition 2.15. *The fixed point property is a topological property.*

For the second, we say that a topological space X is **contractible** if there exists a point $x_0 \in X$ and a continuous function $H : X \times [0,1] \to X$ such that $H(x,0) = x$ and $H(x,1) = x_0$.

Exercise 10. Prove to yourself that $\mathbb{R}^n$ with the standard topology is contractible to the origin, but convince[3] yourself that S^1 is not.

Proposition 2.16. *Contractibility is a topological property.*

We will examine more topological properties later on, and eventually develop some quit sophisticated invariants using group theory towards the end of the course.

3. Closed Sets, Countability, and Separation Axioms

When you studied analysis, open sets were not the only sets of interest — you also studied closed sets. Closed sets were inextricably tied up with the concepts of sequences and limit points. These concepts are more delicate in general topological spaces, but their study leads us to define some interesting topological properties. Throughout this section, you should ask yourself what properties of closed sets, sequences, and limit points generalize from $\mathbb{R}$ to topological spaces. Is the notion of a topological space "too general"? What must we do to bar the monsters that can arise from the definition of a topological space?

3.1. **Closed Sets and Closures.**

Definition 3.1. A set $C \subset (X, \mathcal{T})$ is **closed** if $X \setminus C$ is open.

Exercise 11. Give examples of sets that are closed, both open and closed, and neither open nor closed in $(\mathbb{R}, \mathcal{T}_\ell)$ and $(\mathbb{R}, \mathcal{T}_{FC})$.

Proposition... **3.2.** *In a topological space X,*

(1) *$\emptyset$ and X are closed.*
(2) *If U_α is closed for all $\alpha \in I$, then $\cap U_\alpha$ is closed.*
(3) *If U_0 and U_1 are closed, then $U_0 \cap U_1$ is closed.*

Example 3.3. *The Cantor set C is closed.*

Closed sets have nice interactions with continuous functions as well.

Lemma 3.4. *A function $f : X \to Y$ is continuous if and only if for every closed set $C \subset Y$, $f^{-1}(C)$ is closed in X.*

The following lemma is quite useful in constructing continuous functions.

[3]At an intuitive level — you will not have the tools to prove this until the end of the semester.

Lemma 3.5 (Pasting Lemma)**.** *Let $X = A \cup B$, where A and B are closed sets. Let $f : A \to Y$ and $g : B \to Y$ be continuous functions that agree on $A \cap B$. Then there exists a continuous function $h : X \to Y$ such that*

$$h(x) = \begin{cases} f(x) & x \in A, \\ g(x) & x \in B. \end{cases}$$

To any subset of a space X, we may associate canonical open and closed sets. Informally, the closure of a set A is the "smallest closed set that contains A" and the interior is the "largest open set contained in A". More precisely, we have:

Definition 3.6. Given a set $A \subset X$, its **interior** $\mathring{A}$ is the union of all open sets contained in A, its **closure** $\overline{A}$ is the intersection of all closed sets containing A, and its **boundary** ∂A is $\overline{A} \setminus \mathring{A}$.

Proposition?? 3.7. *If A and B are subsets of a topological space X, then:*

(1) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(2) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

Example 3.8 ($\star$)**.** *In a metric space, the "closed ball" $\overline{B}(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}$ is indeed closed, but, despite the unfortunate notation, is not necessarily the closure of the open ball $B(x, \epsilon)$.*

The definition of closure that we have been using is clean and theoretically useful, but to efficiently examine examples, we need to return to ideas from analysis and introduce the idea of a limit point. Limit points in topological spaces capture the idea of being arbitrarily "close" to a set without using distance.

Definition 3.9. We say that U is a **neighborhood** of $x \in X$ if U is open and $x \in U$. Given a set $A \subset X$, we say that y is a **limit point** of A if every neighborhood of y has non-empty intersection with $A \setminus \{y\}$.

Example 3.10. *The set of limit points of $(0, 1) \subset \mathbb{R}$ is $[0, 1]$ with respect to $\mathcal{T}$, $[0, 1)$ with respect to $\mathcal{T}_\ell$, and all of $\mathbb{R}$ with respect to $\mathcal{T}_{FC}$.*

To examine the notion of limit point more closely, let us introduce a more sophisticated example that seemingly generalizes the product topology. Given a collection $\{X_\alpha\}_{\alpha \in I}$ of sets indexed by a set I, we define the product $\prod_{\alpha \in I} X_\alpha$ to be the set of functions

$$\mathbf{x} : I \to \bigcup_\alpha X_\alpha$$

such that $\mathbf{x}(\alpha) \in X_\alpha$.

This is a tough definition to swallow, so it will be useful for you to think about the following special cases:

Exercise 12. Think about why the elements of $\prod_{i \in \mathbb{N}} \mathbb{R}$ are sequences of real numbers, and why elements of $\prod_{x \in [0,1]} \mathbb{R}$ are functions from $[0, 1]$ to $\mathbb{R}$. How can you graphically represent each of these products?

Now let us use these product sets to examine limit points.

Example 3.11. *Let $A \subset \prod_{k\in\mathbb{N}} \mathbb{R}$ be the set of sequences with positive entries. The sequence $\mathbf{0} = (0,0,0,\dots)$ is a limit point of the set A with respect to the topology $\mathcal{T}_\square$, which has as its basis (for this particular product):*

$$\mathcal{B}_\square = \left\{ \prod_{i\in\mathbb{N}} U_i \;:\; U_i \subset \mathbb{R} \text{ open} \right\}.$$

Let us return to the relationship between limit points and closed sets. Given your experience from analysis, the following proposition should not be surprising.

Proposition 3.12. *Given $A \subset X$, let A' denote the set of limit points of A. Then $\overline{A} = A \cup A'$.*

You may want to think about how to formulate a definition of "interior point" and an analogue of Proposition 3.12 for the interior of a set.

3.2. **Sequences.** Convergence of sequences was a central concept in analysis. In this section, we extend convergence to more general topological spaces, paying careful attention to whether your old analytic intuitions still hold.

Definition 3.13. A sequence (x_n) in a space X converges to $x \in X$ if for all neighborhoods U of x, there is an $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. We call x a **limit** of the sequence (x_n).

This definition of convergence includes the old $\epsilon - N$ definition in $\mathbb{R}^n$, as $B(x,\epsilon)$ is a neighborhood of x. In more general topological spaces, however, some troubling examples can emerge.

Example 3.14. *The sequence $(1,2,3,4,\dots)$ converges to every point in $(\mathbb{R}, \mathcal{T}_{FC})$.*

This shows that the limit of a sequence may not be unique! We will discuss this further in Section 3.3. In fact, it is worthwhile to think about which sequences in $(\mathbb{R}, \mathcal{T}_{FC})$ converge, and to what points they converge.

Example 3.15. *Continuing Example 3.11, no sequence in A converges to $\mathbf{0}$ with respect to the box topology.*

This shows that a point may be a limit point of a set without being the limit of a sequence! We will discuss this further in Section 3.4. The converse is still true, however:

Proposition 3.16. *If there is a sequence (x_n) in $A \subset X$ that converges to x, but is not eventually constant, then x is a limit point of A.*

3.3. **Separation Axioms.** It would be nice to have a condition on a topological space that guarantees that the pathology in Example 3.14 does not occur. The problem in the finite complement topology is that you cannot separate two points by open sets: neighborhoods of distinct points x and y always have non-empty intersection. To remedy this, we impose the Hausdorff condition, which allows us to separate points by neighborhoods. In fact, the Hausdorff condition is one of a series of "separation axioms", the most important of which we define below.

Definition 3.17. Let X be a topological space.

(1) X is T_0 if for every pair of distinct points, there is an open set containing one but not the other.
(2) X is T_1 if for every pair of distinct points x and y, there are neighborhoods U of x that does not contain y and V of y that does not contain x.
(3) X is **Hausdorff** (or T_2) if for each pair of distinct points x and y, there exist disjoint open neighborhoods U of x and V of y.
(4) X is **regular** if for every $x \in X$ and closed set $A \subset X$ with $x \notin A$, there exist disjoint open sets U containing x and V containing A. A T_3 space is both regular and T_1.
(5) X is **normal** if for every pair of disjoint closed sets A and B, there are disjoint open sets U containing A and V containing B. A T_4 space is both normal and T_1.

Exercise 13. Draw cartoon pictures of each of the definitions above.

Exercise 14. Convince yourself that each of the separation axioms is a topological property.

To see that Hausdorff is the condition needed to overcome Example 3.14, we have:

Proposition 3.18. *If X is Hausdorff, then limits of sequences are unique.*

While we are here, let us study these axioms in a bit more detail. Intuitively, a T_i space behaves more "nicely" than a T_{i-1} space.

Lemma 3.19. *A space X is T_1 if and only if for every $x \in X$, $\{x\}$ is closed.*

Proposition 3.20. *The condition T_i implies T_{i-1}.*

Proposition 3.21 ($\star$ for $i = 3$)**.** *The converse of Proposition 3.20 is false for each $i = 1, 2, 3$. (It is also false for $i = 4$, but the construction of the counterexample is completely crazy.)*

The last two propositions essentially tell us that there is a strict hierarchy of separation axioms. One indication that metric spaces are so nice is that they behave well with respect to the separation axioms:

Proposition 3.22 ($\star$)**.** *All metric spaces are normal.*

Aside: If the facts that normal spaces lie at the top of the separation hierarchy and that metric spaces are normal are not enough to convince you that normal spaces are nice, consider the following theorem.

Theorem 3.23 (Urysohn's Lemma). *Let X be a normal space, and let $A, B \subset X$ be disjoint closed sets. There exists a continuous map $f : X \to [0,1]$ so that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.*

You really need normality to do this! Munkres avers that this is the first "deep" theorem that you have encountered in the course. In the interest of spending more time on nicer deep theorems at the end of this course, we will skip a proof of the Urysohn Lemma.

3.4. **Countability Axioms.** Recall that Example 3.15 shows that a limit point of a set may not have any sequence converging to it. To devise a condition that overcomes this pathology, we turn to the notion of countability.

Definition 3.24. Given a point x in a space X, a collection of sets $\{U_\alpha\}_{\alpha \in I}$ is a **neighborhood basis** at x if each U_α is a neighborhood of x and every neighborhood of x contains some U_α. The space X is **first countable** if every point $x \in X$ has a countable neighborhood basis.

Proposition 3.25. *Metric spaces are first countable.*

We now proceed to show that first countability corrects for the pathology of Example 3.15:

Proposition 3.26. *Suppose that X is first countable and $A \subset X$. If x is a limit point of A, then there exists a sequence (x_n) in A converging to x.*

Corollary 3.27. *The box topology on $\prod_{k \in \mathbb{N}} \mathbb{R}$ is not metrizable.*

Aside: Asserting first countability is not the only way to overcome the pathology in Example 3.15. Another method is to generalize the notion of sequence to a **net**. If you are interested, it turns out that the Wikipedia article on nets is quite good, and you should take a peek.

Aside: Corollary 3.27 indicates that the box topology is the "wrong" topology to put on the product $\prod_{i \in \mathbb{N}} \mathbb{R}$. Rather, it is better to use the so-called "product topology", whose basis is given by

$$\mathcal{B}_{\prod} = \left\{ \prod_{i\in\mathbb{N}} U_i \ : \ U_i \subset \mathbb{R} \text{ open and } U_i = \mathbb{R} \text{ for all but finitely many } i \right\}.$$

As with Hausdorff and the separation axioms, first countability is part of a sequence of axioms. In the interest of time, we will discuss the second countability axiom briefly in the notes, but will not explore it in the same depth as the separation axioms of the first countability axiom.

Definition 3.28. A space X is **second countable** if it has a countable basis.

It is not too hard to convince yourself that $(\mathbb{R}, \mathcal{T})$ is second countable, but $(\mathbb{R}, \mathcal{T}_\ell)$ is not. Second countable spaces have countable dense subsets,[4] and the converse is true in a metric space. Since one can prove that $(\mathbb{R}, \mathcal{T}_\ell)$ does not have a countable dense subset, we can conclude that $(\mathbb{R}, \mathcal{T}_\ell)$ is *not* metrizable.

Aside: Second countability also leads to a second "deep" theorem called the Urysohn Metrization theorem (which we will also not prove), which gives sufficient conditions for a space to be metrizable. It is one of several such theorems, but the only one that we can state with our current language.

Theorem 3.29 (Urysohn Metrization Theorem)**.** *Every regular, second countable space is metrizable.*

The proof uses the Urysohn Lemma to find functions $\{f_n : X \to \mathbb{R}\}_{n\in\mathbb{N}}$ that nicely embed a regular, second countable space X into $\prod_{n\in\mathbb{N}} \mathbb{R}$, which has a metric given by:

$$D(\mathbf{x}, \mathbf{y}) = \sup \left\{ \frac{\min(|x_i - y_i|, 1)}{i} \right\}.$$

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4. Compactness

Compactness was one of the most important properties you studied in analysis: it gave you convergence out of chaos, it made continuity uniform, and it was the key idea behind the all-important Extreme Value Theorem. Your definition of compactness, however, was based on sequences, which we have already seen to be untrustworthy. We will recall the sequential definition, but will then move to a definition based on open sets.

[4] Having a countable dense subset is, unfortunately, known as being "separable", even though it has nothing to do with the separation axioms of the previous section.

4.1. **Basic Definitions.** The version of compactness that you studied in analysis used sequences in its definition.

Definition 4.1. A space X is **sequentially compact** if every sequence in X has a convergent subsequence.

Example 4.2 (from Analysis)**.** *The closed interval* $[0,1]$ *(thought of as a subspace of* $(\mathbb{R}, \mathcal{T})$*) is sequentially compact, but* $(0,1)$ *and* $\mathbb{R}$ *are not.*

Theorem 4.3 (Heine-Borel — from Analysis)**.** *A subspace of* $(\mathbb{R}^n, \mathcal{T})$ *is sequentially compact if and only if it is closed and bounded.*

As mentioned above, however, sequences are not reliable objects when our topologies get less nice, i.e. when they don't satisfy certain separation or countability axioms. It would be better to have a definition of compactness that relies only on open sets.

Definition 4.4. An **open cover** of a space X is a collection $\{U_\alpha\}_{\alpha \in I}$ of open sets with the property that $X \subset \bigcup_\alpha U_\alpha$. A space X is **compact** if every open cover of X has a finite subcover, i.e. there exist $\alpha_1, \dots, \alpha_n$ such that $X \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$.

A subset $K \subset X$ is compact if it is compact with respect to the subspace topology.

Example 4.5. *The following sets are not compact:*

(1) $(0,1) \subset (\mathbb{R}, \mathcal{T})$*;*
(2) $(\mathbb{R}, \mathcal{T})$ *itself;*
(3) $[0,1] \subset (\mathbb{R}, \mathcal{T}_\ell)$.

To get a better grip on the definition of compactness, let us investigate how compactness interacts with the subspace topology.

Proposition 4.6. *If* X *is compact and* $K \subset X$ *is closed, the* K *is compact.*

Proposition 4.7. *If* X *is Hausdorff and* $K \subset X$ *is compact, then* K *is closed.*

It would be nice if the new definition of compactness were the same as the old, but by now we should be sufficiently suspicious of sequences not to believe this. There are some things that we can say, however.

Proposition 4.8. *If* X *is first countable and compact, then it is sequentially compact.*

What about a converse? While the counterexamples are somewhat involved,[5] there is one familiar condition under which the converse holds.

[5] If you have taken Analysis II, you can understand the statement of the following counterexample: let X be the space of all functions $f : [0,1] \to \{0,1\}$ with a topology that implies that all pointwise convergent sequences actually converge. This space is compact by the (very deep) Tychonoff Theorem, but it is not sequentially compact, essentially by Cantor's Diagonal Argument.

Proposition 4.9. *If (X, d) is a metric space, then compactness is equivalent to sequential compactness.*

Two steps in one possible proof of Proposition 4.9 (there are others!) are the following:

Lemma 4.10 (Totally Bounded)**.** *If (X, d) is a sequentially compact metric space, then for every $r > 0$, there exists a finite set $\{x_1, \ldots, x_k\} \subset X$ so that $\{B_r(x_i)\}$ is an open cover of X.*

Lemma 4.11 (Lebesgue Number $\star$)**.** *Let (X, d) be a sequentially compact metric space and let $\{U_\alpha\}$ be an open cover of X. There exists $\delta > 0$ such that for any $x \in X$, there exists a U_α such that $B(x, \delta) \subset U_\alpha$.*

In particular, the Heine-Borel theorem from analysis also works for the new definition of compactness. Also note that the Lebesgue Number Lemma is important in its own right.

4.2. Compactness and Continuity.

We would not be so interested in compactness if it were not a topological property; the following proposition proves this.

Proposition 4.12. *If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is also compact.*

Example 4.13. *S^1 is compact.*

Corollary 4.14. *Compactness is a topological property.*

The following lemma is very convenient for checking that a map is a homeomorphism.

Lemma 4.15 ($\star$)**.** *Let X be compact and let Y be Hausdorff. Any bijective continuous map $f : X \to Y$ is a homeomorphism.*

5. Connectedness

5.1. Basic Definitions.

Connectedness is the second of the two foundational topological properties (compactness is the other). We would like a notion of "connected" that allows us to say that an interval $[a, b]$ in $(\mathbb{R}, \mathcal{T})$ is connected, but that $[0, 1) \cup (2, 3]$ or even $[0, 1) \cup (1, 2]$ are not. Do you think that $\mathbb{Q}$ should be connected? What about the closure of the graph of $\sin \frac{1}{x}$ in $\mathbb{R}^2$?

It turns out that it is easier to define when a set is *not* connected.

Definition 5.1. A **separation** of a topological space X is a pair $\{U, V\}$ of open sets such that:

(1) $U, V \neq \emptyset$,
(2) $U \cap V = \emptyset$, and
(3) $U \cup V = X$.

The space X is **connected** if there is no separation of X.

Exercise 15. Given the form of the definition of a connected space, how do you think most proofs involving connectivity will be structured?

Proposition 5.2. *A topological space X is connected if and only if the only sets that are both closed and open are X and $\emptyset$.*

Example 5.3.
(1) $[0,1) \cup (1,2] \subset (\mathbb{R}, \mathcal{T})$ *is not connected.*
(2) $\mathbb{Q} \subset (\mathbb{R}, \mathcal{T})$ *is not connected. In fact, it is* ***totally disconnected*** *(you should supply a definition of "totally disconnected").*
(3) $[0,1] \subset (\mathbb{R}, \mathcal{T}_\ell)$ *is not connected.*

The examples above are really *non*-examples. Finding a non-trivial example of a connected set takes some more work, which you did in your analysis class.

Proposition 5.4 (from Analysis)**.** *Any interval in $(\mathbb{R}, \mathcal{T})$ is connected.*

5.2. **Connectedness and Continuity.** I promised earlier that connectedness is a topological property. Let us prove this!

Proposition 5.5. *If $f : X \to Y$ is continuous and X is connected, then the image $f(X)$ is also connected.*

Example 5.6. *S^1 is connected.*

Corollary 5.7. *Connectedness is a topological property.*

This simple proposition and corollary have several important consequences.

Theorem 5.8 (Intermediate Value Theorem)**.** *Suppose that $f : X \to \mathbb{R}$ is continuous and that X is connected. If $f(a) < r < f(b)$, then there exists $c \in X$ such that $f(c) = r$.*

Theorem 5.9 (Borsuk-Ulam in 1D)**.** *If $f : S^1 \to \mathbb{R}$ is continuous, then there exists $x \in S^1$ such that $f(x) = f(-x)$.*

That is, there are two antipodal points on the equator of the earth that have the same temperature.

Theorem 5.10 (Brouwer Fixed Point Theorem in 1D)**.** *If $f : [0,1] \to [0,1]$ is continuous, then there exists $x \in [0,1]$ such that $f(x) = x$.*

Example 5.11. *No pair of the spaces $(0,1)$, $(0,1]$, $[0,1]$ is homeomorphic.*

Theorem 5.12 (Invariance of Domain in 1D)**.** *If $n > 1$, then there does not exist a homeomorphism $f : \mathbb{R} \to \mathbb{R}^n$.*[6]

---

[6] You may assume that $\mathbb{R}$ is connected in your proof of this theorem; we will prove it in the next section.

Aside: You should notice that the names of the versions above of the Borsuk-Ulam[7] Theorem, the Brouwer Fixed Point Theorem, and Invariance of Domain all hinted that more general statements are true. In fact, they are! For example, the general Invariance of Domain theorem states that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic if $n \neq m$. To prove the more general theorems, we would need a more sophisticated notion of connectedness that, for example, could tell the difference between a disk and a disk with its center removed in much the same way that connectedness can distinguish between an interval and an interval with a point removed. We will take up this question of "higher connectedness" when we study the fundamental group.

5.3. **Path Connectedness.** Connectedness is difficult to verify, so we introduce a more intuitive way of looking at it.

Definition 5.13. A **path** from $x \in X$ to $y \in X$ is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. I will denote such a path by $\gamma : x \rightsquigarrow y$ (though this is nonstandard notation). A space X is **path connected** if for every pair of points $x, y \in X$, there is a path $\gamma : x \rightsquigarrow y$.

Example 5.14. *$\mathbb{R}^n$ is path connected.*

Example 5.15 ($\star$)**.** *If $E \subset \mathbb{R}^2$ is countable, then $\mathbb{R}^2 \setminus E$ is path connected.*

Proposition 5.16. *Every path connected space is connected.*

It is straightforward to prove that path connectedness is a topological property. We should ask: is the converse to Proposition 5.16 true? Let us use the following proposition and example to explore the issue.

Proposition 5.17. *If A is connected and $A \subset B \subset \overline{A}$, then B is also connected.*

Example 5.18 ($\star$)**.** *Let A be the union of the following subsets of $\mathbb{R}^2$:*

(1) *The set $[0,1] \times \{0\}$,*
(2) *The sets $\{\frac{1}{n}\} \times [0,1]$ for every $n \in \mathbb{N}$, and*
(3) *The singleton set $\{(0,1)\}$.*

The space A is called the "flea and comb space" (draw a picture of it to see why). It is connected, but not path connected.

We can define an equivalence relation on a space X: we say that $x \sim_p y$ if there exists a path $\gamma : x \rightsquigarrow y$.

Definition 5.19. The **path components** of X are the equivalence classes of $\sim_p$. The set of path components of X is denoted $\pi_0(X)$.

[7] You should find out a bit about what Ulam did during the Cold War.

Aside: If an object is called π_0, then perhaps there is also a π_1 ... and even a π_n, right? We will study π_1 in the second part of the course.

Proposition 5.20. *The cardinality of the set of path components of a space X is a topological property.*

Example 5.21. *The letters I, X, and Y (with the subspace topology in $\mathbb{R}^2$) are not homeomorphic.*

For fun, you should make a conjecture in the spirit of the previous example about the (capital, sans-serif) letters of the Roman alphabet. For more fun, prove your conjecture!

Part 2. Central Examples

Thus far, we have discussed more than a few examples of topological spaces, from the various topologies that we put on the real line to the topologies that come from the product, metric, and subspace constructions. Some, perhaps many, of these examples are mainly useful for a close reading of the basic definitions and theorems that underlie the language of topology. Modern topologists, however, rarely spend time thinking about, say, the box topology or its brethren. Generally speaking, the topological spaces and constructions that researchers use and investigate either have a rather more geometric flavor to them or arise from analytic (especially functional-analytic) or algebro-geometric concerns.

In this part of the notes, we will build upon the more geometric examples that you have already encountered: the sphere, the torus, and matrix groups. The main tool in our explorations will be the quotient topology; we will then introduce (finite) cell complexes.

6. The Quotient Topology

The quotient topology, like the product and subspace topologies, is a way of defining a topology on a new space using known a known topology. Instead of thinking about products or subspaces, however, the quotient topology deals with *gluing* or *collapsing* pieces of a known space. It will be useful to have a few intuitive examples in mind[8] as we work through the technicalities of the definition. In each example, keep track of the "known" space and the "new" space; the quotient topology will tell you how to put a topology on the "new" space.

Example 6.1. *Consider the unit interval $[0,1]$. Glue the point 0 to the point 1. Convince yourself that the result "is" S^1.*

[8]Unlike other examples in these notes, these examples are not meant to be "proven" ... at least not yet.

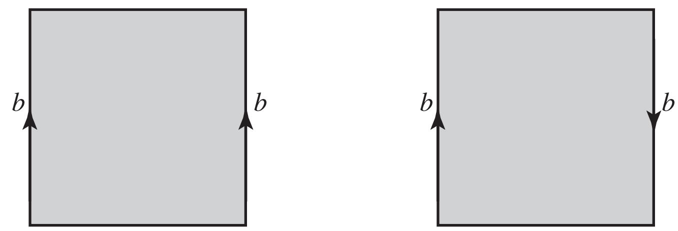

FIGURE 1. (Left) Glue the two sides labeled b together (with the arrows matching up) to obtain a cylinder. (Right) Glue the two sides labeled b together (with the arrows matching up) to obtain a Möbius strip.

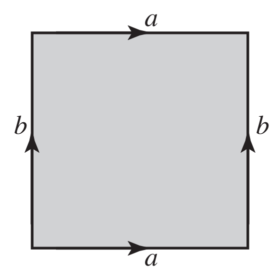

FIGURE 2. Glue the two sides labeled a together (with the arrows matching up) to obtain a cylinder, then continue by gluing the two sides labeled b together to obtain a torus.

Example 6.2. *Consider two disjoint unit* 2*-disks,* D^2_1 *and* D^2_2*. The boundary of each disk is a circle; call them* S^1_1 *and* S^1_2*. Glue* S^1_1 *and* S^1_2*. Convince yourself that the result "is"* S^2*.*

Example 6.3. *Consider the square* $[0,1] \times [0,1]$*. Glue the left and right sides of the square as in Figure 1, with the arrows matching up in direction. Convince yourself that the results "are" a cylinder (or annulus) and a Möbius strip.*

Example 6.4. *Consider the square* $[0,1] \times [0,1]$*. Glue opposite sides of the square as in Figure 2, with the arrows matching up in direction. Convince yourself that the result "is" the torus.*

Example 6.5. *Consider the unit* 2*-disk* D^2*. The boundary is a circle. Collapse that circle to a point. Convince yourself that the result "is"* S^2*.*

Example 6.6. *Consider the torus pictured in Figure 3. Collapse both indicated circles to a single point. Convince yourself that the result "is"* S^2*.*

With these examples in hand, we are ready to work through the technicalities of the quotient topology. There are two closely related ways of thinking

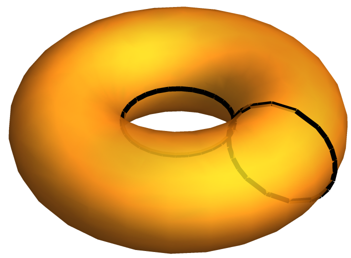

FIGURE 3. Collapse the two thick circles on the torus (one goes around the hole, the other goes through the hole) to a single point to obtain S^2.

about the quotient topology: equivalence relations and quotient maps. We will consider each in turn.

6.1. **Equivalence Relations.** The first way to think about the quotient topology is via *equivalence relations*. Suppose that we have a set X with an equivalence relation $\sim$. Recall that the **equivalence class** of an element $x \in X$ is the set

$$[x] = \{y \in X \ : \ y \sim x\}.$$

Further, equivalence classes *partition* the set X into disjoint sets, and specifying such a partition is equivalent to defining an equivalence relation. Let $X/\sim$ denote the set of all equivalence classes of $\sim$. Define a map $\pi : X \to X/\sim$ by $\pi(x) = [x]$.

Example 6.7. *On the set of real numbers, impose the equivalence relation $x \sim y$ if there is an integer n such that $x = y + n$. There is a natural identification of $\mathbb{R}/\sim$ (which we usually write $\mathbb{R}/\mathbb{Z}$) with the interval $[0, 1)$.*

Example 6.8. *In Example 6.1, the underlying partition that defines the equivalence relation consists of the sets $\{0, 1\}$ and $\{x\}$ for $0 < x < 1$.*[9]

Example 6.9. *In Example 6.5 the underlying partition that defines the equivalence relation consists of the sets $\{(x, y)\}$ if $x^2 + y^2 < 1$ and $\{(x, y) : x^2 + y^2 = 1\}$.*[10]

If X is a topological space, we can put a topology on $X/\sim$ as follows:

Definition 6.10. The **quotient topology** on $X/\sim$ consists of the sets $U \subset X/\sim$ with the property that $\pi^{-1}(U)$ is open in X.

[9]It would not be a bad idea to come up with explicit descriptions of this nature for Examples 6.2 and 6.4

[10]See the previous footnote!

Exercise 16. Describe some open sets in $\mathbb{R}/\mathbb{Z} = [0, 1)$, especially some open sets that contain 0. What do you think this means about the "proximity" between 0 and points near 1? (We'll see later that, indeed, $\mathbb{R}/\mathbb{Z}$ is homeomorphic to S^1.)

Proposition 6.11. *The quotient topology of Definition 6.10 actually defines a topology on $X/\sim$.*

6.2. **Quotient Maps.** The second way to think of the quotient topology is via *functions*. This is usually, though not always, the best way to work through the technicalities of the quotient construction.

Definition 6.12. A surjective map $f : X \to Y$ is a **quotient map** if for all $U \subset Y$, $f^{-1}(U)$ is open if and only if U is open.

To connect this definition to previous ways of thinking about the quotient map, we can use f to put an equivalence relation on X by declaring $x \sim_f x'$ if $f(x) = f(x')$. Thus, if $f : X \to Y$ is a quotient map, then Y is the quotient space $X/\sim_f$. As usual, we note that we can work with bases in this setting, not just full topologies. A useful sufficient (but *not* necessary) condition for determining if a map is a quotient map is:

Lemma 6.13. *If $f : X \to Y$ is a continuous, surjective map with the property that images of open sets are open,*[11] *then it is a quotient map.*

The converse is not true, however. Even better, we have:

Lemma 6.14. *If $f : X \to Y$ is a continuous, surjective map from a compact space to a Hausdorff space, then it is a quotient map.*

Example 6.15 ($\star$)**.** *Define a function $f : \mathbb{R} \to S^1$ by $f(t) = (\cos 2\pi t, \sin 2\pi t)$. This map is a quotient map that induces the equivalence relation on $\mathbb{R}$ given in Example 6.7.*

In order to work with quotient spaces — especially to identify interesting constructions like $\mathbb{R}/\mathbb{Z}$ with familiar spaces like S^1, or to work with rigorously with any of the examples at the beginning of the section — we need to understand how to work with quotient maps.

Lemma 6.16. *Let $f : X \to Z$ be a continuous map and let $\pi : X \to Y$ be a quotient map. If $f(x) = f(x')$ whenever $\pi(x) = \pi(x')$, then there exists a well-defined continuous map $\overline{f} : Y \to Z$:*

$$\begin{tikzcd} X \arrow[d, "\pi"] \arrow[dr, "f"] & \\ Y \arrow[r, dotted, "\overline{f}"] & Z \end{tikzcd}$$

This lemma, together with some earlier labor-saving lemmas, is the key ingredient in understanding many examples.

Example 6.17 ($\star$)**.** $\mathbb{R}/\mathbb{Z}$ *is homeomorphic to* S^1.

[11] Such a map is called an **open map**.

7. Quotient Constructions

The quotient topology allows us to construct many more important examples, along with some useful general constructions.

7.1. Projective Spaces.

The first construction shows how quotients can naturally parametrize interesting sets of objects.

Example 7.1 ($\star$). *The n-dimensional **real projective space** is defined, as a set, to be the set of lines through the origin in $\mathbb{R}^{n+1}$. To put a topology on this set, we note that the set of lines through the origin is in bijective correspondence with the set $S^n/\sim$, where we declare two points to be equivalent if they are antipodal (i.e. $x \sim -x$).*

While $\mathbb{R}P^1 \simeq S^1$, higher dimensional projective spaces are not *homeomorphic to spheres.*[12]

Aside: Since real projective spaces are *lines* in *real* vector spaces, we can generalize real projective spaces in several different directions. First, we could replace the real vector spaces with vector spaces over, say, the complex numbers or the quaternions, yielding complex and quaternionic projective spaces. The space $\mathbb{C}P^1$ turns out to be homeomorphic to S^2 (in fact, proving this would lead you to understanding S^2 as a quotient of S^3 whose equivalence classes consist of linked circles!), while the space $\mathbb{C}P^2$ is an important example of a 4-dimensional manifold (see below) that is not homeomorphic to S^4.

Moving back to n-dimensional real vector spaces, we could replace the set of lines by the set of k-dimensional subspaces. These spaces are called **Grassmanian** spaces $Gr_k(\mathbb{R}^n)$. You may want to think about why $Gr_1(\mathbb{R}^n)$ and $Gr_{n-1}(\mathbb{R}^n)$ are both homeomorphic to $\mathbb{R}P^{n-1}$ — but $Gr_2(\mathbb{R}^4)$, for example, is a whole different animal.

7.2. Attaching Cells.

The second construction is a general one whose usefulness cannot be overstated.

Definition 7.2. Given a topological space X and a continuous map $f : S^{n-1} \to X$, put an equivalence relation on the disjoint union $X \sqcup D^n$ by declaring a point $x \in S^{n-1}$ to be equivalent to its image $f(x) \in X$. The quotient space $X \sqcup D^n/\sim$ (at times denoted $X \sqcup_f D^n$) is the result of **attaching an n-cell to X along f**.

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[12] You should prove the former fact, but we are not yet ready to prove the latter.

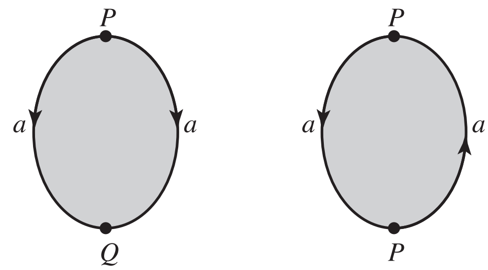

FIGURE 4. A 2-sphere and a real projective plane as cell complexes. The sphere's cell complex consists of two 0-cells, one 1-cell, and one 2-cell. The projective plane's cell complex consists of one 0-cell, one 1-cell, and one 2-cell.

Be careful: there is no reason for f to be injective or surjective in this definition!

The usefulness of this construction is that it breaks a potentially complicated topological space down into simpler pieces. One way to visualize the attachment of a 2-cell to a 1-dimensional space,[13] for example, is to start with a polygon in $\mathbb{R}^2$ and then identify different sides of the polygon with each other. The identification of sides may be thought of as specifying a map from the circle at the boundary of a disk to a circle or a "bouquet" of circles (think of a kid's drawing of a flower), and the result of the identification is the attachment of a single 2-cell. For example, see the quotient spaces in Figures 1, 2 and 4. What other spaces can you create?

Exercise 17. Return to Examples 6.1 through 6.5 and think about how to view them as successively attaching higher and higher cells to a collection of points (0-cells).

Example 7.3. *Let $f : S^1 \to S^1$ be the map that wraps the circle twice around itself. More formally, we may define $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by $\tilde{f}(t) = 2t$ and then feed the map $\pi \circ \tilde{f}$ into Lemma 6.16:*

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & \mathbb{R} \arrow[d, "\pi"] \\ S^1 \arrow[r, dotted, "f"] & S^1 \end{tikzcd}$$

The quotient space $S^1 \sqcup_f D^2$ is homeomorphic to $\mathbb{R}P^2$.

This example may be generalized to a map $f : S^1 \to S^1$ that wraps more than twice, though the result will not be a "familiar" space.

The power of the previous examples is that it has broken down the construction of the seemingly complicated $\mathbb{R}P^2$ into the following process: start

[13]We have *not* defined dimension, so this phrase should be considered intuitively rather than rigorously.

with a point, then attach a 1-cell, and then attach a 2-cell. Many interesting examples that you encounter may be built up this way.

7.3. **Connected Sums and Surfaces.** Perhaps the most-studied class of objects in topology are *manifolds*.[14] We define a manifold as follows:

Definition 7.4. A **(compact)** k**-manifold** X is a topological space with the property that every $x \in X$ has a neighborhood homeomorphic to $B^k(0,1) \subset \mathbb{R}^k$. A dfnsurface is a 2-manifold.

Example 7.5. *The circle* S^1 *is a* 1*-manifold.*

Example 7.6. *The sphere* S^n *is an* n*-manifold.*

Example 7.7. *The* 2*-torus* T^2 *is a* 2*-manifold.*

Example 7.8. *The real projective space* $\mathbb{R}P^2$ *is a* 2*-manifold.*

We can build a new 2-manifold out of two given 2-manifolds M_0 and M_1 by finding two disks $D_i \subset M_i$, $i = 0, 1$ and forming the quotient space

$$M_0 \# M_1 = (M_0 \setminus \mathring{D}_0) \cup (M_1 \setminus \mathring{D}_1)/\partial D_0 \sim \partial D_1.$$

You should find all of the imprecisions / choices made in the foregoing definition. On the other hand, you will have to trust me that the space $M_0 \# M_1$, called the **connected sum** of M_0 and M_1 is independent of those choices up to homeomorphism. For example, you should convince yourself by drawing careful pictures that a two-hold torus is the connected sum of two tori.

In fact, we can classify all compact 2-manifolds using the connected sum! (We will *not* prove the following theorem in this class.)

Theorem 7.9. *Any compact* 2*-manifold is homeomorphic to one of the following:*

1. S^2,
2. $T^2 \# \cdots \# T^2$, *or*
3. $\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2$.

Example 7.10. *Where is* $T^2 \# \mathbb{R}P^2$ *on that list?* HINT: *You should use the polygonal models introduced in the previous section. First, figure out how to make a connect sum between two polygonal models. Next, think about cutting the polygonal models along a line connecting two vertices (recording the identification!) and re-gluing the resulting two polygons along an identified edge.*

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[14] This is the biased view of a low-dimensional topologist. Homotopy theorists would differ, as would analysts and algebraic geometers.

Aside: Are there classification theorems for higher-dimensional (compact) manifolds? In dimension 3, the first step is to understand 3-manifolds M that satisfy the property that every continuous loop in M can be continuously contracted to a point (we'll make this rigorous in the next section). Poincaré conjectured that the only such manifold was S^3; Perelman finally proved this in the mid-2000's. In fact, he proved Thurston's Geometrization Conjecture, which recast the classification of 3-manifolds in terms of the classification of groups that can act as isometries on one of 8 3-dimensional geometries (spherical, Euclidean, hyperbolic, ...).

On the other hand, one can prove that it is not possible to classify 4-manifolds (or higher dimensional manifolds, for that matter)! Further, in higher dimensions, there is a difference between *smooth* manifolds and *topological* manifolds — for example, there are 28 differentiable structures on S^7 (Milnor), and uncountably many on $\mathbb{R}^4$ (Taubes).

Part 3. Algebraic Topology

So far, we have developed a language (point-set topology) that we can use to specify topological problems. We have seen several ways to construct interesting examples of topological spaces, from the subspace and metric topologies through the more interesting quotient constructions in the previous part. We have also considered several types of topological properties, from separability and countability properties to connectedness and compactness. Of these, perhaps connectedness was the most interesting, as it hews closely to the intuitive idea of topology as "rubber sheet geometry" and let us prove one-dimensional versions of some interesting theorems: the Brouwer Fixed Point Theorem, the Borsuk-Ulam theorem, and Invariance of Domain. In order to generalize these theorems, we need to generalize the idea of (path) connectedness. This generalization is accomplished using the fundamental group of a topological space.

The basic idea of algebraic topology is to associate to a topological space X a group, called the fundamental group $\pi_1(X)$ (as we shall see below, the notation "$\pi_1(X)$" is a slight lie). Further, to a continuous map $f : X \to Y$, we will associate a homomorphism $f_* : \pi_1(X) \to \pi_1(Y)$. Thus, we may convert problems in topology to problems in group theory. What is the advantage of this? Computability (at least relative computability)! For example, it is far easier to distinguish two abelian groups (up to isomorphism) than two topological spaces (up to homeomorphism). Similarly, it is far easier to declare that $\mathbb{Z}$ is not isomorphic to the trivial group than it is to prove that a map $f : D^2 \to D^2$ has a fixed point.

Note: You are welcome to consult your algebra textbook while working on this section.

8. The Fundamental Group

The fundamental group is not quite a property of a space X, but a property of a space X together with a chosen basepoint $x_0 \in X$. The set underlying the fundamental group will be the set of paths in X that begin and end at x_0 (i.e. loops "based at" x_0), where we will identify two paths if, intuitively speaking, we can deform one into another while fixing the endpoints.[15] We will multiply two loops together by concatenating them (i.e. following one path and then the next). This operation turns out to be a group multiplication, as we shall see below.

8.1. **Homotopy.**

The first step in defining the fundamental group is to formalize the notion of a deformation of paths. Henceforth, we will denote the unit interval $[0, 1]$ by I. Recall that a path γ in a space X is a continuous map $\gamma : I \to X$.

Definition 8.1. Two paths $\gamma_0, \gamma_1 : x \rightsquigarrow y$ in X (note that they have the same endpoints!) are **path homotopic** if there exists a continuous map $\Gamma : I \times I \to X$ such that:

$$\begin{aligned} \Gamma(s, 0) &= \gamma_0(s) & \qquad \Gamma(0, t) &= x \\ \Gamma(s, 1) &= \gamma_1(s) & \qquad \Gamma(1, t) &= y \end{aligned}$$

The map Γ is called a **homotopy** between γ_0 and γ_1.

We will write $\gamma \simeq \delta$ if the paths γ and δ are path homotopic.

Example 8.2. *Any two paths with the same endpoints are homotopic in $\mathbb{R}^n$. Why does the homotopy you wrote down not work for the paths $\gamma_+(t) = (\cos \pi t, \sin \pi t)$ and $\gamma_-(t) = (\cos \pi t, -\sin \pi t)$ in $\mathbb{R}^2 \setminus \{0\}$)? In fact, no homotopy does, but we cannot prove that yet.*

Proposition 8.3. *Path homotopy is an equivalence relation on the set of paths $\gamma : x \rightsquigarrow y$.*

We will denote the path homotopy class of γ by $[\gamma]$.

8.2. **Definition of the Fundamental Group.**

Definition 8.4. Given a topological space X and a point $x_0 \in X$, the set of all path homotopy classes of paths in X that start and end at x_0 is the **fundamental group** $\pi_1(X, x_0)$ of X.

[15] You may also think of this set as $\pi_0(\Omega_0 X)$, where $\Omega_0 X$ is the set of based loops in X. This is a powerful way of thinking, though it will not be all that useful in this course.

We want to impose a group structure on $\pi_1(X, x_0)$. To do so, we define the composition of two paths $\gamma : x \rightsquigarrow y$ and $\delta : y \rightsquigarrow z$ to be the path $\gamma \cdot \delta : x \rightsquigarrow z$ that is specified by the formula:

$$\gamma \cdot \delta(t) = \begin{cases} \gamma(2t) & t \in [0, 1/2] \\ \delta(2t-1) & t \in [1/2, 1] \end{cases}.$$

Exercise 18. Give a "physical" interpretation of this formula.

Proposition 8.5. *If $\gamma_0 \simeq \gamma_1$ and $\delta_0 \simeq \delta_1$ with $\gamma_i(1) = \delta_i(0)$ for $i = 0, 1$, then $\gamma_0 \cdot \delta_0 \simeq \gamma_1 \cdot \delta_1$.*

It follows that we can define composition at the level of path homotopy classes:

$$[\gamma] \cdot [\delta] = [\gamma \cdot \delta].$$

If we restrict our attention to the set of path homotopy classes of paths that begin and end at the same basepoint $x_0 \in X$, we get a group!

Theorem 8.6. *The fundamental group is, in fact, a group under path composition.*[16]

Proposition 8.7 ($\star$)**.** *If X is path connected, then for any $x_0, x_1 \in X$, we have:*

$$\pi_1(X, x_0) \simeq \pi_1(X, x_1).$$

Let us finish this section by examining a few computations.

Definition 8.8. If $\pi_1(X, x_0)$ is trivial and X is path connected, then we call X **simply connected**.

Example 8.9. *The space $\mathbb{R}^n$ is simply connected.*

Let us save the proofs of the following examples for later (if we get there).[17]

Example 8.10. *The fundamental group of the circle (any basepoint!) is isomorphic to $\mathbb{Z}$.*

Exercise 19. Describe a reasonable guess of a generator of $\pi_1(S^1, (1, 0))$.

Example 8.11. *The fundamental group of the n-sphere is trivial for any $n > 1$.*

Exercise 20. Write down a "proof" of this example using Example 2.12. What hidden assumption did you use?

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[16]There are several axioms you need to verify in this theorem, and each axiom will be a separate presentation in class, though we will not present a proof of associativity. In particular, you will have to define the "identity" path and an "inverse path" for each path. Each part of this problem gets its own $\star$.

[17]The first is actually quite difficult — I usually spend two full lectures on it when I teach this class in the classical format.

9. Continuous Maps and Homomorphisms

Now that we know how to associate the fundamental group to a (based) space, let us explore how to associate a homomorphism between fundamental groups to a continuous map between (based) spaces.

9.1. First Definitions.

The definition of the homomorphism on fundamental groups induced by a continuous map should strike you as being quite natural, though possibly difficult to compute.

Definition 9.1. Let $f : (X, x_0) \to (Y, y_0)$ be continuous.[18] The **homomorphism induced by** f is the map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by:

$$f_*([\gamma]) = [f \circ \gamma].$$

Example 9.2. *Assume the result of Example 8.10 and the generator from the exercise after the example. If $f : S^1 \to S^1$ wraps the circle twice around itself (in polar coordinates, this is $f(\theta) = 2\theta$), then $f_*(n) = 2n$.*

Proposition 9.3. *Given the notation in the definition above, we have:*

(1) *f_* is a homomorphism.*
(2) *If $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$ are continuous maps, then*

$$(g \circ f)_* = g_* \circ f_*.$$

(3) *If $i : X \to X$ is the identity map, then i_* is also the identity map.*

Corollary 9.4. *The fundamental group is a topological property / a topological invariant. That is, if (X, x_0) is homeomorphic to (Y, y_0), then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(Y, y_0)$. (Use only Proposition 9.3 in your proof.)*

Proposition?? 9.5. *Use the same notation as above. If f is injective, then so is f_*. If f is surjective, then so is f_*.*

9.2. Applications.

We can use the notion of induced homomorphism to generalize two of our earlier theorems from the connectedness section. You should feel free to use the (unproved) computations in Examples 8.10 and 8.11.

Definition 9.6. Let $A \subset X$ be a subspace. A continuous map $r : X \to A$ is a **retraction** if $r(a) = a$ for all $a \in A$. In this situation, we say that A is a **retract** of X.

Example 9.7. *S^1 is a retract of $\mathbb{R}^2 \setminus \{0\}$.*

Proposition 9.8. *If $r : (X, a_0) \to (A, a_0)$ is a retraction, then $r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ is surjective and $i_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ is injective.*

Lemma 9.9 ($\star$)**.** *There is no retraction from D^2 to its boundary S^1.*

[18] You should supply your own definition for the notation $f : (X, x_0) \to (Y, y_0)$.

Theorem 9.10 (Brouwer Fixed Point Theorem in 2D)**.** *The disk D^2 has the Fixed Point Property.*

Theorem 9.11 (Invariance of Domain in 2D $\star$)**.** $\mathbb{R}^2$ *is not homeomorphic to* $\mathbb{R}^n$ *for all* $n > 2$.

10. Covering Spaces

I hinted before that the proof that $\pi_1(S^1) \simeq \mathbb{Z}$ is not all that easy. Not only is the result fundamental to the study of the fundamental group, but the machinery behind the proof is of central importance. The machinery is that of covering spaces, and we will begin by examining the basic definitions and lemmas for them.

10.1. The Definition of Covering Spaces.

The central idea of a covering space for a space X is to find a "larger" but "simpler" space related to X.

Suppose that $p : \tilde{X} \to X$ is a continuous, surjective map. An open set $U \subset X$ is **evenly covered by** p if $p^{-1}(U)$ is a disjoint union of open sets $V_\alpha \subset \tilde{X}$ with the property that $p|_{V_\alpha}$ is a homeomorphism of V_α onto U. We call each V_α a **sheet** of $p^{-1}(U)$.

Definition 10.1. If $p : \tilde{X} \to X$ is a continuous, surjective map such that for all $x \in X$, there exists a neighborhood U_x so that U_x is evenly covered by p, then we say that p is a **covering map** and the pair $(\tilde{X}, p)$ is a **covering space** of X.

Example 10.2. *The map* $p : \mathbb{R} \to S^1$ *given by* $p(t) = (\cos 2\pi t, \sin 2\pi t)$ *is a covering map.*

Example 10.3. *The restriction of* p *in the previous example to* $(0, \infty)$ *is continuous, surjective, and a local homeomorphism,*[19] *but is not a covering map.*

Example 10.4. *The map* $p_k : S^1 \to S^1$ *given by*

$$p_k(\cos 2\pi t, \sin 2\pi t) = (\cos 2\pi kt, \sin 2\pi kt)$$

is a covering map.

Example 10.5. *The quotient map* $\pi : S^n \to \mathbb{R}P^n$ *is a covering map.*

Example 10.6. *There exists a covering map* $p : \mathbb{R}^2 \to T^2$.

We now turn to some basic properties of covering spaces.

Lemma 10.7. *If* $p : \tilde{X} \to X$ *is a covering map and* $x \in X$, *then* $p^{-1}(\{x\})$ *has the discrete topology as a subspace of* X.

Lemma 10.8 ($\star$)**.** *Suppose that* $p : \tilde{X} \to X$ *is a covering map and that* $\tilde{X}$ *is connected. Given any* $x_0, x_1 \in X$, *there is a bijection between* $p^{-1}(\{x_0\})$ *and* $p^{-1}(\{x_1\})$.

[19] Supply the definition!

If, in the lemma above, $p^{-1}(\{x\})$ has k elements, then we say that $(\tilde{X}, p)$ is a **k-sheeted covering space**.

Example 10.9. *There is a 2-sheeted cover of the number 8 (thought of as a subspace of $\mathbb{R}^2$). Can you produce other covers?*

10.2. **Lifting Properties.** The fundamental properties of covering spaces are called the "lifting properties". The notion of "lifting" a continuous map to a cover is defined as follows:

Definition 10.10. If $p : \tilde{X} \to X$ is a covering map and $f : A \to X$ is continuous, then a **lift** $\tilde{f} : A \to \tilde{X}$ is a continuous map with the property that $p \circ \tilde{f} = f$:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ A \arrow[ur, dotted, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

We will mostly be concerned with lifting paths. Liftings of paths are not only unique (once you fix a starting point), but also *always exist!*

Proposition 10.11 (Path Lifting $\star$)**.** *Let $(\tilde{X}, p)$ be a covering space of X. Given a path $\gamma : x_0 \rightsquigarrow x_1$ in X and a point $\tilde{x_0} \in \tilde{X}$, there exists a unique lift $\tilde{\gamma}$ of γ with $\tilde{\gamma}(0) = \tilde{x}_0$.*

Exercise 21. Draw pictures of the lifts of the following paths:
- Lift the generator of $\pi_1(S^1)$ to the slinky cover $(\mathbb{R}, p)$.
- Lift the generator of $\pi_1(S^1)$ to the wrapping cover (S^1, p_k).
- Lift the meridian of T^2 to $\mathbb{R}^2$.
- What loop in $\mathbb{R}P^2$ is a longitude of S^2 a lift of? Use the picture in Figure 4.

Proposition 10.12 (Homotopy Lifting)**.** *Let $(\tilde{X}, p)$ be a covering space of X. Given a path homotopy Γ of paths in X that start at x_0 and a point $\tilde{x_0} \in \tilde{X}$, there exists a unique path homotopy $\tilde{\Gamma}$ with $\tilde{\Gamma}(0, 0) = \tilde{x}_0$.*

The proof of the Homotopy Lifting property is not all that different than that of the Path Lifting property, so you need not write it out or present it.

Proposition 10.13. *If $(\tilde{X}, p)$ is a covering space of X and $\tilde{x}_0 \in p^{-1}(\{x_0\})$, then $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.*

10.3. **The Fundamental Group of the Circle.** We are now ready to compute the fundamental group of the circle. Consider the "slinky" covering space $(\mathbb{R}, p)$ of S^1. Let γ be a loop in S^1 based at $(1, 0)$. Let $\tilde{\gamma} : I \to \mathbb{R}$ be a lift of γ to the slinky cover with $\tilde{\gamma}(0) = 0$, which is guaranteed to exist by the Path Lifting property. We define the **degree** of γ by:

$$\deg(\gamma) = \tilde{\gamma}(1).$$

Lemma 10.14. *In the setup above, $\deg(\gamma) \in \mathbb{Z}$.*

Lemma 10.15. *If* $\gamma \simeq \delta$*, then* $\deg(\gamma) = \deg(\delta)$.

The previous two lemmas show that deg can be thought of as a map from $\pi_1(S^1, (1,0))$ to $\mathbb{Z}$. In fact, this map is the desired isomorphism!

Theorem 10.16. *The degree map* $\deg : \pi_1(S^1, (1,0)) \to \mathbb{Z}$ *is an isomorphism.*

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