335 HW

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BEFORE PRESENTATION

Proposition 1.13: Given a set X and a basis $\mathcal{B}$, let

$$\mathcal{T}'_{\mathcal{B}} = \{U \subset X : U = \bigcup_{\alpha \in I} B_\alpha \text{ for some collection } \{B_\alpha\}_{\alpha \in I} \subset \mathcal{B}\}$$

Then $\mathcal{T}_{\mathcal{B}} = \mathcal{T}'_{\mathcal{B}}$

Proof. Proof by double containment.

First prove that $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}'_{\mathcal{B}}$.

For all $U \in \mathcal{T}_B$, it satisfies that $U \subset X$, for all $x \in U$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$. So we know that $U \supset \bigcup_{\alpha \in I} B_\alpha$ for some $\{B_\alpha\}_{\alpha \in I} \subset \mathcal{B}$.

So now we want to show that $U \subset \bigcup_{\alpha \in I} B_\alpha$.

For all $x \in U$, there exist B_α such that $x \in B_\alpha$. So all element in U is in some $B_\alpha \subset U_{\alpha \in I} B_\alpha$. So, $U \subset \bigcup_{\alpha \in I} B_\alpha$. Therefore $U = \bigcup_{\alpha \in I} B_\alpha$, and thus all element in $\mathcal{T}_{\mathcal{B}}$ is in $\mathcal{T}'_{\mathcal{B}}$.

So $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}'_{\mathcal{B}}$.

Now prove that $\mathcal{T}_{\mathcal{B}} \supseteq \mathcal{T}'_{\mathcal{B}}$

For all U in $\mathcal{T}'_{\mathcal{B}}$, we know that $U = \bigcup_\alpha B_\alpha$ for some $\{B_\alpha\}_{\alpha \in I} \in \mathcal{B}$. This implies that $U \supset \bigcup_\alpha B_\alpha$. Since $\mathcal{B}$ is a basis, by B1, for all $x \in U \subset X$, there exist $B_\alpha \in \mathcal{B}$ such that $x \in B_\alpha$. Since U is a contains the union of the B_α's, we know that for all $x \in U$, we can find a $B_\alpha \in \mathcal{B}$, such that $x \in B_\alpha \subset U$. Thus all element in $\mathcal{T}'_{\mathcal{B}} \subseteq \mathcal{T}_{\mathcal{B}}$.

Therefore $\mathcal{T}_{\mathcal{B}} = \mathcal{T}'_{\mathcal{B}}$. $\square$

Example 1.15: in $\mathbb{R}$, the following collections of sets are bases for a topology on $\mathbb{R}$:

(1) $\mathcal{B}_l = \{[a, b) : a < b\}$ is a basis for the lower limit topology $\mathcal{T}_l$.

Proof. Show that $\mathcal{B}_l$ is a basis for $\mathcal{T}_l$.

We want to show that:

(a) for all $x \in l$, there exists $B_x \in \mathcal{B}_l$ such that $x \in B_x$, and
(b) if $x \in B_1 \cap B_2$ for some $B_i \in \mathcal{B}_l$, then there exists $B_3 \in \mathcal{B}_l$ such that $x \in B_3 \subset B_1 \cap B_2$.

Date: September 8, 2021.

Pick an arbitrary $x \in l$. Construct $B_x = [x-1, x+1)$. Since $x-1 < x+1$, B_x is in $\mathcal{B}_l$. Since $x \in [x-1, x+1)$ through observation, condition (a) check.

Still pick an arbitrary $x \in l$. Suppose $x \in B_1 = [a, b)$ and $x \in B_2 = [c, d)$. Then we know that $x \geq a, x \geq c$ and $x < b, x < d$. Find a number $y \geq \max(a, b)$, a number $z \leq \min(b, d)$ such that $y \leq x, z > x$. The reason why we can find such y is that the reals are dense and we can always find a number between x and $\max(a, c)$ (in the case of $x > \max(a, c)$, and $y = x$ in the case $x = \max(a, c)$). Similarly, we can always find a z between x and $\min(b, d)$.

Since $x \geq y, x < z$, we know that $x \in [y, z)$. Let $B_3 = [y, z)$. Since $y \geq \max(a, c)$ and $z \leq \min(b, d)$, we know that $B_3 \subset B_1, B_2 \subset B_2$. Thus, $B_3 \subset B_1 \cap B_2$. Condition (b) check. $\square$

(2) $\mathcal{B}_{l,\mathbb{Q}} = \{[a, b) : a < b \text{ with } a, b \in \mathbb{Q}\}$ is a basis for the rational lower limit topology $\mathcal{T}_{l,\mathbb{Q}}$.

Proof. The proof for this one is similar to the one above, with the exception that we need to use the property of rationals are dense in $\mathbb{R}$.

Pick an arbitrary $x \in l$. Construct $B_x = [x-1, x+1)$. Since $x \in \mathbb{Q}$ by definition of the space, and rationals' sum is still a rational, so we know that $x-1, x+1 \in \mathbb{Q}$. Since $x-1 < x+1$, B_x is in $\mathcal{B}_l$. Since $x \in [x-1, x+1)$ through observation, condition (a) check.

Still pick an arbitrary $x \in l$. Suppose $x \in B_1 = [a, b)$ and $x \in B_2 = [c, d)$, where $a, b, c, d \in \mathbb{Q}$. Then we know that $x \geq a, x \geq c$ and $x < b, x < d$. Find a number $y \geq \max(a, b)$, a number $z \leq \min(b, d)$ such that $y \leq x, z > x$. The reason why we can find such y is that the rationals are dense in the reals and we can always find a number between x and $\max(a, c)$ (in the case of $x > \max(a, c)$, and $y = x$ in the case $x = \max(a, c)$). Similarly, we can always find a z between x and $\min(b, d)$.

Since $x \geq y, x < z$, we know that $x \in [y, z)$. Let $B_3 = [y, z)$. Since $y \geq \max(a, c)$ and $z \leq \min(b, d)$, we know that $B_3 \subset B_1, B_2 \subset B_2$. Thus, $B_3 \subset B_1 \cap B_2$. Condition (b) check. $\square$

(3) $\mathcal{B}_H = \mathcal{B} \cup \{(a, b) \backslash H : a < b\}$, where $H = \{\frac{1}{n}\}_{n \in \mathbb{Z}}$, is a basis for the harmonic topology $\mathcal{T}_H$.

Confused, if $\mathcal{B}$ means the collection of open balls on the reals, there's really nothing need to be proved? It $\mathcal{B}$ doesn't mean that, what does it mean...

Proposition 1.17: Suppose $\mathcal{B}$ and $\mathcal{B}'$ are bases on a set X. Then $\mathcal{T}_{\mathcal{B}}$ is finer than $\mathcal{T}_{\mathcal{B}'}$ if and only if for all $B' \in \mathcal{B}'$, there exists a collection of sets $\{B_\alpha\}_{\alpha \in I}$ in $\mathcal{B}$ such that $B' = \cup_\alpha B_\alpha$ (i.e., $\mathcal{B}' \subset \mathcal{T}_{\mathcal{B}}$).

Proof. First the forward direction.

Suppose $\mathcal{T}_{\mathcal{B}}$ is finer than $\mathcal{T}_{\mathcal{B}'}$. Then we know that $\mathcal{T}_{\mathcal{B}'} \subset \mathcal{T}_{\mathcal{B}}$.

Pick an arbitrary $B' \in \mathcal{B}'$. Since both B and B' are bases on X, for all $x \in X$, there exist $B_1 \in \mathcal{B}$ such that $x \in B_1 \subset X$, and there exist $B'_1 \in \mathcal{B}'$ such that $x \in B'_1 \subset X$. Then, for all $x \in B' \subset X$, there exist $B_\alpha \in B$ such that $x \in B_\alpha \subset X$. By B2, we can choose the "smallest" B_α that contains x into the union $\cup_\alpha B_\alpha$.

Since for all element in B' has an B_α containing it, $B' \subset \cup_\alpha B_\alpha$.

Now we want to show that $B' \supset \cup_\alpha B_\alpha$.

(kind of using proof by contradiction here, real shady.)

Since by construction of B_α's in the union, every x in the union is in the smallest open ball possible. So if there exists some y near x such that $y \notin B', x \in B'$, we can always find an open ball around x, smaller and not containing y. Thus, all element in the union is in B' by construction of the union.

Now backward direction. Suppose that for all $B' \in \mathcal{B}'$, there exists a collection of $\{B_\alpha\}_{\alpha \in I} \in \mathcal{B}$ such that $B' = \cup_\alpha B_\alpha$.

For all open sets U' in $\mathcal{T}_{\mathcal{B}'}$, we know that

$$U' = \bigcup_\alpha B'_2 \text{ for some collection of } B'_\alpha \in \mathcal{B}'$$

Since for all $B' \in \mathcal{B}'$, there exist $\{B_\beta\}_{\beta \in I}$ such that $B' = \cup_\beta B_\beta$, we can write U' as $\cup_\alpha \cup_\beta B_\beta$. We know that $\cup_\beta B_\beta$ is in $\mathcal{T}_{\mathcal{B}}$ by definition of a basis. Therefore all open sets in $\mathcal{T}_{\mathcal{B}'}$ is in $\mathcal{T}_{\mathcal{B}}$, which implies that $\mathcal{T}_{\mathcal{B}'} \subset \mathcal{T}_{\mathcal{B}}$. $\square$