PROOFS FOR TOPOLOGY

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BEFORE PRESENTATION

Example 2.14. The 2-torus described in Example 1.33 is homeomorphic to the product $S^1 \times S^1$.

Version: 1
Comments / Collaborators: I feel this is intuitive.

Proof. For any element in the 2-torus, it can be described with two angles $(\theta_\alpha, \theta_\beta)$ where θ_α is the angle of the point with respect to the x axis, when you are looking down from above, and θ_β is the angle of the point when looking at the cross-section. So, a set notation would be

$$T^2 = \left\{ \left(\begin{pmatrix} \sin\theta_\alpha \\ \cos\theta_\alpha \end{pmatrix}, \begin{pmatrix} \sin\theta_\beta \\ \cos\theta_\beta \end{pmatrix} \right) : \theta_\alpha, \theta_\beta \in [0, 2\pi) \right\} \tag{1}$$

Let f be the function from $S^1 \times S^1$ to T^2,

$$f : \begin{pmatrix} \sin\theta_\alpha \\ \cos\theta_\alpha \end{pmatrix} \times \begin{pmatrix} \sin\theta_\beta \\ \cos\theta_\beta \end{pmatrix} \mapsto \left(\begin{pmatrix} \sin\theta_\alpha \\ \cos\theta_\alpha \end{pmatrix}, \begin{pmatrix} \sin\theta_\beta \\ \cos\theta_\beta \end{pmatrix} \right) \tag{2}$$

For any element in T^2, with θ_1 and θ_2, we can always find and element in $S^1 \times S^1$ such that one is element with θ_1 and the other is with θ_2, that sends to the element in T^2. Thus f is onto.

Pick two arbitrary elements from T^2, say x with θ_1, θ_2, and y with θ_3, θ_4, where $x \neq y$, which implies θ_1, θ_2 differs from θ_3, θ_4. Then we know that the preimage of x would be two elements from S^1 and one with the angle θ_1 and the other with the angle θ_2; the preimage of y would be two elements from S^1 and one with the angle θ_3 and the other with the angle θ_4. Since the elements from S^1 are with different angles (or different positioning of α and β), we know that f is one-to-one.

Thus, f is a bijection.

Let g be the inverse function of f (should have started with g at the beginning, but since we are here already...).

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We know that

$$g : T^2 \to S^1 \times S^1,$$
$$g : \left(\begin{pmatrix} \sin\theta_\alpha \\ \cos\theta_\alpha \end{pmatrix}, \begin{pmatrix} \sin\theta_\beta \\ \cos\theta_\beta \end{pmatrix} \right) \mapsto \begin{pmatrix} \sin\theta_\alpha \\ \cos\theta_\alpha \end{pmatrix} \times \begin{pmatrix} \sin\theta_\beta \\ \cos\theta_\beta \end{pmatrix} \tag{3}$$

By proposition 2.8, g is continuous if and only if $\pi_X \circ g$ and $\pi_Y \circ g$ are continuous. In this case,

$$\begin{aligned} \pi_X &: S^1 \times S^1 \to S^1, \\ \pi_Y &: S^1 \times S^1 \to S^1. \end{aligned} \tag{4}$$

Obviously π_X and π_Y are the same. So we only need to prove $\pi_X \circ g$.

Pick an arbitrary open set from S^1. We know that this open set will be some portion of the unit circle. The preimage of this open set would be, when you looking from above of the 2-torus, the same portion on this "unit circle," and the cross-sections in this portion (it's like a bite/ or bites on a doughnut), which is also open in T^2. Thus, $\pi_X \circ g$ is continuous, g is continuous.

f would be continuous by the virtue of construction. $\left(\begin{pmatrix} \sin\theta_\alpha \\ \cos\theta_\alpha \end{pmatrix}, \begin{pmatrix} \sin\theta_\beta \\ \cos\theta_\beta \end{pmatrix} \right)$ can be seen as basis elements for $S^1 \times S^1$, which requires the sets that are being producted to be open in S^1. Since the preimage of all basis element in T^2 is open in S^1, f is open as well.

Thus f is bijective and bi-continuous.

Therefore, T^2 and $S^1 \times S^1$ is homeomorphic. $\square$

Proposition 2.15. The fixed point property is a topological property

Version: 1
Comments / Collaborators: X has the fixed point property if for any continuous function $f : X \to X$, there exists and $x \in X$ such that $f(x) = x$. Just copying the definition for the fixed point property so that I don't need to look at the notes constantly....

Proof. Suppose X and Y are homeomorphic spaces, and X has the fixed point property. There exists a homeomorphism $g : X \to Y$. Let h be an arbitrary continuous function from Y to Y, we want to show that there exists $y \in Y$ such that $h(y) = y$.

Since g is a homeomorphism, the inverse function of g is continuous as well; call the inverse function f. Let $i = f \circ h \circ g$. We know that $i : X \to X$. So we know that there exists an $x \in X$ such that $i(x) = x$. This implies that $f \circ h \circ g(x) = x$. Since g is bijective, we know that there can be and only be one element $y \in Y$ such that $g(x) = y$. So, $f \circ h \circ (y) = x$. Since f is the inverse function of g, it also have the property that for each x, there can be and only be one element y such that $f(y) = x$. So if we multiply f^{-1} on both sides, we get $h(y) = y$.

Thus, the fixed point property is a topological property. $\square$

Proposition 2.16. Contractibility is a topological property.

Version: 1
Comments / Collaborators: A topological space X is contractible if there exists a point $x_0 \in X$ and a continuous function $H : X \times [0,1] \to X$ such that $H(x,0) = x$ and $H(x,1) = x_0$.

Proof. Suppose there exists two homeomorphic spaces X and Y. Space X is contractible.

We want to show that in the space Y, there exists a point y_0 and a continuous function H such that $Y \times [0,1] \to Y$ such that $H(y,0) = y$ and $H(y,1) = y_0$.

Since there exists a continuous function f from $Y \to X$, let h be the continuous function from $Y \times [0,1]$ to $X \times [0,1]$, by sending (y,a) to (x,a) where $a \in [0,1]$ and $y \mapsto x$ using f.

Since X is contractible, there exists x_0 and G such that $G : X \times [0,1] \to X$, and $G(x,0) = x$ and $G(x,1) = x_0$.

Call the inverse function of f, g.

Let $H = g \circ G \circ h$. We know that H is continuous since all of h, G, g are continuous.

Pick $y_0 = g(x_0)$.

$H(y,0) = g \circ G \circ h(y,0) = g \circ G(x,0) = g(x) = y$.

$H(y,1) = g \circ G \circ h(y,1) = g \circ G(x,1) = g(x_0) = y_0$.

Such H satisfies our conditions.

Thus, contractibility is a topological property. $\square$