335 HW

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Proposition 1.13: Given a set X and a basis $\mathcal{B}$, let

$$\mathcal{T}'_{\mathcal{B}} = \{U \subset X : U = \bigcup_{\alpha \in I} B_\alpha \text{ for some collection } \{B_\alpha\}_{\alpha \in I} \subset \mathcal{B}\}$$

Then $\mathcal{T}_{\mathcal{B}} = \mathcal{T}'_{\mathcal{B}}$

Proof. Proof by double containment.

First prove that $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}'_{\mathcal{B}}$.

For all $U \in \mathcal{T}_{B}$, it satisfies that $U \subset X$, for all $x \in U$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$. So we know that $U \supset \bigcup_{\alpha \in I} B_\alpha$ for some $\{B_\alpha\}_{\alpha \in I} \subset \mathcal{B}$.

So now we want to show that $U \subset \bigcup_{\alpha \in I} B_\alpha$.

For all $x \in U$, there exist B_α such that $x \in B_\alpha$. So all element in U is in some $B_\alpha \subset U_{\alpha \in I} B_\alpha$. So, $U \subset \bigcup_{\alpha \in I} B_\alpha$. Therefore $U = \bigcup_{\alpha \in I} B_\alpha$, and thus all element in $\mathcal{T}_{\mathcal{B}}$ is in $\mathcal{T}'_{\mathcal{B}}$.

So $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}'_{\mathcal{B}}$.

Now prove that $\mathcal{T}_{\mathcal{B}} \supseteq \mathcal{T}'_{\mathcal{B}}$

For all U in $\mathcal{T}'_{\mathcal{B}}$, we know that $U = \bigcup_\alpha B_\alpha$ for some $\{B_\alpha\}_{\alpha \in I} \in \mathcal{B}$. This implies that $U \supset \bigcup_\alpha B_\alpha$. So for all $x \in U$, we can find some $B \in \bigcup_\alpha B_\alpha$ such that $x \in B \subset U$. Thus all element in $\mathcal{T}'_{\mathcal{B}} \subseteq \mathcal{T}_{\mathcal{B}}$.

Therefore $\mathcal{T}_{\mathcal{B}} = \mathcal{T}'_{\mathcal{B}}$. $\square$

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