PROOFS FOR TOPOLOGY

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1. Rewrite of Previous Weeks' problem

Example 2.13. The topological group $SO(2)$ is homeomorphic to S^1

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Proof. The topological group $SO(2)$ is composed by elements in the form of $\begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$, and S^1 is composed of elements in the form of $\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$, with $\theta \in [0, 2\pi)$ for both spaces.

Let f be a function from S^1 to $SO(2)$ such that $f\left(\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}\right) = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$.

Suppose there exists two elements from $SO(2)$ such that they are not equal to each other.

$$\begin{pmatrix} \sin\theta_1 & \cos\theta_1 \\ -\cos\theta_1 & \sin\theta_1 \end{pmatrix} \neq \begin{pmatrix} \sin\theta_2 & \cos\theta_2 \\ -\cos\theta_2 & \sin\theta_2 \end{pmatrix}. \tag{1}$$

Their preimages would be $\begin{pmatrix} \sin\theta_1 \\ \cos\theta_1 \end{pmatrix}$ and $\begin{pmatrix} \sin\theta_2 \\ \cos\theta_2 \end{pmatrix}$ respectively. Since $\theta_1 \neq \theta_2$, we know that

$$\begin{pmatrix} \sin\theta_1 \\ \cos\theta_1 \end{pmatrix} \neq \begin{pmatrix} \sin\theta_2 \\ \cos\theta_2 \end{pmatrix}. \tag{2}$$

For any element in S^1 with the angle being θ, we can find an element in $SO(2)$ with the angle being θ such that it maps to the element in S^1.

Therefore, the function is a bijection.

Since $SO(2)$ is equipped with a subspace topology in $\mathbb{R}^4$, by the same logic in example 1.28, we know that $\mathbb{R}^4$ can be equipped with a product

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topology associated to $\mathbb{R}^2$. Thus, we can see $SO(2)$ as $\begin{pmatrix} \sin\theta \\ \cos\theta \\ -\cos\theta \\ \sin\theta \end{pmatrix}$, which can be further seen as $\begin{pmatrix} X \\ Y \end{pmatrix}$, where $X = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} = S^1$, and $Y = \begin{pmatrix} -\cos\theta \\ \sin\theta \end{pmatrix}$.

Since $\begin{pmatrix} X \\ Y \end{pmatrix}$ is homeomorphic to $X \times Y$, now our function f can be seen as

$$f : S^1 \to S^1 \times Y. \tag{3}$$

By proposition 2.8, f is continuous if and only if $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous. Now we want to show that $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous.

We know that the projection maps are

$$\begin{aligned} \pi_X &: S^1 \times Y \to S^1 \\ \pi_Y &: S^1 \times Y \to Y \end{aligned} \tag{4}$$

We notice that π_X is the inverse function of f, and by lemma 2.7, we know that π_X is continuous. So the inverse function of f is continuous.

We also notice that since π_X is the inverse function of f, so $\pi_X \circ f$ is the "do nothing function," which is also continuous by the virtue of doing nothing.

It's not hard to see that Y is also a description of the unit circle centered at the origin.

Since $\pi_y \circ f$ is sending $-\cos\theta$ to $\sin\theta$, and $\sin\theta$ to $\cos\theta$, it's not hard to see that this essentially is taking the derivatives, and we know that the first derivatives of $-\cos\theta$ and $\sin\theta$ are continuous. Thus f is continuous.

Since f is bijection and bi-continuous, f is a homeomorphism. $\square$

Tarik's corner. (1) Let G be a topological group and let H be a subgroup.

(b) Show that the closure of H is also a subgroup of G. Note: given a topological space X and a subset A of X, the closure of A is the set of points $x \in X$ so that if U is any open subset of X containing x, then there exists some $a \in A$ which is also in U. In other words: the closure of A is the set of points that can not be separated from A by open sets.

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Comments / Collaborators: Only need to rewrite part b

Proof. (b) We know that every element of of H is also in the closure of H since by definition, every open sets containing the element of H for sure has an element from H. So we know that the identity element is in the closure of H.

Since H is a subgroup, so we know that all element in H is invertible and closed under multiplication. Hence, in order to prove that the closure of H is also a subgroup, we need to show that the elements that are not in H but in the closure of H are still closed and invertible in the closure of H.

Suppose g is in the closure of H, but not in H. We want to show that g^{-1} is also in the closure of H. Suppose g^{-1} is in some arbitrary open set U. The preimage of U with respect to the inverse function, ι, will still be an open set since the inverse function is continuous. We know that $g \in \iota^{-1}(U)$, so there exists some $h \in H$ such that $h \in \iota^{-1}(U)$. Take the inverse of h, we know that h^{-1} is in H since H is a subgroup, and we also know that h^{-1} is in U since $h \in \iota^{-1}(U)$. Thus, for any open set containing g^{-1} there exists $h^{-1} \in H$ such that h^{-1} is in that open set.

Suppose g and j are in the closure of H, but not in H. Let U be the open set containing g and V be the open set that contains j. We know that there exists some h in U and some l in V such that h, l are in H.

We know that $g \cdot j \in U \times V$, and $h \cdot l$ is in $U \times V$ as well. Since H is closed, we know that $h \cdot l$ is in H as well. So $g \cdot j$ is in the closure of H by definition. Thus, we know that the closure of H is closed.

Since the identity element is in the closure of H, it's closed, it has all its inverses, and the operation is the same as G, we know that the closure of H is a subgroup of G. $\square$

2. Rewrite of This Week's BP

Example 2.14. The 2-torus described in Example 1.33 is homeomorphic to the product $S^1 \times S^1$.

Version: 2
Comments / Collaborators: I feel this is intuitive.

Proof. For any element in the 2-torus, it can be described with two angles $(\theta_\alpha, \theta_\beta)$ where θ_α is the angle of the point with respect to the x axis, when you are looking down from above, and θ_β is the angle of the point when looking at the cross-section. So, a set notation would be

$$T^2 = \left\{ \left(\begin{pmatrix} \sin\theta_\alpha \\ \cos\theta_\alpha \end{pmatrix}, \begin{pmatrix} \sin\theta_\beta \\ \cos\theta_\beta \end{pmatrix} \right) : \theta_\alpha, \theta_\beta \in [0, 2\pi) \right\} \tag{5}$$

Let f be the function from $S^1 \times S^1$ to T^2,

$$f : \begin{pmatrix} \sin\theta_\alpha \\ \cos\theta_\alpha \end{pmatrix} \times \begin{pmatrix} \sin\theta_\beta \\ \cos\theta_\beta \end{pmatrix} \mapsto \left(\begin{pmatrix} \sin\theta_\alpha \\ \cos\theta_\alpha \end{pmatrix}, \begin{pmatrix} \sin\theta_\beta \\ \cos\theta_\beta \end{pmatrix} \right) \tag{6}$$

For any element in T^2, with θ_1 and θ_2, we can always find and element in $S^1 \times S^1$ such that one is element with θ_1 and the other is with θ_2, that sends to the element in T^2. Thus f is onto.

Pick two arbitrary elements from T^2, say x with θ_1, θ_2, and y with θ_3, θ_4, where $x \neq y$, which implies θ_1, θ_2 differs from θ_3, θ_4. Then we know that the preimage of x would be two elements from S^1 and one with the angle θ_1 and the other with the angle θ_2; the preimage of y would be two elements from S^1 and one with the angle θ_3 and the other with the angle θ_4. Since the elements from S^1 are with different angles (or different positioning of α and β), we know that f is one-to-one.

Thus, f is a bijection.

f would be continuous by the virtue of construction. $\left(\begin{pmatrix} \sin\theta_\alpha \\ \cos\theta_\alpha \end{pmatrix}, \begin{pmatrix} \sin\theta_\beta \\ \cos\theta_\beta \end{pmatrix} \right)$ can be seen as basis elements for $S^1 \times S^1$, which requires the sets that are being producted to be open in S^1. Since the preimage of all basis element in T^2 is open in S^1, f is continuous.

Let g be the inverse function of f.

We know that

$$\begin{aligned} &g : T^2 \to S^1 \times S^1, \\ &g : \left(\begin{pmatrix} \sin\theta_\alpha \\ \cos\theta_\alpha \end{pmatrix}, \begin{pmatrix} \sin\theta_\beta \\ \cos\theta_\beta \end{pmatrix} \right) \mapsto \begin{pmatrix} \sin\theta_\alpha \\ \cos\theta_\alpha \end{pmatrix} \times \begin{pmatrix} \sin\theta_\beta \\ \cos\theta_\beta \end{pmatrix} \end{aligned} \tag{7}$$

By proposition 2.8, g is continuous if and only if $\pi_X \circ g$ and $\pi_Y \circ g$ are continuous. In this case,

$$\begin{aligned} \pi_X &: S^1 \times S^1 \to S^1, \\ \pi_Y &: S^1 \times S^1 \to S^1. \end{aligned} \tag{8}$$

First, $\pi_X \circ g$: Pick an arbitrary open set, U, from S^1. Call the preimage of U, V, we know that $V = (\pi_X \circ g)^{-1}(U) = \{(U, S^1)\}$. Since U and S^1 are open in S^1, we know that V is open in T^2.

Then, $\pi_Y \circ g$: Pick an arbitrary open set, U, from S^1. Call the preimage of U, V, we know that $V = (\pi_Y \circ g)^{-1}(U) = \{(S^1, U)\}$. Since U and S^1 are open S^1, we know that V is open in T^2.

Thus, $\pi_Y \circ g$ and $\pi_Y \circ g$ are continuous, g is continuous.

Thus f is bijective and bi-continuous.

Therefore, T^2 and $S^1 \times S^1$ is homeomorphic. □

3. This Week's problems

Tarik's Corner. Show that metrizability is a topological property. i.e., if X, Y are topological spaces which are homeomorphic to each other, and there exists a metric giving rise to the topology on X, then there exists a metric giving rise to the topology on Y.

Version: 1
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Proof. Suppose X and Y are homeomorphic spaces. Suppose X is a metric topology based on some metric d_x.

Since they are homeomorphic, there exists a homeomorphism $f : X \to Y$.

Construct a metric d_y on Y, such that $d_y(y_1, y_2) = d_x(x_1, x_2)$ where $y_1 = f(x_1), y_2 = f(x_2)$.

Let's show that d_y is a metric.

Since d_x is a metric, we know that $d_x(x_1, x_2) \geq 0$ for all x_1, x_2 in X. And each $d_y(y_1, y_2)$ equals to some $d_x(x_1, x_2)$, we know that d_y can never be negative.

Since $d_x(x_1, x_2)$ is zero if and only if $x_1 = x_2$, and since X and Y are homeomorphic, we know that $d_y(y_1, y_2) = 0$ if and only if $y_1 = f(x_1) = f(x_2) = y_2$.

Suppose $f(x_1) = y_1, f(x_2) = y_2$, then we know that $d_y(y_1, y_2) = d_x(x_1, x_2) = d_x(x_2, x_1) = d_y(y_2, y_1)$.

Suppose $f(x_1) = y_1, f(x_2) = y_2, f(x_3) = y_3$, then we know $d_y(y_1, y_2) + d_y(y_2, y_3) = d_x(x_1, x_2) + d_x(x_2, x_3) \geq d_x(x_1, x_3) = d_y(y_1, y_3)$.

Thus, d_y is a metric on Y. $\square$

Tarik's Corner. Let X be a metric space. A metric space Y is said to contain an isometrically embedded copy of X if there exists a continuous function $f : X \to Y$ satisfying

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)) \text{ for all } x_1, x_2 \in X.$$

(a) Prove that f is a homeomorphism onto its image, i.e., if we change the codomain of f to be $f(X)$, f would be a homeomorphism. (Equivalently, the only reason why f is not a homeomorphism is because it may not be surjective).

(b) Consider the set M of all metric spaces (for reasons we will hopefully eventually discuss in class, this is technically not really a set, but this caveat won't be relevant to the rest of the problem). Say that a given property P is a topological property of metric spaces if whenever $W \in M$ satisfies P and $Y \in M$ is homeomorphic to W, then Y satisfies P as well. Show that if $P =$ "contains an isometrically embedded copy of X" for a given $X \in M$, then P is not a topological property of metric spaces.

Version: 1
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Proof. (a) Since the codomain is the image of f, it's obvious that f is surjective.

Suppose there exists $f(x_1) = f(x_2)$ in the codomain of f. Then we know that $d_Y(f(x_1), f(x_2)) = 0 = d_X(x_1, x_2)$. Since d_X is a metric, we know that $x_1 = x_2$. Thus, f is a bijection.

Since $f : X \to Y$ is a continuous function, restricting Y to only the image of f will not change the continuity of f, thus f is still continuous.

Let $g : f(X) \to X$ be the inverse function of f. Since X is a metric space, we can pick an open ball $B(x, \epsilon)$ where all the elements inside satisfies $d_X(x, x_1) < \epsilon$. We know that there exists $f(x)$ such that it satisfies $d_Y(f(x), f(x_1)) < \epsilon$. Thus, there exists an open ball in Y, $B(f(x), \epsilon)$ such that all element in there are within ϵ distance from $f(x)$. Since $B(f(x), \epsilon)$ is an open ball, it is open in Y, thus open in $f(X)$.

Thus, f is bi-continuous on its image.

(b) We know that $\mathbb{R}$ equipped with the standard topology is a space. We know that $(0, 1)$ equipped with the standard topology is homeomorphic to $\mathbb{R}$. We know that $\mathbb{R}$ has an embedded copy of $(-1, 1)$ if we use the standard metric for both $\mathbb{R}$ and $(-1, 1)$. The $d(-1, 1) = 2$ in $(-1, 1)$, however, the maximum distance in $(0, 1)$ is 1, so $(0, 1)$ cannot have an embedded copy of $(-1, 1)$.

Thus, containing an isometrically embedded copy of X is not a topological property. $\square$