TOPOLOGY NOTES

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1. Problem 1.17 Presented by Justine

1.1. Background Stuff.

Defenition 1.1. *τ is finer than τ' if $\tau' \subseteq \tau$.*

Proposition 1.1. *Prop 113. If we let $\tau_B^* = \{U \subseteq X : U = \bigcup_\alpha B_\alpha$ for some $\{B_\alpha\}_{\alpha \in I} \subseteq B\}$, then $\tau_\mathcal{B} = T_\mathcal{B}^*$.*

1.2. Proposition 1.17.

Proposition 1.2. *$\tau_\mathcal{B}$ is finer than $T_{\mathcal{B}'}$ if and only if for all $B' \in \mathcal{B}'$, there exists a collection $\{B_\alpha\}_{\alpha \in I}$ such that $B' = \bigcup_\alpha B_\alpha$.*

Proof. Forward direction: Suppose $\tau_\mathcal{B}$ is finer than $\tau_{\mathcal{B}'}$. Then $\tau_{\mathcal{B}'} \subseteq \tau_\mathcal{B}$. Let $B' \in \mathcal{B}'$. Then $B' \in \tau_{\mathcal{B}'}$. So, $B' \in \tau_B$ implies that $B' \in \tau_\mathcal{B}^*$. So, $\mathcal{B}' = \bigcup_\alpha B_\alpha$ for some $\{B_\alpha\} \subseteq B$.

Reverse direction: Suppose $\forall B' \in \mathcal{B}'$, we have $B' = \bigcup_\alpha B_\alpha$ for some $\{B_\alpha\} \subseteq B$. We want to show that $\tau_{\mathcal{B}'} \subseteq \tau_\mathcal{B}$. SUppose that $U \in \tau_{\mathcal{B}'}$. Then $U \in \tau_{\mathcal{B}'}^*$ and thus $u = \bigcup_\alpha B'_\alpha$ for some $\{B'_\alpha\} \subseteq \mathcal{B}'$. Each $B'_\alpha = \bigcup_{\gamma \in J_\alpha} B_\gamma$ for some $\{\mathcal{B}_\gamma\} \subseteq B$. So, we have $U = \bigcup_{\alpha \in I} \bigcup_{\gamma \in J_\alpha} B_\gamma$. So, $U \in \tau_\mathcal{B}^*$ and thus $U \in \tau_B$. So, $\tau_{B'} \subseteq \tau_\mathcal{B}$ and we finally get that $\tau_\mathcal{B}$ is finer than $\tau_{\mathcal{B}'}$. $\square$

1.3. Comments.

Martin likes the $*$. Cam thinks it's organized. Tarik says that board organization is stellar and notes it as an example for future presentations, particularly because definitions and theorems that were used were clearly laid out and there is a clear progression from left to right. He also says that the only thing he would recommend (and it's a reach) would be to refer to τ as tau, rather than t.

Tarik's clarification: a topology is not X with a set U of open sets; that would be a topological space. The topology is the set τ generated by unions of elements of a topological basis $\mathcal{B}$. Jess' observation: from a topology, you can deduce the master set by unioning all of the elements in the topology.

The opposite of finer is coarser.

A conversation about alarms ensued. Some people wake up before they go off; some people sleep through. Martin didn't sleep for long last night and says it would be alarming if I did not note this whole conversation.

2. Problem 119 Presented by Yuxuan

2.1. Background Stuff.

Defenition 2.1. *$\tau = \{U \subseteq X : \forall x \in U, \exists B_x \in \mathcal{B}$ s.t. $x \in B_x\}$*

Proposition 2.1. *On $\mathbb{R}$, there exists non-comparable topologies τ_1 and τ_2 such that $\tau_1 \not\subseteq \tau_2$ and $\tau_2 \not\subseteq \tau_1$.*

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Proof. Consider the set $\tau_{FC} = \{U \subseteq X : X \setminus U$ is finite or all of $X\}$. Let $U = X \subseteq \{0\}$. Then $X \setminus U = \{0\}$. For the other topology, consider $\mathcal{B}_{inf} = \{[a, \infty), a \in \mathbb{R}$. Let's check that this basis works. For all $x \in \mathbb{R}$, we have $B = [x, \infty)$ such that $x \in B \in \mathcal{B}_{inf}$. Furthermore, if $x \in [a, \infty) \cap [b, \infty)$, then letting $B_3 = [max\{a, b\}, \infty)$, we have $x \in B_3 \subseteq [a, \infty) \cap [b, \infty)$, as desired.

Let us show that these are not comparable. First, we will show that $\tau_{fc} \nsubseteq \tau_{inf}$ We know that $-1 \in X \setminus \{0\}$, so we want finite $B \in \mathcal{B}_{inf}$ such that $B = [a, \infty)$, where $a \leq -1$ for $x \in \mathcal{B}$. So, $0 \in [a, \infty)$. But we know that $0 \notin X \setminus \{0\}$. Thus $X \setminus \{0\} \nsubseteq \tau_{inf}$. Now, to show that $\tau_{inf} \nsubseteq \tau_{FC}$. We know that $[0, \infty) \in \tau_{inf}$. But we know that $[0, \infty) \notin \tau_{FC}$ because $X \setminus [0, \infty) = (-\infty, 0)$ is infinite. □

3. Tarik on Posets

Defenition 3.1. *A* poset *(partially ordered set) is a set X, equipped with a a partial order, $\leq$ satisfying:*

[I]

(1) $\forall x \in X$, $x \leq x$
(2) $\forall x, y, z$, *if* $x \leq y$ *and* $y \leq z$, *then* $x \leq z$
(3) *If* $x \leq y$ *and* $y \leq x$, *then* $x = y$.

Example 3.1. *Fix X and let $\mathrm{T}_x = \{\tau | \tau$ is a topology on $X\}$. Given τ, τ', we say that $\tau \leq \tau'$ if τ is coarser than τ'*

Defenition 3.2. *A coset $(X, \leq)$ is* partially ordered *if $\forall x, y$, either $x \leq y$ or $y \leq x$.*

Yuxuan just proved that the above example with topologies is *NOT* totally ordered.

Any topology is a basis of itself. The question, then, becomes: given a topology τ is there a way to find a minimal basis.

Lemma 3.1. *Given $(x, \leq)$ a poset, if every totally ordered subset has an upper bound, then there exists a maximal element of $(X, \leq)$, which is some x such if $x \leq y$, $x = y$*

If the set is totally ordered, then this is an if and only if statement. Furthermore, there might be more than one maximal element of $(X, \leq)$, unless the set is totally ordered.

Back to our question: there do no necessarily exist minimal bases of a topology.

4. Proposition 1.25 Presented by Zeus

4.1. Background Stuff.

Defenition 4.1. *A* metric $d : X \times X \to \mathbb{R}_{\geq 0}$ *such that:*

(1) $d(x, y) = 0$ *if and only if* $x = y$
(2) $d(x, y) \geq d(x, z) + d(z, y)$
(3) $d(x, y) = d(y, x)$.

Proposition 4.1. *There exists a topology that is not metrizable.*

Proof. Let $\mathcal{B} = \{(-\epsilon, \epsilon) : \epsilon > 0\}$. Let us make sure that this is a valid basis. We see that $x \in (-(|x| + 1), |x| + 1)$. We also see that $(-\epsilon_1, \epsilon_1) \cap (-\epsilon_2, \epsilon_2) = (-\min(\epsilon_1, \epsilon_2), \min(\epsilon_1, \epsilon_2)) \in \mathcal{B}$, so this is a basis. The topology generated by this basis is $\tau = \mathcal{B} \cup \{\mathbb{R}, \emptyset\}$.

Now suppose that there exists a metric $d : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ such that $\tau = \tau_d$, where τ_d is the topology generated by $\mathcal{B}_d = \{B(x, \epsilon) : x \in \mathbb{R}, \epsilon > 0\}$. Suppose that $1 \in B(1, \epsilon) \in T$. Because 0 is in every non-empty set of τ, the fact that $\tau = \tau_d$ implies that $0 \in B(1, \epsilon)$. Then $d(0, 1) < \epsilon$ for all $\epsilon > 0$. So, $d(0, 1) = 0$, which contradicts the fact that $d(x, y) = 0$ if and only if $x = y$. □