CLASSNOTES FOR SEPT 20TH

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Disclaimer: All the beautiful things below are by the presenters. All the erroneous things below are by me, the note taker.

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Proposition 2.4. If $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ and $g : (Y, \mathcal{T}_Y) \to (Z, \mathcal{T}_Z)$ are continuous, then so is $g \circ f : (X, \mathcal{T}_X) \to (Z, \mathcal{T}_Z)$.

Version: Classnotes
Comments / Collaborators: By Margin

Proof. We want to show that for every open $U \subset Z$, $(g \circ f)^{-1}(U) \subset X$ which is open. We can break the function down into $f^{-1}(g^{-1}(U))$. Since g is continuous, $g^{-1}(U)$ is open. Since f is continuous, $f^{-1}(g^{-1}(U))$ is open.

$\square$

Jess had a question on the notation.

Eric thought the proof was excellent, but pointed out that not every set in X means it's in $\mathcal{T}_X$, then Margin corrected it.

Tarik said that since we already proved on $\mathbb{R}$, the open set definition of continuity is the same as the $\epsilon - \delta$ definition.

Consensus said that it's unclear to take composition of infinite functions.

Martin said take a function from spaces A to B, if a set is open in A, then it's sent to the empty set, if it's not, then it's sent to a non-empty set. Would that be a "piece-wise" discontinuous function?

Tarik didn't understand send an open set in A to the empty set in B.

Alexander's face was very distressed.

Example 2.5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -1 & x < 0, \\ 1 & x \geq 0. \end{cases}$$

Determine which (if any) of the following maps between topological spaces are continuous.

(1) $f : (\mathbb{R}, \mathcal{T}_l) \to (\mathbb{R}, \mathcal{T}_l)$
(2) $f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T}_l)$
(3) $f : (\mathbb{R}, \mathcal{T}_l) \to (\mathbb{R}, \mathcal{T})$

Version: Classnotes
Comments / Collaborators: By Alexander

Proof. Alexander is bothered by the notation in the course notes...

Some backgrounds about the spaces:

$\mathcal{T}_l$ is the topology associated to the basis

$$B_l = \{[a, b) : a < b\} \tag{1}$$

Example 1.18 showed that $\mathcal{T}_l \supset \mathcal{T}$.

we want to know whether given any open set $V \in (\mathcal{T}, \mathcal{T}_l, U = f^{-1}(U)$ is open in $(\mathbb{R}, \mathcal{T}_l)$. We can write $f^{-1}(V) = \cup_{x \in V} f^{-1}(x)$.

(i) Suppose neither 1 nor -1 is in V. Then $f^{-1}(V) = \cup_{x \in V} f^{-1}(x) = \cup_{x \in V} \emptyset = \emptyset$

(ii) Suppose $1 \in V$, but $-1 \notin V$. Then $f^{-1}(V) = \cup_{x \in V} f^{-1}(x) = f^{-1}(1) \cup \bigcup_{x \in V/\{1\}} = [0, \infty) = \bigcup_{n \in \mathbb{N}} [0, z)$

(iii) Suppose $-1 \in V$ but $1 \notin V$. Then $f^{-1}(V) = f^{-1}(-1) \cup \bigcup_{x \in V/\{-1\}} \emptyset = (-\infty, 0) \in \mathcal{T} \subset \mathcal{T}_l$.

(iv) Suppose $1, -1 \in V$. Then $f^{-1}(V) = f^{-1}(-1) \cup f^{-1}(1) \cup \bigcup_{x \in V/\{1,-1\}} = [0, \infty) \cup (-\infty, 0) = \mathbb{R}$

These cases showed that (1) and (3) are continuous and (2) is not. $\square$

Martin liked that Alexander included the union of x and the empty set.

Hikaru enjoyed the forepart of writing down every case, then prove it.

Martin asked if we take the floor function, is it continuous?

Tarik says add more structure to Martin's question: if we were to add more discontinuities to the function, will that function be continuous from $\mathcal{T} \to \mathcal{T}_l$.

Zues suggested a poll, but Tarik neh-ed at it.

Thea asked if the space $\mathbb{R}$ or $\mathbb{Z}$.

Martin wasn't sure how much does that matter.

Thea thought it matters, but don't know how to say.

Eric thought it matters, because any preimage of $\mathbb{R}$ is going to be open in $\mathbb{R}$.

Tarik says if we were to use the upper limit topology instead of the lower limit topology, the floor function should work.

Tarik also proposed that we can come up with topologies so that the discontinuous functions in standard topology becomes continuous in such topology.

Cam said something about choosing the finer and coarser topologies for the domain and co-domain may help us determine the continuity of the function.

Jess wrote on the board:

$$f : \mathbb{R} \to \mathbb{R}$$

which is discontinuous, when equipping the co-domain with the standard topology, with respect to all topologies except the discrete topology.

100 rooms each contain countably many boxes labeled with the natural numbers. Inside of each box is a real number. For any natural number n, all 100 boxes labeled n (one in each room) contain the same real number. In other words, the 100 rooms are identical with respect to the boxes and real numbers. Knowing the rooms are identical, 100 mathematicians play a game. After a time for discussing strategy, the mathematicians will simultaneously be sent to different rooms, not to communicate with one another again. While in the rooms, each mathematician may open up boxes (perhaps countably many) to see the real numbers contained within. Then each mathematician must guess the real number that is contained in a particular unopened box of his choosing. Notice this requires that each leaves at least one box unopened. 99 out of 100 mathematicians must correctly guess their real number for them to (collectively) win the game.

The game mentioned during a recess and the version is quoted from Jrthedawg from BrainDen.com.

Lemma 2.6. (1) Let $f : X \to Y$ be a continuous map. Suppose that $A \subset X$. The restriction of f to A (denoted $f|_A$, and defined by $f|_A(x) = f(x)$ if $x \in A$) is continuous as a function $f|_A : A \to Y$ with respect to the subspace topology on the domain

(2) Let $f : X \to B$ be a map between topological spaces with $B \subset Y$ having the subspace topology. Let $i : B \to Y$ be the inclusion map (i.e., $i(b) = b$ for all $b \in B$). The map f is continuous if and only if $i \circ f$ is continuous.

Version: classnotes
Comments / Collaborators: By Eric

Proof. (1) Let U be a set in Y that is open. As f is a continuous map, there exists a V that is open in X such that $V = f^{-1}(U)$. Now consider the set $W = A \cap V$. $f|_A(W) = f(W) \subset f(V)$. Then we know that $f(W) \subset U$. Then $W \in f|_A^{-1}(U)$. So there doesn't exists an open set $Z \subset A$ such that $W \subset Z$ such that $f(Z) \subset U$. Then we know that $W = f|_A^{-1}(U)$. V is open in X, $W = A \cap V$. Then we know that W is open in A Thus $f|_A$ is continuous.

(2) For any set $U \subset Y$, $i^{-1}(U) = B \cap U$. As $B \subset Y$ has a subspace topology, if V is open in Y, then $i^{-1}(V) = B \cap U$ is open in B. Thus i is continuous.

Assume f is continuous. As $i : B \to Y$, $f : X \to B$ are continuous, $i \circ f : X \to Y$ is continuous.

Assume f is not continuous. There exists $V \subset B$ such that $f^{-1}(V) = W$ is not open in X. $(i \circ f)^{-1}(V) = f^{-1}(i^{-1}(V)) = f^{-1}(V) = W$ is not open in X. Then we know that $i \circ f$ is not continuous.

Thus f is continuous if and only if $i \circ f$ is continuous. □

(1) Justine was not sure what Eric was trying to show. Eric responded that he was trying to show that there exists no set such that is a set in A that contains W, that is not equal to W.

Alexander pointed out that why can't we used the definition of W to achieve this goal.

Eric added "As $W = A \cap V$, and $V = f^{-1}(U)$ in front of Z step.

(2) Martin asked can we say $f(V) = U$. Eric said no, but he said that we can say it's a subset of U.

Martin asked why is it true that W is a subset of $f|_A^{-1}(U)$.

Tarik commented that this is a symbol heavy proof, and asked what's the main idea of the proof.

Alexander said that we can see it as the restriction on the preimage of an open set in Y.

Tarik "summarized" that when looking at the preimage of an open set in Y, we want to show that the preimage is open in A. Since the preimage is open in X, we can take intersection of the preimage and A.

Thea asked is there a direct proof to show the backward direction.

Zues wrote on board: If $i \circ f$ is continuous, then f is continuous. Let U be open in B. Then $f^{-1}(U) = \{x \in X : f(x) \in U\} = \{x \in X : f(x) \in U \cap B\}$. (He sprinted back to his computer, then walked to the board again) By definition of the subspace topology, there exists $U' \in Y$ such that $U = U' \cap B$. (He sprinted back again, but this time came back with his laptop, then erased everything and copied and pasted from his laptop) U' is open in B. Then there exists U open in Y, such that $U' = B \cap U$. $(i \circ f)^{-1}(U) = \{x \in X : i(f(x)) \in U\}$ $= \{x \in X : f(x) \in U$ and $f(x) \in B\} = \{x \in X : f(x) \in U'\}$ $= f^{-1}(U')$.