335 HW 1

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Proposition 1.3: Both $\mathbb{R}$ and the empty set are open.

Proof. Pick an arbitrary element a in $\mathbb{R}$, pick an arbitrary δ_a in $\mathbb{R}$. Then we know that $B(a, \delta_a) = \{x \in \mathbb{R} : |x - a| < \delta_a\}$. Since all x in $B(a, \delta_a)$ is already in $\mathbb{R}$ by definition, so we know that $B(a, \delta_a)$ is a subset of $\mathbb{R}$ by definition, so $\mathbb{R}$ is an open set.

Prove that the empty set is open by contradiction. Suppose there exist an element x in $\emptyset$ that has no $\delta_x > 0$ such that $B(x, \delta_x) \subset \emptyset$. However, the empty set has no element in it, so there cannot be an element that has no $\delta_x > 0$ such that $B(x, \delta_x) \subset \emptyset$. Therefore, the empty set is open. $\square$

Proposition 1.4: If each set in the collection $\{U_\alpha\}_{\alpha \in I}$ of subsets of $\mathbb{R}$ is open, then so is $\bigcup_\alpha U_\alpha$ (i.e. arbitrary unions of open sets are open).

Proof. Pick an arbitrary element, x, from the arbitrary union, $\bigcup_\alpha$. We know that that the element belongs to some arbitrary open set, A. So we know that there exist $\delta_x > 0$ such that $B(x, \delta_x) \subset A$. Since $A \subset \bigcup_\alpha$, by transitivity, we know that $B(x, \delta_x) \subset \bigcup_\alpha$. Thus an arbitrary union of open sets is also open. $\square$

Proposition ???1.5: If $\{U_\alpha\}_{\alpha \in I}$ is a collection of open sets, then $\bigcap_\alpha U_\alpha$ is also open.

The proposition is wrong. The counterexample lies below.

Construct open sets: $(-1/a_i, 1/a_i)$ where a_i are positive integers from 1 to infinity. Since $(-1/a_i, 1/a_i)$ is an open interval, we know that it is also an open set. Let U_a be the intersection of our open sets, $U_a = \bigcap_a (-1/a_i)$. Since all of the open sets are centered at 0, we see that as we take a_i to infinity, the intersection will be 0, so $U_a = 0$, which is not an open set.

Proposition 1.6: A function $f : \mathbb{R} \to \mathbb{R}$ is $\epsilon - \delta$ continuous if and only if $f^{-1}(U)$ is open in $\mathbb{R}$ for any open set $U \subset \mathbb{R}$.

Proof. First the forward direction.

Suppose that f is $\epsilon - \delta$ continuous, and U is an open set in $\mathbb{R}$.

Date: September 7, 2021.

Pick and arbitrary element, y, in U. Since U is open, we know that there exist $\delta_y > 0$ such that $B(y, \delta_y) \subset U$. Since f is $\epsilon - \delta$ continuous, we know that for all $\epsilon > 0$, for $f^{-1}(y)$, there exists a $\delta_x > 0$ such that when $|x - f^{-1}(y)| < \delta_x$, $|f(x) - y| < \epsilon$.

Since f is continuous for all $\epsilon > 0$, we can pick $\epsilon = \delta_y$. So we know that when $|x - f^{-1}(y)| < \delta_x$, $|f(x) - y| < \delta_y$, which implies that $f(x) \in U$, which further implies that $x \in f^{-1}(U)$. For any $f^{-1}(y) \in f^{-1}(U)$, choose $\delta = \delta_x$, we know that all elements in $B(f^{-1}(y), \delta)$ is in $f^{-1}(U)$, thus $f^{-1}(U)$ is open.

Now the backward direction.

Suppose that $f^{-1}(U)$ is open in $\mathbb{R}$, and U is open in $\mathbb{R}$.

Our goal is to show that for any $\epsilon > 0$, there exist a $\delta > 0$ such that $B(x, \delta) \subset f^{-1}(B(f(x), \epsilon))$.

To prove that, we first find a $0 < \delta_y < \epsilon$. Since U is open in $\mathbb{R}$, we know that there exist such δ_y such that $B(f(x), \delta_y) \subset U$. Since $\delta_y < \epsilon$, we know that $B(f(x), \delta_y) \subset B(f(x), \epsilon)$.

If we can show that there exist δ such that $B(x, \delta) \subset f^{-1}(B(f(x), \delta_y))$, we are done.

By the definition of an open ball, and the property of every set is a subset of itself, we know that open balls are open sets as well. So, we know that $f^{-1}(B(f(x), \delta_y))$ is open. So the job now changed to finding a $0 < \delta < \delta_y$ such that $B(x, \delta) \subset f^{-1}(B(f(x), \delta_y))$. Since we already know that $B(x, \delta)$ itself is an open set, so we can always find a qualified δ.

Therefore, we have $B(x, \delta) \subset f^{-1}(B(f(x), \delta_y)) \subset f^{-1}(B(f(x), \epsilon))$. □

Proposition 1.8: Let $(X, \mathcal{T})$ be a topological space, and let $A \subset X$. Suppose that for each $x \in A$, there is an open set U containing x such that $U \subset A$. The set A is open (i.e. that $A \in \mathcal{T}$).

Proof. Since for each $x \in A$, there is an open set U containing x such that $U \subset A$, we know that all of the U's are in the set X. Let B be the union of the U's. By T2 from the definition of a topological space, we know that $B \in \mathcal{T}$.

Now we want to show $A \in \mathcal{T}$ by showing $B = A$.

Since all of the U's are in A, so the union of the U's, B, is a subset of A.

For all element in A, there exist a U such that $x \in U$. Since all the U's are in B, so all elements of A are in B as well. So A is a subset of B.

Since $A = B$, and $B \in \mathcal{T}$, we know that $A \in \mathcal{T}$. □

Proposition 1.12 : $(X, \mathcal{T}_{\mathcal{B}})$ is a topological space.

Proof. $\emptyset \subset X$ by the definition of an empty set. Since there is no any element x in $\emptyset$ such that there is not a $B_x \in \mathcal{B}$ such that $x \in B_x \subset \emptyset$. Thus $\emptyset \in \mathcal{T}_{\mathcal{B}}$.

$X \subset X$ by the definition of a set. For all $x \in X$, there exist a $B_x \in \mathcal{B}$ such that $x \in B_x \subset X$ by $B1$ and the construction of $\mathcal{T}_{\mathcal{B}}$. Thus $X \in \mathcal{T}_{\mathcal{B}}$.

Suppose There exist $U_\alpha \in \mathcal{T}_\mathcal{B}$ for all $\alpha \in I$. Then we know that $U_\alpha \subset X$ for all $\alpha \in I$ by construction of $\mathcal{T}_\mathcal{B}$. So we know that for all x in U_α for all $\alpha \in I$, there exist a $B_x \in \mathcal{B}$ such that $x \in B_x \subset U_\alpha$. Since $U_\alpha \subset X$ for all $\alpha \in I$, we know that the union $\bigcup_\alpha U_\alpha \subset X$. So we know that for all x in the union $\bigcup_\alpha U_\alpha$, there exists a $B_x \in \mathcal{B}$ such that $x \in B_x \subset U_\alpha \subset \bigcup_\alpha U_\alpha$. Thus $\bigcup_\alpha U_\alpha \in \mathcal{T}_\mathcal{B}$.

Suppose that $U_0, U_1 \in \mathcal{T}_\mathcal{B}$. We know that for all element in U_0, U_1, there exist $B_x \in \mathcal{B}$ such that $x \in B_x \subset U_0, U_1$ (which super set B_x belongs to follows which super set x is in). Pick an arbitrary $x \in U_0 \bigcap U_1$. We know that there exist $B_{x_0} \in \mathcal{B}$ such that $x \in B_{x_0} \subset U_0$, and there exist $B_{x_1} \in \mathcal{B}$ such that $x \in B_{x_1} \subset U_1$. By B2, there exist a $B_x \in \mathcal{B}$ such that $x \in B_x \subset B_{x_0} \bigcap B_{x_1}$. Since $B_x \subset B_{x_0} \subset U_0$ and $B_x \subset B_{x_1} \subset U_1$, we know that $B_x \subset U_0 \bigcap U_1$. Thus $U_0 \bigcap U_1 \in \mathcal{T}_\mathcal{B}$. $\square$