PROOFS FOR TOPOLOGY

TONY DENG

1. Rewrite of previous weeks' problems

Lemma 3.5 (Pasting Lemma). Let $X = A \cup B$, where A and B are closed sets. Let $f : A \to Y$ and $g : B \to Y$ be continuous functions that agree on $A \cap B$. Then there exists a continuous function $h : X \to Y$ such that

$$h(x) = \begin{cases} f(x) & x \in A, \\ g(x) & x \in B. \end{cases}$$

Version: 2
Comments / Collaborators: Wonder if it would work when A, B are both open, one open one closed...

Proof. Pick an arbitrary closed set U from Y. We want to show that $h^{-1}(U)$ is closed in X.

There are several possible cases:

(i) $h^{-1}(U) = f^{-1}(U)$. In this case, since f is continuous, $f^{-1}(U)$ is closed in X, so $h^{-1}(U)$ is closed in X.

(ii) $h^{-1}(U) = g^{-1}(U)$. Similar to the above case, since g is continuous, $g^{-1}(U)$ is closed in X, so $h^{-1}(U)$ is closed in X.

(ii) $h^{-1}(U) = f^{-1}(U) \cup g^{-1}(U)$. Since f and g are continuous functions, $f^{-1}(U)$ and $g^{-1}(U)$ are closed sets in X. The union of two closed sets are still closed, so $h^{-1}(U)$ is closed in X.

Thus h is continuous. □

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2. This week's problems

Proposition 3.20. The condition T_i implies T_{i-1}.

Version: 1
Comments / Collaborators:

Proof. Let's start from T_4.

Suppose X is T_4. By lemma 3.19, we know that every singleton in X is a closed set in X. Choose A to be some closed set, B to be a singleton. By normality, we know that there exists U containing A and V containing B such that U and V are disjoint. Since B can be any singleton in X, this is regularity. We get T_1 for free. Thus, T_4 implies T_3.

Suppose X is T_3. By lemma 3.19, we know that every singleton in X is a closed set in X. Choose A to be a singleton that differs from x. By regularity, we know that there exists U containing A and V containing x such that U and V are disjoint. Since A can be any singleton in $X - \{x\}$, we know that for every pair of different points, there exist disjoint open sets that contains them respectively. Thus, regularity together with T_1 implies Hausdorff.

Suppose X is Hausdorff. Since for every pair, we have U containing x not y, and V containing y not x, X is T_1. Thus, T_2 implies T_1.

Suppose X is T_1. Since for every pair, we have U containing x not y, X is T_0. Thus, T_1 implies T_0. $\square$

Proposition 3.22. All metric spaces are normal.

Version: 1
Comments / Collaborators: the hint helped a lot.

Proof. Suppose X is a metric space. So there exists a metric d on X. Suppose A and B are arbitrary disjoint closed sets in X.

Want to show there exists open sets U containing A and V containing B such that $U \cap V = \emptyset$.

Create a set d_{ab} that defined as follow

$$d_{ab} = d(a, b) : a \in A, b \in B. \tag{1}$$

Since we know that A and B are disjoint closed sets, we know that $d_{ab} > 0$ for all possible a and b.

Claim: $\inf d_{ab} \neq 0$.

Proof. Suppose $\inf d_{ab} = 0$. Then by the definition of inf, for all $\epsilon > 0$, we have a d_{ab} such that $d_{ab} < \inf + \epsilon$, which implies $\epsilon > d_{ab} = 0$, which contradicts with our condition $d_{ab} > 0$ for all possible a and b. ☐

Since $\inf d_{ab} > 0$, we can construct U. Construct $U_a = B(a, 1/3 \inf d_{ab}$ for each $a \in A$, take $U = \bigcup_{a \in A} U_a$. Since U_a are open, we know that U is open. We also know that there exists no element of B in U since the radius of the open balls are one third the smallest distance between A and B.

Construct V in the same fashion. $V = \bigcup_{b \in B} V_b$.

We know that there can be no element in both U and V since no element can be in the one third of the shortest distance of A and B at the same time. ☐

Tarik's Corner. Prove that the product of T_3 spaces X and Y is again T_3, in the product topology.

Version: 1
Comments / Collaborators: The subscripts and superscripts are confusing here (it was more confusing on my scratch paper...) I tried stream it down a little here...

Proof. Suppose X and Y are T_3 spaces. Let Z be the product of X and Y.

Want to show that for any (x, y) and a closed C in Z, there exists open sets U containing (x, y) and V containing C such that $U \cap V = \emptyset$.

Pick an arbitrary closed set C from Z, we know that $Z - C$ is an open set. Thus, we know that $Z - C = C^C = \bigcup_{\alpha \in I} B_\alpha$, where $\{B_\alpha\}_{\alpha \in I} \in \mathcal{B}_\times$. Since $\mathcal{B}_\times = \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$, we know that $C^C = \bigcup_{\alpha \in I} B_\alpha = \bigcup_{\alpha \in I} U_\alpha \times V_\alpha$. Hence, we know that $C = \bigcap_{\alpha \in I} U_\alpha^C \times V_\alpha^C$.

Pick an arbitrary (x, y) in $Z - C$.

Since (x, y) is not in $\bigcap_{\alpha \in I} U_\alpha^C \times V_\alpha^C$, there exists at least one α such that (x, y) is not in $U_\alpha^C \times V_\alpha^C$.

Now, there are two situations: (i) neither x nor y is not in U_α^C and V_α^C respectively; (ii) either x in U_α^C or y in V_α^C.

(i) Since X and Y are T_3 spaces, we know that there exists open sets U_X containing x and V_X containing U_α^C, such that $U_X \cap V_X = \emptyset$; there exists open sets U_Y containing y and V_Y containing V_α^C, such that $U_Y \cap V_Y = \emptyset$.

Then we know that $(x, y) \in U_X \times U_Y$ and $U_\alpha^C \times V_\alpha^C \in V_X \times V_Y$. We know that $U_X \times U_Y \cap V_X \times V_Y = \emptyset$, so we know that there exists open sets that contains (x, y) and open sets that contains $U_\alpha^C \times V_\alpha^C$ that are disjoint. Since $V_X \times V_Y$ contains $U_\alpha^C \times V_\alpha^C$, we know that $V_X \times V_Y$ contains $\bigcap_{\alpha \in I} U_\alpha^C \times V_\alpha^C$. Thus, we found two open disjoint sets that contains (x, y) and C respectively.

(ii) Without the loss of generality, suppose $x \in U_\alpha^C$. Replace U_X and V_X with the smallest open set that contains U_α^C in the above proof, the rest still stands. $\square$

Proposition 3.25. Metric spaces are first countable.

Version: 1
Comments / Collaborators:

Proof. Let X be a metric space. Pick an arbitrary x from X. Pick an arbitrary open set U containing x.

Construct the neighborhood basis of x as follow:

$$U_\alpha = \{B(x,\alpha) : \alpha \in \mathbb{Q}\}. \tag{2}$$

Since $\mathbb{Q}$ is countable, we know that U_α is countable.

Now we want to show that it is a neighborhood basis.

Since U is open in X, we know that $U = \bigcup_{\beta \in I} B_\beta$, where $\{B_\beta\}_{\beta \in I} \in \mathcal{B}_d$. Among those B_β, there can be one that is centered at x, say $B_\beta = B(x,\beta)$.

If $\beta \in \mathbb{Q}$, we know that U contains one of our neighborhood basis element. If β is not in $\mathbb{Q}$, find a rational number that is smaller than β, say γ, we know that $B(x,\gamma) \subset B(x,\beta) \subset U$, which again leads to our conclusion.

□

Tarik's Corner. Show that the Cantor set (equipped with the subspace topology) has a countable dense subset.

Version: 1
Comments / Collaborators: I would be happy to rewrite this one with some hint from you...if that's neccessary.

Proof. Since second countability implies countable dense subset, as long as we show that the Cantor set equipped with the subspace topology has second countability, we are done.

Construct the basis element by taking the intersection of open balls with rational radii around the endpoints in C with C.

$$\mathcal{B}_C = \{C \cap B(x, \epsilon) : \epsilon \in \mathbb{Q}, \epsilon > 0, \text{and } x \text{ is an endpoint in } C.\} \tag{3}$$

We know there will be countably many end points since there are 4 end points for C_1, 8 for C_2, thus 2^n for C_n. Since the rationals are countable as well, so the union of countably many countable sets should be countable as well.

So now, what's left is to prove that it is a basis.

For any element, c, in C, we have $c \in B(0, 1.1) \cap C$. Thus, **B1** satisfied.

Suppose c is in two different basis elements $B_1 = B(x, \epsilon_1) \cap C, B_2 = B(y\epsilon_2) \cap C$. We can always find an endpoint that is closer to c and construct an open ball that is smaller than the previous two. Thus, **B2** satisfied. $\square$