PROOFS FOR TOPOLOGY

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Proposition 3.12. Given $A \subset X$, let A' denote the set of limit points of A. Then $\overline{A} = A \cup A'$.

Version: 1
Comments / Collaborators: Although I listened to Nico's presentation on this, but I don't remember anything...First try, wish me good luck...

Proof. When proving equality between two sets, double containment is always the safest choice.

Want to show that $\overline{A}$ is a subset of $A \cup A'$.

Since $\overline{A}$ is the smallest closed set that contains A, as long as we prove that $A \cup A'$ is a closed set, we are done.

The complement of $A \cup A'$ should be $A^C \cap A'^C$.

Pick an arbitrary element, x, in $A^C \cap A'^C$, we know that there exists an open set U such that U contains x but intersect $A \cup A'$ with nothing.

We know that U is a subset of $A^C \cap A'^C$ by definition.

Since U is open, there exists a collection of basis elements that equals to U, so there exists a basis element that contains x.

So for every element in $A^C \cap A'^C$, there exists a basis element, B, such that $x \in B \subset U \subset A^C \cap A'^C$. Thus, $A^C \cap A'^C$ is open, and thus $A \cup A'$ is closed.

Therefore, $\overline{A} \subset A \cup A'$.

Now we want to show that $A \cup A' \subset \overline{A}$.

Since $\overline{A}$ is the smallest closed set that contains A we know that $A \subset \overline{A}$.

So we want to show that A' is a subset of $\overline{A}$. For every element, x, in A', we know that for each open set U containing x, $U \cap A \neq \emptyset$. Since $A \subset \overline{A}$, we know that $U \cup \overline{A} \neq \emptyset$. So every element in A' is also a limit point of $\overline{A}$. Since $\overline{A}$ is closed, it contains all its limit points, thus $A' \subset \overline{A}$.

Thus, $A \cup A' \subset \overline{A}$. $\square$

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Proposition 3.16. If there is a sequence (x_n) in $A \subset X$ that converges to x, but is not eventually constant, then x is a limit point of A.

Version: 1
Comments / Collaborators: Doing this at the end of fall break, completely forgot how the presenter did it...

Proof. Suppose there exists a sequence (x_n) in A such that (x_n) converges to x. Then we know that for every open set U containing x, there is an $N \in \mathbb{N}$, such that for all $n > N$, $x_n \in U$. Therefore, we know that for all open set U, $U \cap A \neq \emptyset$. Thus, x is a limit point of A. $\square$

Proposition 3.18. If X is Hausdorff, then limits of sequences are unique.

Version: 1
Comments / Collaborators: just an innocent line to fill the comment section...

Proof. Proof by contradiction.

Suppose (x_n) is a sequence that converges to two points x, y in X. Then we know that for all neighborhoods U of x, there is an $N_x \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N_x$; for all neighborhoods V of y, there is an $N_y \in \mathbb{N}$ such that $x_n \in V$ for all $n \geq N_y$.

Then we know that for all U containing x and V containing y, choose $N = \max(N_x, N_y)$, then for all $n \geq N$, $x_n \in U \cap V$.

Thus, contradiction with X being Hausdorff. $\square$