PROOFS FOR TOPOLOGY

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1. Rewrite of previous weeks' problems

Tarik's Corner: exercise (1). (a) Consider the function $f : M \to$ (Natural numbers, subspace topology with respect to the standard topology on R) which for each $x \in M$, outputs the minimum n such that there is a neighborhood about x which is homeomorphic to R^n. Use the invariance of domain theorem to prove that f is continuous.

(b) Use the fact that M is connected to show that f must be a constant function.

So, in a connected manifold M, there is indeed one n such that every point has a neighborhood homeomorphic to R^n.

Version: 1
Comments / Collaborators: I assumed that any open ball in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ without proving it...

Proof. (a): We want to show that f is continuous. Showing that for all open sets, U, from $(\mathbb{N}, \mathcal{T})$, $f^{-1}(U)$ is open in M would achieve our goal.

By the definition of subspace topology, we would have singleton sets $\{n\}$, where $n \in N$, open in $\mathbb{N}$.

Take the preimage of $\{n\}$, we get

$$V = \{x \in M : \exists U_x \text{ such that } x \in U_x, \text{ and } U_x \text{ is homeomorphic to } \mathbb{R}^n, \text{ where } n \text{ is the minimum possible integer.}\} \tag{1}$$

Claim: $V = \bigcup_x U_x$.

Proof. Since for all $x \in V$, x is in some U_x, so V is a subset of $\bigcup_x U_x$.

Want to show that there exists no y in U_x such that there exists U_y containing y homeomorphic to some $\mathbb{R}^m$ where $m < n$.

Since M is a manifold, there exists homeomorphisms $\phi_1(U_x) = \mathbb{R}^n$ and $\phi_2(U_y) = \mathbb{R}^m$.

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Let $U = U_x \cap U_y$. Since U_x and U_y are both open sets, U must be open as well.

Take $\phi_1(U)$, we know that $\phi_1(U)$ is open in $\mathbb{R}^n$. We can find an open ball B_1 in $\mathbb{R}^n$ containing $\phi_1(y)$ and a subset of $\phi_1(U)$.

We know that B_1 is homeomorphic to $\mathbb{R}^n$.

Take the preimage of B_1 under ϕ_1, we know that $\phi_1^{-1}(B_1)$ is open in M.

Take the image under ϕ_2, we get $\phi_2(\phi_1^{-1}(B_1))$ which is an open set in $\mathbb{R}^m$.

And since we also have ϕ_1 and ϕ_2 being homeomorphis, we know that $\phi_2(\phi_1^{-1}(B_1))$ is homeomorphic to $\mathbb{R}^n$.

By invariance domain theorem, we reach a contradiction. $\square$

Since U_x's are open, V is open.

Thus f is continuous.

(b) Proof by contrapositive.

Suppose that f is not a constant function, i.e., there exists $n, m \in \mathbb{N}$ such that $f(U) \to n$ or $f(V) \to m$, where U, V are open in M.

We know that $U, V \neq \emptyset$ by the virtue of construction.

We know that $U \cap V = \emptyset$ by the same argument around the invariance domain theorem in the last proof.

We also know that $U \cap V = M$, since otherwise, there would exists some $w \in \mathbb{N}$, such that $f(M - U - V) = w$, which is false by construction.

Thus, we have a separation for M.

Thus, if M is connected, then f has to be a constant function. $\square$

Tarik's Corner: exercise (2). Use the invariance of domain theorem again to show that in a connected manifold M, there is exactly one n such that every point has a neighborhood homeomorphic to R^n.

Exercises (1)+(2) imply that it makes sense to a refer to a connected manifold as an $n-$dimensional manifold (or a $n-$manifold for short).

Version: 1
Comments / Collaborators:

Proof. In part (1), we've shown that for all $x \in M$, the smallest n for x to have a neighborhood that is homeomorphic to $\mathbb{R}^n$ is the same for a connected manifold. Thus, we now need to show that there is no $m \in \mathbb{N}$ such that $m > n$, and all points in M has a neighborhood that is homeomorphic to $\mathbb{R}^m$.

Use proof by contradiction, we suppose that there exists some $m > n$ such that for all $x \in M$, we can find a neighborhood of x, say U_m such that U_m is homeomorphic to $\mathbb{R}^m$.

Since we also know that for all $x \in M$, there also exists U_n such that U_n is homeomorphic to $\mathbb{R}^n$, we can take the intersection of U_n and U_m and still have an open set.

Let $U = U_m \cap U_n$, we know that U is still open in M.

Denote the homeomorphisms from U_n to $\mathbb{R}^n$ as ϕ_1 and U_m to $\mathbb{R}^m$ as ϕ_2.

Take the image of U under ϕ_1, we get an open set in $\mathbb{R}^n$. Find an open ball $\mathcal{B}_1$ in $\phi_1(U)$ containing $\phi_1(x)$.

Take the preimage of that open ball, $\phi_1^{-1}(B_1)$, we have an open set in M.

Now, take the image of this open set under ϕ_2, we get an open set in $\mathbb{R}^m$, $\phi_2(\phi_1^{-1}(B_1))$.

Since ϕ_1 and ϕ_2 are homeomorphisms, we know that $\phi_2(\phi_1^{-1}(B_1))$ is homeomorphic to $\mathbb{R}^n$.

By the invariance domain theorem, we've reached a contradiction. $\square$

(4). The real projective plane is a topological space defined as follows: as a set, it is the set of all lines that go through the origin in R^3. We put a topology on this by saying that a set of lines is open if and only if the set of all points (except for the origin!) in R^3 contained on any of those lines is an open subset of R^3-{origin} (in the subspace topology with respect to the standard topology on R^3). Prove that the real projective plane is compact. (Hint: try to find a continuous onto function from a space that you can easily prove is compact by using the Heine-Borel theorem.)

Version: 2
Comments / Collaborators: I don't know what was I thinking...

Proof. Pick S^2. Since we know that S^2 is closed and bounded, we know that S^2 is a sequentially compact by the Heine-Borel theorem.

Construct a function $\phi : S^2 \to$ the real projective plane, such that each point on S^2 sends to the line containing both that point and the origin.

This function is onto since for all possible lines in $\mathbb{R}^3$ that goes through the origin, there must be and can only be two intersection between the line and the unit sphere.

Now, we need to show that the function is continuous.

For each open set in $\mathbb{RP}^2$, we know that all points are in some open subset of $\mathbb{R} - \{\text{origin}\}$. Take the intersection between the open sets that containing all the points of the lines and the unit sphere, what we get is open in the subspace topology of the unit sphere. Thus, f is continuous.

$\square$

2. This week's problem

1. Consider the standard English alphabet in all capital letters:

$$\{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z\}$$

Consider each as a topological subspace of R^2. And to be concrete, assume that each "extreme point" is actually included, e.g. the letter "I" should be interpreted to include both the lowest and highest point...it's not "open" but closed (I am putting open in quotes here because even without the endpoints, "I" would not be an open subset of R^2). Partition these spaces into homeomorphism classes. For example, C and E are not homeomorphic, because there is a point of E which if removed, produces three connected components and there is no such point on C.

Version: 1
Comments / Collaborators: Exhausting

Proof. The first class would be A.

The second would be B.

The third would be $CGLSIUVWZ$.

Proof. C is homeomorphic to I because one can project C onto I via a projection function.

G is homeomorphic to C because one can get C out of G by simply rotate the returning stroke to the outside.

L is homeomorphic to I because one can get I by rotating the bottom of L ninety degrees clock-wise.

S is homeomorphic to I by the projection function.

U is homeomorphic to C by rotation.

V is homeomorphic to I by projection.

W is homeomorphic to I by projection.

Z is homeomorphic to I by rotating the arms to align with the body. □

The fourth would be DO.

Proof. Put D inside O, we can have a homeomorphism between them by projecting D onto O. □

The fifth would be $EFJTY$.

Proof. E is homeomorphic to F by rotating the bottom of E ninety degrees clock-wise.

F is homeomorphic to T by rotating the top of F ninety degrees counter clock-wise.

J is homeomorphic to T by rotating the bottom of J ninety degrees counter clock-wise.

Y is homeomorphic to T by rotating the right arm of Y 45 degrees clock-wise, and rotating the left arm of Y 45 degrees counter clock-wise. $\square$

The sixth would be H.

The seventh would be KX.

Proof. K is homeomorphic to X by flipping the left leg of X to the right side with respect to the right leg of X. $\square$

The eighth would be P.

The ninth would be QR.

Proof. Q is homeomorphic to R by flip whatever is inside Q to the outside while not overlapping what's already at the outside. $\square$

Now I need to show that neither of these classes is homeomorphic to each other.

A is not homeomorphic to B since there exists a pair of points on A which if removed, produces four connected components and there is no such pair on B.

A is not homeomorphic to C since there exists a pair of points on A which if removed, produces four connected component, where as no such pair exists on C.

A is not homeomorphic to D since exists a pair of points on A which if removed, produces four connected components and there is no such pair on D.

A is not homeomorphic to E since there is a point on E which if removed, produces three connected components and there is no such point on A.

A is not homeomorphic to H since there exists a pair of points on H which if removed, produces five connected components and there is no such pair on A.

A is not homeomorphic to K since there exists a point on K which if removed, produces four connected components and there is no such point on A.

A is not homeomorphic to P since there exists a pair of points on A which if removed, produces four connected components and there is no such pair on P.

A is not homeomorphic to Q since there exists a point on Q which if removed, produces three connected components and there is no such point on A.

B is not homeomorphic to C since there exists a point on C which if removed, produces two connected components, and there is no such point on B.

B is not homeomorphic to D since B is two copies of D minus one point.

B is not homeomorphic to E since there exists a point on E which if removed, produces three connected components, and there is no such point on B.

B is not homeomorphic to H since there exists a pair of points on H which if removed, produces five connected components and there is no such pair on B.

B is not homeomorphic to K since there exists a point on K which if removed, produces four connected components and there is no such point on B.

B is not homeomorphic to P since B is P plus C minus two points.

B is not homeomorphic to R since B is R plus I minus two points.

C is not homeomorphic to D since there exists a point on C which if removed, produces two connected spaces, and there is no such point on D.

C is not homeomorphic to E since there exists a point on E which if removed, produces three connected spaces, and there is no such point on C.

C is not homeomorphic to H since there exists a pair of points on H which if removed, produces five connected components and there is no such pair on C.

C is not homeomorphic to K since there exists a point on K which if removed, produces four connected components and there is no such point on C.

C is not homeomorphic to P since P equals to C plus I minus two points.

C is not homeomorphic to R since there exists a point on R which if removed, produces three connected components, and there is no such point on C.

D is not homeomorphic to E since there exists a point on E which if removed, produces three connected spaces, and there is no such point on D.

D is not homeomorphic to H since there exists a pair of points on H which if removed, produces five connected components and there is no such pair on D.

D is not homeomorphic to K since there exists a point on K which if removed, produces four connected components and there is no such point on D.

D is not homeomorphic to P since there exist a point on P which if removed, produces two connected components, and there is no such point on D.

D is not homeomorphic to R since there exists a point on R which if removed, produces three connected components, and there is no such point on D.

E is not homeomorphic to H since there exists a pair of points on H which if removed, produces five connected components and there is no such pair on E.

E is not homeomorphic to K since there exists a point on K which if removed, produces four connected components and there is no such point on E.

E is not homeomorphic to P since there exists a point on E which if removed, produces three connected spaces, and there is no such point on P.

E is not homeomorphic to R since R is E plus I minus two points.

H is not homeomorphic to K since there exists a point on K which if removed, produces four connected components and there is no such point on H.

H is not homeomorphic to P since there exists a point on K which if removed, produces four connected components and there is no such point on P.

H is not homeomorphic to R since H is R plus one point.

K is not homeomorphic to P since there exists a point on K which if removed, produces four connected components and there is no such point on P.

K is not homeomorphic to R since there exists a point on K which if removed, produces four connected components and there is no such point on R.

P is not homeomorphic to R since there exists a point on R which if removed, produces three connected components and there is no such point on P. $\square$

Example 5.15. If $E \subset \mathbb{R}^2$ is countable, then $\mathbb{R}^2/E$ is path connected.

Version: 1
Comments / Collaborators: lalala

Proof. Pick two points from $\mathbb{R}^2/E$, find a line at the middle of these two points. Find a path from one point to the line, then find a path from the line to the other point. The start of the second path is the end of the first path. Since there are uncountably many points on the line, and E is countable. So there has to be some point on the line which is not in E. Then the paths from one end to that point, and from that point to the other end point form a path between our two points in $\mathbb{R}^2/E$. $\square$

In class, we mentioned that if X is a topological space $\pi_0(X)$ denotes the set of its path components. We also mentioned that there is such a thing as $\pi_1(X)$, $\pi_2(X)$, $\pi_3(X)$, etc... and that we will come back to these at the end of the semester, and that $\pi_n(X)$ is a group for all n at least 1. And that sadly, $\pi_0(X)$ is just a mere set and not a group. *However*, the purpose of the next exercise is to show that $\pi_0(X)$ gets to be a group too when X itself is a topological group:

3. Let H be a topological group.

(a) Let K denote the path component of H which contains the identity element. Prove that K is a normal subgroup.

(b) Show that two elements are in the same coset of K if and only if they're in the same path component of H. Deduce that there is a bijection $\phi : \pi_0(H) \to H/K$. Show that ϕ can be used to put a group structure on $\pi_0(H)$, as desired. Idea: given two path components C_1, C_2, define their product $C_1 \times C_2$ to be $\phi^{-1}(\phi(C_1) \times \phi(C_2))$.

Version: 1
Comments / Collaborators: Confusing

Proof. (a) I want to show that K is a normal subgroup. K is a normal subgroup if and only if for all k in K and all h in H, $hkh^{-1} \in K$. Since $hkh^{-1} \in K$ if and only if there exists a continuous function $r : [0,1] \to H$ such that $r(0) = hkh^{-1}$ and $r(1) = e$, as long as we find this function we are done.

Since k is in K, we know that there exists some continuous function s such that $s(0) = k$ and $s(1) = e$.

Multiply h on both sides of $s(0) = k$ and $s(1) = e$, we get $hs(0) = hk$ and $hs(1) = h$. Notice that the multiplication operator is the same for both sides; also note that the function hs is still continuous since we are effectively multiplying a continuous function by a scalar.

Multiply h^{-1} on both sides of $hs(0) = hk$ and $hs(1) = h$, we get $hs(0)h^{-1} = hkh^{-1}$ and $hs(0)h^{-1} = e$.

Now we've found a continuous function between hkh^{-1} and e.

(b) Suppose that x and y are in the same coset of K.

This implies that $xy^{-1} \in K$, which then implies that there exists continuous function r where $r(0) = xy^{-1}$, $r(1) = e$.

Multiply r by y on the right, we have $r(0)y = x, r(0)y = y$. Since r is a continuous function, ry is a continuous function as well. Thus there exists a path between x and y and thus they are in the same path component of H.

Suppose x and y are in the same path component of H. So there is a path between x and y. Every element in Ky would have a path to y.

Since x has a path to y, y has a path to every ky, then we know that there exists a path between x and every ky. So x is in the same coset as y.

Suppose C_1, C_2 and C_3 are path components in $\pi_0(H)$.

$$\begin{aligned} C_1 \times C_2 \times C_3 &= \phi^{-1}(\phi(C_1) \times \phi(C_2)) \times C_3 \\ &= \phi^{-1}(\phi(C_1) \times \phi(C_2) \times \phi(C_3)), \end{aligned} \tag{2}$$

and

$$\begin{aligned} C_1 \times (C_2 \times C_3) &= C_1 \times (\phi^{-1}(\phi(C_2) \times \phi(C_3))) \\ &= \phi^{-1}(\phi(C_1) \times \phi(C_2) \times \phi(C_3)). \end{aligned} \tag{3}$$

Thus, we know that the operation is associative.

Let C_0 be the path component of e. Then we know that

$$\begin{aligned} C_0 \times C_1 &= \phi^{-1}(\phi(C_0) \times \phi(C_1)) \\ &= \phi^{-1}(e \times \phi(C_1)) \\ &= \phi^{-1}(\phi(C_1)) \\ &= C_1. \end{aligned} \tag{4}$$

The other direction follows the same proof, thus we know there exits an identity element.

Suppose C_1 is a path component. Then there exists x in H/K such that $x = \phi(C_1)$.

Since H/K is a group, there exists inverse of x in H/K, say x^{-1}. Take the preimage of x^{-1} we know that $\phi^{-1}(x^{-1})$ would be the inverse of C_1.

Thus the group is closed under this operation.

Thus ϕ puts a group structure on $\pi_0(H)$. $\square$