PROOFS FOR TOPOLOGY

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BEFORE PRESENTATION

Example 3.8. In a metric space, the "closed ball" $\overline{B}(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}$ is indeed closed, but, despite the unfortunate notation, is not necessarily the closure of the open ball $B(x, \epsilon)$.

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Proof. Let the metric $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

For all $x \neq y$, $d(x, y) = 1 \geq 0$ and only when $x = y$, $d(x, y) = 0$.
$d(x, y) = 1 = d(y, x)$.
$d(x, y) + d(y, z) = 2 \geq 1 = d(x, z)$.
So we know that my d is a metric.
Now, construct the metric topology based on my metric.
Pick $B(0, 1)$. We know that $B(0, 1) = \{0\}$. Thus the closure is $\{0\}$.
Pick $\overline{B}(0, 1)$. We know that $\overline{B}(0, 1) = \mathbb{R}$.
Obviously, $\overline{B}(0, 1) \neq \{0\}$. □

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Example 3.10. The set of limit points of $(0, 1) \subset \mathbb{R}$ is $[0, 1]$ with respect to $\mathcal{T}$, $[0, 1)$ with respect to $\mathcal{T}_l$, and all of $\mathbb{R}$ with respect to $\mathcal{T}_{FC}$.

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Proof. For $\mathcal{T}$:

For every $y \in (0, 1)$, for all open set U that contains y, there exists an open ball $B = B(y, \epsilon)$, where $1 > \epsilon > 0$, such that is a subset of U by the virtue of U being open. Then there exists $y + \epsilon/2$, such that $y + \epsilon/2 \in (0, 1)\backslash\{y\}$. Thus all element in $(0, 1)$ is a limit point of $(0, 1)$.

For $y = 0$, for all open set U containing y, there exists an open ball $B = B(0, \epsilon)$, where $1 > \epsilon > 0$, such that is a subset of U by the virtue of U being open. Then there exists $\epsilon/2$, such that $\epsilon/2 \in (0, 1)$. Thus 0 is a limit point.

Similarly, 1 is a limit point.

For an arbitrary $y > 1$, say $y = 1 + \epsilon_1$. Pick $U = B(y, \epsilon_1/2)$. We know that U shares no element with $(0, 1)$. Thus $y = 1 + \epsilon_1$ cannot be a limit point in $\mathcal{T}$.

Similarly, $0 - \epsilon_1$ can never be a limit point in $\mathcal{T}$.

Thus, the set of limit points in $\mathcal{T}$ is $[0, 1]$.

For $\mathcal{T}_l$:

For all $y \in (0, 1)$, for all open set U containing y, there exists an open ball $B = [y, \epsilon)$, where $1 - y > \epsilon > 0$, such that $y \in B \subset U$, by the virtue of U being open. There always exist $y + \epsilon/2$ that is always in $(0, 1)\backslash\{y\}$. Thus all points in $(0, 1)$ is a limit point in $\mathcal{T}_l$.

For $y = 0$, for all open set U containing y. There exists $B = [0, \epsilon)$, where $1 > \epsilon > 0$, such that $y \in B \subset U$. There always exist $\epsilon/2$ that is always in $(0, 1)$. Thus 0 is a limit point in $\mathcal{T}_l$.

For all $y \geq 1$, pick $U = B = [y, y+\epsilon)$, then we know that $U \cap (0, 1) = \emptyset$. Thus all y greater or equal to 1 is not a limit point.

Similarly, all y smaller than 0 is not a limit point.

Thus, the set of limit points in $\mathcal{T}_l$ is $[0, 1)$.

For $\mathcal{T}_{FC}$:

For all $y \in \mathbb{R}$, for all open set U containing y, the only possibility that y is not a limit point is that $\mathbb{R}\backslash U = (0, 1)\backslash\{y\}$. However, $(0, 1)\backslash\{y\}$ is not a finite set, thus impossible.

Thus, the set of limit points in $\mathcal{T}_{FC}$ is $\mathbb{R}$. $\square$