335 HW

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1. Rewrites

Example 1.15: (3) $\mathcal{B}_H = \mathcal{B} \cup \{(a,b)\backslash H : a < b\}$, where $H = \{\frac{1}{n}\}_{n\in\mathbb{Z}}$, is a basis for the harmonic topology $\mathcal{T}_H$.

Proof. Pick an arbitrary $x \in \mathbb{R}$.

Case 1: If $x = 0$, there exist $\epsilon > 0$ such that $0 \in B(0,\epsilon)$. Suppose $x \in B_1$ and $x \in B_2$.

Sub-case 1: $B_1, B_2 \in \mathcal{B}$, let $B_3 = (0, \min(\epsilon_1, \epsilon_2))$, where ϵ_i is the radius of the open ball B_i.

Sub-case 2: Without the loss of generality, suppose $B_1 \in \mathcal{B}$ and $B_2 = (c,d) \in \{(a,b)\backslash H : a < b\}$. If B_1 is an open ball that is a subset of B_2, it can be our wanted B_3 and satisfies condition B2. If B_1 is not a subset of B_2. Let $\epsilon_3 = \min(|c|,|d|)$ Since we know that there exist no element of H in (c,d), so for every element in $B(0,\epsilon_3)$, they don't exist in H. We know that ϵ_3 will be smaller than ϵ_1 since B_1 is not a subset of B_2.

(Personally, I think this sub-case is impossible since H is oscillating around 0, so it should be impossible to find such B_2.)

Sub-case 3: $B_1 = (a,b)$ and $B_2 = (c,d)$ for some $a < b$ and $c < d$, and B_1, B_2 doesn't have any element in H.

Let $e = \max(a,c)$ and $f = \min(b,d)$. Let $B_3 = (e,f)$, we know that 0 is in B_3 and, there's no element of H in B_3, which is a subset of the intersection of B_1 and B_2 by observation.

Case 2: If $x \notin H$ and $x \neq 0$, by the formation of $\mathcal{B}_H$, we can find such $a, b \in \mathbb{R}$ such that $x \in (a,b)\backslash H$. If such $x \in (a,b)\backslash H$ and in $(c,d)\backslash H$, pick new interval $(e,f) = (\max(a,c), \min(b,d))$. We know that $x \in (e,f)$ for sure and $(e,f) \subset (a,b) \cap (c,d)$.

Case 3: If $x \in H$, find open ball $B(x,\epsilon)$ with $\epsilon > 0$. we know that $x \in B(x,\epsilon)$. If such x is in $B(x,\epsilon_1)$ and $B(x,\epsilon_2)$ for $\epsilon_1, \epsilon_2 \in \mathbb{R}$ and greater than 0, we know that $x \in B(x, \min(\epsilon_1,\epsilon_2)$ and $B(x,\min(\epsilon_1,\epsilon_2) \subset B(x,\epsilon_1) \cap B(x,\epsilon_2)$. $\square$

Proposition 1.17: Suppose $\mathcal{B}$ and $\mathcal{B}'$ are bases on a set X. Then $\mathcal{T}_{\mathcal{B}}$ is finer than $\mathcal{T}_{\mathcal{B}'}$ if and only if for all $B' \in \mathcal{B}'$, there exists a collection of sets $\{B_\alpha\}_{\alpha\in I}$ in $\mathcal{B}$ such that $B' = \cup_\alpha B_\alpha$ (i.e., $\mathcal{B}' \subset \mathcal{T}_{\mathcal{B}}$).

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Proof. First the forward direction.

Suppose $\mathcal{T}_{\mathcal{B}}$ is finer than $\mathcal{T}_{\mathcal{B}'}$. Then we know that $\mathcal{T}_{\mathcal{B}'} \subset \mathcal{T}_{\mathcal{B}}$. We also know that any open set in $\mathcal{T}_{\mathcal{B}'}$ is also an open set in $\mathcal{T}_{\mathcal{B}}$

Pick an arbitrary $B' \in \mathcal{B}'$. Since B' is an open set in $\mathcal{T}_{\mathcal{B}}$, we know that for any element x_α, there exists a $B_\alpha \in \mathcal{B}$ such that $x \in B_\alpha \subset B'$.

Use all of the B_α above construct a union $\cup_\alpha B_\alpha$.

We know that all of the B_α is a subset of B', so $B' \supset \cup_\alpha B_\alpha$.

Since all element of B' is contained in some $B_\alpha \in \cup_\alpha B_\alpha$, so $B' \subset \cup_\alpha B_\alpha$.

So we know that $B' = \cup_\alpha B_\alpha$.

Now, backward direction.

Suppose for all $B' \in \mathcal{B}'$, $B' = \cup_\alpha B_\alpha$ for some collection of $B_\alpha \in \mathcal{B}$.

Pick an arbitrary open set U in $\mathcal{T}_{\mathcal{B}'}$, we can find a collection of $B'_\beta \in \mathcal{B}'$ such that $U = \cup_\beta B'_\beta$. Since for all B' in the union $\cup_\beta B'_\beta$, there exists some collection of $B \in \mathcal{B}$ such that B' is a subset of the collection. So we know that for any open set U in $\mathcal{T}_{\mathcal{B}'}$, there exists a collection of $B_\alpha \in \mathcal{B}$ such that $U = \cup_\beta \cup_\alpha B_\alpha$. Thus, we know that $\mathcal{T}_{\mathcal{B}'} \subset \mathcal{T}_{\mathcal{B}}$. □

Example 1.22: There is a metric on $\mathbb{R}$ whose associated topology is not the same as the standard topology.

Proof. Let the metric $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

For all $x \neq y$, $d(x, y) = 1 \geq 0$ and only when $x = y$, $d(x, y) = 0$.

$d(x, y) = 1 = d(y, x)$.

$d(x, y) + d(y, z) = 2 \geq 1 = d(x, z)$.

So we know that my d is a metric.

Now, construct the metric topology based on my metric.

Pick 0 from $\mathbb{R}$, $B(0, 1/2) = \{0\}$ which is not an open set in the standard topology.

So, this metric topology is not the same as the standard topology. □

2. This week's problem

Proposition 1.19: On $\mathbb{R}$, there exist non-comparable topologies, i.e. topologies $\mathcal{T}$ and $\mathcal{T}'$ such that $\mathcal{T}$ is not finer than $\mathcal{T}'$ and $\mathcal{T}'$ is not finer than $\mathcal{T}$.

Proof. I claim that the metric topology generated by the metric

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

is not comparable with the standard topology.

Pick 0 from $\mathbb{R}$, $B(0, 1/2)$ in the metric topology is $\{0\}$, which is not an open set in the standard topology, thus standard topology is not finer than the metric topology.

Pick open set (-3,3) from the standard topology, since all open sets in the metric topology is either in the form of $(x-1, x+1)$ or $\{x\}$, so $(-3,3)$ is not in the metric topology.

Thus, these two topologies are non-comparable. □

Proposition ??? 1.25: There exists a topology on $\mathbb{R}$ that is not metrizable.

Proof. The indiscrete topology on $\mathbb{R}$ is not metrizable.

Proof by contradiction.

Suppose there exists a metric d on $\mathcal{T}_{indisc}$. Then we know that the metric topology generated by this d is the same as $\mathcal{T}_{indisc}$, namely, $\mathcal{T}_d = \mathcal{T}_{indisc}$.

We also know that $\mathcal{T}_d$ is generated by the metric basis $\mathcal{B}_d$, so $\mathcal{T}_{indisc}$ should also be generated by $\mathcal{B}_d$.

Suppose there exists two elements in $\mathcal{T}_{indisc}$, x and y such that $d(x,y) = \epsilon > 0$.

Take $\epsilon/2$ and construct the open ball $B(x, \epsilon/2)$. Then we know that this open ball is in the metric basis . Since $d(x,y) = \epsilon > \epsilon/2$, we know that y is not in $B(x, \epsilon/2)$.

However, there are only two elements in the indiscrete topology which is the empty set and the real number line, since $B(x, \epsilon/2)$ is obviously not an empty set, it has to be the real number line, but since y is in the real number line, y has to be in $B(x, \epsilon/2)$. Since y cannot be both in and not in $B(x, \epsilon/2)$, there is the contradiction. □

Proposition 1.27: The collection $\mathcal{B}_\times$ defined above is a basis for a topology.

Proof. We want to show that

- for all $(x,y) \in X \times Y$, there exists $U \times V \in \mathcal{B}_\times$ such that $(x,y) \in U \times V$, and
- if $(x,y) \in U_1 \times V_1 \bigcap U_2 \times V_2$, then there exists $U_3 \times V_3 \in \mathcal{B}_\times$ such that $(x,y) \in U_3 \times V_3 \subset U_1 \times V_1 \bigcap U_2 \times V_2$.

Denote $\mathcal{B}_X$ as the basis for $\mathcal{T}_X$, $\mathcal{B}_Y$ as the basis for $\mathcal{T}_Y$.

Pick an arbitrary (x,y) from $X \times Y$. We know that there exist $B_x \in \mathcal{B}_X$ such that $x \in B_x$, and $B_y \in \mathcal{B}_Y$ such that $y \in B_y$. Let $U = B_x$ and $V = B_y$, we know that $(x,y) \in U \times V = B_x \times B_y \in \mathcal{B}_\times$. So for all (x,y) in $X \times Y$, there exists $U \times V \in \mathcal{B}_\times$ such that $(x,y) \in U \times V$.

Suppose $(x,y) \in U_1 \times V_1 \bigcap U_2 \times V_2$. By the commutativity, we can rewrite $U_1 \times V_1 \bigcap U_2 \times V_2$ as $U_1 \bigcap U_2 \times V_1 \bigcap V_2$. By the third property of a topology, we know that there exists $U_3 = U_1 \bigcap U_2$ and $V_3 = V_1 \bigcup V_2$. So we know that there exists $U_3 \times V_3 \subset U_1 \times V_1 \bigcap U_2 \times V_2 \in \mathcal{B}_\times$ such that $(x,y) \in U_3 \times V_3$. $\square$

Example 1.28 The product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the same as the topology induced from the Euclidean metric (i.e. $d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$)

Proof. Denote the basis of the product topology as $\mathcal{B}_\times$, and the basis of the metric topology as $\mathcal{B}_d$.

Firstly, show that $\mathcal{T}_{\mathbb{R}^2} \subseteq \mathcal{T}_d$.

Pick an arbitrary $B \in \mathcal{B}_\times$. Then we know that there exists $U \in \mathbb{R}$ and $V \in \mathbb{R}$ such that $B = U \times V = \{(x,y) : x \in U, y \in V\}$. Pick an arbitrary $(x,y) \in B$.

Since both U and V are open sets in $\mathcal{T}$ the standard topology on $\mathbb{R}$, there exists ϵ_1, ϵ_2 such that $B(x,\epsilon_1) \subset U$ and $B(y,\epsilon_2) \subset V$. Let $\epsilon = \min(\epsilon_1/2, \epsilon_2/2)$. Then we know that $B((x,y),\epsilon) \subset U \times V = B$. Thus B is an open set in $\mathcal{T}_d$. Thus $\mathcal{T}_{\mathbb{R}^2} \subseteq \mathcal{T}_d$.

Now, show that $\mathcal{T}_d \subseteq \mathcal{T}_{\mathbb{R}^2}$.

Pick an arbitrary $(x,y) \in \mathbb{R}^2$, and an arbitrary $\epsilon > 0$. Then we know that there exists $B \in \mathcal{B}_d$ such that $B = B((x,y),\epsilon)$. Since x is on $\mathbb{R}$, there exists open set $B_x \in \mathcal{B}$, where $\mathcal{B}$ is collection of open balls, such that $x \in B_x = (x, \epsilon_1)$; similarly, there exists $B_y \in \mathcal{B}$ such that $y \in B_y = (y, \epsilon_y)$. Choose ϵ_1 and ϵ_2 to be smaller than $\epsilon/2$. Let $U = B_x$ and $V = B_y$, then we know that $U \times V \subset B((x,y),\epsilon)$. Since for all (x,y) in $\mathbb{R}^2$, and for all $\epsilon > 0$ and thus for all open sets, B in $\mathcal{B}_d$, we can find a set, $U \times V$, in $\mathcal{B}_\times$ such that $(x,y) \in U \times V \subset B$, so we know that $B \in \mathcal{T}_{\mathbb{R}^2}$. Thus $\mathcal{T}_d \subset \mathcal{T}_{\mathbb{R}^2}$.

Since both topologies are finer than the other, the are the same topology. $\square$

Lemma... 2.2 In Definition 2.1, suppose that $\mathcal{B}$ is a basis for $\mathcal{T}_Y$. Then f is continuous if $f^{-1}(B)$ is open for all $B \in \mathcal{B}$

Proof. Suppose $f^{-1}(B)$ is open for all $B \in \mathcal{B}$.

Since $\mathcal{T}_Y$ is a topology, we know that all open sets U in the topology equals to the union of some collection, $\{B_\alpha\}_{\alpha \in I}$, in $\mathcal{B}$. Then we know that $f^{-1}(U) = f^{-1}(\bigcup_\alpha B_\alpha) = \bigcup_\alpha f^{-1}(B_\alpha)$. Since $\mathcal{T}_X$ is a topology as well, the union of open sets in $\mathcal{T}_X$ is still open in $\mathcal{T}_x$. Therefore $f^{-1}(U)$ is open. Thus f is continuous. $\square$

Let A be a set, and suppose $\{X_a\}_{a\in A}$ is a collection of topological spaces, indexed by A (i.e., for each element a of A, there is one topological space X_a). Consider the product

$$\prod_{a\in A} X_a,$$

equipped with the product topology – denote this space by X so we don't have to keep writing it out. Given some a, define $p_a : X \to X_a$ be the projection on to the X_a factor, i.e., the map that takes an element of X and sends it to whatever is in the "component" of X corresponding to a.

(i) Show that p_a is continuous.

Proof. For the sake of clarity, I divide the cases with respect to the cardinality of A. One for the set A being a finite set and the other for it being an infinite set.

First case: When $\operatorname{card}(A) \neq \infty$.

So we know that $A = \{a_1, a_2, \dots, a_n\}$. Then it follows that the topological space $X = X_{a_1} \times X_{a_2} \times \cdots \times X_{a_n}$.

Denote the basis for the product topology as $B_\times$.

Pick an arbitrary element from A, say a. Then the function is defined as $p_a = X \to X_a$.

I want to show that for all open sets U in X_a, $p_a^{-1}(U)$ is open in X, or $U \in \mathcal{T}_\times$.

Pick an arbitrary element $x \in U$, we know that there exists corresponding element in X such that they are the preimage of x. We also know that these elements of the preimage of x should take the form

$$(x_{a_1}, x_{a_2}, \dots, x, \dots, x_{a_n}). \tag{1}$$

Note here x_{a_i} can be any element in X_{a_i}.

Let the collection of these elements in X be set V.

Pick an arbitrary element $v \in V$. We know that this v takes the form

$$(x_{a_{1_v}}, x_{a_{2_v}}, \dots, x, \dots, x_{a_{n_v}}). \tag{2}$$

Construct $B \in \mathcal{B}_\times$ such that

$$B = U_{a_{1_v}} \times U_{a_{2_v}} \times \cdots \times U \times \cdots U_{a_{n_v}}, \tag{3}$$

where $U_{a_{i_v}}$ is a set in X_{a_i} and containing $x_{a_{i_v}}$. We know that $v \in B$ by the construction of B. Since x_{a_i} in V can be any element in X_{a_i}, and $U_{a_{i_v}} \subset X_{a_i}$, so any combination of $x_{a_i} \in U_{a_{i_v}}$ (together with x being the a component) will still be in V. So $B \subset V$.

Since for all $v \in V$, there exists $B \in \mathcal{B}_\times$ such that $v \in B \subset V$, we know that $V \in \mathcal{T}_\times$, which implies that V is open in X. Thus, p_a is continuous.

Second case: $\operatorname{card}(A) = \infty$.

In this case, our definition of the basis for the product topology needs a bit edition,

$$B_\times = \{\prod_{a_i\in A} U_{a_i} : U_{a_i} \text{ is open in } X_{a_i} \text{ and } U_{a_i} = X_{a_i} \text{ for all but finitely many } a_i\} \tag{4}$$

The preliminary work is the same as when A is a finite set, until when proving B is a subset of V.

For those finitely many $U_{a_{i_v}}$ which are subsets of X_{a_i}, the logic is the same; for those $U_{a_{i_v}} = X_{a_i}$, we can apply a similar logic and say that since x_{a_i} can be any element of X_{a_i}, so all of the possible combinations of x_{a_i} will still be in V.

Then proceed with the same process and show that V is open in $\mathcal{T}_\times$. Thus p_a is continuous. $\square$

(ii) Show that if $\mathcal{T}$ is any topology on X that is strictly coarser than the product topology, then if we equip X with that topology, there exists some element a in A so that $p_a : X \to X_a$ is discontinuous.

For this reason, the product topology can be characterized as the coarsest topology such that all projection maps are continuous.

Proof. Proof by proving the contrapositive case.

Contrapositive statement: If for all a in A, the function $p_a : X \to X_a$ is continuous, then the topology we equipped with X is not coarser than the product topology.

Suppose p_a is continuous for all $a \in A$.

For any open set, U, in X_a, we know that $p_a^{-1}(U)$ is open in X. Then for this topology that is equipped with X, we can find the basis of product topology in there.

The way to construct the basis of the product topology is that for each U in X_a for each $a \in A$. We take the collection of the $p_a^{-1}(U)$.

Since the equipped topology contains the basis for the product topology, we know that it cannot be coarser than the product topology. $\square$