PROOFS FOR TOPOLOGY

TONY DENG

1. Rewrite of previous weeks' problems

Proposition 1.19. On $\mathbb{R}$, there exist non-comparable topologies, i.e. topologies $\mathcal{T}$ and $\mathcal{T}'$ such that $\mathcal{T}$ is not finer than $\mathcal{T}'$ and $\mathcal{T}'$ is not finer than $\mathcal{T}$.

Version: 2
Comments / Collaborators: Had a chat with Jason Ma.

Proof. The finite complement topology and the harmonic topology are non-comparable.

Pick $B(1/2, 1)$ from $\mathcal{T}_H$, we know that $\mathbb{R} \backslash B(1/2, 1)$ is not finite, nor $\mathbb{R}$. So $B(1/2, 1)$ is not open in the finite complement topology.

Pick $(-\infty, 0) \cup [1/2, \infty)$ from $\mathcal{T}_{FC}$. Since the only open sets in B_H containing $1/2$ are the open balls $B(1/2, \epsilon)$ with $\epsilon > 0$. However, since the open balls will always contain some element that is smaller than $1/2$, we cannot find an open set that doesn't contain element smaller than $1/2$, therefore this open set is not open in $\mathcal{T}_H$.

□

Date: September 27, 2021.

Tarik's corner. Let A be a set, and suppose $\{X_a\}_{a\in A}$ is a collection of topological spaces, indexed by A (i.e., for each element a of A, there is one topological space X_a). Consider the product

$$\prod_{a\in A} X_a,$$

equipped with the product topology – denote this space by X so we don't have to keep writing it out. Given some a, define $p_a : X \to X_a$ be the projection on to the X_a factor, i.e., the map that takes an element of X and sends it to whatever is in the "component" of X corresponding to a.

(i) Show that p_a is continuous.

Version: 2
Comments / Collaborators: Product topologies are complicated lol

Proof. For the sake of clarity, I divide the cases with respect to the cardinality of A. One for the set A being a finite set and the other for it being an infinite set.

First case: When $\mathrm{card}(A) \neq \infty$.

So we know that $A = \{a_1, a_2, \ldots, a_n\}$. Then it follows that the topological space $X = X_{a_1} \times X_{a_2} \times \cdots \times X_{a_n}$.

Denote the basis for the product topology as $B_\times$.

Pick an arbitrary element from A, say a. Then the function is defined as $p_a = X \to X_a$.

I want to show that for all open sets U in X_a, $p_a^{-1}(U)$ is open in X, or $U \in \mathcal{T}_\times$.

Pick an arbitrary element $x \in U$, we know that there exists corresponding element in X such that they are the preimage of x. We also know that these elements of the preimage of x should take the form

$$(x_{a_1}, x_{a_2}, \ldots, x, \ldots, x_{a_n}). \tag{1}$$

Note here x_{a_i} can be any element in X_{a_i}.

Construct $B \in \mathcal{B}_\times$ such that

$$B = U_{a_1} \times U_{a_2} \times \cdots \times U \times \cdots U_{a_n}, \tag{2}$$

where U_{a_i} is a set in X_{a_i} and containing x_{a_i}. We know that the preimages of x is in B by the construction of B. Since x_{a_i} in $p_a^{-1}(U)$ can be any element in X_{a_i}, and $U_{a_i} \subset X_{a_i}$, so any combination of $x_{a_i} \in U_{a_i}$ (together with x being the a component) will still be in $p_a^{-1}(U)$. So $B \subset p_a^{-1}(U)$.

Since for all element in $p_a^{-1}(U)$, there exists $B \in \mathcal{B}_\times$ such that that element is in $B \subset p_a^{-1}(U)$, we know that $p_a^{-1}(U) \in \mathcal{T}_\times$, which implies that $p_a^{-1}(U)$ is open in X. Thus, p_a is continuous.

Second case: $\text{card}(A) = \infty$.

In this case, our definition of the basis for the product topology needs a bit edition,

(3)

$$B_\times = \{\prod_{a_i \in A} U_{a_i} : U_{a_i} \text{ is open in } X_{a_i} \text{ and } U_{a_i} = X_{a_i} \text{ for all but finitely many } a_i\}$$

The preliminary work is the same as when A is a finite set, until when proving B is a subset of $p_a^{-1}(U)$.

For those finitely many U_{a_i} which are subsets of X_{a_i}, the logic is the same; for those $U_{a_i} = X_{a_i}$, we can apply a similar logic and say that since x_{a_i} can be any element of X_{a_i}, so all of the possible combinations of x_{a_i} will still be in $p_a^{-1}(U)$.

Then proceed with the same process and show that $p_a^{-1}(U)$ is open in $\mathcal{T}_\times$. Thus p_a is continuous. $\square$

2. Rewrite of BP's for this week

Example 2.5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -1 & x < 0, \\ 1 & x \geq 0. \end{cases}$$

Determine which (if any) of the following maps between topological spaces are continuous.

(1) $f : (\mathbb{R}, \mathcal{T}_l) \to (\mathbb{R}, \mathcal{T}_l)$
(2) $f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T}_l)$
(3) $f : (\mathbb{R}, \mathcal{T}_l) \to (\mathbb{R}, \mathcal{T})$

Version: 2
Comments / Collaborators: I wrote this with my memories from the presenter.

Proof. Choose an arbitrary open set, U, from $\mathcal{T}_l$ (and $\mathcal{T}$ in another trial).

There are several cases.
(a) Suppose this set doesn't contain ± 1.
Then we know that $f^{-1}(U) = \emptyset$.
(b) Suppose this set contains 1 not -1.
Then we know that $f^{-1}(U) = [0, \infty)$.
(c) Suppose this set contains -1 not 1.
Then we know that $f^{-1}(U) = (-\infty, 0)$
(d) Suppose this set contains both ± 1.
Then we know that $(-\infty, \infty)$.

Now looking at the maps, we see that all possible $f^{-1}(U)$ is open in $\mathcal{T}_l$, and $[0, \infty)$ is not open in $\mathcal{T}$. So we know that the first and third maps are continuous and the second one is not.

$\square$

3. This week's problem

Example 2.10. The identity function $f : (\mathbb{R}, \mathcal{T}_l) \to (\mathbb{R}, \mathcal{T})$ given by $f(x) = x$ is continuous and invertible, but it is not a homeomorphism. Note that this example does not prove that no homeomorphism exists, just that f isn't one.

Version: 1
Comments / Collaborators: I wrote this with the presenter's proof in memory

Proof. Pick an open set from $\mathcal{T}$. Since we know that $\mathcal{T}_l$ is finer than $\mathcal{T}$, and since the preimage of the open set will be the same set by the virtue of this function, we know that the preimage of this open set will be open in $\mathcal{T}_l$ as well. Thus the function is continuous.

Pick two different elements from $\mathcal{T}$, and they are not equal to each other.

$$f(x) \neq f(y).$$

The preimages of these two element from $\mathcal{T}$ will be x and y. Since we know that $f(x) = x$ and $f(y) = y$, and since $f(x) \neq f(y)$, we know that $x \neq y$.

For any element in $\mathcal{T}$, say $f(x)$, we can find an element in $\mathcal{T}_l$ where $x = f(x)$.

Thus the function is a bijection.

Let g be the inverse function of f.

Since we know that $\mathcal{T}_l$ is finer than $\mathcal{T}$, we know that there exists an open set in $\mathcal{T}_l$ such that it is not open in $\mathcal{T}$. Therefore g is not continuous. Thus f is not a homeomorphism.

$\square$

Example 2.13. The topological group $SO(2)$ is homeomorphic to S^1

Version: 1
Comments / Collaborators: I wrote this with the presenter's proof in memory, and chatted with Joao from MQC.

Proof. The topological group $SO(2)$ is composed by elements in the form of $\begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$, and S^1 is composed of elements in the form of $\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$, with $\theta \in [0, 2\pi)$ for both spaces.

Let f be a function from S^1 to $SO(2)$ such that $f\left(\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}\right) = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$.

Suppose there exists two elements from $SO(2)$ such that they are not equal to each other.

$$\begin{pmatrix} \sin\theta_1 & \cos\theta_1 \\ -\cos\theta_1 & \sin\theta_1 \end{pmatrix} \neq \begin{pmatrix} \sin\theta_2 & \cos\theta_2 \\ -\cos\theta_2 & \sin\theta_2 \end{pmatrix}. \tag{4}$$

Their preimages would be $\begin{pmatrix} \sin\theta_1 \\ \cos\theta_1 \end{pmatrix}$ and $\begin{pmatrix} \sin\theta_2 \\ \cos\theta_2 \end{pmatrix}$ respectively. Since $\theta_1 \neq \theta_2$, we know that

$$\begin{pmatrix} \sin\theta_1 \\ \cos\theta_1 \end{pmatrix} \neq \begin{pmatrix} \sin\theta_2 \\ \cos\theta_2 \end{pmatrix}. \tag{5}$$

For any element in S^1 with the angle being θ, we can find an element in $SO(2)$ with the angle being θ such that it maps to the element in S^1.

Therefore, the function is a bijection.

Since $SO(2)$ is a subspace topology in $\mathbb{R}^4$, by the same logic in example 1.28, we know that $\mathbb{R}^4$ is a product topology of $\mathbb{R}^2$.

So now our function f can be seen as

$$f : S^1 \to S^1 \times \left\{ \begin{pmatrix} -\cos\theta \\ \sin\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\}. \tag{6}$$

By proposition 2.8, f is continuous if and only if $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous. Now we want to show that $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous.

We know that the projection maps are

(7)

$$\pi_X : S^1 \times \left\{ \begin{pmatrix} -\cos\theta \\ \sin\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\} \to S^1$$

$$\pi_Y : S^1 \times \left\{ \begin{pmatrix} -\cos\theta \\ \sin\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\} \to \left\{ \begin{pmatrix} -\cos\theta \\ \sin\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\}$$

We notice that π_X is the inverse function of f, and by lemma 2.7, we know that π_X is continuous. So the inverse function of f is continuous.

We also notice that since π_X is the inverse function of f, so $\pi_X \circ f$ is the "do nothing function," which is also continuous by the virtue of doing nothing.

Call the other space in the product topology Y so that we don't need to keep writing its giant definition.

$$\pi_Y = S^1 \times Y \to Y. \tag{8}$$

It's not hard to see that Y is also a description of the unit circle centered at the origin.

Pick an arbitrary open set U from Y. We know that this open set is an open set on the unit circle. Sending $-\cos\theta$ to $\sin\theta$ and $\sin\theta$ to $\cos\theta$ is essentially what π_Y is doing, which can also be seen as rotate 90 degree in counter clock wise direction, then flip around vertical axis. So we know that the preimage of this open set in S^1 would still be an open set. Thus $\pi_Y \circ f$ is also continuous. Therefore f is continuous.

Since f is bijection and bi-continuous, f is a homeomorphism. $\square$

Tarik's corner. (1) Let G be a topological group and let H be a subgroup.

(a) Show that H is also a topological group when equipped with the subspace topology.

(b) Show that the closure of H is also a subgroup of G. Note: given a topological space X and a subset A of X, the closure of A is the set of points $x \in X$ so that if U is any open subset of X containing x, then there exists some $a \in A$ which is also in U. In other words: the closure of A is the set of points that can not be separated from A by open sets.

Version: 1
Comments / Collaborators: I feel like my logic is a bit entangled at the beginning, but I think I got the trick as the proof goes...

Proof. (a) Since H is a subgroup of G, we know that the operations of H is the same as the operations of G. We want to show that the multiplication and the inverse maps are both continuous.

Suppose U is an open set from H. We know that there exists $\tilde{U}$ open in G such that $U = \tilde{U} \cap H$.

Take the preimage of the inverse map on U, we have $\iota^{-1}(U) = \iota^{-1}(\tilde{U} \cap H)$. Since H is a subgroup, it's closed under inverse map. Thus $\iota^{-1}(\tilde{U} \cap H) = \iota^{-1}(\tilde{U}) \cap H$. Since $\iota^{-1}(\tilde{U})$ is open by G being a topological group, $\iota^{-1}(\tilde{U}) \cap H$ is open H. Thus, the inverse map is continuous.

Similarly, take the preimage of U on the multiplication map. We know that $\mu^{-1}(U)$ is a subset of $\mu^{-1}(\tilde{U})$ and $\mu^{-1}(U)$ is a subset of H by closure. We know that $\mu^{-1}(U) = H \cap \mu^{-1}(\tilde{U})$. Since $\mu^{-1}(\tilde{U})$ is open in G, we know that $\mu^{-1}(U)$ is open in H by definition. Thus, the multiplication map is continuous.

Therefore, H is a topological group.

(b) We know that every element of of H is also in the closure of H since by definition, every open sets containing the element of H for sure has an element from H. So we know that the identity element is in the closure of H.

Since H is a subgroup, so we know that all element in H is invertible and closed under multiplication. Hence, in order to prove that the closure of H is also a subgroup, we need to show that the elements that are in H but in the closure of H is still closed and invertible in the closure of H.

Suppose g is in the closure of H, but not in H. Let U be the open set containing g, we know that there exists some h in U such that h in H. Take the preimage of U with respect to the inverse map. We know that $g^{-1} \in \iota^{-1}(U)$, and $h^{-1} \in \iota^{-1}(U)$, and since H is closed, we know that $h^{-1} \in H$. Thus, we know that g^{-1} is in the closure of H.

Suppose g and j are in the closure of H, but not in H. Let U be the open set containing g and V be the open set that contains j. We know that there exists some h in U and some l in V such that h, l are in H.

We know that $g \cdot j \in U \times V$, and $h \cdot l$ is in $U \times V$ as well. Since H is closed, we know that $h \cdot l$ is in H as well. So $g \cdot j$ is in the closure of H by definition. Thus, we know that the closure of H is closed.

Since the identity element is in the closure of H, it's closed, it has all its inverses, and the operation is the same as G, we know that the closure of H is a subgroup of G. $\square$

Tarik's corner. (2) Show that ($\mathbb{R}$, standard topology) and ($\mathbb{R}$, lower limit topology) are not homeomorphic. Hint: an open partitioning of a space X is a way of writing X as a disjoint union of two open sets U, V. Show that if X has an open partitioning and if Y is homeomorphic to X, then Y must also have an open partitioning. Next, show that ($\mathbb{R}$, lower limit topology) has an open partitioning and that ($\mathbb{R}$, standard topology) does not.

Version: 1
Comments / Collaborators: wondered if proof by contradiction works.

Proof. Claim: Suppose X and Y are homeomorphic, if X has an open partitioning, then Y must also have an open partitioning.

Proof. Suppose f is the homeomorphism between X and Y.

Suppose $X = U \cup V$ where U and V are disjoint open sets. Since f is a homeomorphism, we know that it is a bijection and bi-continuous. So for open set U in X, $f^{-1}(U)$ must be open in Y, and for open set V in X, $f^{-1}(V)$ must be open in Y. Since f is a bijection and $U \cap V = \emptyset$, we know that $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ in Y, so $Y = f^{-1}(U) \cap f^{-1}(V)$, which is an open partitioning. $\square$

In the lower limit topology, we can have an open partitioning, $U = (-\infty, 0)$ and $[0, \infty)$.

Suppose there exists an open partitioning in the standard topology, say U and V. Both U and V would be infinite union of open balls on $\mathbb{R}$. Without the loss of generality, suppose U is the open set that has elements greater than all the elements in V. Then we know that $\inf(U) = \sup(V)$. If $\inf(U)$ in U, then we know that there exists an open ball $B(\inf(U), \epsilon)$ in U such that contains elements smaller than $\inf(U)$, which contradicts with the fact that it is the infinum of U. Thus $\inf(U)$ not in U. If $\sup(V) \in V$, similarly, there exists an open ball $B(\sup(V), \epsilon)$ in V, such that it contains elements greater than $\sup(V)$ which contradicts with the fact that it is the supremum of V. Thus $\sup(V) \notin V$.

Since $\inf(U) = \sup(V)$ not in both U or V, it contradicts with the fact that U and V partitions $\mathbb{R}$.

Therefore, there cannot be any partition over the standard topology. $\square$