In the following we describe the accompanying Mathematica codes for computational verification of several important mathemtical facts in the main text and SI.

M1. R-MATRICES, EXCLUSION STATISTICS, AND THE 1D SOLVABLE SPIN MODEL

In RMatricesAnd1DSpinModel.nb we first construct all the nontrivial R-matrices that appear in this paper [Exs. 3 and 4 in Tab. I, and the set-theoretical R-matrix in Eqs. (29,31)] and verify that they satisfy the YBE (5). Then we compute the exclusion statistics and single mode partition function $z_R(x)$ for each R-matrix, using the method in Sec. S2 C. Next we construct the local operators of the 1D solvable spin model for each R-matrix, and verify that they satisfy the defining relations in Eq. (S21). The local operators $\{\hat{x}_a^\pm, \hat{y}_a^\pm\}_{a=1}^m$ and $\hat{T}_{ab}^\pm$ are constructed in the file SpinModelOperators.wl, using the explicit matrix representation in Eq. (S24) and Eq. (S28), respectively. Note that $\{\hat{x}_a^+\}_{a=1}^m$ is stored as a $m \times d \times d$ tensor, where d is the on-site Hilbert space dimension of the 1D spin model, and similarly for $\{\hat{x}_a^-\}_{a=1}^m, \{\hat{y}_a^+\}_{a=1}^m, \{\hat{y}_a^-\}_{a=1}^m$. $\{\hat{T}_{ab}^+ | 1 \le a, b \le m\}$ is stored as a $m \times m \times d \times d$ tensor, for example, Tp[[a, b]] in the code gives $\hat{T}_{ab}^+$, and similarly for $\{\hat{T}_{ab}^- | 1 \le a, b \le m\}$.

M2. THE QUANTUM DOUBLE GROUND STATE OF THE 2D SOLVABLE SPIN MODEL

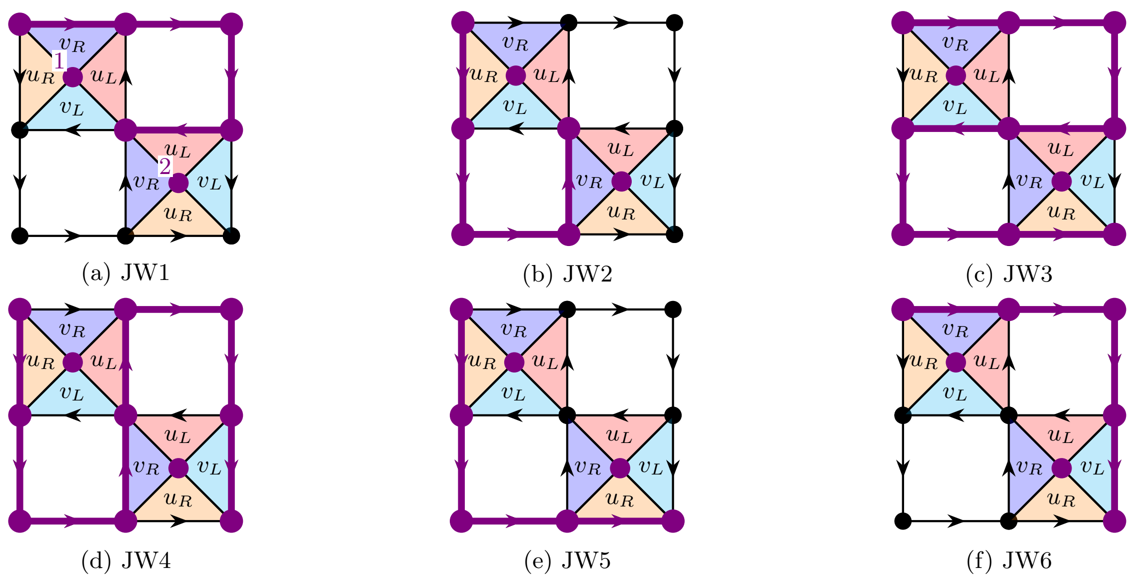


FIG. M1: The six JW strings constructed in the notebook Verify2DSpinModel.nb. We verify that the actions of JW1 and JW2 on the ground state are the same $\hat{W}_{1,bc} |G\rangle = \hat{W}_{2,bc} |G\rangle$, and the actions of JW3, JW4, JW5, and JW6 on the ground state are all equal to a δ-tensor, i.e., $\hat{W}_{bc} |G\rangle = \delta_{bc} |G\rangle$, for $\hat{W}_{bc}$ being JW3, JW4, JW5, and JW6, respectively. The labels 1, 2 indicate the first and second 64-dimensional tensor factor spaces in the computational representation of an operator acting on the system, for example, a local operator $\hat{O}$ acting on site 1 is represented as KroneckerProduct[O,IdentityMatrix[64]].

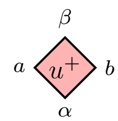
In Verify2DSpinModel.nb, we first load the Hopf algebra $\mathcal{H}_{64}$ from file, and verify that it satisfies the Hopf algebra axioms. The toolbox for finite dimensional Hopf algebras is stored in HopfAlgebraTools.wl. Note that in HopfAlgebraTools.wl we verify the Hopf algebra axioms step by step, for the related definitions such as prebialgebras and weak-Hopf algebras, see Ref. [3]. Then we verify that the set-theoretical R-matrix is indeed constructed from the universal R-matrix of $\mathcal{H}_{64}$ (see, e.g., Sec.8.1.2 of Ref. [2] for the general construction), via the relation

$$R = [(\rho \otimes \rho)\mathcal{R}]X, \tag{M1}$$

where $\mathcal{R}$ is the universal R-matrix and X is the swap operator.

Then we load the tensors u_L, u_R, v_L, v_R of the 2D solvable spin model, and verify they satisfy all the CRs in Fig. S5.

Here u is given as u^+ represented as a $mQ \times mQ$ matrix (where $m = 4, Q = 64$) as defined in Eq. (S31):

$[u^+]^{b\beta}_{a\alpha} =$  $, \quad 1 \le a, b \le m, \quad 1 \le \alpha, \beta \le Q,$

for $u = u_L, u_R, v_L, v_R$. The function VerifyCRuv[R, S, u, v] verifies that u, v satisfies the (R, S) CR, by directly checking the first line of Eq. (S33), and the function VerifyCG[R, X, uL, uR, vL, vR] verifies all the CRs in Fig. S5.

Next we use the tensors u_L, u_R, v_L, v_R to construct $\hat{H}_1$ on a 3×3 lattice, as shown in Fig. M1, using the definition in Eq. (32) in Methods. More precisely, we construct the matrix representation of $\hat{H}_1$ in the subspace where all black sites are in the state $|0\rangle$, as described in Fact. S5.2. Since there are only two white sites in the 3 lattice, the dimension of this subspace is $64^2 = 4096$. We then go on to verify the claim in Fact. S5.2 that $\hat{H}_1$ has a unique ground state on this 3×3 lattice.

The next part of the code verifies the properties of the MPO JW strings [defined in Eq. (34) in Methods] as shown in Fig. M1. Specifically, we verify that the action of the JW string on the ground state is path independent, as claimed in Fact. S5.2, and that the JW string connecting upper left and lower right corners acts on the ground state as a delta function $\hat{W}_{bc} |G\rangle = \delta_{bc} |G\rangle$, a property we used with only a partial proof in Sec. S5 D.

At last, we verify the relation of $\hat{H}_1$ to the definition of Kitaev's quantum double model in the literature [1], as we explain below Fact. S5.2. More precisely, we verify that $\hat{A}'_\nu, \hat{B}'_p$ that appears in the definition of Kitaev's quantum double model $\hat{H}'_1 = -\sum_\nu \hat{A}'_\nu - \sum_p \hat{B}'_p$ in Ref. [1] are indeed projection operators to the highest eigenstates of $\hat{A}_\nu, \hat{B}_p$, and that $\hat{H}_1$ and $\hat{H}'_1$ have the same ground state.

[1] O. Buerschaper, J. M. Mombelli, M. Christandl, and M. Aguado. A hierarchy of topological tensor network states. *J. Math. Phys.*, 54(1):12201, jan 2013.

[2] A. Klimyk and K. Schmüdgen. *Quantum groups and their representations*. Springer-Verlag, 1997.

[3] D. Molnar, Andras and de Alarcón, Alberto Ruiz and Garre-Rubio, José and Schuch, Norbert and Cirac, J Ignacio and Pérez-García. Matrix product operator algebras I: representations of weak Hopf algebras and projected entangled pair states. *arXiv Prepr. arXiv2204.05940*, 2022.