308 FDS6

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6.1. *The shortest path between two points on a curved surface, such as the surface of a sphere, is called a* ***geodesic****. To find a geodesic, one has first to set up an integral that gives the length of a path on the surface in question. This will always be similar to the integral (6.2) but may be more complicated (depending on the nature of the surface) and may involve different coordinates than x and y. To illustrate this, use spherical polar coordinates (r, θ, ϕ) to show that the length of a path joining two points on a sphere of radius R is*

$$L = R\int_{\theta_1}^{\theta_2} \sqrt{1+\sin^2\theta\phi'(\theta)^2}\mathrm{d}\theta \tag{1}$$

if (θ_1, ϕ_1) and (θ_2, ϕ_2) specify the two points and we assume that the path is expressed as $\phi = \phi(\theta)$.

Since we know that $\mathrm{d}\vec{\mathbf{r}} = \mathrm{d}r\hat{\mathbf{r}} + r\mathrm{d}\theta\hat{\theta} + r\sin\theta\mathrm{d}\phi\hat{\phi}$, we know that $\mathrm{d}\|\vec{\mathbf{r}}\|^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$. Since the path is $\phi = \phi(\theta)$, which is independent from the radius, we know that $\mathrm{d}r = 0$ and $r = R$ the radius of the sphere. Hence, now we have $\mathrm{d}\|\vec{\mathbf{r}}\|^2 = R^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$. By the convention that Taylor is using, I will rename my path variable as s, and thus, we have

$$\begin{aligned} \mathrm{d}s^2 &= R^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \\ &= R^2(1+\sin^2\phi'^2)\mathrm{d}\theta^2 \\ \mathrm{d}s &= R\sqrt{1+\sin^2\phi'^2}\mathrm{d}\theta, \end{aligned} \tag{2}$$

with this in mind, we can write our action integral,

$$\begin{aligned} L &= \int_1^2 \mathrm{d}s \\ &= \int_{\theta_1}^{\theta_2} R\sqrt{1+\sin^2\phi'^2}\mathrm{d}\theta \\ L &= R\int_{\theta_1}^{\theta_2} \sqrt{1+\sin^2\phi'(\theta)^2}\mathrm{d}\theta. \quad \text{Q.E.D.} \end{aligned} \tag{3}$$

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We know that we arrived at the correct answer because it matches the integral that we are trying to prove.

6.9. *Find the equation of the path joining the origin O to the point $P(1,1)$ in the xy plane that makes the integral $\int_O^P (y'^2 + yy' + y^2)\mathrm{d}x$ stationary.*

Since we are going from O to P, we know that the bounds for our action integral looks like

$$L = \int_0^1 (y'^2 + yy' + y^2)\mathrm{d}x. \tag{4}$$

By Euler-Lagrange equation, we have

$$\frac{\mathrm{d}f}{\mathrm{d}y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}f}{\mathrm{d}y'} = 0. \tag{5}$$

Since $f = y'^2 + yy' + y^2$, we know that $\frac{\mathrm{d}f}{\mathrm{d}y} = y' + 2y$, and $\frac{\mathrm{d}f}{\mathrm{d}y'} = 2y' + y$. Plug these into equation (5) we get

$$\begin{aligned} y' + 2y - \frac{\mathrm{d}}{\mathrm{d}x}(2y' + y) &= 0 \\ y' + 2y - 2y'' - y' &= 0 \\ y &= y''. \end{aligned} \tag{6}$$

Elementary algebra told us that the only solution to $y = y''$ is $y = A\sinh(x) + B\cosh(x)$.

Now, what's left to do is to plug in the know conditions, $O = (0,0)$ and $P = (1,1)$ to get the constants A and B.

$$\begin{aligned} 0 &= 0 + B \\ B &= 0. \end{aligned} \tag{7}$$

$$\begin{aligned} 1 &= A\sinh(1) \\ A &= 1/\sinh(1). \end{aligned} \tag{8}$$

Therefore, the solution is $y(x) = 1/\sinh(1)\sinh(x)$. Q.E.D.

We know this is correct because we never make mistakes.

6.11. *Find and describe the path $y = y(x)$ for which the integral $\int_{x_1}^{x_2} \sqrt{x}\sqrt{1+y'^2}\mathrm{d}x$ is stationary.*
First we find $\partial f/\partial y$ and $\partial f/\partial y'$.

$$\partial f/\partial y = 0. \tag{9}$$

$$\partial f/\partial y' = y'\sqrt{x}(1+y'^2)^{-1/2}. \tag{10}$$

According to Euler-Lagrange equation, we know that

$$\frac{\mathrm{d}}{\mathrm{d}x}\partial f/\partial y' = \partial f/\partial y = 0. \tag{11}$$

So we know that we can set the partial of y' to some constant k,

$$\begin{aligned}
\partial f/\partial y' = k &= y'\sqrt{x}(1+y'^2)^{-1/2} \\
k\sqrt{1+y'^2} &= \sqrt{x}y' \\
k^2(1+y'^2) &= xy'^2 \\
k^2 + k^2y'^2 &= xy'^2 \\
y'^2 &= \frac{k^2}{x-k^2} \\
y' &= \pm\frac{k}{\sqrt{x-k^2}}
\end{aligned} \tag{12}$$

Q.E.D.

We don't know if this makes sense...

6.16. *Use the result (6.41) of Problem 6.1 to prove that the geodesic (shortest path) between two given points on a sphere is a great circle. [Hint: The integrand $f(\phi, \phi', \theta)$ in (6.41) is independent of ϕ, so the Euler—Lagrange equation reduces to $\partial f/\partial\phi' = c$, a constant. This gives you ϕ' as a function of θ. You can avoid doing the final integral by the following trick: There is no loss of generality in choosing your z axis to pass through the point 1. Show that with this choice the constant c is necessarily zero, and describe the corresponding geodesics.]*

Take the partials with respect to ϕ and ϕ' so that we can use the Euler-Lagrange equation.

$$\partial f/\partial\phi = 0. \tag{13}$$

$$\partial f/\partial\phi' = \sin^2\phi'(1+\sin^2\phi'^2)^{-1/2}. \tag{14}$$

By equation (13), we know that $\frac{\mathrm{d}}{\mathrm{d}\theta}\partial f/\partial\phi' = 0$, so we know that $\partial f/\partial\phi' = C$ for some constant C. Since we are setting the z-axis through the first point, $\theta_1 = 0$ for that matter. So $\partial f/\partial\phi' = 0$ for $\theta = \theta_1$. Since it's true for one θ, it has to be true for all θ, so we have $\partial f/\partial\phi' = 0$. So essentially ϕ is not changing. If ϕ is not changing, the point has to go back to the same ϕ. Thus, it's a great circle like a latitude line.