308 notes 8.1-8.4

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November 15, 2021

This is a wild note

1 The Problem

Any isolated system is translational invariant. If the conservative is central, the potential energy is independent from the direction of the force.

We want to find the possible motions of two bodies whose Lagrangian is

$$\mathcal{L} = \frac{1}{2}m_1\dot{\vec{\mathbf{r}}}_1^2 + \frac{1}{2}m_2\dot{\vec{\mathbf{r}}}_2^2 - U(r). \tag{1}$$

2 CM and Relative Coordinates; Reduced Mass

$$\mu = \frac{m_1 m_2}{M} \equiv \frac{m_1 m_2}{m_1 + m_2}\text{[reduced mass]} \tag{2}$$

has the dimensions of mass and is called the reduced mass.

We can then rewrite the kinetic energy in terms of μ as

$$T = \frac{1}{2}M\dot{\vec{\mathbf{R}}}^2 + \frac{1}{2}\mu\dot{\vec{\mathbf{r}}}^2. \tag{3}$$

This remarkable result shows that the kinetic energy is the same as that of two "fictitious" particles, one of mass M moving with the spped of the CM, and the other of mass μ (the reduced mass) moving with the speed of the relative position $\vec{\mathbf{r}}$. Even more significant is the corresponding result for the Lagrangian:

$$\begin{aligned}\mathcal{L} = T - U &= \frac{1}{2}M\dot{\vec{\mathbf{R}}}^2 + (\frac{1}{2}\mu\dot{\vec{\mathbf{r}}}^2 - U(r)) \\ &= \mathcal{L}_{cm} + \mathcal{L}_{rel}.\end{aligned} \tag{4}$$

3 The Equations of Motion

Since the Lagrangian is independent of $\vec{\mathbf{R}}$, we know that $M\ddot{\vec{\mathbf{R}}} = 0$, so $\dot{\vec{\mathbf{R}}} =$ constant. We also know that $\vec{\mathbf{R}}$ is an ignorable coordinate.

For $\vec{\mathbf{r}}$, applying the Lagrange's equation, we know that $\mu\ddot{\vec{\mathbf{r}}} = -\nabla U\vec{\mathbf{r}}$.

3.1 The CM Reference Frame

Since we know that $\ddot{\vec{\mathbf{R}}}$ is zero, we know that the CM reference frame is an inertial frame. Choosing the CM reference frame, we know that the Lagrangian can be reduced to

$$\mathcal{L} = \mathcal{L}_{rel} = \frac{1}{2}\mu\dot{\vec{\mathbf{r}}}^2 - U(r). \tag{5}$$

3.2 Conservation of Angular Momentum

By algebra, we have

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \mu\dot{\vec{\mathbf{r}}}. \tag{6}$$

Since the angular momentum is conserved, and it doesn't change direction. We know that $\vec{\mathbf{r}}$ and $\dot{\vec{\mathbf{r}}}$ is in one plane and always the same plane.

3.3 The Two Equations of Motion

Some how $\vec{\mathbf{r}}^2 = (\dot{r}^2 + r^2\dot{\phi}^2)$.

Not sure what's going on.

4 The Equivalent One-Dimensional Problem

What the fuck is going on ???????