308 FDS4

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1. Evaluate the work done

$$W = \int_O^P \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_O^P (F_x \mathrm{d}x + F_y \mathrm{d}y) \tag{1}$$

BY THE TWO-DIMENSIONAL FORCE $\vec{\mathbf{F}} = (x^2, 2xy)$ ALONG THE THREE PATHS JOINING THE ORIGIN TO THE POINT $P = (1, 1)$ DEFINED AS FOLLOWS:

1.1. **This path goes along the x axis to $Q = (1, 0)$ and then straight up to P. (Divide the integral into two pieces, $\int_O^P = \int_O^Q + \int_Q^P$.)** First, we need to find the x component and y component of $\vec{\mathbf{F}}$. As one can easily observe from the formula, we can see that the $F_x = x^2$, $F_y = 2xy$. Since from O to Q there is no change in y direction, we know that $\int_O^Q \mathrm{d}y$ is 0. Since from Q to P there is no change in x direction, we know that $\int_Q^P \mathrm{d}x$ is 0. So now we can reduce our equation into

$$\begin{aligned} \int_O^P (F_x \mathrm{d}x + F_y \mathrm{d}y) &= \int_O^Q F_x \mathrm{d}x + \int_Q^P F_y \mathrm{d}y, \\ &= \int_0^1 x^2 \mathrm{d}x + [\int_0^1 2xy \mathrm{d}y]_{x=1} \\ &= \frac{1}{3} + 1 \\ &= \frac{4}{3}. \end{aligned} \tag{2}$$

The work done is $\frac{4}{3}$ unit energy.

1.2. **On this path $y = x^2$, and you can replace the term $\mathrm{d}y$ in (1) by $\mathrm{d}y = 2x\mathrm{d}x$ and convert the whole integral into an**

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integral over x. Be a lamb and listen to the instructions, we turn $\mathrm{d}y$ into $2x\mathrm{d}x$.

$$\begin{aligned}\int_O^P (F_x\mathrm{d}x + F_y\mathrm{d}y) &= \int_0^1 F_x\mathrm{d}x + \int_0^1 F_y\mathrm{d}y \\ &= \int_0^1 x^2\mathrm{d}x + \int_0^1 2x(x^2)2x\mathrm{d}x \\ &= \frac{1}{3} + \frac{4}{5} \\ &= \frac{17}{15}\end{aligned} \tag{3}$$

The work done is $\frac{17}{15}$ unit energy.

1.3. **This path is given parametrically as $x = t^3, y = t^2$. In this case rewrite $x, y, \mathrm{d}x$, and $\mathrm{d}y$ in (1) in terms of t and $\mathrm{d}t$, and convert the integral into an integral over t.** Since we know that $x = t^3, y = t^2$, we know that $\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2, \frac{\mathrm{d}y}{\mathrm{d}t} = 2t$, so $\mathrm{d}x = 3t^2\mathrm{d}t, \mathrm{d}y = 2t\mathrm{d}t$.

Now, plug these into equation (1), we get:

$$\begin{aligned}\int_O^P (F_x\mathrm{d}x + F_y\mathrm{d}y) &= \int_0^1 F_x\mathrm{d}x + \int_0^1 F_y\mathrm{d}y \\ &= \int_0^1 x^2\mathrm{d}x + \int_0^1 2xy\mathrm{d}y \\ &= \int_0^1 (t^3)^2 3t^2\mathrm{d}t + \int_0^1 2(t^3)(t^2)2t\mathrm{d}t \\ &= \int_0^1 3t^8 + 4t^6\mathrm{d}t \\ &= \left(\frac{1}{3}t^9 + \frac{4}{7}t^6\right)\Bigg|_0^1 \\ &= \frac{29}{21}\end{aligned} \tag{4}$$

The work done is $\frac{29}{21}$ unit energy.

Through the results from different paths, we can tell that this force is not conservative.

2. Find the partial derivatives with respect to x, y, and z of the following functions:

2.1. $f(x,y,z) = ax^2 + bxy + cy^2$**.** For the x partial,

$$\frac{\partial}{\partial x}\left(ax^2 + bxy + cy^2\right) = 2ax + by. \tag{5}$$

For the y partial,

$$\frac{\partial}{\partial y}\left(ax^2 + bxy + cy^2\right) = bx + 2cy. \tag{6}$$

For the z partial,

$$\frac{\partial}{\partial z}\left(ax^2 + bxy + cy^2\right) = 0. \tag{7}$$

2.2. $g(x,y,z) = \sin(axyz^2)$**.** For the x partial,

$$\frac{\partial}{\partial x}\left(\sin\left(axyz^2\right)\right) = ayz^2\cos\left(axyz^2\right). \tag{8}$$

For the y partial,

$$\frac{\partial}{\partial y}\left(\sin\left(axyz^2\right)\right) = axz^2\cos\left(axyz^2\right). \tag{9}$$

For the z partial,

$$\frac{\partial}{\partial z}\left(\sin\left(axyz^2\right)\right) = 2axyz\cos\left(axyz^2\right). \tag{10}$$

2.3. $h(x,y,z) = ae^{xy/z^2}$**.** For the x partial,

$$\frac{\partial}{\partial x}\left(ae^{xy/z^2}\right) = aye^{xy/z^2}/z^2. \tag{11}$$

For the y partial,

$$\frac{\partial}{\partial y}\left(ae^{xy/z^2}\right) = axe^{xy/z^2}/z^2. \tag{12}$$

For the z partial,

$$\frac{\partial}{\partial z}\left(ae^{xy/z^2}\right) = -2axye^{xy/z^2}/z^3. \tag{13}$$

No physical insight to this one. The only insight I have is that Taylor is such a ____ to put this many math question in their book.

3. Calculate the gradient ∇f of the following functions, $f(x,y,z)$:

3.1. $f = x^2 + z^3$**.**

$$\nabla x^2 + z^3 = 2x\hat{\mathbf{i}} + 3z^2\hat{\mathbf{k}}. \tag{14}$$

3.2. $f = ky$**, where** k **is a constant.**

$$\nabla ky = k\hat{\mathbf{j}}. \tag{15}$$

3.3. $f = r \equiv -\sqrt{x^2 + y^2 + z^2}$**.**

$$\nabla r = 1\hat{\mathbf{r}}. \tag{16}$$

3.4. $f = 1/r$**.**

$$\nabla 1/r = -1/r^2\hat{\mathbf{r}}. \tag{17}$$

Physical insight, use polar when angles are not involved.

4.

4.1. **Describe the surfaces defined by the equation $f =$ const, where $f = x^2 + 4y^2$.** It's an elliptical paraboloid, where it's skinnier in the y direction and fatter in the x direction.

4.2. **Using the results of Problem 4.18, find a unit normal to the surface $f = 5$ at the point $(1, 1, 1)$. In what direction should one move from this point to maximize the rate of change of f?** To find the maximum rate of change, we need to find the gradient vector of f at that point.

$$\nabla(x^2 + 4y^2) = 2x\hat{\mathbf{i}} + 8y\hat{\mathbf{j}} \tag{18}$$

Plug in the point $(1, 1, 1)$, we get $\nabla f = 2\hat{\mathbf{i}} + 8\hat{\mathbf{j}}$. This should be the direction of which it's moving in the maximum rate of change.

5. Which of the following forces is conservative? For those which are conservative, find the corresponding potential energy U, and verify by direct differentiation that $\vec{\mathbf{F}} = -\nabla U$.

5.1. $\vec{\mathbf{F}} = k(x, 2y, 3z)$ **where** k **is a constant.** We need to calculate the curl of $\vec{\mathbf{F}}$.

$$\begin{aligned} \boldsymbol{\nabla} \times \vec{\mathbf{F}} &= \nabla \times k(x, 2y, 3z) \\ &= 0 - 0\hat{\mathbf{i}} - 0 - 0\hat{\mathbf{j}} + 0 - 0\hat{\mathbf{k}} \\ &= 0 \end{aligned} \tag{19}$$

Since the curl is zero, we know that this force is conservative, thus we need to find the potential energy.

$$\begin{aligned} U &= -\int \vec{\mathbf{F}}(x, y, z) \mathrm{d}x \mathrm{d}y \mathrm{d}z \\ &= -k(x^2/2 + y^2 + 3z^2/2) \end{aligned} \tag{20}$$

Then, we need to verify if our U is the actual U, we need to take the derivative again...

$$\begin{aligned} -\boldsymbol{\nabla} U &= -(-kx, 2ky, 3kz) \\ &= k(x, 2y, 3z) \\ &= \vec{\mathbf{F}} \end{aligned} \tag{21}$$

No physical insight. Normal insight: this is meaningless...

5.2. $\vec{\mathbf{F}} = k(y, x, 0)$**.** We need to calculate the curl of $\vec{\mathbf{F}}$.

$$\begin{aligned} \boldsymbol{\nabla} \times \vec{\mathbf{F}} &= \nabla \times k(y, x, 0) \\ &= 0 - 0\hat{\mathbf{i}} - 0 - 0\hat{\mathbf{j}} + k - k\hat{\mathbf{k}} \\ &= 0 \end{aligned} \tag{22}$$

Since the curl is zero, we know that this force is conservative, thus we need to find the potential energy.

$$\begin{aligned} U &= -\int \vec{\mathbf{F}}(x, y, z) \mathrm{d}x \mathrm{d}y \mathrm{d}z \\ &= -kxy \end{aligned} \tag{23}$$

Then, we need to verify if our U is the actual U, we need to take the derivative again...

$$
\begin{aligned}
-\boldsymbol{\nabla} U &= -(-ky, -kx, 0) \\
&= k(y, x, 0) \\
&= \vec{\mathbf{F}}
\end{aligned} \tag{24}
$$

No physical insight. Normal insight: this is meaningless...

5.3. $\vec{\mathbf{F}} = k(-y, x, 0)$**.** We need to calculate the curl of $\vec{\mathbf{F}}$.

$$
\begin{aligned}
\boldsymbol{\nabla} \times \vec{\mathbf{F}} &= \nabla \times k(-y, x, 0) \\
&= 0 - 0\hat{\mathbf{i}} - 0 - 0\hat{\mathbf{j}} + k - (-k)\hat{\mathbf{k}} \\
&= 2k\hat{\mathbf{k}}
\end{aligned} \tag{25}
$$

Since the curl is non-zero, the force is non-conservative.

No physical insight. Normal insight: this is meaningless...