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1 A shell traveling with speed $\vec{\mathbf{v}}_o$ exactly horizontally and due north explodes into two equal-mass fragments. It is observed that just after the explosion one fragment is traveling vertically up with speed $\vec{\mathbf{v}}_o$. What is the velocity of the other fragment?

Since there is no external force mentioned, we are free to use conservation of momentum.

Call the mass before explosion $2m$, and the masses after explosion each has m. Spin our Cartesian plane so that the north is in $\hat{\mathbf{i}}$ direction. Express the momentum before the explosion:

$$\vec{\mathbf{P}}_i = 2m\vec{\mathbf{v}}_o = 2m\|\vec{\mathbf{v}}_o\|\hat{\mathbf{i}}. \tag{1}$$

The momentum after the explosion would be:

$$\vec{\mathbf{P}}_f = m\|\vec{\mathbf{v}}_o\|\hat{\mathbf{j}} + m\|\vec{\mathbf{v}}\|(a\hat{\mathbf{i}} + b\hat{\mathbf{j}}) = m\|\vec{\mathbf{v}}_o\|\hat{\mathbf{j}} + am\|\vec{\mathbf{v}}\|\hat{\mathbf{i}} + bm\|\vec{\mathbf{v}}\|\hat{\mathbf{j}}, \tag{2}$$

where $(a\hat{\mathbf{i}} + b\hat{\mathbf{j}})$ is a unit vector in the direction of $\vec{\mathbf{v}}$.

By conservation of momentum, we have these two equations being equal to each other. Since there's only $\hat{\mathbf{i}}$ direction momentum before the explosion, there cannot be any $\hat{\mathbf{j}}$ direction momentum after the explosion. Thus, it is not hard to see that

$$m\|\vec{\mathbf{v}}_o\|\hat{\mathbf{j}} + bm\|\vec{\mathbf{v}}\|\hat{\mathbf{j}} = 0, \tag{3}$$

and

$$2m\|\vec{\mathbf{v}}_o\|\hat{\mathbf{i}} = am\|\vec{\mathbf{v}}\|\hat{\mathbf{i}}. \tag{4}$$

Since we also know that $a^2 + b^2 = 1$, we can solve a and b with equations (3) and (4).

$$\begin{cases} \|\vec{\mathbf{v}}_o\| + b\|\vec{\mathbf{v}}\| = 0 \\ 2\|\vec{\mathbf{v}}_o\| = a\|\vec{\mathbf{v}}\| \\ a^2 + b^2 = 1 \end{cases}$$

$$\begin{cases} \|\vec{\mathbf{v}}_o\| = -b\|\vec{\mathbf{v}}\| \\ 2\|\vec{\mathbf{v}}_o\| = a\|\vec{\mathbf{v}}\| \\ a^2 + b^2 = 1 \end{cases}$$

$$\begin{cases} -2b = a \\ a^2 + b^2 = 1 \end{cases} \tag{5}$$

$$\begin{cases} a = 2/\sqrt{5}, b = -1/\sqrt{5} \quad \text{or} \\ a = -2/\sqrt{5}, b = 1/\sqrt{5} \end{cases}$$

Since the velocity before the explosion is

in the positive $\hat{\mathbf{i}}$ direction, so a should be positive.

$a = 2/\sqrt{5}, b = -1/\sqrt{5}.$

Plug a into equation (4), we get that $\|\vec{\mathbf{v}}\| = \sqrt{5}\|\vec{\mathbf{v}}_o\|$. Thus, the velocity of the other fragment after the explosion is $\sqrt{5}\|\vec{\mathbf{v}}_o\|(2/\sqrt{5}\hat{\mathbf{i}} - 1/\sqrt{5}\hat{\mathbf{j}})$.

2 Consider an elastic collision between two equal mass bodies, one of which is initially at rest. Let their velocities be $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2 = 0$ before the collision, and $\vec{\mathbf{v}}_1'$ and $\vec{\mathbf{v}}_2'$ after. Write down the vector equation representing conservation of momentum and the scalar equation which expresses that the collision is elastic. Use these to prove that the angle between $\vec{\mathbf{v}}_1'$ and $\vec{\mathbf{v}}_2'$ is 90°.

Since this is an elastic collision, we are free to use both conservation of momentum and conservation of energy.

By the conservation of momentum, we have

$$\vec{\mathbf{P}}_i = m\vec{\mathbf{v}}_1 + m\vec{\mathbf{v}}_2 = m\vec{\mathbf{v}}_1 = m\vec{\mathbf{v}}_1' + m\vec{\mathbf{v}}_2' = \vec{\mathbf{P}}_f. \tag{6}$$

Since mass is the same for both bodies, we can simplify equation (6) to

$$\vec{\mathbf{v}}_1 = \vec{\mathbf{v}}_1' + \vec{\mathbf{v}}_2'. \tag{7}$$

As one may be clueless to what one could do with equation (7), it is the perfect timing to look at the conservation of energy.

$$K_i = \frac{1}{2}m\|\vec{\mathbf{v}}_1\|^2 + \frac{1}{2}m\|\vec{\mathbf{v}}_2\|^2 = \frac{1}{2}m\|\vec{\mathbf{v}}_1\|^2 = \frac{1}{2}m\|\vec{\mathbf{v}}_1'\|^2 + \frac{1}{2}m\|\vec{\mathbf{v}}_2'\|^2 = K_f \tag{8}$$

Since mass is the same and 1/2 is a constant, we can simplify equation (8) to

$$\|\vec{\mathbf{v}}_1\|^2 = \|\vec{\mathbf{v}}_1'\|^2 + \|\vec{\mathbf{v}}_2'\|^2 \tag{9}$$

As one should be acquainted to dot product between vectors by previous problem sets, we can tell that the left hand side of equation (9) is a dot product between the vector $\vec{\mathbf{v}}_1$ and itself. Hence, the next reasonable move is to dot both sides of equation (7) and compare the result with equation (9).

$$\|\vec{\mathbf{v}}_1\|^2 = \|\vec{\mathbf{v}}_1'\|^2 + \|\vec{\mathbf{v}}_2'\|^2 + 2(\vec{\mathbf{v}}_1' \cdot \vec{\mathbf{v}}_2') \tag{10}$$

Comparing equations (9) and (10), we can see that the last term in equation (10) is 0. Since by the first problem set, we've proven that the dot product between two vectors is zero if and only if the vectors are orthogonal to each other, we know that $\vec{\mathbf{v}}_1'$ and $\vec{\mathbf{v}}_2'$ has a 90 degree angle in between.

3 To illustrate the use of a multistage rocket consider the following:

3.1 (a) A certain rocket carries 60% of its initial mass as fuel. (That is, the mass of fuel is 0.6 m_o.) What is the rocket's final speed, accelerating from rest in free space, if it burns all its fuel in a single stage? Express your answer as a multiple of v_{ex}.

By equation (3.8) from Taylor, we get

$$v - v_o = v_{ex} \ln\left(\frac{m_o}{m}\right), \tag{11}$$

where v_o is the initial speed, m_o is the initial mass. Since we know that the rocket starts from ground, so the initial speed must be zero. Since we know that the fuel mass is $0.6m_o$, so the mass left is $0.4m_o$. Plug all of these know information into equation (11), we have

$$v = v_{ex} \ln\left(\frac{1}{0.4}\right). \tag{12}$$

As a lazy physicist, I will leave this as my final answer.

3.2 (b) Suppose instead it burns the fuel in two stages as follows: In the first stage it burns a mass 0.3 m_o of fuel. It then jettisons the first-stage fuel tank, which has a mass of 0.1 m_o, and then burns the remaining 0.3 m_o of fuel. Find the final speed in this case, assuming the same value of v_{ex} throughout and compare.

In the first stage, we repeat what ever is in part (a), but with a final mass of $0.7m_o$, since this time only 30% mass amount of fuel is burned.

$$v = v_{ex} \ln\left(\frac{1}{0.7}\right). \tag{13}$$

Before we go to the second stage, there's an intermediate stage which the rocket jettisons a fuel tank with mass $0.1m_o$. One would wish to apply part (a) again to this stage, yet unfortunately the initial speed is not zero anymore, it is $v_{ex}\ln(0.7)$. And our initial mass is not m_o anymore, but $0.7m_o$. With these additional information, we plug into equation (3.8) from Taylor

$$\begin{aligned} v - v_{ex}\ln\left(\frac{1}{0.7}\right) &= v_{ex}\ln\left(\frac{7}{6}\right), \\ v &= v_{ex}\left(\ln\left(\frac{1}{0.7}\right) + \ln\left(\frac{7}{6}\right)\right). \end{aligned} \tag{14}$$

Now, it's a good time to get into the second stage since we now know the initial speed for the second stage is $v_{ex}\left(\ln\left(\frac{1}{0.7}\right) + \ln\left(\frac{7}{6}\right)\right)$, and the initial mass for the second stage is $0.6m_o$.

$$\begin{aligned} v - v_{ex}\left(\ln\left(\frac{1}{0.7}\right) + \ln\left(\frac{7}{6}\right)\right) &= v_{ex}\ln\left(\frac{0.6m_o}{0.3m_o}\right), \\ v &= v_{ex}\left(\ln\left(\frac{1}{0.7}\right) + \ln\left(\frac{7}{6}\right) + \ln(2)\right). \end{aligned} \tag{15}$$

Looking at the change in speed between stage one and before stage one and the change in speed between stage two and before stage two, we can see that the latter one is greater than the first one, since the initial mass is lighter than the first one. The insight is that also the change in mass is the same, but that doesn't imply the change in speed is also the same. We need to look at the change in the percentage of mass.

4 The masses of the earth and sun are $M_e \approx 6.0 \times 10^{24}$ and $M_S \approx 2.0 \times 10^{30}$ (both in kg) and their center—to—center distance is 1.5×10^8 km. Find the position of their CM and comment. (The radius of the sun is $R_S \approx 7.0 \times 10^5$ km.)

Choose the center of Sun as the origin, O. Using equation (3.9) from Taylor, we can calculate the center of mass for the earth sun system

$$\vec{\mathbf{R}} = \frac{1}{M}\sum_{\alpha} m_\alpha \cdot \vec{\mathbf{r}}_\alpha = \frac{1}{M_e + M_S}(M_e \cdot R_S + M_s \cdot 0) = \frac{M_e \cdot R_S}{M_e + M_S}. \tag{16}$$

Plug in known values for mass of the earth and sun and the radius of the sun, we get

$$\frac{6.0 \times 10^{24} \cdot 7.0 \times 10^5}{6.0 \times 10^{24} + 2.0 \times 10^{30}} = 2.09999 \text{ km}. \tag{17}$$

So we know that the center of mass is about 2 km away from the center of sun.

I have to say that this result doesn't surprise me since sun has a mass that is ten to the order six bigger than the earth. It makes sense to have the center of mass really close to the center of sun.

5 A particle moves under the influence of a central force directed toward a fixed origin O.

5.1 (a) Explain why the particle's angular momentum about O is constant.

Since the central force is directed toward the origin, the arc of that force is zero, so the torque exerted is zero. With no external torque, the time derivative of the angular momentum is zero, thus the angular momentum should be constant.

5.2 (b) Give in detail the argument that the particle's orbit must lie in a single plane containing O.

Suppose the particle's orbit lies in two planes containing O, and these two plane are not the same. Then we know that there has to be a section of the orbit which is the path the particle takes to travel between the two planes. In order for that to happen, the force must be exerted to the particle. However, if the force is exerted to the particle, whose position is not the origin O, there has to be a torque, thus the angular momentum would not conserve. Therefore, we cannot have the orbit of the particle to be in more than one plane.

6 Show that the moment of inertia of a uniform solid sphere rotating about a diameter is $\frac{2}{5}MR^2$. The sum (3.31) must be replaced by an integral, which is easiest in spherical polar coordinates, with the axis of rotation taken to be the z axis. The element of volume is $\mathrm{d}V = r^2\mathrm{d}r\sin(\theta)\mathrm{d}\theta\mathrm{d}\phi$. (Spherical polar coordinates are defined in Section 4.8. If you are not already familiar with these coordinates, you should probably not try this problem yet.)

We are given the equation (3.31) from Taylor. The integral form of that equation would be

$$I = \int \rho^2 \mathrm{d}m. \tag{18}$$

Let the angle in the x, y plane be ϕ relative to positive y axis, and the angle relative positive z axis be θ. We know that ρ can be expressed as $r\sin(\theta)$, and the volume for a sphere is $\frac{4}{3}\pi R^2$ where R is the radius of the sphere. So the density of the sphere would be $M/V = \frac{3M}{4\pi R^3}$. So $\mathrm{d}m = \frac{3M}{4\pi R^3}\mathrm{d}V$. Since $\mathrm{d}V = r^2\mathrm{d}r\sin(\theta)\mathrm{d}\theta\mathrm{d}\phi$, we get $\mathrm{d}m = \frac{3M}{4\pi R^3}r^2\mathrm{d}r\sin(\theta)\mathrm{d}\theta\mathrm{d}\phi$.

One more important thing to realize before we write out the integral, is that the bounds for different variables are different. For $\mathrm{d}r$ the bound is from zero to R; for $\mathrm{d}\theta$, the bound is from 0 to π; for $\mathrm{d}\phi$, the bound is from 0 to 2π.

Now we are good to go.

$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} (r\sin(\theta))^2 \frac{3M}{4\pi R^3} r^2\mathrm{d}r\sin(\theta)\mathrm{d}\theta\mathrm{d}\phi \tag{19}$$

$$
\begin{aligned}
I &= \frac{3M}{4\pi R^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^4 \mathrm{d}r \sin^3(\theta)\mathrm{d}\theta\mathrm{d}\phi \\
&= \frac{3M}{4\pi R^3}\int_0^{2\pi}\int_0^{\pi}\frac{R^5}{5}\sin(\theta)^3\mathrm{d}\theta\mathrm{d}\phi \\
&= \frac{3M}{4\pi R^3}\frac{R^5}{5}\int_0^{2\pi}\frac{4}{3}\mathrm{d}\phi \\
&= \frac{3M}{4\pi R^3}\frac{R^5}{5}\frac{4}{3}2\pi \\
I &= \frac{2R^2M}{5}
\end{aligned} \tag{20}
$$

And...we are done.

Though the work is rough, but we made it.

This problem is meaningless other than making people feel bad about integrating a really nasty integral.