308 PCS1

Tony Deng

September 14, 2021

1 2

Vectors at play:

$$\vec{\mathbf{b}} = (1, 2, 3) = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}} \tag{1}$$

$$\vec{\mathbf{c}} = (3, 2, 1) = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 1\hat{\mathbf{k}} \tag{2}$$

1.1 $\vec{\mathbf{b}} + \vec{\mathbf{c}}$

For vector addition, we add each component separately.

$$\vec{\mathbf{b}} + \vec{\mathbf{c}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}} + 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 1\hat{\mathbf{k}} = 4\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}} \tag{3}$$

This result has no physical meaning other than being the sum of two vectors. This result is one of the basic vector operations required to conduct more complex calculations in a physic course.

1.2 $5\vec{\mathbf{b}} - 2\vec{\mathbf{c}}$

For vector scalar multiplication, we multiply each component with the scalar; for vector subtraction, we subtract each component separately as we handled the addition.

$$\begin{aligned} 5\vec{\mathbf{b}} - 2\vec{\mathbf{c}} &= 5(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) - 2(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 1\hat{\mathbf{k}}) \\ &= 5\hat{\mathbf{i}} + 10\hat{\mathbf{j}} + 15\hat{\mathbf{k}} - 6\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}} \\ &= -\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 13\hat{\mathbf{k}} \end{aligned} \tag{4}$$

This result has no physical meaning other than being the difference of two scalar multiplied vectors. This result contains two of the basic vector operations required to conduct more complex calculations in a physics course.

1.3 $\vec{\mathbf{b}} \cdot \vec{\mathbf{c}}$

For vector dot product, we multiply the components in two vectors, two at a time, get scalars, then add the scalars together.

$$\begin{aligned} \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) \cdot (3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 1\hat{\mathbf{k}}) \\ = (3 \cdot 1)(\hat{\mathbf{i}} \cdot \hat{\mathbf{i}}) + (2 \cdot 3)(\hat{\mathbf{j}} \cdot \hat{\mathbf{i}}) + (3 \cdot 3)(\hat{\mathbf{k}} \cdot \hat{\mathbf{i}}) + (1 \cdot 2)(\hat{\mathbf{i}} \cdot \hat{\mathbf{j}}) + (1 \cdot 1)(\hat{\mathbf{i}} \cdot \hat{\mathbf{k}}) \\ +(2 \cdot 2)(\hat{\mathbf{j}} \cdot \hat{\mathbf{j}}) + (2 \cdot 1)(\hat{\mathbf{j}} \cdot \hat{\mathbf{k}}) + (3 \cdot 2)(\hat{\mathbf{k}} \cdot \hat{\mathbf{j}}) + (3 \cdot 1)(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}) \end{aligned} \tag{5}$$

By the definition of dot product, if the two vectors being multiplied is parallel, the result is the product of their magnitudes; since $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are unit vectors, their magnitudes are 1. So, we know that $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$.

By the definition of dot product, if the two vectors being multiplied is orthogonal, the result is 0. So, $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0$.

Therefore, equation (5) can be simplified

$$\vec{\mathbf{b}} \cdot \vec{\mathbf{c}} = 3 + 0 + 0 + 0 + 4 + 0 + 0 + 3 = 10 \tag{6}$$

This result has no physical meaning other than being the product of two dot product-ed vectors. This operation is one of the basic vector operations required to conduct more complex calculations in a physics course.

1.4 $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$

The cross product of two vectors is a bit more complicated. As we write the vectors in a matrix form,

$$\begin{array}{ccc} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{array} \tag{7}$$

We then calculate the determinant of this matrix. If one is confused with the concept of determinant, the easiest way to understand it is to see it as a property of a matrix. (The hardest, most abstract, way of understanding determinant is to see as the steps needed to swap a shape with the matrix operation...)

Anyhow...here is how to calculate the determinant of a 3 by 3 matrix

$$\begin{aligned} \vec{\mathbf{b}} \times \vec{\mathbf{c}} &= \hat{\mathbf{i}}(2 \cdot 1 - 3 \cdot 2) - \hat{\mathbf{j}}(1 \cdot 1 - 3 \cdot) + \hat{\mathbf{k}}(1 \cdot -2 \cdot 3) \\ &= \hat{\mathbf{i}}(2 - 6) - \hat{\mathbf{j}}(1 - 9) + \hat{\mathbf{k}}(2 - 6) \\ &= -4\hat{\mathbf{i}} + 8\hat{\mathbf{j}} - 4\hat{\mathbf{k}} \end{aligned} \tag{8}$$

This result surprisingly has a very important physical meaning, that is the resulting vector is orthogonal to both vectors being cross producted. This result is significant in future physics studies since physics involves countless cross product between vectors...

2 6

Vectors at play:

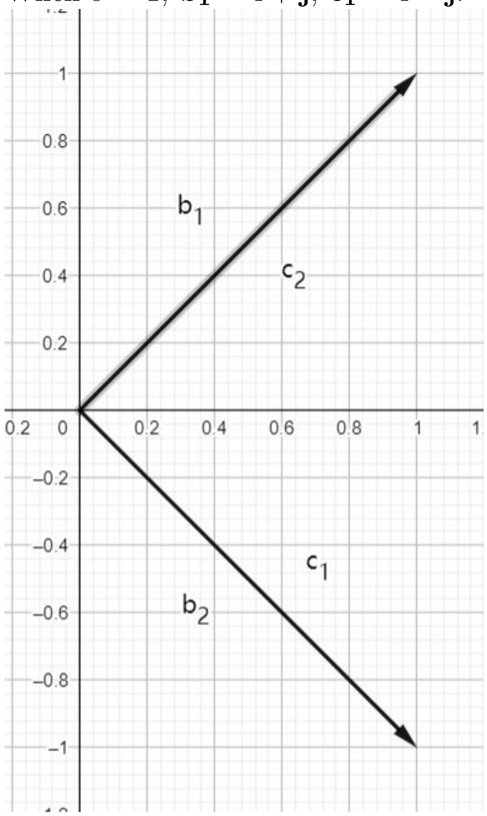
$$\vec{\mathbf{b}} = \hat{\mathbf{i}} + s\hat{\mathbf{j}} \tag{9}$$

$$\vec{\mathbf{c}} = \hat{\mathbf{i}} - s\hat{\mathbf{j}} \tag{10}$$

We want to find out under what s value will these two vectors be orthogonal. Since the dot product of two orthogonal vectors is 0, we can dot $\mathbf{b}, \vec{\mathbf{c}}$ and setting the result to be 0 to get s.

$$\begin{aligned} \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} = (\hat{\mathbf{i}} + s\hat{\mathbf{j}}) \cdot (\hat{\mathbf{i}} - s\hat{\mathbf{j}}) &= 0 \\ 0 &= 1 - s^2 \\ s &= \pm 1 \end{aligned} \tag{11}$$

When $s = 1$, $\vec{\mathbf{b_1}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\vec{\mathbf{c_1}} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$. When $s = -1$, $\vec{\mathbf{b_2}} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$, $\vec{\mathbf{c_2}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$.



It is clear from the sketch that the angles between the vectors and the x-axis is 45 degree, since magnitude in $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are the same. So, it is reasonable to conclude that the angle between the vectors is 90 degree.

This result has no physical meaning other than being two vectors. This result is somewhat important to future studies in physics since we do need to calculate unknown components of vectors using orthogonality.

3 7

Pick arbitrary vectors $\vec{\mathbf{r}}, \vec{\mathbf{s}}$ in the 3D space. Construct our Cartesian system with $\vec{\mathbf{r}}$ lying on the positive $\hat{\mathbf{i}}$ direction, and $\vec{\mathbf{s}}$ in the x-y plane. Then we know that

$$\begin{aligned}\vec{\mathbf{r}} &= r_x\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}} \\ \vec{\mathbf{s}} &= s_x\hat{\mathbf{i}} + s_y\hat{\mathbf{j}} + 0\hat{\mathbf{k}}\end{aligned} \tag{12}$$

The cosine of the angle between the two vectors will be determined by the angle between $\vec{\mathbf{s}}$ and the x-axis. So, the cosine is the determined by the x and y component of $\vec{\mathbf{s}}$.

$$\cos(\theta) = \frac{s_x}{\sqrt{s_x^2 + s_y^2}} \tag{13}$$

Now, we can show the dot product is

$$\begin{aligned} \|\vec{\mathbf{r}}\|\|\vec{\mathbf{s}}\|\cos(\theta) &= r_x\sqrt{s_x^2 + s_y^2}\frac{s_x}{\sqrt{s_x^2 + s_y^2}} \\ &= r_x \cdot s_x \end{aligned} \tag{14}$$

Using the other method, by taking each component of the vectors, we see that

$$\begin{aligned} \sum r_i \cdot s_i &= r_x \cdot s_x + 0 \cdot s_y + 0 \cdot 0 \\ &= r_x \cdot s_x \end{aligned} \tag{15}$$

This result has no physical meaning. However, the mathematical meaning behind this is really important. It shows that the dot product cannot only be view as a projection of one vector on the other, but also the component-wise multiplications. Later, both explanations of dot product can be useful in future physics studies.

4 15

Pick arbitrary vectors $\vec{\mathbf{r}}, \vec{\mathbf{s}}$ in the 3D space. Construct our Cartesian system with $\vec{\mathbf{r}}$ lying on the positive $\hat{\mathbf{i}}$ direction, and $\vec{\mathbf{s}}$ in the x-y plane. Then we know that

$$\begin{aligned} \vec{\mathbf{r}} &= r_x\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}} \\ \vec{\mathbf{s}} &= s_x\hat{\mathbf{i}} + s_y\hat{\mathbf{j}} + 0\hat{\mathbf{k}} \end{aligned} \tag{16}$$

Using the technique we found in equation (8), we can calculate the cross product.

$$\begin{aligned} \vec{\mathbf{p}} = \vec{\mathbf{r}} \times \vec{\mathbf{s}} = (0 \cdot s_x - 0 \cdot s_y)\hat{\mathbf{i}} - (0 \cdot s_x - r_x \cdot 0)\hat{\mathbf{j}} + (r_x \cdot s_y - 0 \cdot s_x)\hat{\mathbf{k}} \\ = r_x s_y\hat{\mathbf{k}} \end{aligned} \tag{17}$$

Using the technique in equation (11) to see if $\vec{\mathbf{p}}$ is perpendicular to $\vec{\mathbf{r}}$ and $\vec{\mathbf{s}}$.

$$\vec{\mathbf{r}} \cdot \vec{\mathbf{p}} = r_x \cdot 0 + 0 \cdot 0 + 0 \cdot r_x s_y = 0 \tag{18}$$

$$\vec{\mathbf{s}} \cdot \vec{\mathbf{p}} = s_x \cdot 0 + s_y \cdot 0 + 0 \cdot r_x s_y = 0 \tag{19}$$

So $\vec{\mathbf{p}}$ is orthogonal to both $\vec{\mathbf{r}}$ and $\vec{\mathbf{s}}$.

Calculate $\sin(\theta)$ using the same logic in equation (13) as we were calculating for the $\cos(\theta)$.

$$\sin(\theta) = \frac{s_y}{\sqrt{s_x^2 + s_y^2}} \tag{20}$$

Compare the magnitude of $\|\vec{\mathbf{r}}\|\|\vec{\mathbf{s}}\| \sin(\theta)$ and the magnitude of $\vec{\mathbf{p}}$.

$$\begin{aligned} \|\vec{\mathbf{r}}\|\|\vec{\mathbf{s}}\| \sin(\theta) &= r_x \cdot \sqrt{s_x^2 + s_y^2} \cdot \frac{s_y}{\sqrt{s_x^2 + s_y^2}} \\ &= r_x s_y = \sqrt{0^2 + 0^2 + (r_x s_y)^2} = \|\vec{\mathbf{p}}\| \end{aligned} \tag{21}$$

This result has no physical meaning but a very significant mathematical meaning. The cross product is always orthogonal to the two vectors being cross product-ed and the magnitude is always equal to the magnitudes of the two vectors multiplied together with the sine of the angle in between. This usage can be very important to future physics studies in a way that it allows us to calculate the cross product with or without a Cartesian system.

5 19

We want to show the following equation is true.

$$\frac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{a}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{r}})] = \dot{\vec{\mathbf{a}}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{r}}) \tag{22}$$

We start with the left hand side and apply product rule to the time differentiation sign.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{a}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{r}})] &= \dot{\vec{\mathbf{a}}} \cdot \vec{\mathbf{v}} \times \vec{\mathbf{r}} + \vec{\mathbf{a}} \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\vec{\mathbf{v}} \times \vec{\mathbf{r}}) \\ &= \dot{\vec{\mathbf{a}}} \cdot \vec{\mathbf{v}} \times \vec{\mathbf{r}} + \vec{\mathbf{a}} \cdot (\vec{\mathbf{a}} \times \vec{\mathbf{v}}) \end{aligned} \tag{23}$$

Let $\vec{\mathbf{b}} = \vec{\mathbf{a}} \times \vec{\mathbf{v}}$. By the conclusion we found in equation (18) and equation (19), we see that $\vec{\mathbf{b}} \perp \vec{\mathbf{a}}$. By the properties of dot product we had in equation (11), we know that $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ will be 0. So, the second term on the right hand side will be 0.

$$\frac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{a}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{r}})] = \dot{\vec{\mathbf{a}}} \cdot \vec{\mathbf{v}} \times \vec{\mathbf{r}} \tag{24}$$

This problem has no physical meaning. Nevertheless, the moral behind this problem is worth extend to future physics studies, such that we need to keep in mind under what circumstances we have 0 out of a dot product and the product of a cross product is always orthogonal to the original vectors.

6 24

We have a first order differential equation.

$$\frac{\mathrm{d}f}{\mathrm{d}t} = f \tag{25}$$

We should solve this by separation of variables.

$$\begin{aligned} \frac{df}{f} &= dt \\ \int \frac{df}{f} &= \int dt \\ \ln(f) &= t + c \\ f &= e^{t+c} \end{aligned} \tag{26}$$

Since this is a first order ordinary differential equation, thus the general solution only has one constant.

7 28

Below is equation 1.25 from Taylor.

$$\vec{\mathbf{F}}_\alpha = \sum_{\beta \neq \alpha} \vec{\mathbf{F}}_{\alpha\beta} + \vec{\mathbf{F}}_\alpha^{ext} \tag{27}$$

Suppose we have 3 particles.

$$\begin{aligned} \vec{\mathbf{F}}_1 &= \vec{\mathbf{F}}_{12} + \vec{\mathbf{F}}_{13} + \vec{\mathbf{F}}_1^{ext} \\ \vec{\mathbf{F}}_2 &= \vec{\mathbf{F}}_{21} + \vec{\mathbf{F}}_{23} + \vec{\mathbf{F}}_2^{ext} \\ \vec{\mathbf{F}}_3 &= \vec{\mathbf{F}}_{31} + \vec{\mathbf{F}}_{32} + \vec{\mathbf{F}}_3^{ext} \end{aligned} \tag{28}$$

Since the time derivative of momentum is equal to the net force, we have

$$\begin{aligned} \dot{\vec{\mathbf{P}}}_1 &= \vec{\mathbf{F}}_1 = \vec{\mathbf{F}}_{12} + \vec{\mathbf{F}}_{13} + \vec{\mathbf{F}}_1^{ext} \\ \dot{\vec{\mathbf{P}}}_2 &= \vec{\mathbf{F}}_2 = \vec{\mathbf{F}}_{21} + \vec{\mathbf{F}}_{23} + \vec{\mathbf{F}}_2^{ext} \\ \dot{\vec{\mathbf{P}}}_3 &= \vec{\mathbf{F}}_3 = \vec{\mathbf{F}}_{31} + \vec{\mathbf{F}}_{32} + \vec{\mathbf{F}}_3^{ext} \end{aligned} \tag{29}$$

The time derivative of the total momentum should equal to the sum of all of the time derivative of momentum.

$$\begin{aligned}\dot{\vec{\mathbf{P}}} &= \dot{\vec{\mathbf{P}}}_1 + \dot{\vec{\mathbf{P}}}_2 + \dot{\vec{\mathbf{P}}}_3 \\ &= \vec{\mathbf{F}}_{12} + \vec{\mathbf{F}}_{13} + \vec{\mathbf{F}}_1^{ext} + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_{21} + \vec{\mathbf{F}}_{23} + \vec{\mathbf{F}}_2^{ext} + \vec{\mathbf{F}}_3 + \vec{\mathbf{F}}_{31} + \vec{\mathbf{F}}_{32} + \vec{\mathbf{F}}_3^{ext}\end{aligned} \tag{30}$$

Since $\vec{\mathbf{F}}_{\alpha\beta} = -\vec{\mathbf{F}}_{\beta\alpha}$, we can cancel the forces in equation (30).

$$\dot{\vec{\mathbf{P}}} = \vec{\mathbf{F}}_1^{ext} + \vec{\mathbf{F}}_2^{ext} + \vec{\mathbf{F}}_3^{ext} = \vec{\mathbf{F}}^{ext} \tag{31}$$

This result surprisingly has a physical meaning, that is the time derivative of the total momentum of the system equals to the net force experienced by the system. This is important for future physics studies, especially for a multi-bodies system.

8 35

Firstly, we need to break the net force experienced by the golf ball into different components.

$$\vec{\mathbf{F}} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}} \tag{32}$$

Since when flying, the golf ball only experiences gravity, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ directions have no force. By Newton's second law, we can write

$$\begin{aligned}0 &= m\ddot{x}\hat{\mathbf{i}} \\ 0 &= m\ddot{y}\hat{\mathbf{j}} \\ m\vec{\mathbf{g}} &= m\ddot{z}\hat{\mathbf{k}}\end{aligned} \tag{33}$$

Since the mass of the golf ball is nonzero, we can divide both side by the mass.

$$\vec{\mathbf{g}} = \ddot{z}\hat{\mathbf{k}} \tag{34}$$

Since the gravity is always downward vertically, we can cancel the direction vector on both sides.

$$-g = \ddot{z} \tag{35}$$

Integrate on both sides,

$$\begin{aligned}-\int g dt &= \int \ddot{z} dt \\ -gt + v_0\sin(\theta) &= \dot{z}\end{aligned} \tag{36}$$

The right hand side of equation (36) is the velocity in the vertical direction. If we set the velocity in the vertical direction to 0, we can get the amount of time required for the ball to fly to the highest point, then double it to get the total time the ball flies.

$$\begin{aligned} -gt + v_0 \sin(\theta) &= 0 \\ t &= \frac{v_0}{g} \sin(theta) \\ t_{total} = 2t &= \frac{2v_0}{g} \sin(theta) \end{aligned} \tag{37}$$

Still looking at equation (36), if we keep integrating, we know that we can get the ball's position over time in the vertical direction.

$$\begin{aligned} \int -gt + v_0 \sin(\theta) dt &= \int \dot{z} dt \\ -gt^2 + v_0 \sin(\theta) t + z_0 &= z(t) \\ \text{since ball initially starts from the ground} \\ -gt^2 + v_0 \sin(\theta) t &= z(t) \end{aligned} \tag{38}$$

Similarly, we can find the ball's position in the x(due east) direction as a function of time.

$$\begin{aligned} \dot{x} = v_x &= v_0 \cos(\theta) \\ \int \dot{x} dt &= \int v_0 \cos(\theta) dt \\ \text{since the ball starts at the origin} \\ x(t) &= v_0 \cos(\theta) t \end{aligned} \tag{39}$$

Since there's no force nor velocity in the y direction, $y(t) = 0$. Now, we can combine our functions into a position vector.

$$\begin{aligned} \vec{\mathbf{r}}(t) &= (x(t), 0, z(t)) \\ &= (v_0 \cos(\theta) t, 0, -gt^2 + v_0 \sin(\theta) t) \end{aligned} \tag{40}$$

Last but not least, we can calculate the distance traveled in the x direction,

$$\begin{aligned} x(t_{total}) - x(0) &= v_0 \cos(\theta) t_{total} - 0 \\ &= 2v_0^2 \frac{\cos(\theta) \sin(\theta)}{g} \end{aligned} \tag{41}$$

The physical meaning of this result is all over the place since it's a physical example of using component to solve for the motion of objects. It is important to physics studies, because this is one of the simplified version of what we will see in the future.