308 PCS4B

TONY DENG

1. Figure 4.25 shows a child's toy, which has the shape of a cylinder mounted on top of a hemisphere. The radius of the hemisphere is R and the CM of the whole toy is at a height h above the floor.

1.1. **(a) Write down the gravitational potential energy when the toy is tipped to an angle θ from the vertical. [You need to find the height of the CM as a function of θ. It helps to think first about the height of the hemisphere's center O as the toy tilts.]** When tilted, the height of the CM above floor would be $R + (h - R)\cos(\theta)$. So the potential energy as a function of θ would be

$$U(\theta) = mg(R + (h - R)\cos(\theta)). \tag{1}$$

Now the question is how do I know that I'm right?

Apart from the fact that I rarely make mistakes, the key here is the height of the CM when $\theta \neq 0$.

Earlier I said that the height is $R+(h-R)\cos(\theta)$ without giving any explanation. (And now I'm giving one...) Although a diagram would make it much more clear to see why this is true, I'm using LaTeXto write this copy and too lazy to attach a picture into it, so from now on, you just have to trust every single word I said.

When the toy is tilted, it is essentially rotating about the center of the hemisphere, and since the CM is above the hemisphere initially, it is reasonable to believe that the height of the CM is never smaller than the radius of the hemisphere. (Think about this, if it height of the CM is smaller than the radius, the upper body of the toy has to go into the floor...) This explained the R term in the height, which indicates that it is always greater than R.

With this in mind, now the question is how much taller is the CM than the center of the hemisphere?

Since we know that initially the CM is $(h - R)$ amount taller than the center when θ is 0, when it tilted, the angle of tiltedness is θ as

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well. The $(h - R)$ line segment becomes the hypotenuse of the right triangle with respect to the vertical line. Since we are looking for the change in height which is in the vertical direction, it is essentially the hypotenuse multiplies with $\cos(\theta)$. This explained the $(h - R)\cos(\theta)$ term in the height.

Combine the two terms, we've got the height to be $R+(h-R)\cos(\theta)$.

In the larger context of this course and physics, this problem reminds us about how to calculate the gravitational potential energy when the center of mass is not the same as rotating center.

1.2. **(b) For what values of R and h is the equilibrium at $\theta = 0$ stable?** For $\theta = 0$ to be an equilibrium, the first derivative at $\theta = 0$ has to be 0.

$$U'(\theta) = -mg(h - R)\sin(\theta). \tag{2}$$

Plug in 0, we got

$$U'(0) = 0. \tag{3}$$

So it is an equilibrium point.

To find out if it is a stable or unstable equilibrium, we need to take the second derivative.

$$U''(\theta) = -mg(h - R)\cos(\theta). \tag{4}$$

Plug in 0, we got

$$U''(0) = -mg(h - R). \tag{5}$$

For the it to be a stable equilibrium, we must have the second derivative to be positive. Since m and g are positive by nature, we need to have $(h - R)$ negative. Thus we need $h < R$ to make sure the toy is stable when $\theta = 0$.

This answer makes sense in a way that if the center of mass is lower than the center of rotation, it is really hard to rotate. An analogy to real life would be people walking on ropes in the air carries a long rod to lower their center of mass to stay stable.

In the larger context of this course and physics, this reminds me of the types of equilibria and how to differentiate them.

Non-physical insight, Taylor actually put in one good physics problem...