# 308 PCS4A

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## 1. Evaluate the work done

$$W = \int_O^P \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_O^P (F_x \mathrm{d}x + F_y \mathrm{d}y) \tag{1}$$

BY THE TWO-DIMENSIONAL FORCE $\vec{\mathbf{F}} = (x^2, 2xy)$ ALONG THE THREE PATHS JOINING THE ORIGIN TO THE POINT $P = (1, 1)$ DEFINED AS FOLLOWS:

1.1. **This path goes along the $x$ axis to $Q = (1, 0)$ and then straight up to $P$. (Divide the integral into two pieces, $\int_O^P = \int_O^Q + \int_Q^P$.)** First, we need to find the $x$ component and $y$ component of $\vec{\mathbf{F}}$. As one can easily observe from the formula, we can see that the $F_x = x^2$, $F_y = 2xy$. Since from $O$ to $Q$ there is no change in $y$ direction, we know that $\int_O^Q \mathrm{d}y$ is 0. Since from $Q$ to $P$ there is no change in $x$ direction, we know that $\int_Q^P \mathrm{d}x$ is 0. So now we can reduce our equation into

$$\begin{aligned}
\int_O^P (F_x \mathrm{d}x + F_y \mathrm{d}y) &= \int_O^Q F_x \mathrm{d}x + \int_Q^P F_y \mathrm{d}y, \\
&= \int_0^1 x^2 \mathrm{d}x + [\int_0^1 2xy \mathrm{d}y]_{x=1} \\
&= \frac{1}{3} + 1 \\
&= \frac{4}{3}.
\end{aligned} \tag{2}$$

The work done is $\frac{4}{3}$ unit energy.

1.2. **On this path $y = x^2$ , and you can replace the term $\mathrm{d}y$ in (1) by $\mathrm{d}y = 2x\mathrm{d}x$ and convert the whole integral into an**

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**integral over $x$.** Be a lamb and listen to the instructions, we turn $\mathrm{d}y$ into $2x\mathrm{d}x$.

$$\begin{aligned}\int_O^P (F_x\mathrm{d}x + F_y\mathrm{d}y) &= \int_0^1 F_x\mathrm{d}x + \int_0^1 F_y\mathrm{d}y \\ &= \int_0^1 x^2\mathrm{d}x + \int_0^1 2x(x^2)2x\mathrm{d}x \\ &= \frac{1}{3} + \frac{4}{5} \\ &= \frac{17}{15}\end{aligned} \tag{3}$$

The work done is $\frac{17}{15}$ unit energy.

1.3. **This path is given parametrically as $x = t^3, y = t^2$ . In this case rewrite $x, y, \mathrm{d}x$, and $\mathrm{d}y$ in (1) in terms of $t$ and $\mathrm{d}t$, and convert the integral into an integral over $t$.** Since we know that $x = t^3, y = t^2$, we know that $\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2, \frac{\mathrm{d}y}{\mathrm{d}t} = 2t$, so $\mathrm{d}x = 3t^2\mathrm{d}t, \mathrm{d}y = 2t\mathrm{d}t$.

Now, plug these into equation (1), we get:

$$\begin{aligned}\int_O^P (F_x\mathrm{d}x + F_y\mathrm{d}y) &= \int_0^1 F_x\mathrm{d}x + \int_0^1 F_y\mathrm{d}y \\ &= \int_0^1 x^2\mathrm{d}x + \int_0^1 2xy\mathrm{d}y \\ &= \int_0^1 (t^3)^2 3t^2\mathrm{d}t + \int_0^1 2(t^3)(t^2)2t\mathrm{d}t \\ &= \int_0^1 3t^8 + 4t^6\mathrm{d}t \\ &= \left(\frac{1}{3}t^9 + \frac{4}{7}t^6\right)\Bigg|_0^1 \\ &= \frac{19}{21}\end{aligned} \tag{4}$$

The work done is $\frac{29}{21}$ unit energy.

Through the results from different paths, we can tell that this force is not conservative.

Is there another way of knowing it's non-conservative? Yes, take the curl of it, and check if it's zero.

Will I do it here? No, I will in the last problem of this set.

## 2. Find the partial derivatives with respect to $x, y$, and $z$ of the following functions:

2.1. $f(x, y, z) = ax^2 + bxy + cy^2$**.** For the $x$ partial,

$$\frac{\partial}{\partial x}\left(ax^2 + bxy + cy^2\right) = 2ax + by. \tag{5}$$

For the $y$ partial,

$$\frac{\partial}{\partial y}\left(ax^2 + bxy + cy^2\right) = bx + 2cy. \tag{6}$$

For the $z$ partial,

$$\frac{\partial}{\partial z}\left(ax^2 + bxy + cy^2\right) = 0. \tag{7}$$

2.2. $g(x, y, z) = \sin(axyz^2)$**.** For the $x$ partial,

$$\frac{\partial}{\partial x}\left(\sin\left(axyz^2\right)\right) = ayz^2\cos\left(axyz^2\right). \tag{8}$$

For the $y$ partial,

$$\frac{\partial}{\partial y}\left(\sin\left(axyz^2\right)\right) = axz^2\cos\left(axyz^2\right). \tag{9}$$

For the $z$ partial,

$$\frac{\partial}{\partial z}\left(\sin\left(axyz^2\right)\right) = 2axyz\cos\left(axyz^2\right). \tag{10}$$

2.3. $h(x, y, z) = ae^{xy/z^2}$**.** For the $x$ partial,

$$\frac{\partial}{\partial x}\left(ae^{xy/z^2}\right) = aye^{xy/z^2}/z^2. \tag{11}$$

For the $y$ partial,

$$\frac{\partial}{\partial y}\left(ae^{xy/z^2}\right) = axe^{xy/z^2}/z^2. \tag{12}$$

For the $z$ partial,

$$\frac{\partial}{\partial z}\left(ae^{xy/z^2}\right) = -2axye^{xy/z^2}/z^3. \tag{13}$$

No physical insight to this one, pure mathematics throughout. The only insight I have is that Taylor is such a ___(for you to fill in) to put this many math questions in their physics text book.

## 3. Calculate the gradient $\nabla f$ of the following functions, $f(x, y, z)$:

**3.1.** $f = x^2 + z^3$**.**

$$\boldsymbol{\nabla}(x^2 + z^3) = 2x\hat{\mathbf{i}} + 3z^2\hat{\mathbf{k}}. \tag{14}$$

**3.2.** $f = ky$**, where** $k$ **is a constant.**

$$\boldsymbol{\nabla}(ky) = k\hat{\mathbf{j}}. \tag{15}$$

**3.3.** $f = r \equiv -\sqrt{x^2 + y^2 + z^2}$**.**

$$\boldsymbol{\nabla} r = 1\hat{\mathbf{r}}. \tag{16}$$

**3.4.** $f = 1/r$**.**

$$\boldsymbol{\nabla} 1/r = -1/r^2\hat{\mathbf{r}}. \tag{17}$$

Physical insight, use polar coordinates when $x, y, z$ are at the same order and with the same coefficients, inasmuch as it's much easier than Cartesian. (I have no idea if I used the word "inasmuch" correctly here, just want to imitate Taylor.)

4.

4.1. **Describe the surfaces defined by the equation $f =$ const, where $f = x^2 + 4y^2$.** It's an elliptical cylinder, where it's skinnier in the $y$ direction and fatter in the $x$ direction. I can hardly call it a surface...

4.2. **Using the results of Problem 4.18, find a unit normal to the surface $f = 5$ at the point $(1, 1, 1)$. In what direction should one move from this point to maximize the rate of change of $f$?** To find the maximum rate of change, we need to find the gradient vector of $f$ at that point.

$$\boldsymbol{\nabla}(x^2 + 4y^2) = 2x\hat{\mathbf{i}} + 8y\hat{\mathbf{j}} \tag{18}$$

Plug in the point $(1, 1, 1)$, we get $\boldsymbol{\nabla} f = 2\hat{\mathbf{i}} + 8\hat{\mathbf{j}}$. Divide the vector by its magnitude to gain the unit normal.

$$\frac{2\hat{\mathbf{i}} + 8\hat{\mathbf{j}}}{\sqrt{2^2 + 8^2}} = \frac{2}{\sqrt{69}}\hat{\mathbf{i}} + \frac{8}{\sqrt{69}}\hat{\mathbf{j}}. \tag{19}$$

This should be the direction of which it's moving in the maximum rate of change.

## 5. Which of the following forces is conservative? For those which are conservative, find the corresponding potential energy $U$, and verify by direct differentiation that $\vec{\mathbf{F}} = -\nabla U$.

5.1. **$\vec{\mathbf{F}} = k(x, 2y, 3z)$ where $k$ is a constant.** We need to calculate the curl of $\vec{\mathbf{F}}$.

$$\begin{aligned} \boldsymbol{\nabla} \times \vec{\mathbf{F}} &= \nabla \times k(x, 2y, 3z) \\ &= 0 - 0\hat{\mathbf{i}} - 0 - 0\hat{\mathbf{j}} + 0 - 0\hat{\mathbf{k}} \\ &= 0 \end{aligned} \tag{20}$$

Since the curl is zero, we know that this force is conservative, thus we need to find the potential energy.

$$\begin{aligned} U = -\int \vec{\mathbf{F}}(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ = -k(x^2/2 + y^2 + 3z^2/2) + c, \text{ for some constant } c. \end{aligned} \tag{21}$$

Then, we need to verify if our $U$ is the actual $U$, we need to take the derivative again...

$$\begin{aligned} -\boldsymbol{\nabla} U &= -(-kx, 2ky, 3kz) + c \\ &= k(x, 2y, 3z) \\ &= \vec{\mathbf{F}} \end{aligned} \tag{22}$$

No physical insight. Normal insight: Who doesn't love take the integral of something, then take the derivative of that integral...

5.2. **$\vec{\mathbf{F}} = k(y, x, 0)$.** We need to calculate the curl of $\vec{\mathbf{F}}$.

$$\begin{aligned} \boldsymbol{\nabla} \times \vec{\mathbf{F}} &= \nabla \times k(y, x, 0) \\ &= 0 - 0\hat{\mathbf{i}} - 0 - 0\hat{\mathbf{j}} + k - k\hat{\mathbf{k}} \\ &= 0 \end{aligned} \tag{23}$$

Since the curl is zero, we know that this force is conservative, thus we need to find the potential energy.

$$\begin{aligned} U = -\int \vec{\mathbf{F}}(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ = -kxy + c, \text{ for some constant } c. \end{aligned} \tag{24}$$

Then, we need to verify if our $U$ is the actual $U$, we need to take the derivative again...

$$
\begin{aligned}
-\boldsymbol{\nabla} U &= -(-ky, -kx, 0) \\
&= k(y, x, 0) \\
&= \vec{\mathbf{F}}
\end{aligned} \tag{25}
$$

No physical insight. Normal insight: Why, Taylor, why...

Is there any other ways of knowing the potential energy without taking integral of the force function? If the answer is no, I genuinely do not understand the point of verifying $U$ is the actual $U$...

5.3. $\vec{\mathbf{F}} = k(-y, x, 0)$**.** We need to calculate the curl of $\vec{\mathbf{F}}$.

$$
\begin{aligned}
\boldsymbol{\nabla} \times \vec{\mathbf{F}} &= \nabla \times k(-y, x, 0) \\
&= 0 - 0\hat{\mathbf{i}} - 0 - 0\hat{\mathbf{j}} + k - (-k)\hat{\mathbf{k}} \\
&= 2k\hat{\mathbf{k}}
\end{aligned} \tag{26}
$$

Since the curl is non-zero, the force is non-conservative.

No physical insight. Normal insight: Finally, no need to take the integral, then differentiate it again...