308 PCS5

TONY DENG

1. A massless spring has unstretched length l_o and force constant k. One end is now attached to the ceiling and a mass m is hung from the other. The equilibrium length of the spring is now l_1.

1.1. **Write down the condition that determines l_1. Suppose now the spring is stretched a further distance x beyond its new equilibrium length. Show that the net force (spring plus gravity) on the mass is $F = -kx$.** Set the zero at the ceiling. Since the new equilibrium length is l_1, we know that it stretched $l_1 - l_o$ amount. Since the mass is at rest, we know that the net force on the mass is 0. The only two forces acting on the mass are gravitational force and spring force. Thus, we have

$$F_{net} = 0 = mg - k(l_1 - l_o). \tag{1}$$

Simplify equation (1), we have

$$l_1 = \frac{mg}{k} + l_o. \tag{2}$$

Now, since the spring is stretched x amount even further, the net force is no longer zero. The gravitational force would stay the same, but the spring force would be greater.

$$\begin{aligned} F_{net} &= mg - k(x + l_1 - l_o) \\ \text{since we have } mg &= k(l_1 - l_o), \\ F_{net} &= k(l_1 - l_o) - k(x + l_1 - l_o) \\ F_{net} &= -kx \end{aligned} \tag{3}$$

The solution means that even our equilibrium position changed, the net forces can still be calculated using the Hooke's law with the new equilibrium point.

In a larger physics context, this showed that for a closed system, we can always use the equilibrium position with the mass at rest, and no need to refer to the old equilibrium position all the time.

Date: October 26, 2021.

1.2. **Prove the same result by showing that the net potential energy (Spring plus gravity) has the form** $U(x) = const + 1/2kx^2$**.** We know that the potential energy the mass has are spring potential and gravity potential energies, so we have

$$U = U_s + U_g. \tag{4}$$

We know that the spring energy is $U_s = 1/2k(x + l_1 - l_o)^2$. And we know that the gravitational energy is $U_g = mg(h - x)$ for some height $(h - x)$. Now plug in, we have

$$\begin{aligned} U &= 1/2k(x^2 + (l_1 - l_0)^2 + 2x(l_1 - l_0)) + k(l_1 - l_0)h - k(l_1 - l_0)x \\ &= 1/2kx^2 + 1/2k(l_1 - l_0)^2 + kx(l_1 - l_0) + k(l_1 - l_0)h - kx(l_1 - l_0) \\ &= constant + 1/2kx^2. \end{aligned} \tag{5}$$

With the potential in hand, take the negative derivative, we get

$$\begin{aligned} F &= -\frac{\mathrm{d}U}{\mathrm{d}x} \\ &= -kx. \end{aligned} \tag{6}$$

We know this is correct since we got the same result from the last subpart, and the context in this course is the same as the last subpart.

2. A mass on the end of a spring is oscillating with angular frequency ω. At $t = 0$, its position is $x_0 > 0$ and I give it a kick so that it moves back toward the origin and executes simple harmonic motion with amplitude $2x_0$. Find its position as a function of time in the form (III).

Form (III) is

$$x(t) = A\cos(\omega t - \delta). \tag{7}$$

The amplitude is $2x_0$, so $A = 2x_0$. Since the initial position is $x_0 > 0$, plug in 0 for t in equation (4), we get

$$x(0) = x_0 = 2x_0\cos(-\delta). \tag{8}$$

Solve for δ, we get $\delta = \pm\pi/3$.

Since Taylor kicked it, so that the mass moves in the negative direction, we know that the initial velocity is negative. Take the derivative of our solution.

$$x'(t) = -\omega 2x_0 \sin(\omega t - \delta). \tag{9}$$

Plug in $t = 0$ and we know that $v_0 < 0$.

$$\begin{aligned} x'(0) = v_0 = -\omega 2x_0 \sin(-\delta), \\ \text{since } v_0 \text{ is negative, } \sin(-\delta) \text{ has to be positive,} \\ \sin(-\delta) > 0 \\ \delta = -\pi/3. \end{aligned} \tag{10}$$

Thus, our solution is $x(t) = 2x_0\cos(\omega t + \pi/3)$.

We know this is correct because if we set the time to zero, we get the initial position of the mass.

This problem uses a lot of premises in SHM. 1) the angular frequency doesn't change even if the amplitude changes; 2) I couldn't thought another one, I used "a lot of" all the time regardless if there are a lot of or not...

I guess another physical insight is that when we know both the angular frequency and the phase shift, it's easier to use form (iii) to solve for the solution.

3. 5.7

3.1. **Solve for the coefficients B_1 and B_2 of the form (II) in terms of the initial position x_0 and velocity v_0 at $t = 0$.** Form (II) looks like this

$$x(t) = B_1 \cos(\omega t) + B_2 \sin(\omega t). \tag{11}$$

Since we know that when $t = 0$, $x = x_0$, we have

$$x(0) = x_0 = B_1. \tag{12}$$

Take the derivative of our solution,

$$x'(t) = -\omega B_1 \sin(\omega t) + \omega B_2 \cos(\omega t). \tag{13}$$

Plug in the initial condition $x'(0) = v_0$, we have

$$\begin{aligned} v_0 &= \omega B_2 \\ B_2 &= v_0/\omega. \end{aligned} \tag{14}$$

Now, with the information on B_1, B_2, the solutions turns into

$$x(t) = x_0 \cos(\omega t) + v_0/\omega \sin(\omega t). \tag{15}$$

3.2. **If the oscillator's mass is $m = 0.5$ kg and the force constant is $k = 50$ N/m, what is the angular frequency ω? If $x_0 = 3.0$ m and $v_0 = 50$ m/s, what are B_1 and B_2? Sketch $x(t)$ for a couple of cycles.** By an equation on Taylor without an equation number, we have

$$\omega = \sqrt{k/m}. \tag{16}$$

Use our known k and m, we get our ω to be 10 rad/s.

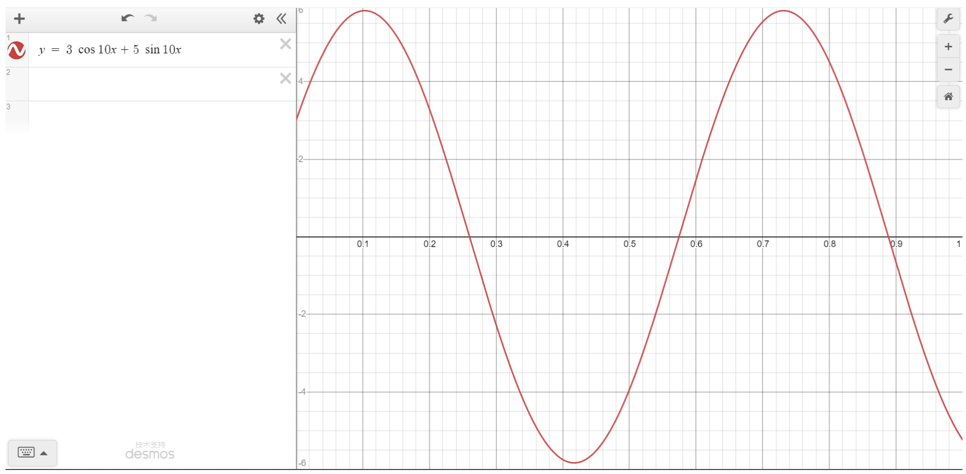
Plug in known x_0 and v_0 for B_1 and B_2, we get

$$\begin{aligned} B_1 &= x_0 = 3, \\ B_2 &= v_0/\omega = 5. \end{aligned} \tag{17}$$

The units are meters.

Now we have the solutions to be

$$x(t) = 3\cos(10t) + 5\sin(10t). \tag{18}$$



3.3. **What are the earliest times at which $x = 0$ and at which $\dot{x} = 0$?** First transfer our solution into form (III). According to equation (5.10) from Taylor, we have

$$\begin{aligned} A &= \sqrt{B_1^2 + B_2^2} \\ &= \sqrt{3^2 + 5^2} \\ A &= \sqrt{34}. \end{aligned} \tag{19}$$

According to equation (5.11) from Taylor, we have

$$\begin{aligned} \delta &= \arctan(B_2/B_1) \\ &= \arctan(5/3) \\ &= 1.03. \end{aligned} \tag{20}$$

Now we know our solution in form (III),

$$x(t) = \sqrt{34}\cos(10t - 1.03). \tag{21}$$

To find out the time for $x = 0$, we plug in 0 on equation (18) on the left side.

$$\begin{aligned} 0 &= \sqrt{34}\cos(10t - 1.03) \\ 0 &= \cos(10t - 1.03) \\ \pi/2 &= 10t - 1.03 \\ t &= (\pi/2 + 1.03)/10. \end{aligned} \tag{22}$$

Unit in seconds, this is the time when $x = 0$.

Take the first derivative of our solution,

$$x'(t) = -10\sqrt{34}\sin(10t - 1.03). \tag{23}$$

Plug in the left side for 0 and solve for t,

$$\begin{aligned}0 &= -10\sqrt{34}\sin(10t - 1.03)\\ 0 &= \sin(10t - 1.03)\\ 0 &= 10t - 1.03\\ t &= .103\end{aligned} \tag{24}$$

Unit in seconds, this is the time when $v = 0$.

We know all of my answers are true because this is just plugging in the numbers for previous questions, would've been a shame to wrong anyone... Also from the sketch we see a sinusoidal function, so we know that the function is at least correct in shape.

One thing I'd mention though not important in this problem is the phase shift. We used calculator to calculate the phase shift here, and both B_1 and B_2 are positive, we are lucky. However, in any case where one of them is negative, or both of them are negative, calculators use different range than human to calculate the arctan, and may give wrong results, so we should be careful in that case.

Putting in a larger physics context, this trained us to switch between two different form of solutions.

4. Consider a particle in two dimensions, subject to a restoring force of the form (5.21). (The two constants k_x and k_y may or may not be equal; if they are, the oscillator is isotropic.) Prove that its potential energy is

$$U = \frac{1}{2}(k_x x^2 + k_y y^2). \tag{25}$$

By equation (5.21), we have two forces act on the particle,

$$\begin{aligned} F_x &= -k_x x, \\ F_y &= -k_y y. \end{aligned} \tag{26}$$

From previous chapters we know that U is the negative integral of F. Thus, we can take the integral on the net force.

$$\begin{aligned} U &= -\int F_{net} \mathrm{d}x\mathrm{d}y \\ &= -\int (F_x + F_y)\mathrm{d}x\mathrm{d}y \\ &= -\int -k_x x \mathrm{d}x - \int -k_y y \mathrm{d}y \\ &= 1/2 k_x x^2 + 1/2 k_y y^2 \\ &= 1/2(k_x x^2 + k_y y^2). \end{aligned} \tag{27}$$

This problem exhibits how the potential energy of two dimensional SHM behaves.

In a larger physics context, this also is one of the base cases for a multi-dimension problem.

Now, the question is, what if the two dimensions are not orthogonal, and somehow their differential equations are coupled? I guess the potential energy would look completely different.