308 notes 5.1-5.5

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This is a lazy note.

1 Hooke's Law

Hooke's law asserts that the force exerted by a spring has the form

$$F_x(x) = -kx. \tag{1}$$

An exactly equivalent way to state Hooke's law is that the potential energy is

$$U(x) = \frac{1}{2}kx^2. \tag{2}$$

Since any reasonable function can be expanded in a Taylor series, we can safely write

$$U(x) = U(0) + U'(0)x + \frac{1}{2}U''(0)x^2 + \cdots . \tag{3}$$

Since it's physics, we can always subtract a constant, so $U(0) = 0$. Since $x = 0$ is an equilibrium point, $U'(0) = 0$. Renaming $U''(0)$ as k, we conclude that for small displacements it is always a good approximation to take.

2 Simple Harmonic Motion

The equation of motion is $m\ddot{x} = F_x = -kx$ or

$$\ddot{x} = -\frac{k}{m}x = -\omega^2 x \tag{4}$$

where Taylor has introduced the constant

$$\omega = \sqrt{\frac{k}{m}}. \tag{5}$$

2.1 The Exponential Solutions

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \tag{6}$$

is a general solution.

2.2 The Sine and Cosine Solutions

Use Euler's formula, we find that

$$x(t) = B_1 \cos(\omega t) + B_2 \sin(\omega t) \tag{7}$$

where

$$B_1 = C_1 + C_2 \text{ and } B_2 = i(C_1 - C_2). \tag{8}$$

By observation, we see that B_1 is the initial position and ωB_2 is the initial velocity.

2.3 The Phase-Shifted Cosine Solution

Let

$$A = \sqrt{B_1^2 + B_2^2}. \tag{9}$$

We can rewrite the solution as

$$x(t) = A \cos(\omega t - \delta). \tag{10}$$

2.4 Solution as the Real Part of a Complex Exponential

C_1 is a complex conjugate of C_2. Define $C = 2C_1$, we see that

$$C = B_1 - iB_2 = Ae^{-i\delta}, \tag{11}$$

and

$$x(t) = ReCE^{i\omega t} = ReAe^{i(\omega t - \delta)} \tag{12}$$

2.5 Energy Considerations

The potential energy is just

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2(\omega t - \delta). \tag{13}$$

The kinetic energy is

$$T = \frac{1}{2}kA^2 \sin^2(\omega t - \delta). \tag{14}$$

The total energy would be

$$E = T + U = \frac{1}{2}kA^2. \tag{15}$$

3 Two-Dimensional Oscillators

The simplest model considered is called isotropic harmonic oscillator, for which the restoring force is proportional to the displacement from equilibrium, with the same constant of proportionality in all directions:

$$\vec{\mathbf{F}} = -k\vec{\mathbf{r}}. \tag{16}$$

In the anisotropic oscillator, the components of the restoring force are proportional to the components of the displacement, but with different constants of proportionality:

$$\begin{aligned} F_x &= -k_x x, \\ F_y &= -k_y y, \\ F_z &= -k_z z. \end{aligned} \tag{17}$$

4 Damped Oscillations

Consider, then, an object in one dimension, such as a cart attached to a spring, that is subject to a Hooke's law force, $-kx$, and a resistive force, $-b\dot{x}$. The net force on the object is $-b\dot{x} - kx$, and Newton's second law reads

$$m\ddot{x} + b\ddot{x} + kx = 0. \tag{18}$$

Rename the constant b/m as 2β, this parameter will be called the damping constant.

5 Driven Damped Oscillations

Now with the drive force,

$$m\ddot{x} + b\dot{x} + kx = F(t). \tag{19}$$

Define the differnetial operator

$$D = \frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\beta\frac{\mathrm{d}}{\mathrm{d}t} + \omega_o^2. \tag{20}$$

5.1 Particular and Homogeneous Solutions

For particular solutions,

$$D_{x_p} = f. \tag{21}$$

For homogeneous solutions,

$$D_{x_h} = 0. \tag{22}$$

5.2 Complex Solutions for a Sinusoidal Driving Force

I got it.