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7.1. *Write down the Lagrangian for a projectile (subject to no air resistance) in terms of its Cartesian coordinates (x, y, z), with z measured vertically upward. Find the three Lagrange equations and show that they are exactly what you would expect for the equations of motion.*
First, we need to write out the Lagrangian for this projectile. Since this is a projectile, there is no change in potential energy in the x and y directions.

The kinetic energy would be

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2. \tag{1}$$

The potential energy would be purely gravitational potential energy, thus only in z direction.

$$U = mgz. \tag{2}$$

Now, we are equipped with enough information to write the Lagrangian for this projectile.

$$\begin{aligned} \mathcal{L} &= T - U \\ \mathcal{L} &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 - mgz. \end{aligned} \tag{3}$$

By Lagrange's equation, we know that

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}. \tag{4}$$

So we would have three equations with our general coordinates.

For x, we have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} \\ 0 &= \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) \\ 0 &= \dot{p_x} = F_x \end{aligned} \tag{5}$$

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For y, we have

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial y} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{y}} \\ 0 &= \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{y}) \\ 0 &= \dot{p_y} = F_y\end{aligned} \tag{6}$$

For z, we have

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial z} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{z}} \\ -mg &= \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{z}) \\ -g &= \ddot{z}\end{aligned} \tag{7}$$

Q.E.D.

These solutions represent the forces in each direction. And they make sense since there should be no force in both x and y directions and gravity in the z direction.

In the larger physics context, this question exhibits the fact that Lagrangian agrees with Newtonian in Cartesian coordinates.

7.2. *Write down the Lagrangian for a one-dimensional particle moving along the x axis and subject to a force $F = -kx$ (with k positive). Find the Lagrange equation of motion and solve it.*
The kinetic energy in this situation would be $T = \frac{1}{2}m\dot{x}^2$. To get the potential energy, we need to take the negative integral of the force with respect to distance, which in this case should be x.

$$\begin{aligned} U &= -\int F\mathrm{d}x \\ U &= -\int -kx\mathrm{d}x \\ U &= k\int x\mathrm{d}x \\ U &= \frac{1}{2}kx^2. \end{aligned} \tag{8}$$

We don't care about the constant after integration because U is the potential energy and thus adding or subtracting a constant to it doesn't matter.

Now, we can write out our Lagrangian.

$$\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2. \tag{9}$$

By the Lagrange's equation, we have

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial x} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}} \\ -kx &= \frac{\mathrm{d}}{\mathrm{d}t}m\dot{x} \\ -kx &= m\ddot{x} \\ \ddot{x} &= -xk/m. \end{aligned} \tag{10}$$

Q.E.D.

Obviously this is an equation for simple harmonic motion. This answer makes sense since we do expect that a particle subject to a spring force move in simple harmonic motion.

In a larger physics context, this problem shows that the Lagrangian agrees with Newtonian for simple harmonic motion.

7.3. *Consider a mass in moving in two dimensions with potential energy* $U(x,y)=\frac{1}{2}kr^2$ *, where* $r^2=x^2+y^2$*. Write down the Lagrangian, using coordinates* x *and* y*, and find the two Lagrange equations of motion. Describe their solutions. [This is the potential energy of an ion in an "ion trap," which can be used to study the properties of individual atomic ions.]*
Without the loss of generality, assume the r here is the position vector with components in both x and y directions. To put in a plain equation,

$$\vec{\mathbf{r}}=\vec{\mathbf{x}}+\vec{\mathbf{y}}. \tag{11}$$

Then the velocity vector should be in the form

$$\dot{\vec{\mathbf{r}}}=\dot{\vec{\mathbf{x}}}+\dot{\vec{\mathbf{y}}}. \tag{12}$$

Now we can write out the kinetic energy and the potential energy.

$$\begin{aligned} T=1/2m\left\|\dot{\vec{\mathbf{r}}}\right\|^2=\frac{1}{2}m\left\|\dot{\vec{\mathbf{x}}}\right\|^2+\frac{1}{2}m\left\|\dot{\vec{\mathbf{y}}}\right\|^2=\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\dot{y}^2, \\ U=\frac{1}{2}kx^2+\frac{1}{2}ky^2. \end{aligned} \tag{13}$$

Consequently, the Lagrangian,

$$\mathcal{L}=T-U=\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\dot{y}^2-\frac{1}{2}kx^2-\frac{1}{2}ky^2. \tag{14}$$

Time for Lagrange's equations.

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial x}&=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}} \\ -kx&=\frac{\mathrm{d}}{\mathrm{d}t}m\dot{x} \\ -kx&=m\ddot{x} \\ x&=A\cos\left(\sqrt{k/m}t\right). \end{aligned} \tag{15}$$

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial y}&=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{y}} \\ -ky&=\frac{\mathrm{d}}{\mathrm{d}t}m\dot{y} \\ -ky&=m\ddot{y} \\ y&=A\cos\left(\sqrt{k/m}t\right). \end{aligned} \tag{16}$$

Q.E.D.

There solutions are solutions for SHM...really, no surprise there.

In a larger physics context, this shows that one can solve for the motion of a particle without knowing anything about the force that particle is experiencing, which is kind of the point of Lagrangian mechanics.

7.15. *A mass m_1 rests on a frictionless horizontal table and is attached to a massless string. The string runs horizontally to the edge of the table, where it passes over a massless, frictionless pulley and then hangs vertically down. A second mass m_2 is now attached to the bottom end of the string. Write down the Lagrangian for the system. Find the Lagrange equation of motion, and solve it for the acceleration of the blocks. For your generalized coordinate, use the distance x of the second mass below the tabletop.*

The kinetic energy for mass 1 is

$$T_1 = \frac{1}{2}m_1\dot{x}^2. \tag{17}$$

The kinetic energy for mass 2 is

$$T_2 = \frac{1}{2}m_2\dot{x}^2. \tag{18}$$

The potential energy for mass 1 is zero since it doesn't depend on x and we don't care about constants. The potential energy for mass 2 is

$$U = -m_2 g x. \tag{19}$$

The reason why there's a negative sign there is that as x increases, the gravitational potential energy decreases.

Now, time for the beloved Lagrangian

$$\mathcal{L} = T_1 + T_2 - U = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + m_2 g x. \tag{20}$$

Without a doubt, the Lagrange's equations

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} \\ m_2 g &= \frac{\mathrm{d}}{\mathrm{d}t}(m_1 + m_2)\dot{x} \\ m_2 g &= (m_1 + m_2)\ddot{x} \\ \ddot{x} &= \frac{m_2}{m_1 + m_2} g \end{aligned} \tag{21}$$

Q.E.D.

The result shows that the acceleration is affected by both masses, which is in our expectation, and the little g, which is also in our expectation. So the result, consequently, is also within our expectation. Since m_2 is obviously less than the sum of the two masses, the acceleration is smaller than the gravitational acceleration, which makes sense since we expect the second mass falling slower than free fall.

In a larger physics context, this example shows that what requires two coordinates in Newtonian can be achieved with only one coordinate in Lagrangian. No surprise there.

7.22. *Using the usual angle ϕ as generalized coordinate, write down the Lagrangian for a simple pendulum of length l suspended from the ceiling of an elevator that is accelerating upward with constant acceleration a. (Be careful when writing T; it is probably safest to write the bob's velocity in component form.) Find the Lagrange equation of motion and show that it is the same as that for a normal, nonaccelerating pendulum, except that g has been replaced by $g+a$. In particular, the angular frequency of small oscillations is $\sqrt{(g+a)/l}$.*
First write out the position vector.

$$\vec{\mathbf{r}} = \langle l\sin\phi, l(1-\cos\phi) + \frac{1}{2}at^2\rangle. \tag{22}$$

Take time derivative of the the position to get the velocity.

$$\vec{\mathbf{v}} = \langle \dot{\phi} l\cos\phi, \dot{\phi}\sin\phi l + at\rangle. \tag{23}$$

The kinetic energy would be

$$\begin{aligned} T &= \frac{1}{2}m\|\vec{\mathbf{v}}\|^2 = \frac{1}{2}((\dot{\phi} l\cos\phi)^2 + (\dot{\phi}\sin\phi l + at)^2) \\ &= \frac{1}{2}m(\dot{\phi}^2 l^2\cos^2\phi + \dot{\phi}^2 l^2 \sin^2\phi + a^2t^2 + 2\dot{\phi}\sin\phi lat) \\ &= \frac{1}{2}m(\dot{\phi}^2 l^2 + a^2t^2 + 2\dot{\phi}\sin\phi lat). \end{aligned} \tag{24}$$

The potential energy would be

$$U = mgh = mg(l(1-\cos\phi) + \frac{1}{2}at^2). \tag{25}$$

Again, our beloved Lagrangian.

$$\begin{aligned} \mathcal{L} &= T - U \\ &= \frac{1}{2}m(\dot{\phi}^2 l^2 + a^2t^2 + 2\dot{\phi}\sin\phi lat) - mg(l(1-\cos\phi) + \frac{1}{2}at^2. \end{aligned} \tag{26}$$

Followed by the second beloved Lagrange's Equation.

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\phi} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} \\ m\dot{\phi}\cos\phi lat + mlg\sin\phi &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(m\dot{\phi}l^2 + m\sin\phi lat\Big) \\ m\dot{\phi}\cos\phi lat + mlg\sin\phi &= m\ddot{\phi}l^2 + \dot{\phi}m\cos\phi lat + m\sin\phi la \\ -mlg\sin\phi &= m\ddot{\phi}l^2 + m\sin\phi la \\ -g\sin\phi &= \ddot{\phi}l + \sin\phi a \\ -\sin\phi(g+a) &= \ddot{\phi}l \end{aligned} \tag{27}$$

Since for small angles we have $\sin\phi \approx \phi$, we get

$$-\frac{g+a}{l}\phi = \ddot{\phi} \tag{28}$$

Obviously a SHM, so the angular frequency is $\sqrt{\frac{g+a}{l}}$. Q.E.D.

Since this is a proof and we did reach at the conclusion, so I'd say this is probably the right answer.

In a larger physics context, we see that this problem gives a non-inertial frame, and it asks us to be careful about non-inertial frames. So the lesson here, I guess humbly, should be no matter what type of mechanics we are doing, doing it in an inertial frame is important.

7.23. *A small cart (mass m) is mounted on rails inside a large cart. The two are attached by a spring (force constant k) in such a way that the small cart is in equilibrium at the midpoint of the large. The distance of the small cart from its equilibrium is denoted x and that of the large one from a fixed point on the ground is X, as shown in Figure 7.13. The large cart is now forced to oscillate such that $X = A\cos\omega t$, with both A and ω fixed. Set up the Lagrangian for the motion of the small cart and show that the Lagrange equation has the form*

$$\ddot{x} + \omega_o^2 x = B\cos\omega t$$

where ω_o is the natural frequency $\omega_o = \sqrt{k/m}$ and B is a constant. This is the form assumed in Section 5.5, Equation (5.57), for driven oscillations (except that we are here ignoring damping). Thus the system described here would be one way to realize the motion discussed there. (We could fill the large cart with molasses to provide some damping.)
First write out the position vector.

$$\vec{\mathbf{r}} = x + Acos\omega t. \tag{29}$$

Then take derivative with respect to time to get the velocity vector.

$$\vec{\mathbf{v}} = \dot{x} - \omega A\sin\omega t. \tag{30}$$

Now, we can write out the kinetic energy.

$$T = \frac{1}{2}m\|\vec{\mathbf{v}}\|^2 = \frac{1}{2}m(\dot{x}^2 + \omega^2 A^2\sin^2\omega t - 2\dot{x}\omega A\sin\omega t). \tag{31}$$

The potential energy is purely the spring potential since the gravitational potential is the same and nobody cares.

$$U = \frac{1}{2}(\omega_o^2 m)x^2. \tag{32}$$

Write out the Lagrangian.

$$\begin{aligned}\mathcal{L} &= T - U \\ &= \frac{1}{2}m(\dot{x}^2 + \omega^2 A^2\sin^2\omega t - 2\dot{x}\omega A\sin\omega t) - \frac{1}{2}(\omega_o^2 m)x^2\end{aligned} \tag{33}$$

The Lagrange's equation.

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial x} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}} \\ -\omega_o^2 mx &= \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x} - m\omega A\sin\omega t) \\ -\omega_o^2 mx &= (m\ddot{x} - m\omega^2 A\cos\omega t) \\ -\omega_o^2 x &= \ddot{x} - \omega^2 A\cos\omega t \\ \ddot{x} + \omega_o^2 x &= \omega^2 A\cos\omega t\end{aligned} \tag{34}$$

Let $B = \omega^2 A$.

$$\ddot{x} + \omega_o^2 x = B \cos \omega t. \tag{35}$$

Q.E.D.

Again, a proof, and again, we reached the conclusion, so the answer is probably right.

In a larger physics context, this problem also presented a non-inertial frame. It differs from the last problem by providing a different acceleration, change of direction.

7.25. *Prove that the potential energy of a central force* $\vec{\mathbf{F}} = -kr^n\hat{\mathbf{r}}$ *(with* $n \neq -1$*) is* $U = kr^{n+1}/(n-1)$*. In particular, if* $n = 1$*, then* $\vec{\mathbf{F}} = -k\vec{\mathbf{r}}$ *and* $U = \frac{1}{2}kr^2$*.*

From Lagrange's equation we know that $\frac{\partial \mathcal{L}}{\partial r}$ equals to the general force in the r direction. We also know that $\frac{\partial \mathcal{L}}{\partial r} = -\frac{\mathrm{d}U}{\mathrm{d}r}$. Thus we know that $-\frac{\mathrm{d}U}{\mathrm{d}r} = F$. To get U, we need to take integral of F with respect to r.

$$\begin{aligned} -U &= \int F\mathrm{d}r \\ -U &= -k\int r^n \mathrm{d}r \\ U &= kr^{n+1}/(n+1). \end{aligned} \tag{36}$$

Again, we don't give a * * about constants in the potential energy.

Another proof...so the answer is probably correct.

In a larger physics context, we see that one important relationship between the Lagrangian and the potential energy is that $\frac{\partial \mathcal{L}}{\partial r} = -\frac{\partial U}{\partial r}$.