308 FDS2

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1 2

We start with the following equation,

$$f_{\text{lin}} = bv \tag{1}$$

Since we know that $b = \beta D$, we plug in b and get

$$f_{\text{lin}} = \beta D v \tag{2}$$

We are given the equation

$$f_{\text{lin}} = 3\pi\eta D v \tag{3}$$

Since the left hand sides of both equation (2) and equation (3) are the linear term of drag force, we can connect the right hand sides together.

$$\beta D v = 3\pi\eta D v \tag{4}$$

For the sake of the question's existence, we must assume that both the velocity of the mass and the diameter of the cross-section of the mass is nonzero, otherwise the problem is meaningless, so we can cancel them on both sides.

$$\beta = 3\pi\eta \tag{5}$$

We are given the value of $\eta = 1.7 \times 10^{-5}\mathrm{N} \cdot \mathrm{s/m}^2$, we can plug into equation (5), and we get

$$\beta = 1.6 \times 10^{-4}\mathrm{N} \cdot \mathrm{s/m}^2 \tag{6}$$

This value of β is the same as the value in Taylor.

2 17

We are given the equations,

$$\begin{aligned} x(t) &= v_{xo}\tau(1 - e^{-t/\tau}) \\ y(t) &= (v_{yo} + v_{ter})\tau(1 - e^{-t/\tau}) - v_{ter}t \end{aligned} \tag{7}$$

We need to eliminate t in $y(t)$ and express y as a function of x. So the first reasonable thing to do is to express t in terms of x.

$$\begin{aligned} \frac{x}{v_{xo}\tau} &= (1 - e^{-t/\tau}) \\ e^{-t/\tau} &= 1 - \frac{x}{v_{xo}\tau} \\ -t/\tau &= \ln\left(1 - \frac{x}{v_{xo}\tau}\right) \\ t &= -\tau \ln\left(1 - \frac{x}{v_{xo}\tau}\right) \end{aligned} \tag{8}$$

Now, the second reasonable step is to plug t into $y(t)$.

$$\begin{aligned} y(x) &= (v_{yo} + v_{ter})\tau(1 - e^{-(-\tau\ln\left(1 - \frac{x}{v_{xo}\tau}\right))/\tau}) - v_{ter}\left(-\tau \ln\left(1 - \frac{x}{v_{xo}\tau}\right)\right) \\ &= (v_{yo} + v_{ter})\tau\left(\frac{x}{v_{xo}\tau}\right) + v_{ter}\tau\ln\left(1 - \frac{x}{v_{xo}\tau}\right) \\ &= \frac{v_{yo} + v_{ter}}{v_{xo}} x + v_{ter}\tau\ln\left(1 - \frac{x}{v_{xo}\tau}\right) \end{aligned} \tag{9}$$

The result is the same as equation (2.37) in Taylor.

3 18

3.1 a

We are given the function $\ln(x)$ and are being asked to approximated $\ln(1 + \delta)$ when δ is small. So the first thing we need to do is to calculate the first, second and third derivatives of $\ln(x)$ with respect to x.

$$\begin{aligned} f(x) &= \ln(x) \\ f'(x) &= x^{-1} \\ f''(x) &= -x^{-2} \\ f'''(x) &= 2x^{-3} \end{aligned} \tag{10}$$

Now, we need to plug in 1 into the derivatives we found.

$$\begin{aligned} f(1) &= \ln(1) = 0 \\ f'(1) &= 1^{-1} = 1 \\ f''(1) &= -1^{-2} = -1 \\ f'''(1) &= 21^{-3} = 2 \end{aligned} \tag{11}$$

Now, we write the Taylor expansion of $\ln(1 + \delta)$,

$$\ln(1 + \delta) \approx 0 + \delta - \delta^2/2 + 2\delta^3/3 + \cdots \tag{12}$$

This result is what we expect from a Taylor expansion.

3.2 b

We are given the function $\cos(x)$ and are being asked to approximated $\cos(0+\delta)$ when δ is small. So the first thing we need to do is to calculate the first, second and third derivatives of $\cos(x)$ with respect to x.

$$\begin{aligned} f(x) &= \cos(x) \\ f'(x) &= -\sin(x) \\ f''(x) &= -\cos(x) \\ f'''(x) &= \sin(x) \end{aligned} \tag{13}$$

Now, we need to plug in 0 into the derivatives we found.

$$\begin{aligned} f(0) &= \cos(0) = 1 \\ f'(0) &= -\sin(0) = 0 \\ f''(0) &= -\cos(0) = -1 \\ f'''(0) &= \sin(0) = 0 \end{aligned} \tag{14}$$

Now, we write the Taylor expansion of $\cos(0+\delta)$,

$$\cos(0+\delta) \approx 1 + 0 - \delta^2/2 + 0 + \cdots \tag{15}$$

This result is what we expect from a Taylor expansion.

3.3 c

We are given the function $\sin(x)$ and are being asked to approximated $\sin(0+\delta)$ when δ is small. So the first thing we need to do is to calculate the first, second and third derivatives of $\sin(x)$ with respect to x.

$$\begin{aligned} f(x) &= \sin(x) \\ f'(x) &= \cos(x) \\ f''(x) &= -\sin(x) \\ f'''(x) &= -\cos(x) \end{aligned} \tag{16}$$

Now, we need to plug in 0 into the derivatives we found.

$$\begin{aligned} f(0) &= \sin(0) = 0 \\ f'(0) &= \cos(0) = 1 \\ f''(0) &= -\sin(0) = 0 \\ f'''(0) &= -\cos(0) = -1 \end{aligned} \tag{17}$$

Now, we write the Taylor expansion of $\sin(0+\delta)$,

$$\sin(0+\delta) \approx 0 + \delta + 0 - \delta^3/3 + \cdots \tag{18}$$

This result is what we expect from a Taylor expansion.

3.4 d

We are given the function e^x and are being asked to approximated $e^{0+\delta}$ when δ is small. So the first thing we need to do is to calculate the first, second and third derivatives of e^x with respect to x.

$$\begin{aligned} f(x) &= e^x \\ f'(x) &= e^x \\ f''(x) &= e^x \\ f'''(x) &= e^x \end{aligned} \tag{19}$$

Now, we need to plug in 0 into the derivatives we found.

$$\begin{aligned} f(0) &= e^0 = 1 \\ f'(0) &= e^0 = 1 \\ f''(0) &= e^0 = 1 \\ f'''(0) &= e^0 = 1 \end{aligned} \tag{20}$$

Now, we write the Taylor expansion of $e^{0+\delta}$,

$$e^{0+\delta} \approx 1 + \delta + \delta^2/2 + \delta^3/3 + \cdots \tag{21}$$

This result is what we expect from a Taylor expansion.

4 19

4.1 a

First, we need to write down the positions of x, y in terms of t when there is no air resistance.

$$\begin{aligned} x(t) &= v_{xo}t \\ y(t) &= v_{yo}t - \frac{1}{2}gt^2 \end{aligned} \tag{22}$$

Then, we need to express t in terms of x.

$$t = x/v_{xo} \tag{23}$$

Now, we need to plug t into $y(t)$.

$$y(x) = v_{yo}x/v_{xo} - \frac{1}{2}g(x/v_{xo})^2 \tag{24}$$

5 b

We are given equation (2.37) from Taylor.

$$y = \frac{v_{yo} + v_{ter}}{v_{xo}} x + v_{ter}\tau \ln\left(1 - \frac{x}{v_{xo}\tau}\right) \tag{25}$$

Let $\epsilon = \frac{x}{v_{xo}\tau}$. From the Taylor expansion, equation (2.40) in Taylor, of the natural log, we can rewrite y with the second term being expanded. Since we are approximating it with really small ϵ values, it is reasonable to only keep the first two terms in the Taylor expansion.

$$-\ln(1-\epsilon) = -\left(\epsilon + \frac{1}{2}\epsilon^2\right) \tag{26}$$

$$y = \frac{v_{yo} + v_{ter}}{v_{xo}} x + v_{ter}\tau \left(-\left(\left(\frac{x}{v_{xo}\tau}\right) + \frac{1}{2}\left(\frac{x}{v_{xo}\tau}\right)^2\right)\right) \tag{27}$$

Since air resistance doesn't exist in this case, we take τ to infinity, and simplify equation (27).

$$\begin{aligned}
y &= \frac{v_{yo} + v_{ter}}{v_{xo}} x + v_{ter}\tau \left(-\left(\frac{x}{v_{xo}\tau}\right) - \frac{1}{2}\left(\frac{x}{v_{xo}\tau}\right)^2\right) \\
&= \frac{v_{yo} + v_{ter}}{v_{xo}} x - v_{ter}\tau \left(\frac{x}{v_{xo}\tau}\right) - v_{ter}\tau\frac{1}{2}\left(\frac{x}{v_{xo}\tau}\right)^2 \\
&= \frac{v_{yo} + v_{ter}}{v_{xo}} x - v_{ter}\frac{x}{v_{xo}} - g\frac{1}{2}\left(\frac{x}{v_{xo}}\right)^2 \\
&= \frac{x}{v_{xo}} v_{yo} - g\frac{1}{2}\left(\frac{x}{v_{xo}}\right)^2
\end{aligned} \tag{28}$$

This expression matches our result found in (a).

6 32

6.1 a

During all time of the projectile, since the speed is much less than the terminal speed, it means that the drag forces are not comparable to the force exerted by gravity and thus the effects of air resistance are usually small.

6.2 b

Equation (2.53) from Taylor is

$$v_{ter} = \sqrt{mg/c} \tag{29}$$

equation (2.26) from Taylor is

$$v_{ter} = mg/b \tag{30}$$

For linear case, we see that the drag force is proportional to the velocity directly,

$$f = bv \tag{31}$$

However, for the the quadratic case, we see that the drag force is proportional to the velocity squared,

$$f = cv^2 \tag{32}$$

When comparing the drag force to gravity, we see that in the linear case,

$$\frac{f}{mg} = \frac{bv}{mg} = \frac{v}{v_{ter}} \tag{33}$$

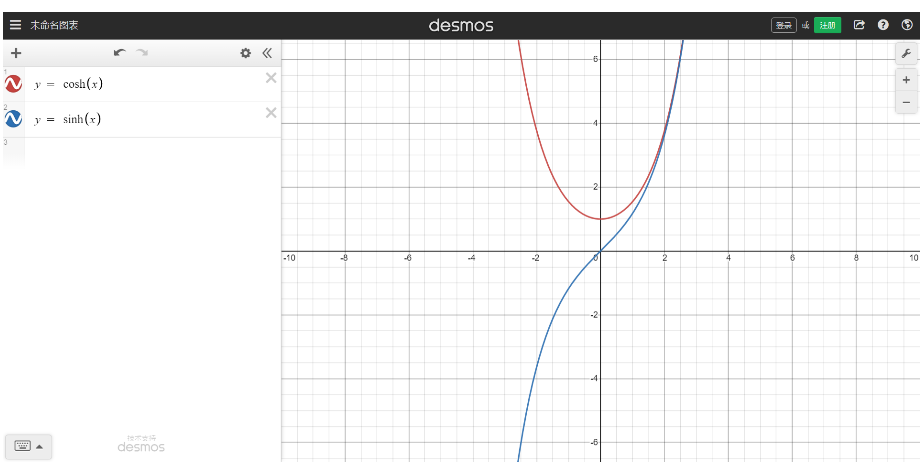
the velocity and the terminal speed have the ratio on the order of 1, whereas in the quadratic case, we see that,

$$\frac{f}{mg} = \frac{cv^2}{mg} = \frac{v^2}{v_{ter}^2} \tag{34}$$

the velocity and terminal velocity has a ratio of the second order.

7 33

7.1 a



7.2 b

We start with Euler's equation

$$e^{iz} = \cos(z) + i\sin(z) \tag{35}$$

If the exponent of e is z, it means that we need to multiply $-i$ to the exponent,

$$e^{z} = \cos(-iz) + i\sin(-iz) \tag{36}$$

By trigonometric identities, we know that $\cos(-x) = \cos(x), \sin(-x) = -\sin(x)$, so we can simplify equation (36)

$$e^{z} = \cos(iz) - i\sin(iz) \tag{37}$$

Similarly, we have

$$e^{-z} = \cos(iz) + i\sin(iz) \tag{38}$$

Since $\cosh(z) = (e^z + e^{-z})/2$, we have

$$\cosh(z) = (\cos(iz) - i\sin(iz) + \cos(iz) + i\sin(iz))/2 = \cos(iz) \tag{39}$$

And since $\sinh(z) = (e^z - e^{-z})/2$, we have

$$\sinh(z) = (\cos(iz) - i\sin(iz) - \cos(iz) - i\sin(iz))/2 = -i\sin(iz) \tag{40}$$

7.3 c

$$\begin{aligned} \frac{\mathrm{d}\cosh(z)}{\mathrm{d}z} &= (e^z - e^{-z})/2 = \sinh(z) \\ \frac{\mathrm{d}\sinh(z)}{\mathrm{d}z} &= (e^z + e^-z)/2 = \cosh(z) \end{aligned} \tag{41}$$

$$\begin{aligned} \int \cosh(z)\mathrm{d}z &= \frac{1}{2}\int e^z + e^{-z}\mathrm{d}z \\ &= \frac{1}{2}(e^z - e^{-z} + c) \\ &= \sinh(z) + c/2 \end{aligned} \tag{42}$$

$$\begin{aligned} \int \sinh(z)\mathrm{d}z &= \frac{1}{2}\int e^z - e^{-z}\mathrm{d}z \\ &= \frac{1}{2}(e^z + e^{-z} + c) \\ &= \cosh(z) + c/2 \end{aligned} \tag{43}$$

Apparently their derivatives and integrals are each other.

7.4 d

$$\begin{aligned}
\cosh^2(z) - \sinh^2(z) &= ((e^z + e^{-z})^2 - (e^z - e^{-z})^2)/4 \\
&= (e^{2z} + e^{-2z} + 2 - e^{2z} - e^{-2z} + 2)/4 \\
&= 4/4 = 1
\end{aligned} \tag{44}$$

7.5 e

$$\begin{aligned}
&\int (1+x^2)^{-1/2} \mathrm{d}x \\
&\text{substitute } x = \sinh(z)\text{, we get } \mathrm{d}x = \cosh(z)\mathrm{d}z \\
&= \int (1+\sinh^2(z))^{-1/2} \cosh(z)\mathrm{d}z \\
&\text{by the identity in part (d), we know that } 1+\sinh^2(z) = \cosh^2(z) \\
&= \int \cosh^{-1}(z)\cosh(z)\mathrm{d}z \\
&= \int 1\mathrm{d}z = z \\
&\text{since } x = \sinh(z), z = arcsinh(x) \\
&= arcsinh(x)
\end{aligned} \tag{45}$$