308 FDS8

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8.3. *Two particles of masses m_1 and m_2 are joined by a massless spring of natural length L and force constant k. Initially, m_2 is resting on a table and I am holding m_1 vertically above m_2 at a height L. At time $t = 0$, I project m_1 vertically upward with initial velocity v_o . Find the positions of the two masses at any subsequent time t (before either mass returns to the table) and describe the motion.*
Define the sum of the masses as $M = m_1 + m_2$. Define the reduced mass as $\mu = m_1 m_2 / m_1 + m_2$.

First, write the Lagrangian in terms of the center of mass and relative position coordinates. The kinetic energy terms are obviously the kinetic energies in terms of the center of mass and the relative position. What may require a bit more articulation are the potential terms. As one may reasonably deduce that there are gravitational potential and spring potential terms, we must be careful about which coordinate to use in each term. For the gravitational potential, we can start with two terms, one in the center of mass coordinate and one in the relative position. Looking at the latter one, we see that as one mass gain gravitational potential, the other mass loses gravitational potential relatively. Thus, it is safe to say that the only term that matters is the term with the center of mass coordinate. For the spring potential, we still start with two terms, one in the center of mass and one in the relative position. Looking at the first one, we see that center of mass term doesn't matter since it doesn't speak for the change in the length of the spring. Therefore, we are now confident to write down the Lagrangian.

$$\mathcal{L} = T - U = \frac{1}{2}M\dot{\vec{\mathbf{R}}}^2 + \frac{1}{2}\mu\dot{\vec{\mathbf{r}}}^2 - Mg\left\|\vec{\mathbf{R}}\right\| - \frac{1}{2}k(\|\vec{\mathbf{r}}\| - L)^2 \tag{1}$$

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Now apply the Lagrange equations.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial R} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{R}} \\
-Mg &= \frac{\mathrm{d}}{\mathrm{d}t}M\dot{R} \\
-Mg &= M\ddot{R} \\
-g &= \ddot{R}
\end{aligned} \tag{2}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial r} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{r}} \\
-k(r-L) &= \frac{\mathrm{d}}{\mathrm{d}t}\mu\dot{r} \\
-k(r-L) &= \mu\ddot{r}
\end{aligned} \tag{3}$$

With the information about the second time derivative of R and r, and the information about the initial conditions, we may solve for the positions for both masses.

$$\begin{aligned}
-g &= \ddot{R} \\
-\int g\mathrm{d}t &= \int \ddot{R}\mathrm{d}t \\
-gt + C &= \dot{R}
\end{aligned} \tag{4}$$

At time $t = 0$, we know that the velocity for m_1 is v_o upward, and the velocity for m_2 is 0. Since $R = \frac{m_1 r_1 + m_2 r_2}{M}$, we know that $\dot{R} = \frac{m_1 \dot{r}_1 + m_2 \dot{r}_2}{M}$. Plug in the initial condition, we know that $\dot{R}(0) = \frac{m_1 v_o + m_2 0}{M}$. Hence we know that $C = \frac{m_1 v_o}{M}$.

$$\begin{aligned}
\dot{R} &= -gt + \frac{m_1 v_o}{M} \\
\int \dot{R}\mathrm{d}t &= \int -gt + \frac{m_1 v_o}{M}\mathrm{d}t \\
R &= -\frac{1}{2}gt^2 + \frac{m_1 v_o t}{M} + D
\end{aligned} \tag{5}$$

At time $t = 0$, we know that the position for m_1 is L above the table, and the position for m_2 is 0. Hence we know that $R(0) = \frac{m_1 L + m_2 0}{M}$.

Thus, we know that

$$R(t) = -\frac{1}{2}gt^2 + \frac{m_1 v_o t}{M} + \frac{m_1 L}{M} \tag{6}$$

Now, time for the relative position. By equation (3), we see that r is a driven non-damped oscillator. Let x equals to $r - L$, we see that $\ddot{x} = \ddot{r}$.

$$\begin{aligned}
-k(r-L) &= \mu\ddot{r} \\
-kx &= \mu\ddot{x} \\
x &= A\cos\left(\sqrt{k/\mu}t\right) + B\sin\left(\sqrt{k/\mu}t\right) \\
r - L &= A\cos\left(\sqrt{k/\mu}t\right) + B\sin\left(\sqrt{k/\mu}t\right) \\
r &= A\cos\left(\sqrt{k/\mu}t\right) + B\sin\left(\sqrt{k/\mu}t\right) + L
\end{aligned} \tag{7}$$

Since we know that when $t = 0$, $r = L$, we can plug into the initial condition to get A.

$$\begin{aligned}
r(0) &= A\cos(0) + B\sin(0) + L \\
0 &= A
\end{aligned} \tag{8}$$

Since $r = r_1 - r_2$, so we know that $\dot{r} = \dot{r}_1 - \dot{r}_2$. When $t = 0$, $\dot{r}_1 = v_o$ and $\dot{r}_2 = 0$.

$$\begin{aligned}
\dot{r} &= \sqrt{k/\mu}B\cos\left(\sqrt{k/\mu}t\right) \\
\dot{r}(0) &= \sqrt{k/\mu}B = v_o \\
B &= v_o\sqrt{\mu/k}
\end{aligned} \tag{9}$$

So we have

$$r(t) = v_o\sqrt{\mu/k}\sin\left(\sqrt{k/\mu}t\right) + L \tag{10}$$

By equation (8.9) from Taylor, we can solve for r_1 and r_2.

(11)

$$\begin{aligned}
r_1(t) &= R(t) + \frac{m_2}{M}r(t) \\
&= -\frac{1}{2}gt^2 + \frac{m_1 v_o t}{M} + \frac{m_1 L}{M} + \frac{m_2}{M}(v_o\sqrt{\mu/k}\sin\left(\sqrt{k/\mu}t\right) + L) \\
r_1(t) &= -\frac{1}{2}gt^2 + \frac{m_1 v_o t}{M} + \frac{m_2}{M}v_o\sqrt{\mu/k}\sin\left(\sqrt{k/\mu}t\right) + L
\end{aligned}$$

(12)

$$
\begin{aligned}
r_2(t) &= R(t) - \frac{m_1}{M} r(t) \\
&= -\frac{1}{2} g t^2 + \frac{m_1 v_o t}{M} + \frac{m_1 L}{M} - \frac{m_1}{M}\left(v_o \sqrt{\mu/k} \sin\left(\sqrt{k/\mu} t\right) + L\right) \\
&= -\frac{1}{2} g t^2 + \frac{m_1 v_o t}{M} - \frac{m_1}{M} v_o \sqrt{\mu/k} \sin\left(\sqrt{k/\mu} t\right)
\end{aligned}
$$

Q.E.D.

8.9. *Consider two particles of equal masses, $m_1 = m_2$, attached to each other by a light straight spring (force constant k, natural length L) and free to slide over a frictionless horizontal table. (a) Write down the Lagrangian in terms of the coordinates $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$, and rewrite it in terms of the CM and relative positions, $\vec{\mathbf{R}}$ and $\vec{\mathbf{r}}$, using polar coordinates (r, ϕ) for $\vec{\mathbf{r}}$. (b) Write down and solve the Lagrange equations for the CM coordinates $\vec{\mathbf{X}}, \vec{\mathbf{Y}}$. (c) Write down the Lagrange equations for r and ϕ. Solve these for the two special cases that r remains constant and that ϕ remains constant. Describe the corresponding motions. In particular, show that the frequency of oscillations in the second case is $\omega = \sqrt{2k/m_1}$.*

(a) Define $M = m_1 + m_2 = 2m_1$. Define $\mu = m_1 m_2/M = \frac{m_1}{2}$.

The only potential term is the spring potential since the gravitational potential is constant and nobody cares about constant.

$$\mathcal{L} = T - U = \frac{1}{2}m_1\dot{\vec{\mathbf{r}}}_1^2 + \frac{1}{2}m_2\dot{\vec{\mathbf{r}}}_2^2 - \frac{1}{2}k(\|\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2\| - L)^2 \tag{13}$$

Use equation (8.9) form Taylor, we can rewrite above in terms of $\vec{\mathbf{R}}$ and $\vec{\mathbf{r}}$.

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}m_1(\dot{\vec{\mathbf{R}}} + \frac{m_2}{M}\dot{\vec{\mathbf{r}}})^2 + \frac{1}{2}m_2(\dot{\vec{\mathbf{R}}} - \frac{m_1}{M}\dot{\vec{\mathbf{r}}})^2 - \frac{1}{2}k(\|\vec{\mathbf{r}}\| - L)^2 \\
&= \frac{1}{2}(m_1 + m_2)\dot{\vec{\mathbf{R}}}^2 + \frac{1}{2}(\frac{m_1 m_2}{m_1 + m_2})\dot{\vec{\mathbf{r}}}^2 - \frac{1}{2}k(\|\vec{\mathbf{r}}\| - L)^2 \\
\mathcal{L} &= \frac{1}{2}M\dot{\vec{\mathbf{R}}}^2 + \frac{1}{2}\mu\dot{\vec{\mathbf{r}}}^2 - \frac{1}{2}k(\|\vec{\mathbf{r}}\| - L)^2
\end{aligned} \tag{14}$$

Let the relative position $\vec{\mathbf{r}} = r\hat{\mathbf{r}} + \phi\hat{\phi}$. Since we know that there is no change in the angle for the relative position, we have $\|\vec{\mathbf{r}}\| = r$.

For the relative position velocity, we need to derive it through following algebra.

$$\begin{aligned}
\vec{\mathbf{v}} &= \frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\langle r\cos\phi, r\sin\phi\rangle \\
&= \langle \dot{r}\cos\phi - r\dot{\phi}\sin\phi, \dot{r}\sin\phi + r\dot{\phi}\cos\phi\rangle \\
v^2 &= (\dot{r}\cos\phi - r\dot{\phi}\sin\phi)^2 + (\dot{r}\sin\phi + r\dot{\phi}\cos\phi)^2 \\
v^2 &= \dot{r}^2\cos^2\phi + r^2\dot{\phi}^2\sin^2\phi - 2\dot{r}r\dot{\phi}\cos\phi\sin\phi \\
&\quad + \dot{r}^2\sin^2\phi + r^2\dot{\phi}^2\cos^2\phi + 2\dot{r}r\dot{\phi}\sin\phi\cos\phi \\
v^2 &= \dot{r}^2 + r^2\dot{\phi}^2
\end{aligned} \tag{15}$$

Now we can substitute the relative position and relative velocity into our Lagrangian.

$$\mathcal{L} = \frac{1}{2}M\dot{\vec{\mathbf{R}}}^2 + \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{2}k(r-L)^2 \tag{16}$$

(b)
Let $\vec{\mathbf{R}} = \vec{\mathbf{X}} + \vec{\mathbf{Y}}$.
Then we can rewrite the Lagrangian as

$$\mathcal{L} = \frac{1}{2}M(\dot{\vec{\mathbf{X}}}^2 + \dot{\vec{\mathbf{Y}}}^2) + \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{2}k(r-L)^2 \tag{17}$$

Since $\mathcal{L}$ is independent from $\vec{\phi}$, $\vec{\mathbf{X}}$ and $\vec{\mathbf{Y}}$, we know that $\dot{\vec{\phi}}$, $\dot{\vec{\mathbf{X}}}$ and $\dot{\vec{\mathbf{Y}}}$ are ignorable coordinates.
Hence we can easily get

$$0 = M\ddot{X} \tag{18}$$

and

$$0 = M\ddot{Y} \tag{19}$$

(c)
By the same argument above, we can easily get

$$0 = \mu r^2\ddot{\phi} + 2\mu r\dot{r}\dot{\phi}^2 \tag{20}$$

Now, apply the happy Lagrange's equation.

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial r} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{r}} \\
\mu r\dot{\phi}^2 - k(r-L) &= \frac{\mathrm{d}}{\mathrm{d}t}\mu\dot{r} \\
\mu r\dot{\phi}^2 - k(r-L) &= \mu\ddot{r}
\end{aligned} \tag{21}
$$

Suppose r is a constant. Then we know that $\dot{r} = 0$ and $\ddot{r} = 0$. So we have

$$
\begin{aligned}
\mu r\dot{\phi}^2 - k(r-L) = 0& \\
\dot{\phi}^2 &= \frac{k(r-L)}{\mu r} \\
\dot{\phi} &= \sqrt{\frac{k(r-L)}{\mu r}} \\
\phi &= \sqrt{\frac{k(r-L)}{\mu r}}t
\end{aligned} \tag{22}
$$

assuming that $\phi = 0$ when $t = 0$.

Suppose ϕ is a constant. Then we have $\dot{\phi} = 0$ and $\ddot{\phi} = 0$. So we have

$$
\begin{aligned}
-k(r-L) &= \mu\ddot{r} \\
r &= A\sin\left(\sqrt{k/\mu}t\right) + L
\end{aligned} \tag{23}
$$

assuming that $r = L$ when $t = 0$. Since $\mu = m_1/2$, we get $\omega = \sqrt{2k/m_1}$.