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7.1. *Write down the Lagrangian for a projectile (subject to no air resistance) in terms of its Cartesian coordinates* (x, y, z)*, with* z *measured vertically upward. Find the three Lagrange equations and show that they are exactly what you would expect for the equations of motion.*
First, we need to write out the Lagrangian for this projectile. Since this is a projectile, there is no change in potential energy in the x and y directions.

The kinetic energy would be

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2. \tag{1}$$

The potential energy would be purely gravitational potential energy, thus only in z direction.

$$U = mgz. \tag{2}$$

Now, we are equipped with enough information to write the Lagrangian for this projectile.

$$\begin{aligned} \mathcal{L} &= T - U \\ \mathcal{L} &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 - mgz. \end{aligned} \tag{3}$$

By Lagrange's equation, we know that

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}. \tag{4}$$

So we would have three equations with our general coordinates.

For x, we have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} \\ 0 &= \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) \\ 0 &= \dot{p_x} = F_x \end{aligned} \tag{5}$$

This makes sense since no force is exerted on the mass in the x direction.

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For y, we have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial y} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{y}} \\ 0 &= \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{y}) \\ 0 &= \dot{p_y} = F_y \end{aligned} \tag{6}$$

This makes sense since no force is exerted on the mass in the y direction.

For z, we have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial z} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{z}} \\ -mg &= \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{z}) \\ -g &= \ddot{z} \end{aligned} \tag{7}$$

This makes sense since the only force exerted on the mass in the z direction is gravity, and the gravitational acceleration is g. Q.E.D.

7.2. *Write down the Lagrangian for a one-dimensional particle moving along the x axis and subject to a force $F = -kx$ (with k positive). Find the Lagrange equation of motion and solve it.*
The kinetic energy in this situation would be $T = \frac{1}{2}m\dot{x}^2$. To get the potential energy, we need to take the negative integral of the force with respect to distance, which in this case should be x.

$$\begin{aligned} U &= -\int F\mathrm{d}x \\ U &= -\int -kx\mathrm{d}x \\ U &= k\int x\mathrm{d}x \\ U &= \frac{1}{2}kx^2. \end{aligned} \tag{8}$$

We don't care about the constant after integration because U is the potential energy and thus adding or subtracting a constant to it doesn't matter.

Now, we can write out our Lagrangian.

$$\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2. \tag{9}$$

By the Lagrange's equation, we have

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial x} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}} \\ -kx &= \frac{\mathrm{d}}{\mathrm{d}t}m\dot{x} \\ -kx &= m\ddot{x} \\ \ddot{x} &= -xk/m. \end{aligned} \tag{10}$$

Obviously this is an equation for simple harmonic motion. Q.E.D.

7.3. *Consider a mass in moving in two dimensions with potential energy $U(x,y) = \frac{1}{2}kr^2$, where $r^2 = x^2 + y^2$. Write down the Lagrangian, using coordinates x and y, and find the two Lagrange equations of motion. Describe their solutions. [This is the potential energy of an ion in an "ion trap," which can be used to study the properties of individual atomic ions.]*

Without the loss of generality, assume the r here is the position vector with components in both x and y directions. To put in a plain equation,

$$\vec{\mathbf{r}} = \vec{\mathbf{x}} + \vec{\mathbf{y}}. \tag{11}$$

Then the velocity vector should be in the form

$$\dot{\vec{\mathbf{r}}} = \dot{\vec{\mathbf{x}}} + \dot{\vec{\mathbf{y}}}. \tag{12}$$

Now we can write out the kinetic energy and the potential energy.

$$\begin{aligned} T = 1/2m\left\|\dot{\vec{\mathbf{r}}}\right\|^2 = \frac{1}{2}m\left\|\dot{\vec{\mathbf{x}}}\right\|^2 + \frac{1}{2}m\left\|\dot{\vec{\mathbf{y}}}\right\|^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2, \\ U = \frac{1}{2}kx^2 + \frac{1}{2}ky^2. \end{aligned} \tag{13}$$

Consequently, the Lagrangian,

$$\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - \frac{1}{2}kx^2 + \frac{1}{2}ky^2. \tag{14}$$

Time for Lagrange's equations.

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial x} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}} \\ -kx &= \frac{\mathrm{d}}{\mathrm{d}t}m\dot{x} \\ -kx &= m\ddot{x} \\ x &= A\cos\left(\sqrt{k/m}t\right). \end{aligned} \tag{15}$$

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial y} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{y}} \\ -ky &= \frac{\mathrm{d}}{\mathrm{d}t}m\dot{y} \\ -ky &= m\ddot{y} \\ y &= A\cos\left(\sqrt{k/m}t\right). \end{aligned} \tag{16}$$

There solutions are solutions for SHM...really, no surprise their. Q.E.D.

7.15. *A mass m_1 rests on a frictionless horizontal table and is attached to a massless string. The string runs horizontally to the edge of the table, where it passes over a massless, frictionless pulley and then hangs vertically down. A second mass m_2 is now attached to the bottom end of the string. Write down the Lagrangian for the system. Find the Lagrange equation of motion, and solve it for the acceleration of the blocks. For your generalized coordinate, use the distance x of the second mass below the tabletop.*

The kinetic energy for mass 1 is

$$T_1 = \frac{1}{2} m_1 \dot{x}^2. \tag{17}$$

The kinetic energy for mass 2 is

$$T_2 = \frac{1}{2} m_2 \dot{x}^2. \tag{18}$$

The potential energy for mass 1 is zero since it doesn't depend on x and we don't care about constants. The potential energy for mass 2 is

$$U = -m_2 g x. \tag{19}$$

The reason why there's a negative sign there is that as x increases, the gravitational potential energy decreases.

Now, time for the beloved Lagrangian

$$\mathcal{L} = T_1 + T_2 - U = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + m_2 g x. \tag{20}$$

Without a doubt, the Lagrange's equations

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{x}} \\ m_2 g &= \frac{\mathrm{d}}{\mathrm{d}t} (m_1 + m_2) \dot{x} \\ m_2 g &= (m_1 + m_2) \ddot{x} \\ \ddot{x} &= \frac{m_2}{m_1 + m_2} g \end{aligned} \tag{21}$$

7.22. *Using the usual angle ϕ as generalized coordinate, write down the Lagrangian for a simple pendulum of length l suspended from the ceiling of an elevator that is accelerating upward with constant acceleration a. (Be careful when writing T; it is probably safest to write the bob's velocity in component form.) Find the Lagrange equation of motion and show that it is the same as that for a normal, nonaccelerating pendulum, except that g has been replaced by $g+a$. In particular, the angular frequency of small oscillations is $\sqrt{(g+a)/l}$.*
First write out the position vector.

$$\vec{\mathbf{r}} = \langle l\sin\phi, l(1-\cos\phi) + \frac{1}{2}at^2\rangle. \tag{22}$$

Take time derivative of the the position to get the velocity.

$$\vec{\mathbf{v}} = \langle \dot\phi l\cos\phi, \dot\phi\sin\phi l + at\rangle. \tag{23}$$

The kinetic energy would be

$$\begin{aligned} T &= \frac{1}{2}m\|\vec{\mathbf{v}}\|^2 = \frac{1}{2}((\dot\phi l\cos\phi)^2 + (\dot\phi\sin\phi l + at)^2) \\ &= \frac{1}{2}m(\dot\phi^2 l^2\cos^2\phi + \dot\phi^2 l^2\sin^2\phi + a^2t^2 + 2\dot\phi\sin\phi lat) \\ &= \frac{1}{2}m(\dot\phi^2 l^2 + a^2t^2 + 2\dot\phi\sin\phi lat). \end{aligned} \tag{24}$$

The potential energy would be

$$U = mgh = mg(l(1-\cos\phi) + \frac{1}{2}at^2. \tag{25}$$

Again, our beloved Lagrangian.

$$\begin{aligned} \mathcal{L} &= T - U \\ &= \frac{1}{2}m(\dot\phi^2 l^2 + a^2t^2 + 2\dot\phi\sin\phi lat) - mg(l(1-\cos\phi) + \frac{1}{2}at^2. \end{aligned} \tag{26}$$

Followed by the second beloved Lagrange's Equation.

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\phi} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\phi} \\ m\dot\phi\cos\phi lat = mlg\sin\phi &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(m\dot\phi l^2 + m\sin\phi lat\Big) \\ m\dot\phi\cos\phi lat = mlg\sin\phi &= m\ddot\phi l^2 + \dot\phi m\cos\phi lat + m\sin\phi la \\ -mlg\sin\phi &= m\ddot\phi l^2 + m\sin\phi la \\ -g\sin\phi &= \ddot\phi l + \sin\phi a \\ -\sin\phi(g+a) &= \ddot\phi l \end{aligned} \tag{27}$$

Since for small angles we have $\sin\phi \approx \phi$, we get

$$-\frac{g+a}{l}\phi = \ddot{\phi} \tag{28}$$

Obviously a SHM, so the angular frequency is $\sqrt{\frac{g+a}{l}}$. Q.E.D.

7.23. *A small cart (mass m) is mounted on rails inside a large cart. The two are attached by a spring (force constant k) in such a way that the small cart is in equilibrium at the midpoint of the large. The distance of the small cart from its equilibrium is denoted x and that of the large one from a fixed point on the ground is X, as shown in Figure 7.13. The large cart is now forced to oscillate such that $X = A\cos\omega t$, with both A and ω fixed. Set up the Lagrangian for the motion of the small cart and show that the Lagrange equation has the form*

$$\ddot{x} + \omega_o^2 x = B\cos\omega t$$

where ω_o is the natural frequency $\omega_o = \sqrt{k/m}$ and B is a constant. This is the form assumed in Section 5.5, Equation (5.57), for driven oscillations (except that we are here ignoring damping). Thus the system described here would be one way to realize the motion discussed there. (We could fill the large cart with molasses to provide some damping.)
First write out the position vector.

$$\vec{\mathbf{r}} = x + Acos\omega t. \tag{29}$$

Then take derivative with respect to time to get the velocity vector.

$$\vec{\mathbf{v}} = \dot{x} - \omega A\sin\omega t. \tag{30}$$

Now, we can write out the kinetic energy.

$$T = \frac{1}{2}m\|\vec{\mathbf{v}}\|^2 = \frac{1}{2}m(\dot{x}^2 + \omega^2 A^2\sin^2\omega t + 2\dot{x}\omega A\sin\omega t). \tag{31}$$

The potential energy is purely the spring potential since the gravitational potential is the same and nobody cares.

$$U = \frac{1}{2}(\omega_o^2 m)x^2. \tag{32}$$

Write out the Lagrangian.

$$\begin{aligned}\mathcal{L} &= T - U \\ &= \frac{1}{2}m(\dot{x}^2 + \omega^2 A^2\sin^2\omega t + 2\dot{x}\omega A\sin\omega t) - \frac{1}{2}(\omega_o^2 m)x^2\end{aligned} \tag{33}$$

The Lagrange's equation.

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial x} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}} \\ -\omega_o^2 mx &= \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x} - m\omega A\sin\omega t) \\ -\omega_o^2 mx &= (m\ddot{x} - m\omega^2 A\cos\omega t) \\ -\omega_o^2 x &= \ddot{x} - \omega^2 A\cos\omega t \\ \ddot{x} + \omega_o^2 x &= \omega^2 A\cos\omega t\end{aligned} \tag{34}$$

Let $B = \omega^2 A$.

(35) $$\ddot{x} + \omega_o^2 x = B \cos \omega t.$$

Q.E.D.

7.25. *Prove that the potential energy of a central force $\vec{\mathbf{F}} = -kr^n\hat{\mathbf{r}}$ (with $n \neq -1$) is $U = kr^{n+1}/(n-1)$. In particular, if $n = 1$, then $\vec{\mathbf{F}} = -k\vec{\mathbf{r}}$ and $U = \frac{1}{2}kr^2$.*

From Lagrange's equation we know that $\frac{\partial \mathcal{L}}{\partial r}$ equals to the general force in the r direction. We also know that $\frac{\partial \mathcal{L}}{\partial r} = \frac{\mathrm{d}U}{\mathrm{d}r}$. Thus we know that $\frac{\mathrm{d}U}{\mathrm{d}r} = F$. To get U, we need to take integral of F with respect to r.

$$\begin{aligned} U &= \int F\mathrm{d}r \\ U &= -k\int r^n\mathrm{d}r \\ U &= -kr^{n+1}/(n+1). \end{aligned} \tag{36}$$

Again, we don't give a * * about constants in the potential energy.