IMO HK (Questions and suggested solutions)

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1 Preface

This paper is being updated in my webpage, and is keep being updated with more questions.

2 IMO HK Preliminary Selection Contest

2.1 IMO HK Preliminary Selection Contest 2020

2.1.1 Q5

2.1.1.1 Question

28 people seat in circle. They all claim that 'the two people next to me are of different genders'.
Given that all boys lie and exactly 3 girls lie, find the **number of girls**.

2.1.1.2 Solution

Denote girl by G and boy by B.
Notice that all boys lie, so it would be reasonable and easier to take girls as variable:
Namely, define $1 \leq B_1 < B_2 < B_3 < ... < B_k$, where $k \in \mathbb{N}$ is the number of boys (disregard of '28' at this moment).
We then define S_i for $i = 1, 2, ..., k$ to be the number of girls between separation of B_i and B_{i+1},
with the convention $B_{k+1} = B_1$.
Assume that for some i, $S_i = 0$, i.e., there are two successive boys with no girls between, namely, BB.
Then, consider the right B, we can extend BB to BBB because in the view of B, two people next to him are **not** of different genders.
Similarly and inductively, we would have only boys in the circle, giving a *contradiction.*
We might consider the following patterns (of course there is no boy between two **successive** boys laterally):

1. BGB: In this case that girl lies.
2. $BGGB$: In this case both girls do not lie.
3. $BG \ldots GB$, where '$\ldots$' means more than one girl: In this case girls in '$\ldots$' lie.

It is worth noticing that since every block $B \ldots B$ has at least one girl between, the number of girls would not have effect on the correctness of the acclaimed '*different genders neighbours*' proposition by any girl **in another block**.
Three girls lie. Therefore, with the constraint that $S_i \geq 1$,

$$\sum_{i=1}^{k} |S_i - 2| = 3$$

We may assume, for the sake of simplicity, that the all $S_i = 2$ at the beginning,

i.e., any two successive boys form the pattern $BGGB$.

Each time (to create) a girl that lies, we add $(+)$ **or** remove $(-)$ **one** girl.

Denote the three operations by $a_1, a_2, a_3 = \pm1, \pm1, \pm1$, and we add the restriction that no two operations **of different sign** should be performed on the same S_i, otherwise their effects would be counteracted and the operation would be flawed.

We now have:

$$\begin{aligned} S_1 + S_2 + S_3 + \ldots + S_k + k &= 28 \\ 2 + 2 + 2 + \ldots + 2 + k + a_1 + a_2 + a_3 &= 28 \\ a_1 + a_2 + a_3 = 28 - 2k - k = 28 - 3k &\equiv 1 \pmod 3 \end{aligned}$$

Let t denotes the number of operations $+$, then $a_1 + a_2 + a_3 = t - (3 - t) = 2t - 3 = 1$.

$$\therefore t = 2 \text{ and } \{a_1, a_2, a_3\} = \{1, 1, -1\}$$

Solving $3k = 28 - (a_1 + a_2 + a_3)$, we get the number of boys $k = 9$, hence the number of girls is 19.

2.1.1.2.1 An example and solution by 'testing'

$$\sum_{i=1}^{k} |S_i - 2| = 3$$

From here, we can exhaust all possible sequence $(S_i)_{i=1}^{k}$

(Disregard of the order of $S_1, S_2, \ldots$ without losing generality):

$$\{S_i\} = \begin{cases} 5, 2, 2, 2, 2, 2, \ldots, 2 \\ 4, 3, 2, 2, 2, 2, \ldots, 2 \\ 4, 1, 2, 2, 2, 2, \ldots, 2 \\ 3, 3, 3, 2, 2, 2, \ldots, 2 \\ 3, 3, 1, 2, 2, 2, \ldots, 2 \\ 3, 1, 1, 2, 2, 2, \ldots, 2 \\ 1, 1, 1, 2, 2, 2, \ldots, 2 \end{cases}$$

$$\sum S_{i\,\textbf{(include left boy of the block)}} = \begin{cases} 6 + 3 + 3 + 3 + \ldots + 3 & \to \text{never} \quad \equiv 28 \pmod 3 \\ 5 + 4 + 3 + 3 + \ldots + 3 & \to \text{never} \quad \equiv 28 \pmod 3 \\ 5 + 2 + 3 + 3 + \ldots + 3 & \to \text{possible} \\ 4 + 4 + 4 + 3 + \ldots + 3 & \to \text{never} \quad \equiv 28 \pmod 3 \\ 4 + 4 + 2 + 3 + \ldots + 3 & \to \text{possible} \\ 4 + 2 + 2 + 3 + \ldots + 3 & \to \text{never} \quad \equiv 28 \pmod 3 \\ 2 + 2 + 2 + 3 + \ldots + 3 & \to \text{never} \quad \equiv 28 \pmod 3 \end{cases}$$

So for the possible two cases we could have example (not unique) respectively (***bold*** for liers):

1.

$$G\boldsymbol{G}\boldsymbol{G}GB\boldsymbol{G}BGGBGGBGGBGGBGGBGGBGGBGGB$$

, which correspond to the solution while two operation '+' are performed on the same block (same S_i);

2.

$$GGGBGGGBGBGGBGGBGGBGGBGGBGGBGGB$$

, which correspond to the solution while two operation '+' are performed on different blocks.

2.1.2 Q7

2.1.2.1 Question

Solve equation $\sqrt{7-x} = 7 - x^2$ for $x \in \mathbb{R}^+$ $(x > 0)$.

2.1.2.2 Remark

There are two methods to solve this question. Observe the similarities between them.

2.1.2.3 Solution 1

Squaring both sides, giving:

$$\begin{aligned} 7 - x &= \left(7 - x^2\right)^2 = x^4 - 14x^2 + 49 \\ x^4 - 14x^2 + x + 42 &= 0 \end{aligned}$$

Then, it is easy to see the factor $(x+2)$ as $(-2)^4 - 14(-2)^2 + (-2) + 42 = 16 - 56 - 2 + 42 = 0$.

$$\begin{aligned} \therefore (x+2)(x^3 - 2x^2 - 10x + 21) &= 0 \\ (x+2)(x-3)(x^2 + x - 7) &= 0 \end{aligned}$$

$$x = -2 \text{ or } 3 \text{ or } \frac{-1 \pm \sqrt{29}}{2}$$

Substituting into the original equation, we notice that the solution root is $\frac{\sqrt{29}-1}{2} \in \mathbb{R}^+$.

2.1.2.3.1 More about this solution

$$x = -2 \text{ or } 3 \text{ or } \frac{-1 \pm \sqrt{29}}{2}$$

We reject $x = -2$ only because it is negative, but $\sqrt{7-(-2)} = 7 - (-2)^2$, so it satisfies the original equation.
On the other hand, we reject $x = 3$ and $x = \frac{-1-\sqrt{29}}{2}$ as they don't even satisfy the original equation.
Why? Because they are artificially created during the process of **squaring both sides**.

You may check:

$$\begin{aligned} \sqrt{7-(3)} &= 2 \neq -2 = 7 - (3)^2 \\ \sqrt{7 - \frac{-1-\sqrt{29}}{2}} &= \sqrt{\frac{30+2\sqrt{29}}{4}} = \frac{\sqrt{29}+1}{2} = \frac{30+2\sqrt{29}}{4} - 7 \\ &\neq 7 - \frac{30+2\sqrt{29}}{4} = 7 - \left(\frac{-1-\sqrt{29}}{2}\right)^2 \end{aligned}$$

Notice that all 4 roots we obtained satisfy $\sqrt{7-x} = |7 - x^2|$, if we allow $x \leq 0$.

2.1.2.4 Solution 2

Notice that the equation can be rewritten as (for $x \in \mathbb{R}^+$):

$$
\begin{aligned}
x^2 &= 7 - \sqrt{7-x} \\
x &= \sqrt{7 - \sqrt{7-x}}
\end{aligned}
$$

We define $f(x) = \sqrt{7-x}$ for $x \in [0, 7]$.
Then, we are actually solving $f(f(x)) = x$.
Observe that if we have $f(x) = x$, $f(f(x)) = f(x) = x$ immediately follows,
and it turns out that $f(x) = x$ is easy to solve:

$$
\begin{aligned}
x &= \sqrt{7-x} \\
x^2 &= 7 - x \\
x^2 + x - 7 &= 0 \\
x &= \frac{-1 \pm \sqrt{29}}{2}
\end{aligned}
$$

Rejecting the root *artificially created when squaring both sides*, we get the only desired solution $\frac{\sqrt{29}-1}{2}$.

2.1.3 Q17

2.1.3.1 Question

Find (the number of) *positive integer solutions* to the following system of equation:

$$
\begin{cases}
\sqrt{2020}\left(\sqrt{a}+\sqrt{b}\right) = \sqrt{(c+2020)(d+2020)} \\
\sqrt{2020}\left(\sqrt{b}+\sqrt{c}\right) = \sqrt{(d+2020)(a+2020)} \\
\sqrt{2020}\left(\sqrt{c}+\sqrt{d}\right) = \sqrt{(a+2020)(b+2020)} \\
\sqrt{2020}\left(\sqrt{d}+\sqrt{a}\right) = \sqrt{(b+2020)(c+2020)}
\end{cases}
$$

2.1.3.2 Solution

WLOG, let's first consider the first equation, dealing always only with $a, b, c, d \in \mathbb{N}$,

$$\sqrt{2020}\left(\sqrt{a}+\sqrt{b}\right) = \sqrt{(c+2020)(d+2020)}$$

Notice that

$$\sqrt{a}+\sqrt{b} = \sqrt{a+b+2\sqrt{ab}},$$

or by squaring both sides in the hope of doing some number theoretical manipulations.

$$2020\left(a+b+2\sqrt{ab}\right) = (c+2020)(d+2020) = cd + 2020(c+d) + 2020^2$$

Notice that

$$a+b+2\sqrt{ab} = \frac{(c+2020)(d+2020)}{2020} \in \mathbb{Q},$$

so we must have $ab = x_1^2$ for some $x_1 \in \mathbb{N}$.
Similarly,

$$(ab, bc, cd, da) = (x_1^2, x_2^2, x_3^2, x_4^2) \text{ for some } (x_1, x_2, x_3, x_4) \in \mathbb{N}^4$$

$$2020(a + b - c - d + 2x_1 - 2020) = 2020(a + b + 2x_1) - 2020(c + d + 2020) = cd$$

$$\therefore \frac{\left(\frac{x_3}{2}\right)^2}{505} = \frac{x_3^2}{505} = \frac{cd}{2020} \in \mathbb{N}, 505|x_3, x_3 = 505y_3, y_3 \in \mathbb{N}$$

Carefully considering 4 cases, we have:

$$(x_1, x_2, x_3, x_4) = (505y_1, 505y_2, 505y_3, 505y_4) \text{ for } (y_1, y_2, y_3, y_4) \in \mathbb{N}^4$$

$$\begin{cases} 2020(a + b - c - d + 1010y_1 - 2020) = x_3^2 = 505^2 y_3^2 \\ 2020(c + d - a - b + 1010y_3 - 2020) = x_1^2 = 505^2 y_1^2 \end{cases}$$

$$\therefore \ 505^2\Big(8(y_1 + y_3) - 32\Big) = 2020\big(1010(y_1 + y_3) - 4040\big) = 505^2(y_1^2 + y_3^2)$$

$$(y_1 - 4)^2 + (y_3 - 4)^2 = 0 \implies y_1 = y_3 = 4$$

$$\therefore \text{(Similarly) } x_1 = x_2 = x_3 = x_4 = 2020 \wedge ab = bc = cd = da = 2020^2$$

Notice that we could not have and only have $\begin{cases} a = c \\ b = d \end{cases}$ (but not $a = b = c = d$).
The four original equations have now been reduced to any one equation of the four, and the question is reduced to finding $(a, b) \in \mathbb{N}^2$ such that $ab = 2020^2 = 2^4 \times 5^2 \times 101^2$, having $(4 + 1)(2 + 1)(2 + 1) = 45$ solutions.
Consider $\sqrt{2020}(\sqrt{a} + \sqrt{b}) = \sqrt{(c + 2020)(d + 2020)}$ such that $ab = 2020^2$, with $(c, d) = (a, b)$:

$$\sqrt{(c + 2020)(d + 2020)} = \sqrt{cd + 2020(c + d) + cd} = \sqrt{2020}\sqrt{a + b + 2\sqrt{ab}} = \sqrt{2020}(\sqrt{a} + \sqrt{b}),$$

and very similarly, we just **justified** that the solution satisfy all 4 original equations.

2.1.3.3 Remark

Simpler methods are to be discovered, but personally I did not manage to think of a way utilizing the *AM-GM inequality*, for example.

2.1.4 Q20

2.1.4.1 Question

Denotes $(F_n)_{n \in \mathbb{N}}$ as the *Fibonacci sequence*. Find $(F_{2020} \bmod 1000)$.

2.1.4.2 Remark

Observe that $1000 = 10^3 = 2^3 \times 5^3 = 8 \times 125$, and it suffices to consider $(F_{2020} \bmod 8)$ and $(F_{2020} \bmod 125)$, which will be shown do-able.
(You may like to refer to the wiki page of Chinese Remainder Theorem.)
Notice that it would be extremely time consuming to write out sequence $(F_n \bmod 125)$ or even $(F_n \bmod 1000)$ until it repeats itself.
For the sequence $(F_n \bmod 5^k)$, I observe that it would repeat itself each 4×5^k terms, i.e., $F_{n+4\times 5^k} = F_n$.

The sequence (F_n mod 125) and (F_n mod 1000) are known to have (smallest) period 500 and 1500 respectively.

(1500 is the least common multiple of 500 and 12 = the period of sequence (F_n mod 8))

For example, $F_{2020} \bmod 1000 = F_{520} \bmod 1000 = 515$.

Notice that (by a small *Python* program),

$$\begin{aligned} F_{2020} = \; & 6390698115594355866513492739852871393819620004374587135576307978726515 \\ & 1995322001898071485437809065533306267303596007174873825024273095647974 \\ & 6498789136160967666042480660086493956623437642602865340590032906541161 \\ & 2695736240674586650793536636971284213893001403888857222739849074228772 \\ & 2358671367235709189166548965693916054419169307250217586108972590100204 \\ & 080980670597873094617540854443783530320723473081393553497241453257464515 \end{aligned}$$

which even a calculator would not do it for you.

However, for small n ($n \leq 250$), you may visit Online Fibonacci Calculator to check.

2.1.4.3 Solution

For (F_{2020} mod 8), you can simply list it out:

$$(F_{2020} \bmod 8) = 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, 1, 1, 2, 3, 5, 0, 5, 5, 2, ...$$

The cyclic sequence has a (smallest) period of 12, and so in general $F_{n+12} \bmod 8 = F_n \bmod 8$, thus:

$$F_{2020} \bmod 8 = F_{2016+4} = F_4 \bmod 8 = 3$$

For (F_{2020} mod 125), however, as stated above, it is hard to wait for the sequence to repeat itself.

Instead, it is well-known that:

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

(or you shall prove by Mathematical induction and remember the formula)
Therefore, by binomial expansion,

$$F_n = \frac{1}{2^n\sqrt{5}}\left((1+\sqrt{5})^n - (1-\sqrt{5})^n\right) = \frac{1}{2^n\sqrt{5}}\left(\sum_{i=0}^{n} C_i^n(\sqrt{5})^i - \sum_{i=0}^{n} C_i^n(-\sqrt{5})^i\right)$$

$$= \frac{1}{2^n\sqrt{5}}\left(\sum_{i=0}^{n} C_i^n\left((\sqrt{5})^i - (-\sqrt{5})^i\right)\right) = \frac{1}{2^n\sqrt{5}}\sum_{i=0}^{n} C_i^n(\sqrt{5})^i\left(1-(-1)^i\right)$$

$$= \frac{1}{2^n\sqrt{5}}\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} C_{2i+1}^n(\sqrt{5})^{2i+1}\left(1-(-1)^{2i+1}\right) = \frac{\sqrt{5}}{2^{n-1}\sqrt{5}}\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} C_{2i+1}^n(\sqrt{5})^{2i}$$

$$\left(\text{Notice that } \left\lfloor\frac{n-1}{2}\right\rfloor \text{ is the least number } m \text{ such that } 2m+1 \le n\right)$$

$$= \frac{\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} C_{2i+1}^n 5^i}{2^{n-1}}$$

$$\therefore F_{2020} = \frac{\sum_{i=0}^{1009} C_{2i+1}^{2020} 5^i}{2^{2019}}$$

$$\therefore 2^{2019}F_{2020} = \sum_{i=0}^{1009} C_{2i+1}^{2020}5^i = C_1^{2020} + 5\times C_3^{2020} + 25\times C_5^{2020} + 125\times(...)$$

$$\equiv C_1^{2020} + 5\times C_3^{2020} + 25\times C_5^{2020} \pmod{125}$$

Now it suffices to find $2^{2019} \pmod{125}$ and $C_1^{2020} + 5\times C_3^{2020} + 25\times C_5^{2020} \pmod{125}$ *or something similar.*
Below is a suggested way to do so, and is not too easy to make careless mistakes:

$$2^5 \equiv 32 \pmod{125}$$
$$2^{10} = 1024 \equiv 24 \pmod{125}$$
$$2^{20} = 1024^2 \equiv 24^2 = 576 \equiv -49 \pmod{125}$$
$$2^{25} = 2^{20}\times 2^5 \equiv (32)\times(-49) = -1568 \equiv 57 \pmod{125}$$
$$2^{50} \equiv (57)^2 = 60^2 - 2\times 3\times 60 + 3^2 = 3600 - 360 + 9 \equiv -1 \pmod{125}$$
$$\left(2^{100} = 2^{\varphi(125)} \equiv 1 \pmod{125} \text{ simply by Euler's Theorem}\right)$$
$$2^{2020} \equiv 2^{20}\times\left((2^{50})^2\right)^{20} \equiv -49\times((-1)^2)^{20} = -49 \pmod{125}$$

$$C_1^{2020} + 5\times C_3^{2020} + 25\times C_5^{2020} = 2020 + \frac{2020\times 2019\times 2018}{3!}\times 5 + \frac{2020\times 2019\times 2018\times 2017\times 2016}{5!}\times 25$$
$$= 2020 + 2020\times 2019\times 2018\div 6\times 5$$
$$+ 2020\times 2019\times 2018\times 2017\times 2016\div 3\div 8\times 5$$
$$= 2020 + 2020\times 673\times 1009\times 5 + 2020\times 673\times 2018\times 2017\times 252\times 5$$
$$\equiv 20 + 20\times 48\times 9\times 5 + 20\times 48\times 18\times 17\times 2\times 5$$
$$= 20 + 100\times 432 + 100\times 96\times 18\times 17$$
$$\equiv 20 + 75 + (-25)\times(-29)\times 306 = 95 + 725\times 306 \equiv -30 - 25\times 56$$
$$= -1430 \equiv 70 \pmod{125}$$

$$\therefore 22F_{2020} \equiv 147F_{2020} = -3(-49F_{2020}) \equiv -6(2^{2019}F_{2020}) \equiv -6 \times 70 = -420 \equiv -45 \pmod{125}$$

To find F_{2020} mod 125, try to find the multiplicative inverse of 22: $22^{-1} \pmod{125}$.
Here is a method without explicitly finding the multiplicative inverse:

Notice that $22 \times 6 = 132 = 125 + 7$

, which $7F_{2020} \equiv 132F_{2020}$ can simplify the expression.

After that, notice that $18 \times 7 = 126 = 125 + 1$ can again simplify the problem.

$$\begin{aligned} F_{2020} &\equiv 126F_{2020} = 18(7F_{2020}) \equiv 18(132F_{2020}) = (18)(6)(22F_{2020}) \equiv (108)(-45) \\ &\equiv 17 \times 45 = 765 \equiv 15 \pmod{125} \end{aligned}$$

After all, use the Chinese Remainder Theorem, or by considering:

$$F_{2020} = \cancel{15}, \cancel{140}, \cancel{265}, \cancel{390}, \mathbf{515}, \cancel{640}, \cancel{765}, \cancel{890} \pmod{\mathbf{1000}}$$

and notice that only 515 provides $F_{2020} \equiv 3 \pmod 8$.

2.1.4.4 Interesting Generalization

Consider a sequence $x, y, x+y, x+2y, 2x+3y, \ldots$.

It is defined similarly as the *Fibonacci sequence* except that the initial two terms are different.
One may generalize *Fibonacci sequence* for other starting values. And may also generalize it for all $n \in \mathbb{Z}$.

(Just define $F_n = F_{n+2} - F_{n+1}$ for all non-negative integers n)

These generalizations are obvious. Instead, here we would like to generalize the property of the *Fibonacci sequence* and discover the relation between the sequence and the number 5 (which is mostly implied by the beautiful relation with the *golden ratio*, see the section More about the general term formula of *Fibonacci sequence* below).

Lemma 2.1. *If $(2i+1)! = 5^g h$ for non-negative integers g, h and $5 \nmid h$, then $i - g \geq 1$.*

Proof.

$$g = \left\lfloor \frac{2i+1}{5^1} \right\rfloor + \left\lfloor \frac{2i+1}{5^2} \right\rfloor + \left\lfloor \frac{2i+1}{5^3} \right\rfloor + \ldots \leq \frac{2i+1}{5^1} + \frac{2i+1}{5^2} + \frac{2i+1}{5^3} + \ldots = \lim_{m\to\infty} \sum_{j=1}^{m} \left(\frac{2i+1}{5^j} \right)$$

$$= \lim_{m\to\infty} \left(\frac{\text{Next term - first term}}{\text{Common ratio} - 1} \right) = \frac{(2i+1) - 0}{5 - 1}$$

$$\frac{2i-1}{4} \leq i - g \in \mathbb{Z}, \quad \therefore i - g \geq 1$$

■

Theorem 2.2. *For non-negative integers m, n, $F_{n \times 5^m} \equiv 0 \pmod{5^m}$.*

Proof.

$$2^{n\times 5^m - 1} F_{n\times 5^m} = \sum_{i=0}^{\left\lfloor \frac{n\times 5^m - 1}{2} \right\rfloor} C_{2i+1}^{n\times 5^m} (5^i) \equiv \sum_{i=0}^{\min\left\{ \left\lfloor \frac{n\times 5^m - 1}{2} \right\rfloor, m-1 \right\}} C_{2i+1}^{n\times 5^m} (5^i) \pmod{5^m}$$

$$\text{For each } i, C_{2i+1}^{n\times 5^m} (5^i) = \frac{(5^m)(5^m - 1)\ldots(5^m - 2i)}{(2i+1)!} (5^i) \equiv 0 \pmod{5^m} \text{ by this lemma.}$$

Then, coping the special case $n = 0$, by considering multiplicative inverse or just $2^{n\times 5^m - 1} F_{n\times 5^m} = 5^k t \ (t \in \mathbb{N})$:

$$F_{n\times 5^m} \equiv 0 \pmod{5^m}$$

■

2.1.4.5 Alternative solution simplified at finding (F_{2020} **mod** 125)

In the above, we have proven that $F_{n+500} \equiv F_n \pmod{125}$ for all $n \in \mathbb{N}$.
We may then compute $F_{2020} \bmod 125 = F_{20} \bmod 125$ by listing out the terms of $(F_n \bmod 125)_{n\in\mathbb{N}}$:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 19, 108, 2, 110, 112, 97, 84, 56, \mathbf{15}$$

Notice that for this solution, as $F_{n+500} \equiv F_n \pmod{125}$ is used, you need to observe that $F_{4\times125} \equiv 0 \pmod{125}$ and $F_{4\times125+1} \equiv 1 \pmod{125}$ in order to think of this solution.

Or, from a competition-directed point of view:
One may suggest or guess F_{2000} to be something simple (mod 125) (just like $\left(F_n \bmod 2\right)_{n\in\mathbb{N}}$ is something easily predictable), and with $F_{2001} \equiv 1 \pmod{125}$ also observed, one can notice that the *Fibonacci sequence* repeats itself every 2000 terms. Though not the least possible period, it's still useful.

$$\begin{aligned} F_{2000} \equiv & \sum_{i=0}^{2} C_{2i+1}^{2000}(5^i) = 2000 + \frac{2000\times1999\times1998}{6}\times5 + \frac{2000\times1999\times1998\times1997\times1996}{5\times24}\times25 \\ =& 125\times16 + 125\times16\times1999\times333\times5 + 125\times16\times1999\times333\times1997\times499\times5 \equiv 0 \pmod{125} \\ F_{2001} \equiv & \sum_{i=0}^{2} C_{2i+1}^{2001}(5^i) = 2001 + \frac{2001\times2000\times1999}{6}\times5 + \frac{2001\times2000\times1999\times1998\times1997}{5\times24}\times25 \\ =& (125\times16+1) + 667\times125\times8\times1999\times5 + 2001\times125\times4\times1999\times333\times1997\times5 \equiv 1 \pmod{125} \end{aligned}$$

So it is just proven that $F_{2020} \equiv F_{20} \pmod{125}$.

2.1.4.6 More about the general term formula of *Fibonacci sequence*

Notice that we have something called the geometric series, for example, $1+2+4+8+...$, and notice that the property that the *Fibonacci sequence* is inductively defined and the fact that it is *addition* of previous two terms make it reasonable to suggest some relations between geometric series and it.
Obviously the *Fibonacci sequence* is not a geometric series, as the ratio between two terms is not a constant.
(Though, the ratio between two consecutive terms actually tends to the *golden ratio* $\phi = \frac{\sqrt{5}+1}{2}$, which provides further justification on the decision to compare geometric series with it)
We assume $F_n = a^n + b^n$ for some $a, b \in \mathbb{R}\setminus\{0\}$ and for all $n \in \mathbb{N}$.
Or, you should let, more generally, $F_n = ca^n + db^n$ for some $a, b, c, d \in \mathbb{R}\setminus\{0\}$.

$$\begin{cases} ca^1 + db^1 = F_1 = 1 \\ ca^0 + db^0 = F_0 = F_2 - F_1 = 0 \end{cases}$$

$$\begin{cases} c = -d \\ c(a-b) = 1 \end{cases}$$

(Using $F_2 = 0$ to set up the equation also works, but it's slightly more complicated.)
We notice that there is not enough information given. However, as the induction condition is $F_{n+2} = F_{n+1} + F_n$, and it would be reasonable to pick a, b to be the roots of $x^{n+2} = x^{n+1} + x^n$, i.e., $x^2 = x + 1$, so it is how the *Fibonacci sequence* is nicely related to the golden ratio.
If we let:

$$(a, b) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$$

We then suggest $c = \frac{1}{\sqrt{5}}$.
As $ca^{n+2} + db^{n+2} = (ca^{n+1} + ca^n) + (db^{n+1} + db^n) = (ca^{n+1} + db^{n+1}) + (ca^n + db^n)$ and

$$\begin{cases} F_0 = 0 = ca^0 + db^0 \\ F_1 = 1 = ca^1 + db^1 = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} \end{cases}$$

We can then conclude

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

$\Bigg($You may compare $_{\text{inductively}}$ *Fibonacci sequence* $(F_n)_{n\in\mathbb{N}}$ with the suggested $\left(\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}\right)_{n\in\mathbb{N}}$ to prove.$\Bigg)$
By the way, for *Mathematical induction* here, it is slightly different than the ordinary one.
You may see *Strong Induction*. Roughly speaking, the induction hypotheses when $n = k+1$ is that $P(1), ...,$ and $P(k)$.
Obviously it's equivalent to the ordinary *Mathematical induction*: just let $Q(n)$ to be '$P(1) \wedge P(2) \wedge ... \wedge P(n)$ $\forall n \in \mathbb{N}$.

2.1.4.6.1 A proof of the general term of *Fibonacci sequence*

Let $P(n)$ be '$\left(F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}\right)$' for non-negative integer n.

$P(0)$ and $P(1)$ are true.

Assume that $P(0) \wedge P(1) \wedge \ldots \wedge P(k)$ for some $k \in \mathbb{N}$,

For $n = k+1, F_{k+1} = F_k + F_{k-1}$ as the golden ratios are the roots of $(x^{k-1})x^2 = (x^{k-1})x + (x^{k-1})$.

$\therefore$ $P(k+1)$ is true.

$\therefore$ By *Mathematical Induction*, $P(n)$ is true for all non-negative integers n.

3 IMO HK Team Selection Test 1

We are thankful for the official solution, and please be acknowledged that we may include direct copies of some parts of the official solution.

3.1 IMO HK 2020 TST 1

3.1.1 Q1

3.1.1.1 Question

Find all $(a, b, c) \in \mathbb{R}^3$ such that $\left(2^{2a} + 1\right)\left(2^{2b} + 2\right)\left(2^{2c} + 8\right) = 2^{a+b+c+5}$

3.1.1.2 Solution

$$\frac{2^{2a}+1}{2^a}\cdot\frac{2^{2b}+2}{2^{b+\frac{1}{2}}}\cdot\frac{2^{2c}+8}{2^{c+\frac{3}{2}}}=\frac{2^{a+b+c+5}}{2^{a+b+\frac{1}{2}+c+\frac{3}{2}}}$$
$$\left(2^a+\frac{1}{2^a}\right)\left(2^{b-\frac{1}{2}}+\frac{1}{2^{b-\frac{1}{2}}}\right)\left(2^{c-\frac{3}{2}}+\frac{1}{2^{c-\frac{3}{2}}}\right)=\frac{2^{a+b+c+5}}{2^{a+b+c+2}}=2\cdot 2\cdot 2$$

Notice that $2^t>0\ \forall t\in\mathbb{R}$, so we can use the fact that $x+\frac{1}{x}\geq 2\ \forall\in\mathbb{R}^+$ **and equality holds iff** $x=1$.
If any of the terms $\left(2^a+\frac{1}{2^a}\right)$, $\left(2^{b-\frac{1}{2}}+\frac{1}{2^{b-\frac{1}{2}}}\right)$, $\left(2^{c-\frac{3}{2}}+\frac{1}{2^{c-\frac{3}{2}}}\right)$ is >2, the equality cannot hold.

$$2^a=2^{b-\frac{1}{2}}=2^{c-\frac{3}{2}}=1$$
$$a=b-\frac{1}{2}=c-\frac{3}{2}=0$$
$$(a,b,c)=\left(0,\frac{1}{2},\frac{3}{2}\right)$$

3.1.2 Q3

3.1.2.1 Original Question

On the table there are 20 coins of weights 1, 2, 3, ..., 15, 37, 38, 39, 40 and 41 grams. They all look alike but their colours are all distinct. Now Miss Adams knows the weight and colour of each coin, but Mr. Bean knows only the weights of the coins. There is also a balance on the table, and each comparison of weights of two groups of coins is called an operation. Miss Adams wants to tell Mr. Bean which coin is the 1-gram coin by performing some operations. What is the minimum number of operations she needs to perform?

3.1.2.2 Question (Rephrased)

- On a table there are **20** coins of weights 1,2,3,4,$\cdots$,15, 37,38,39,40,41 grams respectively.
- The coins all look alike but their colors are distinct (hence distinguishable).
- A knows the weight and color of each coin; B only the weight (knows the first bullet point).
- There is a balance on the table that is only able to compare *L.H.S.* with *R.H.S.*, where both sides must be a non-empty subset of the 20 coins.
- **At least how many comparisons** have to be demonstrated by A so that B knows for sure which the lightest coin (which weighs 1 gram) is, provided that they are not allowed to communicate?

3.1.2.3 Attempts, examples and comments

A possible and helpful demonstration would be like putting the lightest 16 coins on *L.H.S.* and the other on *R.H.S.* The *L.H.S.* would weigh 157 grams while the *R.H.S.* would weigh 158 grams. (The weight of any 16 coins must not be less than 157 while the weight of any 4 coins must not be more than 158.) If the 16 coins are not the lightest 16, then the weight of them must exceeds half of the total weight (**315** grams) of all 20 coins. Therefore, it is for sure that the 16 coins must be the lightest 16 coins if it is seen from the balance that *L.H.S.* < *R.H.S.*
As you can see, the key is to find a comparison so delicate that B knows for sure the exact sets of coins **on *L.H.S.*, on**

***R.H.S.* and on the table**. You may think about the 3-tuple (X, Y, Z), where $X, Y, Z \subseteq \{1, 2, 3, 4, \cdots, 15, 37, 38, 39, 40, 41\}$ (Z the 'unused coins') are disjoint and make up of all 20 coins.
For example, putting the 1-gram coin on *L.H.S.* and the 2-gram coin on *R.H.S.* tells B nothing, as (X, Y, Z) is not uniquely determined by this demonstration.

3.1.2.4 Solution (modified from official)

At least **2** operations have to be done.

3.1.2.4.1 '2 operations' is sufficient - Example

1. Put coins (with weights) $1, 2, 3, 4, \cdots, 15$ on *L.H.S.* and coins $39, 40, 41$ on *R.H.S.* It is a balance $(120 = 120)$. The weight of any other $X \subseteq \{1, 2, 3, 4, \cdots, 15, 37, 38, 39, 40, 41\}$ with 15 elements (coins) would be larger than 120, whereas the weight of any other set Y with 3 elements would be less than 120, equality is then impossible. Hence $(X, Y, Z) = \big(\{1, 2, 3, \cdots, 15\}, \{39, 40, 41\}, \{37, 38\}\big)$ is uniquely determined.

2. Put coins $1, 37$ on *L.H.S.* and coin 38 on *R.H.S.* $(X, Y, Z) = \big(\{1, 37\}, \{38\}, \{\cdots\}\big)$ is also uniquely determined, as the set $\{37, 38\}$ is determined in the first step and can be identified by B with the exclusive colours of coins. (B is aware of that both 'unused' (in step 1) coins 37 and 38 are now used in X or Y but not Z). Now the lightest coin is caught.

3.1.2.4.2 '2 operations' is necessary - Proof

Proof. Assume one operation is enough, then 1 is the **only** coin in either X, Y, Z it belongs to.

- If the 1-gram coin is put on the balance,
  - If the other group on balance contains exactly one coin or no coin, it tells nothing about the weight of any coin ($1 < 2$ can be replaced by $2 < 3$, for instance);
  - If the other group on balance contains more than one coin, the 'situation' is the same when the 1-gram coin is swapped with the 2-gram coin.

- If the 1-gram coin is (the only) unused,
  - If the balance is inclined to one side, then one side is at least 158 grams while the other at most 156 grams (total weight is 315 grams), and thus the situation is the same when the 1-gram coin is swapped with the 2-gram coin (where exactly one side has a change ± 1);
  - If sets X and Y are equally heavy (same weight: 157-gram), WLOG we assume $3 \in X$ as $3 \in X \cup Y$,
    * If $2 \in Y$, we can swap the 2-gram coin with the 3-gram coin, so that now the side with the 3-gram coin is heavier by 2 grams, and we can then swap the 1-gram coin and the 3-gram coin, keeping the balance;
    * Else if for some $k > 3$, we have $k \in X \wedge k + 1 \in Y$, then we can swap those two coins and then swap the 1-gram coin and the 3-gram coin just like the above;
    * (Do **not** combine the two cases above, you need to track the side with the 3-gram coin after first 'swap'.)
    * Else,
      · $1 \notin X \wedge 2, 3 \in X$,
      · $\big(i \in X \cap \{4, 5, \cdots, 15\} \Rightarrow \forall j \in \mathbb{Z} \cap [i, 15], j \in X\big)$ so $\Big(X \cap [4, 15] = \mathbb{Z} \cap \big[\min\big(X \cap [4, 15]\big), 15\big]$ **or** $\{\}\Big)$,
      · $X \cap [37, 41] = \mathbb{Z} \cap \big[\min\big(X \cap [37, 41]\big), 41\big]$ **or** $\{\}$,

but the weight of $X \setminus \{2, 3\}$ cannot be 152, shown by some computations:

- If $38 \in X$, $38 + \cdots + 41 = 158 > 152$
- Else if $39 \in X$, $14 + 15 \ + \ 39 + 40 + 41 = 149 < 152 < 162 = 13 + 14 + 15 \ + \ 39 + 40 + 41$
- Else if $40 \in X$, $11 + 12 + \cdots + 15 \ + \ 40 + 41 = 146 < 152 < 156 = 10 + 11 + \cdots + 15 \ + \ 40 + 41$
- Else if $41 \in X$, $5 + 6 + \cdots + 15 \ + \ 41 = 151 < 152 < 155 = 4 + 5 + \cdots + 15 \ + \ 41$
- Else, $4 + 5 + \cdots + 15 = 114 < 152$

In any case, it is impossible to demonstrate only a **single** comparison so that B can decide which the 1-gram coin is. ■

3.1.2.4.3 Remark of the solution

As seen in the second operation, B needs to remember all the colours of all the coins in their respective set (X, Y or Z). More clarifications may need to be made if such a question is to be generalized or extended.