Roots being square of difference of roots of a depressed cubic

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About

This paper is being updated here.

Introduction

Here we study the *monic* cubic equation with square of difference of roots of a depressed cubic equation as its roots. Namely, given roots x_1, x_2, x_3 of $x^3 + px + q = 0$ (where $p, q \in \mathbb{C}$), we want to form an equation (which the coefficient of the highest term is 1) with roots $y_1 = (x_2 - x_3)^2, y_2 = (x_3 - x_1)^2, y_3 = (x_1 - x_2)^2$.
(Uncountably many polynomials can be formed when we do not require the leading coefficient to be 1.)
Of course, we will do it in a **rigorous** approach, but we shall also appreciate how various methods give the same answer after all.
Notice that the roots $(x_2 - x_3)^2, (x_3 - x_1)^2, (x_1 - x_2)^2$ are symmetrical that for any permutation $\sigma(i) : \{1, 2, 3\} \to \{1, 2, 3\}$, replacing all x_i by $x_{\sigma(i)}$ will not affect the equation formed.
This is, maybe, related to the notion of *symmetric polynomials*.

Lemma

First of all, we shall clearly state the relations between the roots of the original equation:

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_2x_3 + x_3x_1 + x_1x_2 = p \\ x_1x_2x_3 = -q \\ x_1^2 + x_2^2 + x_3^2 = (x_1 + x_2 + x_3)^2 - 2(x_2x_3 + x_3x_1 + x_1x_2) = -2p \end{cases}$$

WLOG, we consider y_1, that is, $(x_2 - x_3)^2$,

$$\begin{aligned} (x_2 - x_3)^2 &= x_2^2 - 2x_2x_3 + x_3^2 + x_1^2 - x_1^2 = -2p - x_1^2 - 2x_2x_3 \\ &= \begin{cases} -2p - x_1^2 - \dfrac{2x_1x_2x_3}{x_1} = -2p - x_1^2 + \dfrac{2q}{x_1} \qquad (\text{if } x_1 \neq 0) & (1) \\ -2p - 3x_1^2 + 2x_1^2 - 2x_2x_3 = -2p + \big(2x_1(-x_2 - x_3) - 2x_2x_3\big) - 3x_1^2 = -4p - 3x_1^2 & (2) \end{cases} \end{aligned}$$

$$\text{From (1),} \quad x_1^3 + (2p + y_1)x_1 - 2q = 0 \tag{3}$$

$$\text{From (2),} \quad x_1^3 + \left(\frac{4p + y_1}{3}\right)x_1 = 0 \tag{4}$$

Recall that $x_1^3 + px_1 + q = 0$, subtracting it from (3) **or** (4) (as to eliminate the term x_1^3) respectively, both giving:

$$x_1 = \frac{3q}{p + y_1} \tag{5}$$

Above holds generally for x_i, y_i $(i = 1, 2, 3)$.

Solution 1 (Using (2) to find elementary symmetrical polynomials of the new equation)

$$\begin{aligned}
&(x_2 - x_3)^2 + (x_3 - x_1)^2 + (x_1 - x_2)^2 = -12p - 3(-2p) = -6p \\
&(x_3 - x_1)^2(x_1 - x_2)^2 + (x_1 - x_2)^2(x_2 - x_3)^2 + (x_2 - x_3)^2(x_3 - x_1)^2 \\
= &(-4p - 3x_2^2)(-4p - 3x_3^2) + (-4p - 3x_3^2)(-4p - 3x_1^2) + (-4p - 3x_1^2)(-4p - 3x_2^2) \\
= &16p^2 + 12px_2^2 + 12px_3^2 + 9x_2^2x_3^2 + 16p^2 + 12px_3^2 + 12px_1^2 + 9x_3^2x_1^2 + 16p^2 + 12px_1^2 + 12px_2^2 + 9x_1^2x_2^2 \\
= &48p^2 - 48p^2 + 9p^2 = 9p^2 \\
&(x_2 - x_3)^2(x_3 - x_1)^2(x_1 - x_2)^2 \\
= &(-4p - 3x_1^2)(-4p - 3x_2^2)(-4p - 3x_3^2) = (16p^2 + 12px_1^2 + 12px_2^2 + 9x_1^2x_2^2)(-4p - 3x_3^2) \\
= &-64p^3 - 48p^2x_1^2 - 48p^2x_2^2 - 36px_1^2x_2^2 - 48p^2x_3^2 - 36px_3^2x_1^2 - 36px_2^2x_3^2 - 27x_1^2x_2^2x_3^2 \\
= &-64p^3 - 48p^2(-2p) - 36p(p^2) - 27q^2 = -(4p^3 + 27q^2)
\end{aligned}$$

Solution 2 (Using (2) to manipulate the equation)

$$\begin{aligned}
(x^6 + 2px^4 + p^2x^2) - q^2 &= (x^3 + px + q)(x^3 + px - q) = (x - x_1)(x - x_2)(x - x_3)(-1)^3(-x - x_1)(-x - x_2)(-x - x_3) \\
&= (x^2 - x_1^2)(x^2 - x_2^2)(x^2 - x_3^2)
\end{aligned}$$

By comparing coefficients, we know that $z^3 + 2pz^2 + p^2z - q^2 = (z - x_1^2)(z - x_2^2)(z - x_3^2)$.
(Coefficients of two polynomials are identical **if** values are all equal on an infinite field.)

$$\begin{aligned}
(y - y_1)(y - y_2)(y - y_3) &= \big(y - (-4p - 3x_1^2)\big)\big(y - (-4p - 3x_2^2)\big)\big(y - (-4p - 3x_3^2)\big) \\
&= -27\left(-\frac{y + 4p}{3} - x_1^2\right)\left(-\frac{y + 4p}{3} - x_2^2\right)\left(-\frac{y + 4p}{3} - x_3^2\right) \\
&= -27\left(\left(-\frac{y + 4p}{3}\right)^3 + 2p\left(-\frac{y + 4p}{3}\right)^2 + p^2\left(-\frac{y + 4p}{3}\right) - q^2\right) \\
&= (y + 4p)^3 - 6p(y + 4p)^2 + 9p^2(y + 4p) + 27q^2 \\
&= y^3 + 12py^2 + 48p^2y + 64p^3 - 6py^2 - 48p^2y - 96p^3 + 9p^2y + 36p^3 + 27q^2 \\
&= y^3 + (6p)y^2 + (9p^2)y + (4p^3 + 27q^2)
\end{aligned}$$

Remark 0.1. Steps should be justified carefully so that the proof is rigorous. For example, it may not be enough proving $(y - y_1)(y - y_2)(y - y_3) = 0 \iff y^3 + 6py^2 + 9p^2y + 4p^3 + 27q^2 = 0$. Consider $y(y - 1)^2 = 0 \iff y^2(y - 1) = 0$, where an '$\iff$' only suggests that the set of roots of two polynomials are identical, but the equality of polynomials should be taken more carefully.
It is understood that the question indeed ask for **expanding** $(y - y_1)(y - y_2)(y - y_3)$, instead of merely a polynomial with roots y_1, y_2, y_3. (Though, of course, if y_1, y_2, y_3 are distinct then there is no such problem.)

Boundary case for solution 3-4

If $q = 0, x(x^2 + p) = 0$:

WLOG we assume that $(x_1, x_2, x_3) = (0, -\sqrt{-p}, \sqrt{-p})$, then

$$(y_1, y_2, y_3) = \big((x_2 - x_3)^2, (x_3 - x_1)^2, (x_1 - x_2)^2\big) = (-4p, -p, -p)$$

$$(y + p)^2(y + 4p) = (y^2 + 2py + p^2)(y + 4p) = y^3 + 6py + 9p^2y + (4p^3 + \ 27(0)^2)$$

Below we shall **assume** $q \neq 0$, hence x_1, x_2, x_3 are all **non-zero** as well.

Solution 3 $\big($Using (5), like solution 1$\big)$

$q \neq 0$.

$$\begin{aligned}
x_i &= \frac{3q}{p + y_i} \\
y_1 + y_2 + y_3 &= (p + y_1) + (p + y_2) + (p + y_3) - 3p = 3q\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right) - 3p \\
&= \frac{3q(x_2x_3 + x_3x_1 + x_1x_2)}{x_1x_2x_3} - 3p = \frac{3pq}{-q} - 3p = -6p \\
y_2y_3 + y_3y_1 + y_1y_2 &= \big((p + y_2)(p + y_3) - p^2 - p(y_2 + y_3)\big) + \big(\cdots\big) + \big(\cdots\big) \\
&= \frac{9p^2}{x_2x_3} + \frac{9p^2}{x_3x_1} + \frac{9p^2}{x_1x_2} - 3p^2 - 2p(y_1 + y_2 + y_3) \\
&= \frac{9p^2(x_1 + x_2 + x_3)}{x_1x_2x_3} - 3p^2 + 12p^2 = 9p^2 \\
-q = x_1x_2x_3 &= \frac{3q}{p + y_1} \cdot \frac{3q}{p + y_2} \cdot \frac{3q}{p + y_3}
\end{aligned}$$

$$(p + y_1)(p + y_2)(p + y_3) = -27q^2 \tag{6}$$

Thinking combinatorically $\big($choosing terms from expanded $(p + y_1)(p + y_2)(p + y_3)\big)$ or anyway,

$$\begin{aligned}
-27q^2 &= p^3 + (y_1 + y_2 + y_3)p^2 + (y_2y_3 + y_3y_1 + y_1y_2)p + y_1y_2y_3 = p^3 - 6p^3 + 9p^3 + y_1y_2y_3 \\
y_1y_2y_3 &= -(4p^3 + 27q^2)
\end{aligned}$$

Solution 4 $\big($Using (5), like solution 2$\big)$

$$
\begin{aligned}
&(y-y_1)(y-y_2)(y-y_3)\\
&=\big((p+y)-(p+y_1)\big)\big((p+y)-(p+y_2)\big)\big((p+y)-(p+y_3)\big)\\
&=\left(\left(\frac{p+y}{3q}\right)^3(p+y_1)(p+y_2)(p+y_3)\right)\left(\frac{3q}{p+y_1}-\frac{3q}{p+y}\right)\left(\frac{3q}{p+y_2}-\frac{3q}{p+y}\right)\left(\frac{3q}{p+y_3}-\frac{3q}{p+y}\right)\\
&=-\left(\left(\frac{p+y}{3q}\right)^3\cdot(-27q^2)\right)\left(\frac{3q}{p+y}-x_1\right)\left(\frac{3q}{p+y}-x_2\right)\left(\frac{3q}{p+y}-x_3\right)\\
&=\frac{(p+y)^3}{q}\cdot\left(\left(\frac{3q}{p+y}\right)^3+p\left(\frac{3q}{p+y}\right)+q\right)\\
&=27q^2+3p(p+y)^2+(p+y)^3\\
&=y^3+(6p)y^2+(9p^2)y+(4p^3+27q^2)
\end{aligned}
$$

Answer

$$y^3+(6p)y^2+(9p^2)y+(4p^3+27q^2)=0$$

Remark

Refer to the remark in solution 2, it is always good to consider things more rigorously.
However, it is not necessary to show all the steps each time, as we know how to justify the subtleties.

More interesting results

About the *Cartan-Tartaglia formula*

The *Cartan-Tartaglia formula* (for cubic equation) is, for *depressed cubic equation* ($x^3+px=q$),

$$x=\sqrt[3]{\sqrt{\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2}+\frac{q}{2}}-\sqrt[3]{\sqrt{\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2}-\frac{q}{2}}$$

(However, it is actually hard to explain which cubic root is to be taken for complex numbers involved.)
If $\alpha,\beta,\gamma\in\mathbb{R}$ are distinct, when applying the formula on the depressed cubic equation formed by reducing the expanded expression $(y-\alpha)(y-\beta)(y-\gamma)$, complex number is **unavoidable** (that is, a negative square root occurs).
To prove this, consider $x^3+px-q=0$ having distinct and real roots $x_1=\alpha+c, x_2=\beta+c, x_3=\gamma+c$, where $c\in\mathbb{R}$ is a constant (Why? because depressed cubic is formed by substitution $y=x-c$, you can find c your own).
Using the results above, as $y_1,y_2,y_3\in\mathbb{R}^+$, $-(4p^3+27q^2)\ >0$,

$$\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2<0$$

Sure, you can solve a cubic equation like $x^3 - x = 0$ by simply finding all the rational factors. The point is, using the method to solve **general** cubic equation (I believe many variations are actually on essence highly similar to the *Cartan-Tartaglia formula*), it is very inconvenient without imaginary numbers being 'invented'.
The amazing thing is that it provides kind of a reason why complex number is so necessary in modern Mathematics.
"*The shortest path between two truths in the real domain passes through the complex domain.*" $---$ *Jacques Hadamard*