IMO HK Preliminary Selection Contest 2020 Suggested Solution

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Preface

This paper is being updated in my webpage, and is not finished until more questions are included.

Q5

Question

28 people seat in circle. They all claim that 'the two people next to me are of different genders'.
Given that all boys lie and exactly 3 girls lie, find the **number of girls**.

Solution

Denote girl by G and boy by B.
Notice that all boys lie, so it would be reasonable and easier to take girls as variable:
Namely, define $1 \leq B_1 < B_2 < B_3 < ... < B_k$, where $k \in \mathbb{N}$ is the number of boys (disregard of '28' at this moment).
We then define S_i for $i = 1, 2, ..., k$ to be the number of girls between separation of B_i and B_{i+1},
with the convention $B_{k+1} = B_1$.
Assume that for some i, $S_i = 0$, i.e., there are two successive boys with no girls between, namely, BB.
Then, consider the right B, we can extend BB to BBB because in the view of B, two people next to him are **not** of different genders.
Similarly and inductively, we would have only boys in the circle, giving a *contradiction*.
We might consider the following patterns (of course there is no boy between two **successive** boys laterally):

1. BGB: In this case that girl lies.
2. $BGGB$: In this case both girls do not lie.
3. $BG\ldots GB$, where '$\ldots$' means more than one girl: In this case girls in '$\ldots$' lie.

It is worth noticing that since every block $B\ldots B$ has at least one girl between, the number of girls would not have effect on the correctness of the acclaimed '*different genders neighbours*' proposition by any girl **in another block**.
Three girls lie. Therefore, with the constraint that $S_i \geq 1$,

$$\sum_{i=1}^{k} |S_i - 2| = 3$$

We may assume, for the sake of simplicity, that the all $S_i = 2$ at the beginning,

i.e., any two successive boys form the pattern $BGGB$.

Each time (to create) a girl that lies, we add (+) **or** remove (−) **one** girl.

Denote the three operations by $a_1, a_2, a_3 = \pm 1, \pm 1, \pm 1$, and we add the restriction that no two operations **of different sign** should be performed on the same S_i, otherwise their effects would be counteracted and the operation would be flawed.

We now have:

$$S_1 + S_2 + S_3 + \ldots + S_k + k = 28$$
$$2 + 2 + 2 + \ldots + 2 + k + a_1 + a_2 + a_3 = 28$$
$$a_1 + a_2 + a_3 = 28 - 2k - k = 28 - 3k \equiv 1 \pmod 3$$

Let t denotes the number of operations +, then $a_1 + a_2 + a_3 = t - (3 - t) = 2t - 3 = 1$.

$$\therefore t = 2 \text{ and } \{a_1, a_2, a_3\} = \{1, 1, -1\}$$

Solving $3k = 28 - (a_1 + a_2 + a_3)$, we get the number of boys $k = 9$, hence the number of girls is 19.

Example and solution by '*testing*'

$$\sum_{i=1}^{k} |S_i - 2| = 3$$

From here, we can exhaust all possible sequence $(S_i)_{i=1}^{k}$

(Disregard of the order of $S_1, S_2, \ldots$ without losing generality):

$$\{S_i\} = \begin{cases} 5, 2, 2, 2, 2, 2, \ldots, 2 \\ 4, 3, 2, 2, 2, 2, \ldots, 2 \\ 4, 1, 2, 2, 2, 2, \ldots, 2 \\ 3, 3, 3, 2, 2, 2, \ldots, 2 \\ 3, 3, 1, 2, 2, 2, \ldots, 2 \\ 3, 1, 1, 2, 2, 2, \ldots, 2 \\ 1, 1, 1, 2, 2, 2, \ldots, 2 \end{cases}$$

$$\sum S_{i\,\textbf{(include left boy of the block)}} = \begin{cases} 6 + 3 + 3 + 3 + \ldots + 3 & \to \text{never} \quad \equiv 28 \pmod 3 \\ 5 + 4 + 3 + 3 + \ldots + 3 & \to \text{never} \quad \equiv 28 \pmod 3 \\ 5 + 2 + 3 + 3 + \ldots + 3 & \to \text{possible} \\ 4 + 4 + 4 + 3 + \ldots + 3 & \to \text{never} \quad \equiv 28 \pmod 3 \\ 4 + 4 + 2 + 3 + \ldots + 3 & \to \text{possible} \\ 4 + 2 + 2 + 3 + \ldots + 3 & \to \text{never} \quad \equiv 28 \pmod 3 \\ 2 + 2 + 2 + 3 + \ldots + 3 & \to \text{never} \quad \equiv 28 \pmod 3 \end{cases}$$

So for the possible two cases we could have example (not unique) respectively (***bold*** for liers):

1. $$G\boldsymbol{GG}GB\boldsymbol{G}BGGBGGBGGBGGBGGBGGBGGBGGB$$

   , which correspond to the solution while two operation '+' are performed on the same block (same S_i);

2.

$$GGGBGGGBGBGGBGGBGGBGGBGGBGGB$$

, which correspond to the solution while two operation '+' are performed on different blocks.

Q7

Question

Solve equation $\sqrt{7-x} = 7 - x^2$ for $x \in \mathbb{R}^+$ $(x > 0)$.

Remark

There are two methods to solve this question. Observe the similarities between them.

Solution 1

Squaring both sides, giving:

$$7 - x = \left(7 - x^2\right)^2 = x^4 - 14x^2 + 49$$
$$x^4 - 14x^2 + x + 42 = 0$$

Then, it is easy to see the factor $(x+2)$ as $(-2)^4 - 14(-2)^2 + (-2) + 42 = 16 - 56 - 2 + 42 = 0$.

$$\therefore (x+2)(x^3 - 2x^2 - 10x + 21) = 0$$
$$(x+2)(x-3)(x^2 + x - 7) = 0$$
$$x = -2 \text{ or } 3 \text{ or } \frac{-1 \pm \sqrt{29}}{2}$$

Substituting into the original equation, we notice that the solution root is $\frac{\sqrt{29}-1}{2} \in \mathbb{R}^+$.

More about this solution

$$x = -2 \text{ or } 3 \text{ or } \frac{-1 \pm \sqrt{29}}{2}$$

We reject $x = -2$ only because it is negative, but $\sqrt{7-(-2)} = 7 - (-2)^2$, so it satisfies the original equation.
On the other hand, we reject $x = 3$ and $x = \frac{-1-\sqrt{29}}{2}$ as they don't even satisfy the original equation.
Why? Because they are artificially created during the process of **squaring both sides**.

You may check:

$$\sqrt{7-(3)} = 2 \neq -2 = 7 - (3)^2$$
$$\sqrt{7 - \frac{-1-\sqrt{29}}{2}} = \sqrt{\frac{30+2\sqrt{29}}{4}} = \frac{\sqrt{29}+1}{2} = \frac{30+2\sqrt{29}}{4} - 7$$
$$\neq 7 - \frac{30+2\sqrt{29}}{4} = 7 - \left(\frac{-1-\sqrt{29}}{2}\right)^2$$

Notice that all 4 roots we obtained satisfy $\sqrt{7-x} = |7 - x^2|$, if we allow $x \leq 0$.

Solution 2

Notice that the equation can be rewritten as (for $x \in \mathbb{R}^+$):

$$x^2 = 7 - \sqrt{7-x}$$
$$x = \sqrt{7 - \sqrt{7-x}}$$

We define $f(x) = \sqrt{7-x}$ for $x \in [0, 7]$.
Then, we are actually solving $f(f(x)) = x$.
Observe that if we have $f(x) = x$, $f(f(x)) = f(x) = x$ immediately follows,
and it turns out that $f(x) = x$ is easy to solve:

$$x = \sqrt{7-x}$$
$$x^2 = 7 - x$$
$$x^2 + x - 7 = 0$$
$$x = \frac{-1 \pm \sqrt{29}}{2}$$

Rejecting the root *artificially created when squaring both sides*, we get the only desired solution $\frac{\sqrt{29}-1}{2}$.

Q17

Question

Find (the number of) *positive integer solutions* to the following system of equation:

$$\begin{cases} \sqrt{2020}\big(\sqrt{a}+\sqrt{b}\big) = \sqrt{(c+2020)(d+2020)} \\ \sqrt{2020}\big(\sqrt{b}+\sqrt{c}\big) = \sqrt{(d+2020)(a+2020)} \\ \sqrt{2020}\big(\sqrt{c}+\sqrt{d}\big) = \sqrt{(a+2020)(b+2020)} \\ \sqrt{2020}\big(\sqrt{d}+\sqrt{a}\big) = \sqrt{(b+2020)(c+2020)} \end{cases}$$

Solution

WLOG, let's first consider the first equation, dealing always only with $a, b, c, d \in \mathbb{N}$,

$$\sqrt{2020}\big(\sqrt{a}+\sqrt{b}\big) = \sqrt{(c+2020)(d+2020)}$$

Notice that

$$\sqrt{a}+\sqrt{b} = \sqrt{a+b+2\sqrt{ab}},$$

or by squaring both sides in the hope of doing some number theoretical manipulations.

$$2020\big(a+b+2\sqrt{ab}\big) = (c+2020)(d+2020) = cd + 2020(c+d) + 2020^2$$

Notice that

$$a+b+2\sqrt{ab} = \frac{(c+2020)(d+2020)}{2020} \in \mathbb{Q},$$

so we must have $ab = x_1^2$ for some $x_1 \in \mathbb{N}$.
Similarly,

$$(ab, bc, cd, da) = (x_1^2, x_2^2, x_3^2, x_4^2) \text{ for some } (x_1, x_2, x_3, x_4) \in \mathbb{N}^4$$

$$2020(a + b - c - d + 2x_1 - 2020) = 2020(a + b + 2x_1) - 2020(c + d + 2020) = cd$$

$$\therefore \frac{\left(\frac{x_3}{2}\right)^2}{505} = \frac{x_3^2}{505} = \frac{cd}{2020} \in \mathbb{N}, 505|x_3, x_3 = 505y_3, y_3 \in \mathbb{N}$$

Carefully considering 4 cases, we have:

$$(x_1, x_2, x_3, x_4) = (505y_1, 505y_2, 505y_3, 505y_4) \text{ for } (y_1, y_2, y_3, y_4) \in \mathbb{N}^4$$

$$\begin{cases} 2020(a + b - c - d + 1010y_1 - 2020) = x_3^2 = 505^2y_3^2 \\ 2020(c + d - a - b + 1010y_3 - 2020) = x_1^2 = 505^2y_1^2 \end{cases}$$

$$\therefore \ 505^2\Big(8(y_1 + y_3) - 32\Big) = 2020\big(1010(y_1 + y_3) - 4040\big) = 505^2(y_1^2 + y_3^2)$$

$$(y_1 - 4)^2 + (y_3 - 4)^2 = 0 \implies y_1 = y_3 = 4$$

$$\therefore \text{(Similarly) } x_1 = x_2 = x_3 = x_4 = 2020 \wedge ab = bc = cd = da = 2020^2$$

Notice that we could not have and only have $\begin{cases} a = c \\ b = d \end{cases}$ (but not $a = b = c = d$).
The four original equations have now been reduced to any one equation of the four, and the question is reduced to finding $(a, b) \in \mathbb{N}^2$ such that $ab = 2020^2 = 2^4 \times 5^2 \times 101^2$, having $(4 + 1)(2 + 1)(2 + 1) = 45$ solutions.
Consider $\sqrt{2020}(\sqrt{a} + \sqrt{b}) = \sqrt{(c + 2020)(d + 2020)}$ such that $ab = 2020^2$, with $(c, d) = (a, b)$:

$$\sqrt{(c + 2020)(d + 2020)} = \sqrt{cd + 2020(c + d) + cd} = \sqrt{2020}\sqrt{a + b + 2\sqrt{ab}} = \sqrt{2020}(\sqrt{a} + \sqrt{b}),$$

and very similarly, we just **justified** that the solution satisfy all 4 original equations.

Remark

Simpler methods are to be discovered, but personally I did not manage to think of a way utilizing the *AM-GM inequality*, for example.

Q20

Question

Denotes $(F_n)_{n\in\mathbb{N}}$ as the *Fibonacci sequence*. Find $(F_{2020} \bmod 1000)$.

Remark

Observe that $1000 = 10^3 = 2^3 \times 5^3 = 8 \times 125$, and it suffices to consider $(F_{2020} \bmod 8)$ and $(F_{2020} \bmod 125)$, which will be shown do-able.

(You may like to refer to the wiki page of Chinese Remainder Theorem.)

Notice that it would be extremely time consuming to write out sequence $(F_n \bmod 125)$ or even $(F_n \bmod 1000)$ until it repeats itself.

For the sequence (F_n mod 5^k), I observe that it would repeat itself each 4×5^k terms, i.e., $F_{n+4\times 5^k} = F_n$.

The sequence (F_n mod 125) and (F_n mod 1000) are known to have (smallest) period 500 and 1500 respectively.

(1500 is the least common multiple of 500 and 12 = the period of sequence (F_n mod 8))

For example, F_{2020} mod 1000 = F_{520} mod 1000 = 515.

Notice that (by a small *Python* program),

$$
\begin{aligned}
F_{2020} = \ & 6390698115594355866513492739852871393819620004374587135576307978726515 \\
& 1995322001898071485437809065533306267303596007174873825024273095647974 \\
& 6498789136160967666042480660086493956623437642602865340590032906541161 \\
& 2695736240674586650793536636971284213893001403888857222739849074228772 \\
& 2358671367235709189166548965693916054419169307250217586108972590100204 \\
& 0809806705978730946175408544437835303207234730813935534972414532574645 15
\end{aligned}
$$

which even a calculator would not do it for you.

However, for small n ($n \leq 250$), you may visit Online Fibonacci Calculator to check.

Solution

For (F_{2020} mod 8), you can simply list it out:

$$(F_{2020} \bmod 8) = 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, 1, 1, 2, 3, 5, 0, 5, 5, 2, \ldots$$

The cyclic sequence has a (smallest) period of 12, and so in general F_{n+12} mod 8 = F_n mod 8, thus:

$$F_{2020} \bmod 8 = F_{2016+4} = F_4 \bmod 8 = 3$$

For (F_{2020} mod 125), however, as stated above, it is hard to wait for the sequence to repeat itself.

Instead, it is well-known that:

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

(or you shall prove by Mathematical induction and remember the formula)

Therefore, by binomial expansion,

$$F_n = \frac{1}{2^n\sqrt{5}}\Big((1+\sqrt{5})^n - (1-\sqrt{5})^n\Big) = \frac{1}{2^n\sqrt{5}}\left(\sum_{i=0}^{n} C_i^n(\sqrt{5})^i - \sum_{i=0}^{n} C_i^n(-\sqrt{5})^i\right)$$

$$= \frac{1}{2^n\sqrt{5}}\left(\sum_{i=0}^{n} C_i^n\Big((\sqrt{5})^i - (-\sqrt{5})^i\Big)\right) = \frac{1}{2^n\sqrt{5}}\sum_{i=0}^{n} C_i^n(\sqrt{5})^i\Big(1-(-1)^i\Big)$$

$$= \frac{1}{2^n\sqrt{5}}\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} C_{2i+1}^n(\sqrt{5})^{2i+1}\Big(1-(-1)^{2i+1}\Big) = \frac{\sqrt{5}}{2^{n-1}\sqrt{5}}\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} C_{2i+1}^n(\sqrt{5})^{2i}$$

$$\left(\text{Notice that } \left\lfloor\frac{n-1}{2}\right\rfloor \text{ is the least number } m \text{ such that } 2m+1 \le n\right)$$

$$= \frac{\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} C_{2i+1}^n 5^i}{2^{n-1}}$$

$$\therefore F_{2020} = \frac{\sum_{i=0}^{1009} C_{2i+1}^{2020}5^i}{2^{2019}}$$

$$\therefore 2^{2019}F_{2020} = \sum_{i=0}^{1009} C_{2i+1}^{2020}5^i = C_1^{2020} + 5\times C_3^{2020} + 25\times C_5^{2020} + 125\times(\ldots)$$

$$\equiv C_1^{2020} + 5\times C_3^{2020} + 25\times C_5^{2020} \pmod{125}$$

Now it suffices to find $2^{2019} \pmod{125}$ and $C_1^{2020} + 5\times C_3^{2020} + 25\times C_5^{2020} \pmod{125}$ *or something similar*. Below is a suggested way to do so, and is not too easy to make careless mistakes:

$$2^5 \equiv 32 \pmod{125}$$
$$2^{10} = 1024 \equiv 24 \pmod{125}$$
$$2^{20} = 1024^2 \equiv 24^2 = 576 \equiv -49 \pmod{125}$$
$$2^{25} = 2^{20}\times 2^5 \equiv (32)\times(-49) = -1568 \equiv 57 \pmod{125}$$
$$2^{50} \equiv (57)^2 = 60^2 - 2\times 3\times 60 + 3^2 = 3600 - 360 + 9 \equiv -1 \pmod{125}$$
$$\left(2^{100} = 2^{\varphi(125)} \equiv 1 \pmod{125} \text{ simply by Euler's Theorem}\right)$$
$$2^{2020} \equiv 2^{20}\times\Big((2^{50})^2\Big)^{20} \equiv -49\times((-1)^2)^{20} = -49 \pmod{125}$$

$$C_1^{2020} + 5\times C_3^{2020} + 25\times C_5^{2020} = 2020 + \frac{2020\times 2019\times 2018}{3!}\times 5 + \frac{2020\times 2019\times 2018\times 2017\times 2016}{5!}\times 25$$
$$= 2020 + 2020\times 2019\times 2018\div 6\times 5$$
$$+ 2020\times 2019\times 2018\times 2017\times 2016\div 3\div 8\times 5$$
$$= 2020 + 2020\times 673\times 1009\times 5 + 2020\times 673\times 2018\times 2017\times 252\times 5$$
$$\equiv 20 + 20\times 48\times 9\times 5 + 20\times 48\times 18\times 17\times 2\times 5$$
$$= 20 + 100\times 432 + 100\times 96\times 18\times 17$$
$$\equiv 20 + 75 + (-25)\times(-29)\times 306 = 95 + 725\times 306 \equiv -30 - 25\times 56$$
$$= -1430 \equiv 70 \pmod{125}$$

$$\therefore 22F_{2020} \equiv 147F_{2020} = -3(-49F_{2020}) \equiv -6(2^{2019}F_{2020}) \equiv -6 \times 70 = -420 \equiv -45 \pmod{125}$$

To find F_{2020} mod 125, try to find the multiplicative inverse of 22: $22^{-1} \pmod{125}$.
Here is a method without explicitly finding the multiplicative inverse:

Notice that $22 \times 6 = 132 = 125 + 7$

, which $7F_{2020} \equiv 132F_{2020}$ can simplify the expression.

After that, notice that $18 \times 7 = 126 = 125 + 1$ can again simplify the problem.

$$\begin{aligned} F_{2020} &\equiv 126F_{2020} = 18(7F_{2020}) \equiv 18(132F_{2020}) = (18)(6)(22F_{2020}) \equiv (108)(-45) \\ &\equiv 17 \times 45 = 765 \equiv 15 \pmod{125} \end{aligned}$$

After all, use the Chinese Remainder Theorem, or by considering:

$$F_{2020} = \cancel{15}, \cancel{140}, \cancel{265}, \cancel{390}, \mathbf{515}, \cancel{640}, \cancel{765}, \cancel{890} \pmod{\mathbf{1000}}$$

and notice that only 515 provides $F_{2020} \equiv 3 \pmod 8$.

Interesting Generalization

Consider a sequence $x, y, x+y, x+2y, 2x+3y, \ldots$.

It is defined similarly as the *Fibonacci sequence* except that the initial two terms are different.

One may generalize *Fibonacci sequence* for other starting values. And may also generalize it for all $n \in \mathbb{Z}$.

(Just define $F_n = F_{n+2} - F_{n+1}$ for all non-negative integers n)

These generalizations are obvious. Instead, here we would like to generalize the property of the *Fibonacci sequence* and discover the relation between the sequence and the number 5 (which is mostly implied by the beautiful relation with the *golden ratio*, see the section More about the general term formula of *Fibonacci sequence* below).

Lemma 0.1. *If $(2i+1)! = 5^g h$ for non-negative integers g, h and $5 \nmid h$, then $i - g \geq 1$.*

Proof.

$$g = \left\lfloor \frac{2i+1}{5^1} \right\rfloor + \left\lfloor \frac{2i+1}{5^2} \right\rfloor + \left\lfloor \frac{2i+1}{5^3} \right\rfloor + \ldots \leq \frac{2i+1}{5^1} + \frac{2i+1}{5^2} + \frac{2i+1}{5^3} + \ldots = \lim_{m\to\infty} \sum_{j=1}^{m} \left(\frac{2i+1}{5^j} \right)$$

$$= \lim_{m\to\infty} \left(\frac{\text{Next term - first term}}{\text{Common ratio} - 1} \right) = \frac{(2i+1) - 0}{5 - 1}$$

$$\frac{2i-1}{4} \leq i - g \in \mathbb{Z}, \quad \therefore i - g \geq 1$$

∎

Theorem 0.2. *For non-negative integers m, n, $F_{n\times 5^m} \equiv 0 \pmod{5^m}$.*

Proof.

$$2^{n\times 5^m - 1} F_{n\times 5^m} = \sum_{i=0}^{\left\lfloor \frac{n\times 5^m - 1}{2} \right\rfloor} C_{2i+1}^{n\times 5^m} \left(5^i\right) \equiv \sum_{i=0}^{\min\left\{\left\lfloor \frac{n\times 5^m - 1}{2} \right\rfloor, m-1\right\}} C_{2i+1}^{n\times 5^m} \left(5^i\right) \pmod{5^m}$$

For each $i, C_{2i+1}^{n\times 5^m}\left(5^i\right) = \dfrac{(5^m)(5^m-1)\ldots(5^m-2i)}{(2i+1)!}\left(5^i\right) \equiv 0 \pmod{5^m}$ by this lemma.

Then, coping the special case $n = 0$, by considering multiplicative inverse or just $2^{n\times 5^m - 1} F_{n\times 5^m} = 5^k t \ (t \in \mathbb{N})$:

$$F_{n\times 5^m} \equiv 0 \pmod{5^m}$$

∎

Alternative solution simplified at finding (F_{2020} mod 125)

In the above, we have proven that $F_{n+500} \equiv F_n \pmod{125}$ for all $n \in \mathbb{N}$.
We may then compute $F_{2020} \bmod 125 = F_{20} \bmod 125$ by listing out the terms of $(F_n \bmod 125)_{n\in\mathbb{N}}$:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 19, 108, 2, 110, 112, 97, 84, 56, \mathbf{15}$$

Notice that for this solution, as $F_{n+500} \equiv F_n \pmod{125}$ is used, you need to observe that $F_{4\times 125} \equiv 0 \pmod{125}$ and $F_{4\times 125+1} \equiv 1 \pmod{125}$ in order to think of this solution.

Or, from a competition-directed point of view:
One may suggest or guess F_{2000} to be something simple (mod 125) (just like $\left(F_n \bmod 2\right)_{n\in\mathbb{N}}$ is something easily predictable), and with $F_{2001} \equiv 1 \pmod{125}$ also observed, one can notice that the *Fibonacci sequence* repeats itself every 2000 terms. Though not the least possible period, it's still useful.

$$\begin{aligned}
F_{2000} \equiv \sum_{i=0}^{2} C_{2i+1}^{2000}(5^i) &= 2000 + \frac{2000 \times 1999 \times 1998}{6} \times 5 + \frac{2000 \times 1999 \times 1998 \times 1997 \times 1996}{5 \times 24} \times 25 \\
&= 125 \times 16 + 125 \times 16 \times 1999 \times 333 \times 5 + 125 \times 16 \times 1999 \times 333 \times 1997 \times 499 \times 5 \equiv 0 \pmod{125} \\
F_{2001} \equiv \sum_{i=0}^{2} C_{2i+1}^{2001}(5^i) &= 2001 + \frac{2001 \times 2000 \times 1999}{6} \times 5 + \frac{2001 \times 2000 \times 1999 \times 1998 \times 1997}{5 \times 24} \times 25 \\
&= (125 \times 16 + 1) + 667 \times 125 \times 8 \times 1999 \times 5 + 2001 \times 125 \times 4 \times 1999 \times 333 \times 1997 \times 5 \equiv 1 \pmod{125}
\end{aligned}$$

So it is just proven that $F_{2020} \equiv F_{20} \pmod{125}$.

More about the general term formula of *Fibonacci sequence*

Notice that we have something called the geometric series, for example, $1+2+4+8+...$, and notice that the property that the *Fibonacci sequence* is inductively defined and the fact that it is *addition* of previous two terms make it reasonable to suggest some relations between geometric series and it.
Obviously the *Fibonacci sequence* is not a geometric series, as the ratio between two terms is not a constant.
(Though, the ratio between two consecutive terms actually tends to the *golden ratio* $\phi = \frac{\sqrt{5}+1}{2}$, which provides further justification on the decision to compare geometric series with it)
We assume $F_n = a^n + b^n$ for some $a, b \in \mathbb{R} \setminus \{0\}$ and for all $n \in \mathbb{N}$.
Or, you should let, more generally, $F_n = ca^n + db^n$ for some $a, b, c, d \in \mathbb{R} \setminus \{0\}$.

$$\begin{cases} ca^1 + db^1 = F_1 = 1 \\ ca^0 + db^0 = F_0 = F_2 - F_1 = 0 \end{cases}$$

$$\begin{cases} c = -d \\ c(a-b) = 1 \end{cases}$$

(Using $F_2 = 0$ to set up the equation also works, but it's slightly more complicated.)
We notice that there is not enough information given. However, as the induction condition is $F_{n+2} = F_{n+1} + F_n$, and it would be reasonable to pick a, b to be the roots of $x^{n+2} = x^{n+1} + x^n$, i.e., $x^2 = x + 1$, so it is how the *Fibonacci sequence* is nicely related to the golden ratio.
If we let:

$$(a, b) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$$

We then suggest $c = \frac{1}{\sqrt{5}}$.
As $ca^{n+2} + db^{n+2} = (ca^{n+1} + ca^n) + (db^{n+1} + db^n) = (ca^{n+1} + db^{n+1}) + (ca^n + db^n)$ and

$$\begin{cases} F_0 = 0 = ca^0 + db^0 \\ F_1 = 1 = ca^1 + db^1 = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} \end{cases}$$

We can then conclude

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

$\Bigg($You may compare $_{\text{inductively}}$ *Fibonacci sequence* $(F_n)_{n\in\mathbb{N}}$ with the suggested $\left(\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}\right)_{n\in\mathbb{N}}\Bigg)$ to prove.
By the way, for *Mathematical induction* here, it is slightly different than the ordinary one.
You may see *Strong Induction*. Roughly speaking, the induction hypotheses when $n = k+1$ is that $P(1), ...,$ and $P(k)$.
Obviously it's equivalent to the ordinary *Mathematical induction*: just let $Q(n)$ to be '$P(1) \wedge P(2) \wedge ... \wedge P(n)$ $\forall n \in \mathbb{N}$.

A proof of the general term of *Fibonacci sequence*

Let $P(n)$ be '$\left(F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}\right)$' for non-negative integer n.

$P(0)$ and $P(1)$ are true.

Assume that $P(0) \wedge P(1) \wedge \ldots \wedge P(k)$ for some $k \in \mathbb{N}$,

For $n = k+1, F_{k+1} = F_k + F_{k-1}$ as the golden ratios are the roots of $(x^{k-1})x^2 = (x^{k-1})x + (x^{k-1})$.

$\therefore$ $P(k+1)$ is true.

$\therefore$ By *Mathematical Induction*, $P(n)$ is true for all non-negative integers n.