Anti-derivative of any polynomial over $(1+x^4)$?

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What is eventually achieved

$$
\begin{pmatrix}
\int \frac{1}{x^4+1} dx \\
\int \frac{x}{x^4+1} dx \\
\int \frac{x^2}{x^4+1} dx \\
\int \frac{x^3}{x^4+1} dx
\end{pmatrix}
=
\begin{pmatrix}
\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{4\sqrt{2}} & -\frac{1}{4\sqrt{2}} \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\
\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{4\sqrt{2}} & \frac{1}{4\sqrt{2}} \\
0 & 0 & \frac{1}{4} & \frac{1}{4}
\end{pmatrix}
\begin{pmatrix}
\arctan\left(\sqrt{2}x+1\right) \\
\arctan\left(\sqrt{2}x-1\right) \\
\ln\left(x^2+\sqrt{2}x+1\right) \\
\ln\left(x^2-\sqrt{2}x+1\right)
\end{pmatrix}
$$

$$
\begin{pmatrix}
\int \frac{1}{(x^4+1)^2} dx \\
\int \frac{x}{(x^4+1)^2} dx \\
\int \frac{x^2}{(x^4+1)^2} dx \\
\int \frac{x^3}{(x^4+1)^2} dx \\
\int \frac{x^4}{(x^4+1)^2} dx \\
\int \frac{x^5}{(x^4+1)^2} dx \\
\int \frac{x^6}{(x^4+1)^2} dx \\
\int \frac{x^7}{(x^4+1)^2} dx
\end{pmatrix}
=
\begin{pmatrix}
\frac{3}{8\sqrt{2}} & \frac{3}{8\sqrt{2}} & \frac{3}{16\sqrt{2}} & -\frac{3}{16\sqrt{2}} & 0 & 0 & -\frac{1}{8\sqrt{2}} & \frac{1}{8\sqrt{2}} \\
-\frac{1}{4} & \frac{1}{4} & 0 & 0 & -\frac{1}{8\sqrt{2}} & \frac{1}{8\sqrt{2}} & 0 & 0 \\
\frac{1}{8\sqrt{2}} & \frac{1}{8\sqrt{2}} & -\frac{1}{16\sqrt{2}} & \frac{1}{16\sqrt{2}} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} \\
0 & 0 & 0 & 0 & -\frac{1}{8\sqrt{2}} & \frac{1}{8\sqrt{2}} & -\frac{1}{8} & -\frac{1}{8} \\
\frac{1}{8\sqrt{2}} & \frac{1}{8\sqrt{2}} & \frac{1}{16\sqrt{2}} & -\frac{1}{16\sqrt{2}} & 0 & 0 & \frac{1}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} \\
-\frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} & 0 & 0 \\
\frac{3}{8\sqrt{2}} & \frac{3}{8\sqrt{2}} & -\frac{3}{16\sqrt{2}} & \frac{3}{16\sqrt{2}} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8\sqrt{2}} & \frac{1}{8\sqrt{2}} \\
0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} & \frac{1}{8} & \frac{1}{8}
\end{pmatrix}
\begin{pmatrix}
\arctan\left(\sqrt{2}x+1\right) \\
\arctan\left(\sqrt{2}x-1\right) \\
\ln\left(x^2+\sqrt{2}x+1\right) \\
\ln\left(x^2-\sqrt{2}x+1\right) \\
\frac{x}{x^2+\sqrt{2}x+1} \\
\frac{x}{x^2-\sqrt{2}x+1} \\
\frac{1}{x^2+\sqrt{2}x+1} \\
\frac{1}{x^2-\sqrt{2}x+1}
\end{pmatrix}
$$

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$$
\begin{pmatrix}
\int \frac{1}{(x^4+1)^3}\,dx \\
\int \frac{x}{(x^4+1)^3}\,dx \\
\int \frac{x^2}{(x^4+1)^3}\,dx \\
\int \frac{x^3}{(x^4+1)^3}\,dx \\
\int \frac{x^4}{(x^4+1)^3}\,dx \\
\int \frac{x^5}{(x^4+1)^3}\,dx \\
\int \frac{x^6}{(x^4+1)^3}\,dx \\
\int \frac{x^7}{(x^4+1)^3}\,dx \\
\int \frac{x^8}{(x^4+1)^3}\,dx \\
\int \frac{x^9}{(x^4+1)^3}\,dx \\
\int \frac{x^{10}}{(x^4+1)^3}\,dx \\
\int \frac{x^{11}}{(x^4+1)^3}\,dx
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\frac{21}{64\sqrt{2}} & \frac{21}{64\sqrt{2}} & \frac{21}{128\sqrt{2}} & -\frac{21}{128\sqrt{2}} & 0 & 0 & -\frac{1}{8\sqrt{2}} & \frac{1}{8\sqrt{2}} & -\frac{1}{64} & -\frac{1}{64} & -\frac{1}{32\sqrt{2}} & \frac{1}{32\sqrt{2}} \\
-\frac{3}{16} & \frac{3}{16} & 0 & 0 & -\frac{7}{64\sqrt{2}} & \frac{7}{64\sqrt{2}} & -\frac{1}{64} & -\frac{1}{64} & 0 & 0 & \frac{1}{64} & \frac{1}{64} \\
\frac{5}{64\sqrt{2}} & \frac{5}{64\sqrt{2}} & -\frac{5}{128\sqrt{2}} & \frac{5}{128\sqrt{2}} & \frac{5}{64} & \frac{5}{64} & \frac{3}{32\sqrt{2}} & -\frac{3}{32\sqrt{2}} & \frac{1}{64} & \frac{1}{64} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{3}{64\sqrt{2}} & \frac{3}{64\sqrt{2}} & -\frac{3}{64} & -\frac{3}{64} & -\frac{1}{32\sqrt{2}} & \frac{1}{32\sqrt{2}} & -\frac{1}{64} & -\frac{1}{64} \\
\frac{3}{64\sqrt{2}} & \frac{3}{64\sqrt{2}} & \frac{3}{128\sqrt{2}} & -\frac{3}{128\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{64} & \frac{1}{64} & \frac{1}{32\sqrt{2}} & -\frac{1}{32\sqrt{2}} \\
-\frac{1}{16} & \frac{1}{16} & 0 & 0 & -\frac{1}{64\sqrt{2}} & \frac{1}{64\sqrt{2}} & \frac{1}{64} & \frac{1}{64} & 0 & 0 & -\frac{1}{64} & -\frac{1}{64} \\
\frac{3}{64\sqrt{2}} & \frac{3}{64\sqrt{2}} & -\frac{3}{128\sqrt{2}} & \frac{3}{128\sqrt{2}} & \frac{3}{64} & \frac{3}{64} & \frac{1}{32\sqrt{2}} & -\frac{1}{32\sqrt{2}} & -\frac{1}{64} & -\frac{1}{64} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{5}{64\sqrt{2}} & \frac{5}{64\sqrt{2}} & -\frac{5}{64} & -\frac{5}{64} & \frac{1}{32\sqrt{2}} & -\frac{1}{32\sqrt{2}} & \frac{1}{64} & \frac{1}{64} \\
\frac{5}{64\sqrt{2}} & \frac{5}{64\sqrt{2}} & \frac{5}{128\sqrt{2}} & -\frac{5}{128\sqrt{2}} & 0 & 0 & \frac{1}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} & -\frac{1}{64} & -\frac{1}{64} & -\frac{1}{32\sqrt{2}} & \frac{1}{32\sqrt{2}} \\
-\frac{3}{16} & \frac{3}{16} & 0 & 0 & \frac{9}{64\sqrt{2}} & -\frac{9}{64\sqrt{2}} & -\frac{1}{64} & -\frac{1}{64} & 0 & 0 & \frac{1}{64} & \frac{1}{64} \\
\frac{21}{64\sqrt{2}} & \frac{21}{64\sqrt{2}} & -\frac{21}{128\sqrt{2}} & \frac{21}{128\sqrt{2}} & -\frac{11}{64} & -\frac{11}{64} & -\frac{5}{32\sqrt{2}} & \frac{5}{32\sqrt{2}} & \frac{1}{64} & \frac{1}{64} & 0 & 0 \\
0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{13}{64\sqrt{2}} & -\frac{13}{64\sqrt{2}} & \frac{13}{64} & \frac{13}{64} & -\frac{1}{32\sqrt{2}} & \frac{1}{32\sqrt{2}} & -\frac{1}{64} & -\frac{1}{64}
\end{pmatrix}
\cdot
\begin{pmatrix}
\arctan\left(\sqrt{2}x+1\right) \\
\arctan\left(\sqrt{2}x-1\right) \\
\ln\left(x^2+\sqrt{2}x+1\right) \\
\ln\left(x^2-\sqrt{2}x+1\right) \\
\frac{x}{x^2+\sqrt{2}x+1} \\
\frac{x}{x^2-\sqrt{2}x+1} \\
\frac{1}{x^2+\sqrt{2}x+1} \\
\frac{1}{x^2-\sqrt{2}x+1} \\
\frac{x}{(x^2+\sqrt{2}x+1)^2} \\
\frac{x}{(x^2-\sqrt{2}x+1)^2} \\
\frac{1}{(x^2+\sqrt{2}x+1)^2} \\
\frac{1}{(x^2-\sqrt{2}x+1)^2}
\end{pmatrix}
$$

But do not forget integration constant!

Here I am not going to deduce $\int \frac{\cdots}{x^4+1}dx$, but only write some **very brief** steps for finding $\int \frac{x^n}{(x^4+1)^2}dx$ and yet even briefer steps for $\int \frac{x^n}{(x^4+1)^3}dx$.

Also, although I did not yet find any, it is possible that some negative signs may not be printed out (I believe it should be fine after zooming in; just occasional problem, maybe with some pdf viewers, I am not sure, but it's better to be more careful) after this *.tex file* is compiled. Better copy the source file to use these matrices more accurately.

Some thoughts and questions for myself

Why for these two things eventually I can summarize the anti-derivatives into linear combinations of $8 = 4 \times 2$ *elementary*(?) functions, with a 'coefficient matrix' of 8×8 (also 4×4 for the first). It is nice to see square matrix,

giving rise of inverse matrix (for non-square matrix why can't us define inverse matrix as well?), but I do not quite understand how and why things can be successfully done (um... interesting / intriguing). Also, these functions are defined for all $x \in \mathbb{R}$, but I do not think I will always be so lucky. These results, though not very tidy and not even known whether are simplest (can I just use 7 functions?), may worth appreciation.
Also, now I can only feel how matrices are going to work and trust online calculators for the way inverse matrix helps finding the solution set to simultaneous equations, but I should be more familiar with such *linear algebra* for deeper understanding.

$$\int \frac{P(x)}{(1+x^4)^2} dx$$

0.1 Manipulating some matrices...

$$A := \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & -\sqrt{2} & 1 & \sqrt{2} & 1 & -2\sqrt{2} & 1 & 2\sqrt{2} \\ -\sqrt{2} & 1 & \sqrt{2} & 1 & -2\sqrt{2} & 4 & 2\sqrt{2} & 4 \\ 1 & 0 & 1 & 0 & 4 & -2\sqrt{2} & 4 & 2\sqrt{2} \\ 0 & 1 & 0 & 1 & -2\sqrt{2} & 1 & 2\sqrt{2} & 1 \\ 1 & -\sqrt{2} & 1 & \sqrt{2} & 1 & 0 & 1 & 0 \\ -\sqrt{2} & 1 & \sqrt{2} & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{3\sqrt{2}}{16} & 0 & \frac{-\sqrt{2}}{16} & 0 & \frac{\sqrt{2}}{16} & 0 & \frac{-3\sqrt{2}}{16} & \frac{1}{2} \\ \frac{3}{8} & \frac{-\sqrt{2}}{16} & \frac{-1}{8} & \frac{\sqrt{2}}{16} & \frac{1}{8} & \frac{-3\sqrt{2}}{16} & \frac{1}{8} & \frac{3\sqrt{2}}{16} \\ \frac{-3\sqrt{2}}{16} & 0 & \frac{\sqrt{2}}{16} & 0 & \frac{-\sqrt{2}}{16} & 0 & \frac{3\sqrt{2}}{16} & \frac{1}{2} \\ \frac{3}{8} & \frac{\sqrt{2}}{16} & \frac{-1}{8} & \frac{-\sqrt{2}}{16} & \frac{1}{8} & \frac{3\sqrt{2}}{16} & \frac{1}{8} & \frac{-3\sqrt{2}}{16} \\ \frac{\sqrt{2}}{8} & \frac{-1}{8} & 0 & \frac{1}{8} & \frac{-\sqrt{2}}{8} & \frac{1}{8} & 0 & \frac{-1}{8} \\ \frac{1}{8} & \frac{-\sqrt{2}}{8} & \frac{1}{8} & 0 & \frac{-1}{8} & \frac{\sqrt{2}}{8} & \frac{-1}{8} & 0 \\ \frac{-\sqrt{2}}{8} & \frac{-1}{8} & 0 & \frac{1}{8} & \frac{\sqrt{2}}{8} & \frac{1}{8} & 0 & \frac{-1}{8} \\ \frac{1}{8} & \frac{\sqrt{2}}{8} & \frac{1}{8} & 0 & \frac{-1}{8} & \frac{-\sqrt{2}}{8} & \frac{-1}{8} & 0 \end{pmatrix}$$

$$B_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, B_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} B_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \cdots, B_8 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(That is, $(B_1|B_2|B_3|B_4|\cdots|B_8) = I_8$)

By evaluating $A^{-1}B_i$ where $i = 1, 2, \cdots, 8$, and transpose each, we get, in ascending order of i, we get:

$$\begin{pmatrix} \frac{3\sqrt{2}}{16} & \frac{3}{8} & \frac{-3\sqrt{2}}{16} & \frac{3}{8} & \frac{\sqrt{2}}{8} & \frac{1}{8} & \frac{-\sqrt{2}}{8} & \frac{1}{8} \end{pmatrix}$$
$$\begin{pmatrix} 0 & \frac{-\sqrt{2}}{16} & 0 & \frac{\sqrt{2}}{16} & \frac{-1}{8} & \frac{-\sqrt{2}}{8} & \frac{-1}{8} & \frac{\sqrt{2}}{8} \end{pmatrix}$$
$$\begin{pmatrix} \frac{-\sqrt{2}}{16} & \frac{-1}{8} & \frac{\sqrt{2}}{16} & \frac{-1}{8} & 0 & \frac{1}{8} & 0 & \frac{1}{8} \end{pmatrix}$$
$$\begin{pmatrix} 0 & \frac{\sqrt{2}}{16} & 0 & \frac{-\sqrt{2}}{16} & \frac{1}{8} & 0 & \frac{1}{8} & 0 \end{pmatrix}$$
$$\begin{pmatrix} \frac{\sqrt{2}}{16} & \frac{1}{8} & \frac{-\sqrt{2}}{16} & \frac{1}{8} & \frac{-\sqrt{2}}{8} & \frac{-1}{8} & \frac{\sqrt{2}}{8} & \frac{-1}{8} \end{pmatrix}$$
$$\begin{pmatrix} 0 & \frac{-3\sqrt{2}}{16} & 0 & \frac{3\sqrt{2}}{16} & \frac{1}{8} & \frac{\sqrt{2}}{8} & \frac{1}{8} & \frac{-\sqrt{2}}{8} \end{pmatrix}$$
$$\begin{pmatrix} \frac{-3\sqrt{2}}{16} & \frac{1}{8} & \frac{3\sqrt{2}}{16} & \frac{1}{8} & 0 & \frac{-1}{8} & 0 & \frac{-1}{8} \end{pmatrix}$$
$$\begin{pmatrix} \frac{1}{2} & \frac{3\sqrt{2}}{16} & \frac{1}{2} & \frac{-3\sqrt{2}}{16} & \frac{-1}{8} & 0 & \frac{-1}{8} & 0 \end{pmatrix}$$

Then if we combine them in one single 8×8 matrix, we get:

$$M := \begin{pmatrix} \frac{3\sqrt{2}}{16} & \frac{3}{8} & \frac{-3\sqrt{2}}{16} & \frac{3}{8} & \frac{\sqrt{2}}{8} & \frac{1}{8} & \frac{-\sqrt{2}}{8} & \frac{1}{8} \\ 0 & \frac{-\sqrt{2}}{16} & 0 & \frac{\sqrt{2}}{16} & \frac{-1}{8} & \frac{-\sqrt{2}}{8} & \frac{-1}{8} & \frac{\sqrt{2}}{8} \\ \frac{-\sqrt{2}}{16} & \frac{-1}{8} & \frac{\sqrt{2}}{16} & \frac{-1}{8} & 0 & \frac{1}{8} & 0 & \frac{1}{8} \\ 0 & \frac{\sqrt{2}}{16} & 0 & \frac{-\sqrt{2}}{16} & \frac{1}{8} & 0 & \frac{1}{8} & 0 \\ \frac{\sqrt{2}}{16} & \frac{1}{8} & \frac{-\sqrt{2}}{16} & \frac{1}{8} & \frac{-\sqrt{2}}{8} & \frac{-1}{8} & \frac{\sqrt{2}}{8} & \frac{-1}{8} \\ 0 & \frac{-3\sqrt{2}}{16} & 0 & \frac{3\sqrt{2}}{16} & \frac{1}{8} & \frac{\sqrt{2}}{8} & \frac{1}{8} & \frac{-\sqrt{2}}{8} \\ \frac{-3\sqrt{2}}{16} & \frac{1}{8} & \frac{3\sqrt{2}}{16} & \frac{1}{8} & 0 & \frac{-1}{8} & 0 & \frac{-1}{8} \\ \frac{1}{2} & \frac{3\sqrt{2}}{16} & \frac{1}{2} & \frac{-3\sqrt{2}}{16} & \frac{-1}{8} & 0 & \frac{-1}{8} & 0 \end{pmatrix}.$$

(You may want to de-rationalize, that is, for example, change $\frac{3\sqrt{2}}{16}$ to $\frac{3}{8\sqrt{2}}$, but it is up to you.)
It turns out that, $M = (A^{-1})^T$ (copying A^{-1} here)

$$= \begin{pmatrix} \frac{3\sqrt{2}}{16} & 0 & \frac{-\sqrt{2}}{16} & 0 & \frac{\sqrt{2}}{16} & 0 & \frac{-3\sqrt{2}}{16} & \frac{1}{2} \\ \frac{3}{8} & \frac{-\sqrt{2}}{16} & \frac{-1}{8} & \frac{\sqrt{2}}{16} & \frac{1}{8} & \frac{-3\sqrt{2}}{16} & \frac{1}{8} & \frac{3\sqrt{2}}{16} \\ \frac{-3\sqrt{2}}{16} & 0 & \frac{\sqrt{2}}{16} & 0 & \frac{-\sqrt{2}}{16} & 0 & \frac{3\sqrt{2}}{16} & \frac{1}{2} \\ \frac{3}{8} & \frac{\sqrt{2}}{16} & \frac{-1}{8} & \frac{-\sqrt{2}}{16} & \frac{1}{8} & \frac{3\sqrt{2}}{16} & \frac{1}{8} & \frac{-3\sqrt{2}}{16} \\ \frac{\sqrt{2}}{8} & \frac{-1}{8} & 0 & \frac{1}{8} & \frac{-\sqrt{2}}{8} & \frac{1}{8} & 0 & \frac{-1}{8} \\ \frac{1}{8} & \frac{-\sqrt{2}}{8} & \frac{1}{8} & 0 & \frac{-1}{8} & \frac{\sqrt{2}}{8} & \frac{-1}{8} & 0 \\ \frac{-\sqrt{2}}{8} & \frac{-1}{8} & 0 & \frac{1}{8} & \frac{\sqrt{2}}{8} & \frac{1}{8} & 0 & \frac{-1}{8} \\ \frac{1}{8} & \frac{\sqrt{2}}{8} & \frac{1}{8} & 0 & \frac{-1}{8} & \frac{-\sqrt{2}}{8} & \frac{-1}{8} & 0 \end{pmatrix}^T.$$

Anyway, now we can say that:

$$\begin{pmatrix} \frac{1}{(x^4+1)^2} \\ \frac{x}{(x^4+1)^2} \\ \frac{x^2}{(x^4+1)^2} \\ \frac{x^3}{(x^4+1)^2} \\ \frac{x^4}{(x^4+1)^2} \\ \frac{x^5}{(x^4+1)^2} \\ \frac{x^6}{(x^4+1)^2} \\ \frac{x^7}{(x^4+1)^2} \end{pmatrix} = M \cdot \begin{pmatrix} \frac{x}{x^2+\sqrt{2}x+1} \\ \frac{1}{x^2+\sqrt{2}x+1} \\ \frac{x}{x^2-\sqrt{2}x+1} \\ \frac{1}{x^2-\sqrt{2}x+1} \\ \frac{x}{\left(x^2+\sqrt{2}x+1\right)^2} \\ \frac{1}{\left(x^2+\sqrt{2}x+1\right)^2} \\ \frac{x}{\left(x^2-\sqrt{2}x+1\right)^2} \\ \frac{1}{\left(x^2-\sqrt{2}x+1\right)^2} \end{pmatrix}$$

$$\begin{pmatrix} \int \frac{1}{(x^4+1)^2} dx \\ \int \frac{x}{(x^4+1)^2} dx \\ \int \frac{x^2}{(x^4+1)^2} dx \\ \int \frac{x^3}{(x^4+1)^2} dx \\ \int \frac{x^4}{(x^4+1)^2} dx \\ \int \frac{x^5}{(x^4+1)^2} dx \\ \int \frac{x^6}{(x^4+1)^2} dx \\ \int \frac{x^7}{(x^4+1)^2} dx \end{pmatrix} = M \cdot \begin{pmatrix} \int \frac{x}{x^2+\sqrt{2}x+1} dx \\ \int \frac{1}{x^2+\sqrt{2}x+1} dx \\ \int \frac{x}{x^2-\sqrt{2}x+1} dx \\ \int \frac{1}{x^2-\sqrt{2}x+1} dx \\ \int \frac{x}{\left(x^2+\sqrt{2}x+1\right)^2} dx \\ \int \frac{1}{\left(x^2+\sqrt{2}x+1\right)^2} dx \\ \int \frac{x}{\left(x^2-\sqrt{2}x+1\right)^2} dx \\ \int \frac{1}{\left(x^2-\sqrt{2}x+1\right)^2} dx \end{pmatrix}.$$

0.2 Some results beforehand

0.2.1 Definition of some functions to be utilized

$$\begin{aligned}
f_1(x) &= \arctan\left(\sqrt{2}x+1\right)\\
f_2(x) = -f_1(-x) &= \arctan\left(\sqrt{2}x-1\right)\\
f_3(x) &= \ln\left(x^2+\sqrt{2}x+1\right)\\
f_4(x) = -f_3(x) &= \ln\left(x^2-\sqrt{2}x+1\right)\\
f_5(x) &= \frac{x}{x^2+\sqrt{2}x+1}\\
f_6(x) = -f_5(-x) &= \frac{x}{x^2-\sqrt{2}x+1}\\
f_7(x) &= \frac{1}{x^2+\sqrt{2}x+1}\\
f_8(x) = f_7(-x) &= \frac{1}{x^2-\sqrt{2}x+1}
\end{aligned}$$

0.2.2 Some anti-derivatives results to be used

$$\begin{aligned}
\int \frac{x}{x^2+\sqrt{2}x+1}dx &= -f_1(x)+\frac{1}{2}f_3(x)+C\\
\int \frac{1}{x^2+\sqrt{2}x+1}dx &= \sqrt{2}f_1(x)+C\\
\int \frac{x}{x^2-\sqrt{2}x+1}dx &= f_2(x)+\frac{1}{2}f_4(x)+C\\
\int \frac{1}{x^2-\sqrt{2}x+1}dx &= \sqrt{2}f_2(x)+C\\
\int \frac{x}{\left(x^2+\sqrt{2}x+1\right)^2}dx &= -f_1(x)-\frac{1}{\sqrt{2}}f_5(x)-f_7(x)+C\\
\int \frac{1}{\left(x^2+\sqrt{2}x+1\right)^2}dx &= \sqrt{2}f_1(x)+f_5(x)+\frac{1}{\sqrt{2}}f_7(x)+C\\
\int \frac{x}{\left(x^2-\sqrt{2}x+1\right)^2}dx &= f_2(x)+\frac{1}{\sqrt{2}}f_6(x)-f_8(x)+C\\
\int \frac{1}{\left(x^2-\sqrt{2}x+1\right)^2}dx &= \sqrt{2}f_2(x)+f_6(x)-\frac{1}{\sqrt{2}}f_8(x)+C
\end{aligned}$$

0.3 Driving to conclusion

Temporarily forgetting about *integration constant*,

$$\begin{pmatrix} \int \frac{x}{x^2+\sqrt{2}x+1}dx \\ \int \frac{1}{x^2+\sqrt{2}x+1}dx \\ \int \frac{x}{x^2-\sqrt{2}x+1}dx \\ \int \frac{1}{x^2-\sqrt{2}x+1}dx \\ \int \frac{x}{\left(x^2+\sqrt{2}x+1\right)^2}dx \\ \int \frac{1}{\left(x^2+\sqrt{2}x+1\right)^2}dx \\ \int \frac{x}{\left(x^2-\sqrt{2}x+1\right)^2}dx \\ \int \frac{1}{\left(x^2-\sqrt{2}x+1\right)^2}dx \end{pmatrix} = \begin{pmatrix} -f_1(x)+\frac{1}{2}f_3(x) \\ \sqrt{2}f_1(x) \\ f_2(x)+\frac{1}{2}f_4(x) \\ \sqrt{2}f_2(x) \\ -f_1(x)-\frac{1}{\sqrt{2}}f_5(x)-f_7(x) \\ \sqrt{2}f_1(x)+f_5(x)+\frac{1}{\sqrt{2}}f_7(x) \\ f_2(x)+\frac{1}{\sqrt{2}}f_6(x)-f_8(x) \\ \sqrt{2}f_2(x)+f_6(x)-\frac{1}{\sqrt{2}}f_8(x) \end{pmatrix} = \begin{pmatrix} -1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -1 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 1 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -1 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 1 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \\ f_5(x) \\ f_6(x) \\ f_7(x) \\ f_8(x) \end{pmatrix}$$

$$\begin{pmatrix} \int \frac{1}{(x^4+1)^2}dx \\ \int \frac{x}{(x^4+1)^2}dx \\ \int \frac{x^2}{(x^4+1)^2}dx \\ \int \frac{x^3}{(x^4+1)^2}dx \\ \int \frac{x^4}{(x^4+1)^2}dx \\ \int \frac{x^5}{(x^4+1)^2}dx \\ \int \frac{x^6}{(x^4+1)^2}dx \\ \int \frac{x^7}{(x^4+1)^2}dx \end{pmatrix} = \begin{pmatrix} \frac{3\sqrt{2}}{16} & \frac{3}{8} & \frac{-3\sqrt{2}}{16} & \frac{3}{8} & \frac{\sqrt{2}}{8} & \frac{1}{8} & \frac{-\sqrt{2}}{8} & \frac{1}{8} \\ 0 & \frac{-\sqrt{2}}{16} & 0 & \frac{\sqrt{2}}{16} & \frac{-1}{8} & \frac{-\sqrt{2}}{8} & \frac{-1}{8} & \frac{\sqrt{2}}{8} \\ \frac{-\sqrt{2}}{16} & \frac{-1}{8} & \frac{\sqrt{2}}{16} & \frac{-1}{8} & 0 & \frac{1}{8} & 0 & \frac{1}{8} \\ 0 & \frac{\sqrt{2}}{16} & 0 & \frac{-\sqrt{2}}{16} & \frac{1}{8} & 0 & \frac{1}{8} & 0 \\ \frac{\sqrt{2}}{16} & \frac{1}{8} & \frac{-\sqrt{2}}{16} & \frac{1}{8} & \frac{-\sqrt{2}}{8} & \frac{-1}{8} & \frac{\sqrt{2}}{8} & \frac{-1}{8} \\ 0 & \frac{-3\sqrt{2}}{16} & 0 & \frac{3\sqrt{2}}{16} & \frac{1}{8} & \frac{\sqrt{2}}{8} & \frac{1}{8} & \frac{-\sqrt{2}}{8} \\ \frac{-3\sqrt{2}}{16} & \frac{1}{8} & \frac{3\sqrt{2}}{16} & \frac{1}{8} & 0 & \frac{-1}{8} & 0 & \frac{-1}{8} \\ \frac{1}{2} & \frac{3\sqrt{2}}{16} & \frac{1}{2} & \frac{-3\sqrt{2}}{16} & \frac{-1}{8} & 0 & \frac{-1}{8} & 0 \end{pmatrix}$$

$$\cdot \begin{pmatrix} -1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -1 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 1 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -1 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 1 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \\ f_5(x) \\ f_6(x) \\ f_7(x) \\ f_8(x) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3\sqrt{2}}{16} & \frac{3\sqrt{2}}{16} & \frac{3\sqrt{2}}{32} & \frac{-3\sqrt{2}}{32} & 0 & 0 & \frac{-\sqrt{2}}{16} & \frac{\sqrt{2}}{16} \\ \frac{-1}{4} & \frac{1}{4} & 0 & 0 & \frac{-\sqrt{2}}{16} & \frac{\sqrt{2}}{16} & 0 & 0 \\ \frac{\sqrt{2}}{16} & \frac{\sqrt{2}}{16} & \frac{-\sqrt{2}}{32} & \frac{\sqrt{2}}{32} & \frac{1}{8} & \frac{1}{8} & \frac{\sqrt{2}}{16} & \frac{-\sqrt{2}}{16} \\ 0 & 0 & 0 & 0 & \frac{-\sqrt{2}}{16} & \frac{\sqrt{2}}{16} & \frac{-1}{8} & \frac{-1}{8} \\ \frac{\sqrt{2}}{16} & \frac{\sqrt{2}}{16} & \frac{\sqrt{2}}{32} & \frac{-\sqrt{2}}{32} & 0 & 0 & \frac{\sqrt{2}}{16} & \frac{-\sqrt{2}}{16} \\ \frac{-1}{4} & \frac{1}{4} & 0 & 0 & \frac{\sqrt{2}}{16} & \frac{-\sqrt{2}}{16} & 0 & 0 \\ \frac{3\sqrt{2}}{16} & \frac{3\sqrt{2}}{16} & \frac{-3\sqrt{2}}{32} & \frac{3\sqrt{2}}{32} & \frac{-1}{8} & \frac{-1}{8} & \frac{-\sqrt{2}}{16} & \frac{\sqrt{2}}{16} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{\sqrt{2}}{16} & \frac{-\sqrt{2}}{16} & \frac{1}{8} & \frac{1}{8} \end{pmatrix} \cdot \begin{pmatrix} \arctan\left(\sqrt{2}x+1\right) \\ \arctan\left(\sqrt{2}x-1\right) \\ \ln\left(x^2+\sqrt{2}x+1\right) \\ \ln\left(x^2-\sqrt{2}x+1\right) \\ \frac{x}{x^2+\sqrt{2}x+1} \\ \frac{x}{x^2-\sqrt{2}x+1} \\ \frac{1}{x^2+\sqrt{2}x+1} \\ \frac{1}{x^2-\sqrt{2}x+1} \end{pmatrix}$$

REMINDER: Do **not** forget integration constant in practice.

0.4 An example

Let's say we are going to find:

$$\int \frac{x^6+x^4+1}{(x^4+1)^2}dx.$$

0.4.1 Using the above result

Consider the purple matrix, we sum up the first, fifth and seventh row (**only here I tranposed it to a column matrix** since otherwise I cannot display the whole row):

$$\begin{pmatrix} \frac{3\sqrt{2}}{16}+\frac{\sqrt{2}}{16}+\frac{3\sqrt{2}}{16} \\ \frac{3\sqrt{2}}{16}+\frac{\sqrt{2}}{16}+\frac{3\sqrt{2}}{16} \\ \frac{3\sqrt{2}}{32}+\frac{\sqrt{2}}{32}+\frac{-3\sqrt{2}}{32} \\ \frac{-3\sqrt{2}}{32}+\frac{-\sqrt{2}}{32}+\frac{3\sqrt{2}}{32} \\ 0+0+\frac{-1}{8} \\ 0+0+\frac{-1}{8} \\ \frac{-\sqrt{2}}{16}+\frac{\sqrt{2}}{16}+\frac{-\sqrt{2}}{16} \\ \frac{\sqrt{2}}{16}+\frac{-\sqrt{2}}{16}+\frac{\sqrt{2}}{16} \end{pmatrix}$$

$$\left(\int \frac{x^6+x^4+1}{(x^4+1)^2}dx\right)=\begin{pmatrix} \frac{7}{8\sqrt{2}} & \frac{7}{8\sqrt{2}} & \frac{1}{16\sqrt{2}} & -\frac{1}{16\sqrt{2}} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8\sqrt{2}} & \frac{1}{8\sqrt{2}} \end{pmatrix}\cdot\begin{pmatrix} \arctan\left(\sqrt{2}x+1\right) \\ \arctan\left(\sqrt{2}x-1\right) \\ \ln\left(x^2+\sqrt{2}x+1\right) \\ \ln\left(x^2-\sqrt{2}x+1\right) \\ \frac{x}{x^2+\sqrt{2}x+1} \\ \frac{x}{x^2-\sqrt{2}x+1} \\ \frac{1}{x^2+\sqrt{2}x+1} \\ \frac{1}{x^2-\sqrt{2}x+1} \end{pmatrix}+(C)$$

$$\begin{aligned}\int \frac{x^6+x^4+1}{(x^4+1)^2}dx = {} & \frac{7}{8\sqrt{2}}\cdot\arctan\left(\sqrt{2}x+1\right)+\frac{7}{8\sqrt{2}}\cdot\arctan\left(\sqrt{2}x-1\right) \\ & +\frac{1}{16\sqrt{2}}\cdot\ln\left(x^2+\sqrt{2}x+1\right)-\frac{1}{16\sqrt{2}}\cdot\ln\left(x^2-\sqrt{2}x+1\right) \\ & -\frac{1}{8}\cdot\frac{x}{x^2+\sqrt{2}x+1}-\frac{1}{8}\cdot\frac{x}{x^2-\sqrt{2}x+1}-\frac{1}{8\sqrt{2}}\cdot\frac{1}{x^2+\sqrt{2}x+1}+\frac{1}{8\sqrt{2}}\cdot\frac{1}{x^2-\sqrt{2}x+1}+C\end{aligned}$$

0.4.1.1 Verifying

$$\frac{d}{dx}\left(\frac{1}{16\sqrt{2}}\left(-\frac{2\sqrt{2}x}{x^2+\sqrt{2}x+1}-\frac{2\sqrt{2}x}{x^2-\sqrt{2}x+1}-\frac{2}{x^2+\sqrt{2}x+1}+\frac{2}{x^2-\sqrt{2}x+1}\right)\right)=\frac{d}{dx}\left(\frac{-x^3}{4(x^4+1)}\right)$$

$$\frac{d}{dx}\left(14\arctan\left(\sqrt{2}x+1\right)+14\arctan\left(\sqrt{2}x-1\right)+\ln\left(x^2+\sqrt{2}x+1\right)-\ln\left(x^2-\sqrt{2}x+1\right)\right)=\frac{4\sqrt{2}(3x^2+4)}{x^4+1}$$

$$\therefore\ \frac{d}{dx}\left(\cdots+C\right)=\frac{-3x^2(x^4+1)+x^3(4x^3)}{4(x^4+1)^2}+\frac{3x^2+4}{4(x^4+1)}=\frac{-3x^6-3x^2+4x^6+3x^6+4x^4+3x^2+4}{4(x^4+1)^2}=\frac{x^6+x^4+1}{(x^4+1)^2}.$$

0.4.1.1.1 Actually I computed by *Microsoft mathsolver*

Refer to here and here.
By the way, it is wise to realize that for such a long anti-derivative like above, relying on any calculator is ridiculous. Instead, use *Microsoft mathsolver* **after** chopping it into several parts and input **by typing**.

0.4.1.2 Simplify the anti-derivative a little bit

$$\int\frac{x^6+x^4+1}{(x^4+1)^2}dx=\frac{7}{8\sqrt{2}}\arctan\left(\sqrt{2}x+1\right)+\frac{7}{8\sqrt{2}}\arctan\left(\sqrt{2}x-1\right)$$
$$+\frac{1}{16\sqrt{2}}\ln\left(x^2+\sqrt{2}x+1\right)-\frac{1}{16\sqrt{2}}\ln\left(x^2-\sqrt{2}x+1\right)-\frac{x^3}{4(x^4+1)}+C$$

0.4.2 Alternative method (kind of)

I know that:

$$\int\frac{1}{x^4+1}dx=\frac{1}{2\sqrt{2}}\arctan\left(\sqrt{2}x+1\right)+\frac{1}{2\sqrt{2}}\arctan\left(\sqrt{2}x-1\right)+\frac{1}{4\sqrt{2}}\ln\left(x^2+\sqrt{2}x+1\right)-\frac{1}{4\sqrt{2}}\ln\left(x^2-\sqrt{2}x+1\right)+C$$

$$\int\frac{x^2}{x^4+1}dx=\frac{1}{2\sqrt{2}}\arctan\left(\sqrt{2}x+1\right)+\frac{1}{2\sqrt{2}}\arctan\left(\sqrt{2}x-1\right)-\frac{1}{4\sqrt{2}}\ln\left(x^2+\sqrt{2}x+1\right)+\frac{1}{4\sqrt{2}}\ln\left(x^2-\sqrt{2}x+1\right)+C$$

$$\therefore\int\frac{x^6+x^4+1}{(x^4+1)^2}dx=\int\frac{x^6}{(x^4+1)^2}dx+\int\frac{1}{x^4+1}dx=\left(-\frac{1}{4}\cdot\frac{x^3}{x^4+1}+\frac{3}{4}\int\frac{x^2}{x^4+1}dx\right)+\int\frac{1}{x^4+1}dx$$
$$=-\frac{1}{4}\cdot\frac{x^3}{x^4+1}+\frac{7}{8\sqrt{2}}\arctan\left(\sqrt{2}x+1\right)+\frac{7}{8\sqrt{2}}\arctan\left(\sqrt{2}x-1\right)$$
$$+\frac{1}{16\sqrt{2}}\ln\left(x^2+\sqrt{2}x+1\right)-\frac{1}{16\sqrt{2}}\ln\left(x^2-\sqrt{2}x+1\right)+C$$

Which you can see it is same as the one we just done.

0.5 Another example

We want to find:

$$\int\frac{x^4-x^6+x^8}{(1+x^4)^2}dx.$$

0.5.1 Solution

By *polynomial division* $(x^8 - x^6 + x^4) \div (x^8 + 2x^4 + 1) = 1 \cdots (-x^6 - x^4 - 1)$,

$$\begin{aligned}\int \frac{x^8 - x^6 + x^4}{(x^4+1)^2}dx = \int \frac{x^8+2x^4+1}{(x^4+1)^2}dx - \int \frac{x^6+x^4+1}{(x^4+1)^2}dx = \int dx - \cdots \\ = x - \frac{7}{8\sqrt{2}}\arctan\left(\sqrt{2}x+1\right) - \frac{7}{8\sqrt{2}}\arctan\left(\sqrt{2}x-1\right) \\ -\frac{1}{16\sqrt{2}}\ln\left(x^2+\sqrt{2}x+1\right) + \frac{1}{16\sqrt{2}}\ln\left(x^2-\sqrt{2}x+1\right) + \frac{x^3}{4(x^4+1)} + C.\end{aligned}$$

0.5.2 Remark

Things like $\int \frac{x^9}{(1+x^4)^2}dx$ can also be handled by using *polynomial division* to reduce the numerator to a polynomial with degree < 8.

$$\int \frac{P(x)}{(1+x^4)^3}dx$$

Reusing $f_1(x)$ to $f_8(x)$ and adding some more functions:

$$\begin{aligned}
f_1(x) &= \arctan\left(\sqrt{2}x+1\right) \\
f_2(x) = -f_1(-x) &= \arctan\left(\sqrt{2}x-1\right) \\
f_3(x) &= \ln\left(x^2+\sqrt{2}x+1\right) \\
f_4(x) = -f_3(x) &= \ln\left(x^2-\sqrt{2}x+1\right) \\
f_5(x) &= \frac{x}{x^2+\sqrt{2}x+1} \\
f_6(x) = -f_5(-x) &= \frac{x}{x^2-\sqrt{2}x+1} \\
f_7(x) &= \frac{1}{x^2+\sqrt{2}x+1} \\
f_8(x) = f_7(-x) &= \frac{1}{x^2-\sqrt{2}x+1} \\
f_9(x) &= \frac{x}{(x^2+\sqrt{2}x+1)^2} \\
f_{10}(x) = -f_9(-x) &= \frac{x}{(x^2-\sqrt{2}x+1)^2} \\
f_{11}(x) &= \frac{1}{(x^2+\sqrt{2}x+1)^2} \\
f_{12}(x) = f_{11}(-x) &= \frac{1}{(x^2-\sqrt{2}x+1)^2}
\end{aligned}$$

Some extended results to be used:

$$\begin{aligned}
\int \frac{x}{x^2+\sqrt{2}x+1}dx &= -f_1(x)+\frac{1}{2}f_3(x)+C \\
\int \frac{1}{x^2+\sqrt{2}x+1}dx &= \sqrt{2}f_1(x)+C \\
\int \frac{x}{x^2-\sqrt{2}x+1}dx &= f_2(x)+\frac{1}{2}f_4(x)+C \\
\int \frac{1}{x^2-\sqrt{2}x+1}dx &= \sqrt{2}f_2(x)+C \\
\int \frac{x}{\left(x^2+\sqrt{2}x+1\right)^2}dx &= -f_1(x)-\frac{1}{\sqrt{2}}f_5(x)-f_7(x)+C \\
\int \frac{1}{\left(x^2+\sqrt{2}x+1\right)^2}dx &= \sqrt{2}f_1(x)+f_5(x)+\frac{1}{\sqrt{2}}f_7(x)+C \\
\int \frac{x}{\left(x^2-\sqrt{2}x+1\right)^2}dx &= f_2(x)+\frac{1}{\sqrt{2}}f_6(x)-f_8(x)+C \\
\int \frac{1}{\left(x^2-\sqrt{2}x+1\right)^2}dx &= \sqrt{2}f_2(x)+f_6(x)-\frac{1}{\sqrt{2}}f_8(x)+C \\
\int \frac{x}{\left(x^2+\sqrt{2}x+1\right)^3}dx &= -\frac{3}{2}f_1(x)-\frac{3}{2\sqrt{2}}f_5(x)-\frac{3}{4}f_7(x)-\frac{1}{2\sqrt{2}}f_9(x)-\frac{1}{2}f_{11}(x)+C \\
\int \frac{1}{\left(x^2+\sqrt{2}x+1\right)^3}dx &= \frac{3}{\sqrt{2}}f_1(x)+\frac{3}{2}f_5(x)+\frac{3}{2\sqrt{2}}f_7(x)+\frac{1}{2}f_9(x)+\frac{1}{2\sqrt{2}}f_{11}(x)+C \\
\int \frac{x}{\left(x^2-\sqrt{2}x+1\right)^3}dx &= \frac{3}{2}f_2(x)+\frac{3}{2\sqrt{2}}f_6(x)-\frac{3}{4}f_8(x)+\frac{1}{2\sqrt{2}}f_{10}(x)-\frac{1}{2}f_{12}(x)+C \\
\int \frac{1}{\left(x^2-\sqrt{2}x+1\right)^3}dx &= \frac{3}{\sqrt{2}}f_2(x)+\frac{3}{2}f_6(x)-\frac{3}{2\sqrt{2}}f_8(x)+\frac{1}{2}f_{10}(x)-\frac{1}{2\sqrt{2}}f_{12}(x)+C
\end{aligned}$$

$$
\begin{pmatrix}
\int \frac{1}{(x^4+1)^3}\,dx \\
\int \frac{x}{(x^4+1)^3}\,dx \\
\int \frac{x^2}{(x^4+1)^3}\,dx \\
\int \frac{x^3}{(x^4+1)^3}\,dx \\
\int \frac{x^4}{(x^4+1)^3}\,dx \\
\int \frac{x^5}{(x^4+1)^3}\,dx \\
\int \frac{x^6}{(x^4+1)^3}\,dx \\
\int \frac{x^7}{(x^4+1)^3}\,dx \\
\int \frac{x^8}{(x^4+1)^3}\,dx \\
\int \frac{x^9}{(x^4+1)^3}\,dx \\
\int \frac{x^{10}}{(x^4+1)^3}\,dx \\
\int \frac{x^{11}}{(x^4+1)^3}\,dx
\end{pmatrix}
=
\left(
\begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & -\sqrt{2} & 1 & \sqrt{2} & 1 & -2\sqrt{2} & 1 & 2\sqrt{2} & 1 & -3\sqrt{2} & 1 & 3\sqrt{2} \\
-\sqrt{2} & 1 & \sqrt{2} & 1 & -2\sqrt{2} & 4 & 2\sqrt{2} & 4 & -3\sqrt{2} & 9 & 3\sqrt{2} & 9 \\
1 & 0 & 1 & 0 & 4 & -2\sqrt{2} & 4 & 2\sqrt{2} & 9 & -8\sqrt{2} & 9 & 8\sqrt{2} \\
0 & 2 & 0 & 2 & -2\sqrt{2} & 2 & 2\sqrt{2} & 2 & -8\sqrt{2} & 9 & 8\sqrt{2} & 9 \\
2 & -2\sqrt{2} & 2 & 2\sqrt{2} & 2 & -2\sqrt{2} & 2 & 2\sqrt{2} & 9 & -3\sqrt{2} & 9 & 3\sqrt{2} \\
-2\sqrt{2} & 2 & 2\sqrt{2} & 2 & -2\sqrt{2} & 4 & 2\sqrt{2} & 4 & -3\sqrt{2} & 1 & 3\sqrt{2} & 1 \\
2 & 0 & 2 & 0 & 4 & -2\sqrt{2} & 4 & 2\sqrt{2} & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & -2\sqrt{2} & 1 & 2\sqrt{2} & 1 & 0 & 0 & 0 & 0 \\
1 & -\sqrt{2} & 1 & \sqrt{2} & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-\sqrt{2} & 1 & \sqrt{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}^{-1}
\right)^{T}
$$

$$
\cdot
\begin{pmatrix}
-1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{2} & 0 & 0 & 0 & 1 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & \sqrt{2} & 0 & 0 & 0 & 1 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
\frac{-3}{2} & 0 & 0 & 0 & \frac{-3}{2\sqrt{2}} & 0 & \frac{-3}{4} & 0 & \frac{-1}{2\sqrt{2}} & 0 & \frac{-1}{2} & 0 \\
\frac{3}{\sqrt{2}} & 0 & 0 & 0 & \frac{3}{2} & 0 & \frac{3}{2\sqrt{2}} & 0 & \frac{1}{2} & 0 & \frac{1}{2\sqrt{2}} & 0 \\
0 & \frac{3}{2} & 0 & 0 & 0 & \frac{3}{2\sqrt{2}} & 0 & \frac{-3}{4} & 0 & \frac{1}{2\sqrt{2}} & 0 & \frac{-1}{2} \\
0 & \frac{3}{\sqrt{2}} & 0 & 0 & 0 & \frac{3}{2} & 0 & \frac{-3}{2\sqrt{2}} & 0 & \frac{1}{2} & 0 & \frac{-1}{2\sqrt{2}}
\end{pmatrix}
\cdot
\begin{pmatrix}
f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \\ f_5(x) \\ f_6(x) \\ f_7(x) \\ f_8(x) \\ f_9(x) \\ f_{10}(x) \\ f_{11}(x) \\ f_{12}(x)
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
\frac{21\sqrt{2}}{128} & \frac{21\sqrt{2}}{128} & \frac{21\sqrt{2}}{256} & \frac{-21\sqrt{2}}{256} & 0 & 0 & \frac{-\sqrt{2}}{16} & \frac{\sqrt{2}}{16} & \frac{-1}{64} & \frac{-1}{64} & \frac{-\sqrt{2}}{64} & \frac{\sqrt{2}}{64} \\
\frac{-3}{16} & \frac{3}{16} & 0 & 0 & \frac{-7\sqrt{2}}{128} & \frac{7\sqrt{2}}{128} & \frac{-1}{64} & \frac{-1}{64} & 0 & 0 & \frac{1}{64} & \frac{1}{64} \\
\frac{5\sqrt{2}}{128} & \frac{5\sqrt{2}}{128} & \frac{-5\sqrt{2}}{256} & \frac{5\sqrt{2}}{256} & \frac{5}{64} & \frac{5}{64} & \frac{3\sqrt{2}}{64} & \frac{-3\sqrt{2}}{64} & \frac{1}{64} & \frac{1}{64} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-3\sqrt{2}}{128} & \frac{3\sqrt{2}}{128} & \frac{-3}{64} & \frac{-3}{64} & \frac{-\sqrt{2}}{64} & \frac{\sqrt{2}}{64} & \frac{-1}{64} & \frac{-1}{64} \\
\frac{3\sqrt{2}}{128} & \frac{3\sqrt{2}}{128} & \frac{3\sqrt{2}}{256} & \frac{-3\sqrt{2}}{256} & 0 & 0 & 0 & 0 & \frac{1}{64} & \frac{1}{64} & \frac{\sqrt{2}}{64} & \frac{-\sqrt{2}}{64} \\
\frac{-1}{16} & \frac{1}{16} & 0 & 0 & \frac{-\sqrt{2}}{128} & \frac{\sqrt{2}}{128} & \frac{1}{64} & \frac{1}{64} & 0 & 0 & \frac{-1}{64} & \frac{-1}{64} \\
\frac{3\sqrt{2}}{128} & \frac{3\sqrt{2}}{128} & \frac{-3\sqrt{2}}{256} & \frac{3\sqrt{2}}{256} & \frac{3}{64} & \frac{3}{64} & \frac{\sqrt{2}}{64} & \frac{-\sqrt{2}}{64} & \frac{-1}{64} & \frac{-1}{64} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-5\sqrt{2}}{128} & \frac{5\sqrt{2}}{128} & \frac{-5}{64} & \frac{-5}{64} & \frac{\sqrt{2}}{64} & \frac{-\sqrt{2}}{64} & \frac{1}{64} & \frac{1}{64} \\
\frac{5\sqrt{2}}{128} & \frac{5\sqrt{2}}{128} & \frac{5\sqrt{2}}{256} & \frac{-5\sqrt{2}}{256} & 0 & 0 & \frac{\sqrt{2}}{16} & \frac{-\sqrt{2}}{16} & \frac{-1}{64} & \frac{-1}{64} & \frac{-\sqrt{2}}{64} & \frac{\sqrt{2}}{64} \\
\frac{-3}{16} & \frac{3}{16} & 0 & 0 & \frac{9\sqrt{2}}{128} & \frac{-9\sqrt{2}}{128} & \frac{-1}{64} & \frac{-1}{64} & 0 & 0 & \frac{1}{64} & \frac{1}{64} \\
\frac{21\sqrt{2}}{128} & \frac{21\sqrt{2}}{128} & \frac{-21\sqrt{2}}{256} & \frac{21\sqrt{2}}{256} & \frac{-11}{64} & \frac{-11}{64} & \frac{-5\sqrt{2}}{64} & \frac{5\sqrt{2}}{64} & \frac{1}{64} & \frac{1}{64} & 0 & 0 \\
0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{13\sqrt{2}}{128} & \frac{-13\sqrt{2}}{128} & \frac{13}{64} & \frac{13}{64} & \frac{-\sqrt{2}}{64} & \frac{\sqrt{2}}{64} & \frac{-1}{64} & \frac{-1}{64}
\end{pmatrix}
\cdot
\begin{pmatrix}
\arctan\left(\sqrt{2}x+1\right) \\
\arctan\left(\sqrt{2}x-1\right) \\
\ln\left(x^2+\sqrt{2}x+1\right) \\
\ln\left(x^2-\sqrt{2}x+1\right) \\
\frac{x}{x^2+\sqrt{2}x+1} \\
\frac{x}{x^2-\sqrt{2}x+1} \\
\frac{1}{x^2+\sqrt{2}x+1} \\
\frac{1}{x^2-\sqrt{2}x+1} \\
\frac{x}{(x^2+\sqrt{2}x+1)^2} \\
\frac{x}{(x^2-\sqrt{2}x+1)^2} \\
\frac{1}{(x^2+\sqrt{2}x+1)^2} \\
\frac{1}{(x^2-\sqrt{2}x+1)^2}
\end{pmatrix}
$$

REMINDER: Do **not** forget integration constant in practice.

0.6 An example to verify the above result

We are going to find:

$$\int \frac{1-x^{12}}{(1-x)(1+x^4)^3}dx.$$

By summing up all rows of the above purple matrix to 'substitute' it back in the equation,

$$\begin{pmatrix} \frac{29\sqrt{2}-28}{64} & \frac{29\sqrt{2}+28}{64} & \frac{1}{4} & \frac{1}{4} & \frac{3\sqrt{2}-3}{64} & \frac{-3\sqrt{2}-3}{64} & \frac{-\sqrt{2}+4}{64} & \frac{\sqrt{2}+4}{64} & \frac{-\sqrt{2}}{64} & \frac{\sqrt{2}}{64} & \frac{-\sqrt{2}}{64} & \frac{\sqrt{2}}{64} \end{pmatrix} \cdot \begin{pmatrix} \arctan\left(\sqrt{2}x+1\right) \\ \arctan\left(\sqrt{2}x-1\right) \\ \ln\left(x^2+\sqrt{2}x+1\right) \\ \ln\left(x^2-\sqrt{2}x+1\right) \\ \frac{x}{x^2+\sqrt{2}x+1} \\ \frac{x}{x^2-\sqrt{2}x+1} \\ \frac{1}{x^2+\sqrt{2}x+1} \\ \frac{1}{x^2-\sqrt{2}x+1} \\ \frac{x}{(x^2+\sqrt{2}x+1)^2} \\ \frac{x}{(x^2-\sqrt{2}x+1)^2} \\ \frac{1}{(x^2+\sqrt{2}x+1)^2} \\ \frac{1}{(x^2-\sqrt{2}x+1)^2} \end{pmatrix},$$

Eventually, we are able to get:

$$\begin{aligned} &\int \frac{x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1}{(1+x^4)^3}dx \\ =& \frac{29\sqrt{2}-28}{64}\arctan\left(\sqrt{2}x+1\right)+\frac{29\sqrt{2}+28}{64}\arctan\left(\sqrt{2}x-1\right)+\frac{1}{4}\ln\left(x^2+\sqrt{2}x+1\right)+\frac{1}{4}\ln\left(x^2-\sqrt{2}x+1\right) \\ &+\frac{3\sqrt{2}-3}{64}\cdot\frac{x}{x^2+\sqrt{2}x+1}-\frac{3\sqrt{2}+3}{64}\cdot\frac{x}{x^2-\sqrt{2}x+1}-\frac{\sqrt{2}-4}{64}\cdot\frac{1}{x^2+\sqrt{2}x+1}+\frac{\sqrt{2}+4}{64}\cdot\frac{1}{x^2-\sqrt{2}x+1} \\ &-\frac{\sqrt{2}}{64}\cdot\frac{x}{(x^2+\sqrt{2}x+1)^2}+\frac{\sqrt{2}}{64}\cdot\frac{x}{(x^2-\sqrt{2}x+1)^2}-\frac{\sqrt{2}}{64}\cdot\frac{1}{(x^2+\sqrt{2}x+1)^2}+\frac{\sqrt{2}}{64}\cdot\frac{1}{(x^2-\sqrt{2}x+1)^2}+C \end{aligned}$$

After all, it would be nice if somebody could tell me whether this anti-derivative can be simplified or not.
By the way, please be reminded that for $\int \frac{1-x^{12}}{(1-x)(1+x^4)^3}dx$, the integrand is **not** defined at $x=1$. That is to say, by the way, the integration constant is not rigorously meaningful when you consider an open set $\subseteq \mathbb{R}$ which is not an interval. If we restrict our attention to $\mathbb{R}\setminus\{0\}$, we should further restrict our attention to any interval $\subseteq \mathbb{R}\setminus\{0\}$ in order that any two anti-derivatives must differ from a constant function.

Remarks

Till now I think that

$$\int \frac{P(x)}{(1+x^4)^n} dx$$

in general, while $n \in \mathbb{N}$, is not so practical. For $n = 2$ and $n = 3$, I already spent a few hours on each, and also relied heavily on calculators or apps. However, the general one, if realized, would be exciting.