Roots being square of difference of roots of a depressed cubic

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Abstract

In this paper, we study the *monic* cubic equation with square of difference of roots of a depressed cubic equation as roots.
Namely, given roots x_1, x_2, x_3 of $x^3 + px + q = 0$ (where $p, q \in \mathbb{C}$), we want to form an equation (which the coefficient of the highest term is 1) with roots $y_1 = (x_2 - x_3)^2, y_2(x_3 - x_1)^2, y_3 = (x_1 - x_2)^2$.
Of course, we will do it in a **rigorous** approach, but we shall also appreciate how various methods give the same answer after all.

Introduction

Notice that the roots $(x_2-x_3)^2, (x_3-x_1)^2, (x_1-x_2)^2$ are symmetric that for any permutation $\sigma(i) : \{1, 2, 3\} \to \{1, 2, 3\}$, replacing all x_i by $x_{\sigma(i)}$ will not affect the equation formed.
This is, maybe, related to the notion of *symmetric polynomials*.
Also, notice that an uncountably many polynomials can be formed when we do not require the leading coefficient is 1).

Lemma / pre-result

To be clear, we first clearly state the relations between roots of the original equation:

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_2x_3 + x_3x_1 + x_1x_2 = p \\ x_1x_2x_3 = -q \\ x_1^2 + x_2^2 + x_3^2 = (x_1 + x_2 + x_3)^2 - 2(x_2x_3 + x_3x_1 + x_1x_2) = -2p \end{cases}$$

WLOG, we consider y_1, that is, $(x_2 - x_3)^2$,

$$\begin{aligned} (x_2 - x_3)^2 &= x_2^2 - 2x_2x_3 + x_3^2 + x_1^2 - x_1^2 = -2p - x_1^2 - 2x_2x_3 \\ &= \begin{cases} -2p - x_1^2 - \dfrac{2x_1x_2x_3}{x_1} = -2p - x_1^2 + \dfrac{2q}{x_1} & (1) \\ -2p - 3x_1^2 + 2x_1^2 - 2x_2x_3 = -2p + \big(2x_1(-x_2 - x_3) - 2x_2x_3\big) - 3x_1^2 = -4p - 3x_1^2 & (2) \end{cases} \end{aligned}$$

$$\text{From (1)}, \quad x_1^3 + (2p + y_1)x_1 - 2q = 0 \tag{3}$$

$$\text{From (2)}, \quad x_1^3 + \left(\frac{4p + y_1}{3}\right)x_1 = 0 \tag{4}$$

Recall that $x_1^3+px_1+q=0$, subtracting it from (3) or (4) (as to eliminate the term x_1^3) respectively, both giving:

$$x_1 = \frac{3q}{p+y_1} \tag{5}$$

Above holds generally for x_i, y_i $(i=1,2,3)$.

Solution 1 $\big($Using (2) to find elementary symmetrical polynomials of the new equation$\big)$

$$\begin{aligned}
&(x_2-x_3)^2+(x_3-x_1)^2+(x_1-x_2)^2 = -12p-3(-2p) = -6p \\
&(x_3-x_1)^2(x_1-x_2)^2+(x_1-x_2)^2(x_2-x_3)^2+(x_2-x_3)^2(x_3-x_1)^2 \\
=\;&(-4p-3x_2^2)(-4p-3x_3^2)+(-4p-3x_3^2)(-4p-3x_1^2)+(-4p-3x_1^2)(-4p-3x_2^2) \\
=\;&16p^2+12px_2^2+12px_3^2+9x_2^2x_3^2+16p^2+12px_3^2+12px_1^2+9x_3^2x_1^2+16p^2+12px_1^2+12px_2^2+9x_1^2x_2^2 \\
=\;&48p^2-48p^2+9p^2 = 9p^2 \\
&(x_2-x_3)^2(x_3-x_1)^2(x_1-x_2)^2 \\
=\;&(-4p-3x_1^2)(-4p-3x_2^2)(-4p-3x_3^2) = (16p^2+12px_1^2+12px_2^2+9x_1^2x_2^2)(-4p-3x_3^2) \\
=\;&-64p^3-48p^2x_1^2-48p^2x_2^2-36px_1^2x_2^2-48p^2x_3^2-36px_3^2x_1^2-36px_2^2x_3^2-27x_1^2x_2^2x_3^2 \\
=\;&-64p^3-48p^2(-2p)-36p(p^2)-27q^2 = -(4p^3+27q^2)
\end{aligned}$$

Solution 1 $\big($Using (2) to manipulate the equation$\big)$

$$\begin{aligned}
(x^6+2px^4+p^2x^2)-q^2 &= (x^3+px+q)(x^3+px-q) = (x-x_1)(x-x_2)(x-x_3)(-1)(-x-x_1)(-x-x_2)(-x-x_3) \\
&= (x^2-x_1^2)(x^2-x_2^2)(x^2-x_3^2)
\end{aligned}$$

By comparing coefficients, we know that $z^3+2pz^2+p^2z-q^2 = (z-x_1^2)(z-x_2^2)(z-x_3^2)$.

$$\begin{aligned}
(y-y_1)(y-y_2)(y-y_3) &= \big(y-(-4p-3x_1^2)\big)\big(y-(-4p-3x_2^2)\big)\big(y-(-4p-3x_3^2)\big) \\
&= -27\left(-\frac{y+4p}{3}-x_1^2\right)\left(-\frac{y+4p}{3}-x_2^2\right)\left(-\frac{y+4p}{3}-x_3^2\right) \\
&= -27\left(\left(-\frac{y+4p}{3}\right)^3+2p\left(-\frac{y+4p}{3}\right)^2-p^2\left(\frac{y+4p}{3}\right)-q^2\right) \\
&= (y+4p)^3-6p(y+4p)^2+9p^2(y+4p)+27q^2 \\
&= y^3+12py^2+48p^2y+64p^3-6py^2-48p^2y-96p^3+9p^2y+36p^3+27q^2 \\
&= y^3+(6p)y^2+(9p^2)y+(4p^3+27q^2)
\end{aligned}$$

Remark 0.1. Steps should be justified carefully so that the proof is rigorous. For example, it may not be enough proving $(y-y_1)(y-y_2)(y-y_3)=0 \iff y^3+6py^2+9p^2y+4p^3+27q^2=0$. Consider $y(y-1)^2=0 \iff y^2(y-1)=0$, an $\iff$ only suggests that the set of roots of two polynomials are identical, but the equality of polynomials should be taken more carefully.
It is understood that the question indeed ask for **expanding** $(y-y_1)(y-y_2)(y-y_3)$, instead of merely a polynomial with roots y_1, y_2, y_3.

Boundary case for solution 3-4

If $t = 0, x(x^2 + p) = 0$:

WLOG we assume that $(x_1, x_2, x_3) = (0, -\sqrt{-p}, -\sqrt{-p})$, then

$$(y_1, y_2, y_3) = \big((x_2 - x_3)^2, (x_3 - x_1)^2, (x_1 - x_2)^2\big) = (-4p, -p, -p)$$

$$(y + p)^2(y + 4p) = (y^2 + 2py + p^2)(y + 4p) = y^3 + 6py + 9p^2y + 4p^3 \;\; +0$$

Below we shall **assume** $t \neq 0$, hence x_1, x_2, x_3 are all **non-zero**.

Solution 3 $\big($Using (5), like solution 1$\big)$

$t \neq 0$.

$$\begin{aligned}
x_i &= \frac{3q}{p + y_i} \\
y_1 + y_2 + y_3 &= (p + y_1) + (p + y_2) + (p + y_3) - 3p = 3q\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right) - 3p \\
&= \frac{3pq}{-q} - 3p = -6p \\
y_2y_3 + y_3y_1 + y_1y_2 &= \big((p + y_2)(p + y_3) - p^2 - p(y_2 + y_3)\big) + (\cdots) + (\cdots) \\
&= \frac{9p^2}{x_2x_3} + \frac{9p^2}{x_3x_1} + \frac{9p^2}{x_1x_2} - 3p^2 - 2p(y_1 + y_2 + y_3) \\
&= \frac{9p^2(x_1 + x_2 + x_3)}{x_1x_2x_3} - 3p + 12p^2 = 9p^2 \\
\frac{3q}{p + y_1} \cdot \frac{3q}{p + y_2} \cdot \frac{3q}{p + y_3} &= x_1x_2x_3 = -q \\
(p + y_1)(p + y_2)(p + y_3) &= -27q^2 \qquad (6)
\end{aligned}$$

Thinking combinatorically $\big($choosing terms from expanded $(p + y_1)(p + y_2)(p + y_3)\big)$,

$$\begin{aligned}
-27q^2 &= p^3 + 9(y_1 + y_2 + y_3)p^2 + (y_2y_3 + y_3y_1 + y_1y_2)p + y_1y_2y_3 = p^3 - 6p^3 + 9p^3 + y_1y_2y_3 \\
y_1y_2y_3 &= -(4p^3 + 27q^2)
\end{aligned}$$

Solution 4 (**Using (5), like solution 2**)

$$
\begin{aligned}
&(y-y_1)(y-y_2)(y-y_3)\\
&=\left(\left(\frac{p+y}{3q}\right)^3\left(p+(x_2-x_3)^2\right)\left(p+(x_3-x_1)^2\right)\left(p+(x_1-x_2)^2\right)\right)\\
&\quad\left(\frac{3q}{p+(x_2-x_3)^2}-\frac{3q}{p+y}\right)\left(\frac{3q}{p+(x_3-x_1)^2}-\frac{3q}{p+y}\right)\left(\frac{3q}{p+(x_1-x_2)^2}-\frac{3q}{p+y}\right)\\
&=-\left(\left(\frac{p+y}{3q}\right)^3\left(\frac{(3q)^3}{-q}\right)\right)\left(\frac{3q}{p+y}-x_1\right)\left(\frac{3q}{p+y}-x_2\right)\left(\frac{3q}{p+y}-x_3\right)\\
&=\left(\frac{(p+y)^3}{q}\right)\left(\left(\frac{3q}{p+y}\right)^3+p\left(\frac{3q}{p+y}\right)+q\right)\\
&=y^3+(6p)y^2+(9p^2)y+(4p^3+27q^2)
\end{aligned}
$$

You may need to notice that

$$
\frac{3t}{p+(x_2-x_3)^2}\times\frac{3t}{p+(x_3-x_1)^2}\times\frac{3t}{p+(x_1-x_2)^2}=x_1x_2x_3=-q
$$

and that the coefficient of y^3 is kept 1 from top to bottom. (Related to (6) in solution 3.)

Answer

$$
y^3+(6p)y^2+(9p^2)y+(4p^3+27q^2)=0
$$

Remark

Refer to the remark in solution 2, it is always good to consider things more rigorously. However, it is not necessary to show all the steps each time, as some justification can also been done to subtleties.

More interesting results

About the *Cartan-Tartaglia formula*

The *Cartan-Tartaglia formula* (for cubic equation) is, for depressed cubic equation $x^3+px+q=0$,

$$
x=\sqrt[3]{\sqrt{\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2}+\frac{q}{2}}-\sqrt[3]{\sqrt{\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2}-\frac{q}{2}}
$$

(However, it is actually hard to explain which cubic root is to be taken for complex numbers appearing.)
If for $\alpha,\beta,\gamma\in\mathbb{R}$ all distinct, apply the formula with the depressed cubic equation formed by reducing the expanded $(y-\alpha)(y-\beta)(y-\gamma)$, complex number is unavoidable (that is, a negative square root occurs).

To prove this, consider $x^3 + px - q = 0$ having distinct and real roots $x_1 = \alpha + c, x_2 = \beta + c, x_3 = \gamma + c$, where $c \in \mathbb{R}$ is a constant (Why? because depressed cubic is formed by substitution $x = y - \cdots$).
Using the results above, as $y_1, y_2, y_3 \in \mathbb{R}^+$, $-(4p^3 + 27q^2) \ > 0$,

$$\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 < 0$$

The amazing thing is that it provides kind of a reason why complex number is so important in modern Mathematics.
"*The shortest path between two truths in the real domain passes through the complex domain.*" $---$ *Jacques Hadamard*