On behavior of Cauchy products

Tony Ma Yixuan

2018-08-22

1 Introduction

In this thesis (kind of... I was told to add more details, so this is indeed notes, but I would rather just call it a thesis), we will consider the convergence (absolute or conditional) of Cauchy product, which is a way to study the product of two series.

Given infinite series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$, it is natural to consider their product $\sum_{m,n\geq 0} a_m b_n$.

Let $d_{(m,n)} = a_m b_n \ \forall m, n \geq 0$, we would like

$$\left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right) = \sum_{m,n\geq 0} a_m b_n = \sum_{s\in\{(m,n):m\geq 0 \text{ and } n\geq 0\}} d_s$$

, while the left hand side is an unordered series:

$$\begin{matrix} a_0b_0 & a_0b_1 & a_0b_2 & a_0b_3 & \dots \\ a_1b_0 & a_1b_1 & a_1b_2 & a_1b_3 & \dots \\ a_2b_0 & a_2b_1 & a_2b_2 & a_2b_3 & \dots \\ a_3b_0 & a_3b_1 & a_3b_2 & a_3b_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

Figure 1.1

However, since evaluating a series depends on the term order, the unordered series $\sum_{m,n\geq 0} a_m b_n$ doesn't always converges.

Although we can avoid this issue if the series are absolutely convergent, we would like another way to view $\sum_{m,n\geq 0} a_m b_n$ even when the convergence is not absolute.

If we view it this way, depending on term order:

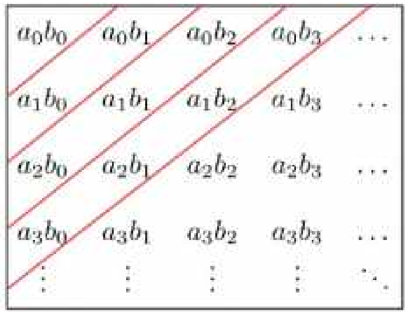

Figure 1.2

We can devide $a_m b_n : m, n \geq 0$ into

$$\begin{aligned}
&\{a_0b_0,\\
&\ a_0b_1, a_1b_0,\\
&\ a_0b_2, a_1b_1, a_2b_0,\\
&\ \dots\}
\end{aligned}$$

This gives the next definition:

Definition 1.1.
Given series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$, their **Cauchy product** is defined as $\sum_{k=0}^{\infty} c_k$, while $(c_n)_{n=0}^{\infty} = \sum_{k=0}^{n} a_k b_{n-k}$.

Convention 1.
Through out this thesis, $(c_n)_{n=0}^{\infty}$ is understood as $\sum_{k=0}^{n} a_k b_{n-k}$ when considering Cauchy product, unless otherwise stated.

Convention 2.
In case it is stated otherwise, we allow the conventions
$+\infty + \alpha = +\infty \quad \forall \alpha > -\infty$ and $-\infty + \alpha = -\infty \quad \forall \alpha < +\infty$ and $+\infty = -(-\infty)$ and $-\infty = -(+\infty)$

Convention 3.
Through out this thesis, $x_{\{\dots\}} = \begin{cases} x, \text{if } \dots \\ 0, \text{if } \neg(\dots) \end{cases}$
In particular, $x_{\{x \geq 0\}}$ and $(-x)_{\{x \leq 0\}}$ gives Definition 2.3, the definition of x^+ and x^- respectively.

Finally, since we aim to study **behavior** of Cauchy products in this thesis, we need to clarify some definitions of behaviors of sequence, which are refered to [3].

Definition 1.2.
Given series $\sum_{n=0}^{\infty} a_n$,
If $\sum_{n=0}^{\infty} |a_n| \in \mathbb{R}$, it is said to converges absolutely.
If $\sum_{n=0}^{\infty} a_n \in \mathbb{R}$ **and** $\sum_{n=0}^{\infty} |a_n| = \infty$, it is said to converges conditionally.
If $\sum_{n=0}^{\infty} a_n = \pm\infty$, it is said to diverges properly.
If $\sum_{n=0}^{\infty} a_n$ doesn't exists, it is said to diverges by oscillation.

2 The theory of unordered series

The theory of unordered series is useful, and this whole section is dedicated for Theorem 3.1 in this thesis.

Below is our definition to unordered series, which is motivated by [4], P.60.

Definition 2.1.
Given any finite set I, if (a_i) is sequence indexed by I,
$\sum_{i\in I} a_i$ is defined in the sense that for any finite set, there is $n \in \mathbb{N}$ such that there is a $1-1$ correspondance between $\{1, 2, ..., n\}$ and it.
Notice the convention that we also let $\sum_{i\in\varnothing} a_i = 0$, as 0 is the identity element in addition.

Definition 2.2. Given any set I, if $(a_i) \geq 0$ is sequence indexed by I,

$$\sum_{i\in I} a_i := \sup\left\{\sum_{i\in J} a_i : J \subseteq I \text{ and } J \text{ is finite}\right\}$$

Definition 2.3. $\forall x \in \mathbb{R}$,

$$x^+ := \begin{cases} x, \text{if } x \geq 0 \\ 0, \text{if } x < 0 \end{cases}$$
$$x^- := \begin{cases} -x, \text{if } x \leq 0 \\ 0, \text{if } x > 0 \end{cases}$$

Lemma 2.4. $\forall x \in \mathbb{R}$,

$$x = x^+ - x^-$$
$$|x| = x^+ + x^-$$

Definition 2.5.
Given any set I, if (a_i) is sequence indexed by I,
If $\left(\sum_{i\in I} a_i^+, \sum_{i\in I} a_i^-\right) \neq (\infty, \infty)$,
$\sum_{i\in I} a_i := \sum_{i\in I} a_i^+ - \sum_{i\in I} a_i^-$, allowing convention 2.
Otherwise, $\sum_{i\in I} a_i$ is left undefined.

Lemma 2.6.
If I is any sets, $J \subseteq I$, $(a_i)_{i\in I} \geq 0$ is a sequence indexed by I,
Then $\sum_{i\in J} a_i \leq \sum_{i\in I} a_i$

Proof. It is obvious by Definition 2.2 and a little bit manipulations on suprema. We will omit the proof here. ∎

Lemma 2.7.
If I is any sets, $J \subseteq I$, $(a_i)_{i\in I} \geq 0$ is a sequence indexed by I,$\sum_{i\in I} a_i \in \mathbb{R}$,
Then $\sum_{i\in J} a_i \in \mathbb{R}$

Proof. It is obvious by Lemma 2.6. We will omit the proof here. ∎

The following definition is referred to [3].

Definition 2.8.
If I is any set, $(a_i)_{i\in I}$ is a sequence indexed by I,

If $s \in \mathbb{R}$,

If $\forall \epsilon > 0, \exists J_0 \subseteq I$ and J_0 is finite : $\left(\forall J_0 \subseteq J \subseteq I \text{ and } J \text{ is finite}, \left|\sum_{i\in J} a_i - s\right| < \epsilon\right)$

Then $\overline{\sum_{i\in I}} a_i := s$

If $s = \infty$,

If $\forall M > 0, \exists J_0 \subseteq I$ and J_0 is finite : $\left(\forall J_0 \subseteq J \subseteq I \text{ and } J \text{ is finite}, \sum_{i\in J} a_i > M\right)$

Then $\overline{\sum_{i\in I}} a_i := \infty$

If $s = -\infty$,

If $\forall M < 0, \exists J_0 \subseteq I$ and J_0 is finite : $\left(\forall J_0 \subseteq J \subseteq I \text{ and } J \text{ is finite}, \sum_{i\in J} a_i < M\right)$

Then $\overline{\sum_{i\in I}} a_i := -\infty$

Theorem 2.9. *Whenever one of* $\sum_{i\in I} a_i, \overline{\sum}_{i\in I} a_i$ *is defined,* $\sum_{i\in I} a_i = \overline{\sum}_{i\in I} a_i$.

Proof. If I is a set, $(a_i)_{i\in I} \geq 0$ is a sequence indexed by I, $\sum_{i\in I} a_i \in \mathbb{R}$,

$$\sum_{i\in I} a_i = \sup\left\{\sum_{i\in J} a_i : J \subseteq I \text{ and } J \text{ is finite}\right\}$$

$$\forall \epsilon > 0, \exists \big(J_0 \subseteq I \text{ and } J_0 \text{ is finite}\big) : \left(\sum_{i\in J_0} a_i > \sum_{i\in I} a_i - \epsilon\right)$$

$$\forall \epsilon > 0, \exists \big(J_0 \subseteq I \text{ and } J_0 \text{ is finite}\big) : \left(\forall J_0 \subseteq J \subseteq I \text{ and } J \text{ is finite}, \sum_{i\in J} a_i > \sum_{i\in I} a_i - \epsilon\right)$$

, by Lemma 2.6.
$\therefore \overline{\sum}_{i\in I} a_i = \sum_{i\in I} a_i$
If I is a set, $(a_i)_{i\in I} \geq 0$ is a sequence indexed by I, $\sum_{i\in I} a_i = \infty$,

$$\sum_{i\in I} a_i = \sup\left\{\sum_{i\in J} a_i : J \subseteq I \text{ and } J \text{ is finite}\right\}$$

$$\forall M > 0, \exists \big(J_0 \subseteq I \text{ and } J_0 \text{ is finite}\big) : \left(\sum_{i\in J_0} a_i > M\right)$$

$$\forall M > 0, \exists\big(J_0 \subseteq I \text{ and } J_0 \text{ is finite}\big) : \left(\forall J_0 \subseteq J \subseteq I \text{ and } J \text{ is finite}, \sum_{i \in J} a_i > M\right)$$

, by Lemma 2.6.

$\therefore \overline{\sum}_{i \in I} a_i = \sum_{i \in I} a_i$

$\therefore$ If I is a set, $(a_i)_{i \in I} \geq 0$ is a sequence indexed by I,

Then $\overline{\sum}_{i \in I} a_i = \sum_{i \in I} a_i$.

If I is a set, $(a_i)_{i \in I}$ is a sequence indexed by I, $\overline{\sum}_{i \in I} a_i = s \in \mathbb{R}$,

Assume $\sum_{i \in I} a_i^+ = \infty$,

$I_1 := \{i \in I : a_i^+ > 0\} \subseteq I, \sum_{i \in I_1} a_i^+ = \infty$

$\therefore I_1$ is infinite.

$$\forall \epsilon, M > 0,$$

$$\exists\Big(J_0 \subseteq I \text{ and } J_0 \text{ is finite and } J_1 \subseteq I_1 \text{ and } J_1 \text{ is finite }\Big) :$$

$$\forall\Big(J_0 \subseteq J \subseteq I \text{ and } J \text{ is finite}\Big),$$

$$\left|\sum_{i \in I} a_i - s\right| < \epsilon$$

$$\sum_{i \in J_1} a_i > M + \sum_{i \in J_0} a_i^-$$

$$\sum_{i \in J_1 \cup J_0} a_i = \sum_{i \in J_1} a_i + \sum_{k \in J_0 \setminus J_1} a_i$$

$$\geq \sum_{i \in J_1} a_i - \sum_{i \in J_0} a_i^- > M$$

$$\sum_{i \in J_1 \cup J_0} a_i > M$$

, implies a contradicion

$\therefore \sum_{i \in I} a_i^+ \in \mathbb{R}, s_1 := \sum_{i \in I} a_i^+$

Similarly, $\sum_{i \in I} a_i^- \in \mathbb{R}, s_2 := \sum_{i \in I} a_i^-$.

Now,

$$\forall \epsilon > 0, \exists\big(J_0 \subseteq I \text{ and } J_1 \subseteq I \text{ and } J_2 \subseteq I \text{ and } J_0, J_1, J_2 \text{are finite}\big) :$$

$$\forall\big(J_0 \subseteq J \subseteq I \text{ and } J \text{ is finite}\big), \left|\sum_{i \in J} a_i^+ - \sum_{i \in J} a_i^- - s\right| < \epsilon$$

$$\forall\big(J_1 \subseteq J \subseteq I \text{ and } J \text{ is finite}\big), \left|-\sum_{i \in J} a_i^- + s_1\right| < \epsilon$$

$$\forall\big(J_2 \subseteq J \subseteq I \text{ and } J \text{ is finite}\big), \left|\sum_{i \in J} a_i^- - s_2\right| < \epsilon$$

$$\therefore \forall\big(J_0 \cup J_1 \cup J_2 \subseteq J \subseteq I \text{ and } J \text{ is finite}\big), |s_1 - s_2 - s| < 3\epsilon$$

$\therefore s_1 - s_2 = s$

$$\begin{aligned}\therefore \overline{\sum}_{i\in I} a_i &= \overline{\sum}_{i\in I} a_i^+ - \overline{\sum}_{i\in I} a_i^- \\ &= \sum_{i\in I} a_i^+ - \sum_{i\in I} a_i^- \\ &= \sum_{i\in I} a_i\end{aligned}$$

If I is a set, $(a_i)_{i\in I}$ is a sequence indexed by I, $\sum_{i\in I} a_i = s \in \mathbb{R}$,

Then $\sum_{i\in I} a_i = \sum_{i\in I} a_i^+ - \sum_{i\in I} a_i^-$, while $\sum_{i\in I} a_i^\pm \in \mathbb{R}$

$$\begin{aligned}&\forall \epsilon > 0, \exists \big(J_0 \subseteq I \text{ and } J_1 \subseteq I \text{ and } J_0, J_1 \text{ are finite}\big): \\ &\qquad \forall \big(J_0 \subseteq J \subseteq I \text{ and } J \text{ is finite}\big), \left|\sum_{i\in J} a_i^+ - \sum_{i\in I} a_i^+\right| < \epsilon \\ &\qquad \forall \big(J_1 \subseteq J \subseteq I \text{ and } J \text{ is finite}\big), \left|-\sum_{i\in J} a_i^- + \sum_{i\in I} a_i^-\right| < \epsilon \\ &\qquad \therefore \forall \big(J_0 \cup J_1 \subseteq J \subseteq I \text{ and } J \text{ is finite}\big), \left|\sum_{i\in J} a_i - \sum_{i\in I} a_i^-\right| < 2\epsilon\end{aligned}$$

$\therefore \sum_{i\in I} a_i = \overline{\sum}_{i\in I} a_i$

If I is a set, $(a_i)_{i\in I}$ is a sequence indexed by I, $\sum_{i\in I} a_i = \infty$,

$\sum_{i\in I} a_i^+ = \infty$ and $\sum_{i\in I} a_i^- = s \in \mathbb{R}$,

$$\begin{aligned}&\forall \epsilon, M > 0, \exists \big(J_0 \subseteq I \text{ and } J_1 \subseteq I \text{ and } J_0, J_1 \text{ are finite}\big): \\ &\qquad \forall \big(J_0 \subseteq J \subseteq I \text{ and } J \text{ is finite}\big), \sum_{i\in J} a_i^+ > M + (s+\epsilon) \\ &\qquad \forall \big(J_1 \subseteq J \subseteq I \text{ and } J \text{ is finite}\big), \sum_{i\in J} a_i^- < s + \epsilon \\ &\qquad \therefore \forall \big(J_0 \cup J_1 \subseteq J \subseteq I \text{ and } J \text{ is finite}\big), \sum_{i\in J} a_i > M\end{aligned}$$

$\therefore \sum_{i\in I} a_i = \overline{\sum}_{i\in I} a_i = \infty$

Similarly, If I is a set, $(a_i)_{i\in I}$ is a sequence indexed by I, $\sum_{i\in I} a_i = -\infty$,

Then $\sum_{i\in I} a_i = \overline{\sum}_{i\in I} a_i = -\infty$

If I is a set, $(a_i)_{i\in I}$ is a sequence indexed by I, $\overline{\sum}_{i\in I} a_i = \infty$,

Assume $\sum_{i\in I} a_i^- = \infty$,

$I_1 := \{i \in I : a_i < 0\}, \sum_{i\in I_1} a_i^- = \infty, \therefore I_1$ is infinite.

$$\forall M,N>0,\exists\big(J_0\subseteq I \text{ and } J_0 \text{ is finite and} J_1\subseteq I_1 \text{ and } J_1 \text{ is finite}\big):$$
$$\forall\big(J_0\subseteq J\subseteq I \text{ and } J \text{ is finite}\big),\sum_{i\in J}a_i>M$$
$$\sum_{i\in J_1}a_i<-N-\sum_{i\in J_0}a_i^+$$
$$\begin{aligned}\sum_{i\in J_0\cup J_1}a_i&=\sum_{k\in J_1}a_i+\sum_{i\in J_0\setminus J_1}a_i\\&\le\sum_{i\in J_1}a_i+\sum_{i\in J_0}a_i^+<-N\end{aligned}$$

,which is a contradiction.

$\therefore\sum_{i\in I}a_i^-\in\mathbb{R}$.

If $\sum_{i\in I}a_i^+\in\mathbb{R}$, then $\sum_{i\in I}a_i\in\mathbb{R}, \therefore\overline{\sum}_{i\in I}a_i\in\mathbb{R}$

$\therefore\sum_{i\in I}a_i^+=\infty$

$\therefore\sum_{i\in I}a_i=\overline{\sum}_{i\in I}a_i=\infty$

Similarly, If I is a set, $(a_i)_{i\in I}$ is a sequence indexed by I, $\overline{\sum}_{i\in I}a_i=-\infty$

Then $\sum_{i\in I}a_i=\overline{\sum}_{i\in I}a_i=-\infty$ ∎

Convention 4.
We allow $\sum_{i\in I}a_i=\overline{\sum}_{i\in I}a_i$ through out the article.

Lemma 2.10.
If $(a_i)_{i\in I},(b_i)_{i\in I}$ are sequences indexed by I;
$\sum_{i\in I}a_i,\sum_{i\in I}b_i\in\mathbb{R}$, $\big\{\sum_{i\in I}a_i,\sum_{i\in I}b_i\big\}\ne\{\infty,\infty\}$, $\big\{\sum_{i\in I}a_i,\sum_{i\in I}b_i\big\}\ne\{-\infty,-\infty\}$
Then $\sum_{i\in I}\big(a_i-b_i\big)=\sum_{i\in I}a_i-\sum_{i\in I}b_i$, allowing convention 2.

Proof. It is obvious using Definition 2.8. We will omit the proof here. ∎

Theorem 2.11. *Regrouping theorem for unorder series.*
If $(S_i)_{i\in I}$ is a collection of pairwise disjoint sets, $S=\bigcup_{i\in I}S_i$.
$\forall i\in I,\sum_{s\in S_i}a_s\in\mathbb{R}$. $\sum_{s\in S}a_s$ is defined.
Then $\sum_{s\in S}a_s=\sum_{i\in I}\sum_{s\in S_i}a_s$.

Proof.

$$\begin{aligned}\sum_{i\in I}\sum_{s\in S_i}a_s^\pm&=\sup\left\{\sum_{i\in T}\sum_{s\in S_i}a_s^\pm: T\subseteq I \text{ and } T \text{ is finite}\right\}\\&=\sup\left\{\sum_{i\in T}\left(\sup\left\{\sum_{s\in J_i}a_s^\pm: J_i\subseteq S_i \text{ and } J_i \text{ is finite}\right\}\right): T\subseteq I \text{ and } T \text{ is finite}\right\}\end{aligned}$$

$$\sum_{i\in I}\sum_{s\in S_i} a_s^{\pm} = \sup\left\{\left(\sup\left\{\sum_{s\in\cup_{i\in T}J_i} a_s^{\pm} : \forall i\in T, \big(J_i\subseteq S_i \text{ and } J_i \text{ is finite}\big)\right\}\right) : T\subseteq I \text{ and } T \text{ is finite}\right\}$$

$\Big($Prove this equality by considering both cases $\leq$ and $\geq$. See [5], P.328, Lemma 35.32 and the paragragh right after its proof for a generalization to $\sup A + \sup B \leq \sup(A+B)$.$\Big)$

$$= \sup\left\{\left(\sup\left\{\sum_{s\in J} a_s^{\pm} : J\subseteq \bigcup_{i\in T} S_i \text{ and } J \text{ is finite}\right\}\right) : T\subseteq I \text{ and } T \text{ is finite}\right\}$$

$$= \sup\left\{\sum_{s\in J} a_s^{\pm} : J\subseteq S \text{ and } J \text{ is finite}\right\}$$

(Prove this equality by considering both cases $\leq$ and $\geq$.)

$$= \sum_{s\in S} a_s^{\pm}$$

$\forall i\in I, \sum_{s\in S_i} a_s = \sum_{s\in S_i} a_s^+ - \sum_{s\in S_i} a_s^-$, and $\sum_{s\in S_i} a_s^+, \sum_{s\in S_i} a_s^- \in \mathbb{R}$

$$\begin{aligned}
\sum_{s\in S} a_s &= \sum_{s\in S} a_s^+ - \sum_{s\in S} a_s^- && \text{, allowing convention 2}\\
&= \sum_{i\in I}\sum_{s\in S_i} a_s^+ - \sum_{i\in I}\sum_{s\in S_i} a_s^- && \text{, allowing convention 2}\\
&= \sum_{i\in I}\left(\sum_{s\in S_i} a_s^+ - \sum_{s\in S_i} a_s^-\right) && \text{, (by Lemma 2.10), allowing convention 2}\\
&= \sum_{i\in I}\sum_{s\in S_i} a_s
\end{aligned}$$

■

3 Behavior of Cauchy products of 2 series

The careful study about behavior of Cauchy products, is kind of interesting, and perhaps a reason why convergence behavior is further divided into absolute and conditional convergence respectively, is the deal with product of more than two series, which the absolute convergence of Cauchy product of first two series, for instance, is needed.

The following theorem is a corollary of a later proven independent Theorem 3.9, however, proven it using the Regrouping theorem for unorder series (Theorem 2.11) gives us a clear image (related to Figure 1.2) of what indeed is a Cauchy product and why indeed we want to invent Cauchy product instead of a just using unordered series $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_i b_j$. Otherwise, we won't be using quite a lot of pages introducing the Regrouping theorem for unordered series.

Theorem 3.1.
If $\sum_{k=0}^{\infty} a_k, \sum_{k=0}^{\infty} b_k$ converge absolutely,

Then unordered series $\sum_{n,m\geq 0} a_n b_m$ converges and their Cauchy product $\sum_{n=0}^{\infty} c_n = \left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right)$.

Further more, $\sum_{n=0}^{\infty} |c_n| \in \mathbb{R}$, i.e. the Cauchy product converges absolutely.

Proof. $\forall n, m \geq 0, d_{(n,m)} := a_m b_n$, $I := \{(m,n) : m, n \geq 0\}$,

$$\begin{aligned}\sum_{s\in I} |d_s| &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |d_{(m,n)}| \\ &= \left|\sum_{k=0}^{\infty} a_k\right| \left|\sum_{k=0}^{\infty} b_k\right| \in \mathbb{R} \qquad \text{, by Theorem 2.11}\end{aligned}$$

$\because \sum_{s\in I} d_s^+$ and $\sum_{s\in I} d_s^- \in \mathbb{R}, \therefore \sum_{s\in I} d_s \in \mathbb{R}$

$$\begin{aligned}\sum_{s\in I} d_s &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} d_{(m,n)} \\ &= \left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right) \qquad \text{, by Theorem 2.11}\end{aligned}$$

$$\sum_{s\in I} d_s = \sum_{n=0}^{\infty}\sum_{k=0}^{n} d_{(k,n-k)}$$, by Theorem 2.11, in the sense of Figure 1.2.

that Cauchy product is just a regrouping of $\sum_{s\in I} d_s$.

$\therefore \left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} d_{(k,n-k)}$

$$\begin{aligned}\sum_{n=0}^{\infty} |c_n| &= \sum_{n=0}^{\infty}\left|\sum_{k=0}^{n} a_k b_{n-k}\right| \\ &\leq \sum_{n=0}^{\infty}\sum_{k=0}^{n} |a_k|\,|b_{n-k}| \in \mathbb{R}\end{aligned}$$

$\therefore \sum_{n=0}^{\infty} |c_n| \in \mathbb{R}$ ■

We emphasize that theorems about behavior of Cauchy product of two series are sometimes proved by considering auxiliary power series and their Cauchy product. To name a few possibilities, Theorem 3.4 and Lemma 3.5, so we would like to have the following nice lemma about power series and their Cauchy products, and Theorem 3.3, which is very useful.

Lemma 3.2. *Power series can be multiplied using Cauchy product under interval of convergence.*
If $\sum_{n=0}^{\infty} a_n x^n, \sum_{n=0}^{\infty} b_n x^n$ are two power series.
Let $R \in \overline{\mathbb{R}}^+$ be the minimum of their radii of convergence (here we assume $R \neq 0$),
i.e.$\forall (x : |x| < R)$, both power series converges.
Then $\forall (x : |x| < R), \sum_{n=0}^{\infty}\sum_{k=0}^{n}(a_k x^k)(b_{n-k}x^{n-k}) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)x^n = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right)$.

Proof. By definition of radius of convergence, (see [5], P.188, Theorem 23.1)
$\forall (x : |x| < R), \sum_{n=0}^{\infty} |a_n| x^n$ and $\sum_{n=0}^{\infty} |b_n| x^n$ converge.
$\forall (x : |x| < R), \sum_{n=0}^{\infty} |a_n| (|x|)^n$ and $\sum_{n=0}^{\infty} |b_n| (|x|)^n$ converge.
i.e. $\forall (x : |x| < R), \sum_{n=0}^{\infty} |a_n x^n|$ and $\sum_{n=0}^{\infty} |b_n x^n|$ converge.
By Theorem 3.1, the proof holds. ■

Theorem 3.3. *Abel's theorem.*
If power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *has radius of convergence* $R \in \mathbb{R}^+$,
Then $\Big(\sum_{n=0}^{\infty} a_n R^n$ *converges* $\implies f$ *is continuous at* $R \Big)$
and $\Big(\sum_{n=0}^{\infty} a_n (-R)^n$ *converges* $\implies f$ *is continuous at* $-R \Big)$.

Proof. See [5], P.212, Theorem 26.6 for a proof. We will omit it here. ∎

The following theorem states that the value of Cauchy product is the product of the values of the two factor series, given all three series converge.

Theorem 3.4.
If $\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$ *converge, and their Cauchy product* $\sum_{n=0}^{\infty} c_n$ *also converges.*
Then $\sum_{n=0}^{\infty} c_n = \Big(\sum_{n=0}^{\infty} a_n \Big) \Big(\sum_{n=0}^{\infty} b_n \Big)$.

Proof. $f(x) := \sum_{n=0}^{\infty} a_n x^n \ \forall x \in (-1,1)$; $g(x) := \sum_{n=0}^{\infty} b_n x^n \ \forall x \in (-1,1)$; $h(x) := \sum_{n=0}^{\infty} c_n x^n \ \forall x \in (-1,1)$;
By Theorem 3.3, $\lim_{x \to 1^-} f(x) = \sum_{n=0}^{\infty} a_n$; $\lim_{x \to 1^-} g(x) = \sum_{n=0}^{\infty} b_n$; $\lim_{x \to 1^-} h(x) = \sum_{n=0}^{\infty} c_n$.
By Lemma 3.2, $\Big(fg \Big)(x) = h(x) \ \forall (-1,1)$
$\therefore \sum_{n=0}^{\infty} c_n = \Big(\sum_{n=0}^{\infty} a_n \Big) \Big(\sum_{n=0}^{\infty} b_n \Big)$ ∎

Lemma 3.5.
If $\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$ *converge, then their Cauchy product* $\sum_{n=0}^{\infty} c_n \neq \infty$.

Proof. Below is refered to the proof in [2] (and is indeed nearly a copy).
$A := \sum_{n=0}^{\infty} a_n; B := \sum_{n=0}^{\infty} b_n$.
$\forall x \in (-1,1), f(x) := \sum_{n=0}^{\infty} a_n x^n; g(x) := \sum_{n=0}^{\infty} b_n x^n$.
By Theorem 3.3, $\lim_{x \to 1^-} f(x) = A; \lim_{x \to 1^-} g(x) = B$.
$\forall x \in (-1,1), h(x) := \Big(fg \Big)(x) = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} a_k b_{n-k} \Big) x^n$
$\therefore \lim_{x \to 1^-} h(x) = AB$

$$(s_n)_{n=-1}^{\infty} = \sum_{k=0}^{n} c_k \qquad \text{, (we include the degenerate case } s_{-1} = 0)$$

Assume $(s_n) \to \infty$,
$\forall x \in (-1,1), H(x) := \sum_{n=0}^{\infty} s_n x^n; h(x) = \sum_{n=0}^{\infty} (s_n - s_{n-1}) x^n = (1-x)H(x)$.

$\forall M > 0,$
$\exists N \in \mathbb{N} : (n > N \implies s_n > M)$

$$\forall x \in (0,1), H(x) = \sum_{n=0}^{N} s_n x^n + \sum_{n=N+1}^{\infty} s_n x^n$$

$$\leq \sum_{n=0}^{N} s_n x^n + \sum_{n=N+1}^{\infty} M x^n = \sum_{n=0}^{N} s_n x^n + \frac{M x^{N+1}}{1-x}$$

$$(1-x)H(x) = (1-x) \sum_{n=0}^{N} s_n x^n + M x^{N+1}$$

$\because \lim_{x\to 1^-} h(x) = AB, \quad \therefore \lim_{x\to 1^-}(1-x)H(x) = AB.$

However, $\lim_{x\to 1^-}(1-x)\sum_{n=0}^{N} s_n x^n = 0$ and $\lim_{x\to 1^-} Mx^{N+1} = M$, implies a contradiction.

$\therefore (s_n) \not\to \infty.$ ∎

Theorem 3.6.
If $\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$ converge, then their Cauchy product $\sum_{n=0}^{\infty} c_n$ never diverges properly.

Proof. Using Lemma 3.5,

If $\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$ converge,
then $\sum_{n=0}^{\infty}(-b_n)$ converges.

$\therefore$ Cauchy product $\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k(-b_{n-k})\right) \neq \infty$.

Using Lemma 3.5 again, the proof holds. ∎

Theorem 3.7.
If $\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$ converges conditionally,
Then their Cauchy product $\sum_{n=0}^{\infty} c_n$ may converges conditionally.

Proof. The following example is given in [6].

$(a_n)_{n=0}^{\infty} = (b_n)_{n=0}^{\infty} := \frac{(-1)^{n+1}}{n+1}$,

$$\begin{aligned}(c_n)_{n=0}^{\infty} &= (-1)^n \sum_{k=0}^{n} \frac{1}{(k+1)(n+1-k)} = (-1)^n \sum_{k=0}^{n}\left(\frac{1}{k+1} + \frac{1}{n+1-k}\right)\frac{1}{n+2} \\ &= \frac{(-1)^n}{n+2} 2H_{n+1} \qquad \text{, while } H_n \text{ represents the } n^{th} \text{ Harmonic number} \\ &\qquad \text{, which will be used through out this thesis.}\end{aligned}$$

$\forall n \in \mathbb{N}$,

$$\begin{aligned} H_n &\geq 1 \\ (n+2)H_n &\geq (n+1)H_{n+1} \\ \frac{H_n}{n+1} &\geq \frac{H_{n+1}}{n+2} \end{aligned}$$

$$\because \lim_{n\to\infty} \frac{H_n}{\ln n} = 1, \quad \therefore \lim_{n\to\infty} \frac{H_n}{n+1} = 0$$

$\therefore \sum_{n=0}^{\infty} c_n \in \mathbb{R}$ by alternate series theorem.

$$\begin{aligned} &\because \sum_{n=2}^{\infty} \frac{\ln n}{n+1} = \infty \\ &\therefore \sum_{n=2}^{\infty} \frac{H_n}{n+1} = \infty, \quad \sum_{n=0}^{\infty} |c_n| = \infty \end{aligned}$$

, i.e. $\sum_{n=0}^{\infty} c_n$ converges conditionally. ∎

Theorem 3.8.
If $\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$ *converge conditionally,*
Then their Cauchy product $\sum_{n=0}^{\infty} c_n$ *may diverges by oscillation.*

Proof. $(a_n)_{n=0}^{\infty} = (b_n)_{n=0}^{\infty} := \frac{(-1)^{n+1}}{\sqrt{n+1}}$,
$\forall n \geq 0, c_n = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\sqrt{n+1-k}}$
$\therefore |c_n| \geq \sum_{k=0}^{n} \frac{1}{n+1} = 1$
$\therefore \sum_{n=0}^{\infty} c_n$ doesn't converge.
By Theorem 3.6, $\sum_{n=0}^{\infty} c_n$ doesn't diverge properly, so it could only diverges by oscillation. ∎

Theorem 3.9. *Merten's theorem.*
If $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *converges absolutely and converges respectively,*
Then Cauchy product $\sum_{n=0}^{\infty} c_n$ *converges.*

Proof. Motivated by [7], P.74, Theorem 3.50,
$S := \max\Big\{|B_j - B| : j \geq 0\Big\}$,

$$\forall N \in \mathbb{N}, \sum_{i=0}^{N} c_i = \sum_{i=0}^{N}\sum_{j=0}^{i} a_j b_{i-j} = \sum_{j=0}^{N}\sum_{i=j}^{N} a_j b_{i-j} = \sum_{j=0}^{N} a_j \sum_{i=0}^{N-j} b_i$$

$$= \sum_{j=0}^{N} a_j B_{N-j}, \text{ which is clear in the sense of Figure 1.1, reading row by row.}$$

$$= \sum_{j=0}^{N} a_{N-j} B_j = \sum_{j=0}^{N} a_{N-j}\big(B_j - B\big) + BA_N$$

$$\forall \epsilon > 0, \exists M_1, M_2 \in \mathbb{N} : \Big(j \geq M_1 \implies |B_j - B| < \epsilon\Big) \bigwedge \Big(n \geq M_2 \implies \textstyle\sum_{j=n}^{\infty} |a_j| < \epsilon\Big)$$

$$\forall N \geq M_1 + M_2, \left|\sum_{j=0}^{N} a_{N-j}\big(B_j - B\big)\right| \leq \sum_{j=0}^{N-M_2} |a_{N-j}|S + \sum_{j=N+1-M_2}^{N} |a_{N-j}|\epsilon$$
$$\leq S\epsilon + A\epsilon$$

$$\therefore \sum_{i=0}^{\infty} c_i = \lim_{N\to\infty} \sum_{j=0}^{N} a_{N-j} B_j = AB$$

∎

Theorem 3.10.
If $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *converges absolutely and converges respectively,*
Then Cauchy product $\sum_{n=0}^{\infty} c_n$ *may not converges absolutely.*

Proof. $(a_n)_{n=0}^{\infty} := \frac{(-1)^{n+1}}{(n+1)\sqrt{n+1}}, \quad (b_n)_{n=0}^{\infty} := \frac{(-1)^{n+1}}{\sqrt{n+1}},$

$$(c_n) = \sum_{k=0}^{n} a_k b_{n-k} = (-1)^n \sum_{k=0}^{n} \frac{1}{k+1} \frac{1}{\sqrt{k+1}\sqrt{n+1-k}}$$
$$|c_n| \geq \sum_{k=0}^{n} \frac{1}{k+1} \frac{2}{n+2} = \frac{2H_{n+1}}{n+2} \geq \frac{2}{n+2}$$

$\therefore \sum_{n=0}^{\infty} |c_n| = \infty$ ∎

Theorem 3.11.
If $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *both converges conditionally,*
Then Cauchy product $\sum_{n=0}^{\infty} c_n$ *may converges absolutely.*

Proof. The following example is given by [1].

$$(a_n)_{n=0}^{\infty} = \begin{cases} \frac{1}{4p+1}, \text{if } n = 4p \\ \frac{-1}{4p+4}, \text{if } n = 4p+1 \\ \frac{1}{4p+1}, \text{if } n = 4p+2 \\ \frac{-1}{4p+4}, \text{if } n = 4p+3 \end{cases}$$
$$(b_n)_{n=0}^{\infty} = \begin{cases} \frac{1}{4p+1}, \text{if } n = 4p \\ \frac{1}{4p+4}, \text{if } n = 4p+1 \\ \frac{-1}{4p+1}, \text{if } n = 4p+2 \\ \frac{-1}{4p+4}, \text{if } n = 4p+3 \end{cases}$$

It is easy to prove $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converges conditionally, and we will omit it here.

In order to prove the absolute convergence of $\sum_{n=0}^{\infty} c_n$, we would like to consider all four cases: $c_{4n}, c_{4n+1}, c_{4n+2}, c_{4n+3}$ for $n \in \mathbb{N}$.

For c_{4n},

$\forall n \in \mathbb{N}$,

$$c_{4n} = \sum_{k=0}^{4n} a_k b_{4n-k}$$
$$= \begin{cases} \sum_{k=0}^{2n-1} a_k b_{4n-k} + \sum_{k=2n+1}^{4n} a_k b_{4n-k} + a_{2n} b_{2n}, \text{if } 2 \mid n \\ \sum_{k=0}^{2n+1} a_k b_{4n-k} + \sum_{k=2n-1}^{4n} a_k b_{4n-k} \\ \quad -a_{2n-1} b_{2n+1} - a_{2n} b_{2n} - a_{2n+1} b_{2n-1}, \text{if } 2 \nmid n \end{cases}$$

$$c_{4n} = \begin{cases} \sum_{k=0}^{2n-1} a_k b_{4n-k} + \sum_{k=0}^{2n-1} a_{4n-k} b_k + \frac{1}{2n+1}\frac{1}{2n+1}, \text{if } 2 \mid n \\ \sum_{k=0}^{2n+1} a_k b_{4n-k} + \sum_{k=0}^{2n+1} a_{4n-k} b_k - (\frac{-1}{2n+2})(\frac{-1}{2n+2}) \\ \qquad -(\frac{1}{2n-1})(\frac{-1}{2n-1}) - (\frac{-1}{2n+2})(\frac{1}{2n+2}), \text{if } 2 \not| \, n \\ \quad \text{(Just take } n = 2p+1) \end{cases}$$

$$= \begin{cases} \sum_{k=0}^{\frac{n}{2}-1} a_{4k} b_{4(n-k)} + \sum_{k=0}^{\frac{n}{2}-1} a_{4k+1} b_{4(n-k+1)+3} + \sum_{k=0}^{\frac{n}{2}-1} a_{4k+2} b_{4(n-k+1)+2} \\ \quad + \sum_{k=0}^{\frac{n}{2}-1} a_{4k+3} b_{4(n-k+1)+1} + \sum_{k=0}^{\frac{n}{2}-1} a_{4(n-k)} b_{4k} + \sum_{k=0}^{\frac{n}{2}-1} a_{4(n-k+1)+3} b_{4k+1} \\ \quad + \sum_{k=0}^{\frac{n}{2}-1} a_{4(n-k+1)+2} b_{4k+2} + \sum_{k=0}^{\frac{n}{2}-1} a_{4(n-k+1)+1} b_{4k+3} + \frac{1}{(2n+1)^2}, \text{if } 2 \mid n \\ \sum_{k=0}^{\frac{n-1}{2}} a_{4k} b_{4(n-k)} + \sum_{k=0}^{\frac{n-1}{2}} a_{4k+1} b_{4(n-k+1)+3} + \sum_{k=0}^{\frac{n-1}{2}} a_{4k+2} b_{4(n-k+1)+2} \\ \quad + \sum_{k=0}^{\frac{n-1}{2}} a_{4k+3} b_{4(n-k+1)+1} + \sum_{k=0}^{\frac{n-1}{2}} a_{4(n-k)} b_{4k} + \sum_{k=0}^{\frac{n-1}{2}} a_{4(n-k+1)+3} b_{4k+1} \\ \quad + \sum_{k=0}^{\frac{n-1}{2}} a_{4(n-k+1)+2} b_{4k+2} + \sum_{k=0}^{\frac{n-1}{2}} a_{4(n-k+1)+1} b_{4k+3} + \frac{1}{(2n-1)^2}, \text{if } 2 \not| \, n \end{cases}$$

$$= \sum_{k=0}^{n'} \left(\frac{1}{4k+1}\frac{1}{4n-4k+1} + \frac{-1}{4k+4}\frac{-1}{4n-4k} + \frac{1}{4k+1}\frac{-1}{4n-4k-3} + \frac{-1}{4k+4}\frac{1}{4n-4k} \right.$$
$$\left. + \frac{1}{4n-4k+1}\frac{1}{4k+1} + \frac{-1}{4n-4k}\frac{1}{4k+4} + \frac{1}{4n-4k-3}\frac{-1}{4k+1} + \frac{-1}{4n-4k}\frac{-1}{4k+4} \right)$$
$$+ \alpha_n$$

$$\text{,while } n' = \begin{cases} \frac{n}{2} - 1, \text{if } 2 \mid n \\ \frac{n-1}{2}, \text{if } 2 \not| \, n \end{cases} \quad \text{and } \alpha_n = \begin{cases} \frac{1}{(2n+1)^2}, \text{if } 2 \mid n \\ \frac{1}{(2n-1)^2}, \text{if } 2 \not| \, n. \end{cases}$$

$$= 2\sum_{k=0}^{n'} \left(\frac{1}{4k+1}\frac{1}{4n-4k+1} - \frac{1}{4k+1}\frac{1}{4n-4k-3} \right) + \alpha_n$$

$$-c_{4n} = 2\sum_{k=0}^{n'} \left[\frac{1}{4k+1}\frac{4}{(4n-4k+1)(4n-4k-3)} \right] + \alpha_n$$

$f_n(x) := (4n-4x+1)(4n-4x+3) = 16x^2 - 4x(8n-2) + (4n+1)(4n-3)$
If $2 \mid n$, $f_n(n') = f_n(\frac{n}{2}-1) = (4n-2n+5)(4n-2n+1) = 4n^2 + 12n + 5$
$f_n(0) = (4n+1)(4n-3) = 16n^2 - 8n - 3$
If $2 \nmid n$, $f_n(n') = f_n(\frac{n-1}{2}) = (2n+3)(2n+1) = 4n^2 + 8n + 3$
$f_n(0) = (4n+1)(4n-3) = 16n^2 - 8n - 3$

$\forall n \in \mathbb{N}$,
If $f_n'(x) = 0$,

$$32x - 4(8n-2) = 0$$
$$4x = 4n - 1$$
$$x = n - \frac{1}{4} > n'$$

$\therefore$ Eventually, $\forall x \in [0, n'], f_n(x) \geq f_n(n') \geq 4n^2$.

$\exists N \in \mathbb{N} : \forall n > N,$

$$
\begin{aligned}
-c_{4n} &\le 2\sum_{k=0}^{n'}\Big(\frac{1}{4k+1}\frac{4}{4n^2}\Big)+\alpha_n \le 2\sum_{k=1}^{4n'+1}\frac{1}{k}\frac{1}{n^2}+\alpha_n\\
&\le \frac{2H_{2n}}{n^2}+\alpha_n \le \frac{2H_{2n}}{n^2}+\frac{1}{(2n-1)^2}\\
-c_{4n} &\ge \frac{1}{(2n+1)^2}
\end{aligned}
$$

$$
\therefore -\frac{2H_{2n}}{n^2}-\frac{1}{(2n-1)^2} \le c_{4n} \le -\frac{1}{(2n+1)^2}
$$

For c_{4n+1},
$\forall n \in \mathbb{N},$

$$
\begin{aligned}
c_{4n+1} &= \sum_{k=0}^{4n+1} a_k b_{4n+1-k}\\
&= \begin{cases} \sum_{k=0}^{2n-1} a_k b_{4n+1-k} + \sum_{k=2n+2}^{4n+1} a_k b_{4n+1-k} + a_{2n}b_{2n+1} + a_{2n+1}b_{2n}, \text{if } 2 \mid n \\ \sum_{k=0}^{2n+1} a_k b_{4n+1-k} + \sum_{k=2n}^{4n+1} a_k b_{4n+1-k} - a_{2n}b_{2n+1} - a_{2n+1}b_{2n}, \text{if } 2 \nmid n \end{cases}\\
&= \begin{cases} \sum_{k=0}^{2n-1} a_k b_{4n+1-k} + \sum_{k=0}^{2n-1} a_{4n+1-k} b_k \\ \qquad + \frac{1}{4(\frac{n}{2})+1}\frac{1}{4(\frac{n}{2})+4} + \frac{-1}{4(\frac{n}{2})+4}\frac{1}{4(\frac{n}{2})+1}, \text{if } 2 \mid n \\ \sum_{k=0}^{2n+1} a_k b_{4n+1-k} + \sum_{k=0}^{2n+1} a_{4n+1-k} b_k - \frac{1}{2n-1}\frac{-1}{2n+2} - \frac{-1}{2n+2}\frac{-1}{2n-1}, \text{if } 2 \nmid n \\ \qquad \text{(Just take } n = 2p+1) \end{cases}\\
&= \begin{cases} \sum_{k=0}^{2n-1} a_k b_{4n+1-k} + \sum_{k=0}^{2n-1} a_{4n+1-k} b_k, \text{if } 2 \mid n \\ \sum_{k=0}^{2n+1} a_k b_{4n+1-k} + \sum_{k=0}^{2n+1} a_{4n+1-k} b_k, \text{if } 2 \nmid n \end{cases}\\
&= \begin{cases} \sum_{k=0}^{\frac{n}{2}-1} a_{4k} b_{4(n-k)+1} + \sum_{k=0}^{\frac{n}{2}-1} a_{4k+1} b_{4(n-k)} + \sum_{k=0}^{\frac{n}{2}-1} a_{4k+2} b_{4(n-k-1)+3} \\ \quad + \sum_{k=0}^{\frac{n}{2}-1} a_{4k+3} b_{4(n-k+1)+2} + \sum_{k=0}^{\frac{n}{2}-1} a_{4(n-k)+1} b_{4k} + \sum_{k=0}^{\frac{n}{2}-1} a_{4(n-k)} b_{4k+1} \\ \quad + \sum_{k=0}^{\frac{n}{2}-1} a_{4(n-k-1)+3} b_{4k+2} + \sum_{k=0}^{\frac{n}{2}-1} a_{4(n-k+1)+2} b_{4k+3}, \text{if } 2 \mid n \\ \sum_{k=0}^{\frac{n-1}{2}} a_{4k} b_{4(n-k)+1} + \sum_{k=0}^{\frac{n-1}{2}} a_{4k+1} b_{4(n-k)} + \sum_{k=0}^{\frac{n-1}{2}} a_{4k+2} b_{4(n-k-1)+3} \\ \quad + \sum_{k=0}^{\frac{n-1}{2}} a_{4k+3} b_{4(n-k+1)+2} + \sum_{k=0}^{\frac{n-1}{2}} a_{4(n-k)+1} b_{4k} + \sum_{k=0}^{\frac{n-1}{2}} a_{4(n-k)} b_{4k+1} \\ \quad + \sum_{k=0}^{\frac{n-1}{2}} a_{4(n-k-1)+3} b_{4k+2} + \sum_{k=0}^{\frac{n-1}{2}} a_{4(n-k+1)+2} b_{4k+3}, \text{if } 2 \nmid n \end{cases}\\
&= \sum_{k=0}^{n'}\Big(\frac{1}{4k+1}\frac{1}{4n-4k+4} + \frac{-1}{4k+1}\frac{1}{4n-4k+1} + \frac{1}{4k+1}\frac{-1}{4n-4k}\\
&\qquad + \frac{-1}{4k+4}\frac{-1}{4n-4k+5} + \frac{-1}{4n-4k+4}\frac{1}{4k+1} + \frac{1}{4n-4k+1}\frac{1}{4k+4}\\
&\qquad\qquad + \frac{-1}{4n-4k}\frac{-1}{4k+1} + \frac{1}{4n-4k+5}\frac{-1}{4k+4}\Big)\\
&\qquad ,\text{while } n' = \begin{cases} \frac{n}{2}-1, \text{if } 2 \mid n \\ \frac{n-1}{2}, \text{if } 2 \nmid n \end{cases}\\
&= 0
\end{aligned}
$$

For c_{4n+2},
$\forall n \in \mathbb{N}$,

$$
\begin{aligned}
c_{4n+2} &= \sum_{k=0}^{4n+2} a_k b_{4n+2-k} \\
&= \begin{cases} \sum_{k=0}^{2n-1} a_k b_{4n+2-k} + \sum_{k=2n+3}^{4n+2} a_k b_{4n+2-k} \\ \qquad + a_{2n} b_{2n+2} + a_{2n+1} b_{2n+1} + a_{2n+2} b_{2n}, \text{if } 2 \mid n \\ \sum_{k=0}^{2n+1} a_k b_{4n+2-k} + \sum_{k=2n+1}^{4n+2} a_k b_{4n+2-k} - a_{2n+1} b_{2n+1}, \text{if } 2 \not| \, n \end{cases} \\
&= \begin{cases} \sum_{k=0}^{2n-1} a_k b_{4n+2-k} + \sum_{k=0}^{2n-1} a_{4n+2-k} b_k \\ \qquad + \frac{1}{4(\frac{n}{2})+1} \frac{-1}{4(\frac{n}{2})+1} + \frac{-1}{4(\frac{n}{2})+4} \frac{1}{4(\frac{n}{2})+4} + \frac{1}{4(\frac{n}{2})+1} \frac{1}{4(\frac{n}{2})+1}, \text{if } 2 \mid n \\ \sum_{k=0}^{2n+1} a_k b_{4n+2-k} + \sum_{k=0}^{2n+1} a_{4n+2-k} b_k - \frac{-1}{2n+2} \frac{-1}{2n+2}, \text{if } 2 \not| \, n \\ \quad \text{(Just take } n = 2p+1) \end{cases} \\
&= \begin{cases} \sum_{k=0}^{2n-1} a_k b_{4n+2-k} + \sum_{k=0}^{2n-1} a_{4n+2-k} b_k - \frac{1}{(2n+4)^2}, \text{if } 2 \mid n \\ \sum_{k=0}^{2n+1} a_k b_{4n+2-k} + \sum_{k=0}^{2n+1} a_{4n+2-k} b_k - \frac{1}{(2n+2)^2}, \text{if } 2 \not| \, n \end{cases} \\
&= \begin{cases} \sum_{k=0}^{\frac{n}{2}-1} a_{4k} b_{4(n-k)+2} + \sum_{k=0}^{\frac{n}{2}-1} a_{4k+1} b_{4(n-k)+1} + \sum_{k=0}^{\frac{n}{2}-1} a_{4k+2} b_{4(n-k)} \\ \quad + \sum_{k=0}^{\frac{n}{2}-1} a_{4k+3} b_{4(n-k+1)+3} + \sum_{k=0}^{\frac{n}{2}-1} a_{4(n-k)+2} b_{4k} + \sum_{k=0}^{\frac{n}{2}-1} a_{4(n-k)+1} b_{4k+1} \\ \quad + \sum_{k=0}^{\frac{n}{2}-1} a_{4(n-k)} b_{4k+2} + \sum_{k=0}^{\frac{n}{2}-1} a_{4(n-k+1)+3} b_{4k+3} - \frac{1}{(2n+4)^2}, \text{if } 2 \mid n \\ \sum_{k=0}^{\frac{n-1}{2}} a_{4k} b_{4(n-k)+2} + \sum_{k=0}^{\frac{n-1}{2}} a_{4k+1} b_{4(n-k)+1} + \sum_{k=0}^{\frac{n-1}{2}} a_{4k+2} b_{4(n-k)} \\ \quad + \sum_{k=0}^{\frac{n-1}{2}} a_{4k+3} b_{4(n-k+1)+3} + \sum_{k=0}^{\frac{n-1}{2}} a_{4(n-k)+2} b_{4k} + \sum_{k=0}^{\frac{n-1}{2}} a_{4(n-k)+1} b_{4k+1} \\ \quad + \sum_{k=0}^{\frac{n-1}{2}} a_{4(n-k)} b_{4k+2} + \sum_{k=0}^{\frac{n-1}{2}} a_{4(n-k+1)+3} b_{4k+3} - \frac{1}{(2n+2)^2}, \text{if } 2 \not| \, n \end{cases} \\
&= \sum_{k=0}^{n'} \left(\frac{1}{4k+1} \frac{-1}{4n-4k+1} + \frac{-1}{4k+4} \frac{1}{4n-4k+4} + \frac{1}{4k+1} \frac{1}{4n-4k+1} \right. \\
&\qquad + \frac{-1}{4k+4} \frac{-1}{4n-4k+8} + \frac{1}{4n-4k+1} \frac{1}{4k+1} + \frac{-1}{4n-4k+4} \frac{1}{4k+4} \\
&\qquad \left. + \frac{1}{4n-4k+1} \frac{-1}{4k+1} + \frac{-1}{4n-4k+8} \frac{-1}{4k+4} \right) + \beta_n \\
&\qquad \text{,while } n' = \begin{cases} \frac{n}{2} - 1, \text{if } 2 \mid n \\ \frac{n-1}{2}, \text{if } 2 \not| \, n \end{cases} \text{ and } \beta_n = \begin{cases} -\frac{1}{(2n+4)^2}, \text{if } 2 \mid n \\ -\frac{1}{(2n+2)^2}, \text{if } 2 \not| \, n. \end{cases} \\
&= 2 \sum_{k=0}^{n'} \left(\frac{1}{4k+4} \frac{1}{4n-4k+8} - \frac{1}{4k+4} \frac{1}{4n-4k+4} \right) + \beta_n \\
&= 2 \sum_{k=0}^{n'} \left(\frac{1}{4k+4} \right) \left(\frac{1}{4n-4k+8} - \frac{1}{4n-4k+4} \right) + \beta_n \\
&= 2 \sum_{k=0}^{n'} \left(\frac{1}{4k+4} \right) \left(\frac{4}{(4n-4k+8)(4n-4k+4)} \right) + \beta_n
\end{aligned}
$$

$$\forall n \in \mathbb{N}, g_n(x) := (4n-4x+4)(4n-4x+8) = 16x^2 - 4x(8n+12) + (4n+4)(4n+8)$$

$$g_n(0) = (4n+4)(4n+8) = 16n^2 + 48n + 32$$
If $2 \mid n, g_n(n') = g_n(\frac{n}{2} - 1) = (2n+8)(2n+12) = 4n^2 + 40n + 96$
If $2 \not| \, n, g_n(n') = g_n(\frac{n-1}{2}) = (2n+6)(2n+10) = 4n^2 + 32n + 60$
$\forall n \in \mathbb{N}, \text{if } g_n'(x) = 0, \text{then } 32x = 4(8n+12) \implies x = \frac{2n+3}{2} = n + \frac{3}{2} > n'$
$\therefore$ Eventually,$\forall x \in [0, n'], g_n(x) \geq g_n(n') \geq 4n^2$

$$\begin{aligned}
\text{Eventually,} -\frac{1}{(2n+2)^2} \leq c_{4n+2} &\leq 2\sum_{k=0}^{n'} \left(\frac{1}{4k+1}\frac{4}{4n^2}\right) + \beta_n \\
&\leq 2\sum_{k=1}^{4n'+1} \frac{1}{k}\frac{1}{n^2} + \beta_n \\
&\leq \frac{2H_{2n}}{n^2} - \frac{1}{(2n+4)^2}
\end{aligned}$$

For c_{4n+3},
$\forall n \in \mathbb{N}$,

$$\begin{aligned}
c_{4n+3} &= \sum_{k=0}^{4n+3} a_k b_{4n+3-k} \\
&= \sum_{k=0}^{n-1} \left(a_{4k}b_{4(n-k)+3} + a_{4k+1}b_{4(n-k)+2} + a_{4k+2}b_{4(n-k)+1} + a_{4k+3}b_{4(n-k)}\right) \\
&= \sum_{k=0}^{n-1} \left(\frac{1}{4k+1}\frac{-1}{4n-4k+4} + \frac{-1}{4k+4}\frac{-1}{4n-4k+1}\right. \\
&\qquad \left. + \frac{1}{4k+1}\frac{1}{4n-4k+4} + \frac{-1}{4k+4}\frac{1}{4n-4k+1}\right) \\
&= 0
\end{aligned}$$

Therefore, for some $N \in \mathbb{N}$,

$$\begin{aligned}
\sum_{n=4N}^{\infty} |c_n| &= \sum_{n=N}^{\infty} |c_{4n}| + \sum_{n=N}^{\infty} |c_{4n+1}| + \sum_{n=N}^{\infty} |c_{4n+2}| + \sum_{n=N}^{\infty} |c_{4n+3}| \\
&= \sum_{n=N}^{\infty} (|c_{4n}| + |c_{4n+1}| + |c_{4n+2}| + |c_{4n+3}|) \\
&= \sum_{n=N}^{\infty} (|c_{4n}| + |c_{4n+2}|) \\
&\leq \sum_{n=N}^{\infty} \left(\frac{2H_{2n}}{n^2} + \frac{1}{(2n-1)^2} + \frac{1}{(2n+1)^2} + \frac{2H_{2n}}{n^2} + \frac{1}{(2n+4)^2} + \frac{1}{(2n+2)^2}\right) \in \mathbb{R}
\end{aligned}$$

$\therefore \sum_{n=0}^{\infty} c_n$ converges absolutely. ■

Conclusion

The following, Figure 1.3, is a conclusion. A, C, P, O stands for absolute convergent, conditional convergent, proper divergent, oscillatory divergent respectively.
Notice when one of $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$ is given absolutely convergent, then the Cauchy product may converges absolutely for sure, to name a possibility, just let all summand be 0.
Also, since $\sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} a_{n-k} b_k$, we will avoid repeated combination.

Figure 1.3: Conclusion

Behavior of		All possible behaviors of Cauchy product $\sum_{n=0}^{\infty} c_n$
$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$	
A	A	A
A	C	A,C
C	C	A,C,O

References

[1] FLORIAN CAJORI. "Divergent and conditionally convergent series whose product is absolutely convergent". In: *Transactions of the American Mathematical Society* 2.1 (1901), pp. 25–36. DOI: https://doi.org/10.1090/S0002-9947-1901-1500553-8.

[2] Daniel Fischer. *Answer to 'Why is the Cauchy product of two convergent (but not absolutely) series either convergent or indeterminate (but does not converge to infinity)?'* URL: https://math.stackexchange.com/questions/1290880/why-is-the-cauchy-product-of-two-convergent-but-not-absolutely-series-either-c?noredirect=1&lq=1#1291335.

[3] Daniel D. Bonar & Michael Khoury Jr. *Real Infinite Series*. Classroom Resource Materials. The Mathematical Association of America, 2006. ISBN: 0883857456.

[4] Graham R. L.& Knuth D. E.& Patashnik O. *Concrete Mathematics : A Foundation for Computer Science*. Addison-Wesley, 2017. ISBN: 0-201-55802-5.

[5] Kenneth A. Ross. *Elementary Analysis : The Theory of Calculus*. Springer, 2013. ISBN: 978-1-4614-6270-5.

[6] RRL. *Answer to 'Conditionally converges $\sum_{k=0}^{\infty} a_n$, $\sum_{k=0}^{\infty} b_n$ and also their Cauchy product'*. URL: https://math.stackexchange.com/questions/2823774/conditionally-converges-sum-k-0-infty-a-n-sum-k-0-infty-b-n-and-als#2824354.

[7] Walter Rudin. *Principles of mathematical analysis*. International series in pure and applied mathematics. McGraw-Hill, Inc., 1976. ISBN: 0-07-054235-X.