Term Paper

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2022-05-19

I shall rely heavily on the [6] for this paper to discuss my understanding on the materials involved.

Contents

1 Construction of $\mathbb{R}$ from the very beginning

1.1 Intuition from the Greeks

There was a natural intuition discussed in Section 7, [6]. Imagine that we are given some line segments of positive lengths x and y respectively, and we are to tell how we could represent or characterize the relative ratio $x : y$ of these two segments, and we assume no familiarity with real numbers. Now, for each segment, we are free to make a certain number (say, m and n respectively) of copies of it, concatenate them, and then we may compare the two whole segments by placing them side by side. Only geometric intuition is required to tell whether $mx = ny$ (if they are observed to coincide regardless of how much we zoom in) or either mx or ny is greater (if it is found that one of the segments exceeds another when fixing the another end). Now we shall note that m and n are arbitrary, as long as they are still positive integers to make our procedure sounds. Consequently, we may partition the set of (m, n) where m, n are positive integers into three disjoint sets: one containing all (m, n) with $mx < ny$, one containing all (zero or just one, actually) (m, n) with $mx = ny$, and the remaining one, necessarily containing all (m, n) with $mx > ny$. Now, note that the first and the last sets must not be empty (we must be able to duplicate a segment once or twice or so to exceed the length of another segment), and we may say, that this kind of 'partition' of the set of pairs of positive integers (m, n) is enough to define $x : y$.

1.2 From intuition to formulation

If we change our perspective just a little bit, it is clear from above that we are basically dividing the set of positive rational numbers n/m (m, n are positive integers, of course) into three cases according to whether n/m is greater than, equal to or less than x/y respectively. Of course, we shall go beyong the Greeks to establish our arguments not on the vague idea of 'line segments of positive length', but instead on set theories and logic.

1.3 From $\mathbb{N}$ and $\mathbb{Z}$ to $\mathbb{Q}$: not enough

First of all, to begin with, the set of positive integers, denoted by $\mathbb{N}$, may simply be assumed to be well-defined, with only some so-called Peano axioms reformulated as below for such a set to obey with,

1. There exists some element in the set, which we shall call it 1, for us to start with, and
2. For every element $n \in \mathbb{N}$ there is associated some unique $m \in \mathbb{N}$, which we shall call it the 'successor of' n and denote it by $f(n)$, in such a manner that the function f we just defined is injective and does not map any $n \in \mathbb{N}$ to 1, and
3. Any set S with $1 \in S$ and $f(S \cap \mathbb{N}) \subseteq S$ must contain $\mathbb{N}$. Particularly, if S is a subset of $\mathbb{N}$, S is $\mathbb{N}$ itself.

We shall see immediately [7] that $S = \{1\} \cup f(\mathbb{N}) = \mathbb{N}$, implying the surjectivity of f in some broad sense. That is to say, we have 1 as a 'starting element' and some bijection $f : \mathbb{N} \to (\mathbb{N} \setminus \{1\})$ giving sense to the 'plus 1' action. Therefore, we may define, as inspired by [6], that $\big(m + 1 = f(m)$ and $m + f(n) = f(m + n)$ for any positive integer $n\big)$ for any positive integer m. Now we may try to define 'integers' from positive integers, so that '$1 - 2$' and '$3 - 5$' make sense. We shall use $(1, 2)$ to denote '$1 - 2$' temporarily and notice that $(1, 2)$, $(2, 3)$, $(3, 4)$ and so on and on basically mean the same '-1' in our mind. Then, $\mathbb{Z}^*$ shall be defined [7] to be the set of equivalence classes in $\mathbb{N} \times \mathbb{N}$ under the equivalence relation $(a, b) \sim (c, d) \iff a + d = c + b$, which means '$a - c = b - d$' in our mind. The set of positive integers is not yet a subset of this set of 'integers', though their names would suggest otherwise. Nevertheless, for any positive integer n we may consider the equivalence class formed by $(n + 1, 1)$ and keep in mind that we would wish to identify this '$(n + 1) - 1$' with this positive integer n. To be precise, let $\mathbb{N}^*$ be the set of equivalence classes formed by $(n + 1, 1)$ for some positive integers n, and we may now consider $\mathbb{Z}^*$ we previously defined and replace $\mathbb{N}^*$ in it by the set of positive integers $\mathbb{N}$ we already know well. That is to say, we may regard $\mathbb{Z} = (\mathbb{Z}^* \setminus \mathbb{N}^*) \cup \mathbb{N}$ to be the set of integers in a way that $\mathbb{N}$ will become embedded in $\mathbb{Z}$. With some more subtle but technical justifications like the compatibility of the addition we defined in $\mathbb{N}$ and the addition we are to define in $\mathbb{Z}$, we may happily realize that we have basically answered the question 'what are integers?' completely.

Next, through an utterly similar definition [7] we are able to rigorously define the set of rational numbers $\mathbb{Q}$. We shall note that the addition, multiplication and order defined along with $\mathbb{Q}$ have some familiar properties (a, b, c are taken to be elements in $\mathbb{Q}$),

- $a + b = b + a$ and $a + (b + c) = (a + b) + c$ (sequence of addition does not matter),
- $a \times b = b \times a$ and $a \times (b \times c) = (a \times b) \times c$ (sequence of multiplication does not matter),
- $a \times (b + c) = a \times b + a \times c$ and (addition and multiplication are 'compatible' in our usual sense),
- $(a > 0 \land b > c \Rightarrow ab > ac)$ (dilation does not affect relative magnitude),
- $(b > c \Rightarrow a + b > a + c)$ (translation does not affect relative magnitude),
- $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$.

But why are we not already satisfied with rational numbers? We, modern people, are fortunately not implemented with the Pythagoreans misbelief [1] that given any two 'natural segments' we were always able to find a line segment of 'unit length' so that it measures both accurately. Take the hypotenuse and either side of an isosceles right-angled triangle ◿ as a well-known counter-example. The question now comes whether $\sqrt{2}$ 'is rational'. To be precise, there shall be no rational number whose square is 2, or equivalently as discussed above, that there is no n/m which *separates* [2] the set of all n/m such that $mx < ny$ and the set of all n/m such that $mx > ny$. Still and all, we wish to determine $x : y$ (which may not be defined yet) by such a 'partition' of $\mathbb{Q}$, giving rise to our construction of real numbers inspired by R. Dedekind. Otherwise, if we are not able to talk about the *least upper bound* of some clearly non-empty and bounded-above set, it is difficult, if not impossible, to talk about what some well-behaving sequence like

$$\frac{1}{1^2}, \frac{1}{1^2}+\frac{1}{2^2}, \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}, \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}, \cdots$$

'approaches to'. Of course, we have to first define such 'partitions' to resolve all ambiguities.

1.4 Real numbers

A 'real number' is identified as a pair of non-empty subsets (A, B) of $\mathbb{Q}$ such that,

1. $A \cap B = \{\}$, $A \cup B = \mathbb{Q}$ ($\mathbb{Q}$ is the disjoint union of A and B), and
2. Any element in A is less than any element in B (A is 'on the left of' B), and
3. B has no smallest element, otherwise we simply consider (A, B) with the smallest element of B migrated to A.

Now, if A has a greatest element r, (A, B) is identified with the rational number r and is customarily written as r^*. In this case, A is the set of all rational numbers not exceeding r and B the rest. If A has no greatest element, we wish to identify (A, B) with an 'irrational number' like $\sqrt{2}, e, \pi$, whatsoever, with the idea that such an irrational number is born to fill in the 'gap' between A and B. Note, however, we shall keep in mind that the irrational numbers are not yet placed on an equal footing with the rational numbers. We shall use $\mathbb{R}^*$ and $\mathbb{Q}^*$ to denote the set of 'real numbers' and among them those identified with some rational numbers r respectively. In fact, we could define addition, multiplication and order of 'real numbers' in such a way that $r^* + s^* = (r+s)^*$, $r^* \times s^* = (r \times s)^*$, so that the addition and multiplication of rational numbers are preserved and $r < s \iff r^* < s^*$ (or equivalently [2], that the mapping $r \mapsto r^*$ is strictly increasing). Now, similar to the case of $\mathbb{N}^* \subseteq \mathbb{Z}^*$ and $\mathbb{N}$, we can identify $\mathbb{Q}^* \subseteq \mathbb{R}^*$ with $\mathbb{Q}$ and define $\mathbb{R} = (\mathbb{R}^* \setminus \mathbb{Q}^*) \cup \mathbb{Q}$. Besides $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, it is worth pointing out that it is our way of defining addition, multiplication and order that makes $\mathbb{R}$ itself an *ordered field*. That is, as its name suggests, a set with these operations obeying certain laws in a way that we shall feel extremely comfortable when manipulating these numbers, and that we are allowed to place the real numbers on a number line. There are several amazing properties regarding such a number line. For an example [5], there is always some (actually infinitely many, otherwise there is an easy contradiction to derive) $r \in \mathbb{Q}$ between any given $x, y \in \mathbb{R}$ with $x < y$, follows immediately from the *Archimedean property*. Moreover, $\mathbb{R}$ satisfies the *completeness axiom* (some equivalent forms of the axiom could be found [3]), which basically means that given any such partition (A, B), we are able to find some fixed real number c that lies between any arbitrarily selected $a \in A$ and $b \in B$ (id est, $a \leq c \leq b$). The partition may be of $\mathbb{Q}$ or of $\mathbb{R}$: now it does not matter (as we may consider the stronger case of $\mathbb{R}$, and therefore, trivially, that any partition of $\mathbb{Q}$ is also 'separated' by some real number).

2 Limiting processes and some 'paradoxes' derived

There were, as we shall acknowledge, quite a few 'paradoxes' derived by the greatest minds of mankind in our history from some limiting processes. Needless to say, they rarely pose a real challenge to modern Mathematics. Nonetheless, these plausible statements are intriguing and each certainly deserves clarification on its own. Here I shall provide my own solutions to two selected 'paradoxes' taken from [**6**].

2.0.1 'Paradox' 1: Zeno's paradox (paraphrased)

'*If the fastest sprinter were left behind a tortoise at some instant, he would never be able to reach it, since we may always consider a subsequent instant where he would have just moved to the original position of the tortoise but the tortoise would have also moved forward by some positive distance, so on and on.*'

2.0.2 Explanation

I have intentionally utilized the phrase 'subsequent instant' to hint the solution to this 'paradox'. In my opinion, capturing an infinite number of instants does not affect the flow of time. Assume that it takes a unit length of time for the sprinter to reach the original position of the tortoise and that the speed of the tortoise is half of that of the sprinter (though makes little sense, it needs only be some positive number smaller than 1). You may, certainly, as you like, consider the instants at

$$T = 0 \text{ s}, 1 \text{ s}, \left(1 + \frac{1}{2}\right) \text{ s}, \left(1 + \frac{1}{2} + \frac{1}{4}\right) \text{ s}, \ldots \tag{1}$$

Indeed, the real numbers need not be introduced for such an infinite sequence of partial sums of geometric series to have some meaningful limit ('do not increase beyond all bounds' [**6**]): 2 s. Or we may also observe that each of the above time instants is strictly before 2 s, but time flies and earth spins just as unaffected and in the twinkling of an eye we have passed all and infinitely many instants in (1).

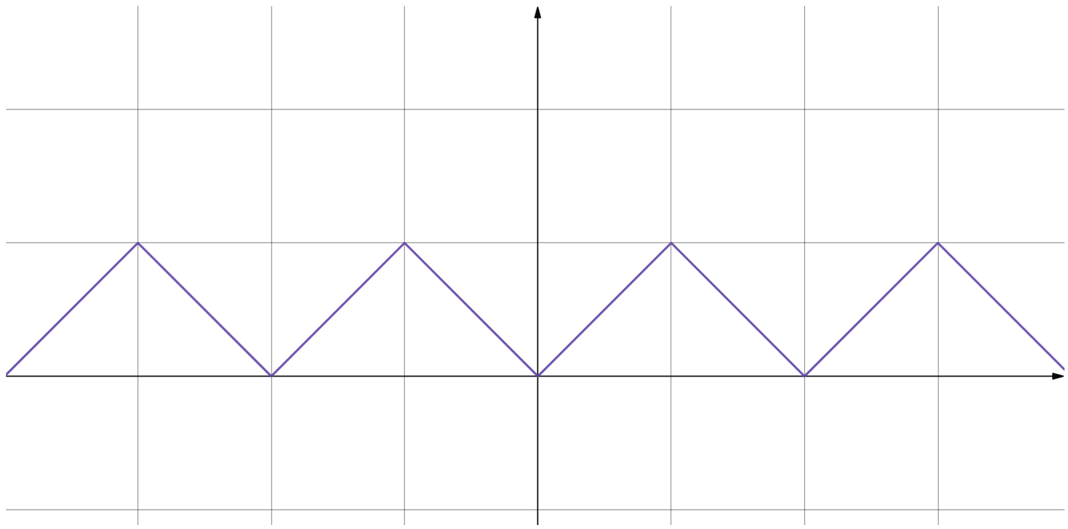
2.0.3 'Paradox' 2

'*If a square is built up of miniature tiles, then there are as many tiles along the diagonal as there are along the side; thus the diagonal should be equal in length to the side.*' [**6**]

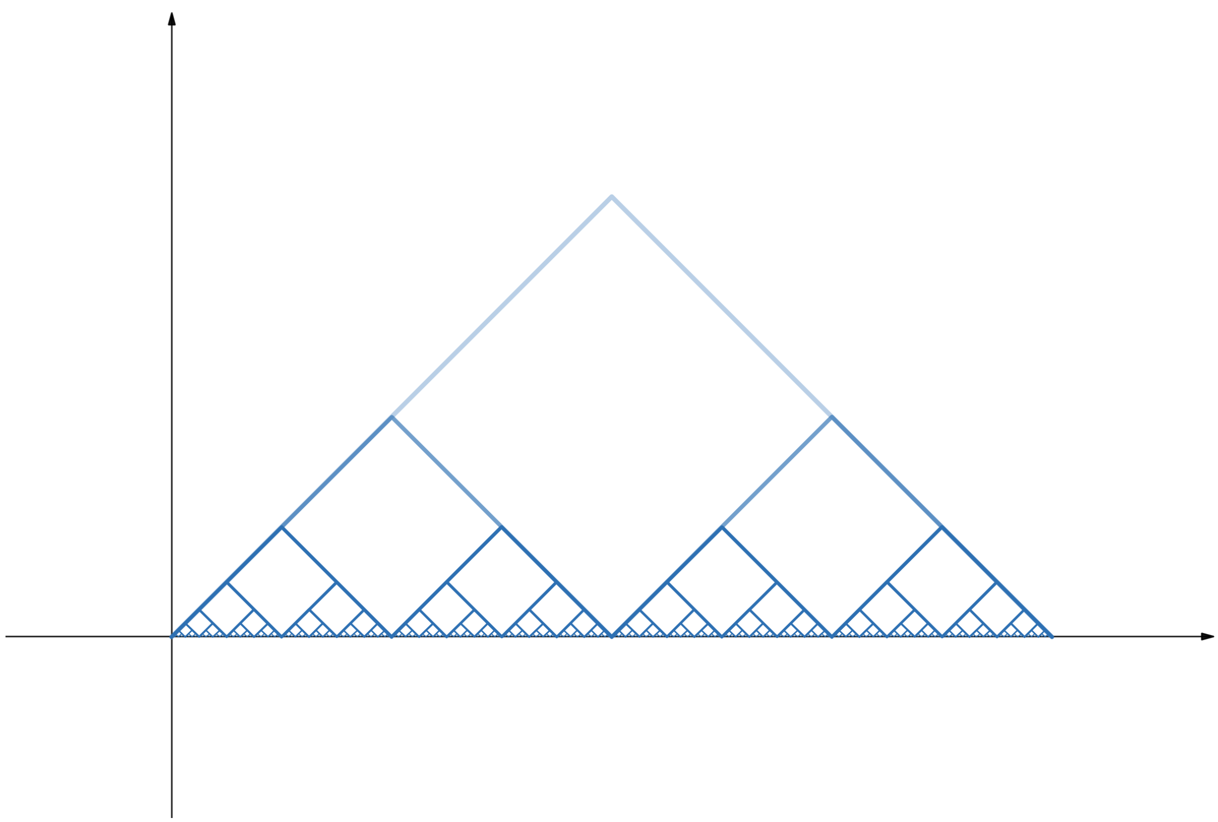
2.0.4 Explanation

Without loss of generality, we may place the diagonal on the interval $[0, 1]$ on the x-axis, let $\lfloor x \rfloor$ be the greatest integer n with $n \leq x$ and consider the following functions (note that f is periodic so it suffices to consider the interval $[0, 2)$),

$$\begin{aligned} f(x) &= \text{`distance from } x \text{ to the closest even integer'} \\ &= \begin{cases} x - \lfloor x \rfloor, & \text{if } \lfloor x \rfloor \text{ is even,} \\ (1 + \lfloor x \rfloor) - x, & \text{if otherwise,} \end{cases} \\ f_n(x) &= \frac{f(2^n x)}{2^n} \text{ for } x \in [0, 1], \text{ where } n \in \mathbb{N}. \end{aligned}$$



$y = f(x)$



$y = f_n(x), x \in [0, 1], n \in \mathbb{N}$ (the larger n, the deeper the color)

We shall observe that the argument is perfectly sound as long as we are still considering the scenario for some positive integers n: the total 'length' of each f_n, presumably the outline of the miniature square tiles arranged along the diagonal of the big square, is constantly $\sqrt{2}$. This does not mean, though, that such an approximate 'diagonal' can in any sense be regarded as the real diagonal of the square. No matter how small the tiles are, in atomic scale or even smaller than a Planck length, as long as they are still recognized as squares in Mathematics, there is something on essence that differentiates them from the real diagonal, the straight line connecting the centers of these miniature tiles. We shall see that the 'length' of a curve with the parametric equation $(t, f_n(t))$ is mainly determined by the derivative of the function [4], which is obvious if we simply recall the definition of arc-length of such a parameterized curve (separate each f_n into 2^n segments to avoid singularities in our case),

$$\int_0^1 \sqrt{1+(f_n'(x))^2}\mathrm{d}x = \sum_{j=1}^{2^n} \int_{\frac{j-1}{2^n}}^{\frac{j}{2^n}} \sqrt{1+1}\mathrm{d}x = \sum_{j=1}^{2^n} \frac{\sqrt{2}}{2^n} = \sqrt{2}. \tag{2}$$

In (2), the fact that $f_n'(x) = 1$ or -1 for any non-integer x is employed. When we consider the function $f_\infty \equiv 0$ where $(f_n)_{n\in\mathbb{N}}$ converges to, however, we shall see that the derivative of the functions jumps from having some fixed non-zero (absolute) value to being completely vanished. This is to say, it is another world regarding the 'result' of the limiting process. In other words, concerning the 'lengths' of the functions, we may never approach the limit in the sense that for any finite n, the absolute value of the derivative of f_n (despite the singularities) is constantly 1.

References

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