

# IMO HK (Questions and suggested solutions)

Charles Fu and Tony Ma

2020-06-14 Onward

## Contents

<b>1</b>	<b>Preface</b>	<b>3</b>
<b>2</b>	<b>IMO HK Preliminary Selection Contest</b>	<b>3</b>
2.1	IMO HK Prelim 2020	3
2.1.1	Q1	3
2.1.1.1	Question	3
2.1.1.2	Solution	3
2.1.2	Q4	3
2.1.2.1	Original Question	3
2.1.2.2	Question (Rephrased)	3
2.1.2.3	Answer	3
2.1.2.4	Solution	4
2.1.2.4.1	Lemma 1	4
2.1.2.4.2	Proof	4
2.1.2.4.3	Lemma 2 and proof	4
2.1.2.4.4	Lemma 3	4
2.1.2.4.5	Proof when $(x, y) \neq (3, 2)$	4
2.1.2.4.6	Proof when $(x, y) = (3, 2)$	4
2.1.2.4.7	Using <i>Bubble sort</i> to complete the proof	4
2.1.3	Q5	5
2.1.3.1	Question	5
2.1.3.2	Solution	5
2.1.3.2.1	A concrete example and a solution by trial and error	6
2.1.4	Q7	6
2.1.4.1	Question	6
2.1.4.2	Remark	7
2.1.4.3	Solution 1	7
2.1.4.3.1	More about this solution	7
2.1.4.4	Solution 2	7
2.1.5	Q17	8
2.1.5.1	Question	8
2.1.5.2	Solution	8

2.1.5.3	Remark . . . . .	9
2.1.6	Q20 . . . . .	9
2.1.6.1	Question . . . . .	9
2.1.6.2	Remark . . . . .	9
2.1.6.3	Solution . . . . .	10
2.1.6.4	Interesting Generalization . . . . .	11
2.1.6.5	Alternative solution simplified at finding ( $F_{2020} \bmod 125$ ) . . . . .	12
2.1.6.6	More about the general term formula of <i>Fibonacci sequence</i> . . . . .	13
2.1.6.6.1	A proof of the general term of <i>Fibonacci sequence</i> . . . . .	14
<b>3</b>	<b>IMO HK Team Selection Test 1</b>	<b>14</b>
3.1	IMO HK 2021 TST 1 . . . . .	14
3.1.1	Q1 . . . . .	14
3.1.1.1	Question . . . . .	14
3.1.1.2	Solution . . . . .	14
3.1.2	Q3 . . . . .	15
3.1.2.1	Original Question . . . . .	15
3.1.2.2	Question (Rephrased) . . . . .	15
3.1.2.3	Attempts, examples and comments . . . . .	15
3.1.2.4	Solution (modified from official) . . . . .	15
3.1.2.4.1	‘2 operations’ is sufficient - Example . . . . .	16
3.1.2.4.2	‘2 operations’ is necessary - Proof . . . . .	16
3.1.2.4.3	Remark of the solution . . . . .	17
3.1.3	Q4 . . . . .	17
3.1.3.1	Question . . . . .	17
3.1.3.2	Solution . . . . .	17
<b>4</b>	<b>IMO HK Team Selection Test 2</b>	<b>17</b>
4.1	IMO HK 2021 TST 2 . . . . .	17
4.1.1	Q2 . . . . .	17
4.1.1.1	Question . . . . .	17
4.1.1.2	Solution . . . . .	17
4.1.1.2.1	Remark - irreducibility . . . . .	18
4.1.2	Q3 . . . . .	18
4.1.2.1	Question . . . . .	18
4.1.2.2	Graph . . . . .	18
4.1.2.3	Hint . . . . .	18
4.1.2.4	Solution . . . . .	20
4.1.2.5	Remark . . . . .	20
<b>5</b>	<b>Hong Kong (China) Mathematical Olympiad</b>	<b>20</b>
5.1	CHKMO 2020 . . . . .	20
5.1.1	Q4 . . . . .	20
5.1.1.1	Question . . . . .	20
5.1.1.2	Solution . . . . .	20

# 1 Preface

This paper is being updated [on my web page](#), and is keep being updated with more questions.

## 2 IMO HK Preliminary Selection Contest

### 2.1 IMO HK Prelim 2020

*Date of test:* **2020-06-13**.

#### 2.1.1 Q1

##### 2.1.1.1 Question

Let  $n = (10^{2020} + 2020)^2$ . Find the sum of all the digits of  $n$  in base 10.

##### 2.1.1.2 Solution

$$n = 1 \times 10^{4040} + 4 \times 10^{2023} + 4 \times 10^{2021} + 4 \times 10^6 + 8 \times 10^4 + 4 \times 10^2.$$

Answer is **25**.

#### 2.1.2 Q4

##### 2.1.2.1 Original Question

*In a game, a participant chooses a nine-digit positive integer  $\overline{ABCDEFGHI}$  with distinct non-zero digits. The score of the participant is  $A^{B^C D^E F^G H^I}$ . Which nine-digit number should be chosen in order to maximise the score?*

##### 2.1.2.2 Question (Rephrased)

We want to find the maximum value (in index form) of

$$A^{B^C D^E F^G H^I}$$

which  $(A, B, C, D, E, F, G, H, I)$  is a permutation of  $(1, 2, 3, 4, 5, 6, 7, 8, 9)$ .

##### 2.1.2.3 Answer

$$2^{3^4 5^6 6^7 8^9 1}$$

([Uniqueness can be proven](#), but not shown below.)

#### 2.1.2.4 Solution

First of all, assume if  $I \neq 1$ , then either  $A, B, C, D, E, F, G$  or  $H$  would be equal to 1. But this number must have a natural number index strictly greater than 1, such as  $1^{2^3} > 1$ . Removing the 1 from the power tower must increase the score the participant get. Hence 1 must only be at  $I$ .

##### 2.1.2.4.1 Lemma 1

$f(x) = \frac{x}{\ln x}$  is strictly increasing on  $(e, \infty)$ .

##### 2.1.2.4.2 Proof

$$f'(x) = \frac{1}{\ln x} - \frac{1}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} > 0 \quad \forall x \in (e, \infty).$$

##### 2.1.2.4.3 Lemma 2 and proof

$\frac{2}{\ln 2} = \frac{4}{\ln 4} \leq \frac{n}{\ln n}$  if  $n \in \mathbb{N} \setminus \{1, 2, 3\}$  by [Lemma 1](#).

##### 2.1.2.4.4 Lemma 3

Given  $x, y, z \in \mathbb{N}$  and  $1 < y < x$ , except if  $(x, y, z) = (3, 2, 1)$ ,

$$x^{y^z} \leq y^{x^z}.$$

##### 2.1.2.4.5 Proof when $(x, y) \neq (3, 2)$

$$\begin{aligned} \left(\frac{y}{x}\right)^z &\leq \frac{y}{x} \leq \frac{\ln y}{\ln x} \\ y^z \ln x &\leq x^z \ln y \\ x^{y^z} &\leq y^{x^z} \end{aligned}$$

##### 2.1.2.4.6 Proof when $(x, y) = (3, 2)$

$$y^z \ln x \leq x^z \ln y \iff z \ln y + \ln \ln x \leq z \ln x + \ln \ln y \iff z \geq \frac{\ln \ln 3 - \ln \ln 2}{\ln 3 - \ln 2}$$

It suffices to prove that this holds when  $z = 2$ .  $2^2 \ln 3 = \ln 3^4 \leq \ln 2^9 = 3^2 \ln 2$  clearly holds. Hence also for  $z > 2$ .

##### 2.1.2.4.7 Using *Bubble sort* to complete the proof

We now conjecture that  $2^{3^{4^{5^{6^{7^{8^9}}}}}}$  is one of the representations of the maximum value (ignore  $I = 1$ ; **uniqueness** need to be proven by a slightly weaker version of [Lemma 3](#) but with  $<$  instead of  $\leq$ ).

Let  $(A, B, C, D, E, F, G, H)$  be a permutation of  $(2, 3, 4, 5, 6, 7, 8, 9)$ . **Bubble sort**  $(A, B, C, D, E, F, G, H)$  in ascending order using the typical procedure, only **swapping adjacent** elements that are found not in relative order desired (i.e., e.g. when  $A > B$ ). We proclaim that in each swapping step we are getting a larger or at least unchanged *score*.

To justify, each time the swapping procedure involve  $x, y, z \in \mathbb{N}$  with  $1 < y < x$  just as described in [Lemma 3](#).

The exception  $(x, y, z) = (3, 2, 1)$  may only happen if we swap  $G = 3, H = 2$  (must be at the top two numbers so that  $z = 1$ ). Consider the worst case where  $H = 2$  at some instance during the procedure, then  $H$  must already be 2 initially. Consider the bubble sort algorithm, bringing the largest element up every time, hence  $H = 2$  will be swapped with  $G = 9$ , not 3.

### 2.1.3 Q5

#### 2.1.3.1 Question

28 people seat in circle. They all claim that ‘the two people next to me are of different genders’. Given that all boys lie and exactly 3 girls lie, find the **number of girls**.

#### 2.1.3.2 Solution

Denote girl by  $G$  and boy by  $B$ .

Notice that all boys lie, so it would be reasonable and easier to take girls as variable:

Namely, define  $1 \leq B_1 < B_2 < B_3 < \dots < B_k$ , where  $k \in \mathbb{N}$  is the number of boys (disregard of ‘28’ at this moment).

We then define  $S_i$  for  $i = 1, 2, \dots, k$  to be the number of girls between separation of  $B_i$  and  $B_{i+1}$ , with the convention  $B_{k+1} = B_1$ .

Assume that for some  $i$ ,  $S_i = 0$ , i.e., there are two successive boys with no girls between, namely,  $BB$ .

Then, consider the right  $B$ , we can extend  $BB$  to  $BBB$  because in the view of  $B$ , two people next to him are **not** of different genders.

Similarly and inductively, we would have only boys in the circle, giving a *contradiction*.

We might consider the following patterns (of course there is no boy between two **successive** boys laterally):

1.  $BGB$ : In this case that girl lies.
2.  $BGGB$ : In this case both girls do not lie.
3.  $BG \dots GB$ , where ‘ $\dots$ ’ means more than one girl: In this case (only those) girls in ‘ $\dots$ ’ lie.

It is worth noticing that since every block  $B \dots B$  has at least one girl between, the number of girls would not have effect on the correctness of the acclaimed ‘*different genders neighbours*’ proposition by any girl **in another block**.

Exactly three girls lie. Therefore, with the constraint  $S_i \geq 1$ ,

$$\sum_{i=1}^k |S_i - 2| = 3$$

We may assume, for the sake of simplicity, all  $S_i$  are equal to 2 at the beginning,

i.e., any boy forms the pattern  $BGGB$  with next boy in the cycle.

Each time to ‘create’ a girl that lies, we add (+) **or** remove (−) **one** girl.

Denote the three operations by  $a_1, a_2, a_3 = \pm 1, \pm 1, \pm 1$ , and we add the restriction that no two operations **of different signs** should be performed on the same  $S_i$ , otherwise they counteract their effects, adding difficulty to the proof below.

We now have:

$$\begin{aligned} S_1 + S_2 + S_3 + \dots + S_k + k &= 28 \\ 2 + 2 + 2 + \dots + 2 + k + a_1 + a_2 + a_3 &= 28 \\ a_1 + a_2 + a_3 &= 28 - 2k - k = 28 - 3k \equiv 1 \pmod{3} \end{aligned}$$

Let  $t$  denotes the number of operations  $+$ , then  $a_1 + a_2 + a_3 = t - (3 - t) = 2t - 3 = 1$ .

$$\therefore t = 2 \text{ and } \{a_1, a_2, a_3\} = \{1, 1, -1\}$$

Solving  $3k = 28 - (a_1 + a_2 + a_3)$ , we get the number of boys  $k = 9$ , hence the number of girls is 19.

#### 2.1.3.2.1 A concrete example and a solution by trial and error

$$\sum_{i=1}^k |S_i - 2| = 3$$

From here, we can exhaust all possible sequences  $(S_i)_{i=1}^k$   
(Disregarding of the order of  $S_1, S_2, \dots$  without loss of generality):

$$\begin{aligned} \{S_i\} &= \begin{cases} 5, 2, 2, 2, 2, \dots, 2 \\ 4, 3, 2, 2, 2, \dots, 2 \\ 4, 1, 2, 2, 2, \dots, 2 \\ 3, 3, 3, 2, 2, \dots, 2 \\ 3, 3, 1, 2, 2, \dots, 2 \\ 3, 1, 1, 2, 2, \dots, 2 \\ 1, 1, 1, 2, 2, \dots, 2 \end{cases} \\ \sum S_{i_{\text{(include left boy of the block)}}} &= \begin{cases} 6 + 3 + 3 + 3 + \dots + 3 & \rightarrow \text{never} \equiv 28 \pmod{3} \\ 5 + 4 + 3 + 3 + \dots + 3 & \rightarrow \text{never} \equiv 28 \pmod{3} \\ 5 + 2 + 3 + 3 + \dots + 3 & \rightarrow \text{possible} \\ 4 + 4 + 4 + 3 + \dots + 3 & \rightarrow \text{never} \equiv 28 \pmod{3} \\ 4 + 4 + 2 + 3 + \dots + 3 & \rightarrow \text{possible} \\ 4 + 2 + 2 + 3 + \dots + 3 & \rightarrow \text{never} \equiv 28 \pmod{3} \\ 2 + 2 + 2 + 3 + \dots + 3 & \rightarrow \text{never} \equiv 28 \pmod{3} \end{cases} \end{aligned}$$

For the two possible cases we have (non-unique) examples respectively (*bold* for *liers*):

- 1.

$G G G G B G B G G B G G B G G B G G B G G B G G B G G B$ ,

which corresponds to the solution which two ‘+’ operations are performed on the same block (same  $S_i$ );

- 2.

*G G B G G B G B G G B G G B G G B G G B G G B*.

which corresponds to the solution which two ‘+’ operations are performed on different blocks.

### 2.1.4 Q7

### 2.1.4.1 Question

Solve equation  $\sqrt{7-x} = 7-x^2$  for  $x \in \mathbb{R}^+$ .

#### 2.1.4.2 Remark

There are two methods to solve this problem. Observe the similarities.

#### 2.1.4.3 Solution 1

Squaring both sides, giving:

$$\begin{aligned}7 - x &= (7 - x^2)^2 = x^4 - 14x^2 + 49 \\x^4 - 14x^2 + x + 42 &= 0\end{aligned}$$

Then, it is easy to see the linear factor  $(x + 2)$  as  $(-2)^4 - 14(-2)^2 + (-2) + 42 = 16 - 56 - 2 + 42 = 0$ .

$$\begin{aligned}\therefore (x + 2)(x^3 - 2x^2 - 10x + 21) &= 0 \\(x + 2)(x - 3)(x^2 + x - 7) &= 0\end{aligned}$$

$$x = -2 \text{ or } 3 \text{ or } \frac{-1 \pm \sqrt{29}}{2}$$

Substituting them into the original equation, we notice that the only solution is  $\frac{\sqrt{29}-1}{2} (\in \mathbb{R}^+)$ .

##### 2.1.4.3.1 More about this solution

$$x = -2 \text{ or } 3 \text{ or } \frac{-1 \pm \sqrt{29}}{2}$$

We reject  $x = -2$  only because it is negative, but  $\sqrt{7 - (-2)} = 7 - (-2)^2$ , so it satisfies the original equation. On the other hand, we reject  $x = 3$  and  $x = \frac{-1 - \sqrt{29}}{2}$  as they don't even satisfy the original equation. Why? Because they are artificially created during the process of **squaring both sides**.

You may check:

$$\begin{aligned}\sqrt{7 - (3)} &= 2 \neq -2 = 7 - (3)^2 \\ \sqrt{7 - \frac{-1 - \sqrt{29}}{2}} &= \sqrt{\frac{30 + 2\sqrt{29}}{4}} = \frac{\sqrt{29} + 1}{2} = \frac{30 + 2\sqrt{29}}{4} - 7 \\ &\neq 7 - \frac{30 + 2\sqrt{29}}{4} = 7 - \left(\frac{-1 - \sqrt{29}}{2}\right)^2\end{aligned}$$

Notice that all 4 roots we obtained satisfy  $\sqrt{7 - x} = |7 - x^2|$ , if we allow  $x \leq 0$ .

#### 2.1.4.4 Solution 2

Notice that the equation can be rewritten as (for  $x \in \mathbb{R}^+$ ):

$$\begin{aligned}x^2 &= 7 - \sqrt{7 - x} \\ x &= \sqrt{7 - \sqrt{7 - x}}\end{aligned}$$

We define  $f(x) = \sqrt{7-x}$  for  $x \in [0, 7]$ .

Then, we are actually solving  $f(f(x)) = x$ .

Observe that if we have  $f(x) = x$  (which is easy to solve),  $f(f(x)) = f(x) = x$  immediately follows,

$$\begin{aligned} x &= \sqrt{7-x} \\ x^2 &= 7-x \\ x^2 + x - 7 &= 0 \\ x &= \frac{-1 \pm \sqrt{29}}{2} \end{aligned}$$

Rejecting the root *artificially created when squaring both sides*, we get the only desired solution  $\frac{\sqrt{29}-1}{2}$ . However, notice that we did not prove that this is the only solution of  $f(f(x)) = x$ .

### 2.1.5 Q17

#### 2.1.5.1 Question

Find (the number of) *positive integer solutions* to the following system of equation:

$$\begin{cases} \sqrt{2020}(\sqrt{a} + \sqrt{b}) = \sqrt{(c+2020)(d+2020)} \\ \sqrt{2020}(\sqrt{b} + \sqrt{c}) = \sqrt{(d+2020)(a+2020)} \\ \sqrt{2020}(\sqrt{c} + \sqrt{d}) = \sqrt{(a+2020)(b+2020)} \\ \sqrt{2020}(\sqrt{d} + \sqrt{a}) = \sqrt{(b+2020)(c+2020)} \end{cases}$$

#### 2.1.5.2 Solution

WLOG, let's first consider the first equation, dealing always only with  $a, b, c, d \in \mathbb{N}$ ,

$$\sqrt{2020}(\sqrt{a} + \sqrt{b}) = \sqrt{(c+2020)(d+2020)}$$

Notice that

$$\sqrt{a} + \sqrt{b} = \sqrt{a+b+2\sqrt{ab}},$$

or by squaring both sides in the hope of doing some number theoretical manipulations.

$$2020(a+b+2\sqrt{ab}) = (c+2020)(d+2020) = cd + 2020(c+d) + 2020^2$$

Notice that

$$a+b+2\sqrt{ab} = \frac{(c+2020)(d+2020)}{2020} \in \mathbb{Q},$$

so we must have  $ab = x_1^2$  for some  $x_1 \in \mathbb{N}$ .

Similarly,

$$\begin{aligned} (ab, bc, cd, da) &= (x_1^2, x_2^2, x_3^2, x_4^2) \text{ for some } (x_1, x_2, x_3, x_4) \in \mathbb{N}^4 \\ 2020(a+b-c-d+2x_1-2020) &= 2020(a+b+2x_1) - 2020(c+d+2020) = cd \\ \therefore \frac{\left(\frac{x_3}{2}\right)^2}{505} &= \frac{x_3^2}{505} = \frac{cd}{2020} \in \mathbb{N}, 505 \mid x_3 \\ x_3 &= 505y_3, y_3 \in \mathbb{N} \end{aligned}$$



Carefully considering 4 cases, we have:

$$\begin{aligned}
(x_1, x_2, x_3, x_4) &= (505y_1, 505y_2, 505y_3, 505y_4) \text{ for } (y_1, y_2, y_3, y_4) \in \mathbb{N}^4 \\
\begin{cases} 2020(a+b-c-d+1010y_1-2020) = x_3^2 = 505^2 y_3^2 \\ 2020(c+d-a-b+1010y_3-2020) = x_1^2 = 505^2 y_1^2 \end{cases} \\
\therefore 505^2(8(y_1+y_3)-32) &= 2020(1010(y_1+y_3)-4040) = 505^2(y_1^2+y_3^2) \\
(y_1-4)^2 + (y_3-4)^2 &= 0 \implies y_1 = y_3 = 4 \\
\therefore (\text{Similarly}) \ x_1 = x_2 = x_3 = x_4 &= 2020 \wedge ab = bc = cd = da = 2020^2
\end{aligned}$$

Notice that we could not have and only have  $\begin{cases} a = c \\ b = d \end{cases}$  (but **not**  $a = b = c = d$ ).

The four original equations have now been reduced to any one equation of the four, and the question is reduced to finding  $(a, b) \in \mathbb{N}^2$  such that  $ab = 2020^2 = 2^4 \times 5^2 \times 101^2$ , having  $(4+1)(2+1)(2+1) = 45$  solutions.

Consider  $\sqrt{2020}(\sqrt{a} + \sqrt{b}) = \sqrt{(c+2020)(d+2020)}$  such that  $ab = 2020^2$ , with  $(c, d) = (a, b)$ :

$$\sqrt{(c+2020)(d+2020)} = \sqrt{cd + 2020(c+d) + cd} = \sqrt{2020}\sqrt{a+b+2\sqrt{ab}} = \sqrt{2020}(\sqrt{a} + \sqrt{b}),$$

and very similarly, we just **justified** that the solution satisfy all 4 original equations.

### 2.1.5.3 Remark

Simpler methods are to be discovered, but personally I did not manage to think of a way utilizing the *AM-GM inequality*, for example.

### 2.1.6 Q20

#### 2.1.6.1 Question

Denotes  $(F_n)_{n \in \mathbb{N}}$  as the *Fibonacci sequence*. Find  $(F_{2020} \bmod 1000)$ .

#### 2.1.6.2 Remark

Observe that  $1000 = 10^3 = 2^3 \times 5^3 = 8 \times 125$ , and it suffices to consider  $(F_{2020} \bmod 8)$  and  $(F_{2020} \bmod 125)$ , which will be shown do-able.

(You may like to refer to the [Wikipedia page of Chinese Remainder Theorem](#).)

Notice that it would be extremely time consuming to write out sequence  $(F_n \bmod 125)$  or even  $(F_n \bmod 1000)$  until it repeats itself.

For the sequence  $(F_n \bmod 5^k)$ , I observe that it would repeat itself each  $4 \times 5^k$  terms, i.e.,  $F_{n+4 \times 5^k} = F_n$ .

The sequence  $(F_n \bmod 125)$  and  $(F_n \bmod 1000)$  are known to have (smallest) period 500 and 1500 respectively.

(1500 is the least common multiple of 500 and 12 = the period of sequence  $(F_n \bmod 8)$ )

For example,  $F_{2020} \bmod 1000 = F_{520} \bmod 1000 = 515$ .

Notice that (by a small *Python* program),

```
F2020 = 6390698115594355866513492739852871393819620004374587135576307978726515
1995322001898071485437809065533306267303596007174873825024273095647974
6498789136160967666042480660086493956623437642602865340590032906541161
2695736240674586650793536636971284213893001403888857222739849074228772
2358671367235709189166548965693916054419169307250217586108972590100204
080980670597873094617540854443783530320723473081393553497241453257464515
```

which a handy calculator probably could not do it for you.

However, for small  $n$  ( $n \leq 250$ ), you may visit [Online Fibonacci Calculator](#) to check.

### 2.1.6.3 Solution

For  $(F_{2020} \bmod 8)$ , you can simply list it out:

$$(F_{2020} \bmod 8) = 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, 1, 1, 2, 3, 5, 0, 5, 5, 2, \dots$$

The cyclic sequence has a (smallest) period of 12, and so in general  $F_{n+12} \bmod 8 = F_n \bmod 8$ , thus:

$$F_{2020} \bmod 8 = F_{2016+4} = F_4 \bmod 8 = 3$$

For  $(F_{2020} \bmod 125)$ , however, as stated above, it is hard to wait for the sequence to repeat itself.

Instead, it is well-known that:

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

(or you shall prove by Mathematical induction and remember the formula)

Therefore, by binomial expansion,

$$\begin{aligned} F_n &= \frac{1}{2^n \sqrt{5}} \left( (1 + \sqrt{5})^n - (1 - \sqrt{5})^n \right) = \frac{1}{2^n \sqrt{5}} \left( \sum_{i=0}^n C_i^n (\sqrt{5})^i - \sum_{i=0}^n C_i^n (-\sqrt{5})^i \right) \\ &= \frac{1}{2^n \sqrt{5}} \left( \sum_{i=0}^n C_i^n \left( (\sqrt{5})^i - (-\sqrt{5})^i \right) \right) = \frac{1}{2^n \sqrt{5}} \sum_{i=0}^n C_i^n (\sqrt{5})^i (1 - (-1)^i) \\ &= \frac{1}{2^n \sqrt{5}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} C_{2i+1}^n (\sqrt{5})^{2i+1} (1 - (-1)^{2i+1}) = \frac{\sqrt{5}}{2^{n-1} \sqrt{5}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} C_{2i+1}^n (\sqrt{5})^{2i} \\ &\quad \left( \text{Notice that } \left\lfloor \frac{n-1}{2} \right\rfloor \text{ is the least number } m \text{ such that } 2m+1 \leq n \right) \\ &= \frac{\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} C_{2i+1}^n 5^i}{2^{n-1}} \\ \therefore F_{2020} &= \frac{\sum_{i=0}^{1009} C_{2i+1}^{2020} 5^i}{2^{2019}} \\ \therefore 2^{2019} F_{2020} &= \sum_{i=0}^{1009} C_{2i+1}^{2020} 5^i = C_1^{2020} + 5 \times C_3^{2020} + 25 \times C_5^{2020} + 125 \times (\dots) \\ &\equiv C_1^{2020} + 5 \times C_3^{2020} + 25 \times C_5^{2020} \pmod{125} \end{aligned}$$

Now it suffices to find  $2^{2019} \pmod{125}$  and  $C_1^{2020} + 5 \times C_3^{2020} + 25 \times C_5^{2020} \pmod{125}$  or something similar. Below is a suggested way to do so, and is not too easy to make careless mistakes:

$$\begin{aligned}
2^5 &\equiv 32 \pmod{125} \\
2^{10} &= 1024 \equiv 24 \pmod{125} \\
2^{20} &= 1024^2 \equiv 24^2 = 576 \equiv -49 \pmod{125} \\
2^{25} &= 2^{20} \times 2^5 \equiv (32) \times (-49) = -1568 \equiv 57 \pmod{125} \\
2^{50} &\equiv (57)^2 = 60^2 - 2 \times 3 \times 60 + 3^2 = 3600 - 360 + 9 \equiv -1 \pmod{125} \\
\left( 2^{100} &= 2^{\varphi(125)} \equiv 1 \pmod{125} \text{ simply by Euler's Theorem} \right) \\
2^{200} &\equiv 2^{20} \times ((2^{50})^2)^{20} \equiv -49 \times ((-1)^2)^{20} = -49 \pmod{125}
\end{aligned}$$

$$\begin{aligned}
C_1^{2020} + 5 \times C_3^{2020} + 25 \times C_5^{2020} &= 2020 + \frac{2020 \times 2019 \times 2018}{3!} \times 5 + \frac{2020 \times 2019 \times 2018 \times 2017 \times 2016}{5!} \times 25 \\
&= 2020 + 2020 \times 2019 \times 2018 \div 6 \times 5 \\
&\quad + 2020 \times 2019 \times 2018 \times 2017 \times 2016 \div 3 \div 8 \times 5 \\
&= 2020 + 2020 \times 673 \times 1009 \times 5 + 2020 \times 673 \times 2018 \times 2017 \times 252 \times 5 \\
&\equiv 20 + 20 \times 48 \times 9 \times 5 + 20 \times 48 \times 18 \times 17 \times 2 \times 5 \\
&= 20 + 100 \times 432 + 100 \times 96 \times 18 \times 17 \\
&\equiv 20 + 75 + (-25) \times (-29) \times 306 = 95 + 725 \times 306 \equiv -30 - 25 \times 56 \\
&= -1430 \equiv 70 \pmod{125}
\end{aligned}$$

$$\therefore 22F_{2020} \equiv 147F_{2020} = -3(-49F_{2020}) \equiv -6(2^{2019}F_{2020}) \equiv -6 \times 70 = -420 \equiv -45 \pmod{125}$$

To find  $F_{2020} \pmod{125}$ , try to find the multiplicative inverse of 22:  $22^{-1} \pmod{125}$ .

Here is a method without explicitly finding the multiplicative inverse:

Notice that  $22 \times 6 = 132 = 125 + 7$

, which  $7F_{2020} \equiv 132F_{2020}$  can simplify the expression.

After that, notice that  $18 \times 7 = 126 = 125 + 1$  can again simplify the problem.

$$\begin{aligned}
F_{2020} &\equiv 126F_{2020} = 18(7F_{2020}) \equiv 18(132F_{2020}) = (18)(6)(22F_{2020}) \equiv (108)(-45) \\
&\equiv 17 \times 45 = 765 \equiv 15 \pmod{125}
\end{aligned}$$

After all, use the Chinese Remainder Theorem, or by considering:

$$F_{2020} = 15, 140, 265, 390, 515, 640, 765, 890 \pmod{1000}$$

and notice that only 515 provides  $F_{2020} \equiv 3 \pmod{8}$ .

#### 2.1.6.4 Interesting Generalization

Consider a sequence  $x, y, x + y, x + 2y, 2x + 3y, \dots$

It is defined similarly as the *Fibonacci sequence* except that the initial two terms are different.

One may generalize *Fibonacci sequence* for other starting values. And may also generalize it for all  $n \in \mathbb{Z}$ .

(Just define  $F_n = F_{n+2} - F_{n+1}$  for all non-negative integers  $n$ )

These generalizations are obvious. Instead, here we would like to generalize the property of the *Fibonacci sequence* and discover the relation between the sequence and the number 5 (which is mostly implied by the beautiful relation with the *golden ratio*, see the section [More about the general term formula of Fibonacci sequence](#) below).

**Lemma 2.1.** *If  $(2i + 1)! = 5^g h$  for non-negative integers  $g, h$  and  $5 \nmid h$ , then  $i - g \geq 1$ .*

*Proof.*

$$\begin{aligned} g &= \left\lfloor \frac{2i+1}{5^1} \right\rfloor + \left\lfloor \frac{2i+1}{5^2} \right\rfloor + \left\lfloor \frac{2i+1}{5^3} \right\rfloor + \dots \leq \frac{2i+1}{5^1} + \frac{2i+1}{5^2} + \frac{2i+1}{5^3} + \dots = \lim_{m \rightarrow \infty} \sum_{j=1}^m \left( \frac{2i+1}{5^j} \right) \\ &= \lim_{m \rightarrow \infty} \left( \frac{\text{Next term} - \text{first term}}{\text{Common ratio} - 1} \right) = \frac{(2i+1) - 0}{5 - 1} \\ \frac{2i-1}{4} &\leq i - g \in \mathbb{Z}, \quad \therefore i - g \geq 1 \end{aligned}$$

■

**Theorem 2.2.** *For non-negative integers  $m, n$ ,  $F_{n \times 5^m} \equiv 0 \pmod{5^m}$ .*

*Proof.*

$$\begin{aligned} 2^{n \times 5^m - 1} F_{n \times 5^m} &= \sum_{i=0}^{\left\lfloor \frac{n \times 5^m - 1}{2} \right\rfloor} C_{2i+1}^{n \times 5^m} (5^i) \equiv \sum_{i=0}^{\min \left\{ \left\lfloor \frac{n \times 5^m - 1}{2} \right\rfloor, m-1 \right\}} C_{2i+1}^{n \times 5^m} (5^i) \pmod{5^m} \\ \text{For each } i, C_{2i+1}^{n \times 5^m} (5^i) &= \frac{(5^m)(5^m - 1) \dots (5^m - 2i)}{(2i+1)!} (5^i) \equiv 0 \pmod{5^m} \text{ by this lemma.} \end{aligned}$$

Then, coping the special case  $n = 0$ , by considering multiplicative inverse or just  $2^{n \times 5^m - 1} F_{n \times 5^m} = 5^k t$  ( $t \in \mathbb{N}$ ):

$$F_{n \times 5^m} \equiv 0 \pmod{5^m}$$

■

#### 2.1.6.5 Alternative solution simplified at finding $(F_{2020} \bmod 125)$

In the above, we have proven that  $F_{n+500} \equiv F_n \pmod{125}$  for all  $n \in \mathbb{N}$ .

We may then compute  $F_{2020} \bmod 125 = F_{20} \bmod 125$  by listing out the terms of  $(F_n \bmod 125)_{n \in \mathbb{N}}$ :

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 19, 108, 2, 110, 112, 97, 84, 56, \mathbf{15}$$

Notice that for this solution, as  $F_{n+500} \equiv F_n \pmod{125}$  is used, you need to observe that  $F_{4 \times 125} \equiv 0 \pmod{125}$  and  $F_{4 \times 125 + 1} \equiv 1 \pmod{125}$  in order to think of this solution.

Or, from a competition-directed point of view:

One may suggest or guess  $F_{2000}$  to be something simple  $\pmod{125}$  (just like  $(F_n \bmod 2)_{n \in \mathbb{N}}$  is something easily predictable), and with  $F_{2001} \equiv 1 \pmod{125}$  also observed, one can notice that the *Fibonacci sequence*

repeats itself every 2000 terms. Though not the least possible period, it's still useful.

$$\begin{aligned}
F_{2000} &\equiv \sum_{i=0}^2 C_{2i+1}^{2000}(5^i) = 2000 + \frac{2000 \times 1999 \times 1998}{6} \times 5 + \frac{2000 \times 1999 \times 1998 \times 1997 \times 1996}{5 \times 24} \times 25 \\
&= 125 \times 16 + 125 \times 16 \times 1999 \times 333 \times 5 + 125 \times 16 \times 1999 \times 333 \times 1997 \times 499 \times 5 \equiv 0 \pmod{125} \\
F_{2001} &\equiv \sum_{i=0}^2 C_{2i+1}^{2001}(5^i) = 2001 + \frac{2001 \times 2000 \times 1999}{6} \times 5 + \frac{2001 \times 2000 \times 1999 \times 1998 \times 1997}{5 \times 24} \times 25 \\
&= (125 \times 16 + 1) + 667 \times 125 \times 8 \times 1999 \times 5 + 2001 \times 125 \times 4 \times 1999 \times 333 \times 1997 \times 5 \equiv 1 \pmod{125}
\end{aligned}$$

So it is just proven that  $F_{2020} \equiv F_{20} \pmod{125}$ .

#### 2.1.6.6 More about the general term formula of *Fibonacci sequence*

Notice that we have something called the geometric series, for example,  $1 + 2 + 4 + 8 + \dots$ , and notice that the property that the *Fibonacci sequence* is inductively defined and the fact that it is *addition* of previous two terms make it reasonable to suggest some relations between geometric series and it.

Obviously the *Fibonacci sequence* is not a geometric series, as the ratio between two terms is not a constant.

(Though, the ratio between two consecutive terms actually tends to the *golden ratio*  $\phi = \frac{\sqrt{5}+1}{2}$ , which provides further justification on the decision to compare geometric series with it)

We assume  $F_n = a^n + b^n$  for some  $a, b \in \mathbb{R} \setminus \{0\}$  and for all  $n \in \mathbb{N}$ .

Or, you should let, more generally,  $F_n = ca^n + db^n$  for some  $a, b, c, d \in \mathbb{R} \setminus \{0\}$ .

$$\begin{cases} ca^1 + db^1 = F_1 = 1 \\ ca^0 + db^0 = F_0 = F_2 - F_1 = 0 \\ c = -d \\ c(a - b) = 1 \end{cases}$$

(Using  $F_2 = 0$  to set up the equation also works, but it's slightly more complicated.)

We notice that there is not enough information given. However, as the induction condition is  $F_{n+2} = F_{n+1} + F_n$ , and it would be reasonable to pick  $a, b$  to be the roots of  $x^{n+2} = x^{n+1} + x^n$ , i.e.,  $x^2 = x + 1$ , so it is how the *Fibonacci sequence* is nicely related to the golden ratio.

If we let:

$$(a, b) = \left( \frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2} \right)$$

We then suggest  $c = \frac{1}{\sqrt{5}}$ .

As  $ca^{n+2} + db^{n+2} = (ca^{n+1} + ca^n) + (db^{n+1} + db^n) = (ca^{n+1} + db^{n+1}) + (ca^n + db^n)$  and

$$\begin{cases} F_0 = 0 = ca^0 + db^0 \\ F_1 = 1 = ca^1 + db^1 = \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} \end{cases}$$

We can then conclude

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

$\left( \text{You may compare inductively Fibonacci sequence } (F_n)_{n \in \mathbb{N}} \text{ with the suggested } \left( \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} \right)_{n \in \mathbb{N}} \right)$  to prove.

By the way, for *Mathematical induction* here, it is slightly different than the ordinary one.

You may see [Strong Induction](#). Roughly speaking, the induction hypotheses when  $n = k + 1$  is that  $P(1), \dots$ , and  $P(k)$ . Obviously it's equivalent to the ordinary *Mathematical induction*: just let  $Q(n)$  to be ' $P(1) \wedge P(2) \wedge \dots \wedge P(n) \quad \forall n \in \mathbb{N}$ '.

#### 2.1.6.6.1 A proof of the general term of *Fibonacci sequence*

Let  $P(n)$  be ' $F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$ ', for non-negative integer  $n$ .

$P(0)$  and  $P(1)$  are true.

Assume that  $P(0) \wedge P(1) \wedge \dots \wedge P(k)$  for some  $k \in \mathbb{N}$ ,

For  $n = k + 1, F_{k+1} = F_k + F_{k-1}$  as the golden ratios are the roots of  $(x^{k-1})x^2 = (x^{k-1})x + (x^{k-1})$ .

$\therefore P(k + 1)$  is true.

$\therefore$  By *Mathematical Induction*,  $P(n)$  is true for all non-negative integers  $n$ .

## 3 IMO HK Team Selection Test 1

We are thankful for the official solution, and please be acknowledged that we may include direct copies of some parts of the official solution.

### 3.1 IMO HK 2021 TST 1

*Date of test: 2020-10-17.*

#### 3.1.1 Q1

##### 3.1.1.1 Question

Find all  $(a, b, c) \in \mathbb{R}^3$  such that  $(2^{2a} + 1)(2^{2b} + 2)(2^{2c} + 8) = 2^{a+b+c+5}$ .

##### 3.1.1.2 Solution

$$\begin{aligned} \frac{2^{2a} + 1}{2^a} \cdot \frac{2^{2b} + 2}{2^{b+\frac{1}{2}}} \cdot \frac{2^{2c} + 8}{2^{c+\frac{3}{2}}} &= \frac{2^{a+b+c+5}}{2^{a+b+\frac{1}{2}+c+\frac{3}{2}}} \\ \left(2^a + \frac{1}{2^a}\right) \left(2^{b-\frac{1}{2}} + \frac{1}{2^{b-\frac{1}{2}}}\right) \left(2^{c-\frac{3}{2}} + \frac{1}{2^{c-\frac{3}{2}}}\right) &= \frac{2^{a+b+c+5}}{2^{a+b+c+2}} = 2 \cdot 2 \cdot 2 \end{aligned}$$

Notice that  $2^t > 0 \forall t \in \mathbb{R}$ , so we can use the fact that  $x + \frac{1}{x} \geq 2 \forall x \in \mathbb{R}^+$  and equality holds iff  $x = 1$ .  
 If any of the terms  $(2^a + \frac{1}{2^a})$ ,  $(2^{b-\frac{1}{2}} + \frac{1}{2^{b-\frac{1}{2}}})$ ,  $(2^{c-\frac{3}{2}} + \frac{1}{2^{c-\frac{3}{2}}})$  is  $> 2$ , the equality cannot hold.

$$\begin{aligned} 2^a &= 2^{b-\frac{1}{2}} = 2^{c-\frac{3}{2}} = 1 \\ a &= b - \frac{1}{2} = c - \frac{3}{2} = 0 \\ (a, b, c) &= \left(0, \frac{1}{2}, \frac{3}{2}\right) \end{aligned}$$

### 3.1.2 Q3

#### 3.1.2.1 Original Question

*On the table there are 20 coins of weights 1, 2, 3, ..., 15, 37, 38, 39, 40 and 41 grams. They all look alike but their colours are all distinct. Now Miss Adams knows the weight and colour of each coin, but Mr. Bean knows only the weights of the coins. There is also a balance on the table, and each comparison of weights of two groups of coins is called an operation. Miss Adams wants to tell Mr. Bean which coin is the 1-gram coin by performing some operations. What is the minimum number of operations she needs to perform?*

#### 3.1.2.2 Question (Rephrased)

- On a table there are **20** coins of weights **1,2,3,4,...,15, 37,38,39,40,41** grams respectively.
- The coins all look alike but their colors are distinct (hence distinguishable).
- A knows the weight and color of each coin; B only the weight (knows the first bullet point).
- There is a balance on the table that is only able to compare *L.H.S.* with *R.H.S.*, where both sides must be a non-empty subset of the 20 coins.
- **At least how many comparisons** have to be demonstrated by A so that B knows for sure which the lightest coin (which weighs 1 gram) is, provided that they are not allowed to communicate?

#### 3.1.2.3 Attempts, examples and comments

A possible and helpful demonstration would be like putting the lightest 16 coins on *L.H.S.* and the other on *R.H.S.* The *L.H.S.* would weigh 157 grams while the *R.H.S.* would weigh 158 grams. (The weight of any 16 coins must not be less than 157 while the weight of any 4 coins must not be more than 158.) If the 16 coins are not the lightest 16, then the weight of them must exceeds half of the total weight (**315** grams) of all 20 coins. Therefore, it is for sure that the 16 coins must be the lightest 16 coins if it is seen from the balance that  $L.H.S. < R.H.S.$

As you can see, the key is to find a comparison so delicate that B knows for sure the exact sets of coins **on L.H.S., on R.H.S. and on the table**. You may think about the 3-tuple  $(X, Y, Z)$ , where  $X, Y, Z \subseteq \{1, 2, 3, 4, \dots, 15, 37, 38, 39, 40, 41\}$  ( $Z$  the ‘unused coins’) are disjoint and make up of all 20 coins.

For example, putting the 1-gram coin on *L.H.S.* and the 2-gram coin on *R.H.S.* tells B nothing, as  $(X, Y, Z)$  is not uniquely determined by this demonstration.

#### 3.1.2.4 Solution (modified from official)

At least **2** operations have to be done.

### 3.1.2.4.1 ‘2 operations’ is sufficient - Example

1. Put coins (with weights)  $1, 2, 3, 4, \dots, 15$  on *L.H.S.* and coins  $39, 40, 41$  on *R.H.S.* It is a balance ( $120 = 120$ ). The weight of any other  $X \subseteq \{1, 2, 3, 4, \dots, 15, 37, 38, 39, 40, 41\}$  with 15 elements (coins) would be larger than 120, whereas the weight of any other set  $Y$  with 3 elements would be less than 120, equality is then impossible. Hence  $(X, Y, Z) = (\{1, 2, 3, \dots, 15\}, \{39, 40, 41\}, \{37, 38\})$  is uniquely determined.
2. Put coins  $1, 37$  on *L.H.S.* and coin  $38$  on *R.H.S.*  $(X, Y, Z) = (\{1, 37\}, \{38\}, \{\dots\})$  is also uniquely determined, as the set  $\{37, 38\}$  is determined in the first step and can be identified by  $B$  with the exclusive colours of coins. ( $B$  is aware of that both ‘unused’ (in step 1) coins  $37$  and  $38$  are now used in  $X$  or  $Y$  but not  $Z$ ). Now the lightest coin is caught.

### 3.1.2.4.2 ‘2 operations’ is necessary - Proof

*Proof.* Assume one operation is enough, then  $1$  is the **only** coin in either  $X, Y, Z$  it belongs to.

- If the 1-gram coin is put on the balance,
    - If the other group on balance contains exactly one coin or no coin, it tells nothing about the weight of any coin ( $1 < 2$  can be replaced by  $2 < 3$ , for instance);
    - If the other group on balance contains more than one coin, the ‘situation’ is the same when the 1-gram coin is swapped with the 2-gram coin.
  - If the 1-gram coin is (the only) unused,
    - If the balance is inclined to one side, then one side is at least 158 grams while the other at most 156 grams (**total weight is 315 grams**), and thus the situation is the same when the 1-gram coin is swapped with the 2-gram coin (where exactly one side has a change  $\pm 1$ );
    - If sets  $X$  and  $Y$  are equally heavy (same weight: 157-gram), WLOG we assume  $3 \in X$  as  $3 \in X \cup Y$ ,
      - \* If  $2 \in Y$ , we can swap the 2-gram coin with the 3-gram coin, so that now the side with the 3-gram coin is heavier by 2 grams, and we can then swap the 1-gram coin and the 3-gram coin, keeping the balance;
      - \* Else if for some  $k > 3$ , we have  $k \in X \wedge k + 1 \in Y$ , then we can swap those two coins and then swap the 1-gram coin and the 3-gram coin just like the above;
      - \* (Do **not** combine the two cases above, you need to track the side with the 3-gram coin after first ‘swap’.)
      - \* Else,
        - $1 \notin X \wedge 2, 3 \in X$ ,
        - $(i \in X \cap \{4, 5, \dots, 15\} \Rightarrow \forall j \in \mathbb{Z} \cap [i, 15], j \in X)$  so  $(X \cap [4, 15] = \mathbb{Z} \cap [\min(X \cap [4, 15]), 15] \text{ or } \{\})$ ,
        - $X \cap [37, 41] = \mathbb{Z} \cap [\min(X \cap [37, 41]), 41] \text{ or } \{\}$ ,
- but the weight of  $X \setminus \{2, 3\}$  cannot be 152, shown by some computations:
- If  $38 \in X$ ,  $38 + \dots + 41 = 158 > 152$
  - Else if  $39 \in X$ ,  $14 + 15 + 39 + 40 + 41 = 149 < 152 < 162 = 13 + 14 + 15 + 39 + 40 + 41$
  - Else if  $40 \in X$ ,  $11 + 12 + \dots + 15 + 40 + 41 = 146 < 152 < 156 = 10 + 11 + \dots + 15 + 40 + 41$
  - Else if  $41 \in X$ ,  $5 + 6 + \dots + 15 + 41 = 151 < 152 < 155 = 4 + 5 + \dots + 15 + 41$
  - Else,  $4 + 5 + \dots + 15 = 114 < 152$

In any case, it is impossible to demonstrate only a **single** comparison so that  $B$  can decide which the 1-gram coin is. ■



### 3.1.2.4.3 Remark of the solution

As seen in [the second operation](#),  $B$  needs to remember all the colours of all the coins in their respective set ( $X$ ,  $Y$  or  $Z$ ). More clarifications may need to be made if such a question is to be generalized or extended.

### 3.1.3 Q4

#### 3.1.3.1 Question

Let  $n \in \mathbb{N}$ . Is it possible to express  $n^2 + 3n + 3$  into the form  $ab$  where  $a, b \in \mathbb{N}$ , such that the difference between  $a$  and  $b$  is smaller than  $2\sqrt{n+1}$ ?

#### 3.1.3.2 Solution

We notice that  $|a - b| < 2\sqrt{n+1}$  is the same as  $(a - b)^2 < 4n + 4$ . We may get some contradiction, or it is similar proving it by  $(a - b)^2 \geq 4n + 4$ .

We may consider *AM-GM inequality* for  $(a + b)^2 \geq 4ab$  but essentially it is  $(a + b)^2 = (a - b)^2 + 4ab$ .

Using this equation for the first time,  $(a + b)^2 \geq 4n^2 + 12n + 12 > (2n + 3)^2$ ,  $\therefore a + b \geq 2n + 4$ .

Using this equation for the second time,  $(2n + 4)^2 \leq (a + b)^2 = 4n^2 + 12n + 12 + (a - b)^2$ ,  $\therefore (a - b)^2 \geq 4n + 4$ .

## 4 IMO HK Team Selection Test 2

We are thankful for the official solution, and please be acknowledged that we may include direct copies of some parts of the official solution.

### 4.1 IMO HK 2021 TST 2

Date of test: 2020-10-31.

#### 4.1.1 Q2

##### 4.1.1.1 Question

Given  $f(x) \in \mathbb{Q}[x]$  and  $\alpha \in \mathbb{R}$  and that

$$\alpha^3 - 2019\alpha = (f(\alpha))^3 - 2019f(\alpha) = 2021,$$

Prove:

$$\forall n \in \mathbb{N}, \left(f^{\circ n}(\alpha)\right)^3 - 2019f^{\circ n}(\alpha) = 2021.$$

(Here I define  $f^{\circ n}(x) = \underbrace{f(f(\cdots f(x)))}_n$ .)

##### 4.1.1.2 Solution

Consider  $g(x) = x^3 - 2019x - 2021 \in \mathbb{Q}[x]$  (FYI,  $g(x)$  actually has 3 real roots), we notice that the question is equivalent to proving  $f^{\circ n}(\alpha)$  is also a root given that  $\alpha, f(\alpha)$  are roots of  $g(x) = 0$ .

First of all, we know that  $g(x)$  has no rational root. Why? Assume there is such a root  $x_0 \in \mathbb{Q}$ , then by *Rational root Theorem* (and  $g(x)$  is *monic*) we know that  $x_0 \in \mathbb{Z}$  such that  $x_0 \mid 2021$ . One may test factors  $\pm 1, \pm 43, \pm 47, \pm 2021$  of

2021 as  $x_0$  and see how all they fail. Or simply,  $x_0^3 - 2019x_0 - 2021 \equiv x_0(x_0 - 1)(x_0 + 1) + 1 \equiv 0 + 1 \not\equiv 0 \pmod{2}$ . Furthermore,  $g(x)$  is *irreducible* over  $\mathbb{Q}[x]$  (otherwise it can be factorized into two factors of degree 1 and 2 respectively, and then  $g(x)$  should have a rational root).

The key observation is that  $g(x)$  is kind of ‘tailor-made’ for  $\alpha$  (which is an irrational number). How? Consider any  $h(x) \in \mathbb{Q}[x]$  with  $\deg h = 1$  or 2, we must have  $h(\alpha) \neq 0$ .

(After proving the case for  $\deg h = 1$ : If  $\deg h = 2$ , use *division algorithm* for  $g \div h$  and the remainder must not be of degree 1, hence is the zero polynomial, which now contradicts the irreducibility of  $g(x)$ .)

We may say,  $g(x)$  is the (only) *minimal polynomial* for  $\alpha$ , that is, among all monic polynomials  $h(x) \in \mathbb{Q}[x]$  with positive degree and  $h(\alpha) = 0$ ,  $g(x)$  is the smallest one. We have  $g \mid h$  as well (see below), so we may also say that  $g(x)$  is the only irreducible monic polynomial with  $\alpha$  as a root.

As  $g(f(\alpha)) = 0$ , and (of course)  $g(f(x)) \in \mathbb{Q}[x]$ , it is impossible for  $g(x) \nmid g(f(x))$  (*division algorithm*).

$$\begin{aligned} g(x) &\mid g(f(x)) \\ g(f(x)) &\mid g(f(f(x))) \\ g(x) &\mid g(f^{\circ 2}(x)) \end{aligned}$$

By induction, we can conclude  $g(x) \mid g(f^{\circ n}(x)) \forall n \in \mathbb{N}$ , and hence the claim.

#### 4.1.1.2.1 Remark - irreducibility

We say a polynomial  $p(x) \in F[x]$  ( $F$  is a *field*) is *irreducible* if:

1.  $\deg p > 0$ , and
2. if  $p = ab$  for  $a, b \in F[x]$ , then either  $a$  or  $b$  is **constant**.

It’s like a ‘prime number’ polynomial. The first condition is like excluding 1 from the set of prime numbers.

#### 4.1.2 Q3

##### 4.1.2.1 Question

Acute  $\triangle ABC$  has circumcircle  $\Gamma$ .  $P$  is the midpoint of minor arc  $\widehat{BC}$  of  $\Gamma$ . Let  $AP$  meets  $BC$  at  $D$  and  $M$  be the midpoint of  $AB$ . Also, let  $E$  be the point such that  $AE \perp AB$  and  $BE \perp MP$ . Prove that  $AE = DE$ .

##### 4.1.2.2 Graph

May refer to the [Desmos graph](#) created by us, illustrating how  $AE = DE$ . [Figure 1](#) is generated from this graph.

##### 4.1.2.3 Hint

The key is to prove a slightly different result. We may let  $E'$  be the point such that  $AE' \perp AB$  and  $E'$  is equidistant from  $A$  and  $D$  (i.e.  $E'$  is on  $\perp$  bisector of  $AD$ ). Then, we may prove that  $BE' \perp MP$  and hence  $E = E'$  (because  $E$  is uniquely determined by the construction: mark the intersection of the line through  $A$  that  $\perp AB$  and the line through  $B$  that  $\perp MP$  as  $E$ ).

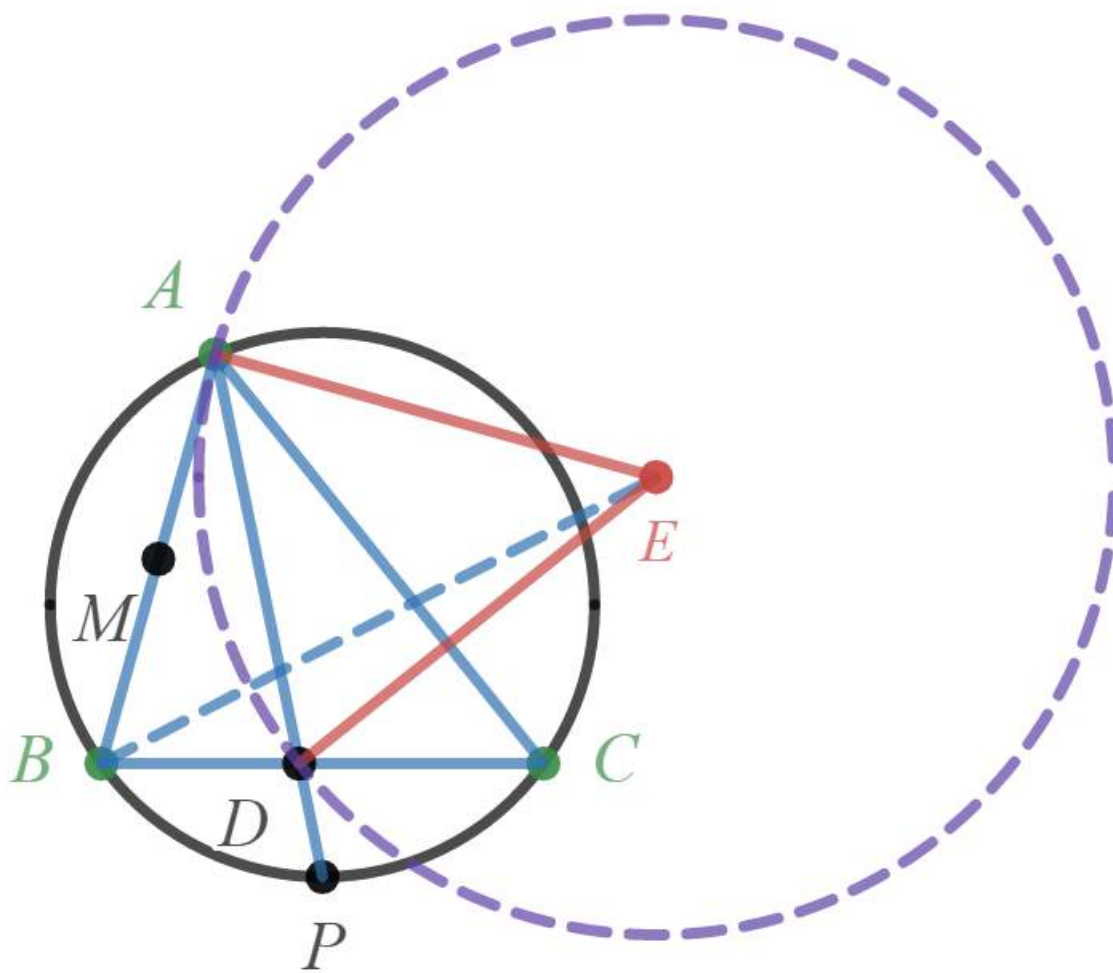


Figure 1 (IMO HK 2021 TST 2 Q3)

#### 4.1.2.4 Solution

Refer to [hint](#) and [Figure 1](#). It is clear that  $E'A = E'D$  and a circle  $\Omega$  with center  $E'$  can thus be drawn.

Notice that the power  $p(M, B) = MB^2 = MA^2 = p(M, \Omega)$  as  $\angle BAE = \frac{\pi}{2}$  ( $MA$  is tangent to  $\Omega$ ).

Also, as  $\angle PBC = \angle PAC = \angle PAB$ ,  $\triangle BPD \sim \triangle APB$ ,  $p(P, B) = BP^2 = PD \cdot PA = p(P, \Omega)$ . (It is worth noticing that here point circle  $B$  is used, and that  $PA \cdot PD = p(P, \Omega)$  because  $P$  is **outside**  $\Omega$ , otherwise they differ by a sign.)

Now,  $PM$  coincide with the *radical axis* of circles  $B$  and  $\Omega$ ,  $PM \perp BE'$ , the line joining centers of the circles.

Hence,  $E'$  coincide with  $E$ .

#### 4.1.2.5 Remark

Without the [hint](#), I think it would be rather difficult (extremely complicated and infeasible within 3 hours) to solve this question using coordinate geometry or complex geometry.

## 5 Hong Kong (China) Mathematical Olympiad

We are thankful for the official solution, and please be acknowledged that we may include direct copies of some parts of the official solution.

### 5.1 CHKMO 2020

Date of test: 2020-12-05.

#### 5.1.1 Q4

##### 5.1.1.1 Question

Given that  $a, b, c \in \mathbb{R}^+$  such that  $abc = 1$ , prove that:

$$\frac{1}{a^3 + 2b^2 + 2b + 4} + \frac{1}{b^3 + 2c^2 + 2c + 4} + \frac{1}{c^3 + 2a^2 + 2a + 4} \leq \frac{1}{3}.$$

##### 5.1.1.2 Solution

$$\begin{aligned} & \frac{1}{a^3 + 2b^2 + 2b + 4} + \frac{1}{b^3 + 2c^2 + 2c + 4} + \frac{1}{c^3 + 2a^2 + 2a + 4} \\ &= \frac{1}{(a^3 + b^2 + b) + (b^2 + b + 1) + 3} + \frac{1}{(b^3 + c^2 + c) + (c^2 + c + 1) + 3} + \frac{1}{(c^3 + a^2 + a) + (a^2 + a + 1) + 3} \\ &\leq \frac{1}{3\sqrt[3]{a^3b^3} + 3\sqrt[3]{b^3} + 3} + \frac{1}{3\sqrt[3]{b^3c^3} + 3\sqrt[3]{c} + 3} + \frac{1}{3\sqrt[3]{c^3a^3} + 3\sqrt[3]{a} + 3} \\ &= \frac{1}{3} \left( \frac{1}{ab + b + 1} + \frac{1}{bc + c + 1} + \frac{1}{ca + a + 1} \right) \\ &= \frac{1}{3} \left( \frac{(bc + c + 1)(ca + a + 1) + (ca + a + 1)(ab + b + 1) + (ab + b + 1)(bc + c + 1)}{(ab + b + 1)(bc + c + 1)(ca + a + 1)} \right) \\ &= \frac{1}{3} \left( \frac{(bc + c + 1)(ca + a + 1) + (ca + a + 1)((bc + c + 1) \div c) + ((ca + a + 1) \cdot b)(bc + c + 1)}{(ab + b + 1)(bc + c + 1)(ca + a + 1)} \right) = \frac{1}{3} \end{aligned}$$

To see how the expression (after applying *AM-GM inequality* on the denominators) becomes an identity, you may also let  $\alpha := bc + c + 1$  and notice that for example  $ca + a + 1 = a \cdot \alpha$ , hence:

$$\frac{1}{\frac{\alpha}{c}} + \frac{1}{\alpha} + \frac{1}{a\alpha} = \frac{1}{\alpha}(bc + c + 1) = 1.$$

Alternatively, you may write  $a, b, c$  into the form  $\frac{x}{y}, \frac{y}{z}, \frac{z}{x}$  respectively (since  $abc = 1$ ):

$$\frac{1}{\frac{x}{z} + \frac{y}{z} + 1} + \frac{1}{\frac{y}{x} + \frac{z}{x} + 1} + \frac{1}{\frac{z}{y} + \frac{x}{y} + 1} = \frac{z}{x + y + z} + \frac{x}{x + y + z} + \frac{y}{x + y + z} = 1.$$

Notice that equality (original inequality) holds iff  $a = b = c = 1$ , where the three fractions becomes  $\frac{1}{9}$ .