Homework 2

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Exercise 1

The exercise requires to state if the following distributions are involutive or not and, if it is possible, to find the annihilator.

A. $\Delta 1 = span\left\{\begin{bmatrix} 3x_1 \\ 0 \\ -1 \end{bmatrix}\right\}, U \in \mathbb{R}\mathbf{3}$

This distribution is one dimensional and so is **involutive** since $[f, f]$ = 0 and the zero vector belongs to any distribution by default.

The next step is to find the annihilator for this distribution. We know that $\mathbf{dim}(\Delta_1) + \mathbf{dim}(\Delta_1^{\perp}) = \boldsymbol{n}$, where n is equal to 3 and $\dim(\Delta_1)$ is equal to 1, so $\dim(\Delta_1^{\perp})$ is equal to 2.

Now we need to define a set of co-vectors such that $\mathbf{w^*F(x) = 0}$.

$$\mathbf{w^*F(x)} = [\boldsymbol{w1}\ \boldsymbol{w2}\ \boldsymbol{w3}]\begin{bmatrix} \mathbf{3}\boldsymbol{x_1} \\ \mathbf{0} \\ \mathbf{-1} \end{bmatrix} = \mathbf{3}\boldsymbol{x}\mathbf{1}\boldsymbol{w}\mathbf{1} - \boldsymbol{w}\mathbf{3} = \mathbf{0}$$

There is one equation, and we must find two linear independent co-vectors field. I can impose for instance these equalities $w1 = w3 = 0$ and $w2 = a1$, where a1 could be any value. The resultant co-vector will be $w1^* = [0\ \ a1\ \ 0]$.

Another solution is choosing $w1 = a2$ and $w3 = 3x1a2$, leaving $w2$ freely chosen and obtain $w2^* = [\mathrm{a}_2\ \ 0\ \ 3x_1]$.

In the end the co-distribution is:

$$\Delta_1^{\perp} = \boldsymbol{span}\,\{[\mathbf{0}\ \ \boldsymbol{a}\mathbf{1}\ \ \mathbf{0}]\quad [\boldsymbol{a_2}\ \ \mathbf{0}\ \ \mathbf{3}\boldsymbol{x}\mathbf{1}]\}$$

B. $\Delta 2 = span\left\{\begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix}, \begin{bmatrix} 0 \\ -a \\ x_1 \end{bmatrix}\right\}, U \in \mathbb{R}\mathbf{3}$

First thing to do is to find the dimension of the distribution, so I construct the $F(x)$ matrix and evaluate the rank.

$F(x) = \begin{bmatrix} 1 & 0 \\ 0 & -4 \\ x_2 & x_1 \end{bmatrix}$ If I select the 2x2 matrix $\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$ the determinant is equal to -4 and so:

$rank\,(F(x)) = 2$, and the dimension of the distribution is equal to 2, it means that the distribution is **not singular**.

The next step is to calculate the lie bracket: $[f1,\ f2] = \frac{\partial f2}{\partial x} f1 - \frac{\partial f1}{\partial x} f2$:

$$[f1, f2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -4 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

If the lie bracket belongs to the span generated by the vectors $f1$, $f2$ it is possible to say that the distribution is involutive. To prove this, we build the following 3x3 matrix which has as columns the vectors $f1, f2, [f1, f2]$:

$$F1(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ x_2 & x_1 & 5 \end{bmatrix}$$

The determinant of this matrix can be evaluated using the Sarrus's rule and it is:

$$\det(F1(x)) = -4 * 5 = -20$$

It means that the rank of F1(x) is equal to 3, so $rank(F(x)) \neq rank(F1(x))$ and so we can state that the distribution is **<u>not involutive</u>**.

The last step is to find the annihilator of the distribution. Firstly, we compute the dimension of the annihilator:

$$\dim(\Delta_2^{\perp}) = \mathrm{n} - \dim\,(\Delta_2) = 3 - 2 = 1$$

Now we need a set of co-vectors such that $\mathbf{w}^*\mathbf{F}(\mathbf{x}) = \mathbf{0}$, so:

$$\mathbf{w}^*\mathbf{F}(\mathbf{x}) = [\boldsymbol{w1}\ \boldsymbol{w2}\ \boldsymbol{w3}] \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-4} \\ \boldsymbol{x_2} & \boldsymbol{x_1} \end{bmatrix} = [\mathbf{0}\quad \mathbf{0}] \rightarrow \begin{matrix} \boldsymbol{w1} + \boldsymbol{w3x2} = \mathbf{0} \\ -\mathbf{4}\ \boldsymbol{w2} + \boldsymbol{w3x1} = \mathbf{0} \end{matrix} \rightarrow \begin{matrix} \boldsymbol{w3} = -\frac{\boldsymbol{w1}}{\boldsymbol{x2}} \\ \boldsymbol{w2} = \frac{\boldsymbol{w3x1}}{\mathbf{4}} = -\frac{\boldsymbol{w1x1}}{\mathbf{4x2}} \end{matrix}$$

If we chose $w1 = a$ we have:

$$\begin{cases} w3 = -a/x_2 \\ w2 = -\dfrac{ax_1}{4x_2} \end{cases}$$

And the co-distribution is:

$$\Delta_2^{\perp} = span\left\{\left[a, -\frac{ax_1}{4x_2}, -\frac{a}{x_2}\right]\right\}$$

C. $\Delta 2 = span\left\{\begin{bmatrix} 2x_3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_1 \\ 1 \end{bmatrix}\right\}, U \in \mathbb{R}\mathbf{3}$

First thing to do is to find the dimension of the distribution, so I construct the $F(x)$ matrix and evaluate the rank.

$F(x) = \begin{bmatrix} 2x_3 & x_2 \\ -1 & x_1 \\ 0 & 1 \end{bmatrix}$ If I select the 2x2 matrix $\begin{bmatrix} -1 & x_1 \\ 0 & 1 \end{bmatrix}$ the determinant is equal to -1 and so:

$rank\,(F(x)) = 2$, and the dimension of the distribution is equal to 2, it means that the distribution is not singular.

The next step is to calculate the lie bracket: $[f1,\ f2] = \frac{\partial f2}{\partial x} f1 - \frac{\partial f1}{\partial x} f2$:

$$[f1, f2] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2x_3 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2x_3 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 2x_3 \\ 0 \end{bmatrix}$$

If the lie bracket belongs to the span generated by the vectors $f1$, $f2$ it is possible to say that the distribution is involutive. To prove this, we build the following 3x3 matrix which has as columns the vectors $f1, f2, [f1, f2]$:

$$F1(x) = \begin{bmatrix} 2x_3 & x_2 & -3 \\ -1 & x_1 & 2x_3 \\ 0 & 1 & 0 \end{bmatrix}$$

The determinant of this matrix can be evaluated using the Sarrus's rule and it is:

$$\det(F1(x)) = 3 - 4x_3^2$$

If $x_3 \neq \pm\frac{\sqrt{3}}{2}$ the matrix F1(x) is full rank (rank(F1(x)) = 3), so $rank(F(x)) \neq rank(F1(x))$ and the distribution is **<u>not involutive</u>**. Otherwhise, if $x_3 = \pm\frac{\sqrt{3}}{2}$ the rank of F1 will be equal to 2 and so the distribution is **<u>involutive</u>**.

The last step is to find the annihilator of the distribution. Firstly, we compute the dimension of the annihilator:

$dim(\Delta_3^{\perp}) = n - dim\,(\Delta_3) = 3 - 2 = 1$

Now we need a set of co-vectors such that $\mathbf{w^*F(x) = 0}$, so:

$$\mathbf{w^*F(x) = [w1\;w2\;w3]} \begin{bmatrix} \mathbf{2x_3} & \mathbf{x_2} \\ \mathbf{-1} & \mathbf{x_1} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \mathbf{= 0} \rightarrow \begin{matrix} \mathbf{2x_3w_1 - w_2 = 0} \\ \mathbf{x_2w_1 + x_1w_2 + w_3 = 0} \end{matrix} \quad \begin{matrix} \mathbf{w_2 = 2x_3w_1} \\ \mathbf{w_3 = -x_2w_1 - 2x_3x_1w_1} \end{matrix}$$

If we chose $w_1 = a$ we have:

$$\begin{cases} w_2 = 2ax_3 \\ w_3 = -ax_2 - 2ax_3x_1 \end{cases}$$

And the co-distribution is:

$$\Delta_3^{\perp} = span\,\{[a, \quad 2ax_3, \quad -ax_2 - 2ax_3x_1]\}$$

Exercise 2

In this exercise we have an omnidirectional mobile robot, provided with 3 Mecanum wheels, which has as vector of generalized coordinates the following one:

$$q = [x\;\;y\;\;\theta\;\;\alpha\;\;\beta\;\;\gamma]^T \rightarrow n = 6$$

Where:

- x and y are the Cartesian coordinates of the center of the robot.
- θ is the vehicle orientation.
- $\alpha,\ \beta,\ \gamma$ represent the angle of rotation of each wheel around its axis.

And it is subject to the following Pfaffian constraints:

$$A^T(q)\dot{q} = \begin{bmatrix} \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta & \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta & l & r & 0 & 0 \\ \sin\theta & -\cos\theta & l & 0 & r & 0 \\ -\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta & \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta & l & 0 & 0 & r \end{bmatrix}\dot{q} = 0_6.$$

Where l is the distance between the wheels and the center of the robot, which I've set equal to 1.2, and r is the radius of the wheels, which I've chosen equal to 0.5.

Now we need to compute the kinematic model of this robot and show if the system is holonomic or not. To accomplish this request is necessary to calculate the G matrix which is a matrix that enters in the Null of A^T, so $G(q) \in \mathcal{N}(\mathrm{A^T(q)})$ and has six rows and three columns.

After that we can compute the kinematic model of the robot:

$$\dot{q} = G(q)u = G(q)\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Where u is the vector of the inputs and $u \in \mathbb{R}^m$.

The next step is to build the accessibility distribution Δ_a which is the distribution generated by the vector fields g_1, g_2, g_3 (which are the three columns of the G matrix) and all the Lie brackets that can be generated by these vector fields. So, we compute the partial derivatives of these vectors using the command jacobian, and then the Lie brackets.

We can notice that the accessibility distribution is a 6x6 matrix but if we compute the rank of this matrix we obtain 5, so there is a vector that is linearly dependent and so v = dim(Δa) = 5.

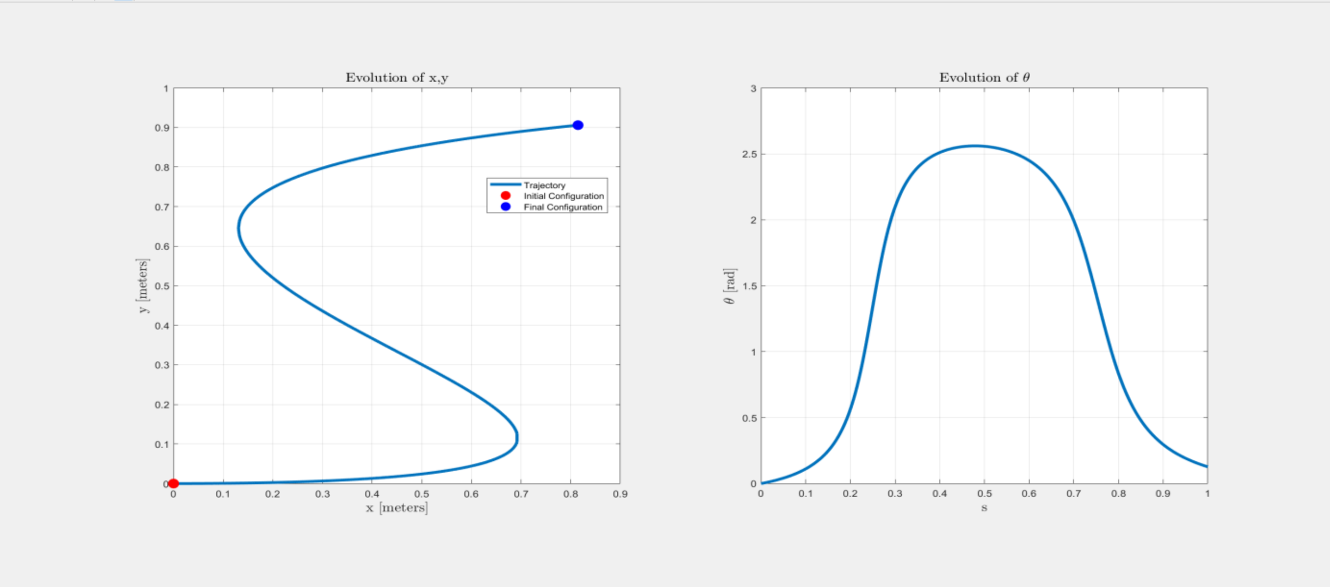
At the end we have: $m = 3$, $v = 5$ and $n = 6$

So, we are in the case of $m < v < n$ and finally we can affirm that the system is **nonholonomic** and **has only partially integrable constraints**.

Exercise 3

The exercise requires to implement via software a path planning algorithm based on a cubic Cartesian polynomial, which takes the unicycle from an initial configuration $q_i = [0\ \ 0\ \ 0]^T$ to a random final configuration $q_f = [x_f\ \ y_f\ \ \theta_f]^T$ such that $\|q_f - q_i\| = 1$. The trajectory must satisfy the following constraints: $\|v(t)\| \leq 2m/s$ and $|\omega(t)| \leq 1\ rad/s$. I'll report here the plots of one of the simulations that has been carried out.

It is possible to observe that in the simulation the robot reaches the random final configuration starting from the initial one with a final orientation almost equal to 2π.



As we said to define the trajectory we chose a cubic polynomial: $s(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ where the four coefficients are computed as follows:

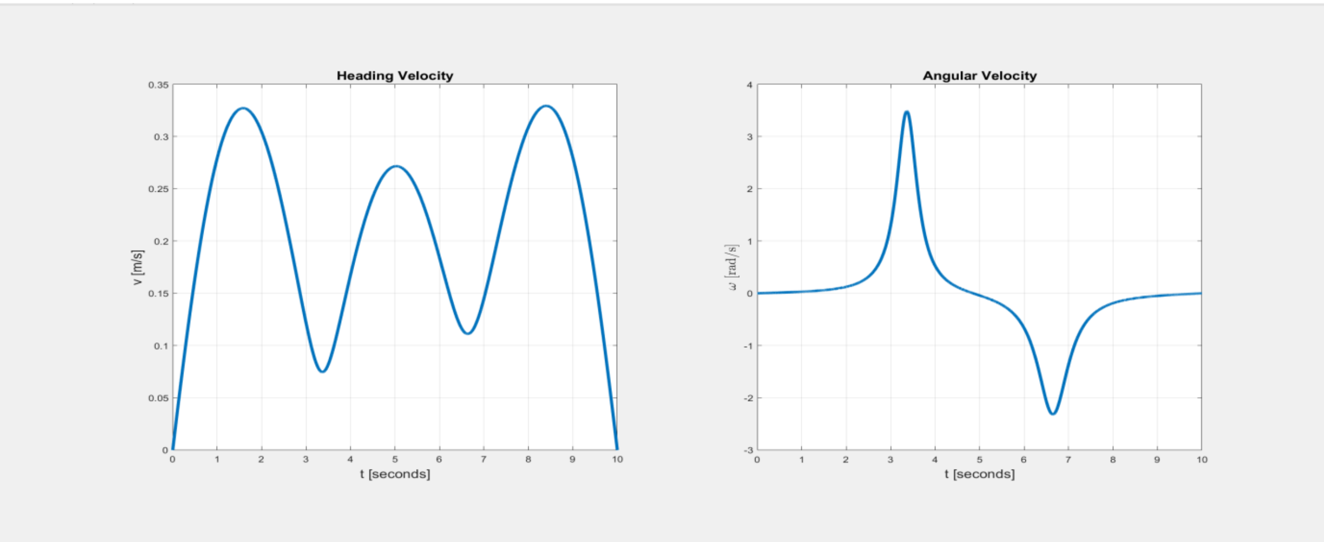
$$a_0 = s_i = 0$$

$$a_1 = \dot{s}_i = 0$$

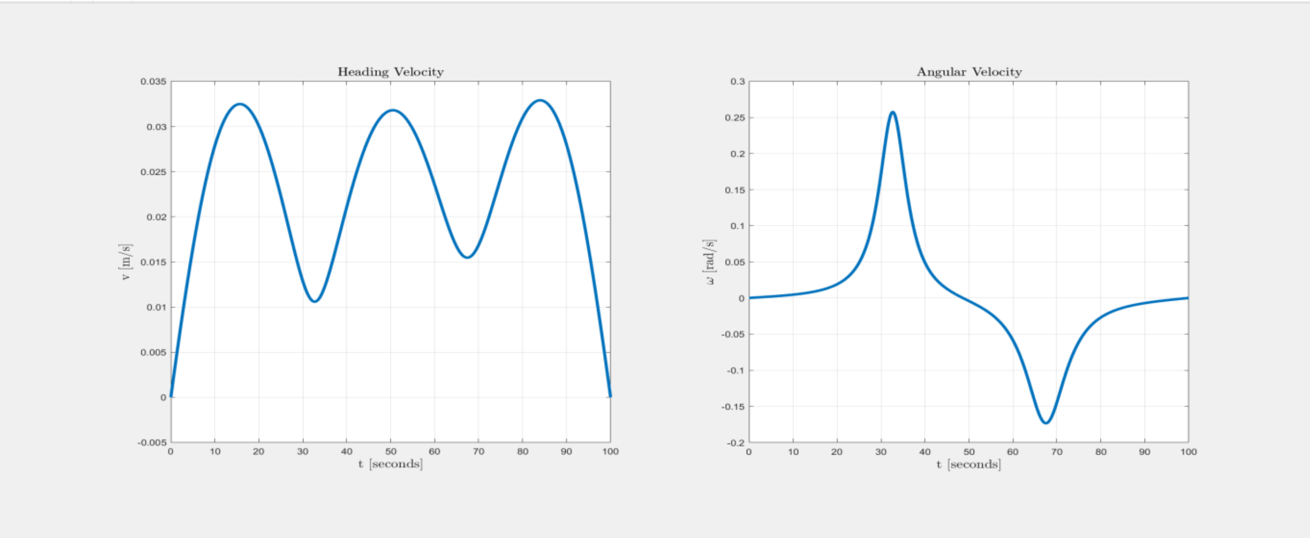
$$a_3 t_f^3 + a_2 t_f^2 + a_1 t_f + a_0 = s_f = 1$$

$$3a_3 t_f^2 + 2a_2 t_f + a1 = \dot{s_f} = 0$$

Solving a_2 from the third equation and substituting inside the fourth one we obtain: $a_2 = 3/t_f^2$ and $a_3 = -2/t_f^3$. Where t_f is a parameter that is set as input in the program. Initially I've set $t_f = 10$ and as we can see the bound on the angular velocity is not satisfied.



Then, if the previous case occurs, the program asks to increase the t_f and, as we can see from the following picture, if I set the value equal to 100, for instance, the angular velocity overcomes the previous issue and remains confined in its bound.



Exercise 4

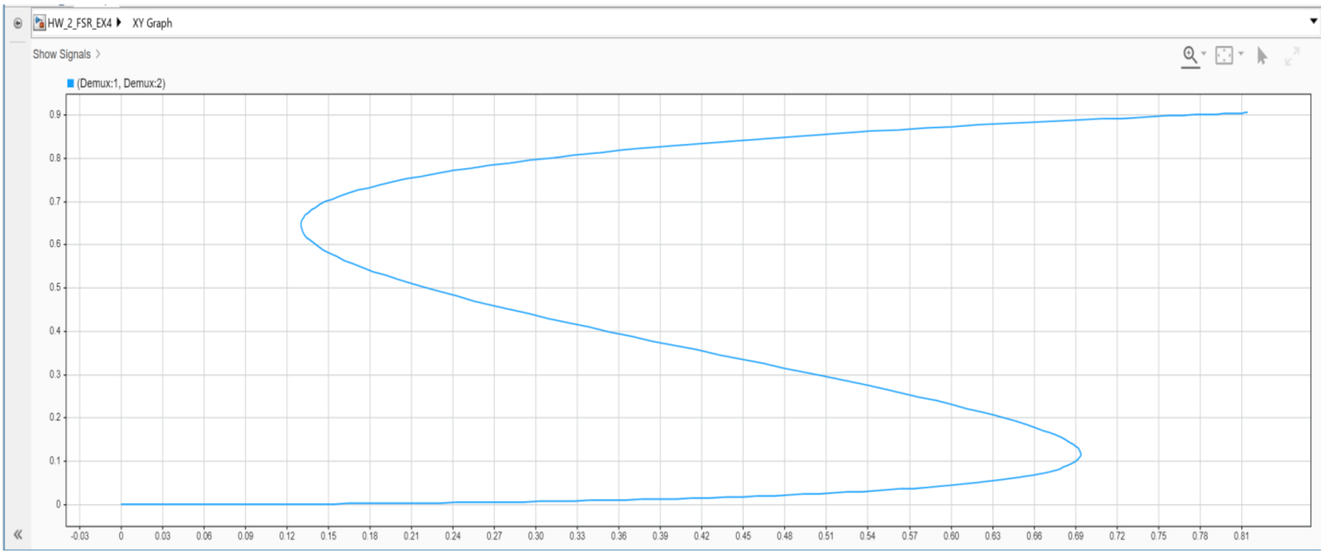
The exercise requires to implement an input/output linearization control approach to control the unicycle's position, given the trajectory of the previous exercise. As first thing we define a point B which is located along the sagittal axis of the unicycle at a distance $|b| \neq 0$ which I assume equal to 0.6 (located ahead).

We choose outputs that represent the coordinates of the point B and (x,y) which are the coordinates of the center of the wheel in such way:

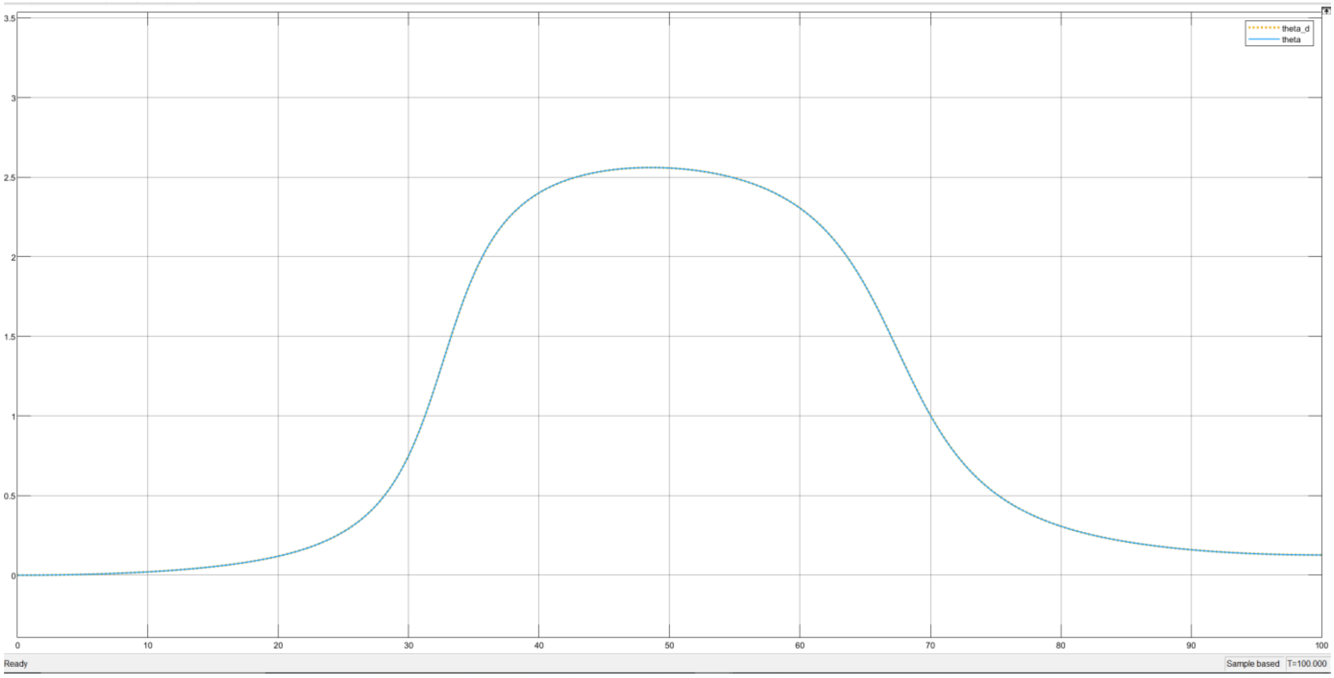
$$y_1 = x + bcos(\theta)$$

$$y_2 = y + bsin(\theta)$$

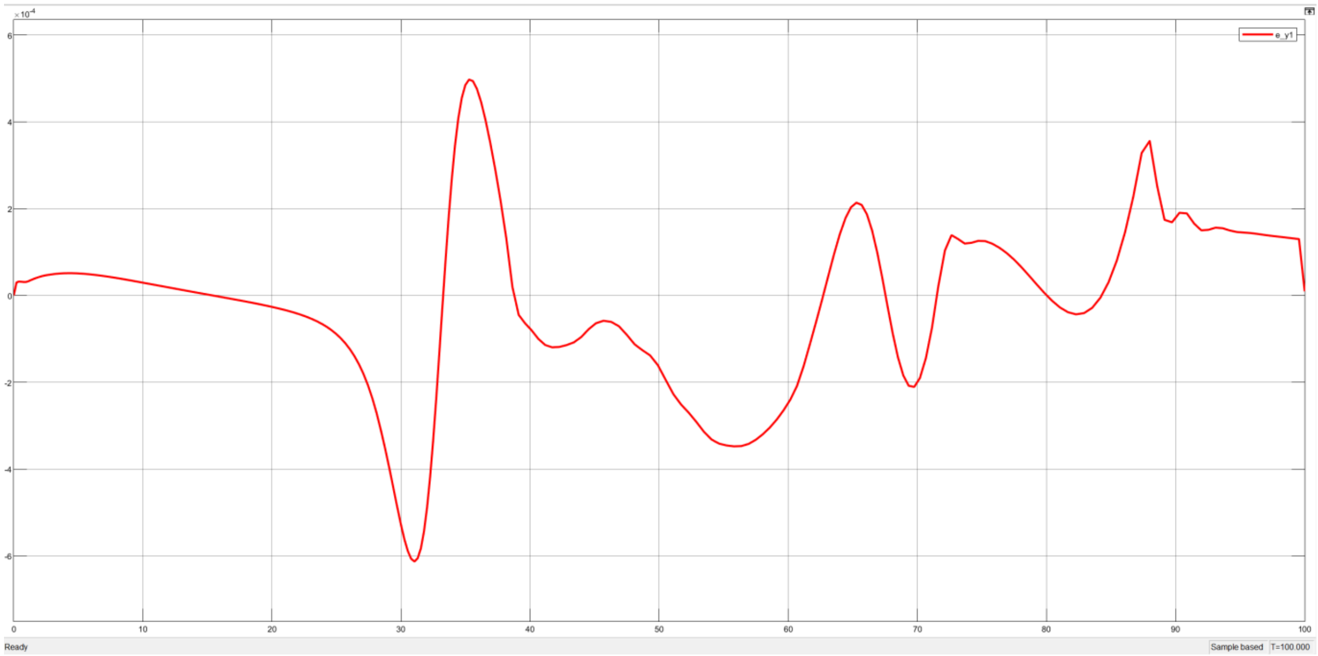
The controller is implemented in Simulink. In one block I calculate the reference ${y_1}_d$ and ${y_2}_d$ which is the trajectory from the previous exercise. In another block I evaluate y_1 and y_2. The next block calculates $u1$ and $u2$ using formulas present on the slides, where I choose the gains $k1$ = $k2$ = 5. The last block calculates heading and angular velocity. As we can see the trajectory is followed correctly by the robot.

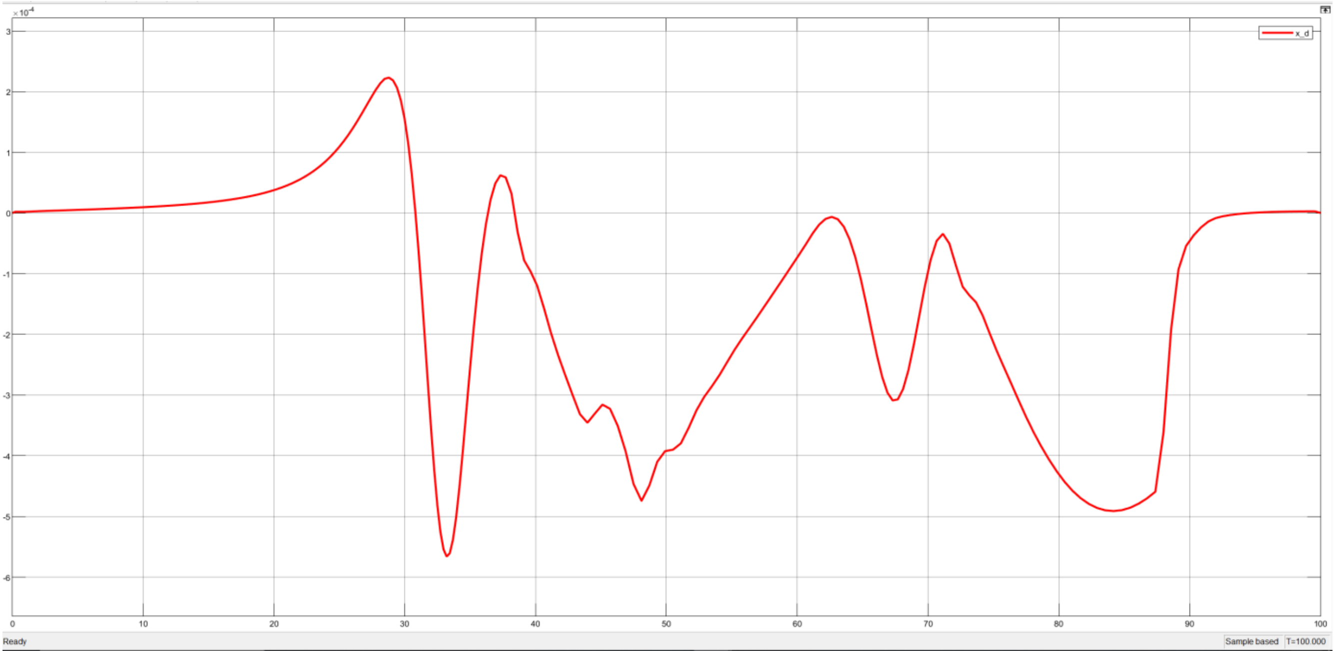


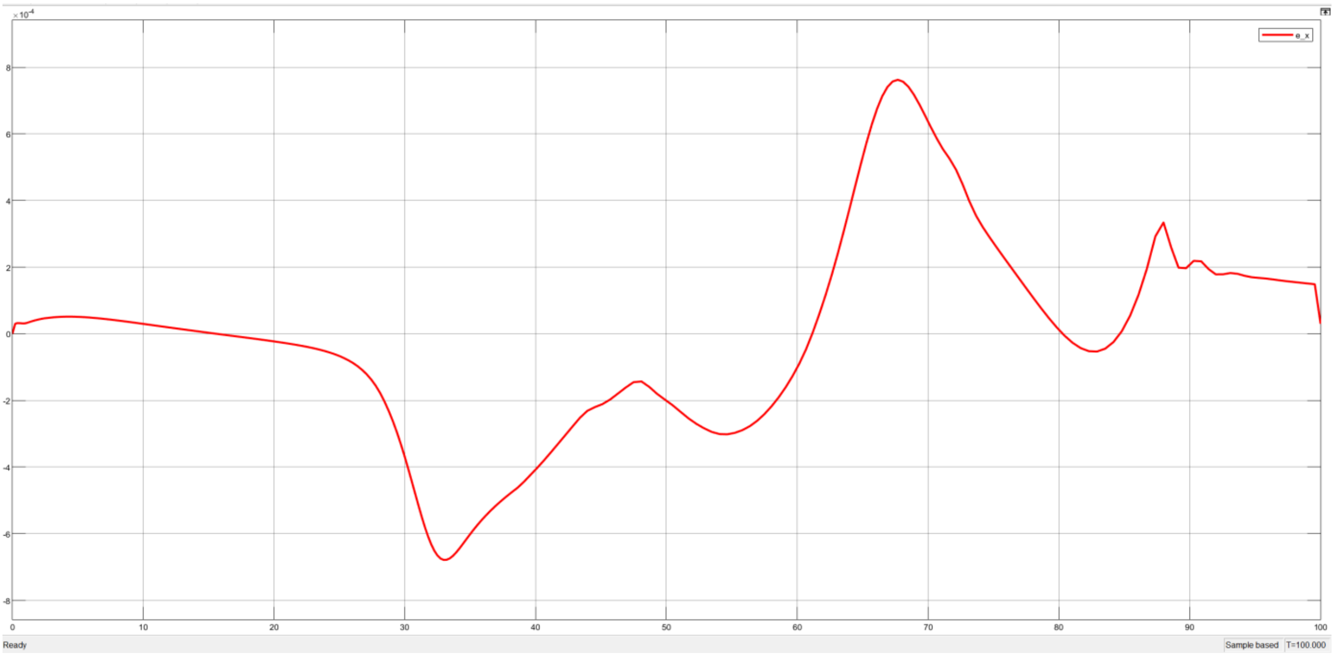
And also the orientation coincides with the one of the previous exercise.

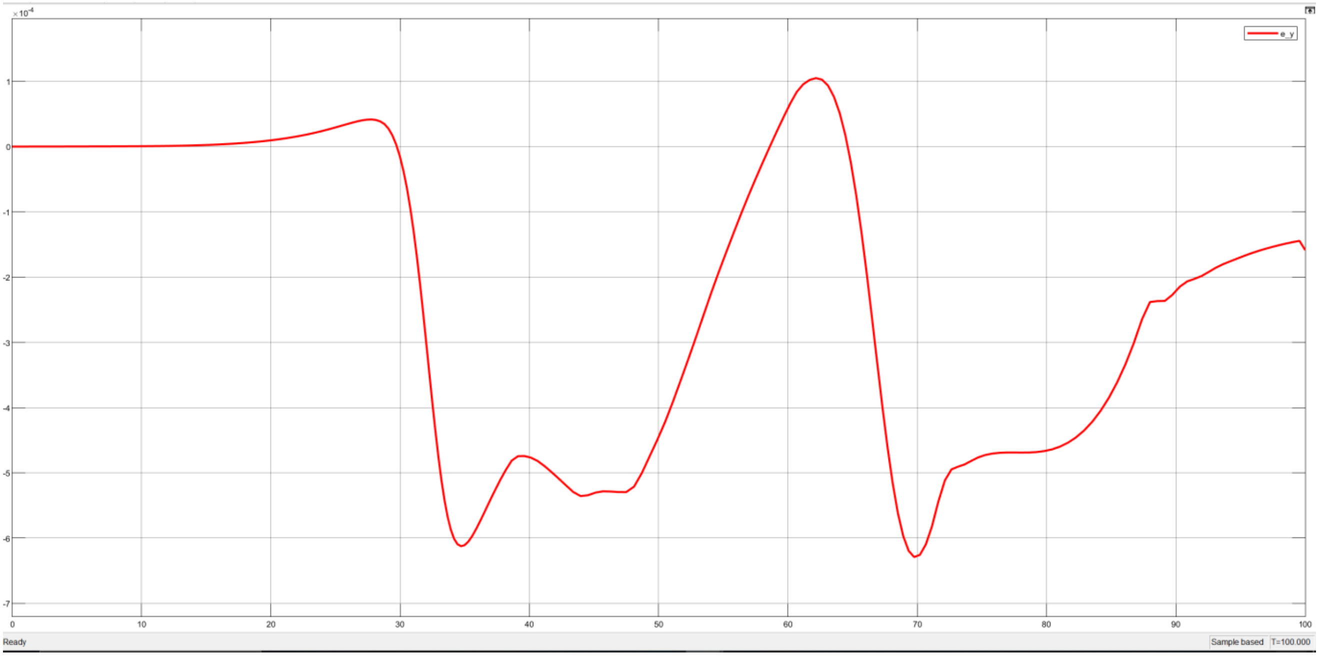


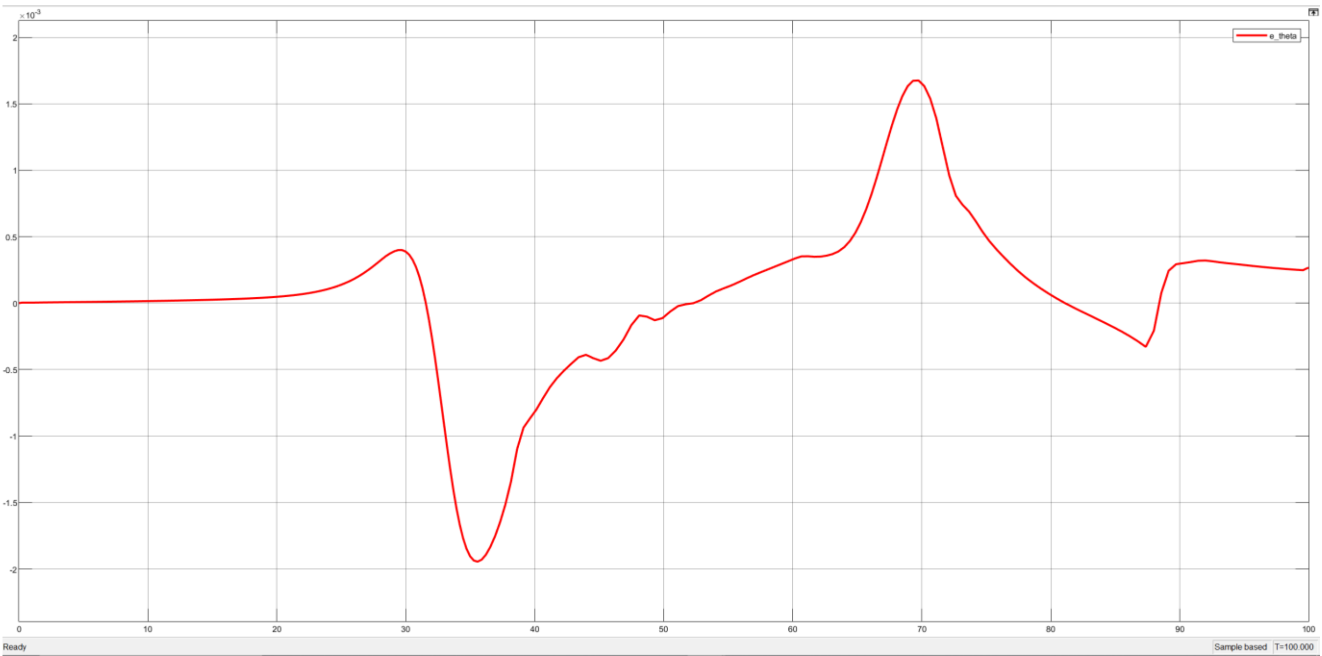
To make sure that the controller works as it should, I plot the errors $e_{y1}, e_{y2}, e_x, e_y, e_\theta$ with respect to the desired values.




× 10⁻⁴
x_d
Ready
Sample based
T=100.000


× 10⁻⁴
e_x
Ready
Sample based
T=100.000

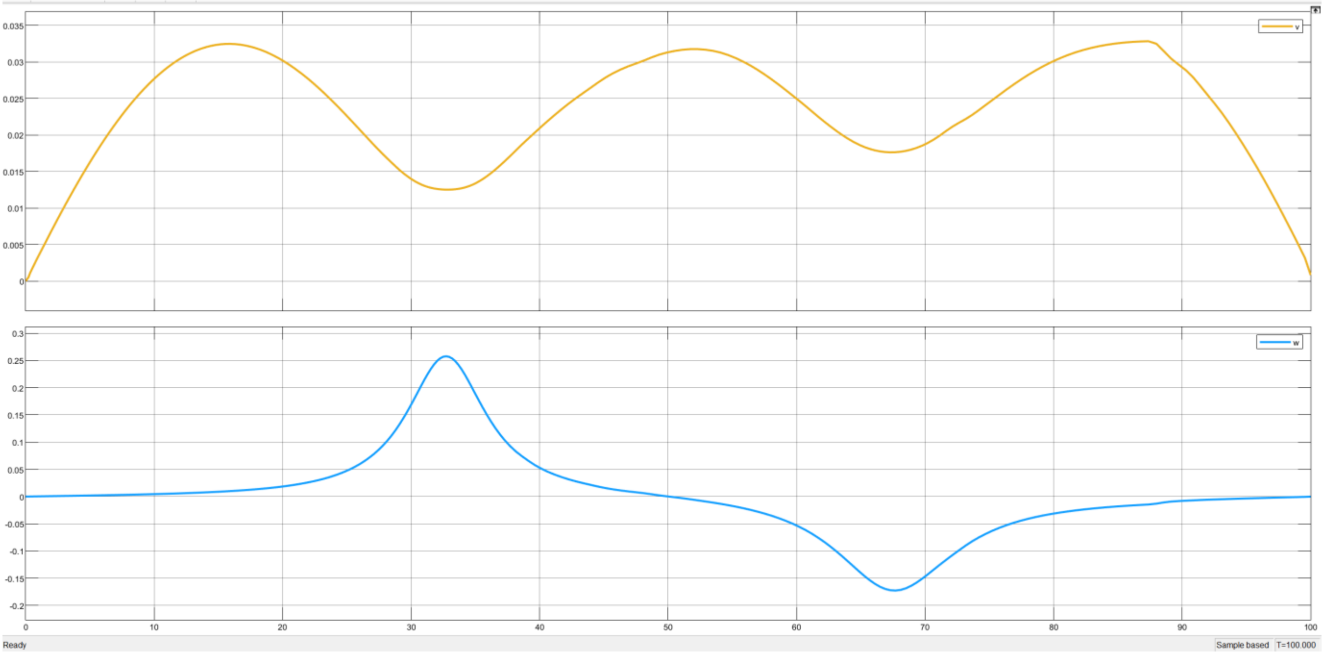




As we can see the errors are of the order of 10^{-4} so we can affirm that the controller works well.

An interesting feature is that the controller left the orientation uncontrolled, but despite that the error on theta is of the order of 10^{-3} which we can assume to be small.

At last, I plot the heading and the angular velocity and as we can see they still maintain their bounds.



Exercise 5

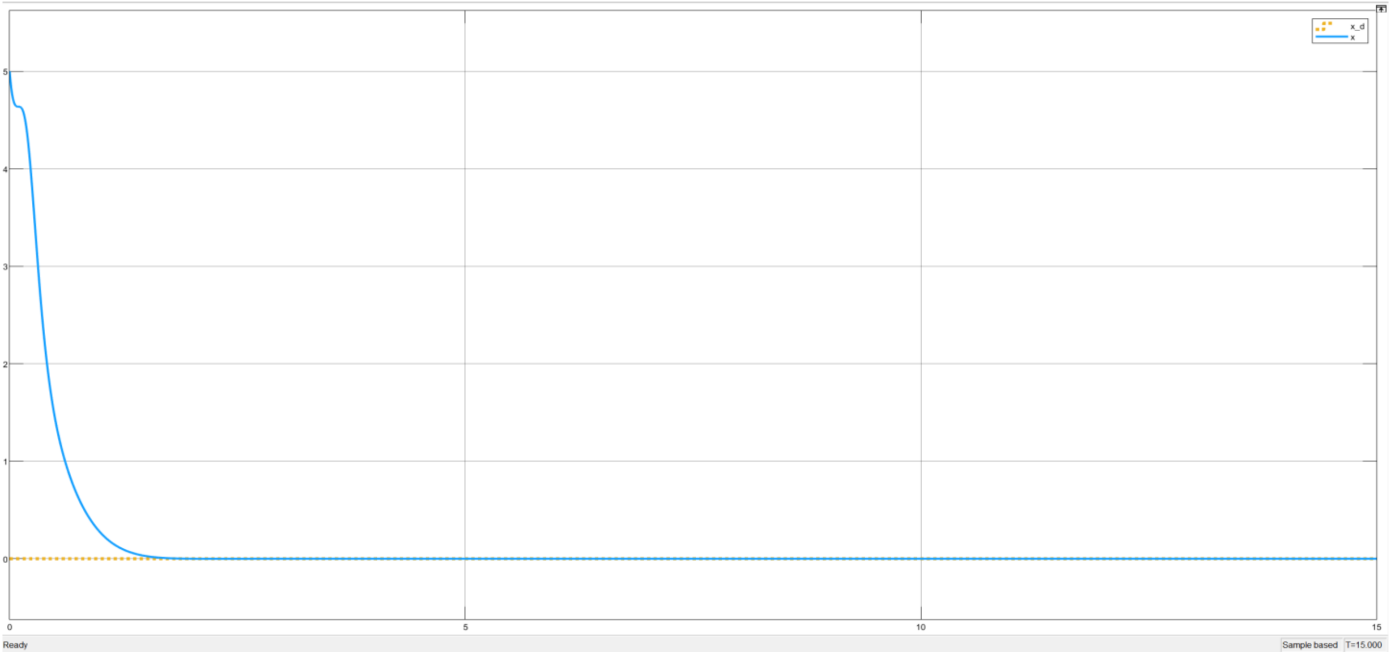
In this exercise our goal is to implement a posture regulator for the unicycle based on polar coordinates, with the state feedback computed through the Runge-Kutta odometric localization method.

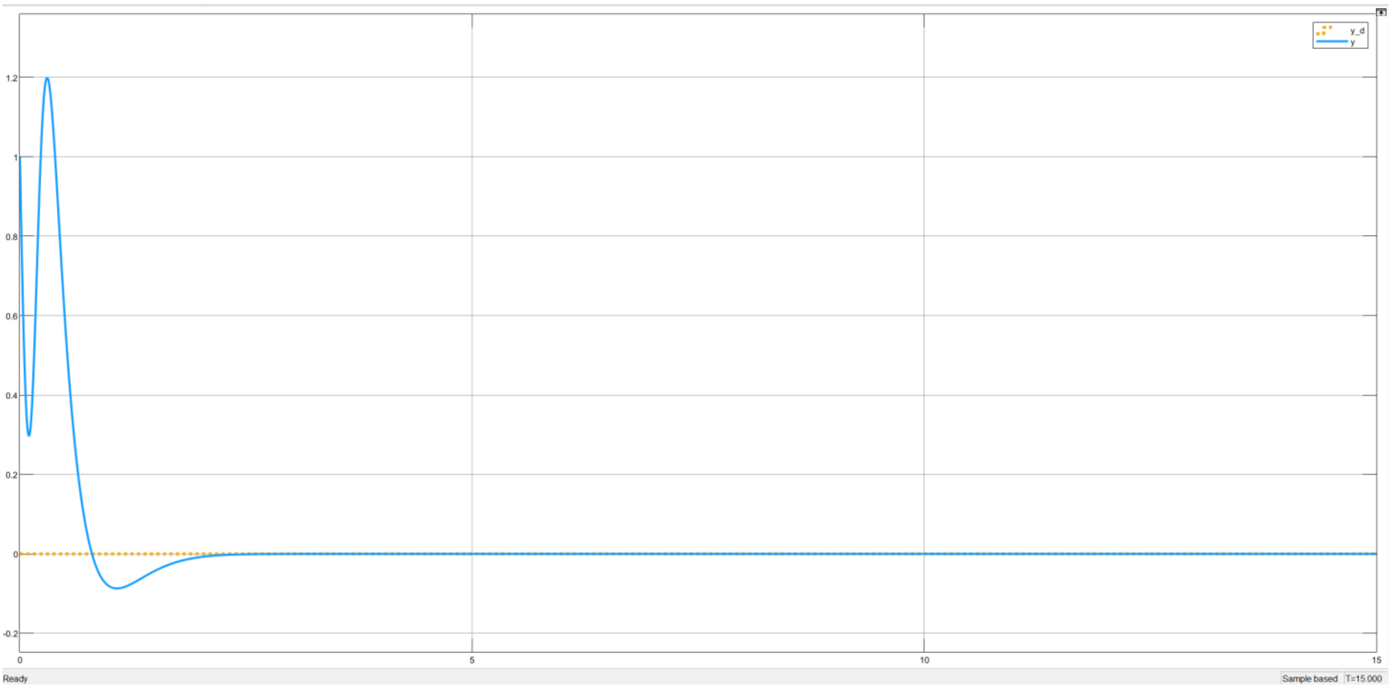
The initial configuration is $q_i = [5 \;\; 1 \;\; \pi/4]^T$ and the final configuration is $q_f = [0 \;\; 0 \;\; 0]^T$. As first thing I discretize $x, y, \theta, v \; and \; \omega$. Then I instantiate a block function which implement $2nd$ Order Runge-Kutta Approximation where I choose a simple time equal to Ts = 0.001 s. The controller works with polar coordinates, so we need a block to convert the cartesian coordinates into the new ones, that will be the input of our regulator:

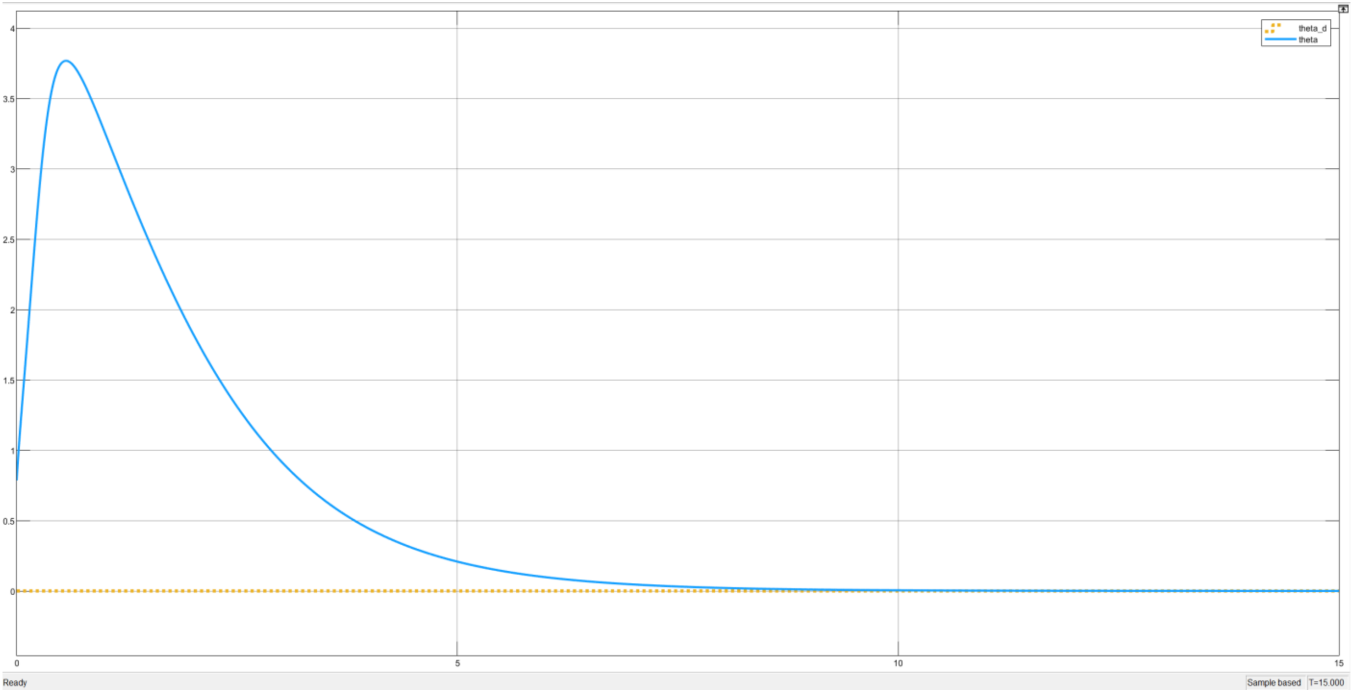
$$\begin{cases} v = k_1 \rho cos(\gamma) \\ \omega = k_2 \gamma + k_1 sin(\gamma) cos(\gamma) \left(1 + k_3 \frac{\delta}{\gamma}\right) \end{cases} \quad with: \quad k_1, k_2, k_3 > 0$$

Where I've chosen as gains the following ones: k1 = 5, k2=6, k3= 0.1.

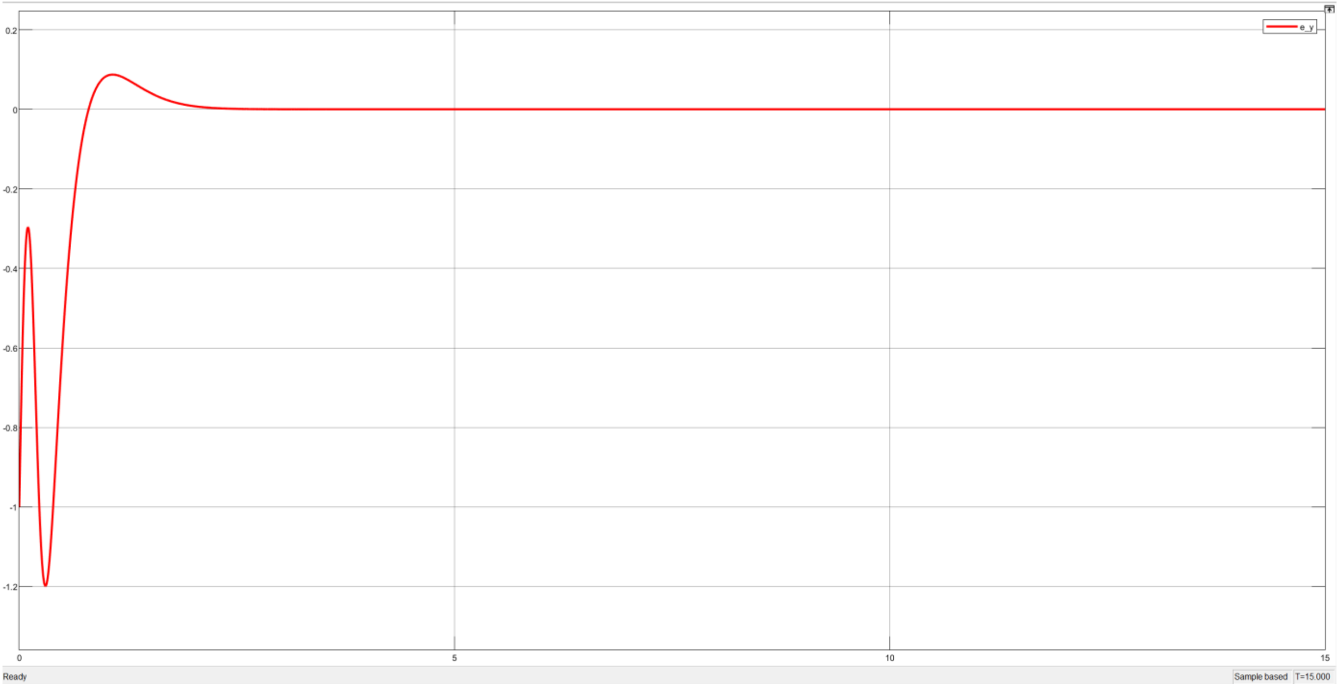
As last thing I plot the graphs for x, y, θ and we can observe that all the variables reaching 0 in a time of 15 seconds, so we can conclude that we have achieved the desired posture.

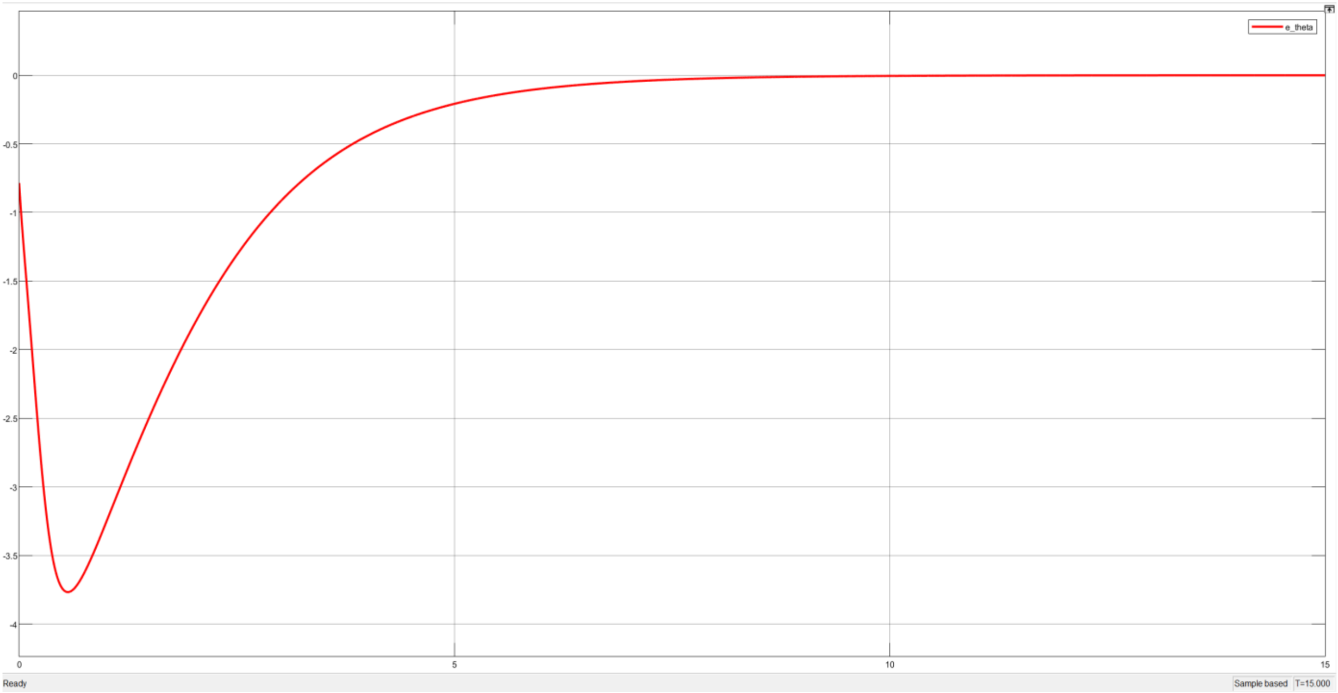

x_d
x
5
4
3
2
1
0
0
5
10
15
Ready
Sample based
T=15.000

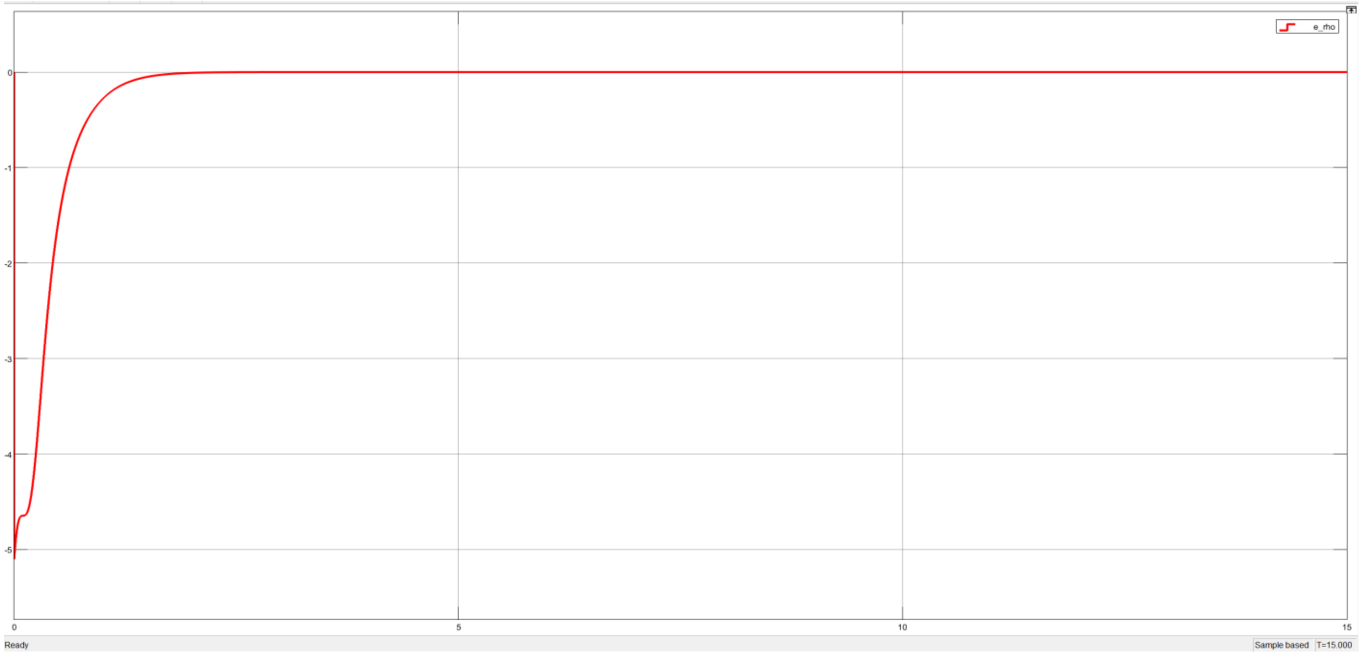

y_d
y
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
5
10
15
Ready
Sample based
T=15.000

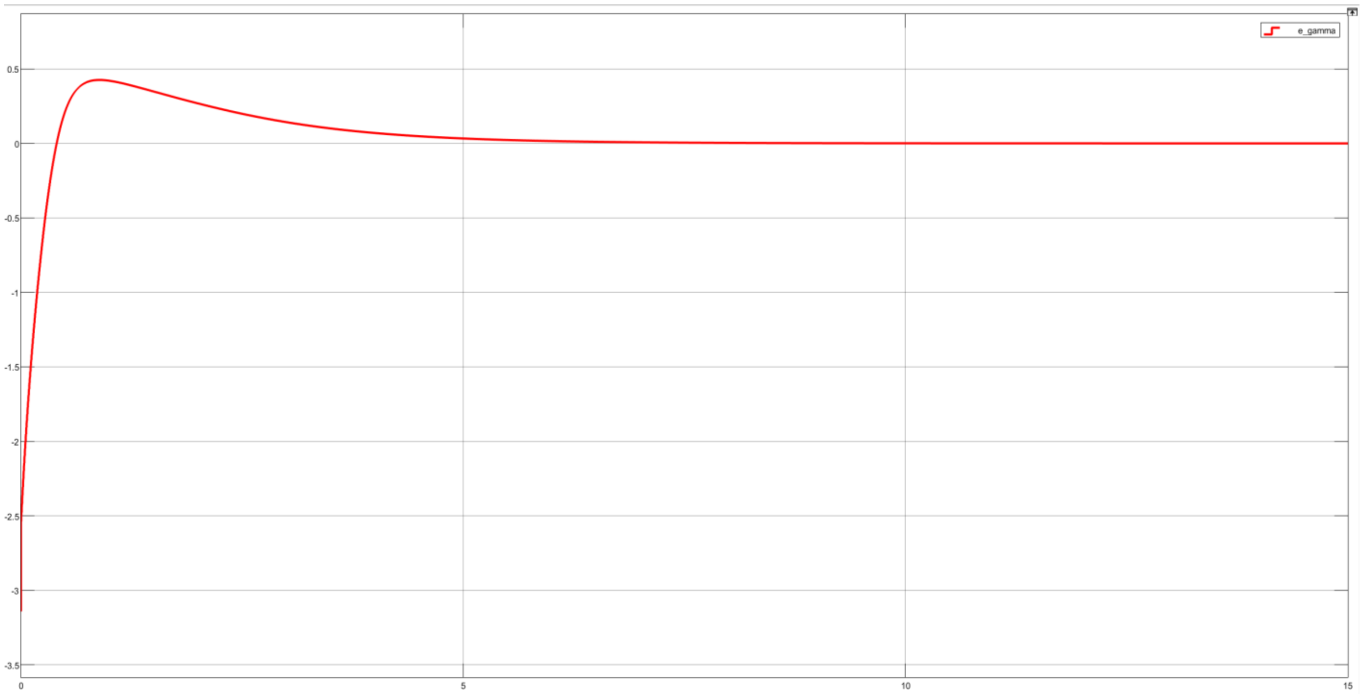


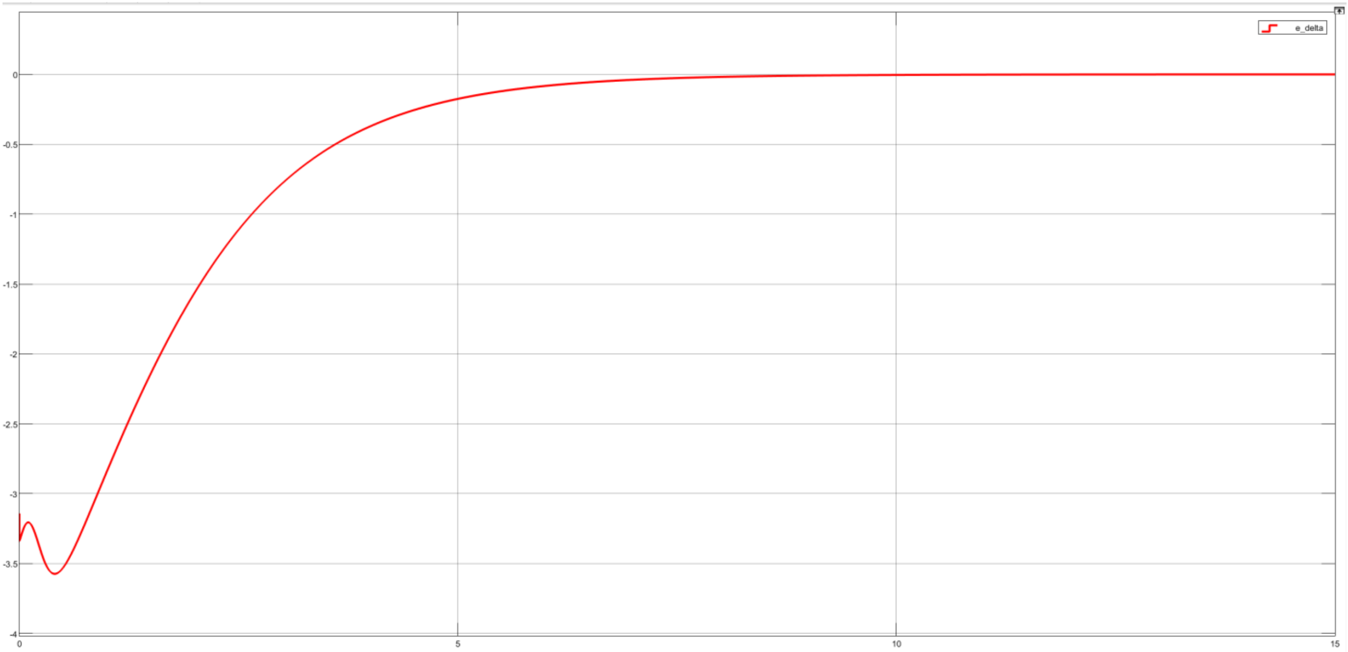
To complete the analysis and to ensure that the controller adjustment is reliable, I show the errors on x, y,θ,ρ,γ,δ


e_y
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
0
5
10
15
Ready
Sample based
T=15.000


e_theta
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
0
5
10
15
Ready
Sample based
T=15.000


e_rho
0
-1
-2
-3
-4
-5
0
5
10
15
Ready
Sample based
T=15.000


e_gamma
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
0
5
10
15



As we can notice all the errors converge to 0, so in the end we can affirm that everything works correctly.