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# **Sage Reference Manual: Functions**

***Release 6.7***

**The Sage Development Team**

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## CONTENTS

<b>1</b>	<b>Logarithmic functions</b>	<b>1</b>
<b>2</b>	<b>Trigonometric Functions</b>	<b>9</b>
<b>3</b>	<b>Hyperbolic Functions</b>	<b>19</b>
<b>4</b>	<b>Transcendental Functions</b>	<b>27</b>
<b>5</b>	<b>Piecewise-defined Functions</b>	<b>33</b>
<b>6</b>	<b>Spike Functions</b>	<b>51</b>
<b>7</b>	<b>Orthogonal Polynomials</b>	<b>53</b>
<b>8</b>	<b>Other functions</b>	<b>65</b>
<b>9</b>	<b>Special Functions</b>	<b>87</b>
<b>10</b>	<b>Hypergeometric Functions</b>	<b>95</b>
<b>11</b>	<b>Jacobi Elliptic Functions</b>	<b>103</b>
<b>12</b>	<b>Airy Functions</b>	<b>109</b>
<b>13</b>	<b>Bessel Functions</b>	<b>115</b>
<b>14</b>	<b>Exponential Integrals</b>	<b>125</b>
<b>15</b>	<b>Wigner, Clebsch-Gordan, Racah, and Gaunt coefficients</b>	<b>137</b>
<b>16</b>	<b>Generalized Functions</b>	<b>145</b>
<b>17</b>	<b>Counting Primes</b>	<b>149</b>
<b>18</b>	<b>Symbolic Minimum and Maximum</b>	<b>153</b>
<b>19</b>	<b>Indices and Tables</b>	<b>155</b>
	<b>Bibliography</b>	<b>157</b>



## LOGARITHMIC FUNCTIONS

## AUTHORS:

- Yoora Yi Tenen (2012-11-16): Add documentation for `log()` ([trac ticket #12113](#))

**class** `sage.functions.log.Function_dilog`

Bases: `sage.symbolic.function.GinacFunction`

The dilogarithm function  $\text{Li}_2(z) = \sum_{k=1}^{\infty} z^k/k^2$ .

This is simply an alias for `polylog(2, z)`.

## EXAMPLES:

**sage:** `dilog(1)`

`1/6*pi^2`

**sage:** `dilog(1/2)`

`1/12*pi^2 - 1/2*log(2)^2`

**sage:** `dilog(x^2+1)`

`dilog(x^2 + 1)`

**sage:** `dilog(-1)`

`-1/12*pi^2`

**sage:** `dilog(-1.1)`

`-0.890838090262283`

**sage:** `float(dilog(1))`

`1.6449340668482262`

**sage:** `var('z')`

`z`

**sage:** `dilog(z).diff(z, 2)`

`log(-z + 1)/z^2 - 1/((z - 1)*z)`

**sage:** `dilog(z).series(z==1/2, 3)`

`(1/12*pi^2 - 1/2*log(2)^2) + (-2*log(1/2))*(z - 1/2) + (2*log(1/2) + 2)*(z - 1/2)^2 + Order(1/8*`

**sage:** `latex(dilog(z))`

`{\rm Li}_2\left(z\right)`

## TESTS:

`conjugate(dilog(x))==dilog(conjugate(x))` unless on the branch cuts which run along the positive real axis beginning at 1.:

**sage:** `conjugate(dilog(x))`

`conjugate(dilog(x))`

**sage:** `var('y', domain='positive')`

`y`

**sage:** `conjugate(dilog(y))`

`conjugate(dilog(y))`

**sage:** `conjugate(dilog(1/19))`

`dilog(1/19)`

```
sage: conjugate(dilog(1/2*I))
dilog(-1/2*I)
sage: dilog(conjugate(1/2*I))
dilog(-1/2*I)
sage: conjugate(dilog(2))
conjugate(dilog(2))
```

**class** sage.functions.log.**Function\_exp**  
Bases: sage.symbolic.function.GinacFunction

The exponential function,  $\exp(x) = e^x$ .

EXAMPLES:

```
sage: exp(-1)
e^(-1)
sage: exp(2)
e^2
sage: exp(2).n(100)
7.3890560989306502272304274606
sage: exp(x^2 + log(x))
e^(x^2 + log(x))
sage: exp(x^2 + log(x)).simplify()
x*e^(x^2)
sage: exp(2.5)
12.1824939607035
sage: exp(float(2.5))
12.182493960703473
sage: exp(RDF('2.5'))
12.182493960703473
```

To prevent automatic evaluation, use the `hold` parameter:

```
sage: exp(I*pi, hold=True)
e^(I*pi)
sage: exp(0, hold=True)
e^0
```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```
sage: exp(0, hold=True).simplify()
1
```

```
sage: exp(pi*I/2)
I
sage: exp(pi*I)
-1
sage: exp(8*pi*I)
1
sage: exp(7*pi*I/2)
-I
```

The precision for the result is deduced from the precision of the input. Convert the input to a higher precision explicitly if a result with higher precision is desired:

```
sage: t = exp(RealField(100)(2)); t
7.3890560989306502272304274606
sage: t.prec()
100
sage: exp(2).n(100)
7.3890560989306502272304274606
```

TEST:

```
sage: latex(exp(x))
e^{x}
sage: latex(exp(sqrt(x)))
e^{\sqrt{x}}
sage: latex(exp)
\exp
sage: latex(exp(sqrt(x))^x)
\left(e^{\sqrt{x}}\right)^{x}
sage: latex(exp(sqrt(x)^x))
e^{\left(\sqrt{x}\right)^{x}}
```

Test conjugates:

```
sage: conjugate(exp(x))
e^conjugate(x)
```

Test simplifications when taking powers of exp, #7264:

```
sage: var('a,b,c,II')
(a, b, c, II)
sage: model_exp = exp(II)**a*(b)
sage: sol1_l={b: 5.0, a: 1.1}
sage: model_exp.subs(sol1_l)
5.000000000000000*(e^II)^1.100000000000000

sage: exp(3)^II*exp(x)
(e^3)^II*e^x
sage: exp(x)*exp(x)
e^(2*x)
sage: exp(x)*exp(a)
e^(a + x)
sage: exp(x)*exp(a)^2
e^(2*a + x)
```

Another instance of the same problem, #7394:

```
sage: 2*sqrt(e)
2*sqrt(e)
```

```
class sage.functions.log.Function_lambert_w
Bases: sage.symbolic.function.BuiltinFunction
```

The integral branches of the Lambert W function  $W_n(z)$ .

This function satisfies the equation

$$z = W_n(z)e^{W_n(z)}$$

INPUT:

- $n$  - an integer.  $n = 0$  corresponds to the principal branch.
- $z$  - a complex number

If called with a single argument, that argument is  $z$  and the branch  $n$  is assumed to be 0 (the principal branch).

ALGORITHM:

Numerical evaluation is handled using the mpmath and SciPy libraries.

## REFERENCES:

- [Wikipedia article Lambert\\_W\\_function](#)

## EXAMPLES:

Evaluation of the principal branch:

```
sage: lambert_w(1.0)
0.567143290409784
sage: lambert_w(-1).n()
-0.318131505204764 + 1.33723570143069*I
sage: lambert_w(-1.5 + 5*I)
1.17418016254171 + 1.10651494102011*I
```

Evaluation of other branches:

```
sage: lambert_w(2, 1.0)
-2.40158510486800 + 10.7762995161151*I
```

Solutions to certain exponential equations are returned in terms of `lambert_w`:

```
sage: S = solve(e^(5*x)+x==0, x, to_poly_solve=True)
sage: z = S[0].rhs(); z
-1/5*lambert_w(5)
sage: N(z)
-0.265344933048440
```

Check the defining equation numerically at  $z = 5$ :

```
sage: N(lambert_w(5)*exp(lambert_w(5)) - 5)
0.000000000000000
```

There are several special values of the principal branch which are automatically simplified:

```
sage: lambert_w(0)
0
sage: lambert_w(e)
1
sage: lambert_w(-1/e)
-1
```

Integration (of the principal branch) is evaluated using Maxima:

```
sage: integrate(lambert_w(x), x)
(lambert_w(x)^2 - lambert_w(x) + 1)*x/lambert_w(x)
sage: integrate(lambert_w(x), x, 0, 1)
(lambert_w(1)^2 - lambert_w(1) + 1)/lambert_w(1) - 1
sage: integrate(lambert_w(x), x, 0, 1.0)
0.3303661247616807
```

Warning: The integral of a non-principal branch is not implemented, neither is numerical integration using GSL. The `numerical_integral()` function does work if you pass a lambda function:

```
sage: numerical_integral(lambda x: lambert_w(x), 0, 1)
(0.33036612476168054, 3.667800782666048e-15)
```

**class** `sage.functions.log.Function_log`

Bases: `sage.symbolic.function.GinacFunction`

The natural logarithm of  $x$ . See *log?* for more information about its behavior.

## EXAMPLES:



```

sage: ln(e^2)
2
sage: ln(2)
log(2)
sage: ln(10)
log(10)

sage: ln(RDF(10))
2.302585092994046
sage: ln(2.718)
0.999896315728952
sage: ln(2.0)
0.693147180559945
sage: ln(float(-1))
3.141592653589793j
sage: ln(complex(-1))
3.141592653589793j

```

The `hold` parameter can be used to prevent automatic evaluation:

```

sage: log(-1, hold=True)
log(-1)
sage: log(-1)
I*pi
sage: I.log(hold=True)
log(I)
sage: I.log(hold=True).simplify()
1/2*I*pi

```

#### TESTS:

```

sage: latex(x.log())
\log\left(x\right)
sage: latex(log(1/4))
\log\left(\frac{1}{4}\right)
sage: loads(dumps(ln(x)+1))
log(x) + 1

```

$\text{conjugate}(\log(x)) == \log(\text{conjugate}(x))$  unless on the branch cut which runs along the negative real axis.:

```

sage: conjugate(log(x))
conjugate(log(x))
sage: var('y', domain='positive')
y
sage: conjugate(log(y))
log(y)
sage: conjugate(log(y+I))
conjugate(log(y + I))
sage: conjugate(log(-1))
-I*pi
sage: log(conjugate(-1))
I*pi

```

Check if float arguments are handled properly.:

```

sage: from sage.functions.log import function_log as log
sage: log(float(5))
1.6094379124341003

```

```
sage: log(float(0))
-inf
sage: log(float(-1))
3.141592653589793j
sage: log(x).subs(x=float(-1))
3.141592653589793j
```

**class** sage.functions.log.**Function\_polylog**  
Bases: sage.symbolic.function.GinacFunction

The polylog function  $\text{Li}_n(z) = \sum_{k=1}^{\infty} z^k/k^n$ .

INPUT:

- n - object
- z - object

EXAMPLES:

```
sage: polylog(1, x)
-log(-x + 1)
sage: polylog(2, 1)
1/6*pi^2
sage: polylog(2, x^2+1)
polylog(2, x^2 + 1)
sage: polylog(4, 0.5)
polylog(4, 0.5000000000000000)

sage: f = polylog(4, 1); f
1/90*pi^4
sage: f.n()
1.08232323371114

sage: polylog(4, 2).n()
2.42786280675470 - 0.174371300025453*I
sage: complex(polylog(4, 2))
(2.4278628067547032-0.17437130002545306j)
sage: float(polylog(4, 0.5))
0.5174790616738993

sage: z = var('z')
sage: polylog(2, z).series(z==0, 5)
1*z + 1/4*z^2 + 1/9*z^3 + 1/16*z^4 + Order(z^5)

sage: loads(dumps(polylog))
polylog

sage: latex(polylog(5, x))
{\rm Li}_{5}(x)
```

TESTS:

Check if #8459 is fixed:

```
sage: t = maxima(polylog(5, x)).sage(); t
polylog(5, x)
sage: t.operator() == polylog
True
sage: t.subs(x=.5).n()
0.508400579242269
```





## TRIGONOMETRIC FUNCTIONS

```
class sage.functions.trig.Function_arccos
    Bases: sage.symbolic.function.GinacFunction
```

The arccosine function.

EXAMPLES:

```
sage: arccos(0.5)
1.04719755119660
sage: arccos(1/2)
1/3*pi
sage: arccos(1 + 1.0*I)
0.904556894302381 - 1.06127506190504*I
sage: arccos(3/4).n(100)
0.72273424781341561117837735264
```

We can delay evaluation using the hold parameter:

```
sage: arccos(0, hold=True)
arccos(0)
```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```
sage: a = arccos(0, hold=True); a.simplify()
1/2*pi
```

$\text{conjugate}(\arccos(x)) == \arccos(\text{conjugate}(x))$ , unless on the branch cuts, which run along the real axis outside the interval  $[-1, +1]$ :

```
sage: conjugate(arccos(x))
conjugate(arccos(x))
sage: var('y', domain='positive')
y
sage: conjugate(arccos(y))
conjugate(arccos(y))
sage: conjugate(arccos(y+I))
conjugate(arccos(y + I))
sage: conjugate(arccos(1/16))
arccos(1/16)
sage: conjugate(arccos(2))
conjugate(arccos(2))
sage: conjugate(arccos(-2))
pi - conjugate(arccos(2))
```

TESTS:

```
sage: arccos(x).operator()  
arccos
```

**class** sage.functions.trig.**Function\_arccot**  
Bases: sage.symbolic.function.BuiltinFunction

The arccotangent function.

EXAMPLES:

```
sage: arccot(1/2)  
arccot(1/2)  
sage: RDF(arccot(1/2))  
1.1071487177940904  
sage: arccot(1 + I)  
arccot(I + 1)
```

We can delay evaluation using the hold parameter:

```
sage: arccot(1,hold=True)  
arccot(1)
```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```
sage: a = arccot(1,hold=True); a.simplify()  
1/4*pi
```

**class** sage.functions.trig.**Function\_arccsc**  
Bases: sage.symbolic.function.BuiltinFunction

The arccosecant function.

EXAMPLES:

```
sage: arccsc(2)  
arccsc(2)  
sage: RDF(arccsc(2)) # rel tol 1e-15  
0.5235987755982988  
sage: arccsc(1 + I)  
arccsc(I + 1)
```

We can delay evaluation using the hold parameter:

```
sage: arccsc(1,hold=True)  
arccsc(1)
```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```
sage: a = arccsc(1,hold=True); a.simplify()  
1/2*pi
```

**class** sage.functions.trig.**Function\_arcsec**  
Bases: sage.symbolic.function.BuiltinFunction

The arcsecant function.

EXAMPLES:

```
sage: arcsec(2)  
arcsec(2)  
sage: arcsec(2.0)  
1.04719755119660
```

```

sage: RDF(arcsec(2)) # abs tol 1e-15
1.0471975511965976
sage: arcsec(1 + I)
arcsec(I + 1)

```

We can delay evaluation using the hold parameter:

```

sage: arcsec(1,hold=True)
arcsec(1)

```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```

sage: a = arcsec(1,hold=True); a.simplify()
0

```

```

class sage.functions.trig.Function_arcsin
Bases: sage.symbolic.function.GinacFunction

```

The arcsine function.

EXAMPLES:

```

sage: arcsin(0.5)
0.523598775598299
sage: arcsin(1/2)
1/6*pi
sage: arcsin(1 + 1.0*I)
0.666239432492515 + 1.06127506190504*I

```

We can delay evaluation using the hold parameter:

```

sage: arcsin(0,hold=True)
arcsin(0)

```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```

sage: a = arcsin(0,hold=True); a.simplify()
0

```

`conjugate(arcsin(x)) == arcsin(conjugate(x))`, unless on the branch cuts which run along the real axis outside the interval  $[-1, +1]$ :

```

sage: conjugate(arcsin(x))
conjugate(arcsin(x))
sage: var('y', domain='positive')
y
sage: conjugate(arcsin(y))
conjugate(arcsin(y))
sage: conjugate(arcsin(y+I))
conjugate(arcsin(y + I))
sage: conjugate(arcsin(1/16))
arcsin(1/16)
sage: conjugate(arcsin(2))
conjugate(arcsin(2))
sage: conjugate(arcsin(-2))
-conjugate(arcsin(2))

```

TESTS:

```

sage: arcsin(x).operator()
arcsin

```

```
class sage.functions.trig.Function_arctan
    Bases: sage.symbolic.function.GinacFunction
```

The arctangent function.

EXAMPLES:

```
sage: arctan(1/2)
arctan(1/2)
sage: RDF(arctan(1/2)) # rel tol 1e-15
0.46364760900080615
sage: arctan(1 + I)
arctan(I + 1)
sage: arctan(1/2).n(100)
0.46364760900080611621425623146
```

We can delay evaluation using the `hold` parameter:

```
sage: arctan(0, hold=True)
arctan(0)
```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```
sage: a = arctan(0, hold=True); a.simplify()
0
```

$\text{conjugate}(\arctan(x)) == \arctan(\text{conjugate}(x))$ , unless on the branch cuts which run along the imaginary axis outside the interval  $[-I, +I]$ :

```
sage: conjugate(arctan(x))
conjugate(arctan(x))
sage: var('y', domain='positive')
y
sage: conjugate(arctan(y))
arctan(y)
sage: conjugate(arctan(y+I))
conjugate(arctan(y + I))
sage: conjugate(arctan(1/16))
arctan(1/16)
sage: conjugate(arctan(-2*I))
conjugate(arctan(-2*I))
sage: conjugate(arctan(2*I))
conjugate(arctan(2*I))
sage: conjugate(arctan(I/2))
arctan(-1/2*I)
```

TESTS:

```
sage: arctan(x).operator()
arctan
```

```
class sage.functions.trig.Function_arctan2
    Bases: sage.symbolic.function.GinacFunction
```

The modified arctangent function.

Returns the arc tangent (measured in radians) of  $y/x$ , where unlike `arctan(y/x)`, the signs of both  $x$  and  $y$  are considered. In particular, this function measures the angle of a ray through the origin and  $(x, y)$ , with the positive  $x$ -axis the zero mark, and with output angle  $\theta$  being between  $-\pi < \theta \leq \pi$ .

Hence,  $\text{arctan2}(y, x) = \text{arctan}(y/x)$  only for  $x > 0$ . One may consider the usual `arctan` to measure angles of lines through the origin, while the modified function measures rays through the origin.



Note that the  $y$ -coordinate is by convention the first input.

#### EXAMPLES:

Note the difference between the two functions:

```
sage: arctan2(1, -1)
3/4*pi
sage: arctan(1/-1)
-1/4*pi
```

This is consistent with Python and Maxima:

```
sage: maxima.atan2(1, -1)
3*%pi/4
sage: math.atan2(1, -1)
2.356194490192345
```

More examples:

```
sage: arctan2(1, 0)
1/2*pi
sage: arctan2(2, 3)
arctan(2/3)
sage: arctan2(-1, -1)
-3/4*pi
```

Of course we can approximate as well:

```
sage: arctan2(-1/2, 1).n(100)
-0.46364760900080611621425623146
sage: arctan2(2, 3).n(100)
0.58800260354756755124561108063
```

We can delay evaluation using the `hold` parameter:

```
sage: arctan2(-1/2, 1, hold=True)
arctan2(-1/2, 1)
```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```
sage: arctan2(-1/2, 1, hold=True).simplify()
-arctan(1/2)
```

The function also works with numpy arrays as input:

```
sage: import numpy
sage: a = numpy.linspace(1, 3, 3)
sage: b = numpy.linspace(3, 6, 3)
sage: atan2(a, b)
array([ 0.32175055,  0.41822433,  0.46364761])

sage: atan2(1, a)
array([ 0.78539816,  0.46364761,  0.32175055])

sage: atan2(a, 1)
array([ 0.78539816,  1.10714872,  1.24904577])
```

#### TESTS:

```
sage: x, y = var('x, y')
sage: arctan2(y, x).operator()
arctan2
```

Check if [trac ticket #8565](#) is fixed:

```
sage: atan2(-pi, 0)
-1/2*pi
```

Check if [trac ticket #8564](#) is fixed:

```
sage: arctan2(x, x).__sympy__()
atan2(x, x)
```

Check if numerical evaluation works [trac ticket #9913](#):

```
sage: arctan2(0, -log(2)).n()
3.14159265358979
```

Check if `atan2(0,0)` throws error of [trac ticket #11423](#):

```
sage: atan2(0, 0)
Traceback (most recent call last):
...
RuntimeError: arctan2_eval(): arctan2(0, 0) encountered

sage: atan2(0, 0, hold=True)
arctan2(0, 0)

sage: atan2(0, 0, hold=True).n()
Traceback (most recent call last):
...
ValueError: arctan2(0, 0) undefined
```

Check if [trac ticket #10062](#) is fixed, this was caused by `(I*I).is_positive()` returning `True`:

```
sage: arctan2(0, I*I)
pi
```

**class** `sage.functions.trig.Function_cos`  
Bases: `sage.symbolic.function.GinacFunction`

The cosine function.

EXAMPLES:

```
sage: cos(pi)
-1
sage: cos(x).subs(x==pi)
-1
sage: cos(2).n(100)
-0.41614683654714238699756822950
sage: loads(dumps(cos))
cos
```

We can prevent evaluation using the `hold` parameter:

```
sage: cos(0, hold=True)
cos(0)
```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```
sage: a = cos(0, hold=True); a.simplify()
1
```

## TESTS:

```

sage: conjugate(cos(x))
cos(conjugate(x))
sage: cos(complex(1,1))      # rel tol 1e-15
(0.8337300251311491-0.9888977057628651j)

```

**class** sage.functions.trig.**Function\_cot**

Bases: sage.symbolic.function.BuiltinFunction

The cotangent function.

## EXAMPLES:

```

sage: cot(pi/4)
1
sage: RR(cot(pi/4))
1.000000000000000
sage: cot(1/2)
cot(1/2)
sage: cot(0.5)
1.83048772171245

sage: latex(cot(x))
\cot\left(x\right)

```

We can prevent evaluation using the hold parameter:

```

sage: cot(pi/4,hold=True)
cot(1/4*pi)

```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```

sage: a = cot(pi/4,hold=True); a.simplify()
1

```

**class** sage.functions.trig.**Function\_csc**

Bases: sage.symbolic.function.BuiltinFunction

The cosecant function.

## EXAMPLES:

```

sage: csc(pi/4)
sqrt(2)
sage: RR(csc(pi/4))
1.41421356237310
sage: n(csc(pi/4),100)
1.4142135623730950488016887242
sage: csc(1/2)
csc(1/2)
sage: csc(0.5)
2.08582964293349

sage: latex(csc(x))
\csc\left(x\right)

```

We can prevent evaluation using the hold parameter:

```

sage: csc(pi/4,hold=True)
csc(1/4*pi)

```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```
sage: a = csc(pi/4, hold=True); a.simplify()
sqrt(2)
```

**class** `sage.functions.trig.Function_sec`  
Bases: `sage.symbolic.function.BuiltinFunction`

The secant function.

EXAMPLES:

```
sage: sec(pi/4)
sqrt(2)
sage: RR(sec(pi/4))
1.41421356237310
sage: n(sec(pi/4), 100)
1.4142135623730950488016887242
sage: sec(1/2)
sec(1/2)
sage: sec(0.5)
1.13949392732455

sage: latex(sec(x))
\sec\left(x\right)
```

We can prevent evaluation using the `hold` parameter:

```
sage: sec(pi/4, hold=True)
sec(1/4*pi)
```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```
sage: a = sec(pi/4, hold=True); a.simplify()
sqrt(2)
```

**class** `sage.functions.trig.Function_sin`  
Bases: `sage.symbolic.function.GinacFunction`

The sine function.

EXAMPLES:

```
sage: sin(0)
0
sage: sin(x).subs(x==0)
0
sage: sin(2).n(100)
0.90929742682568169539601986591
sage: loads(dumps(sin))
sin
```

We can prevent evaluation using the `hold` parameter:

```
sage: sin(0, hold=True)
sin(0)
```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```
sage: a = sin(0, hold=True); a.simplify()
0
```

TESTS:

```
sage: conjugate(sin(x))
sin(conjugate(x))
sage: sin(complex(1,1))      # rel tol 1e-15
(1.2984575814159773+0.6349639147847361j)
```

**class** sage.functions.trig.**Function\_tan**  
Bases: sage.symbolic.function.GinacFunction

The tangent function.

EXAMPLES:

```
sage: tan(pi)
0
sage: tan(3.1415)
-0.0000926535900581913
sage: tan(3.1415/4)
0.999953674278156
sage: tan(pi/4)
1
sage: tan(1/2)
tan(1/2)
sage: RR(tan(1/2))
0.546302489843790
```

We can prevent evaluation using the `hold` parameter:

```
sage: tan(pi/4,hold=True)
tan(1/4*pi)
```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```
sage: a = tan(pi/4,hold=True); a.simplify()
1
```

TESTS:

```
sage: conjugate(tan(x))
tan(conjugate(x))
sage: tan(complex(1,1))      # rel tol 1e-15
(0.2717525853195118+1.0839233273386946j)
```



## HYPERBOLIC FUNCTIONS

```
class sage.functions.hyperbolic.Function_arccosh
    Bases: sage.symbolic.function.GinacFunction
```

The inverse of the hyperbolic cosine function.

EXAMPLES:

```
sage: arccosh(1/2)
arccosh(1/2)
sage: arccosh(1 + I*1.0)
1.06127506190504 + 0.904556894302381*I
sage: float(arccosh(2))
1.3169578969248168
sage: cosh(float(arccosh(2)))
2.0
```

**Warning:** If the input is in the complex field or symbolic (which includes rational and integer input), the output will be complex. However, if the input is a real decimal, the output will be real or *NaN*. See the examples for details.

```
sage: arccosh(0.5)
NaN
sage: arccosh(1/2)
arccosh(1/2)
sage: arccosh(1/2).n()
NaN
sage: arccosh(CC(0.5))
1.04719755119660*I
sage: arccosh(0)
1/2*I*pi
sage: arccosh(-1)
I*pi
```

To prevent automatic evaluation use the `hold` argument:

```
sage: arccosh(-1, hold=True)
arccosh(-1)
```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```
sage: arccosh(-1, hold=True).simplify()
I*pi
```

`conjugate(arccosh(x)) == arccosh(conjugate(x))` unless on the branch cut which runs along the real axis from `+1` to `-inf`:

```
sage: conjugate(arccosh(x))
conjugate(arccosh(x))
sage: var('y', domain='positive')
y
sage: conjugate(arccosh(y))
conjugate(arccosh(y))
sage: conjugate(arccosh(y+I))
conjugate(arccosh(y + I))
sage: conjugate(arccosh(1/16))
conjugate(arccosh(1/16))
sage: conjugate(arccosh(2))
arccosh(2)
sage: conjugate(arccosh(I/2))
arccosh(-1/2*I)
```

**TESTS:**

```
sage: arccosh(x).operator()
arccosh
sage: latex(arccosh(x))
{\rm arccosh}\left(x\right)
```

**class** sage.functions.hyperbolic.**Function\_arccoth**

Bases: sage.functions.hyperbolic.HyperbolicFunction

The inverse of the hyperbolic cotangent function.

**EXAMPLES:**

```
sage: arccoth(2.0)
0.549306144334055
sage: arccoth(2)
arccoth(2)
sage: arccoth(1 + I*1.0)
0.402359478108525 - 0.553574358897045*I
sage: arccoth(2).n(200)
0.54930614433405484569762261846126285232374527891137472586735
```

Using first the `.n(53)` method is slightly more precise than converting directly to a float:

```
sage: float(arccoth(2)) # abs tol 1e-16
0.5493061443340548
sage: float(arccoth(2).n(53)) # Correct result to 53 bits
0.5493061443340549
sage: float(arccoth(2).n(100)) # Compute 100 bits and then round to 53
0.5493061443340549
```

**TESTS:**

```
sage: latex(arccoth(x))
{\rm arccoth}\left(x\right)
```

**class** sage.functions.hyperbolic.**Function\_arccsch**

Bases: sage.functions.hyperbolic.HyperbolicFunction

The inverse of the hyperbolic cosecant function.

**EXAMPLES:**

```
sage: arccsch(2.0)
0.481211825059603
```



```

sage: arccsch(2)
arccsch(2)
sage: arccsch(1 + I*1.0)
0.530637530952518 - 0.452278447151191*I
sage: arccsch(1).n(200)
0.88137358701954302523260932497979230902816032826163541075330
sage: float(arccsch(1))
0.881373587019543

sage: latex(arccsch(x))
{\rm arccsch}\left(x\right)

```

**class** sage.functions.hyperbolic.**Function\_arcsech**  
 Bases: sage.functions.hyperbolic.HyperbolicFunction

The inverse of the hyperbolic secant function.

EXAMPLES:

```

sage: arcsech(0.5)
1.31695789692482
sage: arcsech(1/2)
arcsech(1/2)
sage: arcsech(1 + I*1.0)
0.530637530952518 - 1.11851787964371*I
sage: arcsech(1/2).n(200)
1.3169578969248167086250463473079684440269819714675164797685
sage: float(arcsech(1/2))
1.3169578969248168

sage: latex(arcsech(x))
{\rm arcsech}\left(x\right)

```

**class** sage.functions.hyperbolic.**Function\_arcsinh**  
 Bases: sage.symbolic.function.GinacFunction

The inverse of the hyperbolic sine function.

EXAMPLES:

```

sage: arcsinh
arcsinh
sage: arcsinh(0.5)
0.481211825059603
sage: arcsinh(1/2)
arcsinh(1/2)
sage: arcsinh(1 + I*1.0)
1.06127506190504 + 0.666239432492515*I

```

To prevent automatic evaluation use the hold argument:

```

sage: arcsinh(-2, hold=True)
arcsinh(-2)

```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```

sage: arcsinh(-2, hold=True).simplify()
-arcsinh(2)

```

`conjugate(arcsinh(x)) == arcsinh(conjugate(x))` unless on the branch cuts which run along the imaginary axis outside the interval  $[-I, +I]$ .

```
sage: conjugate(arcsinh(x))
conjugate(arcsinh(x))
sage: var('y', domain='positive')
y
sage: conjugate(arcsinh(y))
arcsinh(y)
sage: conjugate(arcsinh(y+I))
conjugate(arcsinh(y + I))
sage: conjugate(arcsinh(1/16))
arcsinh(1/16)
sage: conjugate(arcsinh(I/2))
arcsinh(-1/2*I)
sage: conjugate(arcsinh(2*I))
conjugate(arcsinh(2*I))
```

**TESTS:**

```
sage: arcsinh(x).operator()
arcsinh
sage: latex(arcsinh(x))
{\rm arcsinh}\left(x\right)
```

**class** sage.functions.hyperbolic.**Function\_arctanh**

Bases: sage.symbolic.function.GinacFunction

The inverse of the hyperbolic tangent function.

**EXAMPLES:**

```
sage: arctanh(0.5)
0.549306144334055
sage: arctanh(1/2)
arctanh(1/2)
sage: arctanh(1 + I*1.0)
0.402359478108525 + 1.01722196789785*I
```

To prevent automatic evaluation use the hold argument:

```
sage: arctanh(-1/2, hold=True)
arctanh(-1/2)
```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```
sage: arctanh(-1/2, hold=True).simplify()
-arctanh(1/2)
```

`conjugate(arctanh(x)) == arctanh(conjugate(x))` unless on the branch cuts which run along the real axis outside the interval  $[-1, +1]$ :

```
sage: conjugate(arctanh(x))
conjugate(arctanh(x))
sage: var('y', domain='positive')
y
sage: conjugate(arctanh(y))
conjugate(arctanh(y))
sage: conjugate(arctanh(y+I))
conjugate(arctanh(y + I))
sage: conjugate(arctanh(1/16))
arctanh(1/16)
sage: conjugate(arctanh(I/2))
arctanh(-1/2*I)
```

```
sage: conjugate(arctanh(-2*I))
arctanh(2*I)
```

**TESTS:**

```
sage: arctanh(x).operator()
arctanh
sage: latex(arctanh(x))
{\rm arctanh}\left(x\right)
```

```
class sage.functions.hyperbolic.Function_cosh
Bases: sage.symbolic.function.GinacFunction
```

The hyperbolic cosine function.

**EXAMPLES:**

```
sage: cosh(pi)
cosh(pi)
sage: cosh(3.1415)
11.5908832931176
sage: float(cosh(pi))
11.591953275521519
sage: RR(cosh(1/2))
1.12762596520638

sage: latex(cosh(x))
\cosh\left(x\right)
```

To prevent automatic evaluation, use the `hold` parameter:

```
sage: cosh(arcsinh(x), hold=True)
cosh(arcsinh(x))
```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```
sage: cosh(arcsinh(x), hold=True).simplify()
sqrt(x^2 + 1)
```

```
class sage.functions.hyperbolic.Function_coth
Bases: sage.functions.hyperbolic.HyperbolicFunction
```

The hyperbolic cotangent function.

**EXAMPLES:**

```
sage: coth(pi)
coth(pi)
sage: coth(3.1415)
1.00374256795520
sage: float(coth(pi))
1.0037418731973213
sage: RR(coth(pi))
1.00374187319732

sage: latex(coth(x))
\coth\left(x\right)
```

```
class sage.functions.hyperbolic.Function_csch
Bases: sage.functions.hyperbolic.HyperbolicFunction
```

The hyperbolic cosecant function.

EXAMPLES:

```
sage: csch(pi)
csch(pi)
sage: csch(3.1415)
0.0865975907592133
sage: float(csch(pi))
0.0865895375300469...
sage: RR(csch(pi))
0.0865895375300470

sage: latex(csch(x))
{\rm csch}\left(x\right)
```

```
class sage.functions.hyperbolic.Function_sech
Bases: sage.functions.hyperbolic.HyperbolicFunction
```

The hyperbolic secant function.

EXAMPLES:

```
sage: sech(pi)
sech(pi)
sage: sech(3.1415)
0.0862747018248192
sage: float(sech(pi))
0.0862667383340544...
sage: RR(sech(pi))
0.0862667383340544

sage: latex(sech(x))
{\rm sech}\left(x\right)
```

```
class sage.functions.hyperbolic.Function_sinh
Bases: sage.symbolic.function.GinacFunction
```

The hyperbolic sine function.

EXAMPLES:

```
sage: sinh(pi)
sinh(pi)
sage: sinh(3.1415)
11.5476653707437
sage: float(sinh(pi))
11.54873935725774...
sage: RR(sinh(pi))
11.5487393572577

sage: latex(sinh(x))
\sinh\left(x\right)
```

To prevent automatic evaluation, use the `hold` parameter:

```
sage: sinh(arccosh(x), hold=True)
sinh(arccosh(x))
```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```
sage: sinh(arccosh(x), hold=True).simplify()
sqrt(x + 1)*sqrt(x - 1)
```

**class** sage.functions.hyperbolic.**Function\_tanh**  
 Bases: sage.symbolic.function.GinacFunction

The hyperbolic tangent function.

EXAMPLES:

```
sage: tanh(pi)
tanh(pi)
sage: tanh(3.1415)
0.996271386633702
sage: float(tanh(pi))
0.99627207622075
sage: tan(3.1415/4)
0.999953674278156
sage: tanh(pi/4)
tanh(1/4*pi)
sage: RR(tanh(1/2))
0.462117157260010

sage: CC(tanh(pi + I*e))
0.997524731976164 - 0.00279068768100315*I
sage: ComplexField(100)(tanh(pi + I*e))
0.99752473197616361034204366446 - 0.0027906876810031453884245163923*I
sage: CDF(tanh(pi + I*e)) # rel tol 4e-16
0.9975247319761636 - 0.002790687681003147*I
```

To prevent automatic evaluation, use the hold parameter:

```
sage: tanh(arcsinh(x), hold=True)
tanh(arcsinh(x))
```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```
sage: tanh(arcsinh(x), hold=True).simplify()
x/sqrt(x^2 + 1)
```

TESTS:

```
sage: latex(tanh(x))
\tanh\left(x\right)
```

**class** sage.functions.hyperbolic.**HyperbolicFunction**(name, latex\_name=None, conversions=None, evalf\_float=None)

Bases: sage.symbolic.function.BuiltinFunction

Abstract base class for the functions defined in this file.

EXAMPLES:

```
sage: from sage.functions.hyperbolic import HyperbolicFunction
sage: f = HyperbolicFunction('foo', latex_name='\\foo', conversions={'mathematica':'Foo'}, evalf_float=1.0)
sage: f(x)
foo(x)
sage: f(0.5r)
1.0
sage: latex(f(x))
\\foo\\left(x\\right)
```

```
sage: f(x)._mathematica_init_()
'Foo[x]'
```

## TRANSCENDENTAL FUNCTIONS

```
class sage.functions.transcendental.DickmanRho
    Bases: sage.symbolic.function.BuiltinFunction
```

Dickman's function is the continuous function satisfying the differential equation

$$x\rho'(x) + \rho(x-1) = 0$$

with initial conditions  $\rho(x) = 1$  for  $0 \leq x \leq 1$ . It is useful in estimating the frequency of smooth numbers as asymptotically

$$\Psi(a, a^{1/s}) \sim a\rho(s)$$

where  $\Psi(a, b)$  is the number of  $b$ -smooth numbers less than  $a$ .

ALGORITHM:

Dickmans's function is analytic on the interval  $[n, n + 1]$  for each integer  $n$ . To evaluate at  $n + t, 0 \leq t < 1$ , a power series is recursively computed about  $n + 1/2$  using the differential equation stated above. As high precision arithmetic may be needed for intermediate results the computed series are cached for later use.

Simple explicit formulas are used for the intervals  $[0,1]$  and  $[1,2]$ .

EXAMPLES:

[illegible]

**AUTHORS:**

- Robert Bradshaw (2008-09)

REFERENCES:

•G. Marsaglia, A. Zaman, J. Marsaglia. “Numerical Solutions to some Classical Differential-Difference Equations.” *Mathematics of Computation*, Vol. 53, No. 187 (1989).

**approximate** ( $x$ ,  $parent=None$ )  
Approximate using de Bruijn's formula

$$\rho(x) \sim \frac{\exp(-x\xi + Ei(\xi))}{\sqrt{2\pi x\xi}}$$

which is asymptotically equal to Dickman's function, and is much faster to compute.

#### REFERENCES:

- N. De Bruijn, "The Asymptotic behavior of a function occurring in the theory of primes." J. Indian Math Soc. v 15. (1951)

#### EXAMPLES:

```
sage: dickman_rho.approximate(10)
2.41739196365564e-11
sage: dickman_rho(10)
2.77017183772596e-11
sage: dickman_rho.approximate(1000)
4.32938809066403e-3464
```

#### **power\_series** (*n*, *abs\_prec*)

This function returns the power series about  $n + 1/2$  used to evaluate Dickman's function. It is scaled such that the interval  $[n, n + 1]$  corresponds to  $x$  in  $[-1, 1]$ .

#### INPUT:

- n* - the lower endpoint of the interval for which this power series holds
- abs\_prec* - the absolute precision of the resulting power series

#### EXAMPLES:

```
sage: f = dickman_rho.power_series(2, 20); f
-9.9376e-8*x^11 + 3.7722e-7*x^10 - 1.4684e-6*x^9 + 5.8783e-6*x^8 - 0.000024259*x^7 + 0.00010
sage: f(-1), f(0), f(1)
(0.30685, 0.13032, 0.048608)
sage: dickman_rho(2), dickman_rho(2.5), dickman_rho(3)
(0.306852819440055, 0.130319561832251, 0.0486083882911316)
```

#### **class** sage.functions.transcendental.**Function\_HurwitzZeta**

Bases: sage.symbolic.function.BuiltinFunction

#### TESTS:

```
sage: latex(hurwitz_zeta(x, 2))
\zeta\left(x, 2\right)
sage: hurwitz_zeta(x, 2)._sympy_()
zeta(x, 2)
```

#### **class** sage.functions.transcendental.**Function\_zeta**

Bases: sage.symbolic.function.GinacFunction

Riemann zeta function at  $s$  with  $s$  a real or complex number.

#### INPUT:

- s* - real or complex number

If  $s$  is a real number the computation is done using the MPFR library. When the input is not real, the computation is done using the PARI C library.

#### EXAMPLES:

```
sage: zeta(x)
zeta(x)
sage: zeta(2)
1/6*pi^2
sage: zeta(2.)
1.64493406684823
```



```

sage: RR = RealField(200)
sage: zeta(RR(2))
1.6449340668482264364724151666460251892189499012067984377356
sage: zeta(I)
zeta(I)
sage: zeta(I).n()
0.00330022368532410 - 0.418155449141322*I

```

It is possible to use the `hold` argument to prevent automatic evaluation:

```

sage: zeta(2, hold=True)
zeta(2)

```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```

sage: a = zeta(2, hold=True); a.simplify()
1/6*pi^2

```

Check that [trac ticket #15846](#) is resolved:

```

sage: zeta(x).series(x==1, 1)
1*(x - 1)^(-1) + (euler_gamma + log(2) + log(pi) + 2*zetaderiv(1, 0)) + Order(x - 1)
sage: zeta(x).residue(x==1)
1

```

TESTS:

```

sage: latex(zeta(x))
\zeta(x)
sage: a = loads(dumps(zeta(x)))
sage: a.operator() == zeta
True

```

```

sage: zeta(1)
Infinity
sage: zeta(x).subs(x=1)
Infinity

```

```

class sage.functions.transcendental.Function_zetaderiv
Bases: sage.symbolic.function.GinacFunction

```

Derivatives of the Riemann zeta function.

EXAMPLES:

```

sage: zetaderiv(1, x)
zetaderiv(1, x)
sage: zetaderiv(1, x).diff(x)
zetaderiv(2, x)
sage: var('n')
n
sage: zetaderiv(n, x)
zetaderiv(n, x)
sage: zetaderiv(1, 4).n()
-0.0689112658961254
sage: import mpmath; mpmath.diff(lambda x: mpmath.zeta(x), 4)
mpf('-0.068911265896125382')

```

TESTS:

```

sage: latex(zetaderiv(2,x))
\zeta^{\prime}\left(2, x\right)
sage: a = loads(dumps(zetaderiv(2,x)))
sage: a.operator() == zetaderiv
True

```

`sage.functions.transcendental.hurwitz_zeta(s, x, prec=None, **kwargs)`

The Hurwitz zeta function  $\zeta(s, x)$ , where  $s$  and  $x$  are complex.

The Hurwitz zeta function is one of the many zeta functions. It defined as

$$\zeta(s, x) = \sum_{k=0}^{\infty} (k+x)^{-s}.$$

When  $x = 1$ , this coincides with Riemann's zeta function. The Dirichlet L-functions may be expressed as a linear combination of Hurwitz zeta functions.

#### EXAMPLES:

Symbolic evaluations:

```

sage: hurwitz_zeta(x, 1)
zeta(x)
sage: hurwitz_zeta(4, 3)
1/90*pi^4 - 17/16
sage: hurwitz_zeta(-4, x)
-1/5*x^5 + 1/2*x^4 - 1/3*x^3 + 1/30*x
sage: hurwitz_zeta(7, -1/2)
127*zeta(7) - 128
sage: hurwitz_zeta(-3, 1)
1/120

```

Numerical evaluations:

```

sage: hurwitz_zeta(3, 1/2).n()
8.41439832211716
sage: hurwitz_zeta(11/10, 1/2).n()
12.1038134956837
sage: hurwitz_zeta(3, x).series(x, 60).subs(x=0.5).n()
8.41439832211716
sage: hurwitz_zeta(3, 0.5)
8.41439832211716

```

#### REFERENCES:

- [Wikipedia article Hurwitz\\_zeta\\_function](#)

`sage.functions.transcendental.zeta_symmetric(s)`

Completed function  $\xi(s)$  that satisfies  $\xi(s) = \xi(1-s)$  and has zeros at the same points as the Riemann zeta function.

#### INPUT:

- $s$  - real or complex number

If  $s$  is a real number the computation is done using the MPFR library. When the input is not real, the computation is done using the PARI C library.

More precisely,

$$xi(s) = \gamma(s/2 + 1) * (s - 1) * \pi^{-s/2} * \zeta(s).$$

## EXAMPLES:

```
sage: zeta_symmetric(0.7)
0.497580414651127
sage: zeta_symmetric(1-0.7)
0.497580414651127
sage: RR = RealField(200)
sage: zeta_symmetric(RR(0.7))
0.49758041465112690357779107525638385212657443284080589766062
sage: C.<i> = ComplexField()
sage: zeta_symmetric(0.5 + i*14.0)
0.000201294444235258 + 1.49077798716757e-19*I
sage: zeta_symmetric(0.5 + i*14.1)
0.0000489893483255687 + 4.40457132572236e-20*I
sage: zeta_symmetric(0.5 + i*14.2)
-0.0000868931282620101 + 7.11507675693612e-20*I
```

## REFERENCE:

- I copied the definition of  $\xi$  from <http://web.viu.ca/pughg/RiemannZeta/RiemannZetaLong.html>



## PIECEWISE-DEFINED FUNCTIONS

Sage implements a very simple class of piecewise-defined functions. Functions may be any type of symbolic expression. Infinite intervals are not supported. The endpoints of each interval must line up.

TODO:

- Implement max/min location and values,
- Need: parent object - ring of piecewise functions
- This class should derive from an element-type class, and should define `__add__`, `__mul__`, etc. That will automatically take care of left multiplication and proper coercion. The coercion mentioned below for scalar mult on right is bad, since it only allows ints and rationals. The right way is to use an element class and only define `__mul__`, and have a parent, so anything gets coerced properly.

AUTHORS:

- David Joyner (2006-04): initial version
- David Joyner (2006-09): added `__eq__`, `extend_by_zero_to`, `unextend`, `convolution`, `trapezoid`, `trapezoid_integral_approximation`, `riemann_sum`, `riemann_sum_integral_approximation`, `tangent_line` fixed bugs in `__mul__`, `__add__`
- David Joyner (2007-03): adding Hann filter for FS, added general FS filter methods for computing and plotting, added options to plotting of FS (eg, specifying rgb values are now allowed). Fixed bug in documentation reported by Pablo De Napoli.
- David Joyner (2007-09): bug fixes due to behaviour of `SymbolicArithmetic`
- David Joyner (2008-04): fixed docstring bugs reported by J Morrow; added support for Laplace transform of functions with infinite support.
- David Joyner (2008-07): fixed a left multiplication bug reported by C. Boncelet (by defining `__rmul__ = __mul__`).
- Paul Butler (2009-01): added indefinite integration and `default_variable`

TESTS:

```
sage: R.<x> = QQ[]
sage: f = Piecewise([[ (0,1), 1*x^0 ]])
sage: 2*f
Piecewise defined function with 1 parts, [[ (0, 1), 2 ]]
```

`sage.functions.piecewise.Piecewise` (*list\_of\_pairs*, *var=None*)

Returns a piecewise function from a list of (interval, function) pairs.

`list_of_pairs` is a list of pairs (I, fcn), where fcn is a Sage function (such as a polynomial over  $\mathbb{R}$ , or functions using the lambda notation), and I is an interval such as  $I = (1,3)$ . Two consecutive intervals must share a common endpoint.

If the optional `var` is specified, then any symbolic expressions in the list will be converted to symbolic functions using `fcn.function(var)`. (This says which variable is considered to be “piecewise”).

We assume that these definitions are consistent (ie, no checking is done).

EXAMPLES:

```
sage: f1(x) = -1
sage: f2(x) = 2
sage: f = Piecewise([[ (0,pi/2), f1], [(pi/2,pi), f2]])
sage: f(1)
-1
sage: f(3)
2
sage: f = Piecewise([[ (0,1), x], [(1,2), x^2]], x); f
Piecewise defined function with 2 parts, [[(0, 1), x |--> x], [(1, 2), x |--> x^2]]
sage: f(0.9)
0.9000000000000000
sage: f(1.1)
1.2100000000000000
```

**class** `sage.functions.piecewise.PiecewisePolynomial(list_of_pairs, var=None)`  
Returns a piecewise function from a list of (interval, function) pairs.

EXAMPLES:

```
sage: f1(x) = -1
sage: f2(x) = 2
sage: f = Piecewise([[ (0,pi/2), f1], [(pi/2,pi), f2]])
sage: f(1)
-1
sage: f(3)
2
```

**base\_ring()**

Returns the base ring of the function pieces. This is useful when this class is extended.

EXAMPLES:

```
sage: f1(x) = 1
sage: f2(x) = 1-x
sage: f3(x) = x^2-5
sage: f = Piecewise([[ (0,1), f1], [(1,2), f2], [(2,3), f3]])
sage: base_ring(f)
Symbolic Ring

sage: R.<x> = QQ[]
sage: f1 = x^0
sage: f2 = 10*x - x^2
sage: f3 = 3*x^4 - 156*x^3 + 3036*x^2 - 26208*x
sage: f = Piecewise([[ (0,3), f1], [(3,10), f2], [(10,20), f3]])
sage: f.base_ring()
Rational Field
```

**convolution** (*other*)

Returns the convolution function,  $f * g(t) = \int_{-\infty}^{\infty} f(u)g(t-u)du$ , for compactly supported  $f, g$ .

EXAMPLES:

```
sage: x = PolynomialRing(QQ, 'x').gen()
sage: f = Piecewise([[ (0,1), 1*x^0]]) ## example 0
sage: g = f.convolution(f)
```

```

sage: h = f.convolution(g)
sage: P = f.plot(); Q = g.plot(rgbcolor=(1,1,0)); R = h.plot(rgbcolor=(0,1,1));
sage: # Type show(P+Q+R) to view
sage: f = Piecewise([(0,1),1*x^0],[1,2),2*x^0],[2,3),1*x^0]]) ## example 1
sage: g = f.convolution(f)
sage: h = f.convolution(g)
sage: P = f.plot(); Q = g.plot(rgbcolor=(1,1,0)); R = h.plot(rgbcolor=(0,1,1));
sage: # Type show(P+Q+R) to view
sage: f = Piecewise([(-1,1),1]]) ## example 2
sage: g = Piecewise([(0,3),x]])
sage: f.convolution(g)
Piecewise defined function with 3 parts, [(-1, 1), 0], [(1, 2), -3/2*x], [(2, 4), -3/2*x]
sage: g = Piecewise([(0,3),1*x^0],[3,4),2*x^0]])
sage: f.convolution(g)
Piecewise defined function with 5 parts, [(-1, 1), x + 1], [(1, 2), 3], [(2, 3), x], [(3, 4), 2*x + 1]

```

**cosine\_series\_coefficient** (*n*, *L*)

Returns the *n*-th cosine series coefficient of  $\cos(n\pi x/L)$ ,  $a_n$ .

INPUT:

- *self* - the function *f*(*x*), defined over 0 *x* *L* (no checking is done to insure this)
- *n* - an integer *n*=0
- *L* - (the period)/2

OUTPUT:  $a_n = \frac{2}{L} \int_{-L}^L f(x) \cos(n\pi x/L) dx$  such that

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad 0 < x < L.$$

EXAMPLES:

```

sage: f(x) = x
sage: f = Piecewise([(0,1),f]])
sage: f.cosine_series_coefficient(2,1)
0
sage: f.cosine_series_coefficient(3,1)
-4/9/pi^2
sage: f1(x) = -1
sage: f2(x) = 2
sage: f = Piecewise([(0,pi/2),f1],[pi/2,pi),f2]])
sage: f.cosine_series_coefficient(2,pi)
0
sage: f.cosine_series_coefficient(3,pi)
2/pi
sage: f.cosine_series_coefficient(111,pi)
2/37/pi
sage: f1 = lambda x: x*(pi-x)
sage: f = Piecewise([(0,pi),f1]])
sage: f.cosine_series_coefficient(0,pi)
1/3*pi^2

```

**critical\_points** ()

Return the critical points of this piecewise function.

**Warning:** Uses maxima, which prints the warning to use results with caution. Only works for piecewise functions whose parts are polynomials with real critical not occurring on the interval endpoints.

## EXAMPLES:

```
sage: R.<x> = QQ[]
sage: f1 = x^0
sage: f2 = 10*x - x^2
sage: f3 = 3*x^4 - 156*x^3 + 3036*x^2 - 26208*x
sage: f = Piecewise([[ (0,3), f1], [(3,10), f2], [(10,20), f3]])
sage: expected = [5, 12, 13, 14]
sage: all(abs(e-a) < 0.001 for e,a in zip(expected, f.critical_points()))
True
```

## TESTS:

Use variables other than x ([trac ticket #13836](#)):

```
sage: R.<y> = QQ[]
sage: f1 = y^0
sage: f2 = 10*y - y^2
sage: f3 = 3*y^4 - 156*y^3 + 3036*y^2 - 26208*y
sage: f = Piecewise([[ (0,3), f1], [(3,10), f2], [(10,20), f3]])
sage: expected = [5, 12, 13, 14]
sage: all(abs(e-a) < 0.001 for e,a in zip(expected, f.critical_points()))
True
```

**default\_variable()**

Return the default variable. The default variable is defined as the first variable in the first piece that has a variable. If no pieces have a variable (each piece is a constant value),  $x$  is returned.

The result is cached.

AUTHOR: Paul Butler

## EXAMPLES:

```
sage: f1(x) = 1
sage: f2(x) = 5*x
sage: p = Piecewise([[ (0,1), f1], [(1,4), f2]])
sage: p.default_variable()
x
```

```
sage: f1 = 3*var('y')
sage: p = Piecewise([[ (0,1), 4], [(1,4), f1]])
sage: p.default_variable()
y
```

**derivative()**

Returns the derivative (as computed by maxima)  $\text{Piecewise}(I, (d/dx)(\text{self}_I))$ , as  $I$  runs over the intervals belonging to self. self must be piecewise polynomial.

## EXAMPLES:

```
sage: f1(x) = 1
sage: f2(x) = 1-x
sage: f = Piecewise([[ (0,1), f1], [(1,2), f2]])
sage: f.derivative()
Piecewise defined function with 2 parts, [[(0, 1), x |--> 0], [(1, 2), x |--> -1]]
sage: f1(x) = -1
sage: f2(x) = 2
sage: f = Piecewise([[ (0,pi/2), f1], [(pi/2,pi), f2]])
sage: f.derivative()
Piecewise defined function with 2 parts, [[(0, 1/2*pi), x |--> 0], [(1/2*pi, pi), x |--> 0]]
```



```
sage: f = Piecewise([[ (0,1), (x * 2) ]], x)
sage: f.derivative()
Piecewise defined function with 1 parts, [[ (0, 1), x |--> 2 ]]
```

**domain()**

Returns the domain of the function.

**EXAMPLES:**

```
sage: f1(x) = 1
sage: f2(x) = 1-x
sage: f3(x) = exp(x)
sage: f4(x) = sin(2*x)
sage: f = Piecewise([[ (0,1), f1 ], [ (1,2), f2 ], [ (2,3), f3 ], [ (3,10), f4 ]])
sage: f.domain()
(0, 10)
```

**end\_points()**

Returns a list of all interval endpoints for this function.

**EXAMPLES:**

```
sage: f1(x) = 1
sage: f2(x) = 1-x
sage: f3(x) = x^2-5
sage: f = Piecewise([[ (0,1), f1 ], [ (1,2), f2 ], [ (2,3), f3 ]])
sage: f.end_points()
[0, 1, 2, 3]
```

**extend\_by\_zero\_to** (*xmin=-1000, xmax=1000*)

This function simply returns the piecewise defined function which is extended by 0 so it is defined on all of (xmin,xmax). This is needed to add two piecewise functions in a reasonable way.

**EXAMPLES:**

```
sage: f1(x) = 1
sage: f2(x) = 1 - x
sage: f = Piecewise([[ (0,1), f1 ], [ (1,2), f2 ]])
sage: f.extend_by_zero_to(-1, 3)
Piecewise defined function with 4 parts, [[ (-1, 0), 0 ], [ (0, 1), x |--> 1 ], [ (1, 2), x |-->
```

**fourier\_series\_cosine\_coefficient** (*n, L*)

Returns the n-th Fourier series coefficient of  $\cos(n\pi x/L)$ ,  $a_n$ .

**INPUT:**

- self - the function f(x), defined over -L x L
- n - an integer n=0
- L - (the period)/2

OUTPUT:  $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos(n\pi x/L) dx$

**EXAMPLES:**

```
sage: f(x) = x^2
sage: f = Piecewise([[ (-1,1), f ]])
sage: f.fourier_series_cosine_coefficient(2,1)
pi^(-2)
sage: f(x) = x^2
sage: f = Piecewise([[ (-pi,pi), f ]])
```

```

sage: f.fourier_series_cosine_coefficient(2,pi)
1
sage: f1(x) = -1
sage: f2(x) = 2
sage: f = Piecewise([[(-pi,pi/2),f1],[(pi/2,pi),f2]])
sage: f.fourier_series_cosine_coefficient(5,pi)
-3/5/pi

```

**fourier\_series\_partial\_sum(N, L)**

Returns the partial sum

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^N [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})],$$

as a string.

EXAMPLE:

```

sage: f(x) = x^2
sage: f = Piecewise([[(-1,1),f]])
sage: f.fourier_series_partial_sum(3,1)
cos(2*pi*x)/pi^2 - 4*cos(pi*x)/pi^2 + 1/3
sage: f1(x) = -1
sage: f2(x) = 2
sage: f = Piecewise([[(-pi,pi/2),f1],[(pi/2,pi),f2]])
sage: f.fourier_series_partial_sum(3,pi)
-3*cos(x)/pi - 3*sin(2*x)/pi + 3*sin(x)/pi - 1/4

```

**fourier\_series\_partial\_sum\_cesaro(N, L)**

Returns the Cesaro partial sum

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^N (1 - n/N) * [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})],$$

as a string. This is a “smoother” partial sum - the Gibbs phenomenon is mollified.

EXAMPLE:

```

sage: f(x) = x^2
sage: f = Piecewise([[(-1,1),f]])
sage: f.fourier_series_partial_sum_cesaro(3,1)
1/3*cos(2*pi*x)/pi^2 - 8/3*cos(pi*x)/pi^2 + 1/3
sage: f1(x) = -1
sage: f2(x) = 2
sage: f = Piecewise([[(-pi,pi/2),f1],[(pi/2,pi),f2]])
sage: f.fourier_series_partial_sum_cesaro(3,pi)
-2*cos(x)/pi - sin(2*x)/pi + 2*sin(x)/pi - 1/4

```

**fourier\_series\_partial\_sum\_filtered(N, L, F)**

Returns the “filtered” partial sum

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^N F_n * [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})],$$

as a string, where  $F = [F_1, F_2, \dots, F_N]$  is a list of length  $N$  consisting of real numbers. This can be used to plot FS solutions to the heat and wave PDEs.

EXAMPLE:

```

sage: f(x) = x^2
sage: f = Piecewise([(-1,1),f])
sage: f.fourier_series_partial_sum_filtered(3,1,[1,1,1])
cos(2*pi*x)/pi^2 - 4*cos(pi*x)/pi^2 + 1/3
sage: f1(x) = -1
sage: f2(x) = 2
sage: f = Piecewise([(-pi,pi/2),f1],[pi/2,pi),f2])
sage: f.fourier_series_partial_sum_filtered(3,pi,[1,1,1])
-3*cos(x)/pi - 3*sin(2*x)/pi + 3*sin(x)/pi - 1/4

```

#### **fourier\_series\_partial\_sum\_hann**( $N, L$ )

Returns the Hann-filtered partial sum (named after von Hann, not Hamming)

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^N H_N(n) * [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})],$$

as a string, where  $H_N(x) = (1 + \cos(\pi x/N))/2$ . This is a “smoother” partial sum - the Gibbs phenomenon is mollified.

EXAMPLE:

```

sage: f(x) = x^2
sage: f = Piecewise([(-1,1),f])
sage: f.fourier_series_partial_sum_hann(3,1)
1/4*cos(2*pi*x)/pi^2 - 3*cos(pi*x)/pi^2 + 1/3
sage: f1(x) = -1
sage: f2(x) = 2
sage: f = Piecewise([(-pi,pi/2),f1],[pi/2,pi),f2])
sage: f.fourier_series_partial_sum_hann(3,pi)
-9/4*cos(x)/pi - 3/4*sin(2*x)/pi + 9/4*sin(x)/pi - 1/4

```

#### **fourier\_series\_sine\_coefficient**( $n, L$ )

Returns the  $n$ -th Fourier series coefficient of  $\sin(n\pi x/L)$ ,  $b_n$ .

INPUT:

- `self` - the function  $f(x)$ , defined over  $-L \times L$
- `n` - an integer  $n_0$
- `L` - (the period)/2

OUTPUT:  $b_n = \frac{1}{L} \int_{-L}^L f(x) \sin(n\pi x/L) dx$

EXAMPLES:

```

sage: f(x) = x^2
sage: f = Piecewise([(-1,1),f])
sage: f.fourier_series_sine_coefficient(2,1) # L=1, n=2
0

```

#### **fourier\_series\_value**( $x, L$ )

Returns the value of the Fourier series coefficient of `self` at  $x$ ,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})], \quad -L < x < L.$$

This method applies to piecewise non-polynomial functions as well.

INPUT:

- `self` - the function  $f(x)$ , defined over  $-L \times L$
- `x` - a real number
- `L` - (the period)/2

OUTPUT:  $(f^*(x+) + f^*(x-))/2$ , where  $f^*$  denotes the function  $f$  extended to  $\mathbb{R}$  with period  $2L$  (Dirichlet's Theorem for Fourier series).

EXAMPLES:

```
sage: f1(x) = 1
sage: f2(x) = 1-x
sage: f3(x) = exp(x)
sage: f4(x) = sin(2*x)
sage: f = Piecewise([(-10,1),f1],[(1,2),f2],[(2,3),f3],[(3,10),f4]])
sage: f.fourier_series_value(101,10)
1/2
sage: f.fourier_series_value(100,10)
1
sage: f.fourier_series_value(10,10)
1/2*sin(20) + 1/2
sage: f.fourier_series_value(20,10)
1
sage: f.fourier_series_value(30,10)
1/2*sin(20) + 1/2
sage: f1(x) = -1
sage: f2(x) = 2
sage: f = Piecewise([(-pi,0),lambda x:0],[(0,pi/2),f1],[(pi/2,pi),f2]])
sage: f.fourier_series_value(-1,pi)
0
sage: f.fourier_series_value(20,pi)
-1
sage: f.fourier_series_value(pi/2,pi)
1/2
```

**functions()**

Returns the list of functions (the “pieces”).

EXAMPLES:

```
sage: f1(x) = 1
sage: f2(x) = 1-x
sage: f3(x) = exp(x)
sage: f4(x) = sin(2*x)
sage: f = Piecewise([[(0,1),f1],[(1,2),f2],[(2,3),f3],[(3,10),f4]])
sage: f.functions()
[x |--> 1, x |--> -x + 1, x |--> e^x, x |--> sin(2*x)]
```

**integral** ( $x=None, a=None, b=None, definite=False$ )

By default, returns the indefinite integral of the function. If `definite=True` is given, returns the definite integral.

AUTHOR:

- Paul Butler

EXAMPLES:

```
sage: f1(x) = 1-x
sage: f = Piecewise([[(0,1),1],[(1,2),f1]])
sage: f.integral(definite=True)
1/2
```

```

sage: f1(x) = -1
sage: f2(x) = 2
sage: f = Piecewise([[ (0,pi/2), f1], [(pi/2,pi), f2]])
sage: f.integral(definite=True)
1/2*pi

sage: f1(x) = 2
sage: f2(x) = 3 - x
sage: f = Piecewise([[ (-2, 0), f1], [(0, 3), f2]])
sage: f.integral()
Piecewise defined function with 2 parts, [[(-2, 0), x |--> 2*x + 4], [(0, 3), x |--> -1/2*x^2 + 3*x + 3]]

sage: f1(y) = -1
sage: f2(y) = y + 3
sage: f3(y) = -y - 1
sage: f4(y) = y^2 - 1
sage: f5(y) = 3
sage: f = Piecewise([[ (-4,-3), f1], [(-3,-2), f2], [(-2,0), f3], [(0,2), f4], [(2,3), f5]])
sage: F = f.integral(y)
sage: F
Piecewise defined function with 5 parts, [[(-4, -3), y |--> -y - 4], [(-3, -2), y |--> 1/2*y^2 + 3*y + 3], [(-2, 0), y |--> -1/2*y^2 - y - 1], [(0, 2), y |--> 1/3*y^3 + 3*y^2 + 9*y + 9], [(2, 3), y |--> 1/2*y^2 + 3*y + 3]]

Ensure results are consistent with FTC:
sage: F(-3) - F(-4)
-1
sage: F(-1) - F(-3)
1
sage: F(2) - F(0)
2/3
sage: f.integral(y, 0, 2)
2/3
sage: F(3) - F(-4)
19/6
sage: f.integral(y, -4, 3)
19/6
sage: f.integral(definite=True)
19/6

sage: f1(y) = (y+3)^2
sage: f2(y) = y+3
sage: f3(y) = 3
sage: f = Piecewise([[(-infinity, -3), f1], [(-3, 0), f2], [(0, infinity), f3]])
sage: f.integral()
Piecewise defined function with 3 parts, [[(-Infinity, -3), y |--> 1/3*y^3 + 3*y^2 + 9*y + 9], [(-3, 0), y |--> 1/2*y^2 + 3*y + 3], [(0, Infinity), y |--> 3*y + 3]]

sage: f1(x) = e^(-abs(x))
sage: f = Piecewise([[(-infinity, infinity), f1]])
sage: f.integral(definite=True)
2
sage: f.integral()
Piecewise defined function with 1 parts, [[(-Infinity, +Infinity), x |--> -1/2*((sgn(x) - 1)*e^(-abs(x)) + 1)]]

sage: f = Piecewise([(0, 5), cos(x)])
sage: f.integral()
Piecewise defined function with 1 parts, [(0, 5), x |--> sin(x)]

```

TESTS:

Verify that piecewise integrals of zero work (trac #10841):

```
sage: f0(x) = 0
sage: f = Piecewise([[ (0,1), f0]])
sage: f.integral(x,0,1)
0
sage: f = Piecewise([[ (0,1), 0]])
sage: f.integral(x,0,1)
0
sage: f = Piecewise([[ (0,1), SR(0)]])
sage: f.integral(x,0,1)
0
```

**intervals()**

A piecewise non-polynomial example.

EXAMPLES:

```
sage: f1(x) = 1
sage: f2(x) = 1-x
sage: f3(x) = exp(x)
sage: f4(x) = sin(2*x)
sage: f = Piecewise([[ (0,1), f1], [(1,2), f2], [(2,3), f3], [(3,10), f4]])
sage: f.intervals()
[(0, 1), (1, 2), (2, 3), (3, 10)]
```

**laplace** ( $x='x', s='t'$ )

Returns the Laplace transform of self with respect to the variable var.

INPUT:

- $x$  - variable of self
- $s$  - variable of Laplace transform.

We assume that a piecewise function is 0 outside of its domain and that the left-most endpoint of the domain is 0.

EXAMPLES:

```
sage: x, s, w = var('x, s, w')
sage: f = Piecewise([[ (0,1), 1], [(1,2), 1-x]])
sage: f.laplace(x, s)
-e^(-s)/s + (s + 1)*e^(-2*s)/s^2 + 1/s - e^(-s)/s^2
sage: f.laplace(x, w)
-e^(-w)/w + (w + 1)*e^(-2*w)/w^2 + 1/w - e^(-w)/w^2

sage: y, t = var('y, t')
sage: f = Piecewise([[ (1,2), 1-y]])
sage: f.laplace(y, t)
(t + 1)*e^(-2*t)/t^2 - e^(-t)/t^2

sage: s = var('s')
sage: t = var('t')
sage: f1(t) = -t
sage: f2(t) = 2
sage: f = Piecewise([[ (0,1), f1], [(1,infinity), f2]])
sage: f.laplace(t, s)
(s + 1)*e^(-s)/s^2 + 2*e^(-s)/s - 1/s^2
```

**length()**

Returns the number of pieces of this function.

## EXAMPLES:

```
sage: f1(x) = 1
sage: f2(x) = 1 - x
sage: f = Piecewise([[ (0,1), f1], [(1,2), f2]])
sage: f.length()
2
```

**list()**

Returns the pieces of this function as a list of functions.

## EXAMPLES:

```
sage: f1(x) = 1
sage: f2(x) = 1 - x
sage: f = Piecewise([[ (0,1), f1], [(1,2), f2]])
sage: f.list()
[[ (0, 1), x |--> 1], [(1, 2), x |--> -x + 1]]
```

**plot(\*args, \*\*kws)**

Returns the plot of self.

Keyword arguments are passed onto the plot command for each piece of the function. E.g., the `plot_points` keyword affects each segment of the plot.

## EXAMPLES:

```
sage: f1(x) = 1
sage: f2(x) = 1-x
sage: f3(x) = exp(x)
sage: f4(x) = sin(2*x)
sage: f = Piecewise([[ (0,1), f1], [(1,2), f2], [(2,3), f3], [(3,10), f4]])
sage: P = f.plot(rgbcolor=(0.7,0.1,0), plot_points=40)
sage: P
Graphics object consisting of 4 graphics primitives
```

Remember: to view this, type `show(P)` or `P.save("path/myplot.png")` and then open it in a graphics viewer such as GIMP.

## TESTS:

We should not add each piece to the legend individually, since this creates duplicates ([trac ticket #12651](#)). This tests that only one of the graphics objects in the plot has a non-None `legend_label`:

```
sage: f1 = sin(x)
sage: f2 = cos(x)
sage: f = piecewise([[-1,0), f1], [(0,1), f2]])
sage: p = f.plot(legend_label='$f(x)$')
sage: lines = [
...     line
...     for line in p._objects
...     if line.options()['legend_label'] is not None ]
sage: len(lines)
1
```

**plot\_fourier\_series\_partial\_sum(N, L, xmin, xmax, \*\*kws)**

Plots the partial sum

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^N [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})],$$

over xmin x xmin.

EXAMPLE:

```
sage: f1(x) = -2
sage: f2(x) = 1
sage: f3(x) = -1
sage: f4(x) = 2
sage: f = Piecewise([(-pi, -pi/2), f1], [(-pi/2, 0), f2], [(0, pi/2), f3], [(pi/2, pi), f4])
sage: P = f.plot_fourier_series_partial_sum(3, pi, -5, 5) # long time
sage: f1(x) = -1
sage: f2(x) = 2
sage: f = Piecewise([(-pi, pi/2), f1], [(pi/2, pi), f2])
sage: P = f.plot_fourier_series_partial_sum(15, pi, -5, 5) # long time
```

Remember, to view this type show(P) or P.save("path/myplot.png") and then open it in a graphics viewer such as GIMP.

**plot\_fourier\_series\_partial\_sum\_cesaro**(*N*, *L*, *xmin*, *xmax*, **\*\*kws**)

Plots the partial sum

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^N (1 - n/N) * [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})],$$

over xmin x xmin. This is a “smoother” partial sum - the Gibbs phenomenon is mollified.

EXAMPLE:

```
sage: f1(x) = -2
sage: f2(x) = 1
sage: f3(x) = -1
sage: f4(x) = 2
sage: f = Piecewise([(-pi, -pi/2), f1], [(-pi/2, 0), f2], [(0, pi/2), f3], [(pi/2, pi), f4])
sage: P = f.plot_fourier_series_partial_sum_cesaro(3, pi, -5, 5) # long time
sage: f1(x) = -1
sage: f2(x) = 2
sage: f = Piecewise([(-pi, pi/2), f1], [(pi/2, pi), f2])
sage: P = f.plot_fourier_series_partial_sum_cesaro(15, pi, -5, 5) # long time
```

Remember, to view this type show(P) or P.save("path/myplot.png") and then open it in a graphics viewer such as GIMP.

**plot\_fourier\_series\_partial\_sum\_filtered**(*N*, *L*, *F*, *xmin*, *xmax*, **\*\*kws**)

Plots the partial sum

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^N F_n * [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})],$$

over xmin x xmin, where  $F = [F_1, F_2, \dots, F_N]$  is a list of length  $N$  consisting of real numbers. This can be used to plot FS solutions to the heat and wave PDEs.

EXAMPLE:

```
sage: f1(x) = -2
sage: f2(x) = 1
sage: f3(x) = -1
sage: f4(x) = 2
sage: f = Piecewise([(-pi, -pi/2), f1], [(-pi/2, 0), f2], [(0, pi/2), f3], [(pi/2, pi), f4])
sage: P = f.plot_fourier_series_partial_sum_filtered(3, pi, [1]*3, -5, 5) # long time
sage: f1(x) = -1
```



```

sage: f2(x) = 2
sage: f = Piecewise([(-pi,-pi/2),f1],[(-pi/2,0),f2],[(0,pi/2),f1],[(pi/2,pi),f2]])
sage: P = f.plot_fourier_series_partial_sum_filtered(15,pi,[1]*15,-5,5) # long time

```

Remember, to view this type show(P) or P.save("path/myplot.png") and then open it in a graphics viewer such as GIMP.

**plot\_fourier\_series\_partial\_sum\_hann** (*N, L, xmin, xmax, \*\*kws*)

Plots the partial sum

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^N H_N(n) * [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})],$$

over  $x_{\min}$  to  $x_{\max}$ , where  $H_N(x) = (0.5) + (0.5) * \cos(x * \pi / N)$  is the  $N$ -th Hann filter.

EXAMPLE:

```

sage: f1(x) = -2
sage: f2(x) = 1
sage: f3(x) = -1
sage: f4(x) = 2
sage: f = Piecewise([(-pi,-pi/2),f1],[(-pi/2,0),f2],[(0,pi/2),f3],[(pi/2,pi),f4]])
sage: P = f.plot_fourier_series_partial_sum_hann(3,pi,-5,5) # long time
sage: f1(x) = -1
sage: f2(x) = 2
sage: f = Piecewise([(-pi,pi/2),f1],[(pi/2,pi),f2]])
sage: P = f.plot_fourier_series_partial_sum_hann(15,pi,-5,5) # long time

```

Remember, to view this type show(P) or P.save("path/myplot.png") and then open it in a graphics viewer such as GIMP.

**riemann\_sum** (*N, mode=None*)

Returns the piecewise line function defined by the Riemann sums in numerical integration based on a subdivision into  $N$  subintervals. Set `mode="midpoint"` for the height of the rectangles to be determined by the midpoint of the subinterval; set `mode="right"` for the height of the rectangles to be determined by the right-hand endpoint of the subinterval; the default is `mode="left"` (the height of the rectangles to be determined by the left-hand endpoint of the subinterval).

EXAMPLES:

```

sage: f1(x) = x^2
sage: f2(x) = 5-x^2
sage: f = Piecewise([(0,1),f1],[(1,2),f2]])
sage: f.riemann_sum(6,mode="midpoint")
Piecewise defined function with 6 parts, [[(0, 1/3), 1/36], [(1/3, 2/3), 1/4], [(2/3, 1), 25/36], [(1, 4/3), 1/3], [(4/3, 5/3), 1/4], [(5/3, 2), 1/36]]

sage: f = Piecewise([(-1,1),(1-x^2).function(x)])
sage: rsf = f.riemann_sum(7)
sage: P = f.plot(rgba=(0.7,0.1,0.5), plot_points=40)
sage: Q = rsf.plot(rgba=(0.7,0.6,0.6), plot_points=40)
sage: L = add([line([a,0],[a,f(x=a)]), rgba=(0.7,0.6,0.6)] for (a,b),f in rsf.list())
sage: P + Q + L
Graphics object consisting of 15 graphics primitives

sage: f = Piecewise([(-1,1),(1/2+x-x^3)], x) ## example 3
sage: rsf = f.riemann_sum(8)
sage: P = f.plot(rgba=(0.7,0.1,0.5), plot_points=40)
sage: Q = rsf.plot(rgba=(0.7,0.6,0.6), plot_points=40)
sage: L = add([line([a,0],[a,f(x=a)]), rgba=(0.7,0.6,0.6)] for (a,b),f in rsf.list())

```

**sage:**  $P + Q + L$   
Graphics object consisting of 17 graphics primitives

**riemann\_sum\_integral\_approximation** ( $N, mode=None$ )

Returns the piecewise line function defined by the Riemann sums in numerical integration based on a subdivision into  $N$  subintervals.

Set `mode="midpoint"` for the height of the rectangles to be determined by the midpoint of the subinterval; set `mode="right"` for the height of the rectangles to be determined by the right-hand endpoint of the subinterval; the default is `mode="left"` (the height of the rectangles to be determined by the left-hand endpoint of the subinterval).

EXAMPLES:

```
sage: f1(x) = x^2                                     ## example 1
sage: f2(x) = 5-x^2
sage: f = Piecewise([(0,1),f1],[(1,2),f2])
sage: f.riemann_sum_integral_approximation(6)
17/6
sage: f.riemann_sum_integral_approximation(6,mode="right")
19/6
sage: f.riemann_sum_integral_approximation(6,mode="midpoint")
3
sage: f.integral(definite=True)
3
```

**sine\_series\_coefficient** ( $n, L$ )

Returns the  $n$ -th sine series coefficient of  $\sin(n\pi x/L)$ ,  $b_n$ .

INPUT:

- `self` - the function  $f(x)$ , defined over  $0 \leq x \leq L$  (no checking is done to insure this)
- $n$  - an integer  $n \geq 0$
- $L$  - (the period)/2

OUTPUT:

$b_n = \frac{2}{L} \int_{-L}^L f(x) \sin(n\pi x/L) dx$  such that

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 < x < L.$$

EXAMPLES:

```
sage: f(x) = 1
sage: f = Piecewise([(0,1),f])
sage: f.sine_series_coefficient(2,1)
0
sage: f.sine_series_coefficient(3,1)
4/3/pi
```

**tangent\_line** ( $pt$ )

Computes the linear function defining the tangent line of the piecewise function self.

EXAMPLES:

```
sage: f1(x) = x^2
sage: f2(x) = 5-x^3+x
sage: f = Piecewise([(0,1),f1],[(1,2),f2])
```

```

sage: tf = f.tangent_line(0.9) ## tangent line at x=0.9
sage: P = f.plot(rgbcolor=(0.7,0.1,0.5), plot_points=40)
sage: Q = tf.plot(rgbcolor=(0.7,0.2,0.2), plot_points=40)
sage: P + Q
Graphics object consisting of 4 graphics primitives

```

**trapezoid(N)**

Returns the piecewise line function defined by the trapezoid rule for numerical integration based on a subdivision into  $N$  subintervals.

**EXAMPLES:**

```

sage: R.<x> = QQ[]
sage: f1 = x^2
sage: f2 = 5-x^2
sage: f = Piecewise([[ (0,1), f1 ], [ (1,2), f2 ]])
sage: f.trapezoid(4)
Piecewise defined function with 4 parts, [[ (0, 1/2), 1/2*x ], [ (1/2, 1), 9/2*x - 2 ], [ (1, 3/2), 3/2*x - 1 ], [ (3/2, 2), 2-x ]]

sage: R.<x> = QQ[]
sage: f = Piecewise([[ (-1,1), 1-x^2 ]])
sage: tf = f.trapezoid(4)
sage: P = f.plot(rgbcolor=(0.7,0.1,0.5), plot_points=40)
sage: Q = tf.plot(rgbcolor=(0.7,0.6,0.6), plot_points=40)
sage: L = add([line([a,0], [a,f(a)]), rgbcolor=(0.7,0.6,0.6)] for (a,b),f in tf.list()))
sage: P+Q+L
Graphics object consisting of 9 graphics primitives

sage: R.<x> = QQ[]
sage: f = Piecewise([[ (-1,1), 1/2+x-x^3 ]]) ## example 3
sage: tf = f.trapezoid(6)
sage: P = f.plot(rgbcolor=(0.7,0.1,0.5), plot_points=40)
sage: Q = tf.plot(rgbcolor=(0.7,0.6,0.6), plot_points=40)
sage: L = add([line([a,0], [a,f(a)]), rgbcolor=(0.7,0.6,0.6)] for (a,b),f in tf.list()))
sage: P+Q+L
Graphics object consisting of 14 graphics primitives

```

**TESTS:**

Use variables other than  $x$  ([trac ticket #13836](#)):

```

sage: R.<y> = QQ[]
sage: f1 = y^2
sage: f2 = 5-y^2
sage: f = Piecewise([[ (0,1), f1 ], [ (1,2), f2 ]])
sage: f.trapezoid(4)
Piecewise defined function with 4 parts, [[ (0, 1/2), 1/2*y ], [ (1/2, 1), 9/2*y - 2 ], [ (1, 3/2), 3/2*y - 1 ], [ (3/2, 2), 2-y ]]

```

**trapezoid\_integral\_approximation(N)**

Returns the approximation given by the trapezoid rule for numerical integration based on a subdivision into  $N$  subintervals.

**EXAMPLES:**

```

sage: f1(x) = x^2 ## example 1
sage: f2(x) = 1-(1-x)^2
sage: f = Piecewise([[ (0,1), f1 ], [ (1,2), f2 ]])
sage: P = f.plot(rgbcolor=(0.7,0.1,0.5), plot_points=40)
sage: tf = f.trapezoid(6)
sage: Q = tf.plot(rgbcolor=(0.7,0.6,0.6), plot_points=40)

```

```
sage: ta = f.trapezoid_integral_approximation(6)
sage: t = text('trapezoid approximation = %s'%ta, (1.5, 0.25))
sage: a = f.integral(definite=True)
sage: tt = text('area under curve = %s'%a, (1.5, -0.5))
sage: P + Q + t + tt
Graphics object consisting of 10 graphics primitives

sage: f = Piecewise([[ (0,1), f1], [(1,2), f2]]) ## example 2
sage: tf = f.trapezoid(4)
sage: ta = f.trapezoid_integral_approximation(4)
sage: Q = tf.plot(rgbcolor=(0.7,0.6,0.6), plot_points=40)
sage: t = text('trapezoid approximation = %s'%ta, (1.5, 0.25))
sage: a = f.integral(definite=True)
sage: tt = text('area under curve = %s'%a, (1.5, -0.5))
sage: P+Q+t+tt
Graphics object consisting of 8 graphics primitives
```

### `unextend()`

This removes any parts in the front or back of the function which is zero (the inverse to `extend_by_zero_to`).

#### EXAMPLES:

```
sage: R.<x> = QQ[]
sage: f = Piecewise([[ (-3,-1), 1+2*x], [ (-1,1), 1-x^2]])
sage: e = f.extend_by_zero_to(-10,10); e
Piecewise defined function with 4 parts, [[(-10, -3), 0], [(-3, -1), x + 3], [(-1, 1), -x^2
sage: d = e.unextend(); d
Piecewise defined function with 2 parts, [[(-3, -1), x + 3], [(-1, 1), -x^2 + 1]]
sage: d==f
True
```

### `which_function(x0)`

Returns the function piece used to evaluate self at `x0`.

#### EXAMPLES:

```
sage: f1(z) = z
sage: f2(x) = 1-x
sage: f3(y) = exp(y)
sage: f4(t) = sin(2*t)
sage: f = Piecewise([[ (0,1), f1], [(1,2), f2], [(2,3), f3], [(3,10), f4]])
sage: f.which_function(3/2)
x |--> -x + 1
```

`sage.functions.piecewise.piecewise(list_of_pairs, var=None)`

Returns a piecewise function from a list of (interval, function) pairs.

`list_of_pairs` is a list of pairs  $(I, \text{fcn})$ , where `fcn` is a Sage function (such as a polynomial over  $\mathbb{R}$ , or functions using the lambda notation), and  $I$  is an interval such as  $I = (1,3)$ . Two consecutive intervals must share a common endpoint.

If the optional `var` is specified, then any symbolic expressions in the list will be converted to symbolic functions using `fcn.function(var)`. (This says which variable is considered to be “piecewise”).

We assume that these definitions are consistent (ie, no checking is done).

#### EXAMPLES:

```
sage: f1(x) = -1
sage: f2(x) = 2
sage: f = Piecewise([[ (0,pi/2), f1], [(pi/2,pi), f2]])
```

```
sage: f(1)
-1
sage: f(3)
2
sage: f = Piecewise([[ (0,1), x], [ (1,2), x^2]], x); f
Piecewise defined function with 2 parts, [[ (0, 1), x |--> x], [ (1, 2), x |--> x^2]]
sage: f(0.9)
0.9000000000000000
sage: f(1.1)
1.2100000000000000
```



## SPIKE FUNCTIONS

## AUTHORS:

- William Stein (2007-07): initial version
- Karl-Dieter Crisman (2009-09): adding documentation and doctests

**class** `sage.functions.spike_function.SpikeFunction` (*v*, *eps*=*1e-07*)  
 Base class for spike functions.

## INPUT:

- *v* - list of pairs (*x*, height)
- *eps* - parameter that determines approximation to a true spike

## OUTPUT:

a function with spikes at each point *x* in *v* with the given height.

## EXAMPLES:

```
sage: spike_function([(-3,4), (-1,1), (2,3)], 0.001)
A spike function with spikes at [-3.0, -1.0, 2.0]
```

Putting the spikes too close together may delete some:

```
sage: spike_function([(1,1), (1.01,4)], 0.1)
Some overlapping spikes have been deleted.
You might want to use a smaller value for eps.
A spike function with spikes at [1.0]
```

Note this should normally be used indirectly via `spike_function`, but one can use it directly:

```
sage: from sage.functions.spike_function import SpikeFunction
sage: S = SpikeFunction([(0,1), (1,2), (pi,-5)])
sage: S
A spike function with spikes at [0.0, 1.0, 3.141592653589793]
sage: S.support
[0.0, 1.0, 3.141592653589793]
```

**plot** (*xmin*=None, *xmax*=None, *\*\*kws*)  
 Special fast plot method for spike functions.

## EXAMPLES:

```
sage: S = spike_function([(-1,1), (1,40)])
sage: P = plot(S)
sage: P[0]
Line defined by 8 points
```

**plot\_fft\_abs** (*samples=4096, xmin=None, xmax=None, \*\*kws*)

Plot of (absolute values of) Fast Fourier Transform of the spike function with given number of samples.

EXAMPLES:

```
sage: S = spike_function([(-3,4), (-1,1), (2,3)]); S
```

A spike function with spikes at [-3.0, -1.0, 2.0]

```
sage: P = S.plot_fft_abs(8)
```

```
sage: p = P[0]; p.ydata
```

```
[5.0, 5.0, 3.367958691924177, 3.367958691924177, 4.123105625617661, 4.123105625617661, 4.759
```

**plot\_fft\_arg** (*samples=4096, xmin=None, xmax=None, \*\*kws*)

Plot of (absolute values of) Fast Fourier Transform of the spike function with given number of samples.

EXAMPLES:

```
sage: S = spike_function([(-3,4), (-1,1), (2,3)]); S
```

A spike function with spikes at [-3.0, -1.0, 2.0]

```
sage: P = S.plot_fft_arg(8)
```

```
sage: p = P[0]; p.ydata
```

```
[0.0, 0.0, -0.211524990023434..., -0.211524990023434..., 0.244978663126864..., 0.24497866312
```

**vector** (*samples=65536, xmin=None, xmax=None*)

Creates a sampling vector of the spike function in question.

EXAMPLES:

```
sage: S = spike_function([(-3,4), (-1,1), (2,3)], 0.001); S
```

A spike function with spikes at [-3.0, -1.0, 2.0]

```
sage: S.vector(16)
```

```
(4.0, 0.0, 0.0, 0.0, 0.0, 0.0, 1.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0)
```

`sage.functions.spike_function.spike_function`

alias of `SpikeFunction`



## ORTHOGONAL POLYNOMIALS

This module wraps some of the orthogonal/special functions in the Maxima package “orthopoly”. This package was written by Barton Willis of the University of Nebraska at Kearney. It is released under the terms of the General Public License (GPL). Send Maxima-related bug reports and comments on this module to [willisb@unk.edu](mailto:willisb@unk.edu). In your report, please include Maxima and specfun version information.

- The Chebyshev polynomial of the first kind arises as a solution to the differential equation

$$(1 - x^2) y'' - x y' + n^2 y = 0$$

and those of the second kind as a solution to

$$(1 - x^2) y'' - 3x y' + n(n + 2) y = 0.$$

The Chebyshev polynomials of the first kind are defined by the recurrence relation

$$T_0(x) = 1, T_1(x) = x, T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

The Chebyshev polynomials of the second kind are defined by the recurrence relation

$$U_0(x) = 1, U_1(x) = 2x, U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).$$

For integers  $m, n$ , they satisfy the orthogonality relations

$$\int_{-1}^1 T_n(x) T_m(x) \frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & : n \neq m \\ \pi & : n = m = 0 \\ \pi/2 & : n = m \neq 0 \end{cases}$$

and

$$\int_{-1}^1 U_n(x) U_m(x) \sqrt{1-x^2} dx = \frac{\pi}{2} \delta_{m,n}.$$

They are named after Pafnuty Chebyshev (alternative transliterations: Tchebyshef or Tschhebyscheff).

- The Hermite polynomials are defined either by

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$$

(the “probabilists’ Hermite polynomials”), or by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

(the “physicists’ Hermite polynomials”). Sage (via Maxima) implements the latter flavor. These satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx = n! 2^n \sqrt{\pi} \delta_{nm}$$

They are named in honor of Charles Hermite.

- Each *Legendre polynomial*  $P_n(x)$  is an  $n$ -th degree polynomial. It may be expressed using Rodrigues' formula:

$$P_n(x) = (2^n n!)^{-1} \frac{d^n}{dx^n} [(x^2 - 1)^n].$$

These are solutions to Legendre's differential equation:

$$\frac{d}{dx} \left[ (1 - x^2) \frac{d}{dx} P(x) \right] + n(n+1)P(x) = 0.$$

and satisfy the orthogonality relation

$$\int_{-1}^1 P_m(x) P_n(x) dx = \frac{2}{2n+1} \delta_{mn}$$

The *Legendre function of the second kind*  $Q_n(x)$  is another (linearly independent) solution to the Legendre differential equation. It is not an "orthogonal polynomial" however.

The associated Legendre functions of the first kind  $P_\ell^m(x)$  can be given in terms of the "usual" Legendre polynomials by

$$\begin{aligned} P_\ell^m(x) &= (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x) \\ &= \frac{(-1)^m}{2^\ell \ell!} (1 - x^2)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2 - 1)^\ell. \end{aligned}$$

Assuming  $0 \leq m \leq \ell$ , they satisfy the orthogonality relation:

$$\int_{-1}^1 P_k^{(m)} P_\ell^{(m)} dx = \frac{2(\ell+m)!}{(2\ell+1)(\ell-m)!} \delta_{k,\ell},$$

where  $\delta_{k,\ell}$  is the Kronecker delta.

The associated Legendre functions of the second kind  $Q_\ell^m(x)$  can be given in terms of the "usual" Legendre polynomials by

$$Q_\ell^m(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} Q_\ell(x).$$

They are named after Adrien-Marie Legendre.

- Laguerre polynomials may be defined by the Rodrigues formula

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^n).$$

They are solutions of Laguerre's equation:

$$x y'' + (1 - x) y' + n y = 0$$

and satisfy the orthogonality relation

$$\int_0^\infty L_m(x) L_n(x) e^{-x} dx = \delta_{mn}.$$

The generalized Laguerre polynomials may be defined by the Rodrigues formula:

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha}).$$

(These are also sometimes called the associated Laguerre polynomials.) The simple Laguerre polynomials are recovered from the generalized polynomials by setting  $\alpha = 0$ .

They are named after Edmond Laguerre.

- Jacobi polynomials are a class of orthogonal polynomials. They are obtained from hypergeometric series in cases where the series is in fact finite:

$$P_n^{(\alpha, \beta)}(z) = \frac{(\alpha + 1)_n}{n!} {}_2F_1 \left( -n, 1 + \alpha + \beta + n; \alpha + 1; \frac{1 - z}{2} \right),$$

where  $(\ )_n$  is Pochhammer's symbol (for the rising factorial), (Abramowitz and Stegun p561.) and thus have the explicit expression

$$P_n^{(\alpha, \beta)}(z) = \frac{\Gamma(\alpha + n + 1)}{n! \Gamma(\alpha + \beta + n + 1)} \sum_{m=0}^n \binom{n}{m} \frac{\Gamma(\alpha + \beta + n + m + 1)}{\Gamma(\alpha + m + 1)} \left( \frac{z - 1}{2} \right)^m.$$

They are named after Carl Jacobi.

- Ultraspherical or Gegenbauer polynomials are given in terms of the Jacobi polynomials  $P_n^{(\alpha, \beta)}(x)$  with  $\alpha = \beta = a - 1/2$  by

$$C_n^{(a)}(x) = \frac{\Gamma(a + 1/2)}{\Gamma(2a)} \frac{\Gamma(n + 2a)}{\Gamma(n + a + 1/2)} P_n^{(a-1/2, a-1/2)}(x).$$

They satisfy the orthogonality relation

$$\int_{-1}^1 (1 - x^2)^{a-1/2} C_m^{(a)}(x) C_n^{(a)}(x) dx = \delta_{mn} 2^{1-2a} \pi \frac{\Gamma(n + 2a)}{(n + a) \Gamma^2(a) \Gamma(n + 1)},$$

for  $a > -1/2$ . They are obtained from hypergeometric series in cases where the series is in fact finite:

$$C_n^{(a)}(z) = \frac{(2a)_n}{n!} {}_2F_1 \left( -n, 2a + n; a + \frac{1}{2}; \frac{1 - z}{2} \right)$$

where  $\underline{n}$  is the falling factorial. (See Abramowitz and Stegun p561)

They are named for Leopold Gegenbauer (1849-1903).

For completeness, the Pochhammer symbol, introduced by Leo August Pochhammer,  $(x)_n$ , is used in the theory of special functions to represent the “rising factorial” or “upper factorial”

$$(x)_n = x(x + 1)(x + 2) \cdots (x + n - 1) = \frac{(x + n - 1)!}{(x - 1)!}.$$

On the other hand, the “falling factorial” or “lower factorial” is

$$x^{\underline{n}} = \frac{x!}{(x - n)!},$$

in the notation of Ronald L. Graham, Donald E. Knuth and Oren Patashnik in their book Concrete Mathematics.

**Note:** The first call of any of these will usually cost a bit extra (it loads “specfun”, but I’m not sure if that is the real reason). The next call is usually faster but not always.

## Todo

Implement associated Legendre polynomials and Zernike polynomials. (Neither is in Maxima.) [Wikipedia article Associated\\_Legendre\\_polynomials](#) [Wikipedia article Zernike\\_polynomials](#)

## REFERENCES:

## AUTHORS:

- David Joyner (2006-06)

- Stefan Reiterer (2010-)

```
class sage.functions.orthogonal_polys.ChebyshevPolynomial(name, nargs=2, latex_name=None, conversions={})
```

Bases: `sage.functions.orthogonal_polys.OrthogonalPolynomial`

Abstract base class for Chebyshev polynomials of the first and second kind.

EXAMPLES:

```
sage: chebyshev_T(3, x)
4*x^3 - 3*x
```

```
class sage.functions.orthogonal_polys.Func_chebyshev_T
Bases: sage.functions.orthogonal_polys.ChebyshevPolynomial
```

Chebyshev polynomials of the first kind.

REFERENCE:

- [ASHandbook] 22.5.31 page 778 and 6.1.22 page 256.

EXAMPLES:

```
sage: chebyshev_T(5, x)
16*x^5 - 20*x^3 + 5*x
sage: var('k')
k
sage: test = chebyshev_T(k, x)
sage: test
chebyshev_T(k, x)
```

**eval\_algebraic**(*n*, *x*)

Evaluate `chebyshev_T` as polynomial, using a recursive formula.

INPUT:

- *n* – an integer
- *x* – a value to evaluate the polynomial at (this can be any ring element)

EXAMPLES:

```
sage: chebyshev_T.eval_algebraic(5, x)
2*(2*(2*x^2 - 1)*x - x)*(2*x^2 - 1) - x
sage: chebyshev_T(-7, x) - chebyshev_T(7, x)
0
sage: R.<t> = ZZ[]
sage: chebyshev_T.eval_algebraic(-1, t)
t
sage: chebyshev_T.eval_algebraic(0, t)
1
sage: chebyshev_T.eval_algebraic(1, t)
t
sage: chebyshev_T(7^100, 1/2)
1/2
sage: chebyshev_T(7^100, Mod(2, 3))
2
sage: n = 97; x = RIF(pi/2/n)
sage: chebyshev_T(n, cos(x)).contains_zero()
True
sage: R.<t> = Zp(2, 8, 'capped-abs')[]
```

```
sage: chebyshev_T(10^6+1, t)
(2^7 + O(2^8))*t^5 + (O(2^8))*t^4 + (2^6 + O(2^8))*t^3 + (O(2^8))*t^2 + (1 + 2^6 + O(2^8))*t
```

**eval\_formula**(*n*, *x*)

Evaluate `chebyshev_T` using an explicit formula. See [ASHandbook] 227 (p. 782) for details for the recursions. See also [EffCheby] for fast evaluation techniques.

INPUT:

- *n* – an integer
- *x* – a value to evaluate the polynomial at (this can be any ring element)

EXAMPLES:

```
sage: chebyshev_T.eval_formula(-1,x)
x
sage: chebyshev_T.eval_formula(0,x)
1
sage: chebyshev_T.eval_formula(1,x)
x
sage: chebyshev_T.eval_formula(2,0.1) == chebyshev_T._evalf_(2,0.1)
True
sage: chebyshev_T.eval_formula(10,x)
512*x^10 - 1280*x^8 + 1120*x^6 - 400*x^4 + 50*x^2 - 1
sage: chebyshev_T.eval_algebraic(10,x).expand()
512*x^10 - 1280*x^8 + 1120*x^6 - 400*x^4 + 50*x^2 - 1
```

**class** `sage.functions.orthogonal_polys.Func_chebyshev_U`

Bases: `sage.functions.orthogonal_polys.ChebyshevPolynomial`

Class for the Chebyshev polynomial of the second kind.

REFERENCE:

- [ASHandbook] 22.8.3 page 783 and 6.1.22 page 256.

EXAMPLES:

```
sage: R.<t> = QQ[]
sage: chebyshev_U(2,t)
4*t^2 - 1
sage: chebyshev_U(3,t)
8*t^3 - 4*t
```

**eval\_algebraic**(*n*, *x*)

Evaluate `chebyshev_U` as polynomial, using a recursive formula.

INPUT:

- *n* – an integer
- *x* – a value to evaluate the polynomial at (this can be any ring element)

EXAMPLES:

```
sage: chebyshev_U.eval_algebraic(5,x)
-2*((2*x + 1)*(2*x - 1)*x - 4*(2*x^2 - 1)*x)*(2*x + 1)*(2*x - 1)
sage: parent(chebyshev_U(3, Mod(8,9)))
Ring of integers modulo 9
sage: parent(chebyshev_U(3, Mod(1,9)))
Ring of integers modulo 9
sage: chebyshev_U(-3,x) + chebyshev_U(1,x)
```

```

0
sage: chebyshev_U(-1, Mod(5, 8))
0
sage: parent(chebyshev_U(-1, Mod(5, 8)))
Ring of integers modulo 8
sage: R.<t> = ZZ[]
sage: chebyshev_U.eval_algebraic(-2, t)
-1
sage: chebyshev_U.eval_algebraic(-1, t)
0
sage: chebyshev_U.eval_algebraic(0, t)
1
sage: chebyshev_U.eval_algebraic(1, t)
2*t
sage: n = 97; x = RIF(pi/n)
sage: chebyshev_U(n-1, cos(x)).contains_zero()
True
sage: R.<t> = Zp(2, 6, 'capped-abs')[]
sage: chebyshev_U(10^6+1, t)
(2 + O(2^6))*t + (O(2^6))

```

**eval\_formula(n, x)**

Evaluate `chebyshev_U` using an explicit formula. See [ASHandbook] 227 (p. 782) for details on the recursions. See also [EffCheby] for the recursion formulas.

INPUT:

- `n` – an integer
- `x` – a value to evaluate the polynomial at (this can be any ring element)

EXAMPLES:

```

sage: chebyshev_U.eval_formula(10, x)
1024*x^10 - 2304*x^8 + 1792*x^6 - 560*x^4 + 60*x^2 - 1
sage: chebyshev_U.eval_formula(-2, x)
-1
sage: chebyshev_U.eval_formula(-1, x)
0
sage: chebyshev_U.eval_formula(0, x)
1
sage: chebyshev_U.eval_formula(1, x)
2*x
sage: chebyshev_U.eval_formula(2, 0.1) == chebyshev_U._evalf_(2, 0.1)
True

```

```

class sage.functions.orthogonal_polys.OrthogonalPolynomial(name, nargs=2, latex_name=None, conversions={})

```

Bases: `sage.symbolic.function.BuiltinFunction`

Base class for orthogonal polynomials.

This class is an abstract base class for all orthogonal polynomials since they share similar properties. The evaluation as a polynomial is either done via maxima, or with pynac.

Convention: The first argument is always the order of the polynomial, the others are other values or parameters where the polynomial is evaluated.

**eval\_formula(\*args)**

Evaluate this polynomial using an explicit formula.

## EXAMPLES:

```
sage: from sage.functions.orthogonal_polys import OrthogonalPolynomial
sage: P = OrthogonalPolynomial('testo_P')
sage: P.eval_formula(1,2.0)
Traceback (most recent call last):
...
NotImplementedError: no explicit calculation of values implemented
```

```
sage.functions.orthogonal_polys.gegenbauer(n, a, x)
```

Returns the ultraspherical (or Gegenbauer) polynomial for integers  $n > -1$ .

Computed using Maxima.

## REFERENCE:

•[ASHandbook] 22.5.27

## EXAMPLES:

```
sage: x = PolynomialRing(QQ, 'x').gen()
sage: ultraspherical(2, 3/2, x)
15/2*x^2 - 3/2
sage: ultraspherical(2, 1/2, x)
3/2*x^2 - 1/2
sage: ultraspherical(1, 1, x)
2*x
sage: t = PolynomialRing(RationalField(), "t").gen()
sage: gegenbauer(3, 2, t)
32*t^3 - 12*t
```

Check that [trac ticket #17192](#) is fixed:: sage: x = PolynomialRing(QQ, 'x').gen() sage: ultraspherical(0,1,x)  
1

```
sage: ultraspherical(-1,1,x)
Traceback (most recent call last): ... ValueError: n must be greater than -1, got n = -1
```

```
sage: ultraspherical(-7,1,x)
Traceback (most recent call last): ... ValueError: n must be greater than -1, got n = -7
```

```
sage.functions.orthogonal_polys.gen_laguerre(n, a, x)
```

Returns the generalized Laguerre polynomial for integers  $n > -1$ . Typically,  $a = 1/2$  or  $a = -1/2$ .

## REFERENCES:

•Table on page 789 in [ASHandbook].

## EXAMPLES:

```
sage: x = PolynomialRing(QQ, 'x').gen()
sage: gen_laguerre(2, 1, x)
1/2*x^2 - 3*x + 3
sage: gen_laguerre(2, 1/2, x)
1/2*x^2 - 5/2*x + 15/8
sage: gen_laguerre(2, -1/2, x)
1/2*x^2 - 3/2*x + 3/8
sage: gen_laguerre(2, 0, x)
1/2*x^2 - 2*x + 1
sage: gen_laguerre(3, 0, x)
-1/6*x^3 + 3/2*x^2 - 3*x + 1
```

Check that [trac ticket #17192](#) is fixed:: sage: x = PolynomialRing(QQ, 'x').gen() sage: gen\_laguerre(0,1,x)  
1

sage: gen\_laguerre(-1,1,x) Traceback (most recent call last): ... ValueError: n must be greater than -1, got n = -1

sage: gen\_laguerre(-7,1,x) Traceback (most recent call last): ... ValueError: n must be greater than -1, got n = -7

sage.functions.orthogonal\_polys.**gen\_legendre\_P**(n, m, x)

Returns the generalized (or associated) Legendre function of the first kind.

The awkward code for when m is odd and 1 results from the fact that Maxima is happy with, for example,  $(1 - t^2)^3/2$ , but Sage is not. For these cases the function is computed from the (m-1)-case using one of the recursions satisfied by the Legendre functions.

REFERENCE:

•Gradshteyn and Ryzhik 8.706 page 1000.

EXAMPLES:

```
sage: P.<t> = QQ[]
sage: gen_legendre_P(2, 0, t)
3/2*t^2 - 1/2
sage: gen_legendre_P(2, 0, t) == legendre_P(2, t)
True
sage: gen_legendre_P(3, 1, t)
-3/2*(5*t^2 - 1)*sqrt(-t^2 + 1)
sage: gen_legendre_P(4, 3, t)
105*(t^2 - 1)*sqrt(-t^2 + 1)*t
sage: gen_legendre_P(7, 3, t).expand()
-16695*sqrt(2)
sage: gen_legendre_P(4, 1, 2.5)
-583.562373654533*I
```

sage.functions.orthogonal\_polys.**gen\_legendre\_Q**(n, m, x)

Returns the generalized (or associated) Legendre function of the second kind.

Maxima restricts  $m = n$ . Hence the cases  $m \neq n$  are computed using the same recursion used for  $\text{gen\_legendre\_P}(n,m,x)$  when m is odd and 1.

EXAMPLES:

```
sage: P.<t> = QQ[]
sage: gen_legendre_Q(2,0,t)
3/4*t^2*log(-(t + 1)/(t - 1)) - 3/2*t - 1/4*log(-(t + 1)/(t - 1))
sage: gen_legendre_Q(2,0,t) - legendre_Q(2, t)
0
sage: gen_legendre_Q(3,1,0.5)
2.49185259170895
sage: gen_legendre_Q(0, 1, x)
-1/sqrt(-x^2 + 1)
sage: gen_legendre_Q(2, 4, x).factor()
48*x/((x + 1)^2*(x - 1)^2)
```

sage.functions.orthogonal\_polys.**hermite**(n, x)

Returns the Hermite polynomial for integers  $n > -1$ .

REFERENCE:

•[\[ASHandbook\]](#) 22.5.40 and 22.5.41, page 779.



## EXAMPLES:

```

sage: x = PolynomialRing(QQ, 'x').gen()
sage: hermite(2,x)
4*x^2 - 2
sage: hermite(3,x)
8*x^3 - 12*x
sage: hermite(3,2)
40
sage: S.<y> = PolynomialRing(RR)
sage: hermite(3,y)
8.000000000000000*y^3 - 12.000000000000000*y
sage: R.<x,y> = QQ[]
sage: hermite(3,y^2)
8*y^6 - 12*y^2
sage: w = var('w')
sage: hermite(3,2*w)
8*(8*w^2 - 3)*w

```

**Check that trac ticket #17192 is fixed::** sage: x = PolynomialRing(QQ, 'x').gen() sage: hermite(0,x) 1

sage: hermite(-1,x) Traceback (most recent call last): ... ValueError: n must be greater than -1, got n = -1

sage: hermite(-7,x) Traceback (most recent call last): ... ValueError: n must be greater than -1, got n = -7

sage.functions.orthogonal\_polys.**jacobi\_P**(n, a, b, x)

Returns the Jacobi polynomial  $P_n^{(a,b)}(x)$  for integers  $n > -1$  and a and b symbolic or  $a > -1$  and  $b > -1$ . The Jacobi polynomials are actually defined for all a and b. However, the Jacobi polynomial weight  $(1-x)^a(1+x)^b$  isn't integrable for  $a \leq -1$  or  $b \leq -1$ .

## REFERENCE:

- Table on page 789 in [ASHandbook].

## EXAMPLES:

```

sage: x = PolynomialRing(QQ, 'x').gen()
sage: jacobi_P(2,0,0,x)
3/2*x^2 - 1/2
sage: jacobi_P(2,1,2,1.2)      # random output of low order bits
5.009999999999998

```

**Check that trac ticket #17192 is fixed::** sage: x = PolynomialRing(QQ, 'x').gen() sage: jacobi\_P(0,0,0,x) 1

sage: jacobi\_P(-1,0,0,x) Traceback (most recent call last): ... ValueError: n must be greater than -1, got n = -1

sage: jacobi\_P(-7,0,0,x) Traceback (most recent call last): ... ValueError: n must be greater than -1, got n = -7

sage.functions.orthogonal\_polys.**laguerre**(n, x)

Return the Laguerre polynomial for integers  $n > -1$ .

## REFERENCE:

- [ASHandbook] 22.5.16, page 778 and page 789.

## EXAMPLES:

```

sage: x = PolynomialRing(QQ, 'x').gen()
sage: laguerre(2,x)
1/2*x^2 - 2*x + 1

```

```
sage: laguerre(3,x)
-1/6*x^3 + 3/2*x^2 - 3*x + 1
sage: laguerre(2,2)
-1
```

**Check that [trac ticket #17192](#) is fixed::** sage: x = PolynomialRing(QQ, 'x').gen() sage: laguerre(0,x) 1

sage: laguerre(-1,x) Traceback (most recent call last): ... ValueError: n must be greater than -1, got n = -1

sage: laguerre(-7,x) Traceback (most recent call last): ... ValueError: n must be greater than -1, got n = -7

sage.functions.orthogonal\_polys.**legendre\_P**(n, x)

Returns the Legendre polynomial of the first kind.

REFERENCE:

•[\[ASHandbook\]](#) 22.5.35 page 779.

EXAMPLES:

```
sage: P.<t> = QQ[]
sage: legendre_P(2,t)
3/2*t^2 - 1/2
sage: legendre_P(3, 1.1)
1.677500000000000
sage: legendre_P(3, 1 + I)
7/2*I - 13/2
sage: legendre_P(3, MatrixSpace(ZZ, 2) ([1, 2, -4, 7]))
[-179 242]
[-484 547]
sage: legendre_P(3, GF(11)(5))
8
```

sage.functions.orthogonal\_polys.**legendre\_Q**(n, x)

Returns the Legendre function of the second kind.

Computed using Maxima.

EXAMPLES:

```
sage: P.<t> = QQ[]
sage: legendre_Q(2, t)
3/4*t^2*log(-(t + 1)/(t - 1)) - 3/2*t - 1/4*log(-(t + 1)/(t - 1))
sage: legendre_Q(3, 0.5)
-0.198654771479482
sage: legendre_Q(4, 2)
443/16*I*pi + 443/16*log(3) - 365/12
sage: legendre_Q(4, 2.0)
0.00116107583162324 + 86.9828465962674*I
```

sage.functions.orthogonal\_polys.**ultraspherical**(n, a, x)

Returns the ultraspherical (or Gegenbauer) polynomial for integers  $n > -1$ .

Computed using Maxima.

REFERENCE:

•[\[ASHandbook\]](#) 22.5.27

EXAMPLES:

```
sage: x = PolynomialRing(QQ, 'x').gen()
sage: ultraspherical(2, 3/2, x)
15/2*x^2 - 3/2
sage: ultraspherical(2, 1/2, x)
3/2*x^2 - 1/2
sage: ultraspherical(1, 1, x)
2*x
sage: t = PolynomialRing(RationalField(), "t").gen()
sage: gegenbauer(3, 2, t)
32*t^3 - 12*t
```

**Check that [trac ticket #17192](#) is fixed::** sage: x = PolynomialRing(QQ, 'x').gen() sage: ultraspherical(0,1,x)  
1

sage: ultraspherical(-1,1,x) Traceback (most recent call last): ... ValueError: n must be greater than -1, got  
n = -1

sage: ultraspherical(-7,1,x) Traceback (most recent call last): ... ValueError: n must be greater than -1, got  
n = -7



## OTHER FUNCTIONS

```
class sage.functions.other.Function_abs
    Bases: sage.symbolic.function.GinacFunction

    The absolute value function.

    EXAMPLES:
    sage: var('x y')
    (x, y)
    sage: abs(x)
    abs(x)
    sage: abs(x^2 + y^2)
    abs(x^2 + y^2)
    sage: abs(-2)
    2
    sage: sqrt(x^2)
    sqrt(x^2)
    sage: abs(sqrt(x))
    sqrt(abs(x))
    sage: complex(abs(3*I))
    (3+0j)

    sage: f = sage.functions.other.Function_abs()
    sage: latex(f)
    \mathrm{abs}
    sage: latex(abs(x))
    {\left| x \right|}
    sage: abs(x)._sympy_()
    Abs(x)

    Test pickling:
    sage: loads(dumps(abs(x)))
    abs(x)
```

```
class sage.functions.other.Function_arg
    Bases: sage.symbolic.function.BuiltinFunction

    The argument function for complex numbers.

    EXAMPLES:
    sage: arg(3+i)
    arctan(1/3)
    sage: arg(-1+i)
    3/4*pi
    sage: arg(2+2*i)
```

```
1/4*pi
sage: arg(2+x)
arg(x + 2)
sage: arg(2.0+i+x)
arg(x + 2.000000000000000 + 1.000000000000000*I)
sage: arg(-3)
pi
sage: arg(3)
0
sage: arg(0)
0
sage: latex(arg(x))
{\rm arg}\left(x\right)
sage: maxima(arg(x))
atan2(0,_SAGE_VAR_x)
sage: maxima(arg(2+i))
atan(1/2)
sage: maxima(arg(sqrt(2)+i))
atan(1/sqrt(2))
sage: arg(2+i)
arctan(1/2)
sage: arg(sqrt(2)+i)
arg(sqrt(2) + I)
sage: arg(sqrt(2)+i).simplify()
arctan(1/2*sqrt(2))
```

**TESTS:**

```
sage: arg(0.0)
0.0000000000000000
sage: arg(3.0)
0.0000000000000000
sage: arg(-2.5)
3.14159265358979
sage: arg(2.0+3*i)
0.982793723247329
```

**class** sage.functions.other.**Function\_beta**

Bases: sage.symbolic.function.GinacFunction

Return the beta function. This is defined by

$$B(p, q) = \int_0^1 t^{p-1} (1-t)^{1-q} dt$$

for complex or symbolic input  $p$  and  $q$ . Note that the order of inputs does not matter:  $B(p, q) = B(q, p)$ .

GiNaC is used to compute  $B(p, q)$ . However, complex inputs are not yet handled in general. When GiNaC raises an error on such inputs, we raise a `NotImplementedError`.

If either input is 1, GiNaC returns the reciprocal of the other. In other cases, GiNaC uses one of the following formulas:

$$B(p, q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$$

or

$$B(p, q) = (-1)^q B(1-p-q, q).$$

For numerical inputs, GiNaC uses the formula

$$B(p, q) = \exp[\log \Gamma(p) + \log \Gamma(q) - \log \Gamma(p + q)]$$

INPUT:

- $p$  - number or symbolic expression
- $q$  - number or symbolic expression

OUTPUT: number or symbolic expression (if input is symbolic)

EXAMPLES:

```
sage: beta(3, 2)
1/12
sage: beta(3, 1)
1/3
sage: beta(1/2, 1/2)
beta(1/2, 1/2)
sage: beta(-1, 1)
-1
sage: beta(-1/2, -1/2)
0
sage: beta(x/2, 3)
beta(3, 1/2*x)
sage: beta(.5, .5)
3.14159265358979
sage: beta(1, 2.0+I)
0.4000000000000000 - 0.2000000000000000*I
sage: beta(3, x+I)
beta(3, x + I)
```

Note that the order of arguments does not matter:

```
sage: beta(1/2, 3*x)
beta(1/2, 3*x)
```

The result is symbolic if exact input is given:

```
sage: beta(2, 1+5*I)
beta(2, 5*I + 1)
sage: beta(2, 2.)
0.1666666666666667
sage: beta(I, 2.)
-0.5000000000000000 - 0.5000000000000000*I
sage: beta(2., 2)
0.1666666666666667
sage: beta(2., I)
-0.5000000000000000 - 0.5000000000000000*I
```

Test pickling:

```
sage: loads(dumps(beta))
beta
```

```
class sage.functions.other.Function_binomial
    Bases: sage.symbolic.function.GinacFunction
```

Return the binomial coefficient

$$\binom{x}{m} = x(x-1) \cdots (x-m+1)/m!$$

which is defined for  $m \in \mathbb{Z}$  and any  $x$ . We extend this definition to include cases when  $x - m$  is an integer but  $m$  is not by

$$\binom{x}{m} = \binom{x}{x-m}$$

If  $m < 0$ , return 0.

INPUT:

- $x, m$  - numbers or symbolic expressions. Either  $m$  or  $x-m$  must be an integer, else the output is symbolic.

OUTPUT: number or symbolic expression (if input is symbolic)

EXAMPLES:

```
sage: binomial(5,2)
10
sage: binomial(2,0)
1
sage: binomial(1/2, 0)
1
sage: binomial(3,-1)
0
sage: binomial(20,10)
184756
sage: binomial(-2, 5)
-6
sage: binomial(RealField()('2.5'), 2)
1.875000000000000
sage: n=var('n'); binomial(n,2)
1/2*(n - 1)*n
sage: n=var('n'); binomial(n,n)
1
sage: n=var('n'); binomial(n,n-1)
n
sage: binomial(2^100, 2^100)
1

sage: k, i = var('k,i')
sage: binomial(k,i)
binomial(k, i)
```

We can use a `hold` parameter to prevent automatic evaluation:

```
sage: SR(5).binomial(3, hold=True)
binomial(5, 3)
sage: SR(5).binomial(3, hold=True).simplify()
10
```

TESTS: We verify that we can convert this function to Maxima and bring it back into Sage.

```
sage: n,k = var('n,k')
sage: maxima(binomial(n,k))
binomial(_SAGE_VAR_n,_SAGE_VAR_k)
sage: _.sage()
binomial(n, k)
sage: binomial._maxima_init_()
'binomial'
```

Test pickling:



1. The `x.ceil()` method is called and returned if it is there. If it is not, then Sage checks if  $x$  is one of Python's native numeric data types. If so, then it calls and returns `Integer(int(math.ceil(x)))`.
2. Sage tries to convert  $x$  into a `RealIntervalField` with 53 bits of precision. Next, the ceilings of the endpoints are computed. If they are the same, then that value is returned. Otherwise, the precision of the `RealIntervalField` is increased until they do match up or it reaches `maximum_bits` of precision.
3. If none of the above work, Sage returns a `Expression` object.

EXAMPLES:

69

```
sage: import numpy
sage: a = numpy.linspace(0,2,6)
sage: ceil(a)
array([ 0.,  1.,  1.,  2.,  2.,  2.]
```

Test pickling:

```
sage: loads(dumps(ceil))
ceil
```

**class** sage.functions.other.**Function\_conjugate**  
Bases: sage.symbolic.function.GinacFunction

Returns the complex conjugate of the input.

It is possible to prevent automatic evaluation using the hold parameter:

```
sage: conjugate(I, hold=True)
conjugate(I)
```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```
sage: conjugate(I, hold=True).simplify()
-I
```

TESTS:

```
sage: x,y = var('x,y')
sage: x.conjugate()
conjugate(x)
sage: latex(conjugate(x))
\overline{x}
sage: f = function('f')
sage: latex(f(x).conjugate())
\overline{f\left(x\right)}
sage: f = function('psi',x,y)
sage: latex(f.conjugate())
\overline{\psi\left(x,y\right)}
sage: x.conjugate().conjugate()
x
sage: x.conjugate().operator()
conjugate
sage: x.conjugate().operator() == conjugate
True
```

Check if #8755 is fixed:

```
sage: conjugate(sqrt(-3))
conjugate(sqrt(-3))
sage: conjugate(sqrt(3))
sqrt(3)
sage: conjugate(sqrt(x))
conjugate(sqrt(x))
sage: conjugate(x^2)
conjugate(x)^2
sage: var('y', domain='positive')
y
sage: conjugate(sqrt(y))
sqrt(y)
```

Check if #10964 is fixed:

```

sage: z= I*sqrt(-3); z
I*sqrt(-3)
sage: conjugate(z)
-I*conjugate(sqrt(-3))
sage: var('a')
a
sage: conjugate(a*sqrt(-2)*sqrt(-3))
conjugate(sqrt(-2))*conjugate(sqrt(-3))*conjugate(a)

```

Test pickling:

```

sage: loads(dumps(conjugate))
conjugate

```

**class** sage.functions.other.**Function\_erf**  
 Bases: sage.symbolic.function.BuiltinFunction

The error function, defined for real values as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

This function is also defined for complex values, via analytic continuation.

EXAMPLES:

We can evaluate numerically:

```

sage: erf(2)
erf(2)
sage: erf(2).n()
0.995322265018953
sage: erf(2).n(100)
0.99532226501895273416206925637
sage: erf(ComplexField(100)(2+3j))
-20.829461427614568389103088452 + 8.6873182714701631444280787545*I

```

Basic symbolic properties are handled by Sage and Maxima:

```

sage: x = var("x")
sage: diff(erf(x), x)
2*e^(-x^2)/sqrt(pi)
sage: integrate(erf(x), x)
x*erf(x) + e^(-x^2)/sqrt(pi)

```

ALGORITHM:

Sage implements numerical evaluation of the error function via the `erf()` function from `mpmath`. Symbolics are handled by Sage and Maxima.

REFERENCES:

- [http://en.wikipedia.org/wiki/Error\\_function](http://en.wikipedia.org/wiki/Error_function)
- <http://mpmath.googlecode.com/svn/trunk/doc/build/functions/expintegrals.html#error-functions>

TESTS:

Check limits:

```

sage: limit(erf(x), x=0)
0
sage: limit(erf(x), x=infinity)
1

```

Check that it's odd::

```
sage: erf(1.0)
0.842700792949715
sage: erf(-1.0)
-0.842700792949715
```

Check against other implementations and against the definition:

```
sage: erf(3).n()
0.999977909503001
sage: maxima.erf(3).n()
0.999977909503001
sage: (1-pari(3).erfc())
0.999977909503001
sage: RR(3).erf()
0.999977909503001
sage: (integrate(exp(-x**2), (x, 0, 3)) * 2/sqrt(pi)).n()
0.999977909503001
```

trac ticket #9044:

```
sage: N(erf(sqrt(2)), 200)
0.95449973610364158559943472566693312505644755259664313203267
```

trac ticket #11626:

```
sage: n(erf(2), 100)
0.99532226501895273416206925637
sage: erf(2).n(100)
0.99532226501895273416206925637
```

Test (indirectly) trac ticket #11885:

```
sage: erf(float(0.5))
0.5204998778130465
sage: erf(complex(0.5))
(0.5204998778130465+0j)
```

Ensure conversion from maxima elements works:

```
sage: merf = maxima(erf(x)).sage().operator()
sage: merf == erf
True
```

Make sure we can dump and load it:

```
sage: loads(dumps(erf(2)))
erf(2)
```

Special-case 0 for immediate evaluation:

```
sage: erf(0)
0
sage: solve(erf(x)==0, x)
[x == 0]
```

Make sure that we can hold:

```

sage: erf(0, hold=True)
erf(0)
sage: simplify(erf(0, hold=True))
0

```

Check that high-precision ComplexField inputs work:

```

sage: CC(erf(ComplexField(1000)(2+3j)))
-20.8294614276146 + 8.68731827147016*I

```

**class** `sage.functions.other.Function_factorial`  
 Bases: `sage.symbolic.function.GinacFunction`

Returns the factorial of  $n$ .

INPUT:

- $n$  - any complex argument (except negative integers) or any symbolic expression

OUTPUT: an integer or symbolic expression

EXAMPLES:

```

sage: x = var('x')
sage: factorial(0)
1
sage: factorial(4)
24
sage: factorial(10)
3628800
sage: factorial(6) == 6*5*4*3*2
True
sage: f = factorial(x + factorial(x)); f
factorial(x + factorial(x))
sage: f(x=3)
362880
sage: factorial(x)^2
factorial(x)^2

```

To prevent automatic evaluation use the `hold` argument:

```

sage: factorial(5, hold=True)
factorial(5)

```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```

sage: factorial(5, hold=True).simplify()
120

```

We can also give input other than nonnegative integers. For other nonnegative numbers, the `gamma()` function is used:

```

sage: factorial(1/2)
1/2*sqrt(pi)
sage: factorial(3/4)
gamma(7/4)
sage: factorial(2.3)
2.68343738195577

```

But negative input always fails:

```
sage: factorial(-32)
Traceback (most recent call last):
...
ValueError: factorial -- self = (-32) must be nonnegative
```

#### TESTS:

We verify that we can convert this function to Maxima and bring it back into Sage.:

```
sage: z = var('z')
sage: factorial._maxima_init_()
'factorial'
sage: maxima(factorial(z))
factorial(_SAGE_VAR_z)
sage: _.sage()
factorial(z)
sage: k = var('k')
sage: factorial(k)
factorial(k)
```

```
sage: factorial(3.14)
7.173269190187...
```

#### Test latex typesetting:

```
sage: latex(factorial(x))
x!
sage: latex(factorial(2*x))
\left(2 \, , \, x\right)!
sage: latex(factorial(sin(x)))
\sin\left(x\right)!
sage: latex(factorial(sqrt(x+1)))
\left(\sqrt{x + 1}\right)!
sage: latex(factorial(sqrt(x)))
\sqrt{x}!
sage: latex(factorial(x^(2/3)))
\left(x^{\frac{2}{3}}\right)!

sage: latex(factorial)
{\rm factorial}
```

#### Check that #11539 is fixed:

```
sage: (factorial(x) == 0).simplify()
factorial(x) == 0
sage: maxima(factorial(x) == 0).sage()
factorial(x) == 0
sage: y = var('y')
sage: (factorial(x) == y).solve(x)
[factorial(x) == y]
```

#### Test pickling:

```
sage: loads(dumps(factorial))
factorial
```

```
class sage.functions.other.Function_floor
    Bases: sage.symbolic.function.BuiltinFunction
    The floor function.
```

- EXAMPLES:

Test pickling:

The Gamma function. This is defined by

It is computed by various libraries within Sage, depending on the input type.

75

```
sage: from sage.functions.other import gamma1
sage: gamma1(CDF(0.5,14))
-4.0537030780372815e-10 - 5.773299834553605e-10*I
sage: gamma1(CDF(I))
-0.15494982830181067 - 0.49801566811835607*I
```

Recall that  $\Gamma(n)$  is  $n - 1$  factorial:

```
sage: gamma1(11) == factorial(10)
True
sage: gamma1(6)
120
sage: gamma1(1/2)
sqrt(pi)
sage: gamma1(-1)
Infinity
sage: gamma1(I)
gamma(I)
sage: gamma1(x/2) (x=5)
3/4*sqrt(pi)

sage: gamma1(float(6)) # For ARM: rel tol 3e-16
120.0
sage: gamma(6.)
120.00000000000000
sage: gamma1(x)
gamma(x)

sage: gamma1(pi)
gamma(pi)
sage: gamma1(i)
gamma(I)
sage: gamma1(i).n()
-0.154949828301811 - 0.498015668118356*I
sage: gamma1(int(5))
24

sage: conjugate(gamma(x))
gamma(conjugate(x))

sage: plot(gamma1(x), (x, 1, 5))
Graphics object consisting of 1 graphics primitive
```

To prevent automatic evaluation use the `hold` argument:

```
sage: gamma1(1/2, hold=True)
gamma(1/2)
```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```
sage: gamma1(1/2, hold=True).simplify()
sqrt(pi)
```

## TESTS:

We verify that we can convert this function to Maxima and convert back to Sage:

```
sage: z = var('z')
sage: maxima(gamma1(z)).sage()
gamma(z)
```



```
sage: latex(gamma1(z))
\Gamma\left(z\right)
```

Test that Trac ticket 5556 is fixed:

```
sage: gamma1(3/4)
gamma(3/4)
```

```
sage: gamma1(3/4).n(100)
1.2254167024651776451290983034
```

Check that negative integer input works:

```
sage: (-1).gamma()
Infinity
sage: (-1.).gamma()
NaN
sage: CC(-1).gamma()
Infinity
sage: RDF(-1).gamma()
NaN
sage: CDF(-1).gamma()
Infinity
```

Check if #8297 is fixed:

```
sage: latex(gamma(1/4))
\Gamma\left(\frac{1}{4}\right)
```

Test pickling:

```
sage: loads(dumps(gamma(x)))
gamma(x)
```

```
class sage.functions.other.Function_gamma_inc
    Bases: sage.symbolic.function.BuiltinFunction
```

The incomplete gamma function.

EXAMPLES:

```
sage: gamma_inc(CDF(0,1), 3)
0.003208574993369116 + 0.012406185811871568*I
sage: gamma_inc(RDF(1), 3)
0.049787068367863944
sage: gamma_inc(3,2)
gamma(3, 2)
sage: gamma_inc(x,0)
gamma(x)
sage: latex(gamma_inc(3,2))
\Gamma\left(3, 2\right)
sage: loads(dumps((gamma_inc(3,2))))
gamma(3, 2)
sage: i = ComplexField(30).0; gamma_inc(2, 1 + i)
0.70709210 - 0.42035364*I
sage: gamma_inc(2., 5)
0.0404276819945128
```

```
class sage.functions.other.Function_imag_part
    Bases: sage.symbolic.function.GinacFunction
```

Returns the imaginary part of the (possibly complex) input.

It is possible to prevent automatic evaluation using the `hold` parameter:

```
sage: imag_part(I, hold=True)
imag_part(I)
```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```
sage: imag_part(I, hold=True).simplify()
1
```

TESTS:

```
sage: z = 1+2*I
sage: imaginary(z)
2
sage: imag(z)
2
sage: imag(complex(3, 4))
4.0
sage: loads(dumps(imag_part))
imag_part
sage: imag_part(x)._sympy_()
im(x)
```

Check if #6401 is fixed:

```
sage: latex(x.imag())
\Im \left( x \right)

sage: f(x) = function('f', x)
sage: latex(f(x).imag())
\Im \left( f\left(x\right) \right)
```

**class** `sage.functions.other.Function_log_gamma`  
Bases: `sage.symbolic.function.GinacFunction`

The principal branch of the logarithm of Gamma function. Gamma is defined for complex input  $z$  with real part greater than zero, and by analytic continuation on the rest of the complex plane (except for negative integers, which are poles).

It is computed by the `log_gamma` function for the number type, or by `lgamma` in Ginac, failing that.

EXAMPLES:

Numerical evaluation happens when appropriate, to the appropriate accuracy (see #10072):

```
sage: log_gamma(6)
log(120)
sage: log_gamma(6.)
4.78749174278205
sage: log_gamma(6).n()
4.78749174278205
sage: log_gamma(RealField(100)(6))
4.7874917427820459942477009345
sage: log_gamma(2.4+i)
-0.0308566579348816 + 0.693427705955790*I
sage: log_gamma(-3.1)
0.400311696703985
```

Symbolic input works (see #10075):

```

sage: log_gamma(3*x)
log_gamma(3*x)
sage: log_gamma(3+i)
log_gamma(I + 3)
sage: log_gamma(3+i+x)
log_gamma(x + I + 3)

```

To get evaluation of input for which gamma is negative and the ceiling is even, we must explicitly make the input complex. This is a known issue, see #12521:

```

sage: log_gamma(-2.1)
NaN
sage: log_gamma(CC(-2.1))
1.53171380819509 + 3.14159265358979*I

```

In order to prevent evaluation, use the *hold* argument; to evaluate a held expression, use the *n()* numerical evaluation method:

```

sage: log_gamma(SR(5), hold=True)
log_gamma(5)
sage: log_gamma(SR(5), hold=True).n()
3.17805383034795

```

#### TESTS:

```

sage: log_gamma(-2.1+i)
-1.90373724496982 - 0.901638463592247*I
sage: log_gamma(pari(6))
4.78749174278205
sage: log_gamma(CC(6))
4.78749174278205
sage: log_gamma(CC(-2.5))
-0.0562437164976740 + 3.14159265358979*I
sage: log_gamma(x)._sympy_()
loggamma(x)

```

`conjugate(log_gamma(x)) == log_gamma(conjugate(x))` unless on the branch cut, which runs along the negative real axis.:

```

sage: conjugate(log_gamma(x))
conjugate(log_gamma(x))
sage: var('y', domain='positive')
y
sage: conjugate(log_gamma(y))
log_gamma(y)
sage: conjugate(log_gamma(y+I))
conjugate(log_gamma(y + I))
sage: log_gamma(-2)
+Infinity
sage: conjugate(log_gamma(-2))
+Infinity

```

**class** sage.functions.other.**Function\_psi1**

Bases: sage.symbolic.function.GinacFunction

The digamma function,  $\psi(x)$ , is the logarithmic derivative of the gamma function.

$$\psi(x) = \frac{d}{dx} \log(\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}$$

## EXAMPLES:

```
sage: from sage.functions.other import psi1
sage: psi1(x)
psi(x)
sage: psi1(x).derivative(x)
psi(1, x)

sage: psi1(3)
-euler_gamma + 3/2

sage: psi(.5)
-1.96351002602142
sage: psi(RealField(100)(.5))
-1.9635100260214234794409763330
```

## TESTS:

```
sage: latex(psi1(x))
\psi\left(x\right)
sage: loads(dumps(psi1(x)+1))
psi(x) + 1

sage: t = psi1(x); t
psi(x)
sage: t.subs(x=.2)
-5.28903989659219
sage: psi(x)._sympy_()
polygamma(0, x)
```

```
class sage.functions.other.Function_psi2
Bases: sage.symbolic.function.GinacFunction
```

Derivatives of the digamma function  $\psi(x)$ . T

## EXAMPLES:

```
sage: from sage.functions.other import psi2
sage: psi2(2, x)
psi(2, x)
sage: psi2(2, x).derivative(x)
psi(3, x)
sage: n = var('n')
sage: psi2(n, x).derivative(x)
psi(n + 1, x)

sage: psi2(0, x)
psi(x)
sage: psi2(-1, x)
log(gamma(x))
sage: psi2(3, 1)
1/15*pi^4

sage: psi2(2, .5).n()
-16.8287966442343
sage: psi2(2, .5).n(100)
-16.828796644234319995596334261
```

## TESTS:

```

sage: psi2(n, x).derivative(n)
Traceback (most recent call last):
...
RuntimeError: cannot diff psi(n,x) with respect to n

sage: latex(psi2(2,x))
\psi\left(2, x\right)
sage: loads(dumps(psi2(2,x)+1))
psi(2, x) + 1
sage: psi(2, x)._sympy_()
polygamma(2, x)

```

```

class sage.functions.other.Function_real_part
Bases: sage.symbolic.function.GinacFunction

```

Returns the real part of the (possibly complex) input.

It is possible to prevent automatic evaluation using the hold parameter:

```

sage: real_part(I, hold=True)
real_part(I)

```

To then evaluate again, we currently must use Maxima via `sage.symbolic.expression.Expression.simplify()`:

```

sage: real_part(I, hold=True).simplify()
0

```

#### EXAMPLES:

```

sage: z = 1+2*I
sage: real(z)
1
sage: real(5/3)
5/3
sage: a = 2.5
sage: real(a)
2.500000000000000
sage: type(real(a))
<type 'sage.rings.real_mpr.RealLiteral'>
sage: real(1.0r)
1.0
sage: real(complex(3, 4))
3.0

```

#### TESTS:

```

sage: loads(dumps(real_part))
real_part
sage: real_part(x)._sympy_()
re(x)

```

Check if #6401 is fixed:

```

sage: latex(x.real())
\Re \left( x \right)

sage: f(x) = function('f', x)
sage: latex(f(x).real())
\Re \left( f\left(x\right) \right)

```

```
class sage.functions.other.Function_sqrt
    Bases: object
```

```
sage.functions.other.gamma(a, *args, **kws)
    Gamma and incomplete gamma functions. This is defined by the integral
```

$$\Gamma(a, z) = \int_z^{\infty} t^{a-1} e^{-t} dt$$

**EXAMPLES:**

Recall that `\Gamma(n)` is `n-1` factorial::

```
sage: gamma(11) == factorial(10)
True
sage: gamma(6)
120
sage: gamma(1/2)
sqrt(pi)
sage: gamma(-4/3)
gamma(-4/3)
sage: gamma(-1)
Infinity
sage: gamma(0)
Infinity

::

sage: gamma_inc(3, 2)
gamma(3, 2)
sage: gamma_inc(x, 0)
gamma(x)

::

sage: gamma(5, hold=True)
gamma(5)
sage: gamma(x, 0, hold=True)
gamma(x, 0)

::

sage: gamma(CDF(0.5, 14))
-4.0537030780372815e-10 - 5.773299834553605e-10*I
sage: gamma(CDF(I))
-0.15494982830181067 - 0.49801566811835607*I
```

The precision for the result is deduced from the precision of the input. Convert the input to a higher precision explicitly if a result with higher precision is desired::

```
sage: t = gamma(RealField(100)(2.5)); t
1.3293403881791370204736256125
sage: t.prec()
100

sage: gamma(6)
120

sage: gamma(pi).n(100)
```

```
2.2880377953400324179595889091
```

```
sage: gamma(3/4).n(100)
1.2254167024651776451290983034
```

The gamma function only works with input that can be coerced to the Symbolic Ring::

```
sage: Q.<i> = NumberField(x^2+1)
sage: gamma(i)
Traceback (most recent call last):
...
TypeError: cannot coerce arguments: no canonical coercion...
```

We make an exception for elements of `AA` or `QQbar`, which cannot be coerced into symbolic expressions to allow this usage::

```
sage: t = QQbar(sqrt(2)) + sqrt(3); t
3.146264369941973?
sage: t.parent()
Algebraic Field
```

Symbolic functions convert the arguments to symbolic expressions if they are in `QQbar` or `AA`::

```
sage: gamma(QQbar(I))
-0.154949828301811 - 0.498015668118356*I
```

`sage.functions.other.psi(x, *args, **kws)`

The digamma function,  $\psi(x)$ , is the logarithmic derivative of the gamma function.

$$\psi(x) = \frac{d}{dx} \log(\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}$$

We represent the  $n$ -th derivative of the digamma function with  $\psi(n, x)$  or  $\text{psi}(n, x)$ .

#### EXAMPLES:

```
sage: psi(x)
psi(x)
sage: psi(.5)
-1.96351002602142
sage: psi(3)
-euler_gamma + 3/2
sage: psi(1, 5)
1/6*pi^2 - 205/144
sage: psi(1, x)
psi(1, x)
sage: psi(1, x).derivative(x)
psi(2, x)

sage: psi(3, hold=True)
psi(3)
sage: psi(1, 5, hold=True)
psi(1, 5)
```

#### TESTS:









## SPECIAL FUNCTIONS

## AUTHORS:

- David Joyner (2006-13-06): initial version
- David Joyner (2006-30-10): bug fixes to pari wrappers of Bessel functions, hypergeometric\_U
- William Stein (2008-02): Impose some sanity checks.
- David Joyner (2008-04-23): addition of elliptic integrals

This module provides easy access to many of Maxima and PARI's special functions.

Maxima's special functions package (which includes spherical harmonic functions, spherical Bessel functions (of the 1st and 2nd kind), and spherical Hankel functions (of the 1st and 2nd kind)) was written by Barton Willis of the University of Nebraska at Kearney. It is released under the terms of the General Public License (GPL).

Support for elliptic functions and integrals was written by Raymond Toy. It is placed under the terms of the General Public License (GPL) that governs the distribution of Maxima.

Next, we summarize some of the properties of the functions implemented here.

- Spherical harmonics: Laplace's equation in spherical coordinates is:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2} = 0.$$

Note that the spherical coordinates  $\theta$  and  $\varphi$  are defined here as follows:  $\theta$  is the colatitude or polar angle, ranging from  $0 \leq \theta \leq \pi$  and  $\varphi$  the azimuth or longitude, ranging from  $0 \leq \varphi < 2\pi$ .

The general solution which remains finite towards infinity is a linear combination of functions of the form

$$r^{-1-\ell} \cos(m\varphi) P_\ell^m(\cos \theta)$$

and

$$r^{-1-\ell} \sin(m\varphi) P_\ell^m(\cos \theta)$$

where  $P_\ell^m$  are the associated Legendre polynomials, and with integer parameters  $\ell \geq 0$  and  $m$  from 0 to  $\ell$ . Put in another way, the solutions with integer parameters  $\ell \geq 0$  and  $-\ell \leq m \leq \ell$ , can be written as linear combinations of:

$$U_{\ell,m}(r, \theta, \varphi) = r^{-1-\ell} Y_\ell^m(\theta, \varphi)$$

where the functions  $Y$  are the spherical harmonic functions with parameters  $\ell, m$ , which can be written as:

$$Y_\ell^m(\theta, \varphi) = \sqrt{\frac{(2\ell+1)}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} \cdot e^{im\varphi} \cdot P_\ell^m(\cos \theta).$$

The spherical harmonics obey the normalisation condition

$$\int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} Y_{\ell}^m Y_{\ell'}^{m'*} d\Omega = \delta_{\ell\ell'} \delta_{mm'} \quad d\Omega = \sin \theta d\varphi d\theta.$$

- When solving for separable solutions of Laplace's equation in spherical coordinates, the radial equation has the form:

$$x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} + [x^2 - n(n+1)]y = 0.$$

The spherical Bessel functions  $j_n$  and  $y_n$ , are two linearly independent solutions to this equation. They are related to the ordinary Bessel functions  $J_n$  and  $Y_n$  by:

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x),$$

$$y_n(x) = \sqrt{\frac{\pi}{2x}} Y_{n+1/2}(x) = (-1)^{n+1} \sqrt{\frac{\pi}{2x}} J_{-n-1/2}(x).$$

- For  $x > 0$ , the confluent hypergeometric function  $y = U(a, b, x)$  is defined to be the solution to Kummer's differential equation

$$xy'' + (b-x)y' - ay = 0,$$

which satisfies  $U(a, b, x) \sim x^{-a}$ , as  $x \rightarrow \infty$ . (There is a linearly independent solution, called Kummer's function  $M(a, b, x)$ , which is not implemented.)

- The incomplete elliptic integrals (of the first kind, etc.) are:

$$\begin{aligned} & \int_0^\phi \frac{1}{\sqrt{1-m\sin(x)^2}} dx, \\ & \int_0^\phi \sqrt{1-m\sin(x)^2} dx, \\ & \int_0^\phi \frac{\sqrt{1-mt^2}}{\sqrt{(1-t^2)}} dx, \\ & \int_0^\phi \frac{1}{\sqrt{1-m\sin(x)^2} \sqrt{1-n\sin(x)^2}} dx, \end{aligned}$$

and the complete ones are obtained by taking  $\phi = \pi/2$ .

#### REFERENCES:

- Abramowitz and Stegun: Handbook of Mathematical Functions, <http://www.math.sfu.ca/~cbm/aands/>
- [http://en.wikipedia.org/wiki/Spherical\\_harmonics](http://en.wikipedia.org/wiki/Spherical_harmonics)
- [http://en.wikipedia.org/wiki/Helmholtz\\_equation](http://en.wikipedia.org/wiki/Helmholtz_equation)
- Online Encyclopedia of Special Function <http://algo.inria.fr/esf/index.html>

TODO: Resolve weird bug in commented out code in hypergeometric\_U below.

#### AUTHORS:

- David Joyner and William Stein

Added 16-02-2008 (wdj): optional calls to scipy and replace all ‘#random’ by ‘...’ (both at the request of William Stein)

**Warning:** SciPy’s versions are poorly documented and seem less accurate than the Maxima and PARI versions; typically they are limited by hardware floats precision.

**class** sage.functions.special.**EllipticE**

Bases: sage.functions.special.MaximaFunction

This returns the value of the “incomplete elliptic integral of the second kind,”

$$\int_0^\phi \sqrt{1 - m \sin(x)^2} dx,$$

i.e., `integrate(sqrt(1 - m*sin(x)^2), x, 0, phi)`. Taking  $\phi = \pi/2$  gives `elliptic_ec`.

EXAMPLES:

```
sage: z = var("z")
sage: # this is still wrong: must be abs(sin(z)) + 2*round(z/pi)
sage: elliptic_e(z, 1)
2*round(z/pi) + sin(z)
sage: elliptic_e(z, 0)
z
sage: elliptic_e(0.5, 0.1) # abs tol 2e-15
0.498011394498832
```

**class** sage.functions.special.**EllipticEC**

Bases: sage.functions.special.MaximaFunction

This returns the value of the “complete elliptic integral of the second kind,”

$$\int_0^{\pi/2} \sqrt{1 - m \sin(x)^2} dx.$$

EXAMPLES:

```
sage: elliptic_ec(0.1)
1.53075763689776
sage: elliptic_ec(x).diff()
1/2*(elliptic_ec(x) - elliptic_kc(x))/x

sage: loads(dumps(elliptic_ec))
elliptic_ec
```

**class** sage.functions.special.**EllipticEU**

Bases: sage.functions.special.MaximaFunction

Return the value of the “incomplete elliptic integral of the second kind,”

$$\int_0^u \operatorname{dn}(x, m)^2 dx = \int_0^\tau \frac{\sqrt{1 - mx^2}}{\sqrt{1 - x^2}} dx.$$

where  $\tau = \operatorname{sn}(u, m)$ .

EXAMPLES:

```
sage: elliptic_eu (0.5, 0.1)
0.496054551286597
```

**class** `sage.functions.special.EllipticF`  
Bases: `sage.functions.special.MaximaFunction`

This returns the value of the “incomplete elliptic integral of the first kind,”

$$\int_0^\phi \frac{dx}{\sqrt{1 - m \sin(x)^2}},$$

i.e., `integrate(1/sqrt(1 - m*sin(x)^2), x, 0, phi)`. Taking  $\phi = \pi/2$  gives `elliptic_kc`.

EXAMPLES:

```
sage: z = var("z")
sage: elliptic_f(z, 0)
z
sage: elliptic_f(z, 1)
log(tan(1/4*pi + 1/2*z))
sage: elliptic_f(0.2, 0.1)
0.200132506747543
```

**class** `sage.functions.special.EllipticKC`  
Bases: `sage.functions.special.MaximaFunction`

This returns the value of the “complete elliptic integral of the first kind,”

$$\int_0^{\pi/2} \frac{dx}{\sqrt{1 - m \sin(x)^2}}.$$

EXAMPLES:

```
sage: elliptic_kc(0.5)
1.85407467730137
sage: elliptic_f(RR(pi/2), 0.5)
1.85407467730137
```

**class** `sage.functions.special.EllipticPi`  
Bases: `sage.functions.special.MaximaFunction`

This returns the value of the “incomplete elliptic integral of the third kind,”

$$\text{elliptic\_pi}(n, t, m) = \int_0^t \frac{dx}{(1 - n \sin(x)^2) \sqrt{1 - m \sin(x)^2}}.$$

INPUT:

- `n` – a real number, called the “characteristic”
- `t` – a real number, called the “amplitude”
- `m` – a real number, called the “parameter”

EXAMPLES:

```
sage: N(elliptic_pi(1, pi/4, 1))
1.14779357469632
```

Compare the value computed by Maxima to the definition as a definite integral (using GSL):

```
sage: elliptic_pi(0.1, 0.2, 0.3)
0.200665068220979
sage: numerical_integral(1/(1-0.1*sin(x)^2)/sqrt(1-0.3*sin(x)^2), 0.0, 0.2)
(0.2006650682209791, 2.227829789769088e-15)
```

## ALGORITHM:

Numerical evaluation and symbolic manipulation are provided by [Maxima](#).

## REFERENCES:

- Abramowitz and Stegun: Handbook of Mathematical Functions, section 17.7  
<http://www.math.sfu.ca/~cbm/aands/>
- Elliptic Functions in [Maxima](#)

```
class sage.functions.special.MaximaFunction(name, nargs=2, conversions={})
    Bases: sage.symbolic.function.BuiltinFunction
```

## EXAMPLES:

```
sage: from sage.functions.special import MaximaFunction
sage: f = MaximaFunction("jacobi_sn")
sage: f(1,1)
tanh(1)
sage: f(1/2,1/2).n()
0.470750473655657
```

```
class sage.functions.special.SphericalHarmonic
    Bases: sage.symbolic.function.BuiltinFunction
```

Returns the spherical harmonic function  $Y_n^m(\theta, \varphi)$ .

For integers  $n > -1$ ,  $|m| \leq n$ , simplification is done automatically. Numeric evaluation is supported for complex  $n$  and  $m$ .

## EXAMPLES:

```
sage: x, y = var('x, y')
sage: spherical_harmonic(3, 2, x, y)
15/4*sqrt(7/30)*cos(x)*e^(2*I*y)*sin(x)^2/sqrt(pi)
sage: spherical_harmonic(3, 2, 1, 2)
15/4*sqrt(7/30)*cos(1)*e^(4*I)*sin(1)^2/sqrt(pi)
sage: spherical_harmonic(3 + I, 2., 1, 2)
-0.351154337307488 - 0.415562233975369*I
sage: latex(spherical_harmonic(3, 2, x, y, hold=True))
Y_{3}^{2}\left(x, y\right)
sage: spherical_harmonic(1, 2, x, y)
0
```

```
sage.functions.special.elliptic_j(z)
    Returns the elliptic modular  $j$ -function evaluated at  $z$ .
```

## INPUT:

- $z$  (complex) – a complex number with positive imaginary part.

## OUTPUT:

(complex) The value of  $j(z)$ .

## ALGORITHM:

Calls the `pari` function `ellj()`.

## AUTHOR:

John Cremona

## EXAMPLES:

```

sage: elliptic_j(CC(i))
1728.000000000000
sage: elliptic_j(sqrt(-2.0))
8000.000000000000
sage: z = ComplexField(100)(1, sqrt(11))/2
sage: elliptic_j(z)
-32768.000...
sage: elliptic_j(z).real().round()
-32768

::

sage: tau = (1 + sqrt(-163))/2
sage: (-elliptic_j(tau.n(100)).real().round())^(1/3)
640320

```

`sage.functions.special.error_fcn(t)`

The complementary error function  $\frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2} dx$  ( $t$  belongs to  $\mathbb{R}$ ). This function is currently always evaluated immediately.

EXAMPLES:

```

sage: error_fcn(6)
2.15197367124989e-17
sage: error_fcn(RealField(100)(1/2))
0.47950012218695346231725334611

```

Note this is literally equal to  $1 - \operatorname{erf}(t)$ :

```

sage: 1 - error_fcn(0.5)
0.520499877813047
sage: erf(0.5)
0.520499877813047

```

`sage.functions.special.hypergeometric_U(alpha, beta, x, algorithm='pari', prec=53)`

Default is a wrap of PARI's `hyperu(alpha,beta,x)` function. Optionally, `algorithm = "scipy"` can be used.

The confluent hypergeometric function  $y = U(a, b, x)$  is defined to be the solution to Kummer's differential equation

$$xy'' + (b - x)y' - ay = 0.$$

This satisfies  $U(a, b, x) \sim x^{-a}$ , as  $x \rightarrow \infty$ , and is sometimes denoted  $x^{-a} {}_2F_0(a, 1+a-b, -1/x)$ . This is not the same as Kummer's  $M$ -hypergeometric function, denoted sometimes as  ${}_1F_1(\alpha, \beta, x)$ , though it satisfies the same DE that  $U$  does.

**Warning:** In the literature, both are called “Kummer confluent hypergeometric” functions.

EXAMPLES:

```

sage: hypergeometric_U(1,1,1,"scipy")
0.596347362323...
sage: hypergeometric_U(1,1,1)
0.59634736232319...
sage: hypergeometric_U(1,1,1,"pari",70)
0.59634736232319407434...

```

`sage.functions.special.maxima_function(name)`

Returns a function which is evaluated both symbolically and numerically via Maxima. In particular, it returns



an instance of `MaximaFunction`.

---

**Note:** This function is cached so that duplicate copies of the same function are not created.

---

**EXAMPLES:**

```
sage: spherical_hankel2(2,i)
-e
```

```
sage.functions.special.meval(x)
```

Return  $x$  evaluated in Maxima, then returned to Sage.

This is used to evaluate several of these special functions.

**TEST:**

```
sage: from sage.functions.special import spherical_bessel_J
sage: spherical_bessel_J(2.,3.) # rel tol 1e-10
0.2986374970757335
```

```
sage.functions.special.spherical_bessel_J(n, var, algorithm='maxima')
```

Returns the spherical Bessel function of the first kind for integers  $n \geq 1$ .

Reference: AS 10.1.8 page 437 and AS 10.1.15 page 439.

**EXAMPLES:**

```
sage: spherical_bessel_J(2,x)
((3/x^2 - 1)*sin(x) - 3*cos(x)/x)/x
sage: spherical_bessel_J(1, 5.2, algorithm='scipy')
-0.12277149950007...
sage: spherical_bessel_J(1, 3, algorithm='scipy')
0.345677499762355...
```

```
sage.functions.special.spherical_bessel_Y(n, var, algorithm='maxima')
```

Returns the spherical Bessel function of the second kind for integers  $n \geq 1$ .

Reference: AS 10.1.9 page 437 and AS 10.1.15 page 439.

**EXAMPLES:**

```
sage: x = PolynomialRing(QQ, 'x').gen()
sage: spherical_bessel_Y(2,x)
-((3/x^2 - 1)*cos(x) + 3*sin(x)/x)/x
```

```
sage.functions.special.spherical_hankel1(n, var)
```

Returns the spherical Hankel function of the first kind for integers  $n > -1$ , written as a string. Reference: AS 10.1.36 page 439.

**EXAMPLES:**

```
sage: spherical_hankel1(2, x)
(I*x^2 - 3*x - 3*I)*e^(I*x)/x^3
```

```
sage.functions.special.spherical_hankel2(n, x)
```

Returns the spherical Hankel function of the second kind for integers  $n > -1$ , written as a string. Reference: AS 10.1.17 page 439.

**EXAMPLES:**

```
sage: spherical_hankel2(2, x)
(-I*x^2 - 3*x + 3*I)*e^(-I*x)/x^3
```

Here  $I = \sqrt{-1}$ .

## HYPERGEOMETRIC FUNCTIONS

This module implements manipulation of infinite hypergeometric series represented in standard parametric form (as  ${}_pF_q$  functions).

AUTHORS:

- Fredrik Johansson (2010): initial version
- Eviatar Bach (2013): major changes

EXAMPLES:

Examples from [trac ticket #9908](#):

```
sage: maxima('integrate(bessel_j(2, x), x)').sage()
1/24*x^3*hypergeometric((3/2,), (5/2, 3), -1/4*x^2)
sage: sum(((2*I)^x/(x^3 + 1)*(1/4)^x), x, 0, oo)
hypergeometric((1, 1, -1/2*I*sqrt(3) - 1/2, 1/2*I*sqrt(3) - 1/2),...
(2, -1/2*I*sqrt(3) + 1/2, 1/2*I*sqrt(3) + 1/2), 1/2*I)
sage: sum((-1)^x/((2*x + 1)*factorial(2*x + 1)), x, 0, oo)
hypergeometric((1/2,), (3/2, 3/2), -1/4)
```

Simplification (note that `simplify_full` does not yet call `simplify_hypergeometric`):

```
sage: hypergeometric([-2], [], x).simplify_hypergeometric()
x^2 - 2*x + 1
sage: hypergeometric([], [], x).simplify_hypergeometric()
e^x
sage: a = hypergeometric((hypergeometric(((), (), x),), (),
.....:                  hypergeometric(((), (), x))
sage: a.simplify_hypergeometric()
1/((-e^x + 1)^e^x)
sage: a.simplify_hypergeometric(algorithm='sage')
(-e^x + 1)^(-e^x)
```

Equality testing:

```
sage: bool(hypergeometric([], [], x).derivative(x) ==
.....:      hypergeometric([], [], x)) # diff(e^x, x) == e^x
True
sage: bool(hypergeometric([], [], x) == hypergeometric([], [1], x))
False
```

Computing terms and series:

```
sage: z = var('z')
sage: hypergeometric([], [], z).series(z, 0)
```

```
Order(1)
sage: hypergeometric([], [], z).series(z, 1)
1 + Order(z)
sage: hypergeometric([], [], z).series(z, 2)
1 + 1*z + Order(z^2)
sage: hypergeometric([], [], z).series(z, 3)
1 + 1*z + 1/2*z^2 + Order(z^3)

sage: hypergeometric([-2], [], z).series(z, 3)
1 + (-2)*z + 1*z^2
sage: hypergeometric([-2], [], z).series(z, 6)
1 + (-2)*z + 1*z^2
sage: hypergeometric([-2], [], z).series(z, 6).is_terminating_series()
True
sage: hypergeometric([-2], [], z).series(z, 2)
1 + (-2)*z + Order(z^2)
sage: hypergeometric([-2], [], z).series(z, 2).is_terminating_series()
False

sage: hypergeometric([1], [], z).series(z, 6)
1 + 1*z + 1*z^2 + 1*z^3 + 1*z^4 + 1*z^5 + Order(z^6)
sage: hypergeometric([], [1/2], -z^2/4).series(z, 11)
1 + (-1/2)*z^2 + 1/24*z^4 + (-1/720)*z^6 + 1/40320*z^8 + ...
(-1/3628800)*z^10 + Order(z^11)

sage: hypergeometric([1], [5], x).series(x, 5)
1 + 1/5*x + 1/30*x^2 + 1/210*x^3 + 1/1680*x^4 + Order(x^5)

sage: sum(hypergeometric([1, 2], [3], 1/3).terms(6)).n()
1.29788359788360
sage: hypergeometric([1, 2], [3], 1/3).n()
1.29837194594696
sage: hypergeometric([], [], x).series(x, 20)(x=1).n() == e.n()
True
```

#### Plotting:

```
sage: plot(hypergeometric([1, 1], [3, 3, 3], x), x, -30, 30)
Graphics object consisting of 1 graphics primitive
sage: complex_plot(hypergeometric([x], [], 2), (-1, 1), (-1, 1))
Graphics object consisting of 1 graphics primitive
```

#### Numeric evaluation:

```
sage: hypergeometric([1], [], 1/10).n() # geometric series
1.111111111111111
sage: hypergeometric([], [], 1).n() # e
2.71828182845905
sage: hypergeometric([], [], 3., hold=True)
hypergeometric((), (), 3.000000000000000)
sage: hypergeometric([1, 2, 3], [4, 5, 6], 1/2).n()
1.02573619590134
sage: hypergeometric([1, 2, 3], [4, 5, 6], 1/2).n(digits=30)
1.02573619590133865036584139535
sage: hypergeometric([5 - 3*I], [3/2, 2 + I, sqrt(2)], 4 + I).n()
5.52605111678805 - 7.86331357527544*I
sage: hypergeometric((10, 10), (50,), 2.)
-1705.75733163554 - 356.749986056024*I
```

Conversions:

```
sage: maxima(hypergeometric([1, 1, 1], [3, 3, 3], x))
hypergeometric([1,1,1],[3,3,3],_SAGE_VAR_x)
sage: hypergeometric((5, 4), (4, 4), 3)._sympy_()
hyper((5, 4), (4, 4), 3)
sage: hypergeometric((5, 4), (4, 4), 3)._mathematica_init_()
'HypergeometricPFQ[{5,4},{4,4},3]'
```

Arbitrary level of nesting for conversions:

```
sage: maxima(nest(lambda y: hypergeometric([y], [], x), 3, 1))
1/(1-_SAGE_VAR_x)^(1/(1-_SAGE_VAR_x)^(1/(1-_SAGE_VAR_x)))
sage: maxima(nest(lambda y: hypergeometric([y], [3], x), 3, 1))._sage_()
hypergeometric(hypergeometric(hypergeometric(1, (3), x), (3), ...
x), (3), x)
sage: nest(lambda y: hypergeometric([y], [], x), 3, 1)._mathematica_init_()
'HypergeometricPFQ[{HypergeometricPFQ[{HypergeometricPFQ[{1},{},x]},...'
```

**class** sage.functions.hypergeometric.**Hypergeometric**

Bases: sage.symbolic.function.BuiltinFunction

Represents a (formal) generalized infinite hypergeometric series. It is defined as

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!},$$

where  $(x)_n$  is the rising factorial.

**class** **EvaluationMethods**

**deflated**(self, a, b, z)

Rewrite as a linear combination of functions of strictly lower degree by eliminating all parameters  $a[i]$  and  $b[j]$  such that  $a[i] = b[i] + m$  for nonnegative integer  $m$ .

EXAMPLES:

```
sage: x = hypergeometric([6, 1], [3, 4, 5], 10)
sage: y = x.deflated()
sage: y
1/252*hypergeometric((4), (7, 8), 10)
+ 1/12*hypergeometric((3), (6, 7), 10)
+ 1/2*hypergeometric((2), (5, 6), 10)
+ hypergeometric((1), (4, 5), 10)
sage: x.n(); y.n()
2.87893612686782
2.87893612686782
```

```
sage: x = hypergeometric([6, 7], [3, 4, 5], 10)
sage: y = x.deflated()
sage: y
25/27216*hypergeometric(( ), (11, ), 10)
+ 25/648*hypergeometric(( ), (10, ), 10)
+ 265/504*hypergeometric(( ), (9, ), 10)
+ 181/63*hypergeometric(( ), (8, ), 10)
+ 19/3*hypergeometric(( ), (7, ), 10)
+ 5*hypergeometric(( ), (6, ), 10)
+ hypergeometric(( ), (5, ), 10)
sage: x.n(); y.n()
```

```
63.0734110716969
63.0734110716969
```

**eliminate\_parameters** (*self*, *a*, *b*, *z*)

Eliminate repeated parameters by pairwise cancellation of identical terms in *a* and *b*.

EXAMPLES:

```
sage: hypergeometric([1, 1, 2, 5], [5, 1, 4],
....:               1/2).eliminate_parameters()
hypergeometric((1, 2), (4,), 1/2)
sage: hypergeometric([x], [x], x).eliminate_parameters()
hypergeometric((), (), x)
sage: hypergeometric((5, 4), (4, 4), 3).eliminate_parameters()
hypergeometric((5,), (4,), 3)
```

**is\_absolutely\_convergent** (*self*, *a*, *b*, *z*)

Determine whether *self* converges absolutely as an infinite series. False is returned if not all terms are finite.

EXAMPLES:

Degree giving infinite radius of convergence:

```
sage: hypergeometric([2, 3], [4, 5],
....:               6).is_absolutely_convergent()
True
sage: hypergeometric([2, 3], [-4, 5],
....:               6).is_absolutely_convergent() # undefined
False
sage: (hypergeometric([2, 3], [-4, 5], Infinity)
....:   .is_absolutely_convergent()) # undefined
False
```

Ordinary geometric series (unit radius of convergence):

```
sage: hypergeometric([1], [], 1/2).is_absolutely_convergent()
True
sage: hypergeometric([1], [], 2).is_absolutely_convergent()
False
sage: hypergeometric([1], [], 1).is_absolutely_convergent()
False
sage: hypergeometric([1], [], -1).is_absolutely_convergent()
False
sage: hypergeometric([1], [], -1).n() # Sum still exists
0.5000000000000000
```

Degree  $p = q + 1$  (unit radius of convergence):

```
sage: hypergeometric([2, 3], [4], 6).is_absolutely_convergent()
False
sage: hypergeometric([2, 3], [4], 1).is_absolutely_convergent()
False
sage: hypergeometric([2, 3], [5], 1).is_absolutely_convergent()
False
sage: hypergeometric([2, 3], [6], 1).is_absolutely_convergent()
True
sage: hypergeometric([-2, 3], [4],
....:               5).is_absolutely_convergent()
True
sage: hypergeometric([2, -3], [4],
....:               5).is_absolutely_convergent()
True
```

```

sage: hypergeometric([2, -3], [-4],
....:                 5).is_absolutely_convergent()
True
sage: hypergeometric([2, -3], [-1],
....:                 5).is_absolutely_convergent()
False

```

Degree giving zero radius of convergence:

```

sage: hypergeometric([1, 2, 3], [4],
....:                 2).is_absolutely_convergent()
False
sage: hypergeometric([1, 2, 3], [4],
....:                 1/2).is_absolutely_convergent()
False
sage: (hypergeometric([1, 2, -3], [4], 1/2)
....:   .is_absolutely_convergent()) # polynomial
True

```

**is\_terminating**(*self*, *a*, *b*, *z*)

Determine whether the series represented by *self* terminates after a finite number of terms, i.e. whether any of the numerator parameters are nonnegative integers (with no preceding nonnegative denominator parameters), or  $z = 0$ .

If terminating, the series represents a polynomial of  $z$ .

EXAMPLES:

```

sage: hypergeometric([1, 2], [3, 4], x).is_terminating()
False
sage: hypergeometric([1, -2], [3, 4], x).is_terminating()
True
sage: hypergeometric([1, -2], [], x).is_terminating()
True

```

**is\_termwise\_finite**(*self*, *a*, *b*, *z*)

Determine whether all terms of *self* are finite. Any infinite terms or ambiguous terms beyond the first zero, if one exists, are ignored.

Ambiguous cases (where a term is the product of both zero and an infinity) are not considered finite.

EXAMPLES:

```

sage: hypergeometric([2], [3, 4], 5).is_termwise_finite()
True
sage: hypergeometric([2], [-3, 4], 5).is_termwise_finite()
False
sage: hypergeometric([-2], [-3, 4], 5).is_termwise_finite()
True
sage: hypergeometric([-3], [-3, 4],
....:                 5).is_termwise_finite() # ambiguous
False

sage: hypergeometric([0], [-1], 5).is_termwise_finite()
True
sage: hypergeometric([0], [0],
....:                 5).is_termwise_finite() # ambiguous
False
sage: hypergeometric([1], [2], Infinity).is_termwise_finite()
False
sage: (hypergeometric([0], [0], Infinity)
....:   .is_termwise_finite()) # ambiguous

```

```
False
sage: (hypergeometric([0], [], Infinity)
....: .is_termwise_finite()) # ambiguous
False
```

**sorted\_parameters** (*self*, *a*, *b*, *z*)

Return with parameters sorted in a canonical order.

EXAMPLES:

```
sage: hypergeometric([2, 1, 3], [5, 4],
....:                1/2).sorted_parameters()
hypergeometric((1, 2, 3), (4, 5), 1/2)
```

**terms** (*self*, *a*, *b*, *z*, *n=None*)

Generate the terms of *self* (optionally only *n* terms).

EXAMPLES:

```
sage: list(hypergeometric([-2, 1], [3, 4], x).terms())
[1, -1/6*x, 1/120*x^2]
sage: list(hypergeometric([-2, 1], [3, 4], x).terms(2))
[1, -1/6*x]
sage: list(hypergeometric([-2, 1], [3, 4], x).terms(0))
[]
```

`sage.functions.hypergeometric.closed_form(hyp)`

Try to evaluate *hyp* in closed form using elementary (and other simple) functions.

It may be necessary to call `Hypergeometric.deflated()` first to find some closed forms.

EXAMPLES:

```
sage: from sage.functions.hypergeometric import closed_form
sage: var('a b c z')
(a, b, c, z)
sage: closed_form(hypergeometric([1], [], 1 + z))
-1/z
sage: closed_form(hypergeometric([], [], 1 + z))
e^(z + 1)
sage: closed_form(hypergeometric([], [1/2], 4))
cosh(4)
sage: closed_form(hypergeometric([], [3/2], 4))
1/4*sinh(4)
sage: closed_form(hypergeometric([], [5/2], 4))
3/16*cosh(4) - 3/64*sinh(4)
sage: closed_form(hypergeometric([], [-3/2], 4))
19/3*cosh(4) - 4*sinh(4)
sage: closed_form(hypergeometric([-3, 1], [var('a')], z))
-3*z/a + 6*z^2/((a + 1)*a) - 6*z^3/((a + 2)*(a + 1)*a) + 1
sage: closed_form(hypergeometric([-3, 1/3], [-4], z))
7/162*z^3 + 1/9*z^2 + 1/4*z + 1
sage: closed_form(hypergeometric([], [], z))
e^z
sage: closed_form(hypergeometric([a], [], z))
(-z + 1)^(-a)
sage: closed_form(hypergeometric([1, 1, 2], [1, 1], z))
(z - 1)^(-2)
sage: closed_form(hypergeometric([2, 3], [1], x))
-1/(x - 1)^3 + 3*x/(x - 1)^4
sage: closed_form(hypergeometric([1/2], [3/2], -5))
1/10*sqrt(5)*sqrt(pi)*erf(sqrt(5))
```



```

sage: closed_form(hypergeometric([2], [5], 3))
4
sage: closed_form(hypergeometric([2], [5], 5))
48/625*e^5 + 612/625
sage: closed_form(hypergeometric([1/2, 7/2], [3/2], z))
1/5*z^2/(-z + 1)^(5/2) + 2/3*z/(-z + 1)^(3/2) + 1/sqrt(-z + 1)
sage: closed_form(hypergeometric([1/2, 1], [2], z))
-2*(sqrt(-z + 1) - 1)/z
sage: closed_form(hypergeometric([1, 1], [2], z))
-log(-z + 1)/z
sage: closed_form(hypergeometric([1, 1], [3], z))
-2*((z - 1)*log(-z + 1)/z - 1)/z
sage: closed_form(hypergeometric([1, 1, 1], [2, 2], x))
hypergeometric(1, 1, 1), (2, 2), x)

```

`sage.functions.hypergeometric.rational_param_as_tuple(x)`

Utility function for converting rational  ${}_pF_q$  parameters to tuples (which mpmath handles more efficiently).

EXAMPLES:

```

sage: from sage.functions.hypergeometric import rational_param_as_tuple
sage: rational_param_as_tuple(1/2)
(1, 2)
sage: rational_param_as_tuple(3)
3
sage: rational_param_as_tuple(pi)
pi

```



## JACOBI ELLIPTIC FUNCTIONS

This module implements the 12 Jacobi elliptic functions, along with their inverses and the Jacobi amplitude function.

Jacobi elliptic functions can be thought of as generalizations of both ordinary and hyperbolic trig functions. There are twelve Jacobian elliptic functions. Each of the twelve corresponds to an arrow drawn from one corner of a rectangle to another.



Each of the corners of the rectangle are labeled, by convention,  $s$ ,  $c$ ,  $d$ , and  $n$ . The rectangle is understood to be lying on the complex plane, so that  $s$  is at the origin,  $c$  is on the real axis, and  $n$  is on the imaginary axis. The twelve Jacobian elliptic functions are then  $pq(x)$ , where  $p$  and  $q$  are one of the letters  $s$ ,  $c$ ,  $d$ ,  $n$ .

The Jacobian elliptic functions are then the unique doubly-periodic, meromorphic functions satisfying the following three properties:

1. There is a simple zero at the corner  $p$ , and a simple pole at the corner  $q$ .
2. The step from  $p$  to  $q$  is equal to half the period of the function  $pq(x)$ ; that is, the function  $pq(x)$  is periodic in the direction  $pq$ , with the period being twice the distance from  $p$  to  $q$ .  $pq(x)$  is periodic in the other two directions as well, with a period such that the distance from  $p$  to one of the other corners is a quarter period.
3. If the function  $pq(x)$  is expanded in terms of  $x$  at one of the corners, the leading term in the expansion has a coefficient of 1. In other words, the leading term of the expansion of  $pq(x)$  at the corner  $p$  is  $x$ ; the leading term of the expansion at the corner  $q$  is  $1/x$ , and the leading term of an expansion at the other two corners is 1.

We can write

$$pq(x) = \frac{pr(x)}{qr(x)}$$

where  $p$ ,  $q$ , and  $r$  are any of the letters  $s$ ,  $c$ ,  $d$ ,  $n$ , with the understanding that  $ss = cc = dd = nn = 1$ .

Let

$$u = \int_0^\phi \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}},$$

then the *Jacobi elliptic function*  $\text{sn}(u)$  is given by

$$\text{sn } u = \sin \phi$$

and  $\text{cn}(u)$  is given by

$$\text{cn } u = \cos \phi$$

and

$$\operatorname{dn} u = \sqrt{1 - m \sin^2 \phi}.$$

To emphasize the dependence on  $m$ , one can write  $\operatorname{sn}(u|m)$  for example (and similarly for  $\operatorname{cn}$  and  $\operatorname{dn}$ ). This is the notation used below.

For a given  $k$  with  $0 < k < 1$  they therefore are solutions to the following nonlinear ordinary differential equations:

- $\operatorname{sn}(x; k)$  solves the differential equations

$$\frac{d^2 y}{dx^2} + (1 + k^2)y - 2k^2 y^3 = 0 \quad \text{and} \quad \left(\frac{dy}{dx}\right)^2 = (1 - y^2)(1 - k^2 y^2).$$

- $\operatorname{cn}(x; k)$  solves the differential equations

$$\frac{d^2 y}{dx^2} + (1 - 2k^2)y + 2k^2 y^3 = 0 \quad \text{and} \quad \left(\frac{dy}{dx}\right)^2 = (1 - y^2)(1 - k^2 + k^2 y^2).$$

- $\operatorname{dn}(x; k)$  solves the differential equations

$$\frac{d^2 y}{dx^2} - (2 - k^2)y + 2y^3 = 0 \quad \text{and} \quad \left(\frac{dy}{dx}\right)^2 = y^2(1 - k^2 - y^2).$$

If  $K(m)$  denotes the complete elliptic integral of the first kind (named `elliptic_kc` in Sage), the elliptic functions  $\operatorname{sn}(x|m)$  and  $\operatorname{cn}(x|m)$  have real periods  $4K(m)$ , whereas  $\operatorname{dn}(x|m)$  has a period  $2K(m)$ . The limit  $m \rightarrow 0$  gives  $K(0) = \pi/2$  and trigonometric functions:  $\operatorname{sn}(x|0) = \sin x$ ,  $\operatorname{cn}(x|0) = \cos x$ ,  $\operatorname{dn}(x|0) = 1$ . The limit  $m \rightarrow 1$  gives  $K(1) \rightarrow \infty$  and hyperbolic functions:  $\operatorname{sn}(x|1) = \tanh x$ ,  $\operatorname{cn}(x|1) = \operatorname{sech} x$ ,  $\operatorname{dn}(x|1) = \operatorname{sech} x$ .

#### REFERENCES:

- [Wikipedia article Jacobi's\\_elliptic\\_functions](#)

#### AUTHORS:

- David Joyner (2006): initial version
- Eviatar Bach (2013): complete rewrite, new numerical evaluation, and addition of the Jacobi amplitude function

**class** `sage.functions.jacobi.InverseJacobi` (*kind*)  
 Bases: `sage.symbolic.function.BuiltinFunction`  
 Base class for the inverse Jacobi elliptic functions.

**class** `sage.functions.jacobi.Jacobi` (*kind*)  
 Bases: `sage.symbolic.function.BuiltinFunction`  
 Base class for the Jacobi elliptic functions.

**class** `sage.functions.jacobi.JacobiAmplitude`  
 Bases: `sage.symbolic.function.BuiltinFunction`

The Jacobi amplitude function  $\operatorname{am}(x|m) = \int_0^x \operatorname{dn}(t|m) dt$  for  $-K(m) \leq x \leq K(m)$ ,  $F(\operatorname{am}(x|m)|m) = x$ .

`sage.functions.jacobi.inverse_jacobi` (*kind*, *x*, *m*, *\*\*kwargs*)  
 The inverses of the 12 Jacobi elliptic functions. They have the property that

$$\operatorname{pq}(\operatorname{arcpq}(x|m)|m) = \operatorname{pq}(\operatorname{pq}^{-1}(x|m)|m) = x.$$

INPUT:

- `kind` – a string of the form `'pq'`, where `p, q` are in `c, d, n, s`
- `x` – a real number
- `m` – a real number; note that  $m = k^2$ , where  $k$  is the elliptic modulus

## EXAMPLES:

```

sage: jacobi('dn', inverse_jacobi('dn', 3, 0.4), 0.4)
3.000000000000000
sage: inverse_jacobi('dn', 10, 1/10).n(digits=50)
2.4777736267904273296523691232988240759001423661683*I
sage: inverse_jacobi_dn(x, 1)
arcsech(x)
sage: inverse_jacobi_dn(1, 3)
0
sage: m = var('m')
sage: z = inverse_jacobi_dn(x, m).series(x, 4).subs(x=0.1, m=0.7)
sage: jacobi_dn(z, 0.7)
0.0999892750039819...
sage: inverse_jacobi_nd(x, 1)
arccosh(x)
sage: inverse_jacobi_nd(1, 2)
0
sage: inverse_jacobi_ns(10^-5, 3).n()
5.77350269202456e-6 + 1.17142008414677*I
sage: jacobi('sn', 1/2, 1/2)
jacobi_sn(1/2, 1/2)
sage: jacobi('sn', 1/2, 1/2).n()
0.470750473655657
sage: inverse_jacobi('sn', 0.47, 1/2)
0.499098231322220
sage: inverse_jacobi('sn', 0.4707504, 0.5)
0.499999911466555
sage: P = plot(inverse_jacobi('sn', x, 0.5), 0, 1)

```

`sage.functions.jacobi.inverse_jacobi_f(kind, x, m)`

Internal function for numerical evaluation of a continuous complex branch of each inverse Jacobi function, as described in [Tee97]. Only accepts real arguments.

## REFERENCES:

## TESTS:

```

sage: from mpmath import ellipfun, chop
sage: from sage.functions.jacobi import inverse_jacobi_f

sage: chop(ellipfun('sn', inverse_jacobi_f('sn', 0.6, 0), 0))
mpf('0.59999999999999998')
sage: chop(ellipfun('sn', inverse_jacobi_f('sn', 0.6, 1), 1))
mpf('0.59999999999999998')
sage: chop(ellipfun('sn', inverse_jacobi_f('sn', 0, -3), -3))
mpf('0.0')
sage: chop(ellipfun('sn', inverse_jacobi_f('sn', -1, 4), 4))
mpf('-1.0')
sage: chop(ellipfun('sn', inverse_jacobi_f('sn', 0.3, 4), 4))
mpf('0.29999999999999999')
sage: chop(ellipfun('sn', inverse_jacobi_f('sn', 0.8, 4), 4))
mpf('0.80000000000000004')

sage: chop(ellipfun('ns', inverse_jacobi_f('ns', 0.8, 0), 0))

```

```
mpf('0.800000000000000004')
sage: chop(ellipfun('ns', inverse_jacobi_f('ns', -0.7, 1), 1))
mpf('-0.69999999999999996')
sage: chop(ellipfun('ns', inverse_jacobi_f('ns', 0.01, 2), 2))
mpf('0.01')
sage: chop(ellipfun('ns', inverse_jacobi_f('ns', 0, 2), 2))
mpf('0.0')
sage: chop(ellipfun('ns', inverse_jacobi_f('ns', -10, 6), 6))
mpf('-10.0')

sage: chop(ellipfun('cn', inverse_jacobi_f('cn', -10, 0), 0))
mpf('-9.9999999999999982')
sage: chop(ellipfun('cn', inverse_jacobi_f('cn', 50, 1), 1))
mpf('50.0000000000000071')
sage: chop(ellipfun('cn', inverse_jacobi_f('cn', 1, 5), 5))
mpf('1.0')
sage: chop(ellipfun('cn', inverse_jacobi_f('cn', 0.5, -5), -5))
mpf('0.5')
sage: chop(ellipfun('cn', inverse_jacobi_f('cn', -0.75, -15), -15))
mpf('-0.750000000000000022')
sage: chop(ellipfun('cn', inverse_jacobi_f('cn', 10, 0.8), 0.8))
mpf('9.9999999999999982')
sage: chop(ellipfun('cn', inverse_jacobi_f('cn', -2, 0.9), 0.9))
mpf('-2.0')

sage: chop(ellipfun('nc', inverse_jacobi_f('nc', -4, 0), 0))
mpf('-3.9999999999999987')
sage: chop(ellipfun('nc', inverse_jacobi_f('nc', 7, 1), 1))
mpf('7.0000000000000009')
sage: chop(ellipfun('nc', inverse_jacobi_f('nc', 7, 3), 3))
mpf('7.0')
sage: chop(ellipfun('nc', inverse_jacobi_f('nc', 0, 2), 2))
mpf('0.0')
sage: chop(ellipfun('nc', inverse_jacobi_f('nc', -18, -4), -4))
mpf('-17.999999999999925')

sage: chop(ellipfun('dn', inverse_jacobi_f('dn', -0.3, 1), 1))
mpf('-0.29999999999999999')
sage: chop(ellipfun('dn', inverse_jacobi_f('dn', 1, -1), -1))
mpf('1.0')
sage: chop(ellipfun('dn', inverse_jacobi_f('dn', 0.8, 0.5), 0.5))
mpf('0.800000000000000004')
sage: chop(ellipfun('dn', inverse_jacobi_f('dn', 5, -4), -4))
mpf('5.0')
sage: chop(ellipfun('dn', inverse_jacobi_f('dn', 0.4, 0.5), 0.5))
mpf('0.400000000000000002')
sage: chop(ellipfun('dn', inverse_jacobi_f('dn', -0.4, 0.5), 0.5))
mpf('-0.400000000000000002')
sage: chop(ellipfun('dn', inverse_jacobi_f('dn', -0.9, 0.5), 0.5))
mpf('-0.900000000000000002')
sage: chop(ellipfun('dn', inverse_jacobi_f('dn', -1.9, 0.2), 0.2))
mpf('-1.8999999999999999')

sage: chop(ellipfun('nd', inverse_jacobi_f('nd', -1.9, 1), 1))
mpf('-1.8999999999999999')
sage: chop(ellipfun('nd', inverse_jacobi_f('nd', 1, -1), -1))
mpf('1.0')
sage: chop(ellipfun('nd', inverse_jacobi_f('nd', 11, -6), -6))
```

```

mpf('11.0')
sage: chop(ellipfun('nd', inverse_jacobi_f('nd', 0, 8), 8))
mpf('0.0')
sage: chop(ellipfun('nd', inverse_jacobi_f('nd', -3, 0.8), 0.8))
mpf('-2.9999999999999996')

sage: chop(ellipfun('sc', inverse_jacobi_f('sc', -3, 0), 0))
mpf('-3.0')
sage: chop(ellipfun('sc', inverse_jacobi_f('sc', 2, 1), 1))
mpf('2.0')
sage: chop(ellipfun('sc', inverse_jacobi_f('sc', 0, 9), 9))
mpf('0.0')
sage: chop(ellipfun('sc', inverse_jacobi_f('sc', -7, 3), 3))
mpf('-7.0')

sage: chop(ellipfun('cs', inverse_jacobi_f('cs', -7, 0), 0))
mpf('-6.9999999999999991')
sage: chop(ellipfun('cs', inverse_jacobi_f('cs', 8, 1), 1))
mpf('8.0')
sage: chop(ellipfun('cs', inverse_jacobi_f('cs', 2, 6), 6))
mpf('2.0')
sage: chop(ellipfun('cs', inverse_jacobi_f('cs', 0, 4), 4))
mpf('0.0')
sage: chop(ellipfun('cs', inverse_jacobi_f('cs', -6, 8), 8))
mpf('-6.00000000000000018')

sage: chop(ellipfun('cd', inverse_jacobi_f('cd', -6, 0), 0))
mpf('-6.0000000000000009')
sage: chop(ellipfun('cd', inverse_jacobi_f('cd', 1, 3), 3))
mpf('1.0')
sage: chop(ellipfun('cd', inverse_jacobi_f('cd', 6, 8), 8))
mpf('6.00000000000000027')

sage: chop(ellipfun('dc', inverse_jacobi_f('dc', 5, 0), 0))
mpf('5.00000000000000018')
sage: chop(ellipfun('dc', inverse_jacobi_f('dc', -4, 2), 2))
mpf('-4.00000000000000018')

sage: chop(ellipfun('sd', inverse_jacobi_f('sd', -4, 0), 0))
mpf('-3.9999999999999991')
sage: chop(ellipfun('sd', inverse_jacobi_f('sd', 7, 1), 1))
mpf('7.0')
sage: chop(ellipfun('sd', inverse_jacobi_f('sd', 0, 9), 9))
mpf('0.0')
sage: chop(ellipfun('sd', inverse_jacobi_f('sd', 8, 0.8), 0.8))
mpf('7.9999999999999991')

sage: chop(ellipfun('ds', inverse_jacobi_f('ds', 4, 0.25), 0.25))
mpf('4.0')

```

`sage.functions.jacobi.jacobi` (*kind*, *z*, *m*, *\*\*kwargs*)

The 12 Jacobi elliptic functions.

INPUT:

- *kind* – a string of the form 'pq', where p, q are in c, d, n, s
- *z* – a complex number

•  $m$  – a complex number; note that  $m = k^2$ , where  $k$  is the elliptic modulus

**EXAMPLES:**

```
sage: jacobi('sn', 1, 1)
tanh(1)
sage: jacobi('cd', 1, 1/2)
jacobi_cd(1, 1/2)
sage: RDF(jacobi('cd', 1, 1/2))
0.7240097216593705
sage: (RDF(jacobi('cn', 1, 1/2)), RDF(jacobi('dn', 1, 1/2)),
....: RDF(jacobi('cn', 1, 1/2) / jacobi('dn', 1, 1/2)))
(0.5959765676721407, 0.8231610016315962, 0.7240097216593705)
sage: jsn = jacobi('sn', x, 1)
sage: P = plot(jsn, 0, 1)
```

sage.functions.jacobi.**jacobi\_am\_f**( $x, m$ )

Internal function for numeric evaluation of the Jacobi amplitude function for real arguments. Procedure described in [Ehrhardt13].

**REFERENCES:****TESTS:**

```
sage: from mpmath import ellipf
sage: from sage.functions.jacobi import jacobi_am_f
sage: ellipf(jacobi_am_f(0.5, 1), 1)
mpf('0.5')
sage: ellipf(jacobi_am(3, 0.3), 0.3)
mpf('3.0')
sage: ellipf(jacobi_am_f(2, -0.5), -0.5)
mpf('2.0')
sage: jacobi_am_f(2, -0.5)
mpf('2.2680930777934176')
sage: jacobi_am_f(-2, -0.5)
mpf('-2.2680930777934176')
sage: jacobi_am_f(-3, 2)
mpf('0.36067407399586108')
```



## AIRY FUNCTIONS

This module implements Airy functions and their generalized derivatives. It supports symbolic functionality through Maxima and numeric evaluation through mpmath and scipy.

Airy functions are solutions to the differential equation  $f''(x) - xf(x) = 0$ .

Four global function symbols are immediately available, please see

- `airy_ai()`: for the Airy Ai function
- `airy_ai_prime()`: for the first differential of the Airy Ai function
- `airy_bi()`: for the Airy Bi function
- `airy_bi_prime()`: for the first differential of the Airy Bi function

AUTHORS:

- Oscar Gerardo Lazo Arjona (2010): initial version
- Douglas McNeil (2012): rewrite

EXAMPLES:

Verify that the Airy functions are solutions to the differential equation:

```
sage: diff(airy_ai(x), x, 2) - x * airy_ai(x)
0
sage: diff(airy_bi(x), x, 2) - x * airy_bi(x)
0
```

```
class sage.functions.airy.FunctionAiryAiGeneral
    Bases: sage.symbolic.function.BuiltinFunction
```

The generalized derivative of the Airy Ai function

INPUT:

- `alpha` – Return the  $\alpha$ -th order fractional derivative with respect to  $z$ . For  $\alpha = n = 1, 2, 3, \dots$  this gives the derivative  $\text{Ai}^{(n)}(z)$ , and for  $\alpha = -n = -1, -2, -3, \dots$  this gives the  $n$ -fold iterated integral.

$$f_0(z) = \text{Ai}(z)$$
$$f_n(z) = \int_0^z f_{n-1}(t) dt$$

- `x` – The argument of the function

EXAMPLES:

```
sage: from sage.functions.airy import airy_ai_general
sage: x, n = var('x n')
sage: airy_ai_general(-2, x)
airy_ai(-2, x)
sage: derivative(airy_ai_general(-2, x), x)
airy_ai(-1, x)
sage: airy_ai_general(n, x)
airy_ai(n, x)
sage: derivative(airy_ai_general(n, x), x)
airy_ai(n + 1, x)
```

**class** sage.functions.airy.**FunctionAiryAiPrime**

Bases: sage.symbolic.function.BuiltinFunction

The derivative of the Airy Ai function; see [airy\\_ai\(\)](#) for the full documentation.

EXAMPLES:

```
sage: x, n = var('x n')
sage: airy_ai_prime(x)
airy_ai_prime(x)
sage: airy_ai_prime(0)
-1/3*3^(2/3)/gamma(1/3)
```

**class** sage.functions.airy.**FunctionAiryAiSimple**

Bases: sage.symbolic.function.BuiltinFunction

The class for the Airy Ai function.

EXAMPLES:

```
sage: from sage.functions.airy import airy_ai_simple
sage: f = airy_ai_simple(x); f
airy_ai(x)
```

**class** sage.functions.airy.**FunctionAiryBiGeneral**

Bases: sage.symbolic.function.BuiltinFunction

The generalized derivative of the Airy Bi function.

INPUT:

- $\alpha$  – Return the  $\alpha$ -th order fractional derivative with respect to  $z$ . For  $\alpha = n = 1, 2, 3, \dots$  this gives the derivative  $\text{Bi}^{(n)}(z)$ , and for  $\alpha = -n = -1, -2, -3, \dots$  this gives the  $n$ -fold iterated integral.

$$f_0(z) = \text{Bi}(z)$$
$$f_n(z) = \int_0^z f_{n-1}(t) dt$$

- $x$  – The argument of the function

EXAMPLES:

```
sage: from sage.functions.airy import airy_bi_general
sage: x, n = var('x n')
sage: airy_bi_general(-2, x)
airy_bi(-2, x)
sage: derivative(airy_bi_general(-2, x), x)
airy_bi(-1, x)
sage: airy_bi_general(n, x)
airy_bi(n, x)
```

```
sage: derivative(airy_bi_general(n, x), x)
airy_bi(n + 1, x)
```

**class** sage.functions.airy.**FunctionAiryBiPrime**

Bases: sage.symbolic.function.BuiltinFunction

The derivative of the Airy Bi function; see `airy_bi()` for the full documentation.

EXAMPLES:

```
sage: x, n = var('x n')
sage: airy_bi_prime(x)
airy_bi_prime(x)
sage: airy_bi_prime(0)
3^(1/6)/gamma(1/3)
```

**class** sage.functions.airy.**FunctionAiryBiSimple**

Bases: sage.symbolic.function.BuiltinFunction

The class for the Airy Bi function.

EXAMPLES:

```
sage: from sage.functions.airy import airy_bi_simple
sage: f = airy_bi_simple(x); f
airy_bi(x)
```

sage.functions.airy.**airy\_ai**(*alpha*, *x=None*, *hold\_derivative=True*, *\*\*kwds*)

The Airy Ai function

The Airy Ai function  $\text{Ai}(x)$  is (along with  $\text{Bi}(x)$ ) one of the two linearly independent standard solutions to the Airy differential equation  $f''(x) - xf(x) = 0$ . It is defined by the initial conditions:

$$\begin{aligned}\text{Ai}(0) &= \frac{1}{2^{2/3}\Gamma(\frac{2}{3})}, \\ \text{Ai}'(0) &= -\frac{1}{2^{1/3}\Gamma(\frac{1}{3})}.\end{aligned}$$

Another way to define the Airy Ai function is:

$$\text{Ai}(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right) dt.$$

INPUT:

- *alpha* – Return the  $\alpha$ -th order fractional derivative with respect to  $z$ . For  $\alpha = n = 1, 2, 3, \dots$  this gives the derivative  $\text{Ai}^{(n)}(z)$ , and for  $\alpha = -n = -1, -2, -3, \dots$  this gives the  $n$ -fold iterated integral.

$$\begin{aligned}f_0(z) &= \text{Ai}(z) \\ f_n(z) &= \int_0^z f_{n-1}(t) dt\end{aligned}$$

- *x* – The argument of the function
- *hold\_derivative* – Whether or not to stop from returning higher derivatives in terms of  $\text{Ai}(x)$  and  $\text{Ai}'(x)$

See also:

`airy_bi()`

EXAMPLES:

```
sage: n, x = var('n x')
sage: airy_ai(x)
airy_ai(x)
```

It can return derivatives or integrals:

```
sage: airy_ai(2, x)
airy_ai(2, x)
sage: airy_ai(1, x, hold_derivative=False)
airy_ai_prime(x)
sage: airy_ai(2, x, hold_derivative=False)
x*airy_ai(x)
sage: airy_ai(-2, x, hold_derivative=False)
airy_ai(-2, x)
sage: airy_ai(n, x)
airy_ai(n, x)
```

It can be evaluated symbolically or numerically for real or complex values:

```
sage: airy_ai(0)
1/3*3^(1/3)/gamma(2/3)
sage: airy_ai(0.0)
0.355028053887817
sage: airy_ai(I)
airy_ai(I)
sage: airy_ai(1.0*I)
0.331493305432141 - 0.317449858968444*I
```

The functions can be evaluated numerically either using `mpmath`, which can compute the values to arbitrary precision, and `scipy`:

```
sage: airy_ai(2).n(prec=100)
0.034924130423274379135322080792
sage: airy_ai(2).n(algorithm='mpmath', prec=100)
0.034924130423274379135322080792
sage: airy_ai(2).n(algorithm='scipy') # rel tol 1e-10
0.03492413042327323
```

And the derivatives can be evaluated:

```
sage: airy_ai(1, 0)
-1/3*3^(2/3)/gamma(1/3)
sage: airy_ai(1, 0.0)
-0.258819403792807
```

Plots:

```
sage: plot(airy_ai(x), (x, -10, 5)) + plot(airy_ai_prime(x),
....: (x, -10, 5), color='red')
Graphics object consisting of 2 graphics primitives
```

## References

- Abramowitz, Milton; Stegun, Irene A., eds. (1965), “Chapter 10”
- [Wikipedia article Airy\\_function](#)

`sage.functions.airy.airy_bi(alpha, x=None, hold_derivative=True, **kws)`  
The Airy Bi function

The Airy Bi function  $\text{Bi}(x)$  is (along with  $\text{Ai}(x)$ ) one of the two linearly independent standard solutions to the

Airy differential equation  $f''(x) - xf(x) = 0$ . It is defined by the initial conditions:

$$\text{Bi}(0) = \frac{1}{3^{1/6}\Gamma\left(\frac{2}{3}\right)},$$

$$\text{Bi}'(0) = \frac{3^{1/6}}{\Gamma\left(\frac{1}{3}\right)}.$$

Another way to define the Airy Bi function is:

$$\text{Bi}(x) = \frac{1}{\pi} \int_0^\infty \left[ \exp\left(xt - \frac{t^3}{3}\right) + \sin\left(xt + \frac{1}{3}t^3\right) \right] dt.$$

INPUT:

- `alpha` – Return the  $\alpha$ -th order fractional derivative with respect to  $z$ . For  $\alpha = n = 1, 2, 3, \dots$  this gives the derivative  $\text{Bi}^{(n)}(z)$ , and for  $\alpha = -n = -1, -2, -3, \dots$  this gives the  $n$ -fold iterated integral.

$$f_0(z) = \text{Bi}(z)$$

$$f_n(z) = \int_0^z f_{n-1}(t) dt$$

- `x` – The argument of the function
- `hold_derivative` – Whether or not to stop from returning higher derivatives in terms of  $\text{Bi}(x)$  and  $\text{Bi}'(x)$

See also:

`airy_ai()`

EXAMPLES:

```
sage: n, x = var('n x')
sage: airy_bi(x)
airy_bi(x)
```

It can return derivatives or integrals:

```
sage: airy_bi(2, x)
airy_bi(2, x)
sage: airy_bi(1, x, hold_derivative=False)
airy_bi_prime(x)
sage: airy_bi(2, x, hold_derivative=False)
x*airy_bi(x)
sage: airy_bi(-2, x, hold_derivative=False)
airy_bi(-2, x)
sage: airy_bi(n, x)
airy_bi(n, x)
```

It can be evaluated symbolically or numerically for real or complex values:

```
sage: airy_bi(0)
1/3*3^(5/6)/gamma(2/3)
sage: airy_bi(0.0)
0.614926627446001
sage: airy_bi(I)
airy_bi(I)
sage: airy_bi(1.0*I)
0.648858208330395 + 0.344958634768048*I
```

The functions can be evaluated numerically using `mpmath`, which can compute the values to arbitrary precision, and `scipy`:

```
sage: airy_bi(2).n(prec=100)
3.2980949999782147102806044252
sage: airy_bi(2).n(algorithm='mpmath', prec=100)
3.2980949999782147102806044252
sage: airy_bi(2).n(algorithm='scipy') # rel tol 1e-10
3.2980949999782134
```

And the derivatives can be evaluated:

```
sage: airy_bi(1, 0)
3^(1/6)/gamma(1/3)
sage: airy_bi(1, 0.0)
0.448288357353826
```

Plots:

```
sage: plot(airy_bi(x), (x, -10, 5)) + plot(airy_bi_prime(x),
....: (x, -10, 5), color='red')
Graphics object consisting of 2 graphics primitives
```

## References

- Abramowitz, Milton; Stegun, Irene A., eds. (1965), “Chapter 10”
- [Wikipedia article Airy\\_function](#)

## BESSEL FUNCTIONS

This module provides symbolic Bessel Functions. These functions use the [mpmath library](#) for numerical evaluation and Maxima, GiNaC, Pynac for symbolics.

The main objects which are exported from this module are:

- `bessel_J` – The Bessel J function
- `bessel_Y` – The Bessel Y function
- `bessel_I` – The Bessel I function
- `bessel_K` – The Bessel K function
- `Bessel` – A factory function for producing Bessel functions of various kinds and orders
- Bessel functions, first defined by the Swiss mathematician Daniel Bernoulli and named after Friedrich Bessel, are canonical solutions  $y(x)$  of Bessel's differential equation:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2) y = 0,$$

for an arbitrary complex number  $\nu$  (the order).

- In this module,  $J_\nu$  denotes the unique solution of Bessel's equation which is non-singular at  $x = 0$ . This function is known as the Bessel Function of the First Kind. This function also arises as a special case of the hypergeometric function  ${}_0F_1$ :

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu + 1)} {}_0F_1(\nu + 1, -\frac{x^2}{4}).$$

- The second linearly independent solution to Bessel's equation (which is singular at  $x = 0$ ) is denoted by  $Y_\nu$  and is called the Bessel Function of the Second Kind:

$$Y_\nu(x) = \frac{J_\nu(x) \cos(\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)}.$$

- There are also two commonly used combinations of the Bessel J and Y Functions. The Bessel I Function, or the Modified Bessel Function of the First Kind, is defined by:

$$I_\nu(x) = i^{-\nu} J_\nu(ix).$$

The Bessel K Function, or the Modified Bessel Function of the Second Kind, is defined by:

$$K_\nu(x) = \frac{\pi}{2} \cdot \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)}.$$

We should note here that the above formulas for Bessel Y and K functions should be understood as limits when  $\nu$  is an integer.

- It follows from Bessel's differential equation that the derivative of  $J_n(x)$  with respect to  $x$  is:

$$\frac{d}{dx} J_n(x) = \frac{1}{x^n} (x^n J_{n-1}(x) - n x^{n-1} J_n(x))$$

- Another important formulation of the two linearly independent solutions to Bessel's equation are the Hankel functions  $H_\nu^{(1)}(x)$  and  $H_\nu^{(2)}(x)$ , defined by:

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x)$$

$$H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x)$$

where  $i$  is the imaginary unit (and  $J_*$  and  $Y_*$  are the usual J- and Y-Bessel functions). These linear combinations are also known as Bessel functions of the third kind; they are also two linearly independent solutions of Bessel's differential equation. They are named for Hermann Hankel.

#### EXAMPLES:

Evaluate the Bessel J function symbolically and numerically:

```
sage: bessell_J(0, x)
bessell_J(0, x)
sage: bessell_J(0, 0)
bessell_J(0, 0)
sage: bessell_J(0, x).diff(x)
-1/2*bessell_J(1, x) + 1/2*bessell_J(-1, x)
```

```
sage: N(bessell_J(0, 0), digits = 20)
1.00000000000000000000
sage: find_root(bessell_J(0,x), 0, 5)
2.404825557695773
```

Plot the Bessel J function:

```
sage: f(x) = Bessel(0)(x); f
x |--> bessell_J(0, x)
sage: plot(f, (x, 1, 10))
Graphics object consisting of 1 graphics primitive
```

Visualize the Bessel Y function on the complex plane (set `plot_points` to a higher value to get more detail):

```
sage: complex_plot(bessell_Y(0, x), (-5, 5), (-5, 5), plot_points=20)
Graphics object consisting of 1 graphics primitive
```

Evaluate a combination of Bessel functions:

```
sage: f(x) = bessell_J(1, x) - bessell_Y(0, x)
sage: f(pi)
bessell_J(1, pi) - bessell_Y(0, pi)
sage: f(pi).n()
-0.0437509653365599
sage: f(pi).n(digits=50)
-0.043750965336559909054985168023342675387737118378169
```

Symbolically solve a second order differential equation with initial conditions  $y(1) = a$  and  $y'(1) = b$  in terms of Bessel functions:

```
sage: y = function('y', x)
sage: a, b = var('a, b')
```



```

sage: diffeq = x^2*diff(y,x,x) + x*diff(y,x) + x^2*y == 0
sage: f = desolve(diffeq, y, [1, a, b]); f
(a*bessel_Y(1, 1) + b*bessel_Y(0, 1))*bessel_J(0, x)/(bessel_J(0,
1)*bessel_Y(1, 1) - bessel_J(1, 1)*bessel_Y(0, 1)) -
(a*bessel_J(1, 1) + b*bessel_J(0, 1))*bessel_Y(0, x)/(bessel_J(0,
1)*bessel_Y(1, 1) - bessel_J(1, 1)*bessel_Y(0, 1))

```

For more examples, see the docstring for `Bessel()`.

#### AUTHORS:

- Benjamin Jones (2012-12-27): initial version
- Some of the documentation here has been adapted from David Joyner's original documentation of Sage's special functions module (2006).

#### REFERENCES:

- Abramowitz and Stegun: Handbook of Mathematical Functions, <http://www.math.sfu.ca/~cbm/aands/>
- [http://en.wikipedia.org/wiki/Bessel\\_function](http://en.wikipedia.org/wiki/Bessel_function)
- mpmath Library [Bessel Functions](#)

`sage.functions.bessel.Bessel(*args, **kwargs)`

A function factory that produces symbolic I, J, K, and Y Bessel functions. There are several ways to call this function:

- `Bessel(order, type)`
- `Bessel(order)` – type defaults to 'J'
- `Bessel(order, typ=T)`
- `Bessel(typ=T)` – order is unspecified, this is a 2-parameter function
- `Bessel()` – order is unspecified, type is 'J'

where `order` can be any integer and `T` must be one of the strings 'I', 'J', 'K', or 'Y'.

See the EXAMPLES below.

#### EXAMPLES:

Construction of Bessel functions with various orders and types:

```

sage: Bessel()
bessel_J
sage: Bessel(1)(x)
bessel_J(1, x)
sage: Bessel(1, 'Y')(x)
bessel_Y(1, x)
sage: Bessel(-2, 'Y')(x)
bessel_Y(-2, x)
sage: Bessel(typ='K')
bessel_K
sage: Bessel(0, typ='I')(x)
bessel_I(0, x)

```

#### Evaluation:

```

sage: f = Bessel(1)
sage: f(3.0)
0.339058958525936
sage: f(3)

```

```
bessel_J(1, 3)
sage: f(3).n(digits=50)
0.33905895852593645892551459720647889697308041819801

sage: g = Bessel(typ='J')
sage: g(1,3)
bessel_J(1, 3)
sage: g(2, 3+I).n()
0.634160370148554 + 0.0253384000032695*I
sage: abs(numerical_integral(1/pi*cos(3*sin(x)), 0.0, pi)[0] - Bessel(0, 'J')(3.0)) < 1e-15
True
```

Symbolic calculus:

```
sage: f(x) = Bessel(0, 'J')(x)
sage: derivative(f, x)
x |--> -1/2*bessel_J(1, x) + 1/2*bessel_J(-1, x)
sage: derivative(f, x, x)
x |--> 1/4*bessel_J(2, x) - 1/2*bessel_J(0, x) + 1/4*bessel_J(-2, x)
```

Verify that  $J_0$  satisfies Bessel's differential equation numerically using the `test_relation()` method:

```
sage: y = bessel_J(0, x)
sage: diffeq = x^2*derivative(y,x,x) + x*derivative(y,x) + x^2*y == 0
sage: diffeq.test_relation(proof=False)
True
```

Conversion to other systems:

```
sage: x,y = var('x,y')
sage: f = maxima(Bessel(typ='K')(x,y))
sage: f.derivative('_SAGE_VAR_x')
%pi*csc(%pi*_SAGE_VAR_x)*(diff(bessel_i(_SAGE_VAR_x,_SAGE_VAR_y),_SAGE_VAR_x,1)-diff(bessel_i
sage: f.derivative('_SAGE_VAR_y')
-(bessel_k(_SAGE_VAR_x+1,_SAGE_VAR_y)+bessel_k(_SAGE_VAR_x-1,_SAGE_VAR_y))/2
```

Compute the particular solution to Bessel's Differential Equation that satisfies  $y(1) = 1$  and  $y'(1) = 1$ , then verify the initial conditions and plot it:

```
sage: y = function('y', x)
sage: diffeq = x^2*diff(y,x,x) + x*diff(y,x) + x^2*y == 0
sage: f = desolve(diffeq, y, [1, 1, 1]); f
(bessel_Y(1, 1) + bessel_Y(0, 1))*bessel_J(0, x)/(bessel_J(0,
1)*bessel_Y(1, 1) - bessel_J(1, 1)*bessel_Y(0, 1)) - (bessel_J(1,
1) + bessel_J(0, 1))*bessel_Y(0, x)/(bessel_J(0, 1)*bessel_Y(1, 1)
- bessel_J(1, 1)*bessel_Y(0, 1))
sage: f.subs(x=1).n() # numerical verification
1.000000000000000
sage: fp = f.diff(x)
sage: fp.subs(x=1).n()
1.000000000000000

sage: f.subs(x=1).simplify_full() # symbolic verification
1
sage: fp = f.diff(x)
sage: fp.subs(x=1).simplify_full()
1

sage: plot(f, (x,0,5))
Graphics object consisting of 1 graphics primitive
```

Plotting:

```
sage: f(x) = Bessel(0)(x); f
x |--> bessell_J(0, x)
sage: plot(f, (x, 1, 10))
Graphics object consisting of 1 graphics primitive

sage: plot([ Bessel(i, 'J') for i in range(5) ], 2, 10)
Graphics object consisting of 5 graphics primitives

sage: G = Graphics()
sage: G += sum([ plot(Bessel(i), 0, 4*pi, rgbcolor=hue(sin(pi*i/10))) for i in range(5) ])
sage: show(G)
```

A recreation of Abramowitz and Stegun Figure 9.1:

```
sage: G = plot(Bessel(0, 'J'), 0, 15, color='black')
sage: G += plot(Bessel(0, 'Y'), 0, 15, color='black')
sage: G += plot(Bessel(1, 'J'), 0, 15, color='black', linestyle='dotted')
sage: G += plot(Bessel(1, 'Y'), 0, 15, color='black', linestyle='dotted')
sage: show(G, ymin=-1, ymax=1)
```

**class** sage.functions.bessel.**Function\_Bessel\_I**

Bases: sage.symbolic.function.BuiltinFunction

The Bessel I function, or the Modified Bessel Function of the First Kind.

DEFINITION:

$$I_{\nu}(x) = i^{-\nu} J_{\nu}(ix)$$

EXAMPLES:

```
sage: bessell_I(1, x)
bessell_I(1, x)
sage: bessell_I(1.0, 1.0)
0.565159103992485
sage: n = var('n')
sage: bessell_I(n, x)
bessell_I(n, x)
sage: bessell_I(2, I).n()
-0.114903484931900
```

Examples of symbolic manipulation:

```
sage: a = bessell_I(pi, bessell_I(1, I))
sage: N(a, digits=20)
0.00026073272117205890528 - 0.0011528954889080572266*I

sage: f = bessell_I(2, x)
sage: f.diff(x)
1/2*bessell_I(3, x) + 1/2*bessell_I(1, x)
```

Special identities that bessell\_I satisfies:

```
sage: bessell_I(1/2, x)
sqrt(2)*sqrt(1/(pi*x))*sinh(x)
sage: eq = bessell_I(1/2, x) == bessell_I(0.5, x)
sage: eq.test_relation()
True
```

```
sage: bessell_I(-1/2, x)
sqrt(2)*sqrt(1/(pi*x))*cosh(x)
sage: eq = bessell_I(-1/2, x) == bessell_I(-0.5, x)
sage: eq.test_relation()
True
```

Examples of asymptotic behavior:

```
sage: limit(bessell_I(0, x), x=oo)
+Infinity
sage: limit(bessell_I(0, x), x=0)
1
```

High precision and complex valued inputs:

```
sage: bessell_I(0, 1).n(128)
1.2660658777520083355982446252147175376
sage: bessell_I(0, RealField(200)(1))
1.2660658777520083355982446252147175376076703113549622068081
sage: bessell_I(0, ComplexField(200)(0.5+I))
0.80644357583493619472428518415019222845373366024179916785502 + 0.226869589879111611413974534014
```

Visualization (set `plot_points` to a higher value to get more detail):

```
sage: plot(bessell_I(1, x), (x, 0, 5), color='blue')
Graphics object consisting of 1 graphics primitive
sage: complex_plot(bessell_I(1, x), (-5, 5), (-5, 5), plot_points=20)
Graphics object consisting of 1 graphics primitive
```

ALGORITHM:

Numerical evaluation is handled by the `mpmath` library. Symbolics are handled by a combination of Maxima and Sage (Ginac/Pynac).

TESTS:

```
sage: N(bessell_I(1, 1), 500)
0.5651591039924850272076960276098633073288996216210920094802944894792556409643711340926649977668
```

Check whether the return value is real whenever the argument is real ([trac ticket #10251](#)):

```
sage: bessell_I(5, 1.5) in RR
True
```

**class** `sage.functions.bessel.Function_Bessel_J`

Bases: `sage.symbolic.function.BuiltinFunction`

The Bessel J Function, denoted by `bessel_J( $\nu$ ,  $x$ )` or  $J_\nu(x)$ . As a Taylor series about  $x = 0$  it is equal to:

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + \nu + 1)} \left(\frac{x}{2}\right)^{2k+\nu}$$

The parameter  $\nu$  is called the order and may be any real or complex number; however, integer and half-integer values are most common. It is defined for all complex numbers  $x$  when  $\nu$  is an integer or greater than zero and it diverges as  $x \rightarrow 0$  for negative non-integer values of  $\nu$ .

For integer orders  $\nu = n$  there is an integral representation:

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(nt - x \sin(t)) dt$$

This function also arises as a special case of the hypergeometric function  ${}_0F_1$ :

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu + 1)} {}_0F_1\left(\nu + 1, -\frac{x^2}{4}\right).$$

#### EXAMPLES:

```
sage: bessell_J(1.0, 1.0)
0.440050585744933
sage: bessell_J(2, I).n(digits=30)
-0.135747669767038281182852569995
```

```
sage: bessell_J(1, x)
bessell_J(1, x)
sage: n = var('n')
sage: bessell_J(n, x)
bessell_J(n, x)
```

Examples of symbolic manipulation:

```
sage: a = bessell_J(pi, bessell_J(1, I)); a
bessell_J(pi, bessell_J(1, I))
sage: N(a, digits=20)
0.00059023706363796717363 - 0.0026098820470081958110*I
sage: f = bessell_J(2, x)
sage: f.diff(x)
-1/2*bessell_J(3, x) + 1/2*bessell_J(1, x)
```

Comparison to a well-known integral representation of  $J_1(1)$ :

```
sage: A = numerical_integral(1/pi*cos(x - sin(x)), 0, pi)
sage: A[0] # abs tol 1e-14
0.44005058574493355
sage: bessell_J(1.0, 1.0) - A[0] < 1e-15
True
```

Integration is supported directly and through Maxima:

```
sage: f = bessell_J(2, x)
sage: f.integrate(x)
1/24*x^3*hypergeometric((3/2,), (5/2, 3), -1/4*x^2)
sage: m = maxima(bessell_J(2, x))
sage: m.integrate(x)
hypergeometric([3/2], [5/2, 3], -_SAGE_VAR_x^2/4) *_SAGE_VAR_x^3/24
```

Visualization (set `plot_points` to a higher value to get more detail):

```
sage: plot(bessell_J(1, x), (x, 0, 5), color='blue')
Graphics object consisting of 1 graphics primitive
sage: complex_plot(bessell_J(1, x), (-5, 5), (-5, 5), plot_points=20)
Graphics object consisting of 1 graphics primitive
```

#### ALGORITHM:

Numerical evaluation is handled by the mpmath library. Symbolics are handled by a combination of Maxima and Sage (Ginac/Pynac).

Check whether the return value is real whenever the argument is real ([trac ticket #10251](#)):

```
sage: bessell_J(5, 1.5) in RR
True
```

**class** sage.functions.bessel.**Function\_Bessel\_K**  
Bases: sage.symbolic.function.BuiltinFunction

The Bessel K function, or the modified Bessel function of the second kind.

DEFINITION:

$$K_{\nu}(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_{\nu}(x)}{\sin(\nu\pi)}$$

EXAMPLES:

```
sage: bessell_K(1, x)
bessell_K(1, x)
sage: bessell_K(1.0, 1.0)
0.601907230197235
sage: n = var('n')
sage: bessell_K(n, x)
bessell_K(n, x)
sage: bessell_K(2, I).n()
-2.59288617549120 + 0.180489972066962*I
```

Examples of symbolic manipulation:

```
sage: a = bessell_K(pi, bessell_K(1, I)); a
bessell_K(pi, bessell_K(1, I))
sage: N(a, digits=20)
3.8507583115005220157 + 0.068528298579883425792*I
```

```
sage: f = bessell_K(2, x)
sage: f.diff(x)
-1/2*bessell_K(3, x) - 1/2*bessell_K(1, x)
```

```
sage: bessell_K(1/2, x)
bessell_K(1/2, x)
sage: bessell_K(1/2, -1)
bessell_K(1/2, -1)
sage: bessell_K(1/2, 1)
sqrt(1/2)*sqrt(pi)*e^(-1)
```

Examples of asymptotic behavior:

```
sage: bessell_K(0, 0.0)
+infinity
sage: limit(bessell_K(0, x), x=0)
+Infinity
sage: limit(bessell_K(0, x), x=oo)
0
```

High precision and complex valued inputs:

```
sage: bessell_K(0, 1).n(128)
0.4210244382407083333562737921260903614
sage: bessell_K(0, RealField(200)(1))
0.4210244382407083333562737921260903613621974822666047229897
sage: bessell_K(0, ComplexField(200)(0.5+I))
0.058365979093103864080375311643360048144715516692187818271179 - 0.67645499731334483535184142196
```

Visualization (set `plot_points` to a higher value to get more detail):

```
sage: plot(bessel_K(1,x), (x,0,5), color='blue')
Graphics object consisting of 1 graphics primitive
sage: complex_plot(bessel_K(1, x), (-5, 5), (-5, 5), plot_points=20)
Graphics object consisting of 1 graphics primitive
```

#### ALGORITHM:

Numerical evaluation is handled by the `mpmath` library. Symbolics are handled by a combination of `Maxima` and `Sage` (`Ginac/Pynac`).

#### TESTS:

Verify that [trac ticket #3426](#) is fixed:

The Bessel K function can be evaluated numerically at complex orders:

```
sage: bessel_K(10 * I, 10).n()
9.82415743819925e-8
```

For a fixed imaginary order and increasing, real, second component the value of Bessel K is exponentially decaying:

```
sage: for x in [10, 20, 50, 100, 200]: print bessel_K(5*I, x).n()
5.27812176514912e-6
3.11005908421801e-10
2.66182488515423e-23 - 8.59622057747552e-58*I
4.11189776828337e-45 - 1.01494840019482e-80*I
1.15159692553603e-88 - 6.75787862113718e-125*I
```

Check whether the return value is real whenever the argument is real ([trac ticket #10251](#)):

```
sage: bessel_K(5, 1.5) in RR
True
```

**class** `sage.functions.bessel.Function_Bessel_Y`

Bases: `sage.symbolic.function.BuiltinFunction`

The Bessel Y functions, also known as the Bessel functions of the second kind, Weber functions, or Neumann functions.

$Y_\nu(z)$  is a holomorphic function of  $z$  on the complex plane, cut along the negative real axis. It is singular at  $z = 0$ . When  $z$  is fixed,  $Y_\nu(z)$  is an entire function of the order  $\nu$ .

#### DEFINITION:

$$Y_n(z) = \frac{J_n(z) \cos(\nu z) - J_{-\nu}(z)}{\sin(\nu z)}$$

Its derivative with respect to  $z$  is:

$$\frac{d}{dz} Y_n(z) = \frac{1}{z^n} (z^n Y_{n-1}(z) - n z^{n-1} Y_n(z))$$

#### EXAMPLES:

```
sage: bessel_Y(1, x)
bessel_Y(1, x)
sage: bessel_Y(1.0, 1.0)
-0.781212821300289
sage: n = var('n')
sage: bessel_Y(n, x)
bessel_Y(n, x)
```

```
sage: bessell_Y(2, I).n()
1.03440456978312 - 0.135747669767038*I
sage: bessell_Y(0, 0).n()
-infinity
sage: bessell_Y(0, 1).n(128)
0.088256964215676957982926766023515162828
```

Examples of symbolic manipulation:

```
sage: a = bessell_Y(pi, bessell_Y(1, I)); a
bessell_Y(pi, bessell_Y(1, I))
sage: N(a, digits=20)
4.2059146571791095708 + 21.307914215321993526*I

sage: f = bessell_Y(2, x)
sage: f.diff(x)
-1/2*bessell_Y(3, x) + 1/2*bessell_Y(1, x)
```

High precision and complex valued inputs (see [trac ticket #4230](#)):

```
sage: bessell_Y(0, 1).n(128)
0.088256964215676957982926766023515162828
sage: bessell_Y(0, RealField(200)(1))
0.088256964215676957982926766023515162827817523090675546711044
sage: bessell_Y(0, ComplexField(200)(0.5+I))
0.077763160184438051408593468823822434235010300228009867784073 + 1.01423360499160691526446776828
```

Visualization (set `plot_points` to a higher value to get more detail):

```
sage: plot(bessell_Y(1,x), (x,0,5), color='blue')
Graphics object consisting of 1 graphics primitive
sage: complex_plot(bessell_Y(1, x), (-5, 5), (-5, 5), plot_points=20)
Graphics object consisting of 1 graphics primitive
```

#### ALGORITHM:

Numerical evaluation is handled by the `mpmath` library. Symbolics are handled by a combination of `Maxima` and `Sage` (`Ginac/Pynac`).

#### TESTS:

Check whether the return value is real whenever the argument is real ([trac ticket #10251](#)):

```
sage: bessell_Y(5, 1.5) in RR
True
```

Coercion works correctly (see [trac ticket #17130](#)):

```
sage: r = bessell_Y(RealField(200)(1), 1.0); r
-0.781212821300289
sage: parent(r)
Real Field with 53 bits of precision
sage: r = bessell_Y(RealField(200)(1), 1); r
-0.78121282130028871654715000004796482054990639071644460784383
sage: parent(r)
Real Field with 200 bits of precision
```



## EXPONENTIAL INTEGRALS

### AUTHORS:

- Benjamin Jones (2011-06-12)

This module provides easy access to many exponential integral special functions. It utilizes Maxima's [special functions package](#) and the [mpmath library](#).

### REFERENCES:

- [AS] Abramowitz and Stegun: *Handbook of Mathematical Functions*
- Wikipedia Entry: [http://en.wikipedia.org/wiki/Exponential\\_integral](http://en.wikipedia.org/wiki/Exponential_integral)
- Online Encyclopedia of Special Function: <http://algo.inria.fr/esf/index.html>
- NIST Digital Library of Mathematical Functions: <http://dlmf.nist.gov/>
- Maxima [special functions package](#)
- [mpmath library](#)

### AUTHORS:

- Benjamin Jones

Implementations of the classes `Function_exp_integral_*`.

- David Joyner and William Stein

Authors of the code which was moved from `special.py` and `trans.py`. Implementation of `exp_int()` (from `sage/functions/special.py`). Implementation of `exponential_integral_1()` (from `sage/functions/transcendental.py`).

**class** `sage.functions.exp_integral.Function_cos_integral`

Bases: `sage.symbolic.function.BuiltinFunction`

The trigonometric integral  $\text{Ci}(z)$  defined by

$$\text{Ci}(z) = \gamma + \log(z) + \int_0^z \frac{\cos(t) - 1}{t} dt,$$

where  $\gamma$  is the Euler gamma constant (`euler_gamma` in Sage), see [AS] 5.2.1.

### EXAMPLES:

```
sage: z = var('z')
sage: cos_integral(z)
cos_integral(z)
sage: cos_integral(3.0)
0.119629786008000
sage: cos_integral(0)
```

```
cos_integral(0)
sage: N(cos_integral(0))
-infinity
```

Numerical evaluation for real and complex arguments is handled using mpmath:

```
sage: cos_integral(3.0)
0.119629786008000
```

The alias *Ci* can be used instead of *cos\_integral*:

```
sage: Ci(3.0)
0.119629786008000
```

Compare `cos_integral(3.0)` to the definition of the value using numerical integration:

```
sage: N(euler_gamma + log(3.0) + integrate((cos(x)-1)/x, x, 0, 3.0) - cos_integral(3.0)) < 1e-14
True
```

Arbitrary precision and complex arguments are handled:

```
sage: N(cos_integral(3), digits=30)
0.119629786008000327626472281177
sage: cos_integral(ComplexField(100)(3+I))
0.078134230477495714401983633057 - 0.37814733904787920181190368789*I
```

The limit  $Ci(z)$  as  $z \rightarrow \infty$  is zero:

```
sage: N(cos_integral(1e23))
-3.24053937643003e-24
```

Symbolic derivatives and integrals are handled by Sage and Maxima:

```
sage: x = var('x')
sage: f = cos_integral(x)
sage: f.diff(x)
cos(x)/x
```

```
sage: f.integrate(x)
x*cos_integral(x) - sin(x)
```

The Nielsen spiral is the parametric plot of  $(Si(t), Ci(t))$ :

```
sage: t=var('t')
sage: f(t) = sin_integral(t)
sage: g(t) = cos_integral(t)
sage: P = parametric_plot([f, g], (t, 0.5, 20))
sage: show(P, frame=True, axes=False)
```

#### ALGORITHM:

Numerical evaluation is handled using mpmath, but symbolics are handled by Sage and Maxima.

#### REFERENCES:

- [http://en.wikipedia.org/wiki/Trigonometric\\_integral](http://en.wikipedia.org/wiki/Trigonometric_integral)
- mpmath documentation: `ci`

```
class sage.functions.exp_integral.Function_cosh_integral
Bases: sage.symbolic.function.BuiltinFunction
```

The trigonometric integral  $\text{Chi}(z)$  defined by

$$\text{Chi}(z) = \gamma + \log(z) + \int_0^z \frac{\cosh(t) - 1}{t} dt,$$

see [AS] 5.2.4.

EXAMPLES:

```
sage: z = var('z')
sage: cosh_integral(z)
cosh_integral(z)
sage: cosh_integral(3.0)
4.96039209476561
```

Numerical evaluation for real and complex arguments is handled using mpmath:

```
sage: cosh_integral(1.0)
0.837866940980208
```

The alias *Chi* can be used instead of *cosh\_integral*:

```
sage: Chi(1.0)
0.837866940980208
```

Here is an example from the mpmath documentation:

```
sage: f(x) = cosh_integral(x)
sage: find_root(f, 0.1, 1.0)
0.523822571389482...
```

Compare `cosh_integral(3.0)` to the definition of the value using numerical integration:

```
sage: N(euler_gamma + log(3.0) + integrate((cosh(x)-1)/x, x, 0, 3.0) -
...     cosh_integral(3.0)) < 1e-14
True
```

Arbitrary precision and complex arguments are handled:

```
sage: N(cosh_integral(3), digits=30)
4.96039209476560976029791763669
sage: cosh_integral(ComplexField(100)(3+I))
3.9096723099686417127843516794 + 3.0547519627014217273323873274*I
```

The limit of  $\text{Chi}(z)$  as  $z \rightarrow \infty$  is  $\infty$ :

```
sage: N(cosh_integral(Infinity))
+infinity
```

Symbolic derivatives and integrals are handled by Sage and Maxima:

```
sage: x = var('x')
sage: f = cosh_integral(x)
sage: f.diff(x)
cosh(x)/x

sage: f.integrate(x)
x*cosh_integral(x) - sinh(x)
```

ALGORITHM:

Numerical evaluation is handled using mpmath, but symbolics are handled by Sage and Maxima.

REFERENCES:

- [http://en.wikipedia.org/wiki/Trigonometric\\_integral](http://en.wikipedia.org/wiki/Trigonometric_integral)

- mpmath documentation: [chi](#)

**class** sage.functions.exp\_integral.**Function\_exp\_integral**

Bases: sage.symbolic.function.BuiltinFunction

The generalized complex exponential integral  $Ei(z)$  defined by

$$Ei(x) = \int_{-\infty}^x \frac{e^t}{t} dt$$

for  $x > 0$  and for complex arguments by analytic continuation, see [AS] 5.1.2.

EXAMPLES:

```
sage: Ei(10)
```

```
Ei(10)
```

```
sage: Ei(I)
```

```
Ei(I)
```

```
sage: Ei(3+I)
```

```
Ei(I + 3)
```

```
sage: Ei(1.3)
```

```
2.72139888023202
```

```
sage: Ei(10r)
```

```
Ei(10)
```

```
sage: Ei(1.3r)
```

```
2.7213988802320235
```

The branch cut for this function is along the negative real axis:

```
sage: Ei(-3 + 0.1*I)
```

```
-0.0129379427181693 + 3.13993830250942*I
```

```
sage: Ei(-3 - 0.1*I)
```

```
-0.0129379427181693 - 3.13993830250942*I
```

The precision for the result is deduced from the precision of the input. Convert the input to a higher precision explicitly if a result with higher precision is desired:

```
sage: Ei(RealField(300)(1.1))
```

```
2.16737827956340282358378734233807621497112737591639704719499002090327541763352339357795426
```

ALGORITHM: Uses mpmath.

TESTS:

Show that the evaluation and limit issue in [trac ticket #13271](#) is fixed:

```
sage: var('Z')
```

```
Z
```

```
sage: (Ei(-Z)).limit(Z=oo)
```

```
0
```

```
sage: (Ei(-Z)).limit(Z=1000)
```

```
Ei(-1000)
```

```
sage: (Ei(-Z)).limit(Z=1000).n()
```

```
-5.07089306023517e-438
```

**class** sage.functions.exp\_integral.**Function\_exp\_integral\_e**

Bases: sage.symbolic.function.BuiltinFunction

The generalized complex exponential integral  $E_n(z)$  defined by

$$E_n(z) = \int_1^{\infty} \frac{e^{-zt}}{t^n} dt$$

for complex numbers  $n$  and  $z$ , see [AS] 5.1.4.

The special case where  $n = 1$  is denoted in Sage by `exp_integral_e1`.

EXAMPLES:

Numerical evaluation is handled using mpmath:

```
sage: N(exp_integral_e(1,1))
0.219383934395520
sage: exp_integral_e(1, RealField(100)(1))
0.21938393439552027367716377546
```

We can compare this to PARI's evaluation of `exponential_integral_1()`:

```
sage: N(exponential_integral_1(1))
0.219383934395520
```

We can verify one case of [AS] 5.1.45, i.e.  $E_n(z) = z^{n-1}\Gamma(1-n, z)$ :

```
sage: N(exp_integral_e(2, 3+I))
0.00354575823814662 - 0.00973200528288687*I
sage: N((3+I)*gamma(-1, 3+I))
0.00354575823814662 - 0.00973200528288687*I
```

Maxima returns the following improper integral as a multiple of `exp_integral_e(1,1)`:

```
sage: uu = integral(e^(-x)*log(x+1), x, 0, oo)
sage: uu
e*exp_integral_e(1, 1)
sage: uu.n(digits=30)
0.596347362323194074341078499369
```

Symbolic derivatives and integrals are handled by Sage and Maxima:

```
sage: x = var('x')
sage: f = exp_integral_e(2,x)
sage: f.diff(x)
-exp_integral_e(1, x)

sage: f.integrate(x)
-exp_integral_e(3, x)

sage: f = exp_integral_e(-1,x)
sage: f.integrate(x)
Ei(-x) - gamma(-1, x)
```

Some special values of `exp_integral_e` can be simplified. [AS] 5.1.23:

```
sage: exp_integral_e(0,x)
e^(-x)/x
```

[AS] 5.1.24:

```
sage: exp_integral_e(6,0)
1/5
sage: nn = var('nn')
sage: assume(nn > 1)
sage: f = exp_integral_e(nn,0)
sage: f.simplify()
1/(nn - 1)
```

ALGORITHM:

Numerical evaluation is handled using mpmath, but symbolics are handled by Sage and Maxima.

**class** sage.functions.exp\_integral.**Function\_exp\_integral\_e1**  
Bases: sage.symbolic.function.BuiltinFunction

The generalized complex exponential integral  $E_1(z)$  defined by

$$E_1(z) = \int_z^{\infty} \frac{e^{-t}}{t} dt$$

see [\[AS\] 5.1.4](#).

EXAMPLES:

```
sage: exp_integral_e1(x)
exp_integral_e1(x)
sage: exp_integral_e1(1.0)
0.219383934395520
```

Numerical evaluation is handled using mpmath:

```
sage: N(exp_integral_e1(1))
0.219383934395520
sage: exp_integral_e1(RealField(100)(1))
0.21938393439552027367716377546
```

We can compare this to PARI's evaluation of `exponential_integral_1()`:

```
sage: N(exp_integral_e1(2.0))
0.0489005107080611
sage: N(exponential_integral_1(2.0))
0.0489005107080611
```

Symbolic derivatives and integrals are handled by Sage and Maxima:

```
sage: x = var('x')
sage: f = exp_integral_e1(x)
sage: f.diff(x)
-e^(-x)/x

sage: f.integrate(x)
-exp_integral_e(2, x)
```

ALGORITHM:

Numerical evaluation is handled using mpmath, but symbolics are handled by Sage and Maxima.

**class** sage.functions.exp\_integral.**Function\_log\_integral**  
Bases: sage.symbolic.function.BuiltinFunction

The logarithmic integral  $\text{li}(z)$  defined by

$$\text{li}(x) = \int_0^x \frac{dt}{\ln(t)} = \text{Ei}(\ln(x))$$

for  $x > 1$  and by analytic continuation for complex arguments  $z$  (see [\[AS\] 5.1.3](#)).

EXAMPLES:

Numerical evaluation for real and complex arguments is handled using mpmath:

```
sage: N(log_integral(3))
2.16358859466719
sage: N(log_integral(3), digits=30)
```

```

2.16358859466719197287692236735
sage: log_integral(ComplexField(100)(3+I))
2.2879892769816826157078450911 + 0.87232935488528370139883806779*I
sage: log_integral(0)
0

```

Symbolic derivatives and integrals are handled by Sage and Maxima:

```

sage: x = var('x')
sage: f = log_integral(x)
sage: f.diff(x)
1/log(x)

sage: f.integrate(x)
x*log_integral(x) - Ei(2*log(x))

```

Here is a test from the mpmath documentation. There are 1,925,320,391,606,803,968,923 many prime numbers less than  $1e23$ . The value of `log_integral(1e23)` is very close to this:

```

sage: log_integral(1e23)
1.92532039161405e21

```

#### ALGORITHM:

Numerical evaluation is handled using mpmath, but symbolics are handled by Sage and Maxima.

#### REFERENCES:

- [http://en.wikipedia.org/wiki/Logarithmic\\_integral\\_function](http://en.wikipedia.org/wiki/Logarithmic_integral_function)
- mpmath documentation: `logarithmic-integral`

```

class sage.functions.exp_integral.Function_log_integral_offset
Bases: sage.symbolic.function.BuiltinFunction

```

The offset logarithmic integral, or Eulerian logarithmic integral,  $\text{Li}(x)$  is defined by

$$\text{Li}(x) = \int_2^x \frac{dt}{\ln(t)} = \text{li}(x) - \text{li}(2)$$

for  $x \geq 2$ .

The offset logarithmic integral should also not be confused with the polylogarithm (also denoted by  $\text{Li}(x)$ ), which is implemented as `sage.functions.log.Function_polylog`.

$\text{Li}(x)$  is identical to  $\text{li}(x)$  except that the lower limit of integration is 2 rather than 0 to avoid the singularity at  $x = 1$  of

$$\frac{1}{\ln(t)}$$

See `Function_log_integral` for details of  $\text{li}(x)$ . Thus  $\text{Li}(x)$  can also be represented by

$$\text{Li}(x) = \text{li}(x) - \text{li}(2)$$

So we have:

```

sage: li(4.5) - li(2.0) - Li(4.5)
0.0000000000000000

```

$\text{Li}(x)$  is extended to complex arguments  $z$  by analytic continuation (see [AS] 5.1.3):

```
sage: Li(6.6+5.4*I)
3.97032201503632 + 2.62311237593572*I
```

The function  $\text{Li}$  is an approximation for the number of primes up to  $x$ . In fact, the famous Riemann Hypothesis is

$$|\pi(x) - \text{Li}(x)| \leq \sqrt{x} \log(x).$$

For “small”  $x$ ,  $\text{Li}(x)$  is always slightly bigger than  $\pi(x)$ . However it is a theorem that there are very large values of  $x$  (e.g., around  $10^{316}$ ), such that  $\exists x : \pi(x) > \text{Li}(x)$ . See “A new bound for the smallest  $x$  with  $\pi(x) > \text{li}(x)$ ”, Bays and Hudson, Mathematics of Computation, 69 (2000) 1285-1296.

---

**Note:** Definite integration returns a part symbolic and part numerical result. This is because when  $\text{Li}(x)$  is evaluated it is passed as  $\text{li}(x)-\text{li}(2)$ .

---

#### EXAMPLES:

Numerical evaluation for real and complex arguments is handled using mpmath:

```
sage: N(log_integral_offset(3))
1.11842481454970
sage: N(log_integral_offset(3), digits=30)
1.11842481454969918803233347815
sage: log_integral_offset(ComplexField(100)(3+I))
1.2428254968641898308632562019 + 0.87232935488528370139883806779*I
sage: log_integral_offset(2)
0
sage: for n in range(1,7):
....:     print '%-10s%-10s%-20s'%(10^n, prime_pi(10^n), N(Li(10^n)))
10          4          5.12043572466980
100         25         29.0809778039621
1000        168        176.564494210035
10000       1229       1245.09205211927
100000      9592      9628.76383727068
1000000     78498     78626.5039956821
```

Here is a test from the mpmath documentation. There are 1,925,320,391,606,803,968,923 prime numbers less than  $1e23$ . The value of  $\text{log\_integral\_offset}(1e23)$  is very close to this:

```
sage: log_integral_offset(1e23)
1.92532039161405e21
```

Symbolic derivatives are handled by Sage and integration by Maxima:

```
sage: x = var('x')
sage: f = log_integral_offset(x)
sage: f.diff(x)
1/log(x)
sage: f.integrate(x)
-x*log_integral(2) + x*log_integral(x) - Ei(2*log(x))
sage: Li(x).integrate(x, 2.0, 4.5)
-2.5*log_integral(2) + 5.799321147411334
sage: N(f.integrate(x, 2.0, 3.0))
0.601621785860587
```

#### ALGORITHM:

Numerical evaluation is handled using mpmath, but symbolics are handled by Sage and Maxima.

#### REFERENCES:



- [http://en.wikipedia.org/wiki/Logarithmic\\_integral\\_function](http://en.wikipedia.org/wiki/Logarithmic_integral_function)

- mpmath documentation: `logarithmic-integral`

**class** `sage.functions.exp_integral.Function_sin_integral`

Bases: `sage.symbolic.function.BuiltinFunction`

The trigonometric integral  $\text{Si}(z)$  defined by

$$\text{Si}(z) = \int_0^z \frac{\sin(t)}{t} dt,$$

see [AS] 5.2.1.

#### EXAMPLES:

Numerical evaluation for real and complex arguments is handled using mpmath:

```
sage: sin_integral(0)
0
sage: sin_integral(0.0)
0.0000000000000000
sage: sin_integral(3.0)
1.84865252799947
sage: N(sin_integral(3), digits=30)
1.84865252799946825639773025111
sage: sin_integral(ComplexField(100)(3+I))
2.0277151656451253616038525998 + 0.015210926166954211913653130271*I
```

The alias  $\text{Si}$  can be used instead of `sin_integral`:

```
sage: Si(3.0)
1.84865252799947
```

The limit of  $\text{Si}(z)$  as  $z \rightarrow \infty$  is  $\pi/2$ :

```
sage: N(sin_integral(1e23))
1.57079632679490
sage: N(pi/2)
1.57079632679490
```

At 200 bits of precision  $\text{Si}(10^{23})$  agrees with  $\pi/2$  up to  $10^{-24}$ :

```
sage: sin_integral(RealField(200)(1e23))
1.5707963267948966192313288218697837425815368604836679189519
sage: N(pi/2, prec=200)
1.5707963267948966192313216916397514420985846996875529104875
```

The exponential sine integral is analytic everywhere:

```
sage: sin_integral(-1.0)
-0.946083070367183
sage: sin_integral(-2.0)
-1.60541297680269
sage: sin_integral(-1e23)
-1.57079632679490
```

Symbolic derivatives and integrals are handled by Sage and Maxima:

```
sage: x = var('x')
sage: f = sin_integral(x)
sage: f.diff(x)
sin(x)/x
```

```
sage: f.integrate(x)
x*sin_integral(x) + cos(x)

sage: integrate(sin(x)/x, x)
-1/2*I*Ei(I*x) + 1/2*I*Ei(-I*x)
```

Compare values of the functions  $\text{Si}(x)$  and  $f(x) = (1/2)i \cdot \text{Ei}(-ix) - (1/2)i \cdot \text{Ei}(ix) - \pi/2$ , which are both anti-derivatives of  $\sin(x)/x$ , at some random positive real numbers:

```
sage: f(x) = 1/2*I*Ei(-I*x) - 1/2*I*Ei(I*x) - pi/2
sage: g(x) = sin_integral(x)
sage: R = [ abs(RDF.random_element()) for i in range(100) ]
sage: all(abs(f(x) - g(x)) < 1e-10 for x in R)
True
```

The Nielsen spiral is the parametric plot of  $(\text{Si}(t), \text{Ci}(t))$ :

```
sage: x=var('x')
sage: f(x) = sin_integral(x)
sage: g(x) = cos_integral(x)
sage: P = parametric_plot([f, g], (x, 0.5, 20))
sage: show(P, frame=True, axes=False)
```

#### ALGORITHM:

Numerical evaluation is handled using `mpmath`, but symbolics are handled by Sage and Maxima.

#### REFERENCES:

- [http://en.wikipedia.org/wiki/Trigonometric\\_integral](http://en.wikipedia.org/wiki/Trigonometric_integral)
- `mpmath` documentation: `si`

```
class sage.functions.exp_integral.Function_sinh_integral
Bases: sage.symbolic.function.BuiltinFunction
```

The trigonometric integral  $\text{Shi}(z)$  defined by

$$\text{Shi}(z) = \int_0^z \frac{\sinh(t)}{t} dt,$$

see [AS] 5.2.3.

#### EXAMPLES:

Numerical evaluation for real and complex arguments is handled using `mpmath`:

```
sage: sinh_integral(3.0)
4.97344047585981
sage: sinh_integral(1.0)
1.05725087537573
sage: sinh_integral(-1.0)
-1.05725087537573
```

The alias *Shi* can be used instead of *sinh<sub>i</sub>integral*:

```
sage: Shi(3.0)
4.97344047585981
```

Compare `sinh_integral(3.0)` to the definition of the value using numerical integration:

```
sage: N(integrate((sinh(x))/x, x, 0, 3.0) - sinh_integral(3.0)) < 1e-14
True
```

Arbitrary precision and complex arguments are handled:

```
sage: N(sinh_integral(3), digits=30)
4.97344047585980679771041838252
sage: sinh_integral(ComplexField(100)(3+I))
3.9134623660329374406788354078 + 3.0427678212908839256360163759*I
```

The limit  $\text{Shi}(z)$  as  $z \rightarrow \infty$  is  $\infty$ :

```
sage: N(sinh_integral(Infinity))
+infinity
```

Symbolic derivatives and integrals are handled by Sage and Maxima:

```
sage: x = var('x')
sage: f = sinh_integral(x)
sage: f.diff(x)
sinh(x)/x

sage: f.integrate(x)
x*sinh_integral(x) - cosh(x)
```

Note that due to some problems with the way Maxima handles these expressions, definite integrals can sometimes give unexpected results (typically when using inexact endpoints) due to inconsistent branching:

```
sage: integrate(sinh_integral(x), x, 0, 1/2)
-cosh(1/2) + 1/2*sinh_integral(1/2) + 1
sage: integrate(sinh_integral(x), x, 0, 1/2).n() # correct
0.125872409703453
sage: integrate(sinh_integral(x), x, 0, 0.5).n() # fixed in maxima 5.29.1
0.125872409703453
```

#### ALGORITHM:

Numerical evaluation is handled using `mpmath`, but symbolics are handled by Sage and Maxima.

#### REFERENCES:

- [http://en.wikipedia.org/wiki/Trigonometric\\_integral](http://en.wikipedia.org/wiki/Trigonometric_integral)
- `mpmath` documentation: [shi](#)

`sage.functions.exp_integral.exponential_integral_1(x, n=0)`

Returns the exponential integral  $E_1(x)$ . If the optional argument  $n$  is given, computes list of the first  $n$  values of the exponential integral  $E_1(xm)$ .

The exponential integral  $E_1(x)$  is

$$E_1(x) = \int_x^\infty e^{-t}/t dt$$

#### INPUT:

- $x$  – a positive real number
- $n$  – (default: 0) a nonnegative integer; if nonzero, then return a list of values  $E_1(x*m)$  for  $m = 1, 2, 3, \dots, n$ . This is useful, e.g., when computing derivatives of L-functions.

#### OUTPUT:

A real number if  $n$  is 0 (the default) or a list of reals if  $n > 0$ . The precision is the same as the input, with a default of 53 bits in case the input is exact.

#### EXAMPLES:

```
sage: exponential_integral_1(2)
0.0489005107080611
sage: exponential_integral_1(2, 4) # abs tol 1e-18
[0.0489005107080611, 0.00377935240984891, 0.000360082452162659, 0.0000376656228439245]
sage: exponential_integral_1(40, 5)
[0.0000000000000000, 2.22854325868847e-37, 6.33732515501151e-55, 2.02336191509997e-72, 6.88522610e-90]
sage: exponential_integral_1(0)
+Infinity
sage: r = exponential_integral_1(RealField(150)(1))
sage: r
0.21938393439552027367716377546012164903104729
sage: parent(r)
Real Field with 150 bits of precision
sage: exponential_integral_1(RealField(150)(100))
3.6835977616820321802351926205081189876552201e-46
```

#### TESTS:

The relative error for a single value should be less than 1 ulp:

```
sage: for prec in [20..1000]: # long time (22s on sage.math, 2013)
.....:     R = RealField(prec)
.....:     S = RealField(prec+64)
.....:     for t in range(8): # Try 8 values for each precision
.....:         a = R.random_element(-15,10).exp()
.....:         x = exponential_integral_1(a)
.....:         y = exponential_integral_1(S(a))
.....:         e = float(abs(S(x) - y)/x.ulp())
.....:         if e >= 1.0:
.....:             print "exponential_integral_1(%s) with precision %s has error of %s ulp"%(a, p
```

The absolute error for a vector should be less than  $c2^{-p}$ , where  $p$  is the precision in bits of  $x$  and  $c = 2\max(1, \text{exponential\_integral}_1(x))$ :

```
sage: for prec in [20..128]: # long time (15s on sage.math, 2013)
.....:     R = RealField(prec)
.....:     S = RealField(prec+64)
.....:     a = R.random_element(-15,10).exp()
.....:     n = 2^ZZ.random_element(14)
.....:     x = exponential_integral_1(a, n)
.....:     y = exponential_integral_1(S(a), n)
.....:     c = RDF(2 * max(1.0, y[0]))
.....:     for i in range(n):
.....:         e = float(abs(S(x[i]) - y[i]) << prec)
.....:         if e >= c:
.....:             print "exponential_integral_1(%s, %s)[%s] with precision %s has error of %s >=
```

ALGORITHM: use the PARI C-library function `eint1`.

#### REFERENCE:

- See Proposition 5.6.12 of Cohen's book "A Course in Computational Algebraic Number Theory".

## WIGNER, CLEBSCH-GORDAN, RACAH, AND GAUNT COEFFICIENTS

Collection of functions for calculating Wigner 3j, 6j, 9j, Clebsch-Gordan, Racah as well as Gaunt coefficients exactly, all evaluating to a rational number times the square root of a rational number [Rasch03].

Please see the description of the individual functions for further details and examples.

REFERENCES:

AUTHORS:

- Jens Rasch (2009-03-24): initial version for Sage
- Jens Rasch (2009-05-31): updated to sage-4.0

`sage.functions.wigner.clebsch_gordan(j_1, j_2, j_3, m_1, m_2, m_3, prec=None)`  
Calculates the Clebsch-Gordan coefficient  $\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle$ .

The reference for this function is [Edmonds74].

INPUT:

- `j_1, j_2, j_3, m_1, m_2, m_3` - integer or half integer
- `prec` - precision, default: None. Providing a precision can drastically speed up the calculation.

OUTPUT:

Rational number times the square root of a rational number (if `prec=None`), or real number if a precision is given.

EXAMPLES:

```
sage: simplify(clebsch_gordan(3/2, 1/2, 2, 3/2, 1/2, 2))
1
sage: clebsch_gordan(1.5, 0.5, 1, 1.5, -0.5, 1)
1/2*sqrt(3)
sage: clebsch_gordan(3/2, 1/2, 1, -1/2, 1/2, 0)
-sqrt(3)*sqrt(1/6)
```

NOTES:

The Clebsch-Gordan coefficient will be evaluated via its relation to Wigner 3j symbols:

$$\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle = (-1)^{j_1 - j_2 + m_3} \sqrt{2j_3 + 1} \text{Wigner3j}(j_1, j_2, j_3, m_1, m_2, -m_3)$$

See also the documentation on Wigner 3j symbols which exhibit much higher symmetry relations than the Clebsch-Gordan coefficient.

AUTHORS:

- Jens Rasch (2009-03-24): initial version

`sage.functions.wigner.gaunt(l_1, l_2, l_3, m_1, m_2, m_3, prec=None)`

Calculate the Gaunt coefficient.

The Gaunt coefficient is defined as the integral over three spherical harmonics:

$$Y(j_1, j_2, j_3, m_1, m_2, m_3) = \int Y_{l_1, m_1}(\Omega) Y_{l_2, m_2}(\Omega) Y_{l_3, m_3}(\Omega) d\Omega = \sqrt{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)/(4\pi)} Y(j_1, j_2, j_3, 0, 0, 0) Y$$

INPUT:

- `l_1, l_2, l_3, m_1, m_2, m_3` - integer
- `prec` - precision, default: None. Providing a precision can drastically speed up the calculation.

OUTPUT:

Rational number times the square root of a rational number (if `prec=None`), or real number if a precision is given.

EXAMPLES:

```
sage: gaunt(1, 0, 1, 1, 0, -1)
-1/2/sqrt(pi)
sage: gaunt(1, 0, 1, 1, 0, 0)
0
sage: gaunt(29, 29, 34, 10, -5, -5)
1821867940156/215552371055153321*sqrt(22134)/sqrt(pi)
sage: gaunt(20, 20, 40, 1, -1, 0)
28384503878959800/74029560764440771/sqrt(pi)
sage: gaunt(12, 15, 5, 2, 3, -5)
91/124062*sqrt(36890)/sqrt(pi)
sage: gaunt(10, 10, 12, 9, 3, -12)
-98/62031*sqrt(6279)/sqrt(pi)
sage: gaunt(1000, 1000, 1200, 9, 3, -12).n(64)
0.00689500421922113448
```

If the sum of the  $l_i$  is odd, the answer is zero, even for Python ints (see [trac ticket #14766](#)):

```
sage: gaunt(1, 2, 2, 1, 0, -1)
0
sage: gaunt(int(1), int(2), int(2), 1, 0, -1)
0
```

It is an error to use non-integer values for  $l$  or  $m$ :

```
sage: gaunt(1.2, 0, 1.2, 0, 0, 0)
Traceback (most recent call last):
...
TypeError: Attempt to coerce non-integral RealNumber to Integer
sage: gaunt(1, 0, 1, 1.1, 0, -1.1)
Traceback (most recent call last):
...
TypeError: Attempt to coerce non-integral RealNumber to Integer
```

NOTES:

The Gaunt coefficient obeys the following symmetry rules:

- invariant under any permutation of the columns

$$Y(j_1, j_2, j_3, m_1, m_2, m_3) = Y(j_3, j_1, j_2, m_3, m_1, m_2) = Y(j_2, j_3, j_1, m_2, m_3, m_1) = Y(j_3, j_2, j_1, m_3, m_2, m_1) = Y(j_1, j_3, j_2, m_1, m_3, m_2)$$

- invariant under space inflection, i.e.

$$Y(j_1, j_2, j_3, m_1, m_2, m_3) = Y(j_1, j_2, j_3, -m_1, -m_2, -m_3)$$

- symmetric with respect to the 72 Regge symmetries as inherited for the  $3j$  symbols [Regge58]
- zero for  $l_1, l_2, l_3$  not fulfilling triangle relation
- zero for violating any one of the conditions:  $l_1 \geq |m_1|, l_2 \geq |m_2|, l_3 \geq |m_3|$
- non-zero only for an even sum of the  $l_i$ , i.e.  $J = l_1 + l_2 + l_3 = 2n$  for  $n$  in  $\mathbb{N}$

#### ALGORITHM:

This function uses the algorithm of [Liberatodebrito82] to calculate the value of the Gaunt coefficient exactly. Note that the formula contains alternating sums over large factorials and is therefore unsuitable for finite precision arithmetic and only useful for a computer algebra system [Rasch03].

#### REFERENCES:

#### AUTHORS:

- Jens Rasch (2009-03-24): initial version for Sage

`sage.functions.wigner.racah(aa, bb, cc, dd, ee, ff, prec=None)`  
Calculate the Racah symbol  $W(a, b, c, d; e, f)$ .

#### INPUT:

- a, ..., f - integer or half integer
- prec - precision, default: None. Providing a precision can drastically speed up the calculation.

#### OUTPUT:

Rational number times the square root of a rational number (if `prec=None`), or real number if a precision is given.

#### EXAMPLES:

**sage:** `racah(3, 3, 3, 3, 3, 3)`  
-1/14

#### NOTES:

The Racah symbol is related to the Wigner  $6j$  symbol:

$$Wigner6j(j_1, j_2, j_3, j_4, j_5, j_6) = (-1)^{j_1+j_2+j_4+j_5} W(j_1, j_2, j_5, j_4, j_3, j_6)$$

Please see the  $6j$  symbol for its much richer symmetries and for additional properties.

#### ALGORITHM:

This function uses the algorithm of [Edmonds74] to calculate the value of the  $6j$  symbol exactly. Note that the formula contains alternating sums over large factorials and is therefore unsuitable for finite precision arithmetic and only useful for a computer algebra system [Rasch03].

#### AUTHORS:

- Jens Rasch (2009-03-24): initial version

`sage.functions.wigner.wigner_3j(j_1, j_2, j_3, m_1, m_2, m_3, prec=None)`  
Calculate the Wigner  $3j$  symbol  $Wigner3j(j_1, j_2, j_3, m_1, m_2, m_3)$ .

#### INPUT:

- j\_1, j\_2, j\_3, m\_1, m\_2, m\_3 - integer or half integer

- `prec` - precision, default: `None`. Providing a precision can drastically speed up the calculation.

OUTPUT:

Rational number times the square root of a rational number (if `prec=None`), or real number if a precision is given.

EXAMPLES:

```
sage: wigner_3j(2, 6, 4, 0, 0, 0)
sqrt(5/143)
sage: wigner_3j(2, 6, 4, 0, 0, 1)
0
sage: wigner_3j(0.5, 0.5, 1, 0.5, -0.5, 0)
sqrt(1/6)
sage: wigner_3j(40, 100, 60, -10, 60, -50)
95608/18702538494885*sqrt(21082735836735314343364163310/220491455010479533763)
sage: wigner_3j(2500, 2500, 5000, 2488, 2400, -4888, prec=64)
7.60424456883448589e-12
```

It is an error to have arguments that are not integer or half integer values:

```
sage: wigner_3j(2.1, 6, 4, 0, 0, 0)
Traceback (most recent call last):
...
ValueError: j values must be integer or half integer
sage: wigner_3j(2, 6, 4, 1, 0, -1.1)
Traceback (most recent call last):
...
ValueError: m values must be integer or half integer
```

NOTES:

The Wigner 3j symbol obeys the following symmetry rules:

- invariant under any permutation of the columns (with the exception of a sign change where  $J := j_1 + j_2 + j_3$ ):

$$Wigner3j(j_1, j_2, j_3, m_1, m_2, m_3) = Wigner3j(j_3, j_1, j_2, m_3, m_1, m_2) = Wigner3j(j_2, j_3, j_1, m_2, m_3, m_1) = (-1)^J Wigner3j(j_1, j_2, j_3, -m_1, -m_2, -m_3)$$

- invariant under space inflection, i.e.

$$Wigner3j(j_1, j_2, j_3, m_1, m_2, m_3) = (-1)^J Wigner3j(j_1, j_2, j_3, -m_1, -m_2, -m_3)$$

- symmetric with respect to the 72 additional symmetries based on the work by [Regge58]
- zero for  $j_1, j_2, j_3$  not fulfilling triangle relation
- zero for  $m_1 + m_2 + m_3 \neq 0$
- zero for violating any one of the conditions  $j_1 \geq |m_1|, j_2 \geq |m_2|, j_3 \geq |m_3|$

ALGORITHM:

This function uses the algorithm of [Edmonds74] to calculate the value of the 3j symbol exactly. Note that the formula contains alternating sums over large factorials and is therefore unsuitable for finite precision arithmetic and only useful for a computer algebra system [Rasch03].

REFERENCES:

AUTHORS:

- Jens Rasch (2009-03-24): initial version



`sage.functions.wigner.wigner_6j(j_1,j_2,j_3,j_4,j_5,j_6,prec=None)`  
 Calculate the Wigner 6j symbol  $Wigner6j(j_1, j_2, j_3, j_4, j_5, j_6)$ .

INPUT:

- `j_1, ..., j_6` - integer or half integer
- `prec` - precision, default: `None`. Providing a precision can drastically speed up the calculation.

OUTPUT:

Rational number times the square root of a rational number (if `prec=None`), or real number if a precision is given.

EXAMPLES:

```
sage: wigner_6j(3,3,3,3,3,3)
-1/14
sage: wigner_6j(5,5,5,5,5,5)
1/52
sage: wigner_6j(6,6,6,6,6,6)
309/10868
sage: wigner_6j(8,8,8,8,8,8)
-12219/965770
sage: wigner_6j(30,30,30,30,30,30)
36082186869033479581/87954851694828981714124
sage: wigner_6j(0.5,0.5,1,0.5,0.5,1)
1/6
sage: wigner_6j(200,200,200,200,200,200, prec=1000)*1.0
0.000155903212413242
```

It is an error to have arguments that are not integer or half integer values or do not fulfill the triangle relation:

```
sage: wigner_6j(2.5,2.5,2.5,2.5,2.5,2.5)
Traceback (most recent call last):
...
ValueError: j values must be integer or half integer and fulfill the triangle relation
sage: wigner_6j(0.5,0.5,1.1,0.5,0.5,1.1)
Traceback (most recent call last):
...
ValueError: j values must be integer or half integer and fulfill the triangle relation
```

NOTES:

The Wigner 6j symbol is related to the Racah symbol but exhibits more symmetries as detailed below.

$$Wigner6j(j_1, j_2, j_3, j_4, j_5, j_6) = (-1)^{j_1+j_2+j_4+j_5} W(j_1, j_2, j_5, j_4, j_3, j_6)$$

The Wigner 6j symbol obeys the following symmetry rules:

- Wigner 6j symbols are left invariant under any permutation of the columns:

$$Wigner6j(j_1, j_2, j_3, j_4, j_5, j_6) = Wigner6j(j_3, j_1, j_2, j_6, j_4, j_5) = Wigner6j(j_2, j_3, j_1, j_5, j_6, j_4) = Wigner6j(j_3, j_2, j_1, j_5, j_6, j_4)$$

- They are invariant under the exchange of the upper and lower arguments in each of any two columns, i.e.

$$Wigner6j(j_1, j_2, j_3, j_4, j_5, j_6) = Wigner6j(j_1, j_5, j_6, j_4, j_2, j_3) = Wigner6j(j_4, j_2, j_6, j_1, j_5, j_3) = Wigner6j(j_4, j_5, j_6, j_1, j_2, j_3)$$

- additional 6 symmetries [Regge59] giving rise to 144 symmetries in total
- only non-zero if any triple of  $j$ 's fulfill a triangle relation



This function uses the algorithm of [\[Edmonds74\]](#) to calculate the value of the  $3j$  symbol exactly. Note that the formula contains alternating sums over large factorials and is therefore unsuitable for finite precision arithmetic and only useful for a computer algebra system [\[Rasch03\]](#).



## GENERALIZED FUNCTIONS

Sage implements several generalized functions (also known as distributions) such as Dirac delta, Heaviside step functions. These generalized functions can be manipulated within Sage like any other symbolic functions.

AUTHORS:

- Golam Mortuza Hossain (2009-06-26): initial version

EXAMPLES:

Dirac delta function:

```
sage: dirac_delta(x)
dirac_delta(x)
```

Heaviside step function:

```
sage: heaviside(x)
heaviside(x)
```

Unit step function:

```
sage: unit_step(x)
unit_step(x)
```

Signum (sgn) function:

```
sage: sgn(x)
sgn(x)
```

Kronecker delta function:

```
sage: m,n=var('m,n')
sage: kronecker_delta(m,n)
kronecker_delta(m, n)
```

```
class sage.functions.generalized.FunctionDiracDelta
```

```
    Bases: sage.symbolic.function.BuiltinFunction
```

```
    The Dirac delta (generalized) function,  $\delta(x)$  (dirac_delta(x)).
```

```
    INPUT:
```

- `x` - a real number or a symbolic expression

```
    DEFINITION:
```

```
    Dirac delta function  $\delta(x)$ , is defined in Sage as:
```

$$\delta(x) = 0 \text{ for real } x \neq 0 \text{ and } \int_{-\infty}^{\infty} \delta(x) dx = 1$$

Its alternate definition with respect to an arbitrary test function  $f(x)$  is

$$\int_{-\infty}^{\infty} f(x) \delta(x - a) dx = f(a)$$

EXAMPLES:

```
sage: dirac_delta(1)
0
sage: dirac_delta(0)
dirac_delta(0)
sage: dirac_delta(x)
dirac_delta(x)
sage: integrate(dirac_delta(x), x, -1, 1, algorithm='sympy')
1
```

REFERENCES:

- [http://en.wikipedia.org/wiki/Dirac\\_delta\\_function](http://en.wikipedia.org/wiki/Dirac_delta_function)

**class** sage.functions.generalized.**FunctionHeaviside**

Bases: sage.symbolic.function.BuiltinFunction

The Heaviside step function,  $H(x)$  (`heaviside(x)`).

INPUT:

- $x$  - a real number or a symbolic expression

DEFINITION:

The Heaviside step function,  $H(x)$  is defined in Sage as:

$$H(x) = 0 \text{ for } x < 0 \text{ and } H(x) = 1 \text{ for } x > 0$$

EXAMPLES:

```
sage: heaviside(-1)
0
sage: heaviside(1)
1
sage: heaviside(0)
heaviside(0)
sage: heaviside(x)
heaviside(x)
```

TESTS:

```
sage: heaviside(x)._sympy_()
Heaviside(x)
```

REFERENCES:

- [Wikipedia article Heaviside\\_function](#)

**class** sage.functions.generalized.**FunctionKroneckerDelta**

Bases: sage.symbolic.function.BuiltinFunction

The Kronecker delta function  $\delta_{m,n}$  (`kronecker_delta(m, n)`).

INPUT:

- $m$  - a number or a symbolic expression
- $n$  - a number or a symbolic expression

## DEFINITION:

Kronecker delta function  $\delta_{m,n}$  is defined as:

$$\delta_{m,n} = 0 \text{ for } m \neq n \text{ and } \delta_{m,n} = 1 \text{ for } m = n$$

## EXAMPLES:

```
sage: kronecker_delta(1,2)
0
sage: kronecker_delta(1,1)
1
sage: m,n=var('m,n')
sage: kronecker_delta(m,n)
kronecker_delta(m, n)
```

## REFERENCES:

- [http://en.wikipedia.org/wiki/Kronecker\\_delta](http://en.wikipedia.org/wiki/Kronecker_delta)

**class** sage.functions.generalized.**FunctionSignum**  
 Bases: sage.symbolic.function.BuiltinFunction

The signum or sgn function  $\text{sgn}(x)$  ( $\text{sgn}(x)$ ).

## INPUT:

- $x$  - a real number or a symbolic expression

## DEFINITION:

The sgn function,  $\text{sgn}(x)$  is defined as:

$$\text{sgn}(x) = 1 \text{ for } x > 0, \text{sgn}(x) = 0 \text{ for } x = 0 \text{ and } \text{sgn}(x) = -1 \text{ for } x < 0$$

## EXAMPLES:

```
sage: sgn(-1)
-1
sage: sgn(1)
1
sage: sgn(0)
0
sage: sgn(x)
sgn(x)
```

We can also use `sign`:

```
sage: sign(1)
1
sage: sign(0)
0
sage: a = AA(-5).nth_root(7)
sage: sign(a)
-1
```

## TESTS:

Check if conversion to sympy works [trac ticket #11921](#):

```
sage: sgn(x).__sympy__()
sign(x)
```

## REFERENCES:

•[http://en.wikipedia.org/wiki/Sign\\_function](http://en.wikipedia.org/wiki/Sign_function)

**class** `sage.functions.generalized.FunctionUnitStep`

Bases: `sage.symbolic.function.BuiltinFunction`

The unit step function,  $u(x)$  (`unit_step(x)`).

INPUT:

• $x$  - a real number or a symbolic expression

DEFINITION:

The unit step function,  $u(x)$  is defined in Sage as:

$$u(x) = 0 \text{ for } x < 0 \text{ and } u(x) = 1 \text{ for } x \geq 0$$

EXAMPLES:

**sage:** `unit_step(-1)`

0

**sage:** `unit_step(1)`

1

**sage:** `unit_step(0)`

1

**sage:** `unit_step(x)`

`unit_step(x)`



## COUNTING PRIMES

### AUTHORS:

- R. Andrew Ohana (2009): initial version of efficient `prime_pi`
- William Stein (2009): fix plot method
- R. Andrew Ohana (2011): complete rewrite, ~5x speedup

### EXAMPLES:

```
sage: z = sage.functions.prime_pi.PrimePi()
sage: loads(dumps(z))
prime_pi
sage: loads(dumps(z)) == z
True
```

```
class sage.functions.prime_pi.PrimePi
Bases: sage.symbolic.function.BuiltinFunction
```

The prime counting function, which counts the number of primes less than or equal to a given value.

### INPUT:

- `x` - a real number
- `prime_bound` - (default 0) a real number  $< 2^{32}$ , `prime_pi` will make sure to use all the primes up to `prime_bound` (although, possibly more) in computing `prime_pi`, this can potentially speedup the time of computation, at a cost to memory usage.

### OUTPUT:

integer – the number of primes  $\leq x$

### EXAMPLES:

These examples test common inputs:

```
sage: prime_pi(7)
4
sage: prime_pi(100)
25
sage: prime_pi(1000)
168
sage: prime_pi(100000)
9592
sage: prime_pi(500509)
41581
```

These examples test a variety of odd inputs:

```
sage: prime_pi(3.5)
2
sage: prime_pi(sqrt(2357))
15
sage: prime_pi(mod(30957, 9750979))
3337
```

We test non-trivial `prime_bound` values:

```
sage: prime_pi(100000, 10000)
9592
sage: prime_pi(500509, 50051)
41581
```

The following test is to verify that ticket #4670 has been essentially resolved:

```
sage: prime_pi(10^10)
455052511
```

The `prime_pi` function also has a special plotting method, so it plots quickly and perfectly as a step function:

```
sage: P = plot(prime_pi, 50, 100)
```

NOTES:

Uses a recursive implementation, using the optimizations described in [\[RAO2011\]](#).

REFERENCES:

AUTHOR:

- R. Andrew Ohana (2011)

**plot** (*xmin=0, xmax=100, vertical\_lines=True, \*\*kws*)

Draw a plot of the prime counting function from *xmin* to *xmax*. All additional arguments are passed on to the line command.

WARNING: we draw the plot of `prime_pi` as a staircase function with explicitly drawn vertical lines where the function jumps. Technically there should not be any vertical lines, but they make the graph look much better, so we include them. Use the option `vertical_lines=False` to turn these off.

EXAMPLES:

```
sage: plot(prime_pi, 1, 100)
Graphics object consisting of 1 graphics primitive
sage: prime_pi.plot(-2, sqrt(2501), thickness=2, vertical_lines=False)
Graphics object consisting of 16 graphics primitives
```

`sage.functions.prime_pi.legendre_phi(x, a)`

Legendre's formula, also known as the partial sieve function, is a useful combinatorial function for computing the prime counting function (the `prime_pi` method in Sage). It counts the number of positive integers  $\leq x$  that are not divisible by the first *a* primes.

INPUT:

- x* – a real number
- a* – a non-negative integer

OUTPUT:

integer – the number of positive integers  $\leq x$  that are not divisible by the first *a* primes

EXAMPLES:

```

sage: legendre_phi(100, 0)
100
sage: legendre_phi(29375, 1)
14688
sage: legendre_phi(91753, 5973)
2893
sage: legendre_phi(7.5, 2)
3
sage: legendre_phi(str(-2^100), 92372)
0
sage: legendre_phi(4215701455, 6450023226)
1

```

## NOTES:

Uses a recursive implementation, using the optimizations described in [RAO2011].

## AUTHOR:

•R. Andrew Ohana (2011)

sage.functions.prime\_pi.**partial\_sieve\_function**( $x, a$ )

Legendre's formula, also known as the partial sieve function, is a useful combinatorial function for computing the prime counting function (the `prime_pi` method in Sage). It counts the number of positive integers  $\leq x$  that are not divisible by the first  $a$  primes.

## INPUT:

- $x$  – a real number
- $a$  – a non-negative integer

## OUTPUT:

integer – the number of positive integers  $\leq x$  that are not divisible by the first  $a$  primes

## EXAMPLES:

```

sage: legendre_phi(100, 0)
100
sage: legendre_phi(29375, 1)
14688
sage: legendre_phi(91753, 5973)
2893
sage: legendre_phi(7.5, 2)
3
sage: legendre_phi(str(-2^100), 92372)
0
sage: legendre_phi(4215701455, 6450023226)
1

```

## NOTES:

Uses a recursive implementation, using the optimizations described in [RAO2011].

## AUTHOR:

•R. Andrew Ohana (2011)



## SYMBOLIC MINIMUM AND MAXIMUM

Sage provides a symbolic maximum and minimum due to the fact that the Python builtin max and min are not able to deal with variables as users might expect. These functions wait to evaluate if there are variables.

Here you can see some differences:

```
sage: max(x, x^2)
x
sage: max_symbolic(x, x^2)
max(x, x^2)
sage: f(x) = max_symbolic(x, x^2); f(1/2)
1/2
```

This works as expected for more than two entries:

```
sage: max(3, 5, x)
5
sage: min(3, 5, x)
3
sage: max_symbolic(3, 5, x)
max(x, 5)
sage: min_symbolic(3, 5, x)
min(x, 3)
```

```
class sage.functions.min_max.MaxSymbolic
Bases: sage.functions.min_max.MinMax_base
```

Symbolic max function.

The Python builtin max function doesn't work as expected when symbolic expressions are given as arguments. This function delays evaluation until all symbolic arguments are substituted with values.

EXAMPLES:

```
sage: max_symbolic(3, x)
max(3, x)
sage: max_symbolic(3, x).subs(x=5)
5
sage: max_symbolic(3, 5, x)
max(x, 5)
sage: max_symbolic([3, 5, x])
max(x, 5)
```

TESTS:

```
sage: loads(dumps(max_symbolic(x, 5)))
max(x, 5)
sage: latex(max_symbolic(x, 5))
```

```
\max\left(x, 5\right)
sage: max_symbolic(x, 5)._sympy_()
Max(5, x)
```

**class** sage.functions.min\_max.MinMax\_base  
Bases: sage.symbolic.function.BuiltinFunction

**eval\_helper** (*this\_f, builtin\_f, initial\_val, args*)  
EXAMPLES:  
**sage:** max\_symbolic(3,5,x) *# indirect doctest*  
max(x, 5)  
**sage:** min\_symbolic(3,5,x)  
min(x, 3)

**class** sage.functions.min\_max.MinSymbolic  
Bases: sage.functions.min\_max.MinMax\_base

Symbolic min function.

The Python builtin min function doesn't work as expected when symbolic expressions are given as arguments. This function delays evaluation until all symbolic arguments are substituted with values.

EXAMPLES:  
**sage:** min\_symbolic(3, x)  
min(3, x)  
**sage:** min\_symbolic(3, x).subs(x=5)  
3  
**sage:** min\_symbolic(3, 5, x)  
min(x, 3)  
**sage:** min\_symbolic([3,5,x])  
min(x, 3)

TESTS:  
**sage:** loads(dumps(min\_symbolic(x,5)))  
min(x, 5)  
**sage:** latex(min\_symbolic(x,5))  
\min\left(x, 5\right)  
**sage:** min\_symbolic(x, 5).\_sympy\_()  
Min(5, x)

## INDICES AND TABLES

- Index
- Module Index
- Search Page





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**f**

`sage.functions.airy`, 109  
`sage.functions.bessel`, 115  
`sage.functions.exp_integral`, 125  
`sage.functions.generalized`, 145  
`sage.functions.hyperbolic`, 19  
`sage.functions.hypergeometric`, 95  
`sage.functions.jacobi`, 103  
`sage.functions.log`, 1  
`sage.functions.min_max`, 153  
`sage.functions.orthogonal_polys`, 53  
`sage.functions.other`, 65  
`sage.functions.piecewise`, 33  
`sage.functions.prime_pi`, 149  
`sage.functions.special`, 87  
`sage.functions.spike_function`, 51  
`sage.functions.transcendental`, 27  
`sage.functions.trig`, 9  
`sage.functions.wigner`, 137



## A

[airy\\_ai\(\)](#) (in module `sage.functions.airy`), [111](#)  
[airy\\_bi\(\)](#) (in module `sage.functions.airy`), [112](#)  
[approximate\(\)](#) (`sage.functions.transcendental.DickmanRho` method), [27](#)

## B

[base\\_ring\(\)](#) (`sage.functions.piecewise.PiecewisePolynomial` method), [34](#)  
[Bessel\(\)](#) (in module `sage.functions.bessel`), [117](#)

## C

[ChebyshevPolynomial](#) (class in `sage.functions.orthogonal_polys`), [56](#)  
[clebsch\\_gordan\(\)](#) (in module `sage.functions.wigner`), [137](#)  
[closed\\_form\(\)](#) (in module `sage.functions.hypergeometric`), [100](#)  
[convolution\(\)](#) (`sage.functions.piecewise.PiecewisePolynomial` method), [34](#)  
[cosine\\_series\\_coefficient\(\)](#) (`sage.functions.piecewise.PiecewisePolynomial` method), [35](#)  
[critical\\_points\(\)](#) (`sage.functions.piecewise.PiecewisePolynomial` method), [35](#)

## D

[default\\_variable\(\)](#) (`sage.functions.piecewise.PiecewisePolynomial` method), [36](#)  
[deflated\(\)](#) (`sage.functions.hypergeometric.Hypergeometric.EvaluationMethods` method), [97](#)  
[derivative\(\)](#) (`sage.functions.piecewise.PiecewisePolynomial` method), [36](#)  
[DickmanRho](#) (class in `sage.functions.transcendental`), [27](#)  
[domain\(\)](#) (`sage.functions.piecewise.PiecewisePolynomial` method), [37](#)

## E

[eliminate\\_parameters\(\)](#) (`sage.functions.hypergeometric.Hypergeometric.EvaluationMethods` method), [98](#)  
[elliptic\\_j\(\)](#) (in module `sage.functions.special`), [91](#)  
[EllipticE](#) (class in `sage.functions.special`), [89](#)  
[EllipticEC](#) (class in `sage.functions.special`), [89](#)  
[EllipticEU](#) (class in `sage.functions.special`), [89](#)  
[EllipticF](#) (class in `sage.functions.special`), [89](#)  
[EllipticKC](#) (class in `sage.functions.special`), [90](#)  
[EllipticPi](#) (class in `sage.functions.special`), [90](#)  
[end\\_points\(\)](#) (`sage.functions.piecewise.PiecewisePolynomial` method), [37](#)  
[error\\_fcn\(\)](#) (in module `sage.functions.special`), [92](#)  
[eval\\_algebraic\(\)](#) (`sage.functions.orthogonal_polys.Func_chebyshev_T` method), [56](#)  
[eval\\_algebraic\(\)](#) (`sage.functions.orthogonal_polys.Func_chebyshev_U` method), [57](#)

`eval_formula()` (sage.functions.orthogonal\_polys.Func\_chebyshev\_T method), 57  
`eval_formula()` (sage.functions.orthogonal\_polys.Func\_chebyshev\_U method), 58  
`eval_formula()` (sage.functions.orthogonal\_polys.OrthogonalPolynomial method), 58  
`eval_helper()` (sage.functions.min\_max.MinMax\_base method), 154  
`exponential_integral_1()` (in module sage.functions.exp\_integral), 135  
`extend_by_zero_to()` (sage.functions.piecewise.PiecewisePolynomial method), 37

## F

`fourier_series_cosine_coefficient()` (sage.functions.piecewise.PiecewisePolynomial method), 37  
`fourier_series_partial_sum()` (sage.functions.piecewise.PiecewisePolynomial method), 38  
`fourier_series_partial_sum_cesaro()` (sage.functions.piecewise.PiecewisePolynomial method), 38  
`fourier_series_partial_sum_filtered()` (sage.functions.piecewise.PiecewisePolynomial method), 38  
`fourier_series_partial_sum_hann()` (sage.functions.piecewise.PiecewisePolynomial method), 39  
`fourier_series_sine_coefficient()` (sage.functions.piecewise.PiecewisePolynomial method), 39  
`fourier_series_value()` (sage.functions.piecewise.PiecewisePolynomial method), 39  
`Func_chebyshev_T` (class in sage.functions.orthogonal\_polys), 56  
`Func_chebyshev_U` (class in sage.functions.orthogonal\_polys), 57  
`Function_abs` (class in sage.functions.other), 65  
`Function_arccos` (class in sage.functions.trig), 9  
`Function_arccosh` (class in sage.functions.hyperbolic), 19  
`Function_arccot` (class in sage.functions.trig), 10  
`Function_arccoth` (class in sage.functions.hyperbolic), 20  
`Function_arccsc` (class in sage.functions.trig), 10  
`Function_arcsch` (class in sage.functions.hyperbolic), 20  
`Function_arcsec` (class in sage.functions.trig), 10  
`Function_arcsech` (class in sage.functions.hyperbolic), 21  
`Function_arcsin` (class in sage.functions.trig), 11  
`Function_arcsinh` (class in sage.functions.hyperbolic), 21  
`Function_arctan` (class in sage.functions.trig), 11  
`Function_arctan2` (class in sage.functions.trig), 12  
`Function_arctanh` (class in sage.functions.hyperbolic), 22  
`Function_arg` (class in sage.functions.other), 65  
`Function_Bessel_I` (class in sage.functions.bessel), 119  
`Function_Bessel_J` (class in sage.functions.bessel), 120  
`Function_Bessel_K` (class in sage.functions.bessel), 122  
`Function_Bessel_Y` (class in sage.functions.bessel), 123  
`Function_beta` (class in sage.functions.other), 66  
`Function_binomial` (class in sage.functions.other), 67  
`Function_ceil` (class in sage.functions.other), 69  
`Function_conjugate` (class in sage.functions.other), 70  
`Function_cos` (class in sage.functions.trig), 14  
`Function_cos_integral` (class in sage.functions.exp\_integral), 125  
`Function_cosh` (class in sage.functions.hyperbolic), 23  
`Function_cosh_integral` (class in sage.functions.exp\_integral), 126  
`Function_cot` (class in sage.functions.trig), 15  
`Function_coth` (class in sage.functions.hyperbolic), 23  
`Function_csc` (class in sage.functions.trig), 15  
`Function_csch` (class in sage.functions.hyperbolic), 23  
`Function_dilog` (class in sage.functions.log), 1  
`Function_erf` (class in sage.functions.other), 71

Function\_exp (class in sage.functions.log), 2  
 Function\_exp\_integral (class in sage.functions.exp\_integral), 128  
 Function\_exp\_integral\_e (class in sage.functions.exp\_integral), 128  
 Function\_exp\_integral\_e1 (class in sage.functions.exp\_integral), 130  
 Function\_factorial (class in sage.functions.other), 73  
 Function\_floor (class in sage.functions.other), 74  
 Function\_gamma (class in sage.functions.other), 75  
 Function\_gamma\_inc (class in sage.functions.other), 77  
 Function\_HurwitzZeta (class in sage.functions.transcendental), 28  
 Function\_imag\_part (class in sage.functions.other), 77  
 Function\_lambert\_w (class in sage.functions.log), 3  
 Function\_log (class in sage.functions.log), 4  
 Function\_log\_gamma (class in sage.functions.other), 78  
 Function\_log\_integral (class in sage.functions.exp\_integral), 130  
 Function\_log\_integral\_offset (class in sage.functions.exp\_integral), 131  
 Function\_polylog (class in sage.functions.log), 6  
 Function\_psi1 (class in sage.functions.other), 79  
 Function\_psi2 (class in sage.functions.other), 80  
 Function\_real\_part (class in sage.functions.other), 81  
 Function\_sec (class in sage.functions.trig), 16  
 Function\_sech (class in sage.functions.hyperbolic), 24  
 Function\_sin (class in sage.functions.trig), 16  
 Function\_sin\_integral (class in sage.functions.exp\_integral), 133  
 Function\_sinh (class in sage.functions.hyperbolic), 24  
 Function\_sinh\_integral (class in sage.functions.exp\_integral), 134  
 Function\_sqrt (class in sage.functions.other), 81  
 Function\_tan (class in sage.functions.trig), 17  
 Function\_tanh (class in sage.functions.hyperbolic), 25  
 Function\_zeta (class in sage.functions.transcendental), 28  
 Function\_zetaderiv (class in sage.functions.transcendental), 29  
 FunctionAiryAiGeneral (class in sage.functions.airy), 109  
 FunctionAiryAiPrime (class in sage.functions.airy), 110  
 FunctionAiryAiSimple (class in sage.functions.airy), 110  
 FunctionAiryBiGeneral (class in sage.functions.airy), 110  
 FunctionAiryBiPrime (class in sage.functions.airy), 111  
 FunctionAiryBiSimple (class in sage.functions.airy), 111  
 FunctionDiracDelta (class in sage.functions.generalized), 145  
 FunctionHeaviside (class in sage.functions.generalized), 146  
 FunctionKroneckerDelta (class in sage.functions.generalized), 146  
 functions() (sage.functions.piecewise.PiecewisePolynomial method), 40  
 FunctionSignum (class in sage.functions.generalized), 147  
 FunctionUnitStep (class in sage.functions.generalized), 148

## G

gamma() (in module sage.functions.other), 82  
 gaunt() (in module sage.functions.wigner), 137  
 gegenbauer() (in module sage.functions.orthogonal\_polys), 59  
 gen\_laguerre() (in module sage.functions.orthogonal\_polys), 59  
 gen\_legendre\_P() (in module sage.functions.orthogonal\_polys), 60  
 gen\_legendre\_Q() (in module sage.functions.orthogonal\_polys), 60

## H

`hermite()` (in module `sage.functions.orthogonal_polys`), 60  
`hurwitz_zeta()` (in module `sage.functions.transcendental`), 30  
`HyperbolicFunction` (class in `sage.functions.hyperbolic`), 25  
`Hypergeometric` (class in `sage.functions.hypergeometric`), 97  
`Hypergeometric.EvaluationMethods` (class in `sage.functions.hypergeometric`), 97  
`hypergeometric_U()` (in module `sage.functions.special`), 92

## I

`integral()` (`sage.functions.piecewise.PiecewisePolynomial` method), 40  
`intervals()` (`sage.functions.piecewise.PiecewisePolynomial` method), 42  
`inverse_jacobi()` (in module `sage.functions.jacobi`), 104  
`inverse_jacobi_f()` (in module `sage.functions.jacobi`), 105  
`InverseJacobi` (class in `sage.functions.jacobi`), 104  
`is_absolutely_convergent()` (`sage.functions.hypergeometric.Hypergeometric.EvaluationMethods` method), 98  
`is_terminating()` (`sage.functions.hypergeometric.Hypergeometric.EvaluationMethods` method), 99  
`is_termwise_finite()` (`sage.functions.hypergeometric.Hypergeometric.EvaluationMethods` method), 99

## J

`Jacobi` (class in `sage.functions.jacobi`), 104  
`jacobi()` (in module `sage.functions.jacobi`), 107  
`jacobi_am_f()` (in module `sage.functions.jacobi`), 108  
`jacobi_P()` (in module `sage.functions.orthogonal_polys`), 61  
`JacobiAmplitude` (class in `sage.functions.jacobi`), 104

## L

`laguerre()` (in module `sage.functions.orthogonal_polys`), 61  
`laplace()` (`sage.functions.piecewise.PiecewisePolynomial` method), 42  
`legendre_P()` (in module `sage.functions.orthogonal_polys`), 62  
`legendre_phi()` (in module `sage.functions.prime_pi`), 150  
`legendre_Q()` (in module `sage.functions.orthogonal_polys`), 62  
`length()` (`sage.functions.piecewise.PiecewisePolynomial` method), 42  
`list()` (`sage.functions.piecewise.PiecewisePolynomial` method), 43

## M

`maxima_function()` (in module `sage.functions.special`), 92  
`MaximaFunction` (class in `sage.functions.special`), 91  
`MaxSymbolic` (class in `sage.functions.min_max`), 153  
`meval()` (in module `sage.functions.special`), 93  
`MinMax_base` (class in `sage.functions.min_max`), 154  
`MinSymbolic` (class in `sage.functions.min_max`), 154

## O

`OrthogonalPolynomial` (class in `sage.functions.orthogonal_polys`), 58

## P

`partial_sieve_function()` (in module `sage.functions.prime_pi`), 151  
`Piecewise()` (in module `sage.functions.piecewise`), 33  
`piecewise()` (in module `sage.functions.piecewise`), 48



[PiecewisePolynomial](#) (class in `sage.functions.piecewise`), 34  
[plot\(\)](#) (`sage.functions.piecewise.PiecewisePolynomial` method), 43  
[plot\(\)](#) (`sage.functions.prime_pi.PrimePi` method), 150  
[plot\(\)](#) (`sage.functions.spike_function.SpikeFunction` method), 51  
[plot\\_fft\\_abs\(\)](#) (`sage.functions.spike_function.SpikeFunction` method), 51  
[plot\\_fft\\_arg\(\)](#) (`sage.functions.spike_function.SpikeFunction` method), 52  
[plot\\_fourier\\_series\\_partial\\_sum\(\)](#) (`sage.functions.piecewise.PiecewisePolynomial` method), 43  
[plot\\_fourier\\_series\\_partial\\_sum\\_cesaro\(\)](#) (`sage.functions.piecewise.PiecewisePolynomial` method), 44  
[plot\\_fourier\\_series\\_partial\\_sum\\_filtered\(\)](#) (`sage.functions.piecewise.PiecewisePolynomial` method), 44  
[plot\\_fourier\\_series\\_partial\\_sum\\_hann\(\)](#) (`sage.functions.piecewise.PiecewisePolynomial` method), 45  
[power\\_series\(\)](#) (`sage.functions.transcendental.DickmanRho` method), 28  
[PrimePi](#) (class in `sage.functions.prime_pi`), 149  
[psi\(\)](#) (in module `sage.functions.other`), 83

## R

[racah\(\)](#) (in module `sage.functions.wigner`), 139  
[rational\\_param\\_as\\_tuple\(\)](#) (in module `sage.functions.hypergeometric`), 101  
[riemann\\_sum\(\)](#) (`sage.functions.piecewise.PiecewisePolynomial` method), 45  
[riemann\\_sum\\_integral\\_approximation\(\)](#) (`sage.functions.piecewise.PiecewisePolynomial` method), 46

## S

[sage.functions.airy](#) (module), 109  
[sage.functions.bessel](#) (module), 115  
[sage.functions.exp\\_integral](#) (module), 125  
[sage.functions.generalized](#) (module), 145  
[sage.functions.hyperbolic](#) (module), 19  
[sage.functions.hypergeometric](#) (module), 95  
[sage.functions.jacobi](#) (module), 103  
[sage.functions.log](#) (module), 1  
[sage.functions.min\\_max](#) (module), 153  
[sage.functions.orthogonal\\_polys](#) (module), 53  
[sage.functions.other](#) (module), 65  
[sage.functions.piecewise](#) (module), 33  
[sage.functions.prime\\_pi](#) (module), 149  
[sage.functions.special](#) (module), 87  
[sage.functions.spike\\_function](#) (module), 51  
[sage.functions.transcendental](#) (module), 27  
[sage.functions.trig](#) (module), 9  
[sage.functions.wigner](#) (module), 137  
[sine\\_series\\_coefficient\(\)](#) (`sage.functions.piecewise.PiecewisePolynomial` method), 46  
[sorted\\_parameters\(\)](#) (`sage.functions.hypergeometric.Hypergeometric.EvaluationMethods` method), 100  
[spherical\\_bessel\\_J\(\)](#) (in module `sage.functions.special`), 93  
[spherical\\_bessel\\_Y\(\)](#) (in module `sage.functions.special`), 93  
[spherical\\_hankel1\(\)](#) (in module `sage.functions.special`), 93  
[spherical\\_hankel2\(\)](#) (in module `sage.functions.special`), 93  
[SphericalHarmonic](#) (class in `sage.functions.special`), 91  
[spike\\_function](#) (in module `sage.functions.spike_function`), 52  
[SpikeFunction](#) (class in `sage.functions.spike_function`), 51  
[sqrt\(\)](#) (in module `sage.functions.other`), 84

## T

`tangent_line()` (`sage.functions.piecewise.PiecewisePolynomial` method), [46](#)  
`terms()` (`sage.functions.hypergeometric.Hypergeometric.EvaluationMethods` method), [100](#)  
`trapezoid()` (`sage.functions.piecewise.PiecewisePolynomial` method), [47](#)  
`trapezoid_integral_approximation()` (`sage.functions.piecewise.PiecewisePolynomial` method), [47](#)

## U

`ultraspherical()` (in module `sage.functions.orthogonal_polys`), [62](#)  
`unextend()` (`sage.functions.piecewise.PiecewisePolynomial` method), [48](#)

## V

`vector()` (`sage.functions.spike_function.SpikeFunction` method), [52](#)

## W

`which_function()` (`sage.functions.piecewise.PiecewisePolynomial` method), [48](#)  
`wigner_3j()` (in module `sage.functions.wigner`), [139](#)  
`wigner_6j()` (in module `sage.functions.wigner`), [140](#)  
`wigner_9j()` (in module `sage.functions.wigner`), [142](#)

## Z

`zeta_symmetric()` (in module `sage.functions.transcendental`), [30](#)