Sage Reference Manual: Modules

Release 6.6

The Sage Development Team

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CHAPTER

ONE

ABSTRACT BASE CLASS FOR MODULES

Two classes for modules are defined here: `Module_old` and `Module`. The former is a legacy class which should not be used for new code anymore as it does not conform to the coercion model. Eventually all occurences shall be ported to `Module`.

AUTHORS:

- William Stein: initial version
- Julian Rueth (2014-05-10): category parameter for Module, doc cleanup

EXAMPLES:

A minimal example of a module:

```
sage: class MyElement(sage.structure.element.ModuleElement):
....:     def __init__(self, parent, x):
....:         self.x = x
....:         sage.structure.element.ModuleElement.__init__(self, parent=parent)
....:     def _rmul_(self, c):
....:         return self.parent()(c*self.x)
....:     def _add_(self, other):
....:         return self.parent()(self.x + other.x)
....:     def __cmp__(self, other):
....:         return cmp(self.x, other.x)
....:     def __hash__(self):
....:         return hash(self.x)
....:     def _repr_(self):
....:         return repr(self.x)

sage: class MyModule(sage.modules.module.Module):
....:     Element = MyElement
....:     def _element_constructor_(self, x):
....:         if isinstance(x, MyElement): x = x.x
....:         return self.element_class(self, self.base_ring()(x))
....:     def __cmp__(self, other):
....:         if not isinstance(other, MyModule): return cmp(type(other),MyModule)
....:         return cmp(self.base_ring(),other.base_ring())

sage: M = MyModule(QQ)
sage: M(1)
1

sage: import __main__
sage: __main__.MyModule = MyModule
sage: __main__.MyElement = MyElement
sage: TestSuite(M).run()
```

class sage.modules.module.**Module**

Bases: sage.structure.parent.Parent

Generic module class.

INPUT:

- base – a ring. The base ring of the module.
- category – a category (default: None), the category for this module. If None, then this is set to the category of modules/vector spaces over base.

EXAMPLES:

```
sage: from sage.modules.module import Module
sage: M = Module(ZZ)
sage: M.base_ring()
Integer Ring
sage: M.category()
Category of modules over Integer Ring
```

Normally the category is set to the category of modules over base. If base is a field, then the category is the category of vector spaces over base:

```
sage: M_QQ = Module(QQ)
sage: M_QQ.category()
Category of vector spaces over Rational Field
```

The category parameter can be used to set a more specific category:

```
   sage: N = Module(ZZ, category=FiniteDimensionalModulesWithBasis(ZZ))
   sage: N.category()
   Category of finite dimensional modules with basis over Integer Ring

TESTS:

We check that :trac:`8119` has been resolved::

   sage: M = ZZ^3
   sage: h = M.__hash__()
   sage: M.rename('toto')
   sage: h == M.__hash__()
   True
```

endomorphism_ring()

Return the endomorphism ring of this module in its category.

EXAMPLES:

```
sage: from sage.modules.module import Module
sage: M = Module(ZZ)
sage: M.endomorphism_ring()
Set of Morphisms from <type 'sage.modules.module.Module'> to <type 'sage.modules.module.Modu
```

class sage.modules.module.**Module_old**

Bases: sage.structure.parent_gens.ParentWithAdditiveAbelianGens

Generic module class.

category()

Return the category to which this module belongs.

endomorphism_ring()

Return the endomorphism ring of this module in its category.

sage.modules.module.**is_Module**(*x*)

Return True if x is a module, False otherwise.

INPUT:

- x – anything.

EXAMPLES:

```
sage: from sage.modules.module import is_Module
sage: M = FreeModule(RationalField(),30)
sage: is_Module(M)
True
sage: is_Module(10)
False
```

sage.modules.module.**is_VectorSpace**(*x*)

Return True if x is a vector space, False otherwise.

INPUT:

- x – anything.

EXAMPLES:

```
sage: from sage.modules.module import is_Module, is_VectorSpace
sage: M = FreeModule(RationalField(),30)
sage: is_VectorSpace(M)
True
sage: M = FreeModule(IntegerRing(),30)
sage: is_Module(M)
True
sage: is_VectorSpace(M)
False
```

CHAPTER

TWO

FREE MODULES

Sage supports computation with free modules over an arbitrary commutative ring. Nontrivial functionality is available over $\mathbf{Z}$, fields, and some principal ideal domains (e.g. $\mathbf{Q}[x]$ and rings of integers of number fields). All free modules over an integral domain are equipped with an embedding in an ambient vector space and an inner product, which you can specify and change.

Create the free module of rank n over an arbitrary commutative ring R using the command `FreeModule(R,n)`. Equivalently, `R^n` also creates that free module.

The following example illustrates the creation of both a vector space and a free module over the integers and a submodule of it. Use the functions `FreeModule`, `span` and member functions of free modules to create free modules. *Do not use the FreeModule_xxx constructors directly.*

EXAMPLES:

```
sage: V = VectorSpace(QQ,3)
sage: W = V.subspace([[1,2,7], [1,1,0]])
sage: W
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1  0 -7]
[ 0  1  7]
sage: C = VectorSpaces(FiniteField(7))
sage: C
Category of vector spaces over Finite Field of size 7
sage: C(W)
Vector space of degree 3 and dimension 2 over Finite Field of size 7
Basis matrix:
[1 0 0]
[0 1 0]
```

```
sage: M = ZZ^3
sage: C = VectorSpaces(FiniteField(7))
sage: C(M)
Vector space of dimension 3 over Finite Field of size 7
sage: W = M.submodule([[1,2,7], [8,8,0]])
sage: C(W)
Vector space of degree 3 and dimension 2 over Finite Field of size 7
Basis matrix:
[1 0 0]
[0 1 0]
```

We illustrate the exponent notation for creation of free modules.

```
sage: ZZ^4
Ambient free module of rank 4 over the principal ideal domain Integer Ring
```

```
sage: QQ^2
Vector space of dimension 2 over Rational Field
sage: RR^3
Vector space of dimension 3 over Real Field with 53 bits of precision
```

Base ring:

```
sage: R.<x,y> = QQ[]
sage: M = FreeModule(R,2)
sage: M.base_ring()
Multivariate Polynomial Ring in x, y over Rational Field
```

```
sage: VectorSpace(QQ, 10).base_ring()
Rational Field
```

TESTS: We intersect a zero-dimensional vector space with a 1-dimension submodule.

```
sage: V = (QQ^1).span([])
sage: W = ZZ^1
sage: V.intersection(W)
Free module of degree 1 and rank 0 over Integer Ring
Echelon basis matrix:
[]
```

We construct subspaces of real and complex double vector spaces and verify that the element types are correct:

```
sage: V = FreeModule(RDF, 3); V
Vector space of dimension 3 over Real Double Field
sage: V.0
(1.0, 0.0, 0.0)
sage: type(V.0)
<type 'sage.modules.vector_real_double_dense.Vector_real_double_dense'>
sage: W = V.span([V.0]); W
Vector space of degree 3 and dimension 1 over Real Double Field
Basis matrix:
[1.0 0.0 0.0]
sage: type(W.0)
<type 'sage.modules.vector_real_double_dense.Vector_real_double_dense'>
sage: V = FreeModule(CDF, 3); V
Vector space of dimension 3 over Complex Double Field
sage: type(V.0)
<type 'sage.modules.vector_complex_double_dense.Vector_complex_double_dense'>
sage: W = V.span_of_basis([CDF.0 * V.1]); W
Vector space of degree 3 and dimension 1 over Complex Double Field
User basis matrix:
[   0.0 1.0*I    0.0]
sage: type(W.0)
<type 'sage.modules.vector_complex_double_dense.Vector_complex_double_dense'>
```

Basis vectors are immutable:

```
sage: A = span([[1,2,3], [4,5,6]], ZZ)
sage: A.0
(1, 2, 3)
sage: A.0[0] = 5
Traceback (most recent call last):
...
ValueError: vector is immutable; please change a copy instead (use copy())
```

Among other things, this tests that we can save and load submodules and elements:

```
sage: M = ZZ^3
sage: TestSuite(M).run()
sage: W = M.span_of_basis([[1,2,3],[4,5,19]])
sage: TestSuite(W).run()
sage: v = W.0 + W.1
sage: TestSuite(v).run()
```

AUTHORS:

- William Stein (2005, 2007)
- David Kohel (2007, 2008)
- Niles Johnson (2010-08): Trac #3893: `random_element()` should pass on `*args` and `**kwds`.
- Simon King (2010-12): Trac #8800: Fixing a bug in `denominator()`.

class `sage.modules.free_module.`**`ComplexDoubleVectorSpace_class`**(*n*)
Bases: `sage.modules.free_module.FreeModule_ambient_field`

`coordinates`(*v*)
x.__init__(...) initializes x; see help(type(x)) for signature

class `sage.modules.free_module.`**`FreeModuleFactory`**
Bases: `sage.structure.factory.UniqueFactory`

Create the free module over the given commutative ring of the given rank.

INPUT:

- `base_ring` - a commutative ring
- `rank` - a nonnegative integer
- `sparse` - bool; (default False)
- `inner_product_matrix` - the inner product matrix (default None)

OUTPUT: a free module

Note: In Sage it is the case that there is only one dense and one sparse free ambient module of rank n over R.

EXAMPLES:

First we illustrate creating free modules over various base fields. The base field affects the free module that is created. For example, free modules over a field are vector spaces, and free modules over a principal ideal domain are special in that more functionality is available for them than for completely general free modules.

```
sage: FreeModule(Integers(8),10)
Ambient free module of rank 10 over Ring of integers modulo 8
sage: FreeModule(QQ,10)
Vector space of dimension 10 over Rational Field
sage: FreeModule(ZZ,10)
Ambient free module of rank 10 over the principal ideal domain Integer Ring
sage: FreeModule(FiniteField(5),10)
Vector space of dimension 10 over Finite Field of size 5
sage: FreeModule(Integers(7),10)
Vector space of dimension 10 over Ring of integers modulo 7
sage: FreeModule(PolynomialRing(QQ,'x'),5)
Ambient free module of rank 5 over the principal ideal domain Univariate Polynomial Ring in x ov
sage: FreeModule(PolynomialRing(ZZ,'x'),5)
Ambient free module of rank 5 over the integral domain Univariate Polynomial Ring in x over Inte
```

Of course we can make rank 0 free modules:

```
sage: FreeModule(RealField(100),0)
Vector space of dimension 0 over Real Field with 100 bits of precision
```

Next we create a free module with sparse representation of elements. Functionality with sparse modules is *identical* to dense modules, but they may use less memory and arithmetic may be faster (or slower!).

```
sage: M = FreeModule(ZZ,200,sparse=True)
sage: M.is_sparse()
True
sage: type(M.0)
<type 'sage.modules.free_module_element.FreeModuleElement_generic_sparse'>
```

The default is dense.

```
sage: M = ZZ^200
sage: type(M.0)
<type 'sage.modules.vector_integer_dense.Vector_integer_dense'>
```

Note that matrices associated in some way to sparse free modules are sparse by default:

```
sage: M = FreeModule(Integers(8), 2)
sage: A = M.basis_matrix()
sage: A.is_sparse()
False
sage: Ms = FreeModule(Integers(8), 2, sparse=True)
sage: M == Ms  # as mathematical objects they are equal
True
sage: Ms.basis_matrix().is_sparse()
True
```

We can also specify an inner product matrix, which is used when computing inner products of elements.

```
sage: A = MatrixSpace(ZZ,2)([[1,0],[0,-1]])
sage: M = FreeModule(ZZ,2,inner_product_matrix=A)
sage: v, w = M.gens()
sage: v.inner_product(w)
0
sage: v.inner_product(v)
1
sage: w.inner_product(w)
-1
sage: (v+2*w).inner_product(w)
-2
```

You can also specify the inner product matrix by giving anything that coerces to an appropriate matrix. This is only useful if the inner product matrix takes values in the base ring.

```
sage: FreeModule(ZZ,2,inner_product_matrix=1).inner_product_matrix()
[1 0]
[0 1]
sage: FreeModule(ZZ,2,inner_product_matrix=[1,2,3,4]).inner_product_matrix()
[1 2]
[3 4]
sage: FreeModule(ZZ,2,inner_product_matrix=[[1,2],[3,4]]).inner_product_matrix()
[1 2]
[3 4]
```

Todo

Refactor modules such that it only counts what category the base ring belongs to, but not what is its Python class.

---

create_key(*base_ring*, *rank*, *sparse=False*, *inner_product_matrix=None*)

TESTS:

```
sage: loads(dumps(ZZ^6)) is ZZ^6
True
sage: loads(dumps(RDF^3)) is RDF^3
True
```

TODO: replace the above by `TestSuite(...).run()`, once `_test_pickling()` will test unique representation and not only equality.

create_object(*version*, *key*)

x.__init__(...) initializes x; see help(type(x)) for signature

class sage.modules.free_module.**FreeModule_ambient**(*base_ring*, *rank*, *sparse=False*, *coordinate_ring=None*)

Bases: `sage.modules.free_module.FreeModule_generic`

Ambient free module over a commutative ring.

ambient_module()

Return self, since self is ambient.

EXAMPLES:

```
sage: A = QQ^5; A.ambient_module()
Vector space of dimension 5 over Rational Field
sage: A = ZZ^5; A.ambient_module()
Ambient free module of rank 5 over the principal ideal domain Integer Ring
```

basis()

Return a basis for this ambient free module.

OUTPUT:

- `Sequence` - an immutable sequence with universe this ambient free module

EXAMPLES:

```
sage: A = ZZ^3; B = A.basis(); B
[
(1, 0, 0),
(0, 1, 0),
(0, 0, 1)
]
sage: B.universe()
Ambient free module of rank 3 over the principal ideal domain Integer Ring
```

change_ring(*R*)

Return the ambient free module over R of the same rank as self.

EXAMPLES:

```
sage: A = ZZ^3; A.change_ring(QQ)
Vector space of dimension 3 over Rational Field
sage: A = ZZ^3; A.change_ring(GF(5))
Vector space of dimension 3 over Finite Field of size 5
```

For ambient modules any change of rings is defined.

```
sage: A = GF(5)**3; A.change_ring(QQ)
Vector space of dimension 3 over Rational Field
```

coordinate_vector (*v*, *check=True*)

Write v in terms of the standard basis for self and return the resulting coefficients in a vector over the fraction field of the base ring.

Returns a vector c such that if B is the basis for self, then

$$\sum c_i B_i = v.$$

If v is not in self, raises an ArithmeticError exception.

EXAMPLES:

```
sage: V = Integers(16)^3
sage: v = V.coordinate_vector([1,5,9]); v
(1, 5, 9)
sage: v.parent()
Ambient free module of rank 3 over Ring of integers modulo 16
```

echelon_coordinate_vector (*v*, *check=True*)

Same as `self.coordinate_vector(v)`, since self is an ambient free module.

INPUT:

- `v` - vector
- `check` - bool (default: True); if True, also verify that v is really in self.

OUTPUT: list

EXAMPLES:

```
sage: V = QQ^4
sage: v = V([-1/2,1/2,-1/2,1/2])
sage: v
(-1/2, 1/2, -1/2, 1/2)
sage: V.coordinate_vector(v)
(-1/2, 1/2, -1/2, 1/2)
sage: V.echelon_coordinate_vector(v)
(-1/2, 1/2, -1/2, 1/2)
sage: W = V.submodule_with_basis([[1/2,1/2,1/2,1/2],[1,0,1,0]])
sage: W.coordinate_vector(v)
(1, -1)
sage: W.echelon_coordinate_vector(v)
(-1/2, 1/2)
```

echelon_coordinates (*v*, *check=True*)

Returns the coordinate vector of v in terms of the echelon basis for self.

EXAMPLES:

```
sage: U = VectorSpace(QQ,3)
sage: [ U.coordinates(v) for v in U.basis() ]
[[1, 0, 0], [0, 1, 0], [0, 0, 1]]
sage: [ U.echelon_coordinates(v) for v in U.basis() ]
[[1, 0, 0], [0, 1, 0], [0, 0, 1]]
sage: V = U.submodule([[1,1,0],[0,1,1]])
sage: V
Vector space of degree 3 and dimension 2 over Rational Field
```

```
Basis matrix:
[ 1  0 -1]
[ 0  1  1]
sage: [ V.coordinates(v) for v in V.basis() ]
[[1, 0], [0, 1]]
sage: [ V.echelon_coordinates(v) for v in V.basis() ]
[[1, 0], [0, 1]]
sage: W = U.submodule_with_basis([[1,1,0],[0,1,1]])
sage: W
Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[1 1 0]
[0 1 1]
sage: [ W.coordinates(v) for v in W.basis() ]
[[1, 0], [0, 1]]
sage: [ W.echelon_coordinates(v) for v in W.basis() ]
[[1, 1], [0, 1]]
```

echelonized_basis()

Return a basis for this ambient free module in echelon form.

EXAMPLES:

```
sage: A = ZZ^3; A.echelonized_basis()
[
(1, 0, 0),
(0, 1, 0),
(0, 0, 1)
]
```

echelonized_basis_matrix()

The echelonized basis matrix of self.

EXAMPLES:

```
sage: V = ZZ^4
sage: W = V.submodule([ V.gen(i)-V.gen(0) for i in range(1,4) ])
sage: W.basis_matrix()
[ 1  0  0 -1]
[ 0  1  0 -1]
[ 0  0  1 -1]
sage: W.echelonized_basis_matrix()
[ 1  0  0 -1]
[ 0  1  0 -1]
[ 0  0  1 -1]
sage: U = V.submodule_with_basis([ V.gen(i)-V.gen(0) for i in range(1,4) ])
sage: U.basis_matrix()
[-1  1  0  0]
[-1  0  1  0]
[-1  0  0  1]
sage: U.echelonized_basis_matrix()
[ 1  0  0 -1]
[ 0  1  0 -1]
[ 0  0  1 -1]
```

gen(*i=0*)

Return the i-th generator for `self`.

Here i is between 0 and rank - 1, inclusive.

INPUT:

- i – an integer (default 0)

OUTPUT: i-th basis vector for self.

EXAMPLES:

```
sage: n = 5
sage: V = QQ^n
sage: B = [V.gen(i) for i in range(n)]
sage: B
[(1, 0, 0, 0, 0),
(0, 1, 0, 0, 0),
(0, 0, 1, 0, 0),
(0, 0, 0, 1, 0),
(0, 0, 0, 0, 1)]
sage: V.gens() == tuple(B)
True
```

TESTS:

```
sage: (QQ^3).gen(4/3)
Traceback (most recent call last):
...
TypeError: rational is not an integer
```

Check that trac ticket #10262 and trac ticket #13304 are fixed (coercions involving FreeModule_ambient used to take quadratic time and space in the rank of the module):

```
sage: vector([0]*50000)/1
(0, 0, 0, ..., 0)
```

is_ambient()

Return True since this module is an ambient module.

EXAMPLES:

```
sage: A = QQ^5; A.is_ambient()
True
sage: A = (QQ^5).span([[1,2,3,4,5]]); A.is_ambient()
False
```

linear_combination_of_basis(*v*)

Return the linear combination of the basis for self obtained from the elements of the list v.

INPUTS:

- v - list

EXAMPLES:

```
sage: V = span([[1,2,3], [4,5,6]], ZZ)
sage: V
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[1 2 3]
[0 3 6]
sage: V.linear_combination_of_basis([1,1])
(1, 5, 9)
```

This should raise an error if the resulting element is not in self:

```
sage: W = span([[2,4]], ZZ)
sage: W.linear_combination_of_basis([1/2])
Traceback (most recent call last):
...
TypeError: element [1, 2] is not in free module
```

random_element (*prob=1.0*, **args*, ***kwds*)

Returns a random element of self.

INPUT:

- **prob - float. Each coefficient will be set to zero with** probability $1 - prob$. Otherwise coefficients will be chosen randomly from base ring (and may be zero).
- ***args, **kwds - passed on to random_element function of base** ring.

EXAMPLES:

```
sage: M = FreeModule(ZZ, 3)
sage: M.random_element()
(-1, 2, 1)
sage: M.random_element()
(-95, -1, -2)
sage: M.random_element()
(-12, 0, 0)
```

Passes extra positional or keyword arguments through:

```
sage: M.random_element(5,10)
(5, 5, 5)

sage: M = FreeModule(ZZ, 16)
sage: M.random_element()
(-6, 5, 0, 0, -2, 0, 1, -4, -6, 1, -1, 1, 1, -1, 1, -1)
sage: M.random_element(prob=0.3)
(0, 0, 0, 0, -3, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, -3)
```

class sage.modules.free_module.**FreeModule_ambient_domain** (*base_ring*, *rank*, *sparse=False*, *coordinate_ring=None*)

Bases: sage.modules.free_module.FreeModule_ambient

Ambient free module over an integral domain.

ambient_vector_space ()

Returns the ambient vector space, which is this free module tensored with its fraction field.

EXAMPLES:

```
sage: M = ZZ^3;
sage: V = M.ambient_vector_space(); V
Vector space of dimension 3 over Rational Field
```

If an inner product on the module is specified, then this is preserved on the ambient vector space.

```
sage: N = FreeModule(ZZ,4,inner_product_matrix=1)
sage: U = N.ambient_vector_space()
sage: U
Ambient quadratic space of dimension 4 over Rational Field
Inner product matrix:
[1 0 0 0]
[0 1 0 0]
```

```
[0 0 1 0]
[0 0 0 1]
sage: P = N.submodule_with_basis([[1,-1,0,0],[0,1,-1,0],[0,0,1,-1]])
sage: P.gram_matrix()
[ 2 -1  0]
[-1  2 -1]
[ 0 -1  2]
sage: U == N.ambient_vector_space()
True
sage: U == V
False
```

coordinate_vector(*v*, *check=True*)

Write v in terms of the standard basis for self and return the resulting coefficients in a vector over the fraction field of the base ring.

INPUT:

- `v` - vector
- `check` - bool (default: True); if True, also verify that v is really in self.

OUTPUT: list

Returns a vector c such that if B is the basis for self, then

$$\sum c_i B_i = v.$$

If v is not in self, raises an ArithmeticError exception.

EXAMPLES:

```
sage: V = ZZ^3
sage: v = V.coordinate_vector([1,5,9]); v
(1, 5, 9)
sage: v.parent()
Vector space of dimension 3 over Rational Field
```

vector_space(*base_field=None*)

Returns the vector space obtained from self by tensoring with the fraction field of the base ring and extending to the field.

EXAMPLES:

```
sage: M = ZZ^3;  M.vector_space()
Vector space of dimension 3 over Rational Field
```

class sage.modules.free_module.**FreeModule_ambient_field**(*base_field*, *dimension*, *sparse=False*)

Bases: sage.modules.free_module.FreeModule_generic_field, sage.modules.free_module.FreeModule_ambient_pid

ambient_vector_space()

Returns self as the ambient vector space.

EXAMPLES:

```
sage: M = QQ^3
sage: M.ambient_vector_space()
Vector space of dimension 3 over Rational Field
```

base_field()

Returns the base field of this vector space.

EXAMPLES:

```
sage: M = QQ^3
sage: M.base_field()
Rational Field
```

class sage.modules.free_module.**FreeModule_ambient_pid**(*base_ring*, *rank*, *sparse=False*, *coordinate_ring=None*)

Bases: sage.modules.free_module.FreeModule_generic_pid, sage.modules.free_module.FreeModule_ambient_domain

Ambient free module over a principal ideal domain.

class sage.modules.free_module.**FreeModule_generic**(*base_ring*, *rank*, *degree*, *sparse=False*, *coordinate_ring=None*)

Bases: sage.modules.module.Module_old

Base class for all free modules.

ambient_module()

Return the ambient module associated to this module.

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: M = FreeModule(R,2)
sage: M.ambient_module()
Ambient free module of rank 2 over the integral domain Multivariate Polynomial Ring in x, y

sage: V = FreeModule(QQ, 4).span([[1,2,3,4], [1,0,0,0]]); V
Vector space of degree 4 and dimension 2 over Rational Field
Basis matrix:
[  1   0   0   0]
[  0   1 3/2   2]
sage: V.ambient_module()
Vector space of dimension 4 over Rational Field
```

are_linearly_dependent(*vecs*)

Return `True` if the vectors `vecs` are linearly dependent and `False` otherwise.

EXAMPLES:

```
sage: M = QQ^3
sage: vecs = [M([1,2,3]), M([4,5,6])]
sage: M.are_linearly_dependent(vecs)
False
sage: vecs.append(M([3,3,3]))
sage: M.are_linearly_dependent(vecs)
True

sage: R.<x> = QQ[]
sage: M = FreeModule(R, 2)
sage: vecs = [M([x^2+1, x+1]), M([x+2, 2*x+1])]
sage: M.are_linearly_dependent(vecs)
False
sage: vecs.append(M([-2*x+1, -2*x^2+1]))
sage: M.are_linearly_dependent(vecs)
True
```

base_extend(*R*)

Return the base extension of self to R. This is the same as `self.change_ring(R)` except that a TypeError is raised if there is no canonical coerce map from the base ring of self to R.

INPUT:

- `R` - ring

EXAMPLES:

```
sage: V = ZZ^7
sage: V.base_extend(QQ)
Vector space of dimension 7 over Rational Field
```

base_field()

Return the base field, which is the fraction field of the base ring of this module.

EXAMPLES:

```
sage: FreeModule(GF(3), 2).base_field()
Finite Field of size 3
sage: FreeModule(ZZ, 2).base_field()
Rational Field
sage: FreeModule(PolynomialRing(GF(7),'x'), 2).base_field()
Fraction Field of Univariate Polynomial Ring in x over Finite Field of size 7
```

basis()

Return the basis of this module.

EXAMPLES:

```
sage: FreeModule(Integers(12),3).basis()
[
(1, 0, 0),
(0, 1, 0),
(0, 0, 1)
]
```

basis_matrix(*ring=None*)

Return the matrix whose rows are the basis for this free module.

INPUT:

- `ring` – (default: `self.coordinate_ring()`) a ring over which the matrix is defined

EXAMPLES:

```
sage: FreeModule(Integers(12),3).basis_matrix()
[1 0 0]
[0 1 0]
[0 0 1]

sage: M = FreeModule(GF(7),3).span([[2,3,4],[1,1,1]]); M
Vector space of degree 3 and dimension 2 over Finite Field of size 7
Basis matrix:
[1 0 6]
[0 1 2]
sage: M.basis_matrix()
[1 0 6]
[0 1 2]
```

```
sage: M = FreeModule(GF(7),3).span_of_basis([[2,3,4],[1,1,1]]);
sage: M.basis_matrix()
[2 3 4]
[1 1 1]

sage: M = FreeModule(QQ,2).span_of_basis([[1,-1],[1,0]]); M
Vector space of degree 2 and dimension 2 over Rational Field
User basis matrix:
[ 1 -1]
[ 1  0]
sage: M.basis_matrix()
[ 1 -1]
[ 1  0]
```

TESTS:

See trac ticket #3699:

```
sage: K = FreeModule(ZZ, 2000)
sage: I = K.basis_matrix()
```

See trac ticket #17585:

```
sage: ((ZZ^2)*2).basis_matrix().parent()
Full MatrixSpace of 2 by 2 dense matrices over Integer Ring
sage: ((ZZ^2)*2).basis_matrix(RDF).parent()
Full MatrixSpace of 2 by 2 dense matrices over Real Double Field

sage: M = (ZZ^2)*(1/2)
sage: M.basis_matrix()
[1/2   0]
[  0 1/2]
sage: M.basis_matrix().parent()
Full MatrixSpace of 2 by 2 dense matrices over Rational Field
sage: M.basis_matrix(QQ).parent()
Full MatrixSpace of 2 by 2 dense matrices over Rational Field
sage: M.basis_matrix(ZZ)
Traceback (most recent call last):
...
TypeError: matrix has denominators so can't change to ZZ.
```

cardinality()

Return the cardinality of the free module.

OUTPUT:

Either an integer or `+Infinity`.

EXAMPLES:

```
sage: k.<a> = FiniteField(9)
sage: V = VectorSpace(k,3)
sage: V.cardinality()
729
sage: W = V.span([[1,2,1],[0,1,1]])
sage: W.cardinality()
81
sage: R = IntegerModRing(12)
sage: M = FreeModule(R,2)
sage: M.cardinality()
144
```

```
sage: (QQ^3).cardinality()
+Infinity
```

construction()

The construction functor and base ring for self.

EXAMPLES:

```
sage: R = PolynomialRing(QQ,3,'x')
sage: V = R^5
sage: V.construction()
(VectorFunctor, Multivariate Polynomial Ring in x0, x1, x2 over Rational Field)
```

coordinate_module(V)

Suppose V is a submodule of self (or a module commensurable with self), and that self is a free module over R of rank n. Let ϕ be the map from self to R^n that sends the basis vectors of self in order to the standard basis of R^n. This function returns the image $\phi(V)$.

> **Warning:** If there is no integer d such that dV is a submodule of self, then this function will give total nonsense.

EXAMPLES:

We illustrate this function with some $\mathbf{Z}$-submodules of $\mathbf{Q}^3$.

```
sage: V = (ZZ^3).span([[1/2,3,5], [0,1,-3]])
sage: W = (ZZ^3).span([[1/2,4,2]])
sage: V.coordinate_module(W)
Free module of degree 2 and rank 1 over Integer Ring
User basis matrix:
[1 4]
sage: V.0 + 4*V.1
(1/2, 4, 2)
```

In this example, the coordinate module isn't even in $\mathbf{Z}^3$.

```
sage: W = (ZZ^3).span([[1/4,2,1]])
sage: V.coordinate_module(W)
Free module of degree 2 and rank 1 over Integer Ring
User basis matrix:
[1/2   2]
```

The following more elaborate example illustrates using this function to write a submodule in terms of integral cuspidal modular symbols:

```
sage: M = ModularSymbols(54)
sage: S = M.cuspidal_subspace()
sage: K = S.integral_structure(); K
Free module of degree 19 and rank 8 over Integer Ring
Echelon basis matrix:
[ 0  1  0  0 -1  0  0  0  0  0  0  0  0  0  0  0  0  0  0]
...
sage: L = M[0].integral_structure(); L
Free module of degree 19 and rank 2 over Integer Ring
Echelon basis matrix:
[ 0  1  1  0 -2  1 -1  1 -1 -2  2  0  0  0  0  0  0  0  0]
[ 0  0  3  0 -3  2 -1  2 -1 -4  2 -1 -2  1  2  0  0 -1  1]
sage: K.coordinate_module(L)
Free module of degree 8 and rank 2 over Integer Ring
```

```
User basis matrix:
[ 1  1  1 -1  1 -1  0  0]
[ 0  3  2 -1  2 -1 -1 -2]
sage: K.coordinate_module(L).basis_matrix() * K.basis_matrix()
[ 0  1  1  0 -2  1 -1  1 -1 -2  2  0  0  0  0  0  0  0]
[ 0  0  3  0 -3  2 -1  2 -1 -4  2 -1 -2  1  2  0  0 -1  1]
```

coordinate_ring()

Return the ring over which the entries of the vectors are defined.

This is the same as `base_ring()` unless an explicit basis was given over the fraction field.

EXAMPLES:

```
sage: M = ZZ^2
sage: M.coordinate_ring()
Integer Ring

sage: M = (ZZ^2) * (1/2)
sage: M.base_ring()
Integer Ring
sage: M.coordinate_ring()
Rational Field

sage: R.<x> = QQ[]
sage: L = R^2
sage: L.coordinate_ring()
Univariate Polynomial Ring in x over Rational Field
sage: L.span([(x,0), (1,x)]).coordinate_ring()
Univariate Polynomial Ring in x over Rational Field
sage: L.span([(x,0), (1,1/x)]).coordinate_ring()
Fraction Field of Univariate Polynomial Ring in x over Rational Field
sage: L.span([]).coordinate_ring()
Univariate Polynomial Ring in x over Rational Field
```

coordinate_vector(*v*, *check=True*)

Return the vector whose coefficients give v as a linear combination of the basis for self.

INPUT:

- `v` - vector
- `check` - bool (default: True); if True, also verify that v is really in self.

OUTPUT: list

EXAMPLES:

```
sage: M = FreeModule(ZZ, 2); M0,M1=M.gens()
sage: W = M.submodule([M0 + M1, M0 - 2*M1])
sage: W.coordinate_vector(2*M0 - M1)
(2, -1)
```

coordinates(*v*, *check=True*)

Write v in terms of the basis for self.

INPUT:

- `v` - vector
- `check` - bool (default: True); if True, also verify that v is really in self.

OUTPUT: list

Returns a list c such that if B is the basis for self, then

$$\sum c_i B_i = v.$$

If v is not in self, raises an `ArithmeticError` exception.

EXAMPLES:

```
sage: M = FreeModule(ZZ, 2); M0,M1=M.gens()
sage: W = M.submodule([M0 + M1, M0 - 2*M1])
sage: W.coordinates(2*M0-M1)
[2, -1]
```

degree()

Return the degree of this free module. This is the dimension of the ambient vector space in which it is embedded.

EXAMPLES:

```
sage: M = FreeModule(ZZ, 10)
sage: W = M.submodule([M.gen(0), 2*M.gen(3) - M.gen(0), M.gen(0) + M.gen(3)])
sage: W.degree()
10
sage: W.rank()
2
```

dense_module()

Return corresponding dense module.

EXAMPLES:

We first illustrate conversion with ambient spaces:

```
sage: M = FreeModule(QQ,3)
sage: S = FreeModule(QQ,3, sparse=True)
sage: M.sparse_module()
Sparse vector space of dimension 3 over Rational Field
sage: S.dense_module()
Vector space of dimension 3 over Rational Field
sage: M.sparse_module() == S
True
sage: S.dense_module() == M
True
sage: M.dense_module() == M
True
sage: S.sparse_module() == S
True
```

Next we create a subspace:

```
sage: M = FreeModule(QQ,3, sparse=True)
sage: V = M.span([ [1,2,3] ] ); V
Sparse vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[1 2 3]
sage: V.sparse_module()
Sparse vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[1 2 3]
```

dimension()

Return the dimension of this free module.

EXAMPLES:

```
sage: M = FreeModule(FiniteField(19), 100)
sage: W = M.submodule([M.gen(50)])
sage: W.dimension()
1
```

direct_sum(*other*)

Return the direct sum of self and other as a free module.

EXAMPLES:

```
sage: V = (ZZ^3).span([[1/2,3,5], [0,1,-3]]); V
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[1/2   0  14]
[  0   1  -3]
sage: W = (ZZ^3).span([[1/2,4,2]]); W
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[1/2   4   2]
sage: V.direct_sum(W)
Free module of degree 6 and rank 3 over Integer Ring
Echelon basis matrix:
[1/2   0  14   0   0   0]
[  0   1  -3   0   0   0]
[  0   0   0 1/2   4   2]
```

discriminant()

Return the discriminant of this free module.

EXAMPLES:

```
sage: M = FreeModule(ZZ, 3)
sage: M.discriminant()
1
sage: W = M.span([[1,2,3]])
sage: W.discriminant()
14
sage: W2 = M.span([[1,2,3], [1,1,1]])
sage: W2.discriminant()
6
```

echelonized_basis_matrix()

The echelonized basis matrix (not implemented for this module).

This example works because M is an ambient module. Submodule creation should exist for generic modules.

EXAMPLES:

```
sage: R = IntegerModRing(12)
sage: S.<x,y> = R[]
sage: M = FreeModule(S,3)
sage: M.echelonized_basis_matrix()
[1 0 0]
[0 1 0]
[0 0 1]
```

TESTS:

```
sage: from sage.modules.free_module import FreeModule_generic
sage: FreeModule_generic.echelonized_basis_matrix(M)
Traceback (most recent call last):
...
NotImplementedError
```

element_class()

The class of elements for this free module.

EXAMPLES:

```
sage: M = FreeModule(ZZ,20,sparse=False)
sage: x = M.random_element()
sage: type(x)
<type 'sage.modules.vector_integer_dense.Vector_integer_dense'>
sage: M.element_class()
<type 'sage.modules.vector_integer_dense.Vector_integer_dense'>
sage: N = FreeModule(ZZ,20,sparse=True)
sage: y = N.random_element()
sage: type(y)
<type 'sage.modules.free_module_element.FreeModuleElement_generic_sparse'>
sage: N.element_class()
<type 'sage.modules.free_module_element.FreeModuleElement_generic_sparse'>
```

free_module()

Return this free module. (This is used by the `FreeModule` functor, and simply returns self.)

EXAMPLES:

```
sage: M = FreeModule(ZZ, 3)
sage: M.free_module()
Ambient free module of rank 3 over the principal ideal domain Integer Ring
```

gen(*i=0*)

Return the i-th generator for `self`.

Here i is between 0 and rank - 1, inclusive.

INPUT:

- i – an integer (default 0)

OUTPUT: i-th basis vector for `self`.

EXAMPLES:

```
sage: n = 5
sage: V = QQ^n
sage: B = [V.gen(i) for i in range(n)]
sage: B
[(1, 0, 0, 0, 0),
(0, 1, 0, 0, 0),
(0, 0, 1, 0, 0),
(0, 0, 0, 1, 0),
(0, 0, 0, 0, 1)]
sage: V.gens() == tuple(B)
True
```

TESTS:

```
sage: (QQ^3).gen(4/3)
Traceback (most recent call last):
...
TypeError: rational is not an integer
```

gram_matrix()

Return the gram matrix associated to this free module, defined to be $G = B * A * B.transpose()$, where A is the inner product matrix (induced from the ambient space), and B the basis matrix.

EXAMPLES:

```
sage: V = VectorSpace(QQ,4)
sage: u = V([1/2,1/2,1/2,1/2])
sage: v = V([0,1,1,0])
sage: w = V([0,0,1,1])
sage: M = span([u,v,w], ZZ)
sage: M.inner_product_matrix() == V.inner_product_matrix()
True
sage: L = M.submodule_with_basis([u,v,w])
sage: L.inner_product_matrix() == M.inner_product_matrix()
True
sage: L.gram_matrix()
[1 1 1]
[1 2 1]
[1 1 2]
```

has_user_basis()

Return `True` if the basis of this free module is specified by the user, as opposed to being the default echelon form.

EXAMPLES:

```
sage: V = QQ^3
sage: W = V.subspace([[2,'1/2', 1]])
sage: W.has_user_basis()
False
sage: W = V.subspace_with_basis([[2,'1/2',1]])
sage: W.has_user_basis()
True
```

inner_product_matrix()

Return the default identity inner product matrix associated to this module.

By definition this is the inner product matrix of the ambient space, hence may be of degree greater than the rank of the module.

TODO: Differentiate the image ring of the inner product from the base ring of the module and/or ambient space. E.g. On an integral module over ZZ the inner product pairing could naturally take values in ZZ, QQ, RR, or CC.

EXAMPLES:

```
sage: M = FreeModule(ZZ, 3)
sage: M.inner_product_matrix()
[1 0 0]
[0 1 0]
[0 0 1]
```

is_ambient()

Returns False since this is not an ambient free module.

EXAMPLES:

```
sage: M = FreeModule(ZZ, 3).span([[1,2,3]]); M
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[1 2 3]
sage: M.is_ambient()
False
sage: M = (ZZ^2).span([[1,0], [0,1]])
sage: M
Free module of degree 2 and rank 2 over Integer Ring
Echelon basis matrix:
[1 0]
[0 1]
sage: M.is_ambient()
False
sage: M == M.ambient_module()
True
```

is_dense()

Return `True` if the underlying representation of this module uses dense vectors, and False otherwise.

EXAMPLES:

```
sage: FreeModule(ZZ, 2).is_dense()
True
sage: FreeModule(ZZ, 2, sparse=True).is_dense()
False
```

is_finite()

Returns True if the underlying set of this free module is finite.

EXAMPLES:

```
sage: FreeModule(ZZ, 2).is_finite()
False
sage: FreeModule(Integers(8), 2).is_finite()
True
sage: FreeModule(ZZ, 0).is_finite()
True
```

is_full()

Return `True` if the rank of this module equals its degree.

EXAMPLES:

```
sage: FreeModule(ZZ, 2).is_full()
True
sage: M = FreeModule(ZZ, 2).span([[1,2]])
sage: M.is_full()
False
```

is_sparse()

Return `True` if the underlying representation of this module uses sparse vectors, and False otherwise.

EXAMPLES:

```
sage: FreeModule(ZZ, 2).is_sparse()
False
sage: FreeModule(ZZ, 2, sparse=True).is_sparse()
True
```

is_submodule(*other*)

Return True if self is a submodule of other.

EXAMPLES:

```
sage: M = FreeModule(ZZ,3)
sage: V = M.ambient_vector_space()
sage: X = V.span([[1/2,1/2,0],[1/2,0,1/2]], ZZ)
sage: Y = V.span([[1,1,1]], ZZ)
sage: N = X + Y
sage: M.is_submodule(X)
False
sage: M.is_submodule(Y)
False
sage: Y.is_submodule(M)
True
sage: N.is_submodule(M)
False
sage: M.is_submodule(N)
True
```

Since basis() is not implemented in general, submodule testing does not work for all PID's. However, trivial cases are already used (and useful) for coercion, e.g.

```
sage: QQ(1/2) * vector(ZZ['x']['y'],[1,2,3,4])
(1/2, 1, 3/2, 2)
sage: vector(ZZ['x']['y'],[1,2,3,4]) * QQ(1/2)
(1/2, 1, 3/2, 2)
```

matrix()

Return the basis matrix of this module, which is the matrix whose rows are a basis for this module.

EXAMPLES:

```
sage: M = FreeModule(ZZ, 2)
sage: M.matrix()
[1 0]
[0 1]
sage: M.submodule([M.gen(0) + M.gen(1), M.gen(0) - 2*M.gen(1)]).matrix()
[1 1]
[0 3]
```

ngens()

Returns the number of basis elements of this free module.

EXAMPLES:

```
sage: FreeModule(ZZ, 2).ngens()
2
sage: FreeModule(ZZ, 0).ngens()
0
sage: FreeModule(ZZ, 2).span([[1,1]]).ngens()
1
```

nonembedded_free_module()

Returns an ambient free module that is isomorphic to this free module.

Thus if this free module is of rank n over a ring R, then this function returns R^n, as an ambient free module.

EXAMPLES:

```
sage: FreeModule(ZZ, 2).span([[1,1]]).nonembedded_free_module()
Ambient free module of rank 1 over the principal ideal domain Integer Ring
```

random_element (*prob=1.0*, **args*, ***kwds*)

Returns a random element of self.

INPUT:

– **prob - float. Each coefficient will be set to zero with** probability $1 - prob$. Otherwise coefficients will be chosen randomly from base ring (and may be zero).

– ***args, **kwds - passed on to random_element() function** of base ring.

EXAMPLES:

```
sage: M = FreeModule(ZZ, 2).span([[1,1]])
sage: M.random_element()
(-1, -1)
sage: M.random_element()
(2, 2)
sage: M.random_element()
(1, 1)
```

Passes extra positional or keyword arguments through:

```
sage: M.random_element(5,10)
(9, 9)
```

rank ()

Return the rank of this free module.

EXAMPLES:

```
sage: FreeModule(Integers(6), 10000000).rank()
10000000
sage: FreeModule(ZZ, 2).span([[1,1], [2,2], [3,4]]).rank()
2
```

sparse_module ()

Return the corresponding sparse module with the same defining data.

EXAMPLES:

We first illustrate conversion with ambient spaces:

```
sage: M = FreeModule(Integers(8),3)
sage: S = FreeModule(Integers(8),3, sparse=True)
sage: M.sparse_module()
Ambient sparse free module of rank 3 over Ring of integers modulo 8
sage: S.dense_module()
Ambient free module of rank 3 over Ring of integers modulo 8
sage: M.sparse_module() is S
True
sage: S.dense_module() is M
True
sage: M.dense_module() is M
True
sage: S.sparse_module() is S
True
```

Next we convert a subspace:

```
sage: M = FreeModule(QQ,3)
sage: V = M.span([ [1,2,3] ] ); V
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[1 2 3]
sage: V.sparse_module()
Sparse vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[1 2 3]
```

uses_ambient_inner_product()

Return `True` if the inner product on this module is the one induced by the ambient inner product.

EXAMPLES:

```
sage: M = FreeModule(ZZ, 2)
sage: W = M.submodule([[1,2]])
sage: W.uses_ambient_inner_product()
True
sage: W.inner_product_matrix()
[1 0]
[0 1]

sage: W.gram_matrix()
[5]
```

zero()

Returns the zero vector in this free module.

EXAMPLES:

```
sage: M = FreeModule(ZZ, 2)
sage: M.zero()
(0, 0)
sage: M.span([[1,1]]).zero()
(0, 0)
sage: M.zero_submodule().zero()
(0, 0)
sage: M.zero_submodule().zero().is_mutable()
False
```

zero_vector()

Returns the zero vector in this free module.

EXAMPLES:

```
sage: M = FreeModule(ZZ, 2)
sage: M.zero_vector()
(0, 0)
sage: M(0)
(0, 0)
sage: M.span([[1,1]]).zero_vector()
(0, 0)
sage: M.zero_submodule().zero_vector()
(0, 0)
```

class sage.modules.free_module.**FreeModule_generic_field**(*base_field*, *dimension*, *degree*, *sparse=False*)

Bases: `sage.modules.free_module.FreeModule_generic_pid`

Base class for all free modules over fields.

complement()

Return the complement of `self` in the `ambient_vector_space()`.

EXAMPLES:

```
sage: V = QQ^3
sage: V.complement()
Vector space of degree 3 and dimension 0 over Rational Field
Basis matrix:
[]
sage: V == V.complement().complement()
True
sage: W = V.span([[1, 0, 1]])
sage: X = W.complement(); X
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1  0 -1]
[ 0  1  0]
sage: X.complement() == W
True
sage: X + W == V
True
```

Even though we construct a subspace of a subspace, the orthogonal complement is still done in the ambient vector space $\mathbf{Q}^3$:

```
sage: V = QQ^3
sage: W = V.subspace_with_basis([[1,0,1],[-1,1,0]])
sage: X = W.subspace_with_basis([[1,0,1]])
sage: X.complement()
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1  0 -1]
[ 0  1  0]
```

All these complements are only done with respect to the inner product in the usual basis. Over finite fields, this means we can get complements which are only isomorphic to a vector space decomposition complement.

```
sage: F2 = GF(2,x)
sage: V = F2^6
sage: W = V.span([[1,1,0,0,0,0]])
sage: W
Vector space of degree 6 and dimension 1 over Finite Field of size 2
Basis matrix:
[1 1 0 0 0 0]
sage: W.complement()
Vector space of degree 6 and dimension 5 over Finite Field of size 2
Basis matrix:
[1 1 0 0 0 0]
[0 0 1 0 0 0]
[0 0 0 1 0 0]
[0 0 0 0 1 0]
[0 0 0 0 0 1]
sage: W.intersection(W.complement())
Vector space of degree 6 and dimension 1 over Finite Field of size 2
Basis matrix:
[1 1 0 0 0 0]
```

echelonized_basis_matrix()

Return basis matrix for self in row echelon form.

EXAMPLES:

```
sage: V = FreeModule(QQ, 3).span_of_basis([[1,2,3],[4,5,6]])
sage: V.basis_matrix()
[1 2 3]
[4 5 6]
sage: V.echelonized_basis_matrix()
[ 1  0 -1]
[ 0  1  2]
```

intersection(*other*)

Return the intersection of self and other, which must be R-submodules of a common ambient vector space.

EXAMPLES:

```
sage: V  = VectorSpace(QQ,3)
sage: W1 = V.submodule([V.gen(0), V.gen(0) + V.gen(1)])
sage: W2 = V.submodule([V.gen(1), V.gen(2)])
sage: W1.intersection(W2)
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[0 1 0]
sage: W2.intersection(W1)
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[0 1 0]
sage: V.intersection(W1)
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[1 0 0]
[0 1 0]
sage: W1.intersection(V)
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[1 0 0]
[0 1 0]
sage: Z = V.submodule([])
sage: W1.intersection(Z)
Vector space of degree 3 and dimension 0 over Rational Field
Basis matrix:
[]
```

is_subspace(*other*)

True if this vector space is a subspace of other.

EXAMPLES:

```
sage: V = VectorSpace(QQ,3)
sage: W = V.subspace([V.gen(0), V.gen(0) + V.gen(1)])
sage: W2 = V.subspace([V.gen(1)])
sage: W.is_subspace(V)
True
sage: W2.is_subspace(V)
True
sage: W.is_subspace(W2)
False
sage: W2.is_subspace(W)
True
```

linear_dependence(*vectors*, *zeros='left'*, *check=True*)

Returns a list of vectors giving relations of linear dependence for the input list of vectors. Can be used to check linear independence of a set of vectors.

INPUT:

- `vectors` - A list of vectors, all from the same vector space.
- `zeros` - default: `'left'` - `'left'` or `'right'` as a general preference for where zeros are located in the returned coefficients
- `check` - default: `True` - if `True` each item in the list `vectors` is checked for membership in `self`. Set to `False` if you can be certain the vectors come from the vector space.

OUTPUT:

Returns a list of vectors. The scalar entries of each vector provide the coefficients for a linear combination of the input vectors that will equal the zero vector in `self`. Furthermore, the returned list is linearly independent in the vector space over the same base field with degree equal to the length of the list `vectors`.

The linear independence of `vectors` is equivalent to the returned list being empty, so this provides a test - see the examples below.

The returned vectors are always independent, and with `zeros` set to `'left'` they have 1's in their first non-zero entries and a qualitative disposition to having zeros in the low-index entries. With `zeros` set to `'right'` the situation is reversed with a qualitative disposition for zeros in the high-index entries.

If the vectors in `vectors` are made the rows of a matrix V and the returned vectors are made the rows of a matrix R, then the matrix product RV is a zero matrix of the proper size. And R is a matrix of full rank. This routine uses kernels of matrices to compute these relations of linear dependence, but handles all the conversions between sets of vectors and matrices. If speed is important, consider working with the appropriate matrices and kernels instead.

EXAMPLES:

We begin with two linearly independent vectors, and add three non-trivial linear combinations to the set. We illustrate both types of output and check a selected relation of linear dependence.

```
sage: v1 = vector(QQ, [2, 1, -4, 3])
sage: v2 = vector(QQ, [1, 5, 2, -2])
sage: V = QQ^4
sage: V.linear_dependence([v1,v2])
[

]

sage: v3 = v1 + v2
sage: v4 = 3*v1 - 4*v2
sage: v5 = -v1 + 2*v2
sage: L = [v1, v2, v3, v4, v5]

sage: relations = V.linear_dependence(L, zeros='left')
sage: relations
[
(1, 0, 0, -1, -2),
(0, 1, 0, -1/2, -3/2),
(0, 0, 1, -3/2, -7/2)
]
sage: v2 + (-1/2)*v4 + (-3/2)*v5
(0, 0, 0, 0)
```

```
sage: relations = V.linear_dependence(L, zeros='right')
sage: relations
[
(-1, -1, 1, 0, 0),
(-3, 4, 0, 1, 0),
(1, -2, 0, 0, 1)
]
sage: z = sum([relations[2][i]*L[i] for i in range(len(L))])
sage: z == zero_vector(QQ, 4)
True
```

A linearly independent set returns an empty list, a result that can be tested.

```
sage: v1 = vector(QQ, [0,1,-3])
sage: v2 = vector(QQ, [4,1,0])
sage: V = QQ^3
sage: relations = V.linear_dependence([v1, v2]); relations
[

]
sage: relations == []
True
```

Exact results result from exact fields. We start with three linearly independent vectors and add in two linear combinations to make a linearly dependent set of five vectors.

```
sage: F = FiniteField(17)
sage: v1 = vector(F, [1, 2, 3, 4, 5])
sage: v2 = vector(F, [2, 4, 8, 16, 15])
sage: v3 = vector(F, [1, 0, 0, 0, 1])
sage: (F^5).linear_dependence([v1, v2, v3]) == []
True
sage: L = [v1, v2, v3, 2*v1+v2, 3*v2+6*v3]
sage: (F^5).linear_dependence(L)
[
(1, 0, 16, 8, 3),
(0, 1, 2, 0, 11)
]
sage: v1 + 16*v3 + 8*(2*v1+v2) + 3*(3*v2+6*v3)
(0, 0, 0, 0, 0)
sage: v2 + 2*v3 + 11*(3*v2+6*v3)
(0, 0, 0, 0, 0)
sage: (F^5).linear_dependence(L, zeros='right')
[
(15, 16, 0, 1, 0),
(0, 14, 11, 0, 1)
]
```

TESTS:

With `check=True` (the default) a mismatch between vectors and the vector space is caught.

```
sage: v1 = vector(RR, [1,2,3])
sage: v2 = vector(RR, [1,2,3,4])
sage: (RR^3).linear_dependence([v1,v2], check=True)
Traceback (most recent call last):
...
ValueError: vector (1.00000000000000, 2.00000000000000, 3.00000000000000, 4.00000000000000)
```

The `zeros` keyword is checked.

```
sage: (QQ^3).linear_dependence([vector(QQ,[1,2,3])], zeros='bogus')
Traceback (most recent call last):
...
ValueError: 'zeros' keyword must be 'left' or 'right', not 'bogus'
```

An empty input set is linearly independent, vacuously.

```
sage: (QQ^3).linear_dependence([]) == []
True
```

quotient (*sub*, *check=True*)

Return the quotient of self by the given subspace sub.

INPUT:

- `sub` - a submodule of self, or something that can be turned into one via self.submodule(sub).
- `check` - (default: True) whether or not to check that sub is a submodule.

EXAMPLES:

```
sage: A = QQ^3; V = A.span([[1,2,3], [4,5,6]])
sage: Q = V.quotient( [V.0 + V.1] ); Q
Vector space quotient V/W of dimension 1 over Rational Field where
V: Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1  0 -1]
[ 0  1  2]
W: Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[1 1 1]
sage: Q(V.0 + V.1)
(0)
```

We illustrate the the base rings must be the same:

```
sage: (QQ^2)/(ZZ^2)
Traceback (most recent call last):
...
ValueError: base rings must be the same
```

quotient_abstract (*sub*, *check=True*)

Returns an ambient free module isomorphic to the quotient space of self modulo sub, together with maps from self to the quotient, and a lifting map in the other direction.

Use `self.quotient(sub)` to obtain the quotient module as an object equipped with natural maps in both directions, and a canonical coercion.

INPUT:

- `sub` - a submodule of self, or something that can be turned into one via self.submodule(sub).
- `check` - (default: True) whether or not to check that sub is a submodule.

OUTPUT:

- `U` - the quotient as an abstract *ambient* free module
- `pi` - projection map to the quotient
- `lift` - lifting map back from quotient

EXAMPLES:

```
sage: V = GF(19)^3
sage: W = V.span_of_basis([ [1,2,3], [1,0,1] ])
sage: U,pi,lift = V.quotient_abstract(W)
sage: pi(V.2)
(18)
sage: pi(V.0)
(1)
sage: pi(V.0 + V.2)
(0)
```

Another example involving a quotient of one subspace by another.

```
sage: A = matrix(QQ,4,4,[0,1,0,0, 0,0,1,0, 0,0,0,1, 0,0,0,0])
sage: V = (A^3).kernel()
sage: W = A.kernel()
sage: U, pi, lift = V.quotient_abstract(W)
sage: [pi(v) == 0 for v in W.gens()]
[True]
sage: [pi(lift(b)) == b for b in U.basis()]
[True, True]
```

scale(*other*)

Return the product of self by the number other, which is the module spanned by other times each basis vector. Since self is a vector space this product equals self if other is nonzero, and is the zero vector space if other is 0.

EXAMPLES:

```
sage: V = QQ^4
sage: V.scale(5)
Vector space of dimension 4 over Rational Field
sage: V.scale(0)
Vector space of degree 4 and dimension 0 over Rational Field
Basis matrix:
[]

sage: W = V.span([[1,1,1,1]])
sage: W.scale(2)
Vector space of degree 4 and dimension 1 over Rational Field
Basis matrix:
[1 1 1 1]
sage: W.scale(0)
Vector space of degree 4 and dimension 0 over Rational Field
Basis matrix:
[]

sage: V = QQ^4; V
Vector space of dimension 4 over Rational Field
sage: V.scale(3)
Vector space of dimension 4 over Rational Field
sage: V.scale(0)
Vector space of degree 4 and dimension 0 over Rational Field
Basis matrix:
[]
```

span(*gens*, *base_ring=None*, *check=True*, *already_echelonized=False*)

Return the K-span of the given list of gens, where K is the base field of self or the user-specified base_ring. Note that this span is a subspace of the ambient vector space, but need not be a subspace of self.

INPUT:

- `gens` - list of vectors
- `check` - bool (default: True): whether or not to coerce entries of gens into base field
- `already_echelonized` - bool (default: False): set this if you know the gens are already in echelon form

EXAMPLES:

```
sage: V = VectorSpace(GF(7), 3)
sage: W = V.subspace([[2,3,4]]); W
Vector space of degree 3 and dimension 1 over Finite Field of size 7
Basis matrix:
[1 5 2]
sage: W.span([[1,1,1]])
Vector space of degree 3 and dimension 1 over Finite Field of size 7
Basis matrix:
[1 1 1]
```

TESTS:

```
sage: V = FreeModule(RDF,3)
sage: W = V.submodule([V.gen(0)])
sage: W.span([V.gen(1)], base_ring=GF(7))
Vector space of degree 3 and dimension 1 over Finite Field of size 7
Basis matrix:
[0 1 0]
sage: v = V((1, pi, e)); v
(1.0, 3.141592653589793, 2.718281828459045)
sage: W.span([v], base_ring=GF(7))
Traceback (most recent call last):
...
ValueError: Argument gens (= [(1.0, 3.141592653589793, 2.718281828459045)]) is not compatibl
sage: W = V.submodule([v])
sage: W.span([V.gen(2)], base_ring=GF(7))
Vector space of degree 3 and dimension 1 over Finite Field of size 7
Basis matrix:
[0 0 1]
```

span_of_basis (*basis*, *base_ring=None*, *check=True*, *already_echelonized=False*)

Return the free K-module with the given basis, where K is the base field of self or user specified base_ring.

Note that this span is a subspace of the ambient vector space, but need not be a subspace of self.

INPUT:

- `basis` - list of vectors
- `check` - bool (default: True): whether or not to coerce entries of gens into base field
- `already_echelonized` - bool (default: False): set this if you know the gens are already in echelon form

EXAMPLES:

```
sage: V = VectorSpace(GF(7), 3)
sage: W = V.subspace([[2,3,4]]); W
Vector space of degree 3 and dimension 1 over Finite Field of size 7
Basis matrix:
[1 5 2]
sage: W.span_of_basis([[2,2,2], [3,3,0]])
```

```
Vector space of degree 3 and dimension 2 over Finite Field of size 7
User basis matrix:
[2 2 2]
[3 3 0]
```

The basis vectors must be linearly independent or an ArithmeticError exception is raised.

```
sage: W.span_of_basis([[2,2,2], [3,3,3]])
Traceback (most recent call last):
...
ValueError: The given basis vectors must be linearly independent.
```

subspace (*gens*, *check=True*, *already_echelonized=False*)

Return the subspace of self spanned by the elements of gens.

INPUT:

- `gens` - list of vectors
- `check` - bool (default: True) verify that gens are all in self.
- `already_echelonized` - bool (default: False) set to True if you know the gens are in Echelon form.

EXAMPLES:

First we create a 1-dimensional vector subspace of an ambient 3-dimensional space over the finite field of order 7.

```
sage: V = VectorSpace(GF(7), 3)
sage: W = V.subspace([[2,3,4]]); W
Vector space of degree 3 and dimension 1 over Finite Field of size 7
Basis matrix:
[1 5 2]
```

Next we create an invalid subspace, but it's allowed since `check=False`. This is just equivalent to computing the span of the element.

```
sage: W.subspace([[1,1,0]], check=False)
Vector space of degree 3 and dimension 1 over Finite Field of size 7
Basis matrix:
[1 1 0]
```

With `check=True` (the default) the mistake is correctly detected and reported with an `ArithmeticError` exception.

```
sage: W.subspace([[1,1,0]], check=True)
Traceback (most recent call last):
...
ArithmeticError: Argument gens (= [[1, 1, 0]]) does not generate a submodule of self.
```

subspace_with_basis (*gens*, *check=True*, *already_echelonized=False*)

Same as `self.submodule_with_basis(...)`.

EXAMPLES:

We create a subspace with a user-defined basis.

```
sage: V = VectorSpace(GF(7), 3)
sage: W = V.subspace_with_basis([[2,2,2], [1,2,3]]); W
Vector space of degree 3 and dimension 2 over Finite Field of size 7
User basis matrix:
```

```
[2 2 2]
[1 2 3]
```

We then create a subspace of the subspace with user-defined basis.

```
sage: W1 = W.subspace_with_basis([[3,4,5]]); W1
Vector space of degree 3 and dimension 1 over Finite Field of size 7
User basis matrix:
[3 4 5]
```

Notice how the basis for the same subspace is different if we merely use the `subspace` command.

```
sage: W2 = W.subspace([[3,4,5]]); W2
Vector space of degree 3 and dimension 1 over Finite Field of size 7
Basis matrix:
[1 6 4]
```

Nonetheless the two subspaces are equal (as mathematical objects):

```
sage: W1 == W2
True
```

subspaces(*dim*)

Iterate over all subspaces of dimension dim.

INPUT:

- `dim` - int, dimension of subspaces to be generated

EXAMPLE:

```
sage: V = VectorSpace(GF(3), 5)
sage: len(list(V.subspaces(0)))
1
sage: len(list(V.subspaces(1)))
121
sage: len(list(V.subspaces(2)))
1210
sage: len(list(V.subspaces(3)))
1210
sage: len(list(V.subspaces(4)))
121
sage: len(list(V.subspaces(5)))
1
```

```
sage: V = VectorSpace(GF(3), 5)
sage: V = V.subspace([V([1,1,0,0,0]),V([0,0,1,1,0])])
sage: list(V.subspaces(1))
[Vector space of degree 5 and dimension 1 over Finite Field of size 3
Basis matrix:
[1 1 0 0 0],
 Vector space of degree 5 and dimension 1 over Finite Field of size 3
Basis matrix:
[1 1 1 1 0],
 Vector space of degree 5 and dimension 1 over Finite Field of size 3
Basis matrix:
[1 1 2 2 0],
 Vector space of degree 5 and dimension 1 over Finite Field of size 3
Basis matrix:
[0 0 1 1 0]]
```

vector_space(*base_field=None*)

Return the vector space associated to self. Since self is a vector space this function simply returns self, unless the base field is different.

EXAMPLES:

```
sage: V = span([[1,2,3]],QQ); V
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[1 2 3]
sage: V.vector_space()
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[1 2 3]
```

zero_submodule()

Return the zero submodule of self.

EXAMPLES:

```
sage: (QQ^4).zero_submodule()
Vector space of degree 4 and dimension 0 over Rational Field
Basis matrix:
[]
```

zero_subspace()

Return the zero subspace of self.

EXAMPLES:

```
sage: (QQ^4).zero_subspace()
Vector space of degree 4 and dimension 0 over Rational Field
Basis matrix:
[]
```

class sage.modules.free_module.**FreeModule_generic_pid**(*base_ring*, *rank*, *degree*, *sparse=False*, *coordinate_ring=None*)

Bases: sage.modules.free_module.FreeModule_generic

Base class for all free modules over a PID.

denominator()

The denominator of the basis matrix of self (i.e. the LCM of the coordinate entries with respect to the basis of the ambient space).

EXAMPLES:

```
sage: V = QQ^3
sage: L = V.span([[1,1/2,1/3], [-1/5,2/3,3]],ZZ)
sage: L
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[ 1/5 19/6 37/3]
[   0 23/6 46/3]
sage: L.denominator()
30
```

index_in(*other*)

Return the lattice index [other:self] of self in other, as an element of the base field. When self is contained in other, the lattice index is the usual index. If the index is infinite, then this function returns infinity.

EXAMPLES:

```
sage: L1 = span([[1,2]], ZZ)
sage: L2 = span([[3,6]], ZZ)
sage: L2.index_in(L1)
3
```

Note that the free modules being compared need not be integral.

```
sage: L1 = span([['1/2','1/3'], [4,5]], ZZ)
sage: L2 = span([[1,2], [3,4]], ZZ)
sage: L2.index_in(L1)
12/7
sage: L1.index_in(L2)
7/12
sage: L1.discriminant() / L2.discriminant()
49/144
```

The index of a lattice of infinite index is infinite.

```
sage: L1 = FreeModule(ZZ, 2)
sage: L2 = span([[1,2]], ZZ)
sage: L2.index_in(L1)
+Infinity
```

index_in_saturation()

Return the index of this module in its saturation, i.e., its intersection with R^n.

EXAMPLES:

```
sage: W = span([[2,4,6]], ZZ)
sage: W.index_in_saturation()
2
sage: W = span([[1/2,1/3]], ZZ)
sage: W.index_in_saturation()
1/6
```

intersection(*other*)

Return the intersection of self and other.

EXAMPLES:

We intersect two submodules one of which is clearly contained in the other.

```
sage: A = ZZ^2
sage: M1 = A.span([[1,1]])
sage: M2 = A.span([[3,3]])
sage: M1.intersection(M2)
Free module of degree 2 and rank 1 over Integer Ring
Echelon basis matrix:
[3 3]
sage: M1.intersection(M2) is M2
True
```

We intersection two submodules of $\mathbf{Z}^3$ of rank 2, whose intersection has rank 1.

```
sage: A = ZZ^3
sage: M1 = A.span([[1,1,1], [1,2,3]])
sage: M2 = A.span([[2,2,2], [1,0,0]])
sage: M1.intersection(M2)
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
```

```
[2 2 2]
```

We compute an intersection of two $\mathbf{Z}$-modules that are not submodules of $\mathbf{Z}^2$.

```
sage: A = ZZ^2
sage: M1 = A.span([[1,2]]).scale(1/6)
sage: M2 = A.span([[1,2]]).scale(1/15)
sage: M1.intersection(M2)
Free module of degree 2 and rank 1 over Integer Ring
Echelon basis matrix:
[1/3 2/3]
```

We intersect a $\mathbf{Z}$-module with a $\mathbf{Q}$-vector space.

```
sage: A = ZZ^3
sage: L = ZZ^3
sage: V = QQ^3
sage: W = L.span([[1/2,0,1/2]])
sage: K = V.span([[1,0,1], [0,0,1]])
sage: W.intersection(K)
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[1/2   0 1/2]
sage: K.intersection(W)
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[1/2   0 1/2]
```

We intersect two modules over the ring of integers of a number field:

```
sage: L.<w> = NumberField(x^2 - x + 2)
sage: OL = L.ring_of_integers()
sage: V = L**3; W1 = V.span([[0,w/5,0], [1,0,-1/17]], OL); W2 = V.span([[0,(1-w)/5,0]], OL)
sage: W1.intersection(W2)
Free module of degree 3 and rank 1 over Maximal Order in Number Field in w with defining pol
Echelon basis matrix:
[  0 2/5   0]
```

is_submodule(*other*)

True if this module is a submodule of other.

EXAMPLES:

```
sage: M = FreeModule(ZZ,2)
sage: M.is_submodule(M)
True
sage: N = M.scale(2)
sage: N.is_submodule(M)
True
sage: M.is_submodule(N)
False
sage: N = M.scale(1/2)
sage: N.is_submodule(M)
False
sage: M.is_submodule(N)
True
```

quotient(*sub*, *check=True*)

Return the quotient of self by the given submodule sub.

INPUT:

- `sub` - a submodule of self, or something that can be turned into one via self.submodule(sub).
- `check` - (default: True) whether or not to check that sub is a submodule.

EXAMPLES:

```
sage: A = ZZ^3; V = A.span([[1,2,3], [4,5,6]])
sage: Q = V.quotient( [V.0 + V.1] ); Q
Finitely generated module V/W over Integer Ring with invariants (0)
```

saturation()

Return the saturated submodule of R^n that spans the same vector space as self.

EXAMPLES:

We create a 1-dimensional lattice that is obviously not saturated and saturate it.

```
sage: L = span([[9,9,6]], ZZ); L
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[9 9 6]
sage: L.saturation()
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[3 3 2]
```

We create a lattice spanned by two vectors, and saturate. Computation of discriminants shows that the index of lattice in its saturation is 3, which is a prime of congruence between the two generating vectors.

```
sage: L = span([[1,2,3], [4,5,6]], ZZ)
sage: L.saturation()
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[ 1  0 -1]
[ 0  1  2]
sage: L.discriminant()
54
sage: L.saturation().discriminant()
6
```

Notice that the saturation of a non-integral lattice L is defined, but the result is integral hence does not contain L:

```
sage: L = span([['1/2',1,3]], ZZ)
sage: L.saturation()
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[1 2 6]
```

scale(*other*)

Return the product of this module by the number other, which is the module spanned by other times each basis vector.

EXAMPLES:

```
sage: M = FreeModule(ZZ, 3)
sage: M.scale(2)
Free module of degree 3 and rank 3 over Integer Ring
Echelon basis matrix:
[2 0 0]
```

```
[0 2 0]
[0 0 2]

sage: a = QQ('1/3')
sage: M.scale(a)
Free module of degree 3 and rank 3 over Integer Ring
Echelon basis matrix:
[1/3   0   0]
[  0 1/3   0]
[  0   0 1/3]
```

span (*gens*, *base_ring=None*, *check=True*, *already_echelonized=False*)

Return the R-span of the given list of gens, where R = base_ring. The default R is the base ring of self. Note that this span need not be a submodule of self, nor even of the ambient space. It must, however, be contained in the ambient vector space, i.e., the ambient space tensored with the fraction field of R.

EXAMPLES:

```
sage: V = FreeModule(ZZ,3)
sage: W = V.submodule([V.gen(0)])
sage: W.span([V.gen(1)])
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[0 1 0]
sage: W.submodule([V.gen(1)])
Traceback (most recent call last):
...
ArithmeticError: Argument gens (= [(0, 1, 0)]) does not generate a submodule of self.
sage: V.span([[1,0,0],[1/5,4,0],[6,3/4,0]])
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[1/5   0   0]
[  0 1/4   0]
```

It also works with other things than integers:

```
sage: R.<x>=QQ[]
sage: L=R^1
sage: a=L.span([(1/x,)])
sage: a
Free module of degree 1 and rank 1 over Univariate Polynomial Ring in x over Rational Field
Echelon basis matrix:
[1/x]
sage: b=L.span([(1/x,)])
sage: a(b.gens()[0])
(1/x)
sage: L2 = R^2
sage: L2.span([[(x^2+x)/(x^2-3*x+2),1/5],[(x^2+2*x)/(x^2-4*x+3),x]])
Free module of degree 2 and rank 2 over Univariate Polynomial Ring in x over Rational Field
Echelon basis matrix:
[x/(x^3 - 6*x^2 + 11*x - 6)  2/15*x^2 - 17/75*x - 1/75]
[                         0 x^3 - 11/5*x^2 - 3*x + 4/5]
```

Note that the `base_ring` can make a huge difference. We repeat the previous example over the fraction field of R and get a simpler vector space.

```
sage: L2.span([[(x^2+x)/(x^2-3*x+2),1/5],[(x^2+2*x)/(x^2-4*x+3),x]],base_ring=R.fraction_fie
Vector space of degree 2 and dimension 2 over Fraction Field of Univariate Polynomial Ring i
Basis matrix:
```

```
[1 0]
[0 1]
```

span_of_basis (*basis*, *base_ring=None*, *check=True*, *already_echelonized=False*)

Return the free R-module with the given basis, where R is the base ring of self or user specified base_ring.

Note that this R-module need not be a submodule of self, nor even of the ambient space. It must, however, be contained in the ambient vector space, i.e., the ambient space tensored with the fraction field of R.

EXAMPLES:

```
sage: M = FreeModule(ZZ,3)
sage: W = M.span_of_basis([M([1,2,3])])
```

Next we create two free $\mathbf{Z}$-modules, neither of which is a submodule of W.

```
sage: W.span_of_basis([M([2,4,0])])
Free module of degree 3 and rank 1 over Integer Ring
User basis matrix:
[2 4 0]
```

The following module isn't in the ambient module $\mathbf{Z}^3$ but is contained in the ambient vector space $\mathbf{Q}^3$:

```
sage: V = M.ambient_vector_space()
sage: W.span_of_basis([ V([1/5,2/5,0]), V([1/7,1/7,0]) ])
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[1/5 2/5   0]
[1/7 1/7   0]
```

Of course the input basis vectors must be linearly independent.

```
sage: W.span_of_basis([ [1,2,0], [2,4,0] ])
Traceback (most recent call last):
...
ValueError: The given basis vectors must be linearly independent.
```

submodule (*gens*, *check=True*, *already_echelonized=False*)

Create the R-submodule of the ambient vector space with given generators, where R is the base ring of self.

INPUT:

- `gens` - a list of free module elements or a free module
- `check` - (default: True) whether or not to verify that the gens are in self.

OUTPUT:

- `FreeModule` - the submodule spanned by the vectors in the list gens. The basis for the subspace is always put in reduced row echelon form.

EXAMPLES:

We create a submodule of $\mathbf{Z}^3$:

```
sage: M = FreeModule(ZZ, 3)
sage: B = M.basis()
sage: W = M.submodule([B[0]+B[1], 2*B[1]-B[2]])
sage: W
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
```

```
[ 1  1  0]
[ 0  2 -1]
```

We create a submodule of a submodule.

```
sage: W.submodule([3*B[0] + 3*B[1]])
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[3 3 0]
```

We try to create a submodule that isn't really a submodule, which results in an ArithmeticError exception:

```
sage: W.submodule([B[0] - B[1]])
Traceback (most recent call last):
...
ArithmeticError: Argument gens (= [(1, -1, 0)]) does not generate a submodule of self.
```

Next we create a submodule of a free module over the principal ideal domain $\mathbf{Q}[x]$, which uses the general Hermite normal form functionality:

```
sage: R = PolynomialRing(QQ, 'x'); x = R.gen()
sage: M = FreeModule(R, 3)
sage: B = M.basis()
sage: W = M.submodule([x*B[0], 2*B[1]- x*B[2]]); W
Free module of degree 3 and rank 2 over Univariate Polynomial Ring in x over Rational Field
Echelon basis matrix:
[ x  0  0]
[ 0  2 -x]
sage: W.ambient_module()
Ambient free module of rank 3 over the principal ideal domain Univariate Polynomial Ring in
```

submodule_with_basis (*basis*, *check=True*, *already_echelonized=False*)

Create the R-submodule of the ambient vector space with given basis, where R is the base ring of self.

INPUT:

- `basis` - a list of linearly independent vectors
- `check` - whether or not to verify that each gen is in the ambient vector space

OUTPUT:

- `FreeModule` - the R-submodule with given basis

EXAMPLES:

First we create a submodule of $\mathbf{Z}^3$:

```
sage: M = FreeModule(ZZ, 3)
sage: B = M.basis()
sage: N = M.submodule_with_basis([B[0]+B[1], 2*B[1]-B[2]])
sage: N
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[ 1  1  0]
[ 0  2 -1]
```

A list of vectors in the ambient vector space may fail to generate a submodule.

```
sage: V = M.ambient_vector_space()
sage: X = M.submodule_with_basis([ V(B[0]+B[1])/2, V(B[1]-B[2])/2])
Traceback (most recent call last):
```

```
...
ArithmeticError: The given basis does not generate a submodule of self.
```

However, we can still determine the R-span of vectors in the ambient space, or over-ride the submodule check by setting check to False.

```
sage: X = V.span([ V(B[0]+B[1])/2, V(B[1]-B[2])/2 ], ZZ)
sage: X
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[ 1/2    0  1/2]
[   0  1/2 -1/2]
sage: Y = M.submodule([ V(B[0]+B[1])/2, V(B[1]-B[2])/2 ], check=False)
sage: X == Y
True
```

Next we try to create a submodule of a free module over the principal ideal domain $\mathbf{Q}[x]$, using our general Hermite normal form implementation:

```
sage: R = PolynomialRing(QQ, 'x'); x = R.gen()
sage: M = FreeModule(R, 3)
sage: B = M.basis()
sage: W = M.submodule_with_basis([x*B[0], 2*B[0]- x*B[2]]); W
Free module of degree 3 and rank 2 over Univariate Polynomial Ring in x over Rational Field
User basis matrix:
[ x  0  0]
[ 2  0 -x]
```

vector_space_span (*gens*, *check=True*)

Create the vector subspace of the ambient vector space with given generators.

INPUT:

- `gens` - a list of vector in self
- `check` - whether or not to verify that each gen is in the ambient vector space

OUTPUT: a vector subspace

EXAMPLES:

We create a 2-dimensional subspace of $\mathbf{Q}^3$.

```
sage: V = VectorSpace(QQ, 3)
sage: B = V.basis()
sage: W = V.vector_space_span([B[0]+B[1], 2*B[1]-B[2]])
sage: W
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[   1    0  1/2]
[   0    1 -1/2]
```

We create a subspace of a vector space over $\mathbf{Q}(i)$.

```
sage: R.<x> = QQ[]
sage: K = NumberField(x^2 + 1, 'a'); a = K.gen()
sage: V = VectorSpace(K, 3)
sage: V.vector_space_span([2*V.gen(0) + 3*V.gen(2)])
Vector space of degree 3 and dimension 1 over Number Field in a with defining polynomial x^2
Basis matrix:
[  1   0 3/2]
```

We use the `vector_space_span` command to create a vector subspace of the ambient vector space of a submodule of $\mathbf{Z}^3$.

```
sage: M = FreeModule(ZZ,3)
sage: W = M.submodule([M([1,2,3])])
sage: W.vector_space_span([M([2,3,4])])
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[  1 3/2   2]
```

vector_space_span_of_basis (*basis*, *check=True*)

Create the vector subspace of the ambient vector space with given basis.

INPUT:

- `basis` - a list of linearly independent vectors
- `check` - whether or not to verify that each gen is in the ambient vector space

OUTPUT: a vector subspace with user-specified basis

EXAMPLES:

```
sage: V = VectorSpace(QQ, 3)
sage: B = V.basis()
sage: W = V.vector_space_span_of_basis([B[0]+B[1], 2*B[1]-B[2]])
sage: W
Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[ 1  1  0]
[ 0  2 -1]
```

zero_submodule ()

Return the zero submodule of this module.

EXAMPLES:

```
sage: V = FreeModule(ZZ,2)
sage: V.zero_submodule()
Free module of degree 2 and rank 0 over Integer Ring
Echelon basis matrix:
[]
```

class sage.modules.free_module.**FreeModule_submodule_field** (*ambient*, *gens*, *check=True*, *already_echelonized=False*)

Bases: `sage.modules.free_module.FreeModule_submodule_with_basis_field`

An embedded vector subspace with echelonized basis.

EXAMPLES:

Since this is an embedded vector subspace with echelonized basis, the echelon_coordinates() and user coordinates() agree:

```
sage: V = QQ^3
sage: W = V.span([[1,2,3],[4,5,6]])
sage: W
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1  0 -1]
[ 0  1  2]
```

```
sage: v = V([1,5,9])
sage: W.echelon_coordinates(v)
[1, 5]
sage: vector(QQ, W.echelon_coordinates(v)) * W.basis_matrix()
(1, 5, 9)
sage: v = V([1,5,9])
sage: W.coordinates(v)
[1, 5]
sage: vector(QQ, W.coordinates(v)) * W.basis_matrix()
(1, 5, 9)
```

coordinate_vector(*v*, *check=True*)

Write v in terms of the user basis for self.

INPUT:

- `v` - vector
- `check` - bool (default: True); if True, also verify that v is really in self.

OUTPUT: list

Returns a list c such that if B is the basis for self, then

$$\sum c_i B_i = v.$$

If v is not in self, raises an `ArithmeticError` exception.

EXAMPLES:

```
sage: V = QQ^3
sage: W = V.span([[1,2,3],[4,5,6]]); W
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1  0 -1]
[ 0  1  2]
sage: v = V([1,5,9])
sage: W.coordinate_vector(v)
(1, 5)
sage: W.coordinates(v)
[1, 5]
sage: vector(QQ, W.coordinates(v)) * W.basis_matrix()
(1, 5, 9)

sage: V = VectorSpace(QQ,5, sparse=True)
sage: W = V.subspace([[0,1,2,0,0], [0,-1,0,0,-1/2]])
sage: W.coordinate_vector([0,0,2,0,-1/2])
(0, 2)
```

echelon_coordinates(*v*, *check=True*)

Write v in terms of the echelonized basis of self.

INPUT:

- `v` - vector
- `check` - bool (default: True); if True, also verify that v is really in self.

OUTPUT: list

Returns a list c such that if B is the basis for self, then

$$\sum c_i B_i = v.$$

If v is not in self, raises an `ArithmeticError` exception.

EXAMPLES:

```
sage: V = QQ^3
sage: W = V.span([[1,2,3],[4,5,6]])
sage: W
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1  0 -1]
[ 0  1  2]

sage: v = V([1,5,9])
sage: W.echelon_coordinates(v)
[1, 5]
sage: vector(QQ, W.echelon_coordinates(v)) * W.basis_matrix()
(1, 5, 9)
```

has_user_basis()

Return `True` if the basis of this free module is specified by the user, as opposed to being the default echelon form.

EXAMPLES:

```
sage: V = QQ^3
sage: W = V.subspace([[2,'1/2', 1]])
sage: W.has_user_basis()
False
sage: W = V.subspace_with_basis([[2,'1/2',1]])
sage: W.has_user_basis()
True
```

class sage.modules.free_module.**FreeModule_submodule_pid**(*ambient*, *gens*, *check=True*, *already_echelonized=False*)

Bases: `sage.modules.free_module.FreeModule_submodule_with_basis_pid`

An R-submodule of K^n where K is the fraction field of a principal ideal domain R.

EXAMPLES:

```
sage: M = ZZ^3
sage: W = M.span_of_basis([[1,2,3],[4,5,19]]); W
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[ 1  2  3]
[ 4  5 19]
```

Generic tests, including saving and loading submodules and elements:

```
sage: TestSuite(W).run()
sage: v = W.0 + W.1
sage: TestSuite(v).run()
```

coordinate_vector(*v*, *check=True*)

Write v in terms of the user basis for self.

INPUT:

- `v` - vector
- `check` - bool (default: True); if True, also verify that v is really in self.

OUTPUT: list

Returns a list c such that if B is the basis for self, then

$$\sum c_i B_i = v.$$

If v is not in self, raises an `ArithmeticError` exception.

EXAMPLES:

```
sage: V = ZZ^3
sage: W = V.span_of_basis([[1,2,3],[4,5,6]])
sage: W.coordinate_vector([1,5,9])
(5, -1)
```

has_user_basis()

Return `True` if the basis of this free module is specified by the user, as opposed to being the default echelon form.

EXAMPLES:

```
sage: A = ZZ^3; A
Ambient free module of rank 3 over the principal ideal domain Integer Ring
sage: A.has_user_basis()
False
sage: W = A.span_of_basis([[2,'1/2',1]])
sage: W.has_user_basis()
True
sage: W = A.span([[2,'1/2',1]])
sage: W.has_user_basis()
False
```

class sage.modules.free_module.**FreeModule_submodule_with_basis_field**(*ambient*, *basis*, *check=True*, *echelonize=False*, *echelonized_basis=None*, *already_echelonized=False*)

Bases: `sage.modules.free_module.FreeModule_generic_field`, `sage.modules.free_module.FreeModule_submodule_with_basis_pid`

An embedded vector subspace with a distinguished user basis.

EXAMPLES:

```
sage: M = QQ^3; W = M.submodule_with_basis([[1,2,3], [4,5,19]]); W
Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[ 1  2  3]
[ 4  5 19]
```

Since this is an embedded vector subspace with a distinguished user basis possibly different than the echelonized basis, the echelon_coordinates() and user coordinates() do not agree:

```
sage: V = QQ^3

sage: W = V.submodule_with_basis([[1,2,3], [4,5,6]])
sage: W
Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
```

```
[1 2 3]
[4 5 6]

sage: v = V([1,5,9])
sage: W.echelon_coordinates(v)
[1, 5]
sage: vector(QQ, W.echelon_coordinates(v)) * W.echelonized_basis_matrix()
(1, 5, 9)

sage: v = V([1,5,9])
sage: W.coordinates(v)
[5, -1]
sage: vector(QQ, W.coordinates(v)) * W.basis_matrix()
(1, 5, 9)
```

Generic tests, including saving and loading submodules and elements:

```
sage: TestSuite(W).run()

sage: K.<x> = FractionField(PolynomialRing(QQ,'x'))
sage: M = K^3; W = M.span_of_basis([[1,1,x]])
sage: TestSuite(W).run()
```

is_ambient()

Return False since this is not an ambient module.

EXAMPLES:

```
sage: V = QQ^3
sage: V.is_ambient()
True
sage: W = V.span_of_basis([[1,2,3],[4,5,6]])
sage: W.is_ambient()
False
```

class sage.modules.free_module.**FreeModule_submodule_with_basis_pid**(*ambient, basis, check=True, echelonize=False, echelonized_basis=None, already_echelonized=False*)

Bases: sage.modules.free_module.FreeModule_generic_pid

Construct a submodule of a free module over PID with a distinguished basis.

INPUT:

- `ambient` – ambient free module over a principal ideal domain R, i.e. R^n;
- `basis` – list of elements of K^n, where K is the fraction field of R. These elements must be linearly independent and will be used as the default basis of the constructed submodule;
- `check` – (default: `True`) if `False`, correctness of the input will not be checked and type conversion may be omitted, use with care;
- `echelonize` – (default:`False`) if `True`, `basis` will be echelonized and the result will be used as the default basis of the constructed submodule;

- `` echelonized_basis`` – (default: `None`) if not `None`, must be the echelonized basis spanning the same submodule as `basis`;
- `already_echelonized` – (default: `False`) if `True`, `basis` must be already given in the echelonized form.

OUTPUT:

- R-submodule of K^n with the user-specified `basis`.

EXAMPLES:

```
sage: M = ZZ^3
sage: W = M.span_of_basis([[1,2,3],[4,5,6]]); W
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[1 2 3]
[4 5 6]
```

Now we create a submodule of the ambient vector space, rather than `M` itself:

```
sage: W = M.span_of_basis([[1,2,3/2],[4,5,6]]); W
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[  1   2 3/2]
[  4   5   6]
```

ambient_module()

Return the ambient module related to the R-module self, which was used when creating this module, and is of the form R^n. Note that self need not be contained in the ambient module, though self will be contained in the ambient vector space.

EXAMPLES:

```
sage: A = ZZ^3
sage: M = A.span_of_basis([[1,2,'3/7'],[4,5,6]])
sage: M
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[  1   2 3/7]
[  4   5   6]
sage: M.ambient_module()
Ambient free module of rank 3 over the principal ideal domain Integer Ring
sage: M.is_submodule(M.ambient_module())
False
```

ambient_vector_space()

Return the ambient vector space in which this free module is embedded.

EXAMPLES:

```
sage: M = ZZ^3;  M.ambient_vector_space()
Vector space of dimension 3 over Rational Field

sage: N = M.span_of_basis([[1,2,'1/5']])
sage: N
Free module of degree 3 and rank 1 over Integer Ring
User basis matrix:
[  1   2 1/5]
sage: M.ambient_vector_space()
Vector space of dimension 3 over Rational Field
sage: M.ambient_vector_space() is N.ambient_vector_space()
```

```
True
```

If an inner product on the module is specified, then this is preserved on the ambient vector space.

```
sage: M = FreeModule(ZZ,4,inner_product_matrix=1)
sage: V = M.ambient_vector_space()
sage: V
Ambient quadratic space of dimension 4 over Rational Field
Inner product matrix:
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
sage: N = M.submodule([[1,-1,0,0],[0,1,-1,0],[0,0,1,-1]])
sage: N.gram_matrix()
[2 1 1]
[1 2 1]
[1 1 2]
sage: V == N.ambient_vector_space()
True
```

basis()

Return the user basis for this free module.

EXAMPLES:

```
sage: V = ZZ^3
sage: V.basis()
[
(1, 0, 0),
(0, 1, 0),
(0, 0, 1)
]
sage: M = V.span_of_basis([['1/8',2,1]])
sage: M.basis()
[
(1/8, 2, 1)
]
```

change_ring(R)

Return the free module over R obtained by coercing each element of the basis of `self` into a vector space over the fraction field of R, then taking the resulting R-module.

INPUT:

- `R` - a principal ideal domain

EXAMPLES:

```
sage: V = QQ^3
sage: W = V.subspace([[2, 1/2, 1]])
sage: W.change_ring(GF(7))
Vector space of degree 3 and dimension 1 over Finite Field of size 7
Basis matrix:
[1 2 4]

sage: M = (ZZ^2) * (1/2)
sage: N = M.change_ring(QQ)
sage: N
Vector space of degree 2 and dimension 2 over Rational Field
Basis matrix:
```

```
[1 0]
[0 1]
sage: N = M.change_ring(QQ['x'])
sage: N
Free module of degree 2 and rank 2 over Univariate Polynomial Ring in x over Rational Field
Echelon basis matrix:
[1/2   0]
[  0 1/2]
sage: N.coordinate_ring()
Univariate Polynomial Ring in x over Rational Field
```

The ring must be a principal ideal domain:

```
sage: M.change_ring(ZZ['x'])
Traceback (most recent call last):
...
TypeError: the new ring Univariate Polynomial Ring in x over Integer Ring should be a princi
```

construction()

Returns the functorial construction of self, namely, the subspace of the ambient module spanned by the given basis.

EXAMPLE:

```
sage: M = ZZ^3
sage: W = M.span_of_basis([[1,2,3],[4,5,6]]); W
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[1 2 3]
[4 5 6]
sage: c, V = W.construction()
sage: c(V) == W
True
```

coordinate_vector(*v*, *check=True*)

Write v in terms of the user basis for self.

INPUT:

- `v` - vector
- `check` - bool (default: True); if True, also verify that v is really in self.

OUTPUT: list

Returns a vector c such that if B is the basis for self, then

If v is not in self, raises an ArithmeticError exception.

EXAMPLES:

```
sage: V = ZZ^3
sage: M = V.span_of_basis([['1/8',2,1]])
sage: M.coordinate_vector([1,16,8])
(8)
```

echelon_coordinate_vector(*v*, *check=True*)

Write v in terms of the echelonized basis for self.

INPUT:

- `v` - vector

- `check` - bool (default: True); if True, also verify that v is really in self.

Returns a list c such that if B is the echelonized basis for self, then

$$\sum c_i B_i = v.$$

If v is not in self, raises an `ArithmeticError` exception.

EXAMPLES:

```
sage: V = ZZ^3
sage: M = V.span_of_basis([['1/2',3,1], [0,'1/6',0]])
sage: B = M.echelonized_basis(); B
[
(1/2, 0, 1),
(0, 1/6, 0)
]
sage: M.echelon_coordinate_vector(['1/2', 3, 1])
(1, 18)
```

echelon_coordinates(*v*, *check=True*)

Write v in terms of the echelonized basis for self.

INPUT:

- `v` - vector
- `check` - bool (default: True); if True, also verify that v is really in self.

OUTPUT: list

Returns a list c such that if B is the basis for self, then

$$\sum c_i B_i = v.$$

If v is not in self, raises an `ArithmeticError` exception.

EXAMPLES:

```
sage: A = ZZ^3
sage: M = A.span_of_basis([[1,2,'3/7'],[4,5,6]])
sage: M.coordinates([8,10,12])
[0, 2]
sage: M.echelon_coordinates([8,10,12])
[8, -2]
sage: B = M.echelonized_basis(); B
[
(1, 2, 3/7),
(0, 3, -30/7)
]
sage: 8*B[0] - 2*B[1]
(8, 10, 12)
```

We do an example with a sparse vector space:

```
sage: V = VectorSpace(QQ,5, sparse=True)
sage: W = V.subspace_with_basis([[0,1,2,0,0], [0,-1,0,0,-1/2]])
sage: W.echelonized_basis()
[
(0, 1, 0, 0, 1/2),
(0, 0, 1, 0, -1/4)
]
sage: W.echelon_coordinates([0,0,2,0,-1/2])
[0, 2]
```

echelon_to_user_matrix()

Return matrix that transforms the echelon basis to the user basis of self. This is a matrix A such that if v is a vector written with respect to the echelon basis for self then vA is that vector written with respect to the user basis of self.

EXAMPLES:

```
sage: V = QQ^3
sage: W = V.span_of_basis([[1,2,3],[4,5,6]])
sage: W.echelonized_basis()
[
(1, 0, -1),
(0, 1, 2)
]
sage: A = W.echelon_to_user_matrix(); A
[-5/3  2/3]
[ 4/3 -1/3]
```

The vector $(1, 1, 1)$ has coordinates $v = (1, 1)$ with respect to the echelonized basis for self. Multiplying vA we find the coordinates of this vector with respect to the user basis.

```
sage: v = vector(QQ, [1,1]); v
(1, 1)
sage: v * A
(-1/3, 1/3)
sage: u0, u1 = W.basis()
sage: (-u0 + u1)/3
(1, 1, 1)
```

echelonized_basis()

Return the basis for self in echelon form.

EXAMPLES:

```
sage: V = ZZ^3
sage: M = V.span_of_basis([['1/2',3,1], [0,'1/6',0]])
sage: M.basis()
[
(1/2, 3, 1),
(0, 1/6, 0)
]
sage: B = M.echelonized_basis(); B
[
(1/2, 0, 1),
(0, 1/6, 0)
]
sage: V.span(B) == M
True
```

echelonized_basis_matrix()

Return basis matrix for self in row echelon form.

EXAMPLES:

```
sage: V = FreeModule(ZZ, 3).span_of_basis([[1,2,3],[4,5,6]])
sage: V.basis_matrix()
[1 2 3]
[4 5 6]
sage: V.echelonized_basis_matrix()
[1 2 3]
[0 3 6]
```

has_user_basis()

Return True if the basis of this free module is specified by the user, as opposed to being the default echelon form.

EXAMPLES:

```
sage: V = ZZ^3; V.has_user_basis()
False
sage: M = V.span_of_basis([[1,3,1]]); M.has_user_basis()
True
sage: M = V.span([[1,3,1]]); M.has_user_basis()
False
```

linear_combination_of_basis(*v*)

Return the linear combination of the basis for self obtained from the coordinates of v.

INPUTS:

- v - list

EXAMPLES:

```
sage: V = span([[1,2,3], [4,5,6]], ZZ); V
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[1 2 3]
[0 3 6]
sage: V.linear_combination_of_basis([1,1])
(1, 5, 9)
```

This should raise an error if the resulting element is not in self:

```
sage: W = (QQ**2).span([[2, 0], [0, 8]], ZZ)
sage: W.linear_combination_of_basis([1, -1/2])
Traceback (most recent call last):
...
TypeError: element [2, -4] is not in free module
```

user_to_echelon_matrix()

Return matrix that transforms a vector written with respect to the user basis of self to one written with respect to the echelon basis. The matrix acts from the right, as is usual in Sage.

EXAMPLES:

```
sage: A = ZZ^3
sage: M = A.span_of_basis([[1,2,3],[4,5,6]])
sage: M.echelonized_basis()
[
(1, 2, 3),
(0, 3, 6)
]
sage: M.user_to_echelon_matrix()
[ 1  0]
[ 4 -1]
```

The vector $v = (5, 7, 9)$ in M is $(1, 1)$ with respect to the user basis. Multiplying the above matrix on the right by this vector yields $(5, -1)$, which has components the coordinates of v with respect to the echelon basis.

```
sage: v0,v1 = M.basis(); v = v0+v1
sage: e0,e1 = M.echelonized_basis()
sage: v
```

```
(5, 7, 9)
sage: 5*e0 + (-1)*e1
(5, 7, 9)
```

vector_space (*base_field=None*)

Return the vector space associated to this free module via tensor product with the fraction field of the base ring.

EXAMPLES:

```
sage: A = ZZ^3; A
Ambient free module of rank 3 over the principal ideal domain Integer Ring
sage: A.vector_space()
Vector space of dimension 3 over Rational Field
sage: M = A.span_of_basis([['1/3',2,'3/7'],[4,5,6]]); M
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[1/3   2 3/7]
[  4   5   6]
sage: M.vector_space()
Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[1/3   2 3/7]
[  4   5   6]
```

class sage.modules.free_module.**RealDoubleVectorSpace_class** (*n*)

Bases: sage.modules.free_module.FreeModule_ambient_field

coordinates (*v*)

x.__init__(...) initializes x; see help(type(x)) for signature

sage.modules.free_module.**VectorSpace** (*K, dimension, sparse=False, inner_product_matrix=None*)

EXAMPLES:

The base can be complicated, as long as it is a field.

```
sage: V = VectorSpace(FractionField(PolynomialRing(ZZ,'x')),3)
sage: V
Vector space of dimension 3 over Fraction Field of Univariate Polynomial Ring in x over Integer
sage: V.basis()
[
(1, 0, 0),
(0, 1, 0),
(0, 0, 1)
]
```

The base must be a field or a TypeError is raised.

```
sage: VectorSpace(ZZ,5)
Traceback (most recent call last):
...
TypeError: Argument K (= Integer Ring) must be a field.
```

sage.modules.free_module.**basis_seq** (*V, vecs*)

This converts a list vecs of vectors in V to an Sequence of immutable vectors.

Should it? I.e. in most other parts of the system the return type of basis or generators is a tuple.

EXAMPLES:

```
sage: V = VectorSpace(QQ,2)
sage: B = V.gens()
sage: B
((1, 0), (0, 1))
sage: v = B[0]
sage: v[0] = 0 # immutable
Traceback (most recent call last):
...
ValueError: vector is immutable; please change a copy instead (use copy())
sage: sage.modules.free_module.basis_seq(V, V.gens())
[
(1, 0),
(0, 1)
]
```

sage.modules.free_module.**element_class**(*R*, *is_sparse*)

The class of the vectors (elements of a free module) with base ring R and boolean is_sparse.

EXAMPLES:

```
sage: FF = FiniteField(2)
sage: P = PolynomialRing(FF,'x')
sage: sage.modules.free_module.element_class(QQ, is_sparse=True)
<type 'sage.modules.free_module_element.FreeModuleElement_generic_sparse'>
sage: sage.modules.free_module.element_class(QQ, is_sparse=False)
<type 'sage.modules.vector_rational_dense.Vector_rational_dense'>
sage: sage.modules.free_module.element_class(ZZ, is_sparse=True)
<type 'sage.modules.free_module_element.FreeModuleElement_generic_sparse'>
sage: sage.modules.free_module.element_class(ZZ, is_sparse=False)
<type 'sage.modules.vector_integer_dense.Vector_integer_dense'>
sage: sage.modules.free_module.element_class(FF, is_sparse=True)
<type 'sage.modules.free_module_element.FreeModuleElement_generic_sparse'>
sage: sage.modules.free_module.element_class(FF, is_sparse=False)
<type 'sage.modules.vector_mod2_dense.Vector_mod2_dense'>
sage: sage.modules.free_module.element_class(GF(7), is_sparse=False)
<type 'sage.modules.vector_modn_dense.Vector_modn_dense'>
sage: sage.modules.free_module.element_class(P, is_sparse=True)
<type 'sage.modules.free_module_element.FreeModuleElement_generic_sparse'>
sage: sage.modules.free_module.element_class(P, is_sparse=False)
<type 'sage.modules.free_module_element.FreeModuleElement_generic_dense'>
```

sage.modules.free_module.**is_FreeModule**(*M*)

Return True if M inherits from from FreeModule_generic.

EXAMPLES:

```
sage: from sage.modules.free_module import is_FreeModule
sage: V = ZZ^3
sage: is_FreeModule(V)
True
sage: W = V.span([ V.random_element() for i in range(2) ])
sage: is_FreeModule(W)
True
```

sage.modules.free_module.**span**(*gens*, *base_ring=None*, *check=True*, *already_echelonized=False*)

Return the span of the vectors in `gens` using scalars from `base_ring`.

INPUT:

- `gens` - a list of either vectors or lists of ring elements used to generate the span

- base_ring - default: None - a principal ideal domain for the ring of scalars
- check - default: True - passed to the span() method of the ambient module
- already_echelonized - default: False - set to True if the vectors form the rows of a matrix in echelon form, in order to skip the computation of an echelonized basis for the span.

OUTPUT:

A module (or vector space) that is all the linear combinations of the free module elements (or vectors) with scalars from the ring (or field) given by base_ring. See the examples below describing behavior when the base ring is not specified and/or the module elements are given as lists that do not carry explicit base ring information.

EXAMPLES:

The vectors in the list of generators can be given as lists, provided a base ring is specified and the elements of the list are in the ring (or the fraction field of the ring). If the base ring is a field, the span is a vector space.

```
sage: V = span([[1,2,5], [2,2,2]], QQ); V
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1  0 -3]
[ 0  1  4]

sage: span([V.gen(0)], QuadraticField(-7,'a'))
Vector space of degree 3 and dimension 1 over Number Field in a with defining polynomial x^2 + 7
Basis matrix:
[ 1  0 -3]

sage: span([[1,2,3], [2,2,2], [1,2,5]], GF(2))
Vector space of degree 3 and dimension 1 over Finite Field of size 2
Basis matrix:
[1 0 1]
```

If the base ring is not a field, then a module is created. The entries of the vectors can lie outside the ring, if they are in the fraction field of the ring.

```
sage: span([[1,2,5], [2,2,2]], ZZ)
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[ 1  0 -3]
[ 0  2  8]

sage: span([[1,1,1], [1,1/2,1]], ZZ)
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[  1   0   1]
[  0 1/2   0]

sage: R.<x> = QQ[]
sage: M= span( [[x, x^2+1], [1/x, x^3]], R); M
Free module of degree 2 and rank 2 over
Univariate Polynomial Ring in x over Rational Field
Echelon basis matrix:
[          1/x           x^3]
[            0 x^5 - x^2 - 1]
sage: M.basis()[0][0].parent()
Fraction Field of Univariate Polynomial Ring in x over Rational Field
```

A base ring can be inferred if the generators are given as a list of vectors.

```
sage: span([vector(QQ, [1,2,3]), vector(QQ, [4,5,6])])
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1  0 -1]
[ 0  1  2]
sage: span([vector(QQ, [1,2,3]), vector(ZZ, [4,5,6])])
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1  0 -1]
[ 0  1  2]
sage: span([vector(ZZ, [1,2,3]), vector(ZZ, [4,5,6])])
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[1 2 3]
[0 3 6]
```

TESTS:

```
sage: span([[1,2,3], [2,2,2], [1,2/3,5]], ZZ)
Free module of degree 3 and rank 3 over Integer Ring
Echelon basis matrix:
[  1   0  13]
[  0 2/3   6]
[  0   0  14]
sage: span([[1,2,3], [2,2,2], [1,2,QQ['x'].gen()]], ZZ)
Traceback (most recent call last):
...
ValueError: The elements of gens (= [[1, 2, 3], [2, 2, 2], [1, 2, x]]) must be defined over base
```

For backwards compatibility one can also give the base ring as the first argument.

```
sage: span(QQ,[[1,2],[3,4]])
Vector space of degree 2 and dimension 2 over Rational Field
Basis matrix:
[1 0]
[0 1]
```

The base ring must be a principal ideal domain (PID).

```
sage: span([[1,2,3]], Integers(6))
Traceback (most recent call last):
...
TypeError: The base_ring (= Ring of integers modulo 6)
must be a principal ideal domain.
```

Fix trac ticket #5575:

```
sage: V = QQ^3
sage: span([V.0, V.1])
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[1 0 0]
[0 1 0]
```

Improve error message from trac ticket #12541:

```
sage: span({0:vector([0,1])}, QQ)
Traceback (most recent call last):
...
TypeError: generators must be lists of ring elements
```

```
or free module elements!
```

CHAPTER

THREE

DISCRETE SUBGROUPS OF $\mathbb{Z}^N$.

AUTHORS:

- Martin Albrecht (2014-03): initial version
- Jan Pöschko (2012-08): some code in this module was taken from Jan Pöschko's 2012 GSoC project

TESTS:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: L = IntegerLattice(random_matrix(ZZ, 10, 10))
sage: TestSuite(L).run()
```

class sage.modules.free_module_integer.**FreeModule_submodule_with_basis_integer**(*ambient*, *basis*, *check=True*, *echelonize=False*, *echelonized_basis=None*, *already_echelonized=False*, *lll_reduce=True*)

Bases: sage.modules.free_module.FreeModule_submodule_with_basis_pid

This class represents submodules of $\mathbf{Z}^n$ with a distinguished basis.

However, most functionality in excess of standard submodules over PID is for these submodules considered as discrete subgroups of $\mathbf{Z}^n$, i.e. as lattices. That is, this class provides functions for computing LLL and BKZ reduced bases for this free module with respect to the standard Euclidean norm.

EXAMPLE:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: L = IntegerLattice(sage.crypto.gen_lattice(type='modular', m=10, seed=42, dual=True)); L
Free module of degree 10 and rank 10 over Integer Ring
User basis matrix:
[ 0  1  2  0  1  2 -1  0 -1 -1]
[ 0  1  0 -3  0  0  0  0  3 -1]
[ 1  1 -1  0 -3  0  0  1  2 -2]
```

```
[-1  2 -1 -1  2 -2  1 -1  0 -1]
[ 1  0 -4  2  0  1 -2 -1  0  0]
[ 2  3  0  1  1  0 -2  3  0  0]
[-2 -3 -2  0  0  1 -1  1  3 -2]
[-3  0 -1  0 -2 -1 -2  1 -1  1]
[ 1  4 -1  1  2  2  1  0  3  1]
[-1 -1  0 -3 -1  2  2  3 -1  0]
sage: L.shortest_vector()
(0, 1, 2, 0, 1, 2, -1, 0, -1, -1)
```

BKZ (**args*, ***kwds*)

Return a Block Korkine-Zolotareff reduced basis for `self`.

INPUT:

- `*args` – passed through to `sage.matrix.matrix_integer_dense.Matrix_integer_dense.BKZ()`
- `*kwds` – passed through to `sage.matrix.matrix_integer_dense.Matrix_integer_dense.BKZ()`

OUTPUT:

An integer matrix which is a BKZ-reduced basis for this lattice.

EXAMPLES:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: A = sage.crypto.gen_lattice(type='random', n=1, m=60, q=2^60, seed=42)
sage: L = IntegerLattice(A, lll_reduce=False)
sage: min(v.norm().n() for v in L.reduced_basis)
4.17330740711759e15

sage: L.LLL()
60 x 60 dense matrix over Integer Ring (use the '.str()' method to see the entries)

sage: min(v.norm().n() for v in L.reduced_basis)
5.19615242270663

sage: L.BKZ(block_size=10)
60 x 60 dense matrix over Integer Ring (use the '.str()' method to see the entries)

sage: min(v.norm().n() for v in L.reduced_basis)
4.12310562561766
```

Note: If `block_size == L.rank()` where `L` is this latice, then this function performs Hermite-Korkine-Zolotareff (HKZ) reduction.

HKZ (**args*, ***kwds*)

Hermite-Korkine-Zolotarev (HKZ) reduce the basis.

A basis B of a lattice L, with orthogonalized basis B^* such that $B = M \cdot B^*$ is HKZ reduced, if and only if, the following properties are satisfied:

1. The basis B is size-reduced, i.e., all off-diagonal coefficients of M satisfy $|\mu_{i,j}| \leq 1/2$
2. The vector b_1 realizes the first minimum $\lambda_1(L)$.
3. The projection of the vectors $b_2, \ldots, b_r$ orthogonally to b_1 form an HKZ reduced basis.

Note: This is realised by calling `sage.modules.free_module_integer.FreeModule_submodule_with_ba` with `block_size == self.rank()`.

INPUT:

- `*args` – passed through to `BKZ()`
- `*kwds` – passed through to `BKZ()`

OUTPUT:

An integer matrix which is a HKZ-reduced basis for this lattice.

EXAMPLE:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: L = sage.crypto.gen_lattice(type='random', n=1, m=40, q=2^60, seed=42, lattice=True)
sage: L.HKZ()
40 x 40 dense matrix over Integer Ring (use the '.str()' method to see the entries)

sage: L.reduced_basis[0]
(-1, 0, 0, 0, 0, 0, 1, 0, -1, 0, 0, 0, 0, 0, -2, 0, 0, 0, 0, 0,
 -1, 1, 0, 1, 1, 0, 0, 0, 3, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0)
```

LLL(**args*, ***kwds*)

Return an LLL reduced basis for `self`.

A lattice basis $(b_1, b_2, ..., b_d)$ is (δ, η)-LLL-reduced if the two following conditions hold:

- For any $i > j$, we have $|\mu_{i,j}| \leq \eta$.
- For any $i < d$, we have $\delta |b_i^*|^2 \leq |b_{i+1}^* + \mu_{i+1,i} b_i^*|^2$,

where $\mu_{i,j} = \langle b_i, b_j^* \rangle / \langle b_j^*, b_j^* \rangle$ and b_i^* is the i-th vector of the Gram-Schmidt orthogonalisation of $(b_1, b_2, \ldots, b_d)$.

The default reduction parameters are $\delta = 3/4$ and $\eta = 0.501$.

The parameters δ and η must satisfy: $0.25 < \delta \leq 1.0$ and $0.5 \leq \eta < \sqrt{\delta}$. Polynomial time complexity is only guaranteed for $\delta < 1$.

INPUT:

- `*args` – passed through to `sage.matrix.matrix_integer_dense.Matrix_integer_dense.LLL()`
- `**kwds` – passed through to `sage.matrix.matrix_integer_dense.Matrix_integer_dense.LLL()`

OUTPUT:

An integer matrix which is an LLL-reduced basis for this lattice.

EXAMPLES:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: A = random_matrix(ZZ, 10, 10, x=-2000, y=2000)
sage: L = IntegerLattice(A, lll_reduce=False); L
Free module of degree 10 and rank 10 over Integer Ring
User basis matrix:
[ -645 -1037 -1775 -1619  1721 -1434  1766  1701  1669  1534]
[ 1303   960  1998 -1838  1683 -1332   149   327  -849 -1562]
[-1113 -1366  1379   669    54  1214 -1750  -605 -1566  1626]
[-1367  1651   926  1731  -913   627   669 -1437  -132  1712]
[ -549  1327 -1353    68  1479 -1803  -456  1090  -606  -317]
[ -221 -1920 -1361  1695  1139   111 -1792  1925  -656  1992]
[-1934   -29    88   890  1859  1820 -1912 -1614 -1724  1606]
[ -590 -1380  1768   774   656   760  -746  -849  1977 -1576]
[  312  -242 -1732  1594  -439 -1069   458 -1195  1715    35]
[  391  1229 -1815   607  -413  -860  1408  1656  1651  -628]
sage: min(v.norm().n() for v in L.reduced_basis)
```

```
3346.57...

sage: L.LLL()
[ -888    53  -274   243   -19   431   710   -83   928   347]
[  448  -330   370  -511   242  -584    -8  1220   502   183]
[ -524  -460   402  1338  -247  -279 -1038   -28  -159  -794]
[  166  -190  -162  1033  -340   -77 -1052  1134  -843   651]
[  -47 -1394  1076  -132   854  -151   297  -396  -580  -220]
[-1064   373  -706   601  -587 -1394   424   796   -22  -133]
[-1126   398   565 -1418  -446  -890  -237  -378   252   247]
[ -339   799   295   800   425  -605  -730 -1160   808   666]
[  755 -1206  -918  -192 -1063   -37  -525   -75   338   400]
[  382  -199 -1839  -482   984   -15  -695   136   682   563]
sage: L.reduced_basis[0].norm().n()
1613.74...
```

closest_vector(*t*)

Compute the closest vector in the embedded lattice to a given vector.

INPUT:

- `t` – the target vector to compute the closest vector to

OUTPUT:

The vector in the lattice closest to `t`.

EXAMPLES:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: L = IntegerLattice([[1, 0], [0, 1]])
sage: L.closest_vector((-6, 5/3))
(-6, 2)
```

ALGORITHM:

Uses the algorithm from [Mic2010].

REFERENCES:

discriminant()

Return $|\det(G)|$, i.e. the absolute value of the determinant of the Gram matrix $B \cdot B^T$ for any basis B.

OUTPUT:

An integer.

EXAMPLE:

```
sage: L = sage.crypto.gen_lattice(m=10, seed=42, lattice=True)
sage: L.discriminant()
214358881
```

is_unimodular()

Return `True` if this lattice is unimodular.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: L = IntegerLattice([[1, 0], [0, 1]])
```

```
sage: L.is_unimodular()
True
sage: IntegerLattice([[2, 0], [0, 3]]).is_unimodular()
False
```

reduced_basis

This attribute caches the currently best known reduced basis for `self`, where "best" is defined by the Euclidean norm of the first row vector.

EXAMPLE:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: L = IntegerLattice(random_matrix(ZZ, 10, 10), lll_reduce=False)
sage: L.reduced_basis
[   -8     2     0     0     1    -1     2     1   -95    -1]
[   -2   -12     0     0     1    -1     1    -1    -2    -1]
[    4    -4    -6     5     0     0    -2     0     1    -4]
[   -6     1    -1     1     1    -1     1    -1    -3     1]
[    1     0     0    -3     2    -2     0    -2     1     0]
[   -1     1     0     0     1    -1     4    -1     1    -1]
[   14     1    -5     4    -1     0     2     4     1     1]
[   -2    -1     0     4    -3     1    -5     0    -2    -1]
[   -9    -1    -1     3     2     1    -1     1    -2     1]
[   -1     2    -7     1     0     2     3 -1955   -22    -1]

sage: _ = L.LLL()
sage: L.reduced_basis
[   1    0    0   -3    2   -2    0   -2    1    0]
[  -1    1    0    0    1   -1    4   -1    1   -1]
[  -2    0    0    1    0   -2   -1   -3    0   -2]
[  -2   -2    0   -1    3    0   -2    0    2    0]
[   1    1    1    2    3   -2   -2    0    3    1]
[  -4    1   -1    0    1    1    2    2   -3    3]
[   1   -3   -7    2    3   -1    0    0   -1   -1]
[   1   -9    1    3    1   -3    1   -1   -1    0]
[   8    5   19    3   27    6   -3    8  -25  -22]
[ 172  -25   57  248  261  793   76 -839  -41  376]
```

shortest_vector(*update_reduced_basis=True*, *algorithm='fplll'*, **args*, ***kwds*)

Return a shortest vector.

INPUT:

- `update_reduced_basis` – (default: `True`) set this flag if the found vector should be used to improve the basis
- `algorithm` – (default: `"fplll"`) either `"fplll"` or `"pari"`
- `*args` – passed through to underlying implementation
- `**kwds` – passed through to underlying implementation

OUTPUT:

A shortest non-zero vector for this lattice.

EXAMPLES:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: A = sage.crypto.gen_lattice(type='random', n=1, m=30, q=2^40, seed=42)
sage: L = IntegerLattice(A, lll_reduce=False)
sage: min(v.norm().n() for v in L.reduced_basis)
```

```
6.03890756700000e10

sage: L.shortest_vector().norm().n()
3.74165738677394

sage: L = IntegerLattice(A, lll_reduce=False)
sage: min(v.norm().n() for v in L.reduced_basis)
6.03890756700000e10

sage: L.shortest_vector(algorithm="pari").norm().n()
3.74165738677394

sage: L = IntegerLattice(A, lll_reduce=True)
sage: L.shortest_vector(algorithm="pari").norm().n()
3.74165738677394
```

update_reduced_basis(*w*)

Inject the vector w and run LLL to update the basis.

INPUT:

- w – a vector

OUTPUT:

Nothing is returned but the internal state is modified.

EXAMPLE:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: A = sage.crypto.gen_lattice(type='random', n=1, m=30, q=2^40, seed=42)
sage: L = IntegerLattice(A)
sage: B = L.reduced_basis
sage: v = L.shortest_vector(update_reduced_basis=False)
sage: L.update_reduced_basis(v)
sage: bool(L.reduced_basis[0].norm() < B[0].norm())
True
```

volume()

Return $vol(L)$ which is $\sqrt{\det(B \cdot B^T)}$ for any basis B.

OUTPUT:

An integer.

EXAMPLE:

```
sage: L = sage.crypto.gen_lattice(m=10, seed=42, lattice=True)
sage: L.volume()
14641
```

voronoi_cell(*radius=None*)

Compute the Voronoi cell of a lattice, returning a Polyhedron.

INPUT:

- radius – (default: automatic determination) radius of ball containing considered vertices

OUTPUT:

The Voronoi cell as a Polyhedron instance.

The result is cached so that subsequent calls to this function return instantly.

EXAMPLES:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: L = IntegerLattice([[1, 0], [0, 1]])
sage: V = L.voronoi_cell()
sage: V.Vrepresentation()
(A vertex at (1/2, -1/2), A vertex at (1/2, 1/2), A vertex at (-1/2, 1/2), A vertex at (-1/2
```

The volume of the Voronoi cell is the square root of the discriminant of the lattice:

```
sage: L = IntegerLattice(Matrix(ZZ, 4, 4, [[0,0,1,-1],[1,-1,2,1],[-6,0,3,3,],[-6,-24,-6,-5]]
Free module of degree 4 and rank 4 over Integer Ring
User basis matrix:
[  0   0   1  -1]
[  1  -1   2   1]
[ -6   0   3   3]
[ -6 -24  -6  -5]
sage: V = L.voronoi_cell() # long time
sage: V.volume()           # long time
678
sage: sqrt(L.discriminant())
678
```

Lattices not having full dimension are handled as well:

```
sage: L = IntegerLattice([[2, 0, 0], [0, 2, 0]])
sage: V = L.voronoi_cell()
sage: V.Hrepresentation()
(An inequality (-1, 0, 0) x + 1 >= 0, An inequality (0, -1, 0) x + 1 >= 0, An inequality (1,
```

ALGORITHM:

Uses parts of the algorithm from [Vit1996].

REFERENCES:

voronoi_relevant_vectors()

Compute the embedded vectors inducing the Voronoi cell.

OUTPUT:

The list of Voronoi relevant vectors.

EXAMPLES:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: L = IntegerLattice([[3, 0], [4, 0]])
sage: L.voronoi_relevant_vectors()
[(-1, 0), (1, 0)]
```

sage.modules.free_module_integer.**IntegerLattice**(*basis*, *lll_reduce=True*)

Construct a new integer lattice from `basis`.

INPUT:

- `basis` – can be one of the following:
  - a list of vectors
  - a matrix over the integers
  - an element of an absolute order
- `lll_reduce` – (default: `True`) run LLL reduction on the basis on construction.

EXAMPLES:

We construct a lattice from a list of rows:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: IntegerLattice([[1,0,3], [0,2,1], [0,2,7]])
Free module of degree 3 and rank 3 over Integer Ring
User basis matrix:
[-2  0  0]
[ 0  2  1]
[ 1 -2  2]
```

Sage includes a generator for hard lattices from cryptography:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: A = sage.crypto.gen_lattice(type='modular', m=10, seed=42, dual=True)
sage: IntegerLattice(A)
Free module of degree 10 and rank 10 over Integer Ring
User basis matrix:
[ 0  1  2  0  1  2 -1  0 -1 -1]
[ 0  1  0 -3  0  0  0  0  3 -1]
[ 1  1 -1  0 -3  0  0  1  2 -2]
[-1  2 -1 -1  2 -2  1 -1  0 -1]
[ 1  0 -4  2  0  1 -2 -1  0  0]
[ 2  3  0  1  1  0 -2  3  0  0]
[-2 -3 -2  0  0  1 -1  1  3 -2]
[-3  0 -1  0 -2 -1 -2  1 -1  1]
[ 1  4 -1  1  2  2  1  0  3  1]
[-1 -1  0 -3 -1  2  2  3 -1  0]
```

You can also construct the lattice directly:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: sage.crypto.gen_lattice(type='modular', m=10, seed=42, dual=True, lattice=True)
Free module of degree 10 and rank 10 over Integer Ring
User basis matrix:
[ 0  1  2  0  1  2 -1  0 -1 -1]
[ 0  1  0 -3  0  0  0  0  3 -1]
[ 1  1 -1  0 -3  0  0  1  2 -2]
[-1  2 -1 -1  2 -2  1 -1  0 -1]
[ 1  0 -4  2  0  1 -2 -1  0  0]
[ 2  3  0  1  1  0 -2  3  0  0]
[-2 -3 -2  0  0  1 -1  1  3 -2]
[-3  0 -1  0 -2 -1 -2  1 -1  1]
[ 1  4 -1  1  2  2  1  0  3  1]
[-1 -1  0 -3 -1  2  2  3 -1  0]
```

We construct an ideal lattice from an element of an absolute order:

```
sage: K.<a>  = CyclotomicField(17)
sage: O = K.ring_of_integers()
sage: f = O.random_element(); f
-a^15 - a^12 - a^10 - 8*a^9 - a^8 - 4*a^7 + 3*a^6 + a^5 + 2*a^4 + 8*a^3 - a^2 + a + 1

sage: from sage.modules.free_module_integer import IntegerLattice
sage: IntegerLattice(f)
Free module of degree 16 and rank 16 over Integer Ring
User basis matrix:
[ 1  1 -1  8  2  1  3 -4 -1 -8 -1  0 -1  0  0 -1]
[-1  0  1  1 -1  8  2  1  3 -4 -1 -8 -1  0 -1  0]
[ 1  1  0  1  2  2  0  9  3  2  4 -3  0 -7  0  1]
```

```
[ 1  0  1  1  0  1  2  2  0  9  3  2  4 -3  0 -7]
[ 2 -5 -2 -9 -2 -1 -2 -1 -1 -2 -1  0  0 -2  7  1]
[ 1  4  0 -5 -3  0  3  5 -2  2  0 -7  4  0 -6 -2]
[-7  4  0 -6 -2  0  1  4  0 -5 -3  0  3  5 -2  2]
[-1  0  0 -1  0  1  1 -1  8  2  1  3 -4 -1 -8 -1]
[-1 -1 -2  4  1  9  1 -1  0  1 -8 -1 -1 -4  3  2]
[-1 -2  6 -6  8  1 -3  5  3  1  1  0 -2  4  3  2]
[ 4  8  2  7 -3  2 -1  2  0 -4  0  3  6  3  0 -4]
[ 0 -1  0  1  1 -1  8  2  1  3 -4 -1 -8 -1  0 -1]
[ 2 -7 -1  0 -2  5  2  9  2  1  2  1  1  2  1  0]
[-1  7 -5  9  2 -2  6  4  2  2  1 -1  5  4  3  1]
[ 3  1 -6  0  3  1 -5  2  2  8 -4  4 -3  2 -6 -7]
[ 2  6 -4  4  0 -1  7  0 -6  3  9  1 -3 -1  4  3]
```

We construct $\mathbf{Z}^n$:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: IntegerLattice(ZZ^10)
Free module of degree 10 and rank 10 over Integer Ring
User basis matrix:
[1 0 0 0 0 0 0 0 0 0]
[0 1 0 0 0 0 0 0 0 0]
[0 0 1 0 0 0 0 0 0 0]
[0 0 0 1 0 0 0 0 0 0]
[0 0 0 0 1 0 0 0 0 0]
[0 0 0 0 0 1 0 0 0 0]
[0 0 0 0 0 0 1 0 0 0]
[0 0 0 0 0 0 0 1 0 0]
[0 0 0 0 0 0 0 0 1 0]
[0 0 0 0 0 0 0 0 0 1]
```

Sage also interfaces with fpLLL's lattice generator:

```
sage: from sage.modules.free_module_integer import IntegerLattice
sage: from sage.libs.fplll.fplll import gen_simdioph
sage: IntegerLattice(gen_simdioph(8, 20, 10), lll_reduce=False)
Free module of degree 8 and rank 8 over Integer Ring
User basis matrix:
[   1024  829556  161099   11567  521155  769480  639201  689979]
[      0 1048576       0       0       0       0       0       0]
[      0       0 1048576       0       0       0       0       0]
[      0       0       0 1048576       0       0       0       0]
[      0       0       0       0 1048576       0       0       0]
[      0       0       0       0       0 1048576       0       0]
[      0       0       0       0       0       0 1048576       0]
[      0       0       0       0       0       0       0 1048576]
```

CHAPTER

FOUR

ELEMENTS OF FREE MODULES

AUTHORS:

- William Stein
- Josh Kantor
- Thomas Feulner (2012-11): Added `FreeModuleElement.hamming_weight()` and `FreeModuleElement_generic_sparse.hamming_weight()`

TODO: Change to use a get_unsafe / set_unsafe, etc., structure exactly like with matrices, since we'll have to define a bunch of special purpose implementations of vectors easily and systematically.

EXAMPLES: We create a vector space over $\mathbf{Q}$ and a subspace of this space.

```
sage: V = QQ^5
sage: W = V.span([V.1, V.2])
```

Arithmetic operations always return something in the ambient space, since there is a canonical map from W to V but not from V to W.

```
sage: parent(W.0 + V.1)
Vector space of dimension 5 over Rational Field
sage: parent(V.1 + W.0)
Vector space of dimension 5 over Rational Field
sage: W.0 + V.1
(0, 2, 0, 0, 0)
sage: W.0 - V.0
(-1, 1, 0, 0, 0)
```

Next we define modules over $\mathbf{Z}$ and a finite field.

```
sage: K = ZZ^5
sage: M = GF(7)^5
```

Arithmetic between the $\mathbf{Q}$ and $\mathbf{Z}$ modules is defined, and the result is always over $\mathbf{Q}$, since there is a canonical coercion map to $\mathbf{Q}$.

```
sage: K.0 + V.1
(1, 1, 0, 0, 0)
sage: parent(K.0 + V.1)
Vector space of dimension 5 over Rational Field
```

Since there is no canonical coercion map to the finite field from $\mathbf{Q}$ the following arithmetic is not defined:

```
sage: V.0 + M.0
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for '+': 'Vector space of dimension 5 over Rational Field' a
```

However, there is a map from **Z** to the finite field, so the following is defined, and the result is in the finite field.

```
sage: w = K.0 + M.0; w
(2, 0, 0, 0, 0)
sage: parent(w)
Vector space of dimension 5 over Finite Field of size 7
sage: parent(M.0 + K.0)
Vector space of dimension 5 over Finite Field of size 7
```

Matrix vector multiply:

```
sage: MS = MatrixSpace(QQ,3)
sage: A = MS([0,1,0,1,0,0,0,0,1])
sage: V = QQ^3
sage: v = V([1,2,3])
sage: v * A
(2, 1, 3)
```

TESTS:

```
sage: D = 46341
sage: u = 7
sage: R = Integers(D)
sage: p = matrix(R,[[84, 97, 55, 58, 51]])
sage: 2*p.row(0)
(168, 194, 110, 116, 102)
```

class `sage.modules.free_module_element.`**`FreeModuleElement`**

Bases: `sage.structure.element.Vector`

An element of a generic free module.

`Mod`(*p*)

EXAMPLES:

```
sage: V = vector(ZZ, [5, 9, 13, 15])
sage: V.Mod(7)
(5, 2, 6, 1)
sage: parent(V.Mod(7))
Vector space of dimension 4 over Ring of integers modulo 7
```

`additive_order`()

Return the additive order of self.

EXAMPLES:

```
sage: v = vector(Integers(4), [1,2])
sage: v.additive_order()
4

sage: v = vector([1,2,3])
sage: v.additive_order()
+Infinity
```

```
sage: v = vector(Integers(30), [6, 15]); v
(6, 15)
sage: v.additive_order()
10
sage: 10*v
(0, 0)
```

apply_map (*phi*, *R=None*, *sparse=None*)

Apply the given map phi (an arbitrary Python function or callable object) to this free module element. If R is not given, automatically determine the base ring of the resulting element.

INPUT:

sparse – True or False will control whether the result is sparse. By default, the result is sparse iff self is sparse.

- `phi` - arbitrary Python function or callable object
- `R` - (optional) ring

OUTPUT: a free module element over R

EXAMPLES:

```
sage: m = vector([1,x,sin(x+1)])
sage: m.apply_map(lambda x: x^2)
(1, x^2, sin(x + 1)^2)
sage: m.apply_map(sin)
(sin(1), sin(x), sin(sin(x + 1)))
```

```
sage: m = vector(ZZ, 9, range(9))
sage: k.<a> = GF(9)
sage: m.apply_map(k)
(0, 1, 2, 0, 1, 2, 0, 1, 2)
```

In this example, we explicitly specify the codomain.

```
sage: s = GF(3)
sage: f = lambda x: s(x)
sage: n = m.apply_map(f, k); n
(0, 1, 2, 0, 1, 2, 0, 1, 2)
sage: n.parent()
Vector space of dimension 9 over Finite Field in a of size 3^2
```

If your map sends 0 to a non-zero value, then your resulting vector is not mathematically sparse:

```
sage: v = vector([0] * 6 + [1], sparse=True); v
(0, 0, 0, 0, 0, 0, 1)
sage: v2 = v.apply_map(lambda x: x+1); v2
(1, 1, 1, 1, 1, 1, 2)
```

but it's still represented with a sparse data type:

```
sage: parent(v2)
Ambient sparse free module of rank 7 over the principal ideal domain Integer Ring
```

This data type is inefficient for dense vectors, so you may want to specify sparse=False:

```
sage: v2 = v.apply_map(lambda x: x+1, sparse=False); v2
(1, 1, 1, 1, 1, 1, 2)
```

```
sage: parent(v2)
Ambient free module of rank 7 over the principal ideal domain Integer Ring
```

Or if you have a map that will result in mostly zeroes, you may want to specify sparse=True:

```
sage: v = vector(srange(10))
sage: v2 = v.apply_map(lambda x: 0 if x else 1, sparse=True); v2
(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)
sage: parent(v2)
Ambient sparse free module of rank 10 over the principal ideal domain Integer Ring
```

TESTS:

```
sage: m = vector(SR,[])
sage: m.apply_map(lambda x: x*x) == m
True
```

Check that we don't unnecessarily apply phi to 0 in the sparse case:

```
sage: m = vector(ZZ, range(1, 4), sparse=True)
sage: m.apply_map(lambda x: 1/x)
(1, 1/2, 1/3)

sage: parent(vector(RDF, (), sparse=True).apply_map(lambda x: x, sparse=True))
Sparse vector space of dimension 0 over Real Double Field
sage: parent(vector(RDF, (), sparse=True).apply_map(lambda x: x, sparse=False))
Vector space of dimension 0 over Real Double Field
sage: parent(vector(RDF, (), sparse=False).apply_map(lambda x: x, sparse=True))
Sparse vector space of dimension 0 over Real Double Field
sage: parent(vector(RDF, (), sparse=False).apply_map(lambda x: x, sparse=False))
Vector space of dimension 0 over Real Double Field
```

Check that the bug in trac ticket #14558 has been fixed:

```
sage: F.<a> = GF(9)
sage: v = vector([a, 0,0,0], sparse=True)
sage: f = F.hom([a**3])
sage: v.apply_map(f)
(2*a + 1, 0, 0, 0)
```

change_ring(*R*)

Change the base ring of this vector.

EXAMPLES:

```
sage: v = vector(QQ['x,y'], [1..5]); v.change_ring(GF(3))
(1, 2, 0, 1, 2)
```

column()

Return a matrix with a single column and the same entries as the vector `self`.

OUTPUT:

A matrix over the same ring as the vector (or free module element), with a single column. The entries of the column are identical to those of the vector, and in the same order.

EXAMPLES:

```
sage: v = vector(ZZ, [1,2,3])
sage: w = v.column(); w
[1]
```

```
[2]
[3]
sage: w.parent()
Full MatrixSpace of 3 by 1 dense matrices over Integer Ring

sage: x = vector(FiniteField(13), [2,4,8,16])
sage: x.column()
[2]
[4]
[8]
[3]
```

There is more than one way to get one-column matrix from a vector. The `column` method is about equally efficient to making a row and then taking a transpose. Notice that supplying a vector to the matrix constructor demonstrates Sage's preference for rows.

```
sage: x = vector(RDF, [sin(i*pi/20) for i in range(10)])
sage: x.column() == matrix(x).transpose()
True
sage: x.column() == x.row().transpose()
True
```

Sparse or dense implementations are preserved.

```
sage: d = vector(RR, [1.0, 2.0, 3.0])
sage: s = vector(CDF, {2:5.0+6.0*I})
sage: dm = d.column()
sage: sm = s.column()
sage: all([d.is_dense(), dm.is_dense(), s.is_sparse(), sm.is_sparse()])
True
```

TESTS:

The `column()` method will return a specified column of a matrix as a vector. So here are a couple of round-trips.

```
sage: A = matrix(ZZ, [[1],[2],[3]])
sage: A == A.column(0).column()
True
sage: v = vector(ZZ, [4,5,6])
sage: v == v.column().column(0)
True
```

And a very small corner case.

```
sage: v = vector(ZZ, [])
sage: w = v.column()
sage: w.parent()
Full MatrixSpace of 0 by 1 dense matrices over Integer Ring
```

conjugate()

Returns a vector where every entry has been replaced by its complex conjugate.

OUTPUT:

A vector of the same length, over the same ring, but with each entry replaced by the complex conjugate, as implemented by the `conjugate()` method for elements of the base ring, which is presently always complex conjugation.

EXAMPLES:

```
sage: v = vector(CDF, [2.3 - 5.4*I, -1.7 + 3.6*I])
sage: w = v.conjugate(); w
(2.3 + 5.4*I, -1.7 - 3.6*I)
sage: w.parent()
Vector space of dimension 2 over Complex Double Field
```

Even if conjugation seems nonsensical over a certain ring, this method for vectors cooperates silently.

```
sage: u = vector(ZZ, range(6))
sage: u.conjugate()
(0, 1, 2, 3, 4, 5)
```

Sage implements a few specialized subfields of the complex numbers, such as the cyclotomic fields. This example uses such a field containing a primitive 7-th root of unity named `a`.

```
sage: F.<a> = CyclotomicField(7)
sage: v = vector(F, [a^i for i in range(7)])
sage: v
(1, a, a^2, a^3, a^4, a^5, -a^5 - a^4 - a^3 - a^2 - a - 1)
sage: v.conjugate()
(1, -a^5 - a^4 - a^3 - a^2 - a - 1, a^5, a^4, a^3, a^2, a)
```

Sparse vectors are returned as such.

```
sage: v = vector(CC, {1: 5 - 6*I, 3: -7*I}); v
(0.000000000000000, 5.00000000000000 - 6.00000000000000*I, 0.000000000000000, -7.00000000000
sage: v.is_sparse()
True
sage: vc = v.conjugate(); vc
(0.000000000000000, 5.00000000000000 + 6.00000000000000*I, 0.000000000000000, 7.000000000000
sage: vc.conjugate()
(0.000000000000000, 5.00000000000000 - 6.00000000000000*I, 0.000000000000000, -7.00000000000
```

TESTS:

```
sage: n = 15
sage: x = vector(CDF, [sin(i*pi/n)+cos(i*pi/n)*I for i in range(n)])
sage: x + x.conjugate() in RDF^n
True
sage: I*(x - x.conjugate()) in RDF^n
True
```

The parent of the conjugate is the same as that of the original vector. We test this by building a specialized vector space with a non-standard inner product, and constructing a test vector in this space.

```
sage: V = VectorSpace(CDF, 2, inner_product_matrix = [[2,1],[1,5]])
sage: v = vector(CDF, [2-3*I, 4+5*I])
sage: w = V(v)
sage: w.parent()
Ambient quadratic space of dimension 2 over Complex Double Field
Inner product matrix:
[2.0 1.0]
[1.0 5.0]
sage: w.conjugate().parent()
Ambient quadratic space of dimension 2 over Complex Double Field
Inner product matrix:
[2.0 1.0]
[1.0 5.0]
```

coordinate_ring()

Return the ring from which the coefficients of this vector come.

This is different from `base_ring()`, which returns the ring of scalars.

EXAMPLES:

```
sage: M = (ZZ^2) * (1/2)
sage: v = M([0,1/2])
sage: v.base_ring()
Integer Ring
sage: v.coordinate_ring()
Rational Field
```

cross_product (*right*)

Return the cross product of self and right, which is only defined for vectors of length 3 or 7.

INPUT:

- `right` - A vector of the same size as `self`, either degree three or degree seven.

OUTPUT:

The cross product (vector product) of `self` and `right`, a vector of the same size of `self` and `right`.

This product is performed under the assumption that the basis vectors are orthonormal.

EXAMPLES:

```
sage: v = vector([1,2,3]); w = vector([0,5,-9])
sage: v.cross_product(v)
(0, 0, 0)
sage: u = v.cross_product(w); u
(-33, 9, 5)
sage: u.dot_product(v)
0
sage: u.dot_product(w)
0
```

The cross product is defined for degree seven vectors as well. [WIKIPEDIA:CROSSPRODUCT] The 3-D cross product is achieved using the quaternians, whereas the 7-D cross product is achieved using the octions.

```
sage: u = vector(QQ, [1, -1/3, 57, -9, 56/4, -4,1])
sage: v = vector(QQ, [37, 55, -99/57, 9, -12, 11/3, 4/98])
sage: u.cross_product(v)
(1394815/2793, -2808401/2793, 39492/49, -48737/399, -9151880/2793, 62513/2793, -326603/171)
```

The degree seven cross product is anticommutative.

```
sage: u.cross_product(v) + v.cross_product(u)
(0, 0, 0, 0, 0, 0, 0)
```

The degree seven cross product is distributive across addition.

```
sage: v = vector([-12, -8/9, 42, 89, -37, 60/99, 73])
sage: u = vector([31, -42/7, 97, 80, 30/55, -32, 64])
sage: w = vector([-25/4, 40, -89, -91, -72/7, 79, 58])
sage: v.cross_product(u + w) - (v.cross_product(u) + v.cross_product(w))
(0, 0, 0, 0, 0, 0, 0)
```

The degree seven cross product respects scalar multiplication.

```
sage: v = vector([2, 17, -11/5, 21, -6, 2/17, 16])
sage: u = vector([-8, 9, -21, -6, -5/3, 12, 99])
```

```
sage: (5*v).cross_product(u) - 5*(v.cross_product(u))
(0, 0, 0, 0, 0, 0, 0)
sage: v.cross_product(5*u) - 5*(v.cross_product(u))
(0, 0, 0, 0, 0, 0, 0)
sage: (5*v).cross_product(u) - (v.cross_product(5*u))
(0, 0, 0, 0, 0, 0, 0)
```

The degree seven cross product respects the scalar triple product.

```
sage: v = vector([2,6,-7/4,-9/12,-7,12,9])
sage: u = vector([22,-7,-9/11,12,15,15/7,11])
sage: w = vector([-11,17,19,-12/5,44,21/56,-8])
sage: v.dot_product(u.cross_product(w)) - w.dot_product(v.cross_product(u))
0
```

TESTS:

Both vectors need to be of length three or both vectors need to be of length seven.

```
sage: u = vector(range(7))
sage: v = vector(range(3))
sage: u.cross_product(v)
Traceback (most recent call last):
...
ArithmeticError: Cross product only defined for vectors of length three or seven, not (7 and
```

REFERENCES:

AUTHOR:

Billy Wonderly (2010-05-11), Added 7-D Cross Product

curl(*variables=None*)

Return the curl of this two-dimensional or three-dimensional vector function.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: vector([-y, x, 0]).curl()
(0, 0, 2)
sage: vector([y, -x, x*y*z]).curl()
(x*z, -y*z, -2)
sage: vector([y^2, 0, 0]).curl()
(0, 0, -2*y)
sage: (R^3).random_element().curl().div()
0
```

For rings where the variable order is not well defined, it must be defined explicitly:

```
sage: v = vector(SR, [-y, x, 0])
sage: v.curl()
Traceback (most recent call last):
...
ValueError: Unable to determine ordered variable names for Symbolic Ring
sage: v.curl([x, y, z])
(0, 0, 2)
```

Note that callable vectors have well defined variable orderings:

```
sage: v(x, y, z) = (-y, x, 0)
sage: v.curl()
(x, y, z) |--> (0, 0, 2)
```

In two-dimensions, this returns a scalar value:

```
sage: R.<x,y> = QQ[]
sage: vector([-y, x]).curl()
2
```

degree()

Return the degree of this vector, which is simply the number of entries.

EXAMPLES:

```
sage: sage.modules.free_module_element.FreeModuleElement(QQ^389).degree()
389
sage: vector([1,2/3,8]).degree()
3
```

denominator()

Return the least common multiple of the denominators of the entries of self.

EXAMPLES:

```
sage: v = vector([1/2,2/5,3/14])
sage: v.denominator()
70
sage: 2*5*7
70

sage: M = (ZZ^2)*(1/2)
sage: M.basis()[0].denominator()
2
```

TESTS:

The following was fixed in trac ticket #8800:

```
sage: M = GF(5)^3
sage: v = M((4,0,2))
sage: v.denominator()
1
```

dense_vector()

Return dense version of self. If self is dense, just return self; otherwise, create and return correspond dense vector.

EXAMPLES:

```
sage: vector([-1,0,3,0,0,0]).dense_vector().is_dense()
True
sage: vector([-1,0,3,0,0,0],sparse=True).dense_vector().is_dense()
True
sage: vector([-1,0,3,0,0,0],sparse=True).dense_vector()
(-1, 0, 3, 0, 0, 0)
```

derivative(**args*)

Derivative with respect to variables supplied in args.

Multiple variables and iteration counts may be supplied; see documentation for the global derivative() function for more details.

`diff()` is an alias of this function.

EXAMPLES:

```
sage: v = vector([1,x,x^2])
sage: v.derivative(x)
(0, 1, 2*x)
sage: type(v.derivative(x)) == type(v)
True
sage: v = vector([1,x,x^2], sparse=True)
sage: v.derivative(x)
(0, 1, 2*x)
sage: type(v.derivative(x)) == type(v)
True
sage: v.derivative(x,x)
(0, 0, 2)
```

dict (*copy=True*)

Return dictionary of nonzero entries of self.

INPUT:

- `copy` – bool (default: True)

OUTPUT:

- Python dictionary

EXAMPLES:

```
sage: v = vector([0,0,0,0,1/2,0,3/14])
sage: v.dict()
{4: 1/2, 6: 3/14}
```

In some cases when copy=False, we get back a dangerous reference:

```
sage: v = vector({0:5, 2:3/7}, sparse=True)
sage: v.dict(copy=False)
{0: 5, 2: 3/7}
sage: v.dict(copy=False)[0] = 18
sage: v
(18, 0, 3/7)
```

diff (**args*)

Derivative with respect to variables supplied in args.

Multiple variables and iteration counts may be supplied; see documentation for the global derivative() function for more details.

`diff()` is an alias of this function.

EXAMPLES:

```
sage: v = vector([1,x,x^2])
sage: v.derivative(x)
(0, 1, 2*x)
sage: type(v.derivative(x)) == type(v)
True
sage: v = vector([1,x,x^2], sparse=True)
sage: v.derivative(x)
(0, 1, 2*x)
sage: type(v.derivative(x)) == type(v)
True
sage: v.derivative(x,x)
(0, 0, 2)
```

div (*variables=None*)

Return the divergence of this vector function.

EXAMPLES:

```
sage: R.<x,y,z> = QQ[]
sage: vector([x, y, z]).div()
3
sage: vector([x*y, y*z, z*x]).div()
x + y + z

sage: R.<x,y,z,w> = QQ[]
sage: vector([x*y, y*z, z*x]).div([x, y, z])
x + y + z
sage: vector([x*y, y*z, z*x]).div([z, x, y])
0
sage: vector([x*y, y*z, z*x]).div([x, y, w])
y + z

sage: vector(SR, [x*y, y*z, z*x]).div()
Traceback (most recent call last):
...
ValueError: Unable to determine ordered variable names for Symbolic Ring
sage: vector(SR, [x*y, y*z, z*x]).div([x, y, z])
x + y + z
```

dot_product (*right*)

Return the dot product of `self` and `right`, which is the sum of the product of the corresponding entries.

INPUT:

- `right` – a vector of the same degree as `self`. It does not need to belong to the same parent as `self`, so long as the necessary products and sums are defined.

OUTPUT:

If `self` and `right` are the vectors $\vec{x}$ and $\vec{y}$, of degree n, then this method returns

$$\sum_{i=1}^{n} x_i y_i$$

Note: The `inner_product()` is a more general version of this method, and the `hermitian_inner_product()` method may be more appropriate if your vectors have complex entries.

EXAMPLES:

```
sage: V = FreeModule(ZZ, 3)
sage: v = V([1,2,3])
sage: w = V([4,5,6])
sage: v.dot_product(w)
32

sage: R.<x> = QQ[]
sage: v = vector([x,x^2,3*x]); w = vector([2*x,x,3+x])
sage: v*w
x^3 + 5*x^2 + 9*x
sage: (x*2*x) + (x^2*x) + (3*x*(3+x))
x^3 + 5*x^2 + 9*x
```

```
sage: w*v
x^3 + 5*x^2 + 9*x
```

The vectors may be from different vector spaces, provided the necessary operations make sense. Notice that coercion will generate a result of the same type, even if the order of the arguments is reversed.:

```
sage: v = vector(ZZ, [1,2,3])
sage: w = vector(FiniteField(3), [0,1,2])
sage: ip = w.dot_product(v); ip
2
sage: ip.parent()
Finite Field of size 3

sage: ip = v.dot_product(w); ip
2
sage: ip.parent()
Finite Field of size 3
```

The dot product of a vector with itself is the 2-norm, squared.

```
sage: v = vector(QQ, [3, 4, 7])
sage: v.dot_product(v) - v.norm()^2
0
```

TESTS:

The second argument must be a free module element.

```
sage: v = vector(QQ, [1,2])
sage: v.dot_product('junk')
Traceback (most recent call last):
...
TypeError: Cannot convert str to sage.modules.free_module_element.FreeModuleElement
```

The degrees of the arguments must match.

```
sage: v = vector(QQ, [1,2])
sage: w = vector(QQ, [1,2,3])
sage: v.dot_product(w)
Traceback (most recent call last):
...
ArithmeticError: degrees (2 and 3) must be the same
```

Check that vectors with different base rings play out nicely (trac ticket #3103):

```
sage: vector(CDF, [2, 2]) * vector(ZZ, [1, 3])
8.0
```

Zero-dimensional vectors work:

```
sage: v = vector(ZZ, [])
sage: v.dot_product(v)
0
```

element()

Simply returns self. This is useful, since for many objects, self.element() returns a vector corresponding to self.

EXAMPLES:

```
sage: v = vector([1/2,2/5,0]); v
(1/2, 2/5, 0)
sage: v.element()
(1/2, 2/5, 0)
```

get (*i*)

The get method is in some cases more efficient (and more dangerous) than __getitem__, because it is not guaranteed to do any error checking.

EXAMPLES:

```
sage: vector([1/2,2/5,0]).get(0)
1/2
sage: vector([1/2,2/5,0]).get(3)
Traceback (most recent call last):
...
IndexError: index out of range
```

hamming_weight ()

Return the number of positions `i` such that `self[i] != 0`.

EXAMPLES:

```
sage: vector([-1,0,3,0,0,0,0.01]).hamming_weight()
3
```

hermitian_inner_product (*right*)

Returns the dot product, but with the entries of the first vector conjugated beforehand.

INPUT:

- `right` - a vector of the same degree as `self`

OUTPUT:

If `self` and `right` are the vectors $\vec{x}$ and $\vec{y}$ of degree n then this routine computes

$$\sum_{i=1}^{n} \overline{x}_i y_i$$

where the bar indicates complex conjugation.

Note: If your vectors do not contain complex entries, then `dot_product()` will return the same result without the overhead of conjugating elements of `self`.

If you are not computing a weighted inner product, and your vectors do not have complex entries, then the `dot_product()` will return the same result.

EXAMPLES:

```
sage: v = vector(CDF, [2+3*I, 5-4*I])
sage: w = vector(CDF, [6-4*I, 2+3*I])
sage: v.hermitian_inner_product(w)
-2.0 - 3.0*I
```

Sage implements a few specialized fields over the complex numbers, such as cyclotomic fields and quadratic number fields. So long as the base rings have a conjugate method, then the Hermitian inner product will be available.

```
sage: Q.<a> = QuadraticField(-7)
sage: a^2
-7
sage: v = vector(Q, [3+a, 5-2*a])
sage: w = vector(Q, [6, 4+3*a])
sage: v.hermitian_inner_product(w)
17*a - 4
```

The Hermitian inner product should be additive in each argument (we only need to test one), linear in each argument (with conjugation on the first scalar), and anti-commutative.

```
sage: alpha = CDF(5.0 + 3.0*I)
sage: u = vector(CDF, [2+4*I, -3+5*I, 2-7*I])
sage: v = vector(CDF, [-1+3*I, 5+4*I, 9-2*I])
sage: w = vector(CDF, [8+3*I, -4+7*I, 3-6*I])
sage: (u+v).hermitian_inner_product(w) == u.hermitian_inner_product(w) + v.hermitian_inner_p
True
sage: (alpha*u).hermitian_inner_product(w) == alpha.conjugate()*u.hermitian_inner_product(w)
True
sage: u.hermitian_inner_product(alpha*w) == alpha*u.hermitian_inner_product(w)
True
sage: u.hermitian_inner_product(v) == v.hermitian_inner_product(u).conjugate()
True
```

For vectors with complex entries, the Hermitian inner product has a more natural relationship with the 2-norm (which is the default for the `norm()` method). The norm squared equals the Hermitian inner product of the vector with itself.

```
sage: v = vector(CDF, [-0.66+0.47*I, -0.60+0.91*I, -0.62-0.87*I, 0.53+0.32*I])
sage: abs(v.norm()^2 - v.hermitian_inner_product(v)) < 1.0e-10
True
```

TESTS:

This method is built on the `dot_product()` method, which allows for a wide variety of inputs. Any error handling happens there.

```
sage: v = vector(CDF, [2+3*I])
sage: w = vector(CDF, [5+2*I, 3+9*I])
sage: v.hermitian_inner_product(w)
Traceback (most recent call last):
...
ArithmeticError: degrees (1 and 2) must be the same
```

inner_product(*right*)

Returns the inner product of `self` and `right`, possibly using an inner product matrix from the parent of `self`.

INPUT:

- `right` - a vector of the same degree as `self`

OUTPUT:

If the parent vector space does not have an inner product matrix defined, then this is the usual dot product (`dot_product()`). If `self` and `right` are considered as single column matrices, $\vec{x}$ and $\vec{y}$, and A is the inner product matrix, then this method computes

$$(\vec{x})^t A\vec{y}$$

where t indicates the transpose.

Note: If your vectors have complex entries, the `hermitian_inner_product()` may be more appropriate for your purposes.

EXAMPLES:

```
sage: v = vector(QQ, [1,2,3])
sage: w = vector(QQ, [-1,2,-3])
sage: v.inner_product(w)
-6
sage: v.inner_product(w) == v.dot_product(w)
True
```

The vector space or free module that is the parent to `self` can have an inner product matrix defined, which will be used by this method. This matrix will be passed through to subspaces.

```
sage: ipm = matrix(ZZ,[[2,0,-1], [0,2,0], [-1,0,6]])
sage: M = FreeModule(ZZ, 3, inner_product_matrix = ipm)
sage: v = M([1,0,0])
sage: v.inner_product(v)
2
sage: K = M.span_of_basis([[0/2,-1/2,-1/2], [0,1/2,-1/2],[2,0,0]])
sage: (K.0).inner_product(K.0)
2
sage: w = M([1,3,-1])
sage: v = M([2,-4,5])
sage: w.row()*ipm*v.column() == w.inner_product(v)
True
```

Note that the inner product matrix comes from the parent of `self`. So if a vector is not an element of the correct parent, the result could be a source of confusion.

```
sage: V = VectorSpace(QQ, 2, inner_product_matrix=[[1,2],[2,1]])
sage: v = V([12, -10])
sage: w = vector(QQ, [10,12])
sage: v.inner_product(w)
88
sage: w.inner_product(v)
0
sage: w = V(w)
sage: w.inner_product(v)
88
```

Note: The use of an inner product matrix makes no restrictions on the nature of the matrix. In particular, in this context it need not be Hermitian and positive-definite (as it is in the example above).

TESTS:

Most error handling occurs in the `dot_product()` method. But with an inner product defined, this method will check that the input is a vector or free module element.

```
sage: W = VectorSpace(RDF, 2, inner_product_matrix = matrix(RDF, 2, [1.0,2.0,3.0,4.0]))
sage: v = W([2.0, 4.0])
sage: v.inner_product(5)
Traceback (most recent call last):
...
TypeError: right must be a free module element
```

integral(**args*, ***kwds*)

Returns a symbolic integral of the vector, component-wise.

`integrate()` is an alias of the function.

EXAMPLES:

```
sage: t=var('t')
sage: r=vector([t,t^2,sin(t)])
sage: r.integral(t)
(1/2*t^2, 1/3*t^3, -cos(t))
sage: integrate(r,t)
(1/2*t^2, 1/3*t^3, -cos(t))
sage: r.integrate(t,0,1)
(1/2, 1/3, -cos(1) + 1)
```

integrate(**args*, ***kwds*)

Returns a symbolic integral of the vector, component-wise.

`integrate()` is an alias of the function.

EXAMPLES:

```
sage: t=var('t')
sage: r=vector([t,t^2,sin(t)])
sage: r.integral(t)
(1/2*t^2, 1/3*t^3, -cos(t))
sage: integrate(r,t)
(1/2*t^2, 1/3*t^3, -cos(t))
sage: r.integrate(t,0,1)
(1/2, 1/3, -cos(1) + 1)
```

is_dense()

Return True if this is a dense vector, which is just a statement about the data structure, not the number of nonzero entries.

EXAMPLES:

```
sage: vector([1/2,2/5,0]).is_dense()
True
sage: vector([1/2,2/5,0],sparse=True).is_dense()
False
```

is_immutable()

Return True if this vector is immutable, i.e., the entries cannot be changed.

EXAMPLES:

```
sage: v = vector(QQ['x,y'], [1..5]); v.is_immutable()
False
sage: v.set_immutable()
sage: v.is_immutable()
True
```

is_mutable()

Return True if this vector is mutable, i.e., the entries can be changed.

EXAMPLES:

```
sage: v = vector(QQ['x,y'], [1..5]); v.is_mutable()
True
sage: v.set_immutable()
sage: v.is_mutable()
False
```

is_sparse()

Return True if this is a sparse vector, which is just a statement about the data structure, not the number of nonzero entries.

EXAMPLES:

```
sage: vector([1/2,2/5,0]).is_sparse()
False
sage: vector([1/2,2/5,0],sparse=True).is_sparse()
True
```

is_vector()

Return True, since this is a vector.

EXAMPLES:

```
sage: vector([1/2,2/5,0]).is_vector()
True
```

iteritems()

Return iterator over self.

EXAMPLES:

```
sage: v = vector([1,2/3,pi])
sage: v.iteritems()
<dictionary-itemiterator object at ...>
sage: list(v.iteritems())
[(0, 1), (1, 2/3), (2, pi)]
```

lift()

EXAMPLES:

```
sage: V = vector(Integers(7), [5, 9, 13, 15]) ; V
(5, 2, 6, 1)
sage: V.lift()
(5, 2, 6, 1)
sage: parent(V.lift())
Ambient free module of rank 4 over the principal ideal domain Integer Ring
```

list(*copy=True*)

Return list of elements of self.

INPUT:

- copy – bool, whether returned list is a copy that is safe to change, is ignored.

EXAMPLES:

```
sage: P.<x,y,z> = QQ[]
sage: v = vector([x,y,z], sparse=True)
sage: type(v)
<type 'sage.modules.free_module_element.FreeModuleElement_generic_sparse'>
sage: a = v.list(); a
[x, y, z]
sage: a[0] = x*y; v
(x, y, z)
```

The optional argument `copy` is ignored:

```
sage: a = v.list(copy=False); a
[x, y, z]
```

```
sage: a[0] = x*y; v
(x, y, z)
```

list_from_positions (*positions*)

Return list of elements chosen from this vector using the given positions of this vector.

INPUT:

- positions – iterable of ints

EXAMPLES:

```
sage: v = vector([1,2/3,pi])
sage: v.list_from_positions([0,0,0,2,1])
[1, 1, 1, pi, 2/3]
```

monic ()

Return this vector divided through by the first nonzero entry of this vector.

EXAMPLES:

```
sage: v = vector(QQ, [0, 4/3, 5, 1, 2])
sage: v.monic()
(0, 1, 15/4, 3/4, 3/2)
sage: v = vector(QQ, [])
sage: v.monic()
()
```

nintegral (**args*, ***kwds*)

Returns a numeric integral of the vector, component-wise, and the result of the nintegral command on each component of the input.

`nintegrate()` is an alias of the function.

EXAMPLES:

```
sage: t=var('t')
sage: r=vector([t,t^2,sin(t)])
sage: vec,answers=r.nintegral(t,0,1)
sage: vec
(0.5, 0.3333333333333334, 0.4596976941318602)
sage: type(vec)
<type 'sage.modules.vector_real_double_dense.Vector_real_double_dense'>
sage: answers
[(0.5, 5.551115123125784e-15, 21, 0), (0.3333333333333..., 3.70074341541719e-15, 21, 0), (0.

sage: r=vector([t,0,1], sparse=True)
sage: r.nintegral(t,0,1)
((0.5, 0.0, 1.0), {0: (0.5, 5.551115123125784e-15, 21, 0), 2: (1.0, 1.11022302462515...e-14,
```

nintegrate (**args*, ***kwds*)

Returns a numeric integral of the vector, component-wise, and the result of the nintegral command on each component of the input.

`nintegrate()` is an alias of the function.

EXAMPLES:

```
sage: t=var('t')
sage: r=vector([t,t^2,sin(t)])
sage: vec,answers=r.nintegral(t,0,1)
sage: vec
```

```
(0.5, 0.3333333333333334, 0.4596976941318602)
sage: type(vec)
<type 'sage.modules.vector_real_double_dense.Vector_real_double_dense'>
sage: answers
[(0.5, 5.551115123125784e-15, 21, 0), (0.3333333333333..., 3.70074341541719e-15, 21, 0), (0.

sage: r=vector([t,0,1], sparse=True)
sage: r.nintegral(t,0,1)
((0.5, 0.0, 1.0), {0: (0.5, 5.551115123125784e-15, 21, 0), 2: (1.0, 1.11022302462515...e-14,
```

nonzero_positions()

Return the sorted list of integers i such that `self[i] != 0`.

EXAMPLES:

```
sage: vector([-1,0,3,0,0,0,0.01]).nonzero_positions()
[0, 2, 6]
```

norm(*p='__two__'*)

Return the p-norm of `self`.

INPUT:

- p - default: 2 - p can be a real number greater than 1, infinity (`oo` or `Infinity`), or a symbolic expression.
  - $p = 1$: the taxicab (Manhattan) norm
  - $p = 2$: the usual Euclidean norm (the default)
  - $p = \infty$: the maximum entry (in absolute value)

Note: See also `sage.misc.functional.norm()`

EXAMPLES:

```
sage: v = vector([1,2,-3])
sage: v.norm(5)
276^(1/5)
```

The default is the usual Euclidean norm.

```
sage: v.norm()
sqrt(14)
sage: v.norm(2)
sqrt(14)
```

The infinity norm is the maximum size (in absolute value) of the entries.

```
sage: v.norm(Infinity)
3
sage: v.norm(oo)
3
```

Real or symbolic values may be used for p.

```
sage: v=vector(RDF,[1,2,3])
sage: v.norm(5)
3.077384885394063
sage: v.norm(pi/2)
4.216595864704748
```

```
sage: _=var('a b c d p'); v=vector([a, b, c, d])
sage: v.norm(p)
(abs(a)^p + abs(b)^p + abs(c)^p + abs(d)^p)^(1/p)
```

Notice that the result may be a symbolic expression, owing to the necessity of taking a square root (in the default case). These results can be converted to numerical values if needed.

```
sage: v = vector(ZZ, [3,4])
sage: nrm = v.norm(); nrm
5
sage: nrm.parent()
Rational Field

sage: v = vector(QQ, [3, 5])
sage: nrm = v.norm(); nrm
sqrt(34)
sage: nrm.parent()
Symbolic Ring
sage: numeric = N(nrm); numeric
5.83095189484...
sage: numeric.parent()
Real Field with 53 bits of precision
```

TESTS:

The value of p must be greater than, or equal to, one.

```
sage: v = vector(QQ, [1,2])
sage: v.norm(0.99)
Traceback (most recent call last):
...
ValueError: 0.990000000000000 is not greater than or equal to 1
```

Norm works with Python integers (see trac ticket #13502).

```
sage: v = vector(QQ, [1,2])
sage: v.norm(int(2))
sqrt(5)
```

normalized(*p='__two__'*)

Return the input vector divided by the p-norm.

INPUT:

- "p" - default: 2 - p value for the norm

EXAMPLES:

```
sage: v = vector(QQ, [4, 1, 3, 2])
sage: v.normalized()
(2/15*sqrt(30), 1/30*sqrt(30), 1/10*sqrt(30), 1/15*sqrt(30))
sage: sum(v.normalized(1))
1
```

Note that normalizing the vector may change the base ring:

```
sage: v.base_ring() == v.normalized().base_ring()
False
sage: u = vector(RDF, [-3, 4, 6, 9])
sage: u.base_ring() == u.normalized().base_ring()
True
```

numpy (*dtype='object'*)

Converts self to a numpy array.

INPUT:

- **dtype – the numpy dtype** of the returned array

EXAMPLES:

```
sage: v = vector([1,2,3])
sage: v.numpy()
array([1, 2, 3], dtype=object)
sage: v.numpy() * v.numpy()
array([1, 4, 9], dtype=object)

sage: vector(QQ, [1, 2, 5/6]).numpy()
array([1, 2, 5/6], dtype=object)
```

By default the `object` dtype is used. Alternatively, the desired dtype can be passed in as a parameter:

```
sage: v = vector(QQ, [1, 2, 5/6])
sage: v.numpy()
array([1, 2, 5/6], dtype=object)
sage: v.numpy(dtype=float)
array([ 1.        ,  2.        ,  0.83333333])
sage: v.numpy(dtype=int)
array([1, 2, 0])
sage: import numpy
sage: v.numpy(dtype=numpy.uint8)
array([1, 2, 0], dtype=uint8)
```

Passing a dtype of None will let numpy choose a native type, which can be more efficient but may have unintended consequences:

```
sage: v.numpy(dtype=None)
array([ 1.        ,  2.        ,  0.83333333])

sage: w = vector(ZZ, [0, 1, 2^63 -1]); w
(0, 1, 9223372036854775807)
sage: wn = w.numpy(dtype=None); wn
array([                  0,                    1, 9223372036854775807]...)
sage: wn.dtype
dtype('int64')
sage: w.dot_product(w)
85070591730234615847396907784232501250
sage: wn.dot(wn)          # overflow
2
```

Numpy can give rather obscure errors; we wrap these to give a bit of context:

```
sage: vector([1, 1/2, QQ['x'].0]).numpy(dtype=float)
Traceback (most recent call last):
...
ValueError: Could not convert vector over Univariate Polynomial Ring in x over Rational Fiel
```

outer_product (*right*)

Returns a matrix, the outer product of two vectors `self` and `right`.

INPUT:

- `right` - a vector (or free module element) of any size, whose elements are compatible (with regard to multiplication) with the elements of `self`.

OUTPUT:

The outer product of two vectors x and y (respectively `self` and `right`) can be described several ways. If we interpret x as a $m \times 1$ matrix and interpret y as a $1 \times n$ matrix, then the outer product is the $m \times n$ matrix from the usual matrix product xy. Notice how this is the "opposite" in some ways from an inner product (which would require $m = n$).

If we just consider vectors, use each entry of x to create a scalar multiples of the vector y and use these vectors as the rows of a matrix. Or use each entry of y to create a scalar multiples of x and use these vectors as the columns of a matrix.

EXAMPLES:

```
sage: u = vector(QQ, [1/2, 1/3, 1/4, 1/5])
sage: v = vector(ZZ, [60, 180, 600])
sage: u.outer_product(v)
[ 30  90 300]
[ 20  60 200]
[ 15  45 150]
[ 12  36 120]
sage: M = v.outer_product(u); M
[ 30  20  15  12]
[ 90  60  45  36]
[300 200 150 120]
sage: M.parent()
Full MatrixSpace of 3 by 4 dense matrices over Rational Field
```

The more general `sage.matrix.matrix2.tensor_product()` is an operation on a pair of matrices. If we construe a pair of vectors as a column vector and a row vector, then an outer product and a tensor product are identical. Thus $tensor_product$ is a synonym for this method.

```
sage: u = vector(QQ, [1/2, 1/3, 1/4, 1/5])
sage: v = vector(ZZ, [60, 180, 600])
sage: u.tensor_product(v) == (u.column()).tensor_product(v.row())
True
```

The result is always a dense matrix, no matter if the two vectors are, or are not, dense.

```
sage: d = vector(ZZ,[4,5], sparse=False)
sage: s = vector(ZZ, [1,2,3], sparse=True)
sage: dd = d.outer_product(d)
sage: ds = d.outer_product(s)
sage: sd = s.outer_product(d)
sage: ss = s.outer_product(s)
sage: all([dd.is_dense(), ds.is_dense(), sd.is_dense(), dd.is_dense()])
True
```

Vectors with no entries do the right thing.

```
sage: v = vector(ZZ, [])
sage: z = v.outer_product(v)
sage: z.parent()
Full MatrixSpace of 0 by 0 dense matrices over Integer Ring
```

There is a fair amount of latitude in the value of the `right` vector, and the matrix that results can have entries from a new ring large enough to contain the result. If you know better, you can sometimes bring the result down to a less general ring.

```
sage: R.<t> = ZZ[]
sage: v = vector(R, [12, 24*t])
sage: w = vector(QQ, [1/2, 1/3, 1/4])
```

```
sage: op = v.outer_product(w)
sage: op
[   6    4    3]
[12*t  8*t  6*t]
sage: op.base_ring()
Univariate Polynomial Ring in t over Rational Field
sage: m = op.change_ring(R); m
[   6    4    3]
[12*t  8*t  6*t]
sage: m.base_ring()
Univariate Polynomial Ring in t over Integer Ring
```

But some inputs are not compatible, even if vectors.

```
sage: w = vector(GF(5), [1,2])
sage: v = vector(GF(7), [1,2,3,4])
sage: z = w.outer_product(v)
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for '*': 'Full MatrixSpace of 2 by 1 dense matrices
```

And some inputs don't make any sense at all.

```
sage: w=vector(QQ, [5,10])
sage: z=w.outer_product(6)
Traceback (most recent call last):
...
TypeError: right operand in an outer product must be a vector, not an element of Integer Rin
```

pairwise_product (*right*)

Return the pairwise product of self and right, which is a vector of the products of the corresponding entries.

INPUT:

- `right` - vector of the same degree as self. It need not be in the same vector space as self, as long as the coefficients can be multiplied.

EXAMPLES:

```
sage: V = FreeModule(ZZ, 3)
sage: v = V([1,2,3])
sage: w = V([4,5,6])
sage: v.pairwise_product(w)
(4, 10, 18)
sage: sum(v.pairwise_product(w)) == v.dot_product(w)
True
```

```
sage: W = VectorSpace(GF(3),3)
sage: w = W([0,1,2])
sage: w.pairwise_product(v)
(0, 2, 0)
sage: w.pairwise_product(v).parent()
Vector space of dimension 3 over Finite Field of size 3
```

Implicit coercion is well defined (regardless of order), so we get 2 even if we do the dot product in the other order.

```
sage: v.pairwise_product(w).parent()
Vector space of dimension 3 over Finite Field of size 3
```

TESTS:

```
sage: x, y = var('x, y')

sage: parent(vector(ZZ,[1,2]).pairwise_product(vector(ZZ,[1,2])))
Ambient free module of rank 2 over the principal ideal domain Integer Ring
sage: parent(vector(ZZ,[1,2]).pairwise_product(vector(QQ,[1,2])))
Vector space of dimension 2 over Rational Field
sage: parent(vector(QQ,[1,2]).pairwise_product(vector(ZZ,[1,2])))
Vector space of dimension 2 over Rational Field
sage: parent(vector(QQ,[1,2]).pairwise_product(vector(QQ,[1,2])))
Vector space of dimension 2 over Rational Field

sage: parent(vector(QQ,[1,2,3,4]).pairwise_product(vector(ZZ['x'],[1,2,3,4])))
Ambient free module of rank 4 over the principal ideal domain Univariate Polynomial Ring in
sage: parent(vector(ZZ[x],[1,2,3,4]).pairwise_product(vector(QQ,[1,2,3,4])))
Ambient free module of rank 4 over the principal ideal domain Univariate Polynomial Ring in

sage: parent(vector(QQ,[1,2,3,4]).pairwise_product(vector(ZZ['x']['y'],[1,2,3,4])))
Ambient free module of rank 4 over the integral domain Univariate Polynomial Ring in y over
sage: parent(vector(ZZ[x][y],[1,2,3,4]).pairwise_product(vector(QQ,[1,2,3,4])))
Ambient free module of rank 4 over the integral domain Univariate Polynomial Ring in y over

sage: parent(vector(QQ['x'],[1,2,3,4]).pairwise_product(vector(ZZ['x']['y'],[1,2,3,4])))
Ambient free module of rank 4 over the integral domain Univariate Polynomial Ring in y over
sage: parent(vector(ZZ[x][y],[1,2,3,4]).pairwise_product(vector(QQ['x'],[1,2,3,4])))
Ambient free module of rank 4 over the integral domain Univariate Polynomial Ring in y over

sage: parent(vector(QQ['y'],[1,2,3,4]).pairwise_product(vector(ZZ['x']['y'],[1,2,3,4])))
Ambient free module of rank 4 over the integral domain Univariate Polynomial Ring in y over
sage: parent(vector(ZZ[x][y],[1,2,3,4]).pairwise_product(vector(QQ['y'],[1,2,3,4])))
Ambient free module of rank 4 over the integral domain Univariate Polynomial Ring in y over

sage: parent(vector(ZZ['x'],[1,2,3,4]).pairwise_product(vector(ZZ['y'],[1,2,3,4])))
Traceback (most recent call last):
...
TypeError: no common canonical parent for objects with parents: 'Ambient free module of rank
sage: parent(vector(ZZ['x'],[1,2,3,4]).pairwise_product(vector(QQ['y'],[1,2,3,4])))
Traceback (most recent call last):
...
TypeError: no common canonical parent for objects with parents: 'Ambient free module of rank
sage: parent(vector(QQ['x'],[1,2,3,4]).pairwise_product(vector(ZZ['y'],[1,2,3,4])))
Traceback (most recent call last):
...
TypeError: no common canonical parent for objects with parents: 'Ambient free module of rank
sage: parent(vector(QQ['x'],[1,2,3,4]).pairwise_product(vector(QQ['y'],[1,2,3,4])))
Traceback (most recent call last):
...
TypeError: no common canonical parent for objects with parents: 'Ambient free module of rank
sage: v = vector({1: 1, 3: 2})  # test sparse vectors
sage: w = vector({0: 6, 3: -4})
sage: v.pairwise_product(w)
(0, 0, 0, -8)
sage: w.pairwise_product(v) == v.pairwise_product(w)
True
```

plot(*plot_type=None*, *start=None*, ***kwds*)

INPUT:

- **plot_type - (default: 'arrow' if v has 3 or fewer components,** otherwise 'step') type of plot.

Options are:

– 'arrow' to draw an arrow

– 'point' to draw a point at the coordinates specified by the vector

– 'step' to draw a step function representing the coordinates of the vector.

Both 'arrow' and 'point' raise exceptions if the vector has more than 3 dimensions.

•start - (default: origin in correct dimension) may be a tuple, list, or vector.

EXAMPLES:

The following both plot the given vector:

```
sage: v = vector(RDF, (1,2))
sage: A = plot(v)
sage: B = v.plot()
sage: A+B # should just show one vector
Graphics object consisting of 2 graphics primitives
```

Examples of the plot types:

```
sage: A = plot(v, plot_type='arrow')
sage: B = plot(v, plot_type='point', color='green', size=20)
sage: C = plot(v, plot_type='step') # calls v.plot_step()
sage: A+B+C
Graphics object consisting of 3 graphics primitives
```

You can use the optional arguments for plot_step():

```
sage: eps = 0.1
sage: plot(v, plot_type='step', eps=eps, xmax=5, hue=0)
Graphics object consisting of 1 graphics primitive
```

Three-dimensional examples:

```
sage: v = vector(RDF, (1,2,1))
sage: plot(v) # defaults to an arrow plot
Graphics3d Object

sage: plot(v, plot_type='arrow')
Graphics3d Object

sage: from sage.plot.plot3d.shapes2 import frame3d
sage: plot(v, plot_type='point')+frame3d((0,0,0), v.list())
Graphics3d Object

sage: plot(v, plot_type='step') # calls v.plot_step()
Graphics object consisting of 1 graphics primitive

sage: plot(v, plot_type='step', eps=eps, xmax=5, hue=0)
Graphics object consisting of 1 graphics primitive
```

With greater than three coordinates, it defaults to a step plot:

```
sage: v = vector(RDF, (1,2,3,4))
sage: plot(v)
Graphics object consisting of 1 graphics primitive
```

One dimensional vectors are plotted along the horizontal axis of the coordinate plane:

```
sage: plot(vector([1]))
Graphics object consisting of 1 graphics primitive
```

An optional start argument may also be specified by a tuple, list, or vector:

```
sage: u = vector([1,2]); v = vector([2,5])
sage: plot(u, start=v)
Graphics object consisting of 1 graphics primitive
```

TESTS:

```
sage: u = vector([1,1]); v = vector([2,2,2]); z=(3,3,3)
sage: plot(u) #test when start=None
Graphics object consisting of 1 graphics primitive

sage: plot(u, start=v) #test when coordinate dimension mismatch exists
Traceback (most recent call last):
...
ValueError: vector coordinates are not of the same dimension
sage: P = plot(v, start=z) #test when start coordinates are passed as a tuple
sage: P = plot(v, start=list(z)) #test when start coordinates are passed as a list
```

plot_step (*xmin=0, xmax=1, eps=None, res=None, connect=True, **kwds*)

INPUT:

- `xmin` - (default: 0) start x position to start plotting
- `xmax` - (default: 1) stop x position to stop plotting
- `eps` - (default: determined by xmax) we view this vector as defining a function at the points xmin, xmin + eps, xmin + 2*eps, ...,
- `res` - (default: all points) total number of points to include in the graph
- `connect` - (default: True) if True draws a line; otherwise draw a list of points.

EXAMPLES:

```
sage: eps=0.1
sage: v = vector(RDF, [sin(n*eps) for n in range(100)])
sage: v.plot_step(eps=eps, xmax=5, hue=0)
Graphics object consisting of 1 graphics primitive
```

row ()

Return a matrix with a single row and the same entries as the vector `self`.

OUTPUT:

A matrix over the same ring as the vector (or free module element), with a single row. The entries of the row are identical to those of the vector, and in the same order.

EXAMPLES:

```
sage: v = vector(ZZ, [1,2,3])
sage: w = v.row(); w
[1 2 3]
sage: w.parent()
Full MatrixSpace of 1 by 3 dense matrices over Integer Ring

sage: x = vector(FiniteField(13), [2,4,8,16])
sage: x.row()
[2 4 8 3]
```

There is more than one way to get one-row matrix from a vector, but the `row` method is more efficient than making a column and then taking a transpose. Notice that supplying a vector to the matrix constructor demonstrates Sage's preference for rows.

```
sage: x = vector(RDF, [sin(i*pi/20) for i in range(10)])
sage: x.row() == matrix(x)
True
sage: x.row() == x.column().transpose()
True
```

Sparse or dense implementations are preserved.

```
sage: d = vector(RR, [1.0, 2.0, 3.0])
sage: s = vector(CDF, {2:5.0+6.0*I})
sage: dm = d.row()
sage: sm = s.row()
sage: all([d.is_dense(), dm.is_dense(), s.is_sparse(), sm.is_sparse()])
True
```

TESTS:

The `row()` method will return a specified row of a matrix as a vector. So here are a couple of round-trips.

```
sage: A = matrix(ZZ, [[1,2,3]])
sage: A == A.row(0).row()
True
sage: v = vector(ZZ, [4,5,6])
sage: v == v.row().row(0)
True
```

And a very small corner case.

```
sage: v = vector(ZZ, [])
sage: w = v.row()
sage: w.parent()
Full MatrixSpace of 1 by 0 dense matrices over Integer Ring
```

set (*i*, *x*)

The set method is meant to be more efficient than __setitem__, because it need not be guaranteed to do any error checking or coercion. Use with great, great care.

EXAMPLES:

```
sage: v = vector([1/2,2/5,0]); v
(1/2, 2/5, 0)
sage: v.set(2, -15/17); v
(1/2, 2/5, -15/17)
```

set_immutable ()

Make this vector immutable. This operation can't be undone.

EXAMPLES:

```
sage: v = vector([1..5]); v
(1, 2, 3, 4, 5)
sage: v[1] = 10
sage: v.set_immutable()
sage: v[1] = 10
Traceback (most recent call last):
...
ValueError: vector is immutable; please change a copy instead (use copy())
```

sparse_vector()

Return sparse version of self. If self is sparse, just return self; otherwise, create and return correspond sparse vector.

EXAMPLES:

```
sage: vector([-1,0,3,0,0,0]).sparse_vector().is_sparse()
True
sage: vector([-1,0,3,0,0,0]).sparse_vector().is_sparse()
True
sage: vector([-1,0,3,0,0,0]).sparse_vector()
(-1, 0, 3, 0, 0, 0)
```

subs(*in_dict=None*, ***kwds*)

EXAMPLES:

```
sage: var('a,b,d,e')
(a, b, d, e)
sage: v = vector([a, b, d, e])
sage: v.substitute(a=1)
(1, b, d, e)
sage: v.subs(a=b, b=d)
(b, d, d, e)
```

support()

Return the integers `i` such that `self[i] != 0`. This is the same as the `nonzero_positions` function.

EXAMPLES:

```
sage: vector([-1,0,3,0,0,0,0.01]).support()
[0, 2, 6]
```

tensor_product(*right*)

Returns a matrix, the outer product of two vectors `self` and `right`.

INPUT:

- `right` - a vector (or free module element) of any size, whose elements are compatible (with regard to multiplication) with the elements of `self`.

OUTPUT:

The outer product of two vectors x and y (respectively `self` and `right`) can be described several ways. If we interpret x as a $m \times 1$ matrix and interpret y as a $1 \times n$ matrix, then the outer product is the $m \times n$ matrix from the usual matrix product xy. Notice how this is the "opposite" in some ways from an inner product (which would require $m = n$).

If we just consider vectors, use each entry of x to create a scalar multiples of the vector y and use these vectors as the rows of a matrix. Or use each entry of y to create a scalar multiples of x and use these vectors as the columns of a matrix.

EXAMPLES:

```
sage: u = vector(QQ, [1/2, 1/3, 1/4, 1/5])
sage: v = vector(ZZ, [60, 180, 600])
sage: u.outer_product(v)
[ 30  90 300]
[ 20  60 200]
[ 15  45 150]
[ 12  36 120]
sage: M = v.outer_product(u); M
```

```
[ 30  20  15  12]
[ 90  60  45  36]
[300 200 150 120]
sage: M.parent()
Full MatrixSpace of 3 by 4 dense matrices over Rational Field
```

The more general `sage.matrix.matrix2.tensor_product()` is an operation on a pair of matrices. If we construe a pair of vectors as a column vector and a row vector, then an outer product and a tensor product are identical. Thus $tensor_product$ is a synonym for this method.

```
sage: u = vector(QQ, [1/2, 1/3, 1/4, 1/5])
sage: v = vector(ZZ, [60, 180, 600])
sage: u.tensor_product(v) == (u.column()).tensor_product(v.row())
True
```

The result is always a dense matrix, no matter if the two vectors are, or are not, dense.

```
sage: d = vector(ZZ,[4,5], sparse=False)
sage: s = vector(ZZ, [1,2,3], sparse=True)
sage: dd = d.outer_product(d)
sage: ds = d.outer_product(s)
sage: sd = s.outer_product(d)
sage: ss = s.outer_product(s)
sage: all([dd.is_dense(), ds.is_dense(), sd.is_dense(), dd.is_dense()])
True
```

Vectors with no entries do the right thing.

```
sage: v = vector(ZZ, [])
sage: z = v.outer_product(v)
sage: z.parent()
Full MatrixSpace of 0 by 0 dense matrices over Integer Ring
```

There is a fair amount of latitude in the value of the `right` vector, and the matrix that results can have entries from a new ring large enough to contain the result. If you know better, you can sometimes bring the result down to a less general ring.

```
sage: R.<t> = ZZ[]
sage: v = vector(R, [12, 24*t])
sage: w = vector(QQ, [1/2, 1/3, 1/4])
sage: op = v.outer_product(w)
sage: op
[   6    4    3]
[12*t  8*t  6*t]
sage: op.base_ring()
Univariate Polynomial Ring in t over Rational Field
sage: m = op.change_ring(R); m
[   6    4    3]
[12*t  8*t  6*t]
sage: m.base_ring()
Univariate Polynomial Ring in t over Integer Ring
```

But some inputs are not compatible, even if vectors.

```
sage: w = vector(GF(5), [1,2])
sage: v = vector(GF(7), [1,2,3,4])
sage: z = w.outer_product(v)
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for '*': 'Full MatrixSpace of 2 by 1 dense matrices
```

And some inputs don't make any sense at all.

```
sage: w=vector(QQ, [5,10])
sage: z=w.outer_product(6)
Traceback (most recent call last):
...
TypeError: right operand in an outer product must be a vector, not an element of Integer Rin
```

class sage.modules.free_module_element.**FreeModuleElement_generic_dense**

Bases: sage.modules.free_module_element.FreeModuleElement

A generic dense element of a free module.

TESTS:

```
sage: V = ZZ^3
sage: loads(dumps(V)) == V
True
sage: v = V.0
sage: loads(dumps(v)) == v
True
sage: v = (QQ['x']^3).0
sage: loads(dumps(v)) == v
True
```

function(**args*)

Returns a vector over a callable symbolic expression ring.

EXAMPLES:

```
sage: x,y=var('x,y')
sage: v=vector([x,y,x*sin(y)])
sage: w=v.function([x,y]); w
(x, y) |--> (x, y, x*sin(y))
sage: w.coordinate_ring()
Callable function ring with arguments (x, y)
sage: w(1,2)
(1, 2, sin(2))
sage: w(2,1)
(2, 1, 2*sin(1))
sage: w(y=1,x=2)
(2, 1, 2*sin(1))

sage: x,y=var('x,y')
sage: v=vector([x,y,x*sin(y)])
sage: w=v.function([x]); w
x |--> (x, y, x*sin(y))
sage: w.coordinate_ring()
Callable function ring with argument x
sage: w(4)
(4, y, 4*sin(y))
```

list(*copy=True*)

Return list of elements of self.

INPUT:

- copy – bool, return list of underlying entries

EXAMPLES:

```
sage: P.<x,y,z> = QQ[]
sage: v = vector([x,y,z])
sage: type(v)
<type 'sage.modules.free_module_element.FreeModuleElement_generic_dense'>
sage: a = v.list(); a
[x, y, z]
sage: a[0] = x*y; v
(x, y, z)
sage: a = v.list(copy=False); a
[x, y, z]
sage: a[0] = x*y; v
(x*y, y, z)
```

class `sage.modules.free_module_element.`**`FreeModuleElement_generic_sparse`**

Bases: `sage.modules.free_module_element.FreeModuleElement`

A generic sparse free module element is a dictionary with keys ints i and entries in the base ring.

EXAMPLES:

Pickling works:

```
sage: v = FreeModule(ZZ, 3, sparse=True).0
sage: loads(dumps(v)) == v
True
sage: v = FreeModule(Integers(8)['x,y'], 5, sparse=True).1
sage: loads(dumps(v)) - v
(0, 0, 0, 0, 0)
```

```
sage: a = vector([-1,0,1/1],sparse=True); b = vector([-1/1,0,0],sparse=True)
sage: a.parent()
Sparse vector space of dimension 3 over Rational Field
sage: b - a
(0, 0, -1)
sage: (b-a).dict()
{2: -1}
```

`denominator`()

Return the least common multiple of the denominators of the entries of self.

EXAMPLES:

```
sage: v = vector([1/2,2/5,3/14], sparse=True)
sage: v.denominator()
70
```

`dict`(*copy=True*)

Return dictionary of nonzero entries of self.

INPUT:

- `copy` – bool (default: True)

OUTPUT:

- Python dictionary

EXAMPLES:

```
sage: v = vector([0,0,0,0,1/2,0,3/14], sparse=True)
sage: v.dict()
{4: 1/2, 6: 3/14}
```

get (*i*)

Like __getitem__ but with no guaranteed type or bounds checking. Returns 0 if access is out of bounds.

EXAMPLES:

```
sage: v = vector([1,2/3,pi], sparse=True)
sage: v.get(1)
2/3
sage: v.get(10)
0
```

hamming_weight ()

Returns the number of positions `i` such that `self[i] != 0`.

EXAMPLES:

```
sage: v = vector({1: 1, 3: -2})
sage: w = vector({1: 4, 3: 2})
sage: v+w
(0, 5, 0, 0)
sage: (v+w).hamming_weight()
1
```

iteritems ()

Return iterator over the entries of self.

EXAMPLES:

```
sage: v = vector([1,2/3,pi], sparse=True)
sage: v.iteritems()
<dictionary-itemiterator object at ...>
sage: list(v.iteritems())
[(0, 1), (1, 2/3), (2, pi)]
```

list (*copy=True*)

Return list of elements of self.

INPUT:

- `copy` – ignored for sparse vectors

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: M = FreeModule(R, 3, sparse=True) * (1/x)
sage: v = M([-x^2, 3/x, 0])
sage: type(v)
<type 'sage.modules.free_module_element.FreeModuleElement_generic_sparse'>
sage: a = v.list()
sage: a
[-x^2, 3/x, 0]
sage: [parent(c) for c in a]
[Fraction Field of Univariate Polynomial Ring in x over Rational Field,
 Fraction Field of Univariate Polynomial Ring in x over Rational Field,
 Fraction Field of Univariate Polynomial Ring in x over Rational Field]
```

nonzero_positions ()

Returns the list of numbers `i` such that `self[i] != 0`.

EXAMPLES:

```
sage: v = vector({1: 1, 3: -2})
sage: w = vector({1: 4, 3: 2})
```

```
sage: v+w
(0, 5, 0, 0)
sage: (v+w).nonzero_positions()
[1]
```

set (*i*, *x*)

Like __setitem__ but with no guaranteed type or bounds checking.

EXAMPLES:

```
sage: v = vector([1,2/3,pi], sparse=True)
sage: v.set(1, pi^3)
sage: v
(1, pi^3, pi)
```

No bounds checking:

```
sage: v.set(10, pi)
```

This lack of bounds checking causes trouble later:

```
sage: v
<repr(<sage.modules.free_module_element.FreeModuleElement_generic_sparse at 0x...>) failed:
```

sage.modules.free_module_element.**free_module_element** (*arg0*, *arg1=None*, *arg2=None*, *sparse=None*)

Return a vector or free module element with specified entries.

CALL FORMATS:

This constructor can be called in several different ways. In each case, `sparse=True` or `sparse=False` can be supplied as an option. `free_module_element()` is an alias for `vector()`.

1. vector(object)
2. vector(ring, object)
3. vector(object, ring)
4. vector(ring, degree, object)
5. vector(ring, degree)

INPUT:

- `object` – a list, dictionary, or other iterable containing the entries of the vector, including any object that is palatable to the `Sequence` constructor
- `ring` – a base ring (or field) for the vector space or free module, which contains all of the elements
- `degree` – an integer specifying the number of entries in the vector or free module element
- `sparse` – boolean, whether the result should be a sparse vector

In call format 4, an error is raised if the `degree` does not match the length of `object` so this call can provide some safeguards. Note however that using this format when `object` is a dictionary is unlikely to work properly.

OUTPUT:

An element of the ambient vector space or free module with the given base ring and implied or specified dimension or rank, containing the specified entries and with correct degree.

In call format 5, no entries are specified, so the element is populated with all zeros.

If the sparse option is not supplied, the output will generally have a dense representation. The exception is if object is a dictionary, then the representation will be sparse.

EXAMPLES:

```
sage: v = vector([1,2,3]); v
(1, 2, 3)
sage: v.parent()
Ambient free module of rank 3 over the principal ideal domain Integer Ring
sage: v = vector([1,2,3/5]); v
(1, 2, 3/5)
sage: v.parent()
Vector space of dimension 3 over Rational Field
```

All entries must *canonically* coerce to some common ring:

```
sage: v = vector([17, GF(11)(5), 19/3]); v
Traceback (most recent call last):
...
TypeError: unable to find a common ring for all elements

sage: v = vector([17, GF(11)(5), 19]); v
(6, 5, 8)
sage: v.parent()
Vector space of dimension 3 over Finite Field of size 11
sage: v = vector([17, GF(11)(5), 19], QQ); v
(17, 5, 19)
sage: v.parent()
Vector space of dimension 3 over Rational Field
sage: v = vector((1,2,3), QQ); v
(1, 2, 3)
sage: v.parent()
Vector space of dimension 3 over Rational Field
sage: v = vector(QQ, (1,2,3)); v
(1, 2, 3)
sage: v.parent()
Vector space of dimension 3 over Rational Field
sage: v = vector(vector([1,2,3])); v
(1, 2, 3)
sage: v.parent()
Ambient free module of rank 3 over the principal ideal domain Integer Ring
```

You can also use free_module_element, which is the same as vector.

```
sage: free_module_element([1/3, -4/5])
(1/3, -4/5)
```

We make a vector mod 3 out of a vector over $\mathbf{Z}$.

```
sage: vector(vector([1,2,3]), GF(3))
(1, 2, 0)
```

The degree of a vector may be specified:

```
sage: vector(QQ, 4, [1,1/2,1/3,1/4])
(1, 1/2, 1/3, 1/4)
```

But it is an error if the degree and size of the list of entries are mismatched:

```
sage: vector(QQ, 5, [1,1/2,1/3,1/4])
Traceback (most recent call last):
```

```
...
ValueError: incompatible degrees in vector constructor
```

Providing no entries populates the vector with zeros, but of course, you must specify the degree since it is not implied. Here we use a finite field as the base ring.

```
sage: w = vector(FiniteField(7), 4); w
(0, 0, 0, 0)
sage: w.parent()
Vector space of dimension 4 over Finite Field of size 7
```

The fastest method to construct a zero vector is to call the `zero_vector()` method directly on a free module or vector space, since vector(...) must do a small amount of type checking. Almost as fast as the `zero_vector()` method is the `zero_vector()` constructor, which defaults to the integers.

```
sage: vector(ZZ, 5)          # works fine
(0, 0, 0, 0, 0)
sage: (ZZ^5).zero_vector()   # very tiny bit faster
(0, 0, 0, 0, 0)
sage: zero_vector(ZZ, 5)     # similar speed to vector(...)
(0, 0, 0, 0, 0)
sage: z = zero_vector(5); z
(0, 0, 0, 0, 0)
sage: z.parent()
Ambient free module of rank 5 over
the principal ideal domain Integer Ring
```

Here we illustrate the creation of sparse vectors by using a dictionary.

```
sage: vector({1:1.1, 3:3.14})
(0.000000000000000, 1.10000000000000, 0.000000000000000, 3.14000000000000)
```

With no degree given, a dictionary of entries implicitly declares a degree by the largest index (key) present. So you can provide a terminal element (perhaps a zero?) to set the degree. But it is probably safer to just include a degree in your construction.

```
sage: v = vector(QQ, {0:1/2, 4:-6, 7:0}); v
(1/2, 0, 0, 0, -6, 0, 0, 0)
sage: v.degree()
8
sage: v.is_sparse()
True
sage: w = vector(QQ, 8, {0:1/2, 4:-6})
sage: w == v
True
```

It is an error to specify a negative degree.

```
sage: vector(RR, -4, [1.0, 2.0, 3.0, 4.0])
Traceback (most recent call last):
...
ValueError: cannot specify the degree of a vector as a negative integer (-4)
```

It is an error to create a zero vector but not provide a ring as the first argument.

```
sage: vector('junk', 20)
Traceback (most recent call last):
...
TypeError: first argument must be base ring of zero vector, not junk
```

And it is an error to specify an index in a dictionary that is greater than or equal to a requested degree.

```
sage: vector(ZZ, 10, {3:4, 7:-2, 10:637})
Traceback (most recent call last):
...
ValueError: dictionary of entries has a key (index) exceeding the requested degree
```

A 1-dimensional numpy array of type float or complex may be passed to vector. Unless an explicit ring is given, the result will be a vector in the appropriate dimensional vector space over the real double field or the complex double field. The data in the array must be contiguous, so column-wise slices of numpy matrices will raise an exception.

```
sage: import numpy
sage: x = numpy.random.randn(10)
sage: y = vector(x)
sage: parent(y)
Vector space of dimension 10 over Real Double Field
sage: parent(vector(RDF, x))
Vector space of dimension 10 over Real Double Field
sage: parent(vector(CDF, x))
Vector space of dimension 10 over Complex Double Field
sage: parent(vector(RR, x))
Vector space of dimension 10 over Real Field with 53 bits of precision
sage: v = numpy.random.randn(10) * numpy.complex(0,1)
sage: w = vector(v)
sage: parent(w)
Vector space of dimension 10 over Complex Double Field
```

Multi-dimensional arrays are not supported:

```
sage: import numpy as np
sage: a = np.array([[1, 2, 3], [4, 5, 6]], np.float64)
sage: vector(a)
Traceback (most recent call last):
...
TypeError: cannot convert 2-dimensional array to a vector
```

If any of the arguments to vector have Python type int, long, real, or complex, they will first be coerced to the appropriate Sage objects. This fixes trac ticket #3847.

```
sage: v = vector([int(0)]); v
(0)
sage: v[0].parent()
Integer Ring
sage: v = vector(range(10)); v
(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)
sage: v[3].parent()
Integer Ring
sage: v = vector([float(23.4), int(2), complex(2+7*I), long(1)]); v
(23.4, 2.0, 2.0 + 7.0*I, 1.0)
sage: v[1].parent()
Complex Double Field
```

If the argument is a vector, it doesn't change the base ring. This fixes trac ticket #6643:

```
sage: K.<sqrt3> = QuadraticField(3)
sage: u = vector(K, (1/2, sqrt3/2) )
sage: vector(u).base_ring()
Number Field in sqrt3 with defining polynomial x^2 - 3
sage: v = vector(K, (0, 1) )
```

```
sage: vector(v).base_ring()
Number Field in sqrt3 with defining polynomial x^2 - 3
```

Constructing a vector from a numpy array behaves as expected:

```
sage: import numpy
sage: a=numpy.array([1,2,3])
sage: v=vector(a); v
(1, 2, 3)
sage: parent(v)
Ambient free module of rank 3 over the principal ideal domain Integer Ring
```

Complex numbers can be converted naturally to a sequence of length 2. And then to a vector.

```
sage: c = CDF(2 + 3*I)
sage: v = vector(c); v
(2.0, 3.0)
```

A generator, or other iterable, may also be supplied as input. Anything that can be converted to a `Sequence` is a possible input.

```
sage: type(i^2 for i in range(3))
<type 'generator'>
sage: v = vector(i^2 for i in range(3)); v
(0, 1, 4)
```

An empty list, without a ring given, will default to the integers.

```
sage: x = vector([]); x
()
sage: x.parent()
Ambient free module of rank 0 over the principal ideal domain Integer Ring
```

`sage.modules.free_module_element.`**`is_FreeModuleElement`**(*x*)

EXAMPLES:

```
sage: sage.modules.free_module_element.is_FreeModuleElement(0)
False
sage: sage.modules.free_module_element.is_FreeModuleElement(vector([1,2,3]))
True
```

`sage.modules.free_module_element.`**`make_FreeModuleElement_generic_dense`**(*parent*, *entries*, *degree*)

EXAMPLES:

```
sage: sage.modules.free_module_element.make_FreeModuleElement_generic_dense(QQ^3, [1,2,-3/7], 3)
(1, 2, -3/7)
```

`sage.modules.free_module_element.`**`make_FreeModuleElement_generic_dense_v1`**(*parent*, *entries*, *degree*, *is_mutable*)

EXAMPLES:

```
sage: v = sage.modules.free_module_element.make_FreeModuleElement_generic_dense_v1(QQ^3, [1,2,-3
(1, 2, -3/7)
sage: v[0] = 10; v
(10, 2, -3/7)
sage: v = sage.modules.free_module_element.make_FreeModuleElement_generic_dense_v1(QQ^3, [1,2,-3
(1, 2, -3/7)
sage: v[0] = 10
Traceback (most recent call last):
...
ValueError: vector is immutable; please change a copy instead (use copy())
```

sage.modules.free_module_element.**make_FreeModuleElement_generic_sparse**(*parent*, *entries*, *degree*)

EXAMPLES:

```
sage: v = sage.modules.free_module_element.make_FreeModuleElement_generic_sparse(QQ^3, {2:5/2},
(0, 0, 5/2)
```

sage.modules.free_module_element.**make_FreeModuleElement_generic_sparse_v1**(*parent*, *entries*, *degree*, *is_mutable*)

EXAMPLES:

```
sage: v = sage.modules.free_module_element.make_FreeModuleElement_generic_sparse_v1(QQ^3, {2:5/2
(0, 0, 5/2)
sage: v.is_mutable()
False
```

sage.modules.free_module_element.**prepare**(*v*, *R*, *degree=None*)

Converts an object describing elements of a vector into a list of entries in a common ring.

INPUT:

- `v` - a dictionary with non-negative integers as keys, or a list or other object that can be converted by the `Sequence` constructor
- `R` - a ring containing all the entries, possibly given as `None`
- `degree` - a requested size for the list when the input is a dictionary, otherwise ignored

OUTPUT:

A pair.

The first item is a list of the values specified in the object `v`. If the object is a dictionary , entries are placed in the list according to the indices that were their keys in the dictionary, and the remainder of the entries are zero. The value of `degree` is assumed to be larger than any index provided in the dictionary and will be used as the number of entries in the returned list.

The second item returned is a ring that contains all of the entries in the list. If `R` is given, the entries are coerced in. Otherwise a common ring is found. For more details, see the `Sequence` object. When `v` has no elements and `R` is `None`, the ring returned is the integers.

EXAMPLES:

```
sage: from sage.modules.free_module_element import prepare
sage: prepare([1,2/3,5],None)
([1, 2/3, 5], Rational Field)

sage: prepare([1,2/3,5],RR)
([1.00000000000000, 0.666666666666667, 5.00000000000000], Real Field with 53 bits of precision)

sage: prepare({1:4, 3:-2}, ZZ, 6)
([0, 4, 0, -2, 0, 0], Integer Ring)

sage: prepare({3:1, 5:3}, QQ, 6)
([0, 0, 0, 1, 0, 3], Rational Field)

sage: prepare([1,2/3,'10',5],RR)
([1.00000000000000, 0.666666666666667, 10.0000000000000, 5.00000000000000], Real Field with 53 b

sage: prepare({},QQ, 0)
([], Rational Field)

sage: prepare([1,2/3,'10',5],None)
Traceback (most recent call last):
...
TypeError: unable to find a common ring for all elements
```

Some objects can be converted to sequences even if they are not always thought of as sequences.

```
sage: c = CDF(2+3*I)
sage: prepare(c, None)
([2.0, 3.0], Real Double Field)
```

This checks a bug listed at Trac #10595. Without good evidence for a ring, the default is the integers.

```
sage: prepare([], None)
([], Integer Ring)
```

`sage.modules.free_module_element.`**`random_vector`**`(`*ring*, *degree=None*, **args*, ***kwds*`)`

Returns a vector (or module element) with random entries.

INPUT:

- ring - default: `ZZ` - the base ring for the entries
- degree - a non-negative integer for the number of entries in the vector
- sparse - default: `False` - whether to use a sparse implementation
- args, kwds - additional arguments and keywords are passed to the `random_element()` method of the ring

OUTPUT:

A vector, or free module element, with `degree` elements from `ring`, chosen randomly from the ring according to the ring's `random_element()` method.

Note: See below for examples of how random elements are generated by some common base rings.

EXAMPLES:

First, module elements over the integers. The default distribution is tightly clustered around -1, 0, 1. Uniform distributions can be specified by giving bounds, though the upper bound is never met. See `sage.rings.integer_ring.IntegerRing_class.random_element()` for several other variants.

```
sage: random_vector(10)
(-8, 2, 0, 0, 1, -1, 2, 1, -95, -1)

sage: sorted(random_vector(20))
[-12, -6, -4, -4, -2, -2, -2, -1, -1, -1, 0, 0, 0, 0, 0, 1, 1, 1, 4, 5]

sage: random_vector(ZZ, 20, x=4)
(2, 0, 3, 0, 1, 0, 2, 0, 2, 3, 0, 3, 1, 2, 2, 2, 1, 3, 2, 3)

sage: random_vector(ZZ, 20, x=-20, y=100)
(43, 47, 89, 31, 56, -20, 23, 52, 13, 53, 49, -12, -2, 94, -1, 95, 60, 83, 28, 63)

sage: random_vector(ZZ, 20, distribution="1/n")
(0, -1, -2, 0, -1, -2, 0, 0, 27, -1, 1, 1, 0, 2, -1, 1, -1, -2, -1, 3)
```

If the ring is not specified, the default is the integers, and parameters for the random distribution may be passed without using keywords. This is a random vector with 20 entries uniformly distributed between -20 and 100.

```
sage: random_vector(20, -20, 100)
(70, 19, 98, 2, -18, 88, 36, 66, 76, 52, 82, 99, 55, -17, 82, -15, 36, 28, 79, 18)
```

Now over the rationals. Note that bounds on the numerator and denominator may be specified. See `sage.rings.rational_field.RationalField.random_element()` for documentation.

```
sage: random_vector(QQ, 10)
(0, -1, -4/3, 2, 0, -13, 2/3, 0, -4/5, -1)

sage: random_vector(QQ, 10, num_bound = 15, den_bound = 5)
(-12/5, 9/4, -13/3, -1/3, 1, 5/4, 4, 1, -15, 10/3)
```

Inexact rings may be used as well. The reals have uniform distributions, with the range $(-1, 1)$ as the default. More at: `sage.rings.real_mpfr.RealField_class.random_element()`

```
sage: random_vector(RR, 5)
(0.248997268533725, -0.112200126330480, 0.776829203293064, -0.899146461031406, 0.534465018743125

sage: random_vector(RR, 5, min = 8, max = 14)
(8.43260944052606, 8.34129413391087, 8.92391495103829, 11.5784799413416, 11.0973561568002)
```

Any ring with a `random_element()` method may be used.

```
sage: F = FiniteField(23)
sage: hasattr(F, 'random_element')
True
sage: random_vector(F, 10)
(21, 6, 5, 2, 6, 2, 18, 9, 9, 7)
```

The default implementation is a dense representation, equivalent to setting `sparse=False`.

```
sage: v = random_vector(10)
sage: v.is_sparse()
False

sage: w = random_vector(ZZ, 20, sparse=True)
sage: w.is_sparse()
True
```

Inputs get checked before constructing the vector.

```
sage: random_vector('junk')
Traceback (most recent call last):
...
TypeError: degree of a random vector must be an integer, not None

sage: random_vector('stuff', 5)
Traceback (most recent call last):
...
TypeError: elements of a vector, or module element, must come from a ring, not stuff

sage: random_vector(ZZ, -9)
Traceback (most recent call last):
...
ValueError: degree of a random vector must be non-negative, not -9
```

`sage.modules.free_module_element.`**`vector`**(*arg0*, *arg1=None*, *arg2=None*, *sparse=None*)

Return a vector or free module element with specified entries.

CALL FORMATS:

This constructor can be called in several different ways. In each case, `sparse=True` or `sparse=False` can be supplied as an option. `free_module_element()` is an alias for `vector()`.

1. vector(object)
2. vector(ring, object)
3. vector(object, ring)
4. vector(ring, degree, object)
5. vector(ring, degree)

INPUT:

- `object` – a list, dictionary, or other iterable containing the entries of the vector, including any object that is palatable to the `Sequence` constructor
- `ring` – a base ring (or field) for the vector space or free module, which contains all of the elements
- `degree` – an integer specifying the number of entries in the vector or free module element
- `sparse` – boolean, whether the result should be a sparse vector

In call format 4, an error is raised if the `degree` does not match the length of `object` so this call can provide some safeguards. Note however that using this format when `object` is a dictionary is unlikely to work properly.

OUTPUT:

An element of the ambient vector space or free module with the given base ring and implied or specified dimension or rank, containing the specified entries and with correct degree.

In call format 5, no entries are specified, so the element is populated with all zeros.

If the `sparse` option is not supplied, the output will generally have a dense representation. The exception is if `object` is a dictionary, then the representation will be sparse.

EXAMPLES:

```
sage: v = vector([1,2,3]); v
(1, 2, 3)
sage: v.parent()
Ambient free module of rank 3 over the principal ideal domain Integer Ring
sage: v = vector([1,2,3/5]); v
```

```
(1, 2, 3/5)
sage: v.parent()
Vector space of dimension 3 over Rational Field
```

All entries must *canonically* coerce to some common ring:

```
sage: v = vector([17, GF(11)(5), 19/3]); v
Traceback (most recent call last):
...
TypeError: unable to find a common ring for all elements
```

```
sage: v = vector([17, GF(11)(5), 19]); v
(6, 5, 8)
sage: v.parent()
Vector space of dimension 3 over Finite Field of size 11
sage: v = vector([17, GF(11)(5), 19], QQ); v
(17, 5, 19)
sage: v.parent()
Vector space of dimension 3 over Rational Field
sage: v = vector((1,2,3), QQ); v
(1, 2, 3)
sage: v.parent()
Vector space of dimension 3 over Rational Field
sage: v = vector(QQ, (1,2,3)); v
(1, 2, 3)
sage: v.parent()
Vector space of dimension 3 over Rational Field
sage: v = vector(vector([1,2,3])); v
(1, 2, 3)
sage: v.parent()
Ambient free module of rank 3 over the principal ideal domain Integer Ring
```

You can also use `free_module_element`, which is the same as `vector`.

```
sage: free_module_element([1/3, -4/5])
(1/3, -4/5)
```

We make a vector mod 3 out of a vector over $\mathbf{Z}$.

```
sage: vector(vector([1,2,3]), GF(3))
(1, 2, 0)
```

The degree of a vector may be specified:

```
sage: vector(QQ, 4, [1,1/2,1/3,1/4])
(1, 1/2, 1/3, 1/4)
```

But it is an error if the degree and size of the list of entries are mismatched:

```
sage: vector(QQ, 5, [1,1/2,1/3,1/4])
Traceback (most recent call last):
...
ValueError: incompatible degrees in vector constructor
```

Providing no entries populates the vector with zeros, but of course, you must specify the degree since it is not implied. Here we use a finite field as the base ring.

```
sage: w = vector(FiniteField(7), 4); w
(0, 0, 0, 0)
sage: w.parent()
```

```
Vector space of dimension 4 over Finite Field of size 7
```

The fastest method to construct a zero vector is to call the `zero_vector()` method directly on a free module or vector space, since vector(...) must do a small amount of type checking. Almost as fast as the `zero_vector()` method is the `zero_vector()` constructor, which defaults to the integers.

```
sage: vector(ZZ, 5)          # works fine
(0, 0, 0, 0, 0)
sage: (ZZ^5).zero_vector()   # very tiny bit faster
(0, 0, 0, 0, 0)
sage: zero_vector(ZZ, 5)     # similar speed to vector(...)
(0, 0, 0, 0, 0)
sage: z = zero_vector(5); z
(0, 0, 0, 0, 0)
sage: z.parent()
Ambient free module of rank 5 over
the principal ideal domain Integer Ring
```

Here we illustrate the creation of sparse vectors by using a dictionary.

```
sage: vector({1:1.1, 3:3.14})
(0.000000000000000, 1.10000000000000, 0.000000000000000, 3.14000000000000)
```

With no degree given, a dictionary of entries implicitly declares a degree by the largest index (key) present. So you can provide a terminal element (perhaps a zero?) to set the degree. But it is probably safer to just include a degree in your construction.

```
sage: v = vector(QQ, {0:1/2, 4:-6, 7:0}); v
(1/2, 0, 0, 0, -6, 0, 0, 0)
sage: v.degree()
8
sage: v.is_sparse()
True
sage: w = vector(QQ, 8, {0:1/2, 4:-6})
sage: w == v
True
```

It is an error to specify a negative degree.

```
sage: vector(RR, -4, [1.0, 2.0, 3.0, 4.0])
Traceback (most recent call last):
...
ValueError: cannot specify the degree of a vector as a negative integer (-4)
```

It is an error to create a zero vector but not provide a ring as the first argument.

```
sage: vector('junk', 20)
Traceback (most recent call last):
...
TypeError: first argument must be base ring of zero vector, not junk
```

And it is an error to specify an index in a dictionary that is greater than or equal to a requested degree.

```
sage: vector(ZZ, 10, {3:4, 7:-2, 10:637})
Traceback (most recent call last):
...
ValueError: dictionary of entries has a key (index) exceeding the requested degree
```

A 1-dimensional numpy array of type float or complex may be passed to vector. Unless an explicit ring is given, the result will be a vector in the appropriate dimensional vector space over the real double field or the complex double field. The data in the array must be contiguous, so column-wise slices of numpy matrices will raise an exception.

```
sage: import numpy
sage: x = numpy.random.randn(10)
sage: y = vector(x)
sage: parent(y)
Vector space of dimension 10 over Real Double Field
sage: parent(vector(RDF, x))
Vector space of dimension 10 over Real Double Field
sage: parent(vector(CDF, x))
Vector space of dimension 10 over Complex Double Field
sage: parent(vector(RR, x))
Vector space of dimension 10 over Real Field with 53 bits of precision
sage: v = numpy.random.randn(10) * numpy.complex(0,1)
sage: w = vector(v)
sage: parent(w)
Vector space of dimension 10 over Complex Double Field
```

Multi-dimensional arrays are not supported:

```
sage: import numpy as np
sage: a = np.array([[1, 2, 3], [4, 5, 6]], np.float64)
sage: vector(a)
Traceback (most recent call last):
...
TypeError: cannot convert 2-dimensional array to a vector
```

If any of the arguments to vector have Python type int, long, real, or complex, they will first be coerced to the appropriate Sage objects. This fixes trac ticket #3847.

```
sage: v = vector([int(0)]); v
(0)
sage: v[0].parent()
Integer Ring
sage: v = vector(range(10)); v
(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)
sage: v[3].parent()
Integer Ring
sage: v = vector([float(23.4), int(2), complex(2+7*I), long(1)]); v
(23.4, 2.0, 2.0 + 7.0*I, 1.0)
sage: v[1].parent()
Complex Double Field
```

If the argument is a vector, it doesn't change the base ring. This fixes trac ticket #6643:

```
sage: K.<sqrt3> = QuadraticField(3)
sage: u = vector(K, (1/2, sqrt3/2) )
sage: vector(u).base_ring()
Number Field in sqrt3 with defining polynomial x^2 - 3
sage: v = vector(K, (0, 1) )
sage: vector(v).base_ring()
Number Field in sqrt3 with defining polynomial x^2 - 3
```

Constructing a vector from a numpy array behaves as expected:

```
sage: import numpy
sage: a=numpy.array([1,2,3])
sage: v=vector(a); v
(1, 2, 3)
sage: parent(v)
```

```
Ambient free module of rank 3 over the principal ideal domain Integer Ring
```

Complex numbers can be converted naturally to a sequence of length 2. And then to a vector.

```
sage: c = CDF(2 + 3*I)
sage: v = vector(c); v
(2.0, 3.0)
```

A generator, or other iterable, may also be supplied as input. Anything that can be converted to a `Sequence` is a possible input.

```
sage: type(i^2 for i in range(3))
<type 'generator'>
sage: v = vector(i^2 for i in range(3)); v
(0, 1, 4)
```

An empty list, without a ring given, will default to the integers.

```
sage: x = vector([]); x
()
sage: x.parent()
Ambient free module of rank 0 over the principal ideal domain Integer Ring
```

`sage.modules.free_module_element.`**`zero_vector`**(*arg0*, *arg1=None*)

Returns a vector or free module element with a specified number of zeros.

CALL FORMATS:

1. zero_vector(degree)
2. zero_vector(ring, degree)

INPUT:

- `degree` - the number of zero entries in the vector or free module element
- `ring` - default `ZZ` - the base ring of the vector space or module containing the constructed zero vector

OUTPUT:

A vector or free module element with `degree` entries, all equal to zero and belonging to the ring if specified. If no ring is given, a free module element over `ZZ` is returned.

EXAMPLES:

A zero vector over the field of rationals.

```
sage: v = zero_vector(QQ, 5); v
(0, 0, 0, 0, 0)
sage: v.parent()
Vector space of dimension 5 over Rational Field
```

A free module zero element.

```
sage: w = zero_vector(Integers(6), 3); w
(0, 0, 0)
sage: w.parent()
Ambient free module of rank 3 over Ring of integers modulo 6
```

If no ring is given, the integers are used.

```
sage: u = zero_vector(9); u
(0, 0, 0, 0, 0, 0, 0, 0, 0)
```

```
sage: u.parent()
Ambient free module of rank 9 over the principal ideal domain Integer Ring
```

Non-integer degrees produce an error.

```
sage: zero_vector(5.6)
Traceback (most recent call last):
...
TypeError: Attempt to coerce non-integral RealNumber to Integer
```

Negative degrees also give an error.

```
sage: zero_vector(-3)
Traceback (most recent call last):
...
ValueError: rank (=-3) must be nonnegative
```

Garbage instead of a ring will be recognized as such.

```
sage: zero_vector(x^2, 5)
Traceback (most recent call last):
...
TypeError: first argument must be a ring
```

CHAPTER

FIVE

FREE MODULES OF FINITE RANK

The class `FiniteRankFreeModule` implements free modules of finite rank over a commutative ring.

A *free module of finite rank* over a commutative ring R is a module M over R that admits a *finite basis*, i.e. a finite familly of linearly independent generators. Since R is commutative, it has the invariant basis number property, so that the rank of the free module M is defined uniquely, as the cardinality of any basis of M.

No distinguished basis of M is assumed. On the contrary, many bases can be introduced on the free module along with change-of-basis rules (as module automorphisms). Each module element has then various representations over the various bases.

Note: The class `FiniteRankFreeModule` does not inherit from class `FreeModule_generic` nor from class `CombinatorialFreeModule`, since both classes deal with modules with a *distinguished basis* (see details *below*). Accordingly, the class `FiniteRankFreeModule` inherits directly from the generic class `Parent` with the category set to `Modules` (and not to `ModulesWithBasis`).

Todo

- implement submodules
- create a FreeModules category (cf. the *TODO* statement in the documentation of `Modules`: *Implement a ‘‘FreeModules(R)‘‘ category, when so prompted by a concrete use case*)

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2014-2015): initial version

REFERENCES:

- Chap. 10 of R. Godement : *Algebra*, Hermann (Paris) / Houghton Mifflin (Boston) (1968)
- Chap. 3 of S. Lang : *Algebra*, 3rd ed., Springer (New York) (2002)

EXAMPLES:

Let us define a free module of rank 2 over **Z**:

```
sage: M = FiniteRankFreeModule(ZZ, 2, name='M') ; M
Rank-2 free module M over the Integer Ring
sage: M.category()
Category of modules over Integer Ring
```

We introduce a first basis on `M`:

```
sage: e = M.basis('e') ; e
Basis (e_0,e_1) on the Rank-2 free module M over the Integer Ring
```

The elements of the basis are of course module elements:

```
sage: e[0]
Element e_0 of the Rank-2 free module M over the Integer Ring
sage: e[1]
Element e_1 of the Rank-2 free module M over the Integer Ring
sage: e[0].parent()
Rank-2 free module M over the Integer Ring
```

We define a module element by its components w.r.t. basis `e`:

```
sage: u = M([2,-3], basis=e, name='u')
sage: u.display(e)
u = 2 e_0 - 3 e_1
```

Module elements can be also be created by arithmetic expressions:

```
sage: v = -2*u + 4*e[0] ; v
Element of the Rank-2 free module M over the Integer Ring
sage: v.display(e)
6 e_1
sage: u == 2*e[0] - 3*e[1]
True
```

We define a second basis on `M` from a family of linearly independent elements:

```
sage: f = M.basis('f', from_family=(e[0]-e[1], -2*e[0]+3*e[1])) ; f
Basis (f_0,f_1) on the Rank-2 free module M over the Integer Ring
sage: f[0].display(e)
f_0 = e_0 - e_1
sage: f[1].display(e)
f_1 = -2 e_0 + 3 e_1
```

We may of course express the elements of basis `e` in terms of basis `f`:

```
sage: e[0].display(f)
e_0 = 3 f_0 + f_1
sage: e[1].display(f)
e_1 = 2 f_0 + f_1
```

as well as any module element:

```
sage: u.display(f)
u = -f_1
sage: v.display(f)
12 f_0 + 6 f_1
```

The two bases are related by a module automorphism:

```
sage: a = M.change_of_basis(e,f) ; a
Automorphism of the Rank-2 free module M over the Integer Ring
sage: a.parent()
General linear group of the Rank-2 free module M over the Integer Ring
sage: a.matrix(e)
[ 1 -2]
[-1  3]
```

Let us check that basis `f` is indeed the image of basis `e` by `a`:

```
sage: f[0] == a(e[0])
True
sage: f[1] == a(e[1])
True
```

The reverse change of basis is of course the inverse automorphism:

```
sage: M.change_of_basis(f,e) == a^(-1)
True
```

We introduce a new module element via its components w.r.t. basis `f`:

```
sage: v = M([2,4], basis=f, name='v')
sage: v.display(f)
v = 2 f_0 + 4 f_1
```

The sum of the two module elements `u` and `v` can be performed even if they have been defined on different bases, thanks to the known relation between the two bases:

```
sage: s = u + v ; s
Element u+v of the Rank-2 free module M over the Integer Ring
```

We can display the result in either basis:

```
sage: s.display(e)
u+v = -4 e_0 + 7 e_1
sage: s.display(f)
u+v = 2 f_0 + 3 f_1
```

Tensor products of elements are implemented:

```
sage: t = u*v ; t
Type-(2,0) tensor u*v on the Rank-2 free module M over the Integer Ring
sage: t.parent()
Free module of type-(2,0) tensors on the
 Rank-2 free module M over the Integer Ring
sage: t.display(e)
u*v = -12 e_0*e_0 + 20 e_0*e_1 + 18 e_1*e_0 - 30 e_1*e_1
sage: t.display(f)
u*v = -2 f_1*f_0 - 4 f_1*f_1
```

We can access to tensor components w.r.t. to a given basis via the square bracket operator:

```
sage: t[e,0,1]
20
sage: t[f,1,0]
-2
sage: u[e,0]
2
sage: u[e,:]
[2, -3]
sage: u[f,:]
[0, -1]
```

The parent of the automorphism `a` is the group $\mathrm{GL}(M)$, but `a` can also be considered as a tensor of type $(1,1)$ on `M`:

```
sage: a.parent()
General linear group of the Rank-2 free module M over the Integer Ring
```

```
sage: a.tensor_type()
(1, 1)
sage: a.display(e)
e_0*e^0 - 2 e_0*e^1 - e_1*e^0 + 3 e_1*e^1
sage: a.display(f)
f_0*f^0 - 2 f_0*f^1 - f_1*f^0 + 3 f_1*f^1
```

As such, we can form its tensor product with `t`, yielding a tensor of type $(3, 1)$:

```
sage: t*a
Type-(3,1) tensor on the Rank-2 free module M over the Integer Ring
sage: (t*a).display(e)
-12 e_0*e_0*e_0*e^0 + 24 e_0*e_0*e_0*e^1 + 12 e_0*e_0*e_1*e^0
 - 36 e_0*e_0*e_1*e^1 + 20 e_0*e_1*e_0*e^0 - 40 e_0*e_1*e_0*e^1
 - 20 e_0*e_1*e_1*e^0 + 60 e_0*e_1*e_1*e^1 + 18 e_1*e_0*e_0*e^0
 - 36 e_1*e_0*e_0*e^1 - 18 e_1*e_0*e_1*e^0 + 54 e_1*e_0*e_1*e^1
 - 30 e_1*e_1*e_0*e^0 + 60 e_1*e_1*e_0*e^1 + 30 e_1*e_1*e_1*e^0
 - 90 e_1*e_1*e_1*e^1
```

The parent of $t \otimes a$ is itself a free module of finite rank over $\mathbf{Z}$:

```
sage: T = (t*a).parent() ; T
Free module of type-(3,1) tensors on the Rank-2 free module M over the
 Integer Ring
sage: T.base_ring()
Integer Ring
sage: T.rank()
16
```

Differences between `FiniteRankFreeModule` and `FreeModule` (or `VectorSpace`)

To illustrate the differences, let us create two free modules of rank 3 over $\mathbf{Z}$, one with `FiniteRankFreeModule` and the other one with `FreeModule`:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M') ; M
Rank-3 free module M over the Integer Ring
sage: N = FreeModule(ZZ, 3) ; N
Ambient free module of rank 3 over the principal ideal domain Integer Ring
```

The main difference is that `FreeModule` returns a free module with a distinguished basis, while `FiniteRankFreeModule` does not:

```
sage: N.basis()
[
(1, 0, 0),
(0, 1, 0),
(0, 0, 1)
]
sage: M.bases()
[]
sage: M.print_bases()
No basis has been defined on the Rank-3 free module M over the Integer Ring
```

This is also revealed by the category of each module:

```
sage: M.category()
Category of modules over Integer Ring
sage: N.category()
Category of modules with basis over Integer Ring
```

In other words, the module created by `FreeModule` is actually $\mathbf{Z}^3$, while, in the absence of any distinguished basis, no *canonical* isomorphism relates the module created by `FiniteRankFreeModule` to $\mathbf{Z}^3$:

```
sage: N is ZZ^3
True
sage: M is ZZ^3
False
sage: M == ZZ^3
False
```

Because it is $\mathbf{Z}^3$, `N` is unique, while there may be various modules of the same rank over the same ring created by `FiniteRankFreeModule`; they are then distinguished by their names (actually by the complete sequence of arguments of `FiniteRankFreeModule`):

```
sage: N1 = FreeModule(ZZ, 3) ; N1
Ambient free module of rank 3 over the principal ideal domain Integer Ring
sage: N1 is N  # FreeModule(ZZ, 3) is unique
True
sage: M1 = FiniteRankFreeModule(ZZ, 3, name='M_1') ; M1
Rank-3 free module M_1 over the Integer Ring
sage: M1 is M  # M1 and M are different rank-3 modules over ZZ
False
sage: M1b = FiniteRankFreeModule(ZZ, 3, name='M_1') ; M1b
Rank-3 free module M_1 over the Integer Ring
sage: M1b is M1  # because M1b and M1 have the same name
True
```

As illustrated above, various bases can be introduced on the module created by `FiniteRankFreeModule`:

```
sage: e = M.basis('e') ; e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: f = M.basis('f', from_family=(-e[0], e[1]-e[2], -2*e[1]+3*e[2])) ; f
Basis (f_0,f_1,f_2) on the Rank-3 free module M over the Integer Ring
sage: M.bases()
[Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring,
 Basis (f_0,f_1,f_2) on the Rank-3 free module M over the Integer Ring]
```

Each element of a basis is accessible via its index:

```
sage: e[0]
Element e_0 of the Rank-3 free module M over the Integer Ring
sage: e[0].parent()
Rank-3 free module M over the Integer Ring
sage: f[1]
Element f_1 of the Rank-3 free module M over the Integer Ring
sage: f[1].parent()
Rank-3 free module M over the Integer Ring
```

while on module `N`, the element of the (unique) basis is accessible directly from the module symbol:

```
sage: N.0
(1, 0, 0)
sage: N.1
```

```
(0, 1, 0)
sage: N.0.parent()
Ambient free module of rank 3 over the principal ideal domain Integer Ring
```

The arithmetic of elements is similar; the difference lies in the display: a basis has to be specified for elements of M, while elements of N are displayed directly as elements of $\mathbf{Z}^3$:

```
sage: u = 2*e[0] - 3*e[2] ; u
Element of the Rank-3 free module M over the Integer Ring
sage: u.display(e)
2 e_0 - 3 e_2
sage: u.display(f)
-2 f_0 - 6 f_1 - 3 f_2
sage: u[e,:]
[2, 0, -3]
sage: u[f,:]
[-2, -6, -3]
sage: v = 2*N.0 - 3*N.2 ; v
(2, 0, -3)
```

For the case of M, in order to avoid to specify the basis if the user is always working with the same basis (e.g. only one basis has been defined), the concept of *default basis* has been introduced:

```
sage: M.default_basis()
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: M.print_bases()
Bases defined on the Rank-3 free module M over the Integer Ring:
 - (e_0,e_1,e_2) (default basis)
 - (f_0,f_1,f_2)
```

This is different from the *distinguished basis* of N: it simply means that the mention of the basis can be omitted in function arguments:

```
sage: u.display()  # equivalent to u.display(e)
2 e_0 - 3 e_2
sage: u[:]         # equivalent to u[e,:]
[2, 0, -3]
```

At any time, the default basis can be changed:

```
sage: M.set_default_basis(f)
sage: u.display()
-2 f_0 - 6 f_1 - 3 f_2
```

Another difference between FiniteRankFreeModule and FreeModule is that for the former the range of indices can be specified (by default, it starts from 0):

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1) ; M
Rank-3 free module M over the Integer Ring
sage: e = M.basis('e') ; e  # compare with (e_0,e_1,e_2) above
Basis (e_1,e_2,e_3) on the Rank-3 free module M over the Integer Ring
sage: e[1], e[2], e[3]
(Element e_1 of the Rank-3 free module M over the Integer Ring,
 Element e_2 of the Rank-3 free module M over the Integer Ring,
 Element e_3 of the Rank-3 free module M over the Integer Ring)
```

All the above holds for VectorSpace instead of FreeModule: the object created by VectorSpace is actually a Cartesian power of the base field:

```
sage: V = VectorSpace(QQ,3) ; V
Vector space of dimension 3 over Rational Field
sage: V.category()
Category of vector spaces over Rational Field
sage: V is QQ^3
True
sage: V.basis()
[
(1, 0, 0),
(0, 1, 0),
(0, 0, 1)
]
```

To create a vector space without any distinguished basis, one has to use `FiniteRankFreeModule`:

```
sage: V = FiniteRankFreeModule(QQ, 3, name='V') ; V
3-dimensional vector space V over the Rational Field
sage: V.category()
Category of vector spaces over Rational Field
sage: V.bases()
[]
sage: V.print_bases()
No basis has been defined on the 3-dimensional vector space V over the
 Rational Field
```

The class `FiniteRankFreeModule` has been created for the needs of the SageManifolds project, where free modules do not have any distinguished basis. Too kinds of free modules occur in the context of differentiable manifolds (see here for more details):

- the tangent vector space at any point of the manifold
- the set of vector fields on a parallelizable open subset U of the manifold, which is a free module over the algebra of scalar fields on U.

For instance, without any specific coordinate choice, no basis can be distinguished in a tangent space.

On the other side, the modules created by `FreeModule` have much more algebraic functionalities than those created by `FiniteRankFreeModule`. In particular, submodules have not been implemented yet in `FiniteRankFreeModule`. Moreover, modules resulting from `FreeModule` are tailored to the specific kind of their base ring:

- free module over a commutative ring that is not an integral domain ($\mathbf{Z}/6\mathbf{Z}$):

```
sage: R = IntegerModRing(6) ; R
Ring of integers modulo 6
sage: FreeModule(R, 3)
Ambient free module of rank 3 over Ring of integers modulo 6
sage: type(FreeModule(R, 3))
<class 'sage.modules.free_module.FreeModule_ambient_with_category'>
```

- free module over an integral domain that is not principal ($\mathbf{Z}[X]$):

```
sage: R.<X> = ZZ[] ; R
Univariate Polynomial Ring in X over Integer Ring
sage: FreeModule(R, 3)
Ambient free module of rank 3 over the integral domain Univariate
 Polynomial Ring in X over Integer Ring
sage: type(FreeModule(R, 3))
<class 'sage.modules.free_module.FreeModule_ambient_domain_with_category'>
```

- free module over a principal ideal domain ($\mathbf{Z}$):

```
sage: R = ZZ ; R
Integer Ring
sage: FreeModule(R,3)
Ambient free module of rank 3 over the principal ideal domain Integer Ring
sage: type(FreeModule(R, 3))
<class 'sage.modules.free_module.FreeModule_ambient_pid_with_category'>
```

On the contrary, all objects constructed with `FiniteRankFreeModule` belong to the same class:

```
sage: R = IntegerModRing(6)
sage: type(FiniteRankFreeModule(R, 3))
<class 'sage.tensor.modules.finite_rank_free_module.FiniteRankFreeModule_with_category'>
sage: R.<X> = ZZ[]
sage: type(FiniteRankFreeModule(R, 3))
<class 'sage.tensor.modules.finite_rank_free_module.FiniteRankFreeModule_with_category'>
sage: R = ZZ
sage: type(FiniteRankFreeModule(R, 3))
<class 'sage.tensor.modules.finite_rank_free_module.FiniteRankFreeModule_with_category'>
```

Differences between `FiniteRankFreeModule` and `CombinatorialFreeModule`

An alternative to construct free modules in Sage is `CombinatorialFreeModule`. However, as `FreeModule`, it leads to a module with a distinguished basis:

```
sage: N = CombinatorialFreeModule(ZZ, [1,2,3]) ; N
Free module generated by {1, 2, 3} over Integer Ring
sage: N.category()
Category of modules with basis over Integer Ring
```

The distinguished basis is returned by the method `basis()`:

```
sage: b = N.basis() ; b
Finite family {1: B[1], 2: B[2], 3: B[3]}
sage: b[1]
B[1]
sage: b[1].parent()
Free module generated by {1, 2, 3} over Integer Ring
```

For the free module `M` created above with `FiniteRankFreeModule`, the method `basis` has at least one argument: the symbol string that specifies which basis is required:

```
sage: e = M.basis('e') ; e
Basis (e_1,e_2,e_3) on the Rank-3 free module M over the Integer Ring
sage: e[1]
Element e_1 of the Rank-3 free module M over the Integer Ring
sage: e[1].parent()
Rank-3 free module M over the Integer Ring
```

The arithmetic of elements is similar:

```
sage: u = 2*e[1] - 5*e[3] ; u
Element of the Rank-3 free module M over the Integer Ring
sage: v = 2*b[1] - 5*b[3] ; v
2*B[1] - 5*B[3]
```

One notices that elements of N are displayed directly in terms of their expansions on the distinguished basis. For elements of M, one has to use the method `display()` in order to specify the basis:

```
sage: u.display(e)
2 e_1 - 5 e_3
```

The components on the basis are returned by the square bracket operator for M and by the method `coefficient` for N:

```
sage: [u[e,i] for i in {1,2,3}]
[2, 0, -5]
sage: u[e,:]  # a shortcut for the above
[2, 0, -5]
sage: [v.coefficient(i) for i in {1,2,3}]
[2, 0, -5]
```

class `sage.tensor.modules.finite_rank_free_module.`**`FiniteRankFreeModule`**(*ring*, *rank*, *name=None*, *latex_name=None*, *start_index=0*, *output_formatter=None*)

Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.parent.Parent`

Free module of finite rank over a commutative ring.

A *free module of finite rank* over a commutative ring R is a module M over R that admits a *finite basis*, i.e. a finite familly of linearly independent generators. Since R is commutative, it has the invariant basis number property, so that the rank of the free module M is defined uniquely, as the cardinality of any basis of M.

No distinguished basis of M is assumed. On the contrary, many bases can be introduced on the free module along with change-of-basis rules (as module automorphisms). Each module element has then various representations over the various bases.

Note: The class `FiniteRankFreeModule` does not inherit from class `FreeModule_generic` nor from class `CombinatorialFreeModule`, since both classes deal with modules with a *distinguished basis* (see details *above*). Moreover, following the recommendation exposed in trac ticket #16427 the class `FiniteRankFreeModule` inherits directly from `Parent` (with the category set to `Modules`) and not from the Cython class `Module`.

The class `FiniteRankFreeModule` is a Sage *parent* class, the corresponding *element* class being `FiniteRankFreeModuleElement`.

INPUT:

- `ring` – commutative ring R over which the free module is constructed
- `rank` – positive integer; rank of the free module
- `name` – (default: `None`) string; name given to the free module
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the freemodule; if none is provided, it is set to `name`
- `start_index` – (default: 0) integer; lower bound of the range of indices in bases defined on the free module

- `output_formatter` – (default: `None`) function or unbound method called to format the output of the tensor components; `output_formatter` must take 1 or 2 arguments: the first argument must be an element of the ring R and the second one, if any, some format specification

EXAMPLES:

Free module of rank 3 over $\mathbf{Z}$:

```
sage: FiniteRankFreeModule._clear_cache_() # for doctests only
sage: M = FiniteRankFreeModule(ZZ, 3) ; M
Rank-3 free module over the Integer Ring
sage: M = FiniteRankFreeModule(ZZ, 3, name='M') ; M  # declaration with a name
Rank-3 free module M over the Integer Ring
sage: M.category()
Category of modules over Integer Ring
sage: M.base_ring()
Integer Ring
sage: M.rank()
3
```

If the base ring is a field, the free module is in the category of vector spaces:

```
sage: V = FiniteRankFreeModule(QQ, 3, name='V') ; V
3-dimensional vector space V over the Rational Field
sage: V.category()
Category of vector spaces over Rational Field
```

The LaTeX output is adjusted via the parameter `latex_name`:

```
sage: latex(M)  # the default is the symbol provided in the string ''name''
M
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', latex_name=r'\mathcal{M}')
sage: latex(M)
\mathcal{M}
```

The free module M has no distinguished basis:

```
sage: M in ModulesWithBasis(ZZ)
False
sage: M in Modules(ZZ)
True
```

In particular, no basis is initialized at the module construction:

```
sage: M.print_bases()
No basis has been defined on the Rank-3 free module M over the Integer Ring
sage: M.bases()
[]
```

Bases have to be introduced by means of the method `basis()`, the first defined basis being considered as the *default basis*, meaning it can be skipped in function arguments required a basis (this can be changed by means of the method `set_default_basis()`):

```
sage: e = M.basis('e') ; e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: M.default_basis()
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
```

A second basis can be created from a family of linearly independent elements expressed in terms of basis `e`:

```
sage: f = M.basis('f', from_family=(-e[0], e[1]+e[2], 2*e[1]+3*e[2]))
sage: f
```

```
Basis (f_0,f_1,f_2) on the Rank-3 free module M over the Integer Ring
sage: M.print_bases()
Bases defined on the Rank-3 free module M over the Integer Ring:
 - (e_0,e_1,e_2) (default basis)
 - (f_0,f_1,f_2)
sage: M.bases()
[Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring,
 Basis (f_0,f_1,f_2) on the Rank-3 free module M over the Integer Ring]
```

M is a *parent* object, whose elements are instances of `FiniteRankFreeModuleElement` (actually a dynamically generated subclass of it):

```
sage: v = M.an_element() ; v
Element of the Rank-3 free module M over the Integer Ring
sage: from sage.tensor.modules.free_module_tensor import FiniteRankFreeModuleElement
sage: isinstance(v, FiniteRankFreeModuleElement)
True
sage: v in M
True
sage: M.is_parent_of(v)
True
sage: v.display() # expansion w.r.t. the default basis (e)
e_0 + e_1 + e_2
sage: v.display(f)
-f_0 + f_1
```

The test suite of the category of modules is passed:

```
sage: TestSuite(M).run()
```

Constructing an element of `M` from (the integer) 0 yields the zero element of `M`:

```
sage: M(0)
Element zero of the Rank-3 free module M over the Integer Ring
sage: M(0) is M.zero()
True
```

Non-zero elements are constructed by providing their components in a given basis:

```
sage: v = M([-1,0,3]) ; v  # components in the default basis (e)
Element of the Rank-3 free module M over the Integer Ring
sage: v.display() # expansion w.r.t. the default basis (e)
-e_0 + 3 e_2
sage: v.display(f)
f_0 - 6 f_1 + 3 f_2
sage: v = M([-1,0,3], basis=f) ; v  # components in a specific basis
Element of the Rank-3 free module M over the Integer Ring
sage: v.display(f)
-f_0 + 3 f_2
sage: v.display()
e_0 + 6 e_1 + 9 e_2
sage: v = M([-1,0,3], basis=f, name='v') ; v
Element v of the Rank-3 free module M over the Integer Ring
sage: v.display(f)
v = -f_0 + 3 f_2
sage: v.display()
v = e_0 + 6 e_1 + 9 e_2
```

An alternative is to construct the element from an empty list of componentsand to set the nonzero components afterwards:

```
sage: v = M([], name='v')
sage: v[e,0] = -1
sage: v[e,2] = 3
sage: v.display(e)
v = -e_0 + 3 e_2
```

Indices on the free module, such as indices labelling the element of a basis, are provided by the generator method `irange()`. By default, they range from 0 to the module's rank minus one:

```
sage: for i in M.irange(): print i,
0 1 2
```

This can be changed via the parameter `start_index` in the module construction:

```
sage: M1 = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: for i in M1.irange(): print i,
1 2 3
```

The parameter `output_formatter` in the constructor of the free module is used to set the output format of tensor components:

```
sage: N = FiniteRankFreeModule(QQ, 3, output_formatter=Rational.numerical_approx)
sage: e = N.basis('e')
sage: v = N([1/3, 0, -2], basis=e)
sage: v[e,:]
[0.333333333333333, 0.000000000000000, -2.00000000000000]
sage: v.display(e)  # default format (53 bits of precision)
0.333333333333333 e_0 - 2.00000000000000 e_2
sage: v.display(e, format_spec=10)  # 10 bits of precision
0.33 e_0 - 2.0 e_2
```

Element
alias of `FiniteRankFreeModuleElement`

alternating_form(*degree*, *name=None*, *latex_name=None*)
Construct an alternating form on the free module.

INPUT:

- `degree` – the degree of the alternating form (i.e. its tensor rank)
- `name` – (default: `None`) string; name given to the alternating form
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the alternating form; if none is provided, the LaTeX symbol is set to `name`

OUTPUT:

- instance of `FreeModuleAltForm`

EXAMPLES:

Alternating forms on a rank-3 module:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: a = M.alternating_form(2, 'a') ; a
Alternating form a of degree 2 on the
 Rank-3 free module M over the Integer Ring
```

The nonzero components in a given basis have to be set in a second step, thereby fully specifying the alternating form:

```
sage: e = M.basis('e') ; e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: a.set_comp(e)[0,1] = 2
sage: a.set_comp(e)[1,2] = -3
sage: a.display(e)
a = 2 e^0/\e^1 - 3 e^1/\e^2
```

An alternating form of degree 1 is a linear form:

```
sage: a = M.alternating_form(1, 'a') ; a
Linear form a on the Rank-3 free module M over the Integer Ring
```

To construct such a form, it is preferable to call the method `linear_form()` instead:

```
sage: a = M.linear_form('a') ; a
Linear form a on the Rank-3 free module M over the Integer Ring
```

See `FreeModuleAltForm` for more documentation.

automorphism(*matrix=None*, *basis=None*, *name=None*, *latex_name=None*)

Construct a module automorphism of `self`.

Denoting `self` by M, an automorphism of `self` is an element of the general linear group $\mathrm{GL}(M)$.

INPUT:

- `matrix` – (default: `None`) matrix of size rank(M)*rank(M) representing the automorphism with respect to `basis`; this entry can actually be any material from which a matrix of elements of `self` base ring can be constructed; the *columns* of `matrix` must be the components w.r.t. `basis` of the images of the elements of `basis`. If `matrix` is `None`, the automorphism has to be initialized afterwards by method `set_comp()` or via the operator [].
- `basis` – (default: `None`) basis of `self` defining the matrix representation; if `None` the default basis of `self` is assumed.
- `name` – (default: `None`) string; name given to the automorphism
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the automorphism; if none is provided, the LaTeX symbol is set to `name`

OUTPUT:

- instance of `FreeModuleAutomorphism`

EXAMPLES:

Automorphism of a rank-2 free $\mathbf{Z}$-module:

```
sage: M = FiniteRankFreeModule(ZZ, 2, name='M')
sage: e = M.basis('e')
sage: a = M.automorphism(matrix=[[1,2],[1,3]], basis=e, name='a') ; a
Automorphism a of the Rank-2 free module M over the Integer Ring
sage: a.parent()
General linear group of the Rank-2 free module M over the Integer Ring
sage: a.matrix(e)
[1 2]
[1 3]
```

An automorphism is a tensor of type (1,1):

```
sage: a.tensor_type()
(1, 1)
```

```
sage: a.display(e)
a = e_0*e^0 + 2 e_0*e^1 + e_1*e^0 + 3 e_1*e^1
```

The automorphism components can be specified in a second step, as components of a type-(1,1) tensor:

```
sage: a1 = M.automorphism(name='a')
sage: a1[e,:] = [[1,2],[1,3]]
sage: a1.matrix(e)
[1 2]
[1 3]
sage: a1 == a
True
```

Component by component specification:

```
sage: a2 = M.automorphism(name='a')
sage: a2[0,0] = 1  # component set in the module's default basis (e)
sage: a2[0,1] = 2
sage: a2[1,0] = 1
sage: a2[1,1] = 3
sage: a2.matrix(e)
[1 2]
[1 3]
sage: a2 == a
True
```

See `FreeModuleAutomorphism` for more documentation.

bases()

Return the list of bases that have been defined on the free module `self`.

Use the method `print_bases()` to get a formatted output with more information.

OUTPUT:

- list of instances of class `FreeModuleBasis`

EXAMPLES:

Bases on a rank-3 free module:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M_3', start_index=1)
sage: M.bases()
[]
sage: e = M.basis('e')
sage: M.bases()
[Basis (e_1,e_2,e_3) on the Rank-3 free module M_3 over the Integer Ring]
sage: f = M.basis('f')
sage: M.bases()
[Basis (e_1,e_2,e_3) on the Rank-3 free module M_3 over the Integer Ring,
 Basis (f_1,f_2,f_3) on the Rank-3 free module M_3 over the Integer Ring]
```

basis(*symbol*, *latex_symbol=None*, *from_family=None*)

Define or return a basis of the free module `self`.

Let M denotes the free module `self` and n its rank.

The basis can be defined from a set of n linearly independent elements of M by means of the argument `from_family`. If `from_family` is not specified, the basis is created from scratch and, at this stage, is unrelated to bases that could have been defined previously on M. It can be related afterwards by means of the method `set_change_of_basis()`.

If the basis specified by the given symbol already exists, it is simply returned, whatever the value of the arguments `latex_symbol` or `from_family`.

Note that another way to construct a basis of `self` is to use the method `new_basis()` on an existing basis, with the automorphism relating the two bases as an argument.

INPUT:

- `symbol` – string; a letter (of a few letters) to denote a generic element of the basis
- `latex_symbol` – (default: `None`) string; symbol to denote a generic element of the basis; if `None`, the value of `symbol` is used
- `from_family` – (default: `None`) a tuple of n linearly independent elements of the free module `self` (n being the rank of `self`)

OUTPUT:

- instance of `FreeModuleBasis` representing a basis on `self`

EXAMPLES:

Bases on a rank-3 free module:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e') ; e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: e[0]
Element e_0 of the Rank-3 free module M over the Integer Ring
sage: latex(e)
\left(e_0,e_1,e_2\right)
```

The LaTeX symbol can be set explicitely, as the second argument of `basis()`:

```
sage: eps = M.basis('eps', r'\epsilon') ; eps
Basis (eps_0,eps_1,eps_2) on the Rank-3 free module M
 over the Integer Ring
sage: latex(eps)
\left(\epsilon_0,\epsilon_1,\epsilon_2\right)
```

If the provided symbol is that of an already defined basis, the latter is returned (no new basis is created):

```
sage: M.basis('e') is e
True
sage: M.basis('eps') is eps
True
```

The individual elements of the basis are labelled according the parameter `start_index` provided at the free module construction:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: e = M.basis('e') ; e
Basis (e_1,e_2,e_3) on the Rank-3 free module M over the Integer Ring
sage: e[1]
Element e_1 of the Rank-3 free module M over the Integer Ring
```

Construction of a basis from a family of linearly independent module elements:

```
sage: f1 = -e[2]
sage: f2 = 4*e[1] + 3*e[3]
sage: f3 = 7*e[1] + 5*e[3]
sage: f = M.basis('f', from_family=(f1,f2,f3))
sage: f[1].display()
```

```
f_1 = -e_2
sage: f[2].display()
f_2 = 4 e_1 + 3 e_3
sage: f[3].display()
f_3 = 7 e_1 + 5 e_3
```

The change-of-basis automorphisms have been registered:

```
sage: M.change_of_basis(e,f).matrix(e)
[ 0  4  7]
[-1  0  0]
[ 0  3  5]
sage: M.change_of_basis(f,e).matrix(e)
[ 0 -1  0]
[-5  0  7]
[ 3  0 -4]
sage: M.change_of_basis(f,e) == M.change_of_basis(e,f).inverse()
True
```

Check of the change-of-basis e –> f:

```
sage: a = M.change_of_basis(e,f) ; a
Automorphism of the Rank-3 free module M over the Integer Ring
sage: all( f[i] == a(e[i]) for i in M.irange() )
True
```

For more documentation on bases see `FreeModuleBasis`.

change_of_basis(*basis1*, *basis2*)

Return a module automorphism linking two bases defined on the free module `self`.

If the automorphism has not been recorded yet (in the internal dictionary `self._basis_changes`), it is computed by transitivity, i.e. by performing products of recorded changes of basis.

INPUT:

- `basis1` – a basis of `self`, denoted (e_i) below
- `basis2` – a basis of `self`, denoted (f_i) below

OUTPUT:

- instance of `FreeModuleAutomorphism` describing the automorphism P that relates the basis (e_i) to the basis (f_i) according to $f_i = P(e_i)$

EXAMPLES:

Changes of basis on a rank-2 free module:

```
sage: FiniteRankFreeModule._clear_cache_() # for doctests only
sage: M = FiniteRankFreeModule(ZZ, 2, name='M', start_index=1)
sage: e = M.basis('e')
sage: f = M.basis('f', from_family=(e[1]+2*e[2], e[1]+3*e[2]))
sage: P = M.change_of_basis(e,f) ; P
Automorphism of the Rank-2 free module M over the Integer Ring
sage: P.matrix(e)
[1 1]
[2 3]
```

Note that the columns of this matrix contain the components of the elements of basis `f` w.r.t. to basis `e`:

```
sage: f[1].display(e)
f_1 = e_1 + 2 e_2
sage: f[2].display(e)
f_2 = e_1 + 3 e_2
```

The change of basis is cached:

```
sage: P is M.change_of_basis(e,f)
True
```

Check of the change-of-basis automorphism:

```
sage: f[1] == P(e[1])
True
sage: f[2] == P(e[2])
True
```

Check of the reverse change of basis:

```
sage: M.change_of_basis(f,e) == P^(-1)
True
```

We have of course:

```
sage: M.change_of_basis(e,e)
Identity map of the Rank-2 free module M over the Integer Ring
sage: M.change_of_basis(e,e) is M.identity_map()
True
```

Let us introduce a third basis on M:

```
sage: h = M.basis('h', from_family=(3*e[1]+4*e[2], 5*e[1]+7*e[2]))
```

The change of basis e –> h has been recorded directly from the definition of h:

```
sage: Q = M.change_of_basis(e,h) ; Q.matrix(e)
[3 5]
[4 7]
```

The change of basis f –> h is computed by transitivity, i.e. from the changes of basis f –> e and e –> h:

```
sage: R = M.change_of_basis(f,h) ; R
Automorphism of the Rank-2 free module M over the Integer Ring
sage: R.matrix(e)
[-1  2]
[-2  3]
sage: R.matrix(f)
[ 5  8]
[-2 -3]
```

Let us check that R is indeed the change of basis f –> h:

```
sage: h[1] == R(f[1])
True
sage: h[2] == R(f[2])
True
```

A related check is:

```
sage: R == Q*P^(-1)
True
```

default_basis()

Return the default basis of the free module `self`.

The *default basis* is simply a basis whose name can be skipped in methods requiring a basis as an argument. By default, it is the first basis introduced on the module. It can be changed by the method `set_default_basis()`.

OUTPUT:

- instance of `FreeModuleBasis`

EXAMPLES:

At the module construction, no default basis is assumed:

```
sage: M = FiniteRankFreeModule(ZZ, 2, name='M', start_index=1)
sage: M.default_basis()
No default basis has been defined on the
 Rank-2 free module M over the Integer Ring
```

The first defined basis becomes the default one:

```
sage: e = M.basis('e') ; e
Basis (e_1,e_2) on the Rank-2 free module M over the Integer Ring
sage: M.default_basis()
Basis (e_1,e_2) on the Rank-2 free module M over the Integer Ring
sage: f =  M.basis('f') ; f
Basis (f_1,f_2) on the Rank-2 free module M over the Integer Ring
sage: M.default_basis()
Basis (e_1,e_2) on the Rank-2 free module M over the Integer Ring
```

dual()

Return the dual module of `self`.

EXAMPLE:

Dual of a free module over $\mathbf{Z}$:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: M.dual()
Dual of the Rank-3 free module M over the Integer Ring
sage: latex(M.dual())
M^*
```

The dual is a free module of the same rank as M:

```
sage: isinstance(M.dual(), FiniteRankFreeModule)
True
sage: M.dual().rank()
3
```

It is formed by alternating forms of degree 1, i.e. linear forms:

```
sage: M.dual() is M.dual_exterior_power(1)
True
sage: M.dual().an_element()
Linear form on the Rank-3 free module M over the Integer Ring
sage: a = M.linear_form()
sage: a in M.dual()
True
```

The elements of a dual basis belong of course to the dual module:

```
sage: e = M.basis('e')
sage: e.dual_basis()[0] in M.dual()
True
```

dual_exterior_power(*p*)

Return the p-th exterior power of the dual of `self`.

If M stands for the free module `self`, the *p-th exterior power* of the dual of M is the set $\Lambda^p(M^*)$ of all *alternating forms of degree* p on M, i.e. of all multilinear maps

$$\underbrace{M \times \cdots \times M}_{p \text{ times}} \longrightarrow R$$

that vanish whenever any of two of their arguments are equal. $\Lambda^p(M^*)$ is a free module of rank $\binom{n}{p}$ over the same ring as M, where n is the rank of M.

INPUT:

- `p` – non-negative integer

OUTPUT:

- for $p \geq 1$, instance of `ExtPowerFreeModule` representing the free module $\Lambda^p(M^*)$; for $p = 0$, the base ring R is returned instead

EXAMPLES:

Exterior powers of the dual of a free **Z**-module of rank 3:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: M.dual_exterior_power(0)  # return the base ring
Integer Ring
sage: M.dual_exterior_power(1)  # return the dual module
Dual of the Rank-3 free module M over the Integer Ring
sage: M.dual_exterior_power(1) is M.dual()
True
sage: M.dual_exterior_power(2)
2nd exterior power of the dual of the Rank-3 free module M over the Integer Ring
sage: M.dual_exterior_power(2).an_element()
Alternating form of degree 2 on the Rank-3 free module M over the Integer Ring
sage: M.dual_exterior_power(2).an_element().display()
e^0/\e^1
sage: M.dual_exterior_power(3)
3rd exterior power of the dual of the Rank-3 free module M over the Integer Ring
sage: M.dual_exterior_power(3).an_element()
Alternating form of degree 3 on the Rank-3 free module M over the Integer Ring
sage: M.dual_exterior_power(3).an_element().display()
e^0/\e^1/\e^2
```

See `ExtPowerFreeModule` for more documentation.

endomorphism(*matrix_rep*, *basis=None*, *name=None*, *latex_name=None*)

Contruct an endomorphism of the free module `self`.

The returned object is a module morphism $\phi : M \to M$, where M is `self`.

INPUT:

- `matrix_rep` – matrix of size rank(M)*rank(M) representing the endomorphism with respect to `basis`; this entry can actually be any material from which a matrix of elements of `self` base ring can be constructed; the *columns* of `matrix_rep` must be the components w.r.t. `basis` of the images of the elements of `basis`.

- `basis` – (default: `None`) basis of `self` defining the matrix representation; if None the default basis of `self` is assumed.
- `name` – (default: `None`) string; name given to the endomorphism
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the endomorphism; if none is provided, `name` will be used.

OUTPUT:

- the endomorphism $\phi : M \to M$ corresponding to the given specifications, as an instance of `FiniteRankFreeModuleMorphism`

EXAMPLES:

Construction of an endomorphism with minimal data (module's default basis and no name):

```
sage: M = FiniteRankFreeModule(ZZ, 2, name='M')
sage: e = M.basis('e')
sage: phi = M.endomorphism([[1,-2], [-3,4]]) ; phi
Generic endomorphism of Rank-2 free module M over the Integer Ring
sage: phi.matrix()  # matrix w.r.t the default basis
[ 1 -2]
[-3  4]
```

Construction with full list of arguments (matrix given a basis different from the default one):

```
sage: a = M.automorphism() ; a[0,1], a[1,0] = 1, -1
sage: ep = e.new_basis(a, 'ep', latex_symbol="e'")
sage: phi = M.endomorphism([[1,-2], [-3,4]], basis=ep, name='phi',
....:                       latex_name=r'\phi')
sage: phi
Generic endomorphism of Rank-2 free module M over the Integer Ring
sage: phi.matrix(ep)  # the input matrix
[ 1 -2]
[-3  4]
sage: phi.matrix()  # matrix w.r.t the default basis
[4 3]
[2 1]
```

See `FiniteRankFreeModuleMorphism` for more documentation.

general_linear_group()

Return the general linear group of `self`.

If `self` is the free module M, the *general linear group* is the group $\mathrm{GL}(M)$ of automorphisms of M.

OUTPUT:

- instance of class `FreeModuleLinearGroup` representing $\mathrm{GL}(M)$

EXAMPLES:

The general linear group of a rank-3 free module:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: GL = M.general_linear_group() ; GL
General linear group of the Rank-3 free module M over the Integer Ring
sage: GL.category()
Category of groups
```

```
sage: type(GL)
<class 'sage.tensor.modules.free_module_linear_group.FreeModuleLinearGroup_with_category'>
```

There is a unique instance of the general linear group:

```
sage: M.general_linear_group() is GL
True
```

The group identity element:

```
sage: GL.one()
Identity map of the Rank-3 free module M over the Integer Ring
sage: GL.one().matrix(e)
[1 0 0]
[0 1 0]
[0 0 1]
```

An element:

```
sage: GL.an_element()
Automorphism of the Rank-3 free module M over the Integer Ring
sage: GL.an_element().matrix(e)
[ 1  0  0]
[ 0 -1  0]
[ 0  0  1]
```

See `FreeModuleLinearGroup` for more documentation.

hom(*codomain*, *matrix_rep*, *bases=None*, *name=None*, *latex_name=None*)

Homomorphism from `self` to a free module.

Define a module homomorphism

$$\phi :\ M \longrightarrow N,$$

where M is `self` and N is a free module of finite rank over the same ring R as `self`.

Note: This method is a redefinition of `sage.structure.parent.Parent.hom()` because the latter assumes that `self` has some privileged generators, while an instance of `FiniteRankFreeModule` has no privileged basis.

INPUT:

- `codomain` – the target module N
- `matrix_rep` – matrix of size rank(N)*rank(M) representing the homomorphism with respect to the pair of bases defined by `bases`; this entry can actually be any material from which a matrix of elements of R can be constructed; the *columns* of `matrix_rep` must be the components w.r.t. `basis_N` of the images of the elements of `basis_M`.
- `bases` – (default: `None`) pair `(basis_M, basis_N)` defining the matrix representation, `basis_M` being a basis of `self` and `basis_N` a basis of module N ; if None the pair formed by the default bases of each module is assumed.
- `name` – (default: `None`) string; name given to the homomorphism
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the homomorphism; if None, `name` will be used.

OUTPUT:

- the homomorphism $\phi : M \to N$ corresponding to the given specifications, as an instance of `FiniteRankFreeModuleMorphism`

EXAMPLES:

Homomorphism between two free modules over **Z**:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: N = FiniteRankFreeModule(ZZ, 2, name='N')
sage: e = M.basis('e')
sage: f = N.basis('f')
sage: phi = M.hom(N, [[-1,2,0], [5,1,2]]) ; phi
Generic morphism:
  From: Rank-3 free module M over the Integer Ring
  To:   Rank-2 free module N over the Integer Ring
```

Homomorphism defined by a matrix w.r.t. bases that are not the default ones:

```
sage: ep = M.basis('ep', latex_symbol=r"e'")
sage: fp = N.basis('fp', latex_symbol=r"f'")
sage: phi = M.hom(N, [[3,2,1], [1,2,3]], bases=(ep, fp)) ; phi
Generic morphism:
  From: Rank-3 free module M over the Integer Ring
  To:   Rank-2 free module N over the Integer Ring
```

Call with all arguments specified:

```
sage: phi = M.hom(N, [[3,2,1], [1,2,3]], bases=(ep, fp),
....:             name='phi', latex_name=r'\phi')
```

The parent:

```
sage: phi.parent() is Hom(M,N)
True
```

See class `FiniteRankFreeModuleMorphism` for more documentation.

identity_map (*name='Id'*, *latex_name=None*)

Return the identity map of the free module `self`.

INPUT:

- `name` – (string; default: 'Id') name given to the identity identity map
- `latex_name` – (string; default: `None`) LaTeX symbol to denote the identity map; if none is provided, the LaTeX symbol is set to 'mathrm{Id}' if `name` is 'Id' and to `name` otherwise

OUTPUT:

- the identity map of `self` as an instance of `FreeModuleAutomorphism`

EXAMPLES:

Identity map of a rank-3 **Z**-module:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: Id = M.identity_map() ; Id
Identity map of the Rank-3 free module M over the Integer Ring
sage: Id.parent()
General linear group of the Rank-3 free module M over the Integer Ring
sage: Id.matrix(e)
[1 0 0]
```

```
[0 1 0]
[0 0 1]
```

The default LaTeX symbol:

```
sage: latex(Id)
\mathrm{Id}
```

It can be changed by means of the method `set_name()`:

```
sage: Id.set_name(latex_name=r'\mathrm{1}_M')
sage: latex(Id)
\mathrm{1}_M
```

The identity map is actually the identity element of GL(M):

```
sage: Id is M.general_linear_group().one()
True
```

It is also a tensor of type-(1,1) on M:

```
sage: Id.tensor_type()
(1, 1)
sage: Id.comp(e)
Kronecker delta of size 3x3
sage: Id[:]
[1 0 0]
[0 1 0]
[0 0 1]
```

Example with a LaTeX symbol different from the default one and set at the creation of the object:

```
sage: N = FiniteRankFreeModule(ZZ, 3, name='N')
sage: f = N.basis('f')
sage: Id = N.identity_map(name='Id_N', latex_name=r'\mathrm{Id}_N')
sage: Id
Identity map of the Rank-3 free module N over the Integer Ring
sage: latex(Id)
\mathrm{Id}_N
```

irange(*start=None*)

Single index generator, labelling the elements of a basis of `self`.

INPUT:

- `start` – (default: `None`) integer; initial value of the index; if none is provided, `self._sindex` is assumed

OUTPUT:

- an iterable index, starting from `start` and ending at `self._sindex + self.rank() - 1`

EXAMPLES:

Index range on a rank-3 module:

```
sage: M = FiniteRankFreeModule(ZZ, 3)
sage: for i in M.irange(): print i,
0 1 2
sage: for i in M.irange(start=1): print i,
1 2
```

The default starting value corresponds to the parameter `start_index` provided at the module construction (the default value being 0):

```
sage: M1 = FiniteRankFreeModule(ZZ, 3, start_index=1)
sage: for i in M1.irange(): print i,
1 2 3
sage: M2 = FiniteRankFreeModule(ZZ, 3, start_index=-4)
sage: for i in M2.irange(): print i,
-4 -3 -2
```

linear_form(*name=None*, *latex_name=None*)

Construct a linear form on the free module `self`.

A *linear form* on a free module M over a ring R is a map $M \to R$ that is linear. It can be viewed as a tensor of type $(0, 1)$ on M.

INPUT:

- `name` – (default: `None`) string; name given to the linear form
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the linear form; if none is provided, the LaTeX symbol is set to `name`

OUTPUT:

- instance of `FreeModuleAltForm`

EXAMPLES:

Linear form on a rank-3 free module:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e')
sage: a = M.linear_form('A') ; a
Linear form A on the Rank-3 free module M over the Integer Ring
sage: a[:] = [2,-1,3]  # components w.r.t. the module's default basis (e)
sage: a.display()
A = 2 e^0 - e^1 + 3 e^2
```

A linear form maps module elements to ring elements:

```
sage: v = M([1,1,1])
sage: a(v)
4
```

Test of linearity:

```
sage: u = M([-5,-2,7])
sage: a(3*u - 4*v) == 3*a(u) - 4*a(v)
True
```

See `FreeModuleAltForm` for more documentation.

print_bases()

Display the bases that have been defined on the free module `self`.

Use the method `bases()` to get the raw list of bases.

EXAMPLES:

Bases on a rank-4 free module:

```
sage: M = FiniteRankFreeModule(ZZ, 4, name='M', start_index=1)
sage: M.print_bases()
No basis has been defined on the
```

```
 Rank-4 free module M over the Integer Ring
sage: e = M.basis('e')
sage: M.print_bases()
Bases defined on the Rank-4 free module M over the Integer Ring:
 - (e_1,e_2,e_3,e_4) (default basis)
sage: f = M.basis('f')
sage: M.print_bases()
Bases defined on the Rank-4 free module M over the Integer Ring:
 - (e_1,e_2,e_3,e_4) (default basis)
 - (f_1,f_2,f_3,f_4)
sage: M.set_default_basis(f)
sage: M.print_bases()
Bases defined on the Rank-4 free module M over the Integer Ring:
 - (e_1,e_2,e_3,e_4)
 - (f_1,f_2,f_3,f_4) (default basis)
```

rank()

Return the rank of the free module `self`.

Since the ring over which `self` is built is assumed to be commutative (and hence has the invariant basis number property), the rank is defined uniquely, as the cardinality of any basis of `self`.

EXAMPLES:

Rank of free modules over $\mathbf{Z}$:

```
sage: M = FiniteRankFreeModule(ZZ, 3)
sage: M.rank()
3
sage: M.tensor_module(0,1).rank()
3
sage: M.tensor_module(0,2).rank()
9
sage: M.tensor_module(1,0).rank()
3
sage: M.tensor_module(1,1).rank()
9
sage: M.tensor_module(1,2).rank()
27
sage: M.tensor_module(2,2).rank()
81
```

set_change_of_basis(*basis1*, *basis2*, *change_of_basis*, *compute_inverse=True*)

Relates two bases by an automorphism of `self`.

This updates the internal dictionary `self._basis_changes`.

INPUT:

- `basis1` – basis 1, denoted (e_i) below
- `basis2` – basis 2, denoted (f_i) below
- `change_of_basis` – instance of class `FreeModuleAutomorphism` describing the automorphism P that relates the basis (e_i) to the basis (f_i) according to $f_i = P(e_i)$
- `compute_inverse` (default: `True`) – if set to `True`, the inverse automorphism is computed and the change from basis (f_i) to (e_i) is set to it in the internal dictionary `self._basis_changes`

EXAMPLES:

Defining a change of basis on a rank-2 free module:

```
sage: M = FiniteRankFreeModule(QQ, 2, name='M')
sage: e = M.basis('e')
sage: f = M.basis('f')
sage: a = M.automorphism()
sage: a[:] = [[1, 2], [-1, 3]]
sage: M.set_change_of_basis(e, f, a)
```

The change of basis and its inverse have been recorded:

```
sage: M.change_of_basis(e,f).matrix(e)
[ 1  2]
[-1  3]
sage: M.change_of_basis(f,e).matrix(e)
[ 3/5 -2/5]
[ 1/5  1/5]
```

and are effective:

```
sage: f[0].display(e)
f_0 = e_0 - e_1
sage: e[0].display(f)
e_0 = 3/5 f_0 + 1/5 f_1
```

set_default_basis(*basis*)

Sets the default basis of `self`.

The *default basis* is simply a basis whose name can be skipped in methods requiring a basis as an argument. By default, it is the first basis introduced on the module.

INPUT:

- `basis` – instance of `FreeModuleBasis` representing a basis on `self`

EXAMPLES:

Changing the default basis on a rank-3 free module:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M', start_index=1)
sage: e = M.basis('e') ; e
Basis (e_1,e_2,e_3) on the Rank-3 free module M over the Integer Ring
sage: f =  M.basis('f') ; f
Basis (f_1,f_2,f_3) on the Rank-3 free module M over the Integer Ring
sage: M.default_basis()
Basis (e_1,e_2,e_3) on the Rank-3 free module M over the Integer Ring
sage: M.set_default_basis(f)
sage: M.default_basis()
Basis (f_1,f_2,f_3) on the Rank-3 free module M over the Integer Ring
```

sym_bilinear_form(*name=None*, *latex_name=None*)

Construct a symmetric bilinear form on the free module `self`.

INPUT:

- `name` – (default: `None`) string; name given to the symmetric bilinear form
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the symmetric bilinear form; if none is provided, the LaTeX symbol is set to `name`

OUTPUT:

- instance of `FreeModuleTensor` of tensor type $(0, 2)$ and symmetric

EXAMPLES:

Symmetric bilinear form on a rank-3 free module:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: a = M.sym_bilinear_form('A') ; a
Symmetric bilinear form A on the
 Rank-3 free module M over the Integer Ring
```

A symmetric bilinear form is a type-(0,2) tensor that is symmetric:

```
sage: a.parent()
Free module of type-(0,2) tensors on the
 Rank-3 free module M over the Integer Ring
sage: a.tensor_type()
(0, 2)
sage: a.tensor_rank()
2
sage: a.symmetries()
symmetry: (0, 1);  no antisymmetry
```

Components with respect to a given basis:

```
sage: e = M.basis('e')
sage: a[0,0], a[0,1], a[0,2] = 1, 2, 3
sage: a[1,1], a[1,2] = 4, 5
sage: a[2,2] = 6
```

Only independent components have been set; the other ones are deduced by symmetry:

```
sage: a[1,0], a[2,0], a[2,1]
(2, 3, 5)
sage: a[:]
[1 2 3]
[2 4 5]
[3 5 6]
```

A symmetric bilinear form acts on pairs of module elements:

```
sage: u = M([2,-1,3]) ; v = M([-2,4,1])
sage: a(u,v)
61
sage: a(v,u) == a(u,v)
True
```

The sum of two symmetric bilinear forms is another symmetric bilinear form:

```
sage: b = M.sym_bilinear_form('B')
sage: b[0,0], b[0,1], b[1,2] = -2, 1, -3
sage: s = a + b ; s
Symmetric bilinear form A+B on the
 Rank-3 free module M over the Integer Ring
sage: a[:], b[:], s[:]
(
[1 2 3]  [-2  1  0]  [-1  3  3]
[2 4 5]  [ 1  0 -3]  [ 3  4  2]
[3 5 6], [ 0 -3  0], [ 3  2  6]
)
```

Adding a symmetric bilinear from with a non-symmetric one results in a generic type-$(0, 2)$ tensor:

```
sage: c = M.tensor((0,2), name='C')
sage: c[0,1] = 4
sage: s = a + c ; s
Type-(0,2) tensor A+C on the Rank-3 free module M over the Integer Ring
sage: s.symmetries()
no symmetry;  no antisymmetry
sage: s[:]
[1 6 3]
[2 4 5]
[3 5 6]
```

See `FreeModuleTensor` for more documentation.

tensor (*tensor_type*, *name=None*, *latex_name=None*, *sym=None*, *antisym=None*)

Construct a tensor on the free module `self`.

INPUT:

- `tensor_type` – pair `(k, l)` with `k` being the contravariant rank and `l` the covariant rank
- `name` – (default: `None`) string; name given to the tensor
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the tensor; if none is provided, the LaTeX symbol is set to `name`
- `sym` – (default: `None`) a symmetry or a list of symmetries among the tensor arguments: each symmetry is described by a tuple containing the positions of the involved arguments, with the convention `position = 0` for the first argument. For instance:
  - `sym = (0,1)` for a symmetry between the 1st and 2nd arguments
  - `sym = [(0,2), (1,3,4)]` for a symmetry between the 1st and 3rd arguments and a symmetry between the 2nd, 4th and 5th arguments.
- `antisym` – (default: `None`) antisymmetry or list of antisymmetries among the arguments, with the same convention as for `sym`

OUTPUT:

- instance of `FreeModuleTensor` representing the tensor defined on `self` with the provided characteristics

EXAMPLES:

Tensors on a rank-3 free module:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: t = M.tensor((1,0), name='t') ; t
Element t of the Rank-3 free module M over the Integer Ring
sage: t = M.tensor((0,1), name='t') ; t
Linear form t on the Rank-3 free module M over the Integer Ring
sage: t = M.tensor((1,1), name='t') ; t
Type-(1,1) tensor t on the Rank-3 free module M over the Integer Ring
sage: t = M.tensor((0,2), name='t', sym=(0,1)) ; t
Symmetric bilinear form t on the
 Rank-3 free module M over the Integer Ring
sage: t = M.tensor((0,2), name='t', antisym=(0,1)) ; t
Alternating form t of degree 2 on the
 Rank-3 free module M over the Integer Ring
sage: t = M.tensor((1,2), name='t') ; t
Type-(1,2) tensor t on the Rank-3 free module M over the Integer Ring
```

See `FreeModuleTensor` for more examples and documentation.

tensor_from_comp (*tensor_type*, *comp*, *name=None*, *latex_name=None*)

Construct a tensor on `self` from a set of components.

The tensor symmetries are deduced from those of the components.

INPUT:

- `tensor_type` – pair `(k, l)` with `k` being the contravariant rank and `l` the covariant rank
- `comp` – instance of `Components` representing the tensor components in a given basis
- `name` – (default: `None`) string; name given to the tensor
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the tensor; if none is provided, the LaTeX symbol is set to `name`

OUTPUT:

- instance of `FreeModuleTensor` representing the tensor defined on `self` with the provided characteristics.

EXAMPLES:

Construction of a tensor of rank 1:

```
sage: from sage.tensor.modules.comp import Components, CompWithSym, CompFullySym, CompFullyA
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: e = M.basis('e') ; e
Basis (e_0,e_1,e_2) on the Rank-3 free module M over the Integer Ring
sage: c = Components(ZZ, e, 1)
sage: c[:]
[0, 0, 0]
sage: c[:] = [-1,4,2]
sage: t = M.tensor_from_comp((1,0), c)
sage: t
Element of the Rank-3 free module M over the Integer Ring
sage: t.display(e)
-e_0 + 4 e_1 + 2 e_2
sage: t = M.tensor_from_comp((0,1), c) ; t
Linear form on the Rank-3 free module M over the Integer Ring
sage: t.display(e)
-e^0 + 4 e^1 + 2 e^2
```

Construction of a tensor of rank 2:

```
sage: c = CompFullySym(ZZ, e, 2)
sage: c[0,0], c[1,2] = 4, 5
sage: t = M.tensor_from_comp((0,2), c) ; t
Symmetric bilinear form on the
 Rank-3 free module M over the Integer Ring
sage: t.symmetries()
symmetry: (0, 1);  no antisymmetry
sage: t.display(e)
4 e^0*e^0 + 5 e^1*e^2 + 5 e^2*e^1
sage: c = CompFullyAntiSym(ZZ, e, 2)
sage: c[0,1], c[1,2] = 4, 5
sage: t = M.tensor_from_comp((0,2), c) ; t
Alternating form of degree 2 on the
 Rank-3 free module M over the Integer Ring
sage: t.display(e)
4 e^0/\e^1 + 5 e^1/\e^2
```

tensor_module(k, l)

Return the free module of all tensors of type (k, l) defined on `self`.

INPUT:

- `k` – non-negative integer; the contravariant rank, the tensor type being (k, l)
- `l` – non-negative integer; the covariant rank, the tensor type being (k, l)

OUTPUT:

- instance of `TensorFreeModule` representing the free module $T^{(k,l)}(M)$ of type-(k, l) tensors on the free module `self`

EXAMPLES:

Tensor modules over a free module over $\mathbf{Z}$:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: T = M.tensor_module(1,2) ; T
Free module of type-(1,2) tensors on the Rank-3 free module M
 over the Integer Ring
sage: T.an_element()
Type-(1,2) tensor on the Rank-3 free module M over the Integer Ring
```

Tensor modules are unique:

```
sage: M.tensor_module(1,2) is T
True
```

The base module is itself the module of all type-$(1, 0)$ tensors:

```
sage: M.tensor_module(1,0) is M
True
```

See `TensorFreeModule` for more documentation.

zero()

Return the zero element of `self`.

EXAMPLES:

Zero elements of free modules over $\mathbf{Z}$:

```
sage: M = FiniteRankFreeModule(ZZ, 3, name='M')
sage: M.zero()
Element zero of the Rank-3 free module M over the Integer Ring
sage: M.zero().parent() is M
True
sage: M.zero() is M(0)
True
sage: T = M.tensor_module(1,1)
sage: T.zero()
Type-(1,1) tensor zero on the Rank-3 free module M over the Integer Ring
sage: T.zero().parent() is T
True
sage: T.zero() is T(0)
True
```

Components of the zero element with respect to some basis:

```
sage: e = M.basis('e')
sage: M.zero()[e,:]
[0, 0, 0]
```

```
sage: all(M.zero()[e,i] == M.base_ring().zero() for i in M.irange())
True
sage: T.zero()[e,:]
[0 0 0]
[0 0 0]
[0 0 0]
sage: M.tensor_module(1,2).zero()[e,:]
[[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
 [[0, 0, 0], [0, 0, 0], [0, 0, 0]],
 [[0, 0, 0], [0, 0, 0], [0, 0, 0]]]
```

CHAPTER

SIX

PICKLING FOR THE OLD CDF VECTOR CLASS

AUTHORS:

- Jason Grout

TESTS:

```
sage: v = vector(CDF,[(1,-1), (2,pi), (3,5)])
sage: v
(1.0 - 1.0*I, 2.0 + 3.141592653589793*I, 3.0 + 5.0*I)
sage: loads(dumps(v)) == v
True
```

CHAPTER SEVEN

PICKLING FOR THE OLD RDF VECTOR CLASS

AUTHORS:

- Jason Grout

TESTS:

```
sage: v = vector(RDF, [1,2,3,4])
sage: loads(dumps(v)) == v
True
```

CHAPTER

EIGHT

VECTORS OVER CALLABLE SYMBOLIC RINGS

AUTHOR: – Jason Grout (2010)

EXAMPLES:

```
sage: f(r, theta, z) = (r*cos(theta), r*sin(theta), z)
sage: f.parent()
Vector space of dimension 3 over Callable function ring with arguments (r, theta, z)
sage: f
(r, theta, z) |--> (r*cos(theta), r*sin(theta), z)
sage: f[0]
(r, theta, z) |--> r*cos(theta)
sage: f+f
(r, theta, z) |--> (2*r*cos(theta), 2*r*sin(theta), 2*z)
sage: 3*f
(r, theta, z) |--> (3*r*cos(theta), 3*r*sin(theta), 3*z)
sage: f*f # dot product
(r, theta, z) |--> r^2*cos(theta)^2 + r^2*sin(theta)^2 + z^2
sage: f.diff()(0,1,2) # the matrix derivative
[cos(1)      0      0]
[sin(1)      0      0]
[     0      0      1]
```

TESTS:

```
sage: f(u,v,w) = (2*u+v,u-w,w^2+u)
sage: loads(dumps(f)) == f
True
```

class sage.modules.vector_callable_symbolic_dense.**Vector_callable_symbolic_dense**

Bases: sage.modules.free_module_element.FreeModuleElement_generic_dense

EXAMPLES:

```
sage: type(vector(RR, [-1,0,2/3,pi,oo]))
<type 'sage.modules.free_module_element.FreeModuleElement_generic_dense'>
```

We can initialize with lists, tuples and derived types:

```
sage: from sage.modules.free_module_element import FreeModuleElement_generic_dense
sage: FreeModuleElement_generic_dense(RR^5, [-1,0,2/3,pi,oo])
(-1.00000000000000, 0.000000000000000, 0.666666666666667, 3.14159265358979, +infinity)
sage: FreeModuleElement_generic_dense(RR^5, (-1,0,2/3,pi,oo))
(-1.00000000000000, 0.000000000000000, 0.666666666666667, 3.14159265358979, +infinity)
sage: FreeModuleElement_generic_dense(RR^5, Sequence([-1,0,2/3,pi,oo]))
(-1.00000000000000, 0.000000000000000, 0.666666666666667, 3.14159265358979, +infinity)
```

```
sage: FreeModuleElement_generic_dense(RR^0, 0)
()
```

TESTS:

Disabling coercion can lead to illegal objects:

```
sage: FreeModuleElement_generic_dense(RR^5, [-1,0,2/3,pi,oo], coerce=False)
(-1, 0, 2/3, pi, +Infinity)
```

We test the `copy` flag:

```
sage: from sage.modules.free_module_element import FreeModuleElement_generic_dense
sage: L = [RR(x) for x in (-1,0,2/3,pi,oo)]
sage: FreeModuleElement_generic_dense(RR^5, tuple(L), coerce=False, copy=False)
(-1.00000000000000, 0.000000000000000, 0.666666666666667, 3.14159265358979, +infinity)
sage: v = FreeModuleElement_generic_dense(RR^5, L, coerce=False, copy=False)
sage: L[4] = 42.0
sage: v  # last entry changed since we didn't copy
(-1.00000000000000, 0.000000000000000, 0.666666666666667, 3.14159265358979, 42.0000000000000)

sage: L = [RR(x) for x in (-1,0,2/3,pi,oo)]
sage: v = FreeModuleElement_generic_dense(RR^5, L, coerce=False, copy=True)
sage: L[4] = 42.0
sage: v  # last entry did not change
(-1.00000000000000, 0.000000000000000, 0.666666666666667, 3.14159265358979, +infinity)
```

Check that trac ticket #11751 is fixed:

```
sage: K.<x> = QQ[]
sage: M = K^1
sage: N = M.span([[1/x]]); N
Free module of degree 1 and rank 1 over Univariate Polynomial Ring in x over Rational Field
Echelon basis matrix:
[1/x]
sage: N([1/x]) # this used to fail prior to #11751
(1/x)
sage: N([1/x^2])
Traceback (most recent call last):
...
TypeError: element [1/x^2] is not in free module

sage: L=K^2
sage: R=L.span([[x,0],[0,1/x]], check=False, already_echelonized=True)
sage: R.basis()[0][0].parent()
Fraction Field of Univariate Polynomial Ring in x over Rational Field
sage: R=L.span([[x,x^2]])
sage: R.basis()[0][0].parent()
Univariate Polynomial Ring in x over Rational Field
```

CHAPTER

NINE

SPACE OF MORPHISMS OF VECTOR SPACES (LINEAR TRANSFORMATIONS)

AUTHOR:

- Rob Beezer: (2011-06-29)

A `VectorSpaceHomspace` object represents the set of all possible homomorphisms from one vector space to another. These mappings are usually known as linear transformations.

For more information on the use of linear transformations, consult the documentation for vector space morphisms at `sage.modules.vector_space_morphism`. Also, this is an extremely thin veneer on free module homspaces (`sage.modules.free_module_homspace`) and free module morphisms (`sage.modules.free_module_morphism`) - objects which might also be useful, and places where much of the documentation resides.

EXAMPLES:

Creation and basic examination is simple.

```
sage: V = QQ^3
sage: W = QQ^2
sage: H = Hom(V, W)
sage: H
Set of Morphisms (Linear Transformations) from
Vector space of dimension 3 over Rational Field to
Vector space of dimension 2 over Rational Field
sage: H.domain()
Vector space of dimension 3 over Rational Field
sage: H.codomain()
Vector space of dimension 2 over Rational Field
```

Homspaces have a few useful properties. A basis is provided by a list of matrix representations, where these matrix representatives are relative to the bases of the domain and codomain.

```
sage: K = Hom(GF(3)^2, GF(3)^2)
sage: B = K.basis()
sage: for f in B:
...     print f, "\n"
Vector space morphism represented by the matrix:
[1 0]
[0 0]
Domain: Vector space of dimension 2 over Finite Field of size 3
Codomain: Vector space of dimension 2 over Finite Field of size 3

Vector space morphism represented by the matrix:
[0 1]
```

```
[0 0]
Domain: Vector space of dimension 2 over Finite Field of size 3
Codomain: Vector space of dimension 2 over Finite Field of size 3

Vector space morphism represented by the matrix:
[0 0]
[1 0]
Domain: Vector space of dimension 2 over Finite Field of size 3
Codomain: Vector space of dimension 2 over Finite Field of size 3

Vector space morphism represented by the matrix:
[0 0]
[0 1]
Domain: Vector space of dimension 2 over Finite Field of size 3
Codomain: Vector space of dimension 2 over Finite Field of size 3
```

The zero and identity mappings are properties of the space. The identity mapping will only be available if the domain and codomain allow for endomorphisms (equal vector spaces with equal bases).

```
sage: H = Hom(QQ^3, QQ^3)
sage: g = H.zero()
sage: g([1, 1/2, -3])
(0, 0, 0)
sage: f = H.identity()
sage: f([1, 1/2, -3])
(1, 1/2, -3)
```

The homspace may be used with various representations of a morphism in the space to create the morphism. We demonstrate three ways to create the same linear transformation between two two-dimensional subspaces of `QQ^3`. The `V.n` notation is a shortcut to the generators of each vector space, better known as the basis elements. Note that the matrix representations are relative to the bases, which are purposely fixed when the subspaces are created ("user bases").

```
sage: U = QQ^3
sage: V = U.subspace_with_basis([U.0+U.1, U.1-U.2])
sage: W = U.subspace_with_basis([U.0, U.1+U.2])
sage: H = Hom(V, W)
```

First, with a matrix. Note that the matrix representation acts by matrix multiplication with the vector on the left. The input to the linear transformation, `(3, 1, 2)`, is converted to the coordinate vector `(3, -2)`, then matrix multiplication yields the vector `(-3, -2)`, which represents the vector `(-3, -2, -2)` in the codomain.

```
sage: m = matrix(QQ, [[1, 2], [3, 4]])
sage: f1 = H(m)
sage: f1
Vector space morphism represented by the matrix:
[1 2]
[3 4]
Domain: Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[ 1  1  0]
[ 0  1 -1]
Codomain: Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[1 0 0]
[0 1 1]
sage: f1([3,1,2])
(-3, -2, -2)
```

Second, with a list of images of the domain's basis elements.

```
sage: img = [1*(U.0) + 2*(U.1+U.2), 3*U.0 + 4*(U.1+U.2)]
sage: f2 = H(img)
sage: f2
Vector space morphism represented by the matrix:
[1 2]
[3 4]
Domain: Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[ 1  1  0]
[ 0  1 -1]
Codomain: Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[1 0 0]
[0 1 1]
sage: f2([3,1,2])
(-3, -2, -2)
```

Third, with a linear function taking the domain to the codomain.

```
sage: g = lambda x: vector(QQ, [-2*x[0]+3*x[1], -2*x[0]+4*x[1], -2*x[0]+4*x[1]])
sage: f3 = H(g)
sage: f3
Vector space morphism represented by the matrix:
[1 2]
[3 4]
Domain: Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[ 1  1  0]
[ 0  1 -1]
Codomain: Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[1 0 0]
[0 1 1]
sage: f3([3,1,2])
(-3, -2, -2)
```

The three linear transformations look the same, and are the same.

```
sage: f1 == f2
True
sage: f2 == f3
True
```

TESTS:

```
sage: V = QQ^2
sage: W = QQ^3
sage: H = Hom(QQ^2, QQ^3)
sage: loads(dumps(H))
Set of Morphisms (Linear Transformations) from
Vector space of dimension 2 over Rational Field to
Vector space of dimension 3 over Rational Field
sage: loads(dumps(H)) == H
True
```

class sage.modules.vector_space_homspace.**VectorSpaceHomspace**(*X*, *Y*, *category=None*, *check=True*, *base=None*)

Bases: sage.modules.free_module_homspace.FreeModuleHomspace

TESTS:

```
sage: X = ZZ['x']; X.rename("X")
sage: Y = ZZ['y']; Y.rename("Y")
sage: class MyHomset(HomsetWithBase):
...       def my_function(self, x):
...           return Y(x[0])
...       def _an_element_(self):
...           return sage.categories.morphism.SetMorphism(self, self.my_function)
...
sage: import __main__; __main__.MyHomset = MyHomset # fakes MyHomset being defined in a Python m
sage: H = MyHomset(X, Y, category=Monoids())
sage: H
Set of Morphisms from X to Y in Category of monoids
sage: H.base()
Integer Ring
sage: TestSuite(H).run()
```

sage.modules.vector_space_homspace.**is_VectorSpaceHomspace**(*x*)

Return True if x is a vector space homspace.

INPUT:

x - anything

EXAMPLES:

To be a vector space morphism, the domain and codomain must both be vector spaces, in other words, modules over fields. If either set is just a module, then the Hom() constructor will build a space of free module morphisms.

```
sage: H = Hom(QQ^3, QQ^2)
sage: type(H)
<class 'sage.modules.vector_space_homspace.VectorSpaceHomspace_with_category'>
sage: sage.modules.vector_space_homspace.is_VectorSpaceHomspace(H)
True

sage: K = Hom(QQ^3, ZZ^2)
sage: type(K)
<class 'sage.modules.free_module_homspace.FreeModuleHomspace_with_category'>
sage: sage.modules.vector_space_homspace.is_VectorSpaceHomspace(K)
False

sage: L = Hom(ZZ^3, QQ^2)
sage: type(L)
<class 'sage.modules.free_module_homspace.FreeModuleHomspace_with_category'>
sage: sage.modules.vector_space_homspace.is_VectorSpaceHomspace(L)
False

sage: sage.modules.vector_space_homspace.is_VectorSpaceHomspace('junk')
False
```

CHAPTER

TEN

VECTOR SPACE MORPHISMS (AKA LINEAR TRANSFORMATIONS)

AUTHOR:

- Rob Beezer: (2011-06-29)

A vector space morphism is a homomorphism between vector spaces, better known as a linear transformation. These are a specialization of Sage's free module homomorphisms. (A free module is like a vector space, but with scalars from a ring that may not be a field.) So references to free modules in the documentation or error messages should be understood as simply reflectng a more general situation.

10.1 Creation

The constructor `linear_transformation()` is designed to accept a variety of inputs that can define a linear transformation. See the documentation of the function for all the possibilities. Here we give two.

First a matrix representation. By default input matrices are understood to act on vectors placed to left of the matrix. Optionally, an input matrix can be described as acting on vectors placed to the right.

```
sage: A = matrix(QQ, [[-1, 2, 3], [4, 2, 0]])
sage: phi = linear_transformation(A)
sage: phi
Vector space morphism represented by the matrix:
[-1  2  3]
[ 4  2  0]
Domain: Vector space of dimension 2 over Rational Field
Codomain: Vector space of dimension 3 over Rational Field
sage: phi([2, -3])
(-14, -2, 6)
```

A symbolic function can be used to specify the "rule" for a linear transformation, along with explicit descriptions of the domain and codomain.

```
sage: F = Integers(13)
sage: D = F^3
sage: C = F^2
sage: x, y, z = var('x y z')
sage: f(x, y, z) = [2*x + 3*y + 5*z, x + z]
sage: rho = linear_transformation(D, C, f)
sage: f(1, 2, 3)
(23, 4)
sage: rho([1, 2, 3])
(10, 4)
```

A "vector space homspace" is the set of all linear transformations between two vector spaces. Various input can be coerced into a homspace to create a linear transformation. See `sage.modules.vector_space_homspace` for more.

```
sage: D = QQ^4
sage: C = QQ^2
sage: hom_space = Hom(D, C)
sage: images = [[1, 3], [2, -1], [4, 0], [3, 7]]
sage: zeta = hom_space(images)
sage: zeta
Vector space morphism represented by the matrix:
[ 1  3]
[ 2 -1]
[ 4  0]
[ 3  7]
Domain: Vector space of dimension 4 over Rational Field
Codomain: Vector space of dimension 2 over Rational Field
```

A homomorphism may also be created via a method on the domain.

```
sage: F = QQ[sqrt(3)]
sage: a = F.gen(0)
sage: D = F^2
sage: C = F^2
sage: A = matrix(F, [[a, 1], [2*a, 2]])
sage: psi = D.hom(A, C)
sage: psi
Vector space morphism represented by the matrix:
[  sqrt3       1]
[2*sqrt3       2]
Domain: Vector space of dimension 2 over Number Field in sqrt3 with defining polynomial x^2 - 3
Codomain: Vector space of dimension 2 over Number Field in sqrt3 with defining polynomial x^2 - 3
sage: psi([1, 4])
(9*sqrt3, 9)
```

10.2 Properties

Many natural properties of a linear transformation can be computed. Some of these are more general methods of objects in the classes `sage.modules.free_module_morphism.FreeModuleMorphism` and `sage.modules.matrix_morphism.MatrixMorphism`.

Values are computed in a natural way, an inverse image of an element can be computed with the `lift()` method, when the inverse image actually exists.

```
sage: A = matrix(QQ, [[1,2], [2,4], [3,6]])
sage: phi = linear_transformation(A)
sage: phi([1,2,0])
(5, 10)
sage: phi.lift([10, 20])
(10, 0, 0)
sage: phi.lift([100, 100])
Traceback (most recent call last):
...
ValueError: element is not in the image
```

Images and pre-images can be computed as vector spaces.

```
sage: A = matrix(QQ, [[1,2], [2,4], [3,6]])
sage: phi = linear_transformation(A)
sage: phi.image()
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[1 2]

sage: phi.inverse_image( (QQ^2).span([[1,2]]) )
Vector space of degree 3 and dimension 3 over Rational Field
Basis matrix:
[1 0 0]
[0 1 0]
[0 0 1]

sage: phi.inverse_image( (QQ^2).span([[1,1]]) )
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[   1    0 -1/3]
[   0    1 -2/3]
```

Injectivity and surjectivity can be checked.

```
sage: A = matrix(QQ, [[1,2], [2,4], [3,6]])
sage: phi = linear_transformation(A)
sage: phi.is_injective()
False
sage: phi.is_surjective()
False
```

10.3 Restrictions and Representations

It is possible to restrict the domain and codomain of a linear transformation to make a new linear transformation. We will use those commands to replace the domain and codomain by equal vector spaces, but with alternate bases. The point here is that the matrix representation used to represent linear transformations are relative to the bases of both the domain and codomain.

```
sage: A = graphs.PetersenGraph().adjacency_matrix()
sage: V = QQ^10
sage: phi = linear_transformation(V, V, A)
sage: phi
Vector space morphism represented by the matrix:
[0 1 0 0 1 1 0 0 0 0]
[1 0 1 0 0 0 1 0 0 0]
[0 1 0 1 0 0 0 1 0 0]
[0 0 1 0 1 0 0 0 1 0]
[1 0 0 1 0 0 0 0 0 1]
[1 0 0 0 0 0 0 1 1 0]
[0 1 0 0 0 0 0 0 1 1]
[0 0 1 0 0 1 0 0 0 1]
[0 0 0 1 0 1 1 0 0 0]
[0 0 0 0 1 0 1 1 0 0]
Domain: Vector space of dimension 10 over Rational Field
Codomain: Vector space of dimension 10 over Rational Field

sage: B1 = [V.gen(i) + V.gen(i+1) for i in range(9)] + [V.gen(9)]
```

```
sage: B2 = [V.gen(0)] + [-V.gen(i-1) + V.gen(i) for i in range(1,10)]
sage: D = V.subspace_with_basis(B1)
sage: C = V.subspace_with_basis(B2)
sage: rho = phi.restrict_codomain(C)
sage: zeta = rho.restrict_domain(D)
sage: zeta
Vector space morphism represented by the matrix:
[6 5 4 3 3 2 1 0 0 0]
[6 5 4 3 2 2 2 1 0 0]
[6 6 5 4 3 2 2 2 1 0]
[6 5 5 4 3 2 2 2 2 1]
[6 4 4 4 3 3 3 3 2 1]
[6 5 4 4 4 4 4 4 3 1]
[6 6 5 4 4 4 3 3 3 2]
[6 6 6 5 4 4 2 1 1 1]
[6 6 6 6 5 4 3 1 0 0]
[3 3 3 3 3 2 2 1 0 0]
Domain: Vector space of degree 10 and dimension 10 over Rational Field
User basis matrix:
[1 1 0 0 0 0 0 0 0 0]
[0 1 1 0 0 0 0 0 0 0]
[0 0 1 1 0 0 0 0 0 0]
[0 0 0 1 1 0 0 0 0 0]
[0 0 0 0 1 1 0 0 0 0]
[0 0 0 0 0 1 1 0 0 0]
[0 0 0 0 0 0 1 1 0 0]
[0 0 0 0 0 0 0 1 1 0]
[0 0 0 0 0 0 0 0 1 1]
[0 0 0 0 0 0 0 0 0 1]
Codomain: Vector space of degree 10 and dimension 10 over Rational Field
User basis matrix:
[ 1  0  0  0  0  0  0  0  0  0]
[-1  1  0  0  0  0  0  0  0  0]
[ 0 -1  1  0  0  0  0  0  0  0]
[ 0  0 -1  1  0  0  0  0  0  0]
[ 0  0  0 -1  1  0  0  0  0  0]
[ 0  0  0  0 -1  1  0  0  0  0]
[ 0  0  0  0  0 -1  1  0  0  0]
[ 0  0  0  0  0  0 -1  1  0  0]
[ 0  0  0  0  0  0  0 -1  1  0]
[ 0  0  0  0  0  0  0  0 -1  1]
```

An endomorphism is a linear transformation with an equal domain and codomain, and here each needs to have the same basis. We are using a matrix that has well-behaved eigenvalues, as part of showing that these do not change as the representation changes.

```
sage: A = graphs.PetersenGraph().adjacency_matrix()
sage: V = QQ^10
sage: phi = linear_transformation(V, V, A)
sage: phi.eigenvalues()
[3, -2, -2, -2, -2, 1, 1, 1, 1, 1]

sage: B1 = [V.gen(i) + V.gen(i+1) for i in range(9)] + [V.gen(9)]
sage: C = V.subspace_with_basis(B1)
sage: zeta = phi.restrict(C)
sage: zeta
Vector space morphism represented by the matrix:
[ 1  0  1 -1  2 -1  2 -2  2 -2]
```

```
[ 1  0  1  0  0  0  1  0  0  0]
[ 0  1  0  1  0  0  0  1  0  0]
[ 1 -1  2 -1  2 -2  2 -2  3 -2]
[ 2 -2  2 -1  1 -1  1  0  1  0]
[ 1  0  0  0  0  0  0  1  1  0]
[ 0  1  0  0  0  1 -1  1  0  2]
[ 0  0  1  0  0  2 -1  1 -1  2]
[ 0  0  0  1  0  1  1  0  0  0]
[ 0  0  0  0  1 -1  2 -1  1 -1]
Domain: Vector space of degree 10 and dimension 10 over Rational Field
User basis matrix:
[1 1 0 0 0 0 0 0 0 0]
[0 1 1 0 0 0 0 0 0 0]
[0 0 1 1 0 0 0 0 0 0]
[0 0 0 1 1 0 0 0 0 0]
[0 0 0 0 1 1 0 0 0 0]
[0 0 0 0 0 1 1 0 0 0]
[0 0 0 0 0 0 1 1 0 0]
[0 0 0 0 0 0 0 1 1 0]
[0 0 0 0 0 0 0 0 1 1]
[0 0 0 0 0 0 0 0 0 1]
Codomain: Vector space of degree 10 and dimension 10 over Rational Field
User basis matrix:
[1 1 0 0 0 0 0 0 0 0]
[0 1 1 0 0 0 0 0 0 0]
[0 0 1 1 0 0 0 0 0 0]
[0 0 0 1 1 0 0 0 0 0]
[0 0 0 0 1 1 0 0 0 0]
[0 0 0 0 0 1 1 0 0 0]
[0 0 0 0 0 0 1 1 0 0]
[0 0 0 0 0 0 0 1 1 0]
[0 0 0 0 0 0 0 0 1 1]
[0 0 0 0 0 0 0 0 0 1]

sage: zeta.eigenvalues()
[3, -2, -2, -2, -2, 1, 1, 1, 1, 1]
```

10.4 Equality

Equality of linear transformations is a bit nuanced. The equality operator == tests if two linear transformations have equal matrix representations, while we determine if two linear transformations are the same function with the `.is_equal_function()` method. Notice in this example that the function never changes, just the representations.

```
sage: f = lambda x: vector(QQ, [x[1], x[0]+x[1], x[0]])
sage: H = Hom(QQ^2, QQ^3)
sage: phi = H(f)

sage: rho = linear_transformation(QQ^2, QQ^3, matrix(QQ,2, 3, [[0,1,1], [1,1,0]]))

sage: phi == rho
True

sage: U = (QQ^2).subspace_with_basis([[1, 2], [-3, 1]])
sage: V = (QQ^3).subspace_with_basis([[0, 1, 0], [2, 3, 1], [-1, 1, 6]])
sage: K = Hom(U, V)
```

```
sage: zeta = K(f)

sage: zeta == phi
False
sage: zeta.is_equal_function(phi)
True
sage: zeta.is_equal_function(rho)
True
```

TESTS:

```
sage: V = QQ^2
sage: H = Hom(V, V)
sage: f = H([V.1,-2*V.0])
sage: loads(dumps(f))
Vector space morphism represented by the matrix:
[ 0  1]
[-2  0]
Domain: Vector space of dimension 2 over Rational Field
Codomain: Vector space of dimension 2 over Rational Field
sage: loads(dumps(f)) == f
True
```

class sage.modules.vector_space_morphism.**VectorSpaceMorphism**(*homspace*, *A*)

Bases: sage.modules.free_module_morphism.FreeModuleMorphism

Create a linear transformation, a morphism between vector spaces.

INPUT:

- homspace - a homspace (of vector spaces) to serve as a parent for the linear transformation and a home for the domain and codomain of the morphism
- A - a matrix representing the linear transformation, which will act on vectors placed to the left of the matrix

EXAMPLES:

Nominally, we require a homspace to hold the domain and codomain and a matrix representation of the morphism (linear transformation).

```
sage: from sage.modules.vector_space_homspace import VectorSpaceHomspace
sage: from sage.modules.vector_space_morphism import VectorSpaceMorphism
sage: H = VectorSpaceHomspace(QQ^3, QQ^2)
sage: A = matrix(QQ, 3, 2, range(6))
sage: zeta = VectorSpaceMorphism(H, A)
sage: zeta
Vector space morphism represented by the matrix:
[0 1]
[2 3]
[4 5]
Domain: Vector space of dimension 3 over Rational Field
Codomain: Vector space of dimension 2 over Rational Field
```

See the constructor, sage.modules.vector_space_morphism.linear_transformation() for another way to create linear transformations.

The .hom() method of a vector space will create a vector space morphism.

```
sage: V = QQ^3; W = V.subspace_with_basis([[1,2,3], [-1,2,5/3], [0,1,-1]])
sage: phi = V.hom(matrix(QQ, 3, range(9)), codomain=W) # indirect doctest
```

```
sage: type(phi)
<class 'sage.modules.vector_space_morphism.VectorSpaceMorphism'>
```

A matrix may be coerced into a vector space homspace to create a vector space morphism.

```
sage: from sage.modules.vector_space_homspace import VectorSpaceHomspace
sage: H = VectorSpaceHomspace(QQ^3, QQ^2)
sage: A = matrix(QQ, 3, 2, range(6))
sage: rho = H(A)  # indirect doctest
sage: type(rho)
<class 'sage.modules.vector_space_morphism.VectorSpaceMorphism'>
```

is_invertible()

Determines if the vector space morphism has an inverse.

OUTPUT:

`True` if the vector space morphism is invertible, otherwise `False`.

EXAMPLES:

If the dimension of the domain does not match the dimension of the codomain, then the morphism cannot be invertible.

```
sage: V = QQ^3
sage: U = V.subspace_with_basis([V.0 + V.1, 2*V.1 + 3*V.2])
sage: phi = V.hom([U.0, U.0 + U.1, U.0 - U.1], U)
sage: phi.is_invertible()
False
```

An invertible linear transformation.

```
sage: A = matrix(QQ, 3, [[-3, 5, -5], [4, -7, 7], [6, -8, 10]])
sage: A.determinant()
2
sage: H = Hom(QQ^3, QQ^3)
sage: rho = H(A)
sage: rho.is_invertible()
True
```

A non-invertible linear transformation, an endomorphism of a vector space over a finite field.

```
sage: F.<a> = GF(11^2)
sage: A = matrix(F, [[6*a + 3,   8*a +  2, 10*a + 3],
...                  [2*a + 7,   4*a +  3,  2*a + 3],
...                  [9*a + 2,  10*a + 10,  3*a + 3]])
sage: A.nullity()
1
sage: E = End(F^3)
sage: zeta = E(A)
sage: zeta.is_invertible()
False
```

sage.modules.vector_space_morphism.**is_VectorSpaceMorphism**(*x*)

Returns `True` if `x` is a vector space morphism (a linear transformation).

INPUT:

`x` - anything

OUTPUT:

True only if x is an instance of a vector space morphism, which are also known as linear transformations.

EXAMPLES:

```
sage: V = QQ^2; f = V.hom([V.1,-2*V.0])
sage: sage.modules.vector_space_morphism.is_VectorSpaceMorphism(f)
True
sage: sage.modules.vector_space_morphism.is_VectorSpaceMorphism('junk')
False
```

sage.modules.vector_space_morphism.**linear_transformation**(*arg0*, *arg1=None*, *arg2=None*, *side='left'*)

Create a linear transformation from a variety of possible inputs.

FORMATS:

In the following, D and C are vector spaces over the same field that are the domain and codomain (respectively) of the linear transformation.

side is a keyword that is either 'left' or 'right'. When a matrix is used to specify a linear transformation, as in the first two call formats below, you may specify if the function is given by matrix multiplication with the vector on the left, or the vector on the right. The default is 'left'. Internally representations are always carried as the 'left' version, and the default text representation is this version. However, the matrix representation may be obtained as either version, no matter how it is created.

- linear_transformation(A, side='left')

  Where A is a matrix. The domain and codomain are inferred from the dimension of the matrix and the base ring of the matrix. The base ring must be a field, or have its fraction field implemented in Sage.

- linear_transformation(D, C, A, side='left')

  A is a matrix that behaves as above. However, now the domain and codomain are given explicitly. The matrix is checked for compatibility with the domain and codomain. Additionally, the domain and codomain may be supplied with alternate ("user") bases and the matrix is interpreted as being a representation relative to those bases.

- linear_transformation(D, C, f)

  f is any function that can be applied to the basis elements of the domain and that produces elements of the codomain. The linear transformation returned is the unique linear transformation that extends this mapping on the basis elements. f may come from a function defined by a Python def statement, or may be defined as a lambda function.

  Alternatively, f may be specified by a callable symbolic function, see the examples below for a demonstration.

- linear_transformation(D, C, images)

  images is a list, or tuple, of codomain elements, equal in number to the size of the basis of the domain. Each basis element of the domain is mapped to the corresponding element of the images list, and the linear transformation returned is the unique linear transfromation that extends this mapping.

OUTPUT:

A linear transformation described by the input. This is a "vector space morphism", an object of the class sage.modules.vector_space_morphism.

EXAMPLES:

We can define a linear transformation with just a matrix, understood to act on a vector placed on one side or the other. The field for the vector spaces used as domain and codomain is obtained from the base ring of the matrix, possibly promoting to a fraction field.

```
sage: A = matrix(ZZ, [[1, -1, 4], [2, 0, 5]])
sage: phi = linear_transformation(A)
sage: phi
Vector space morphism represented by the matrix:
[ 1 -1  4]
[ 2  0  5]
Domain: Vector space of dimension 2 over Rational Field
Codomain: Vector space of dimension 3 over Rational Field
sage: phi([1/2, 5])
(21/2, -1/2, 27)

sage: B = matrix(Integers(7), [[1, 2, 1], [3, 5, 6]])
sage: rho = linear_transformation(B, side='right')
sage: rho
Vector space morphism represented by the matrix:
[1 3]
[2 5]
[1 6]
Domain: Vector space of dimension 3 over Ring of integers modulo 7
Codomain: Vector space of dimension 2 over Ring of integers modulo 7
sage: rho([2, 4, 6])
(2, 6)
```

We can define a linear transformation with a matrix, while explicitly giving the domain and codomain. Matrix entries will be coerced into the common field of scalars for the vector spaces.

```
sage: D = QQ^3
sage: C = QQ^2
sage: A = matrix([[1, 7], [2, -1], [0, 5]])
sage: A.parent()
Full MatrixSpace of 3 by 2 dense matrices over Integer Ring
sage: zeta = linear_transformation(D, C, A)
sage: zeta.matrix().parent()
Full MatrixSpace of 3 by 2 dense matrices over Rational Field
sage: zeta
Vector space morphism represented by the matrix:
[ 1  7]
[ 2 -1]
[ 0  5]
Domain: Vector space of dimension 3 over Rational Field
Codomain: Vector space of dimension 2 over Rational Field
```

Matrix representations are relative to the bases for the domain and codomain.

```
sage: u = vector(QQ, [1, -1])
sage: v = vector(QQ, [2, 3])
sage: D = (QQ^2).subspace_with_basis([u, v])
sage: x = vector(QQ, [2, 1])
sage: y = vector(QQ, [-1, 4])
sage: C = (QQ^2).subspace_with_basis([x, y])
sage: A = matrix(QQ, [[2, 5], [3, 7]])
sage: psi = linear_transformation(D, C, A)
sage: psi
Vector space morphism represented by the matrix:
[2 5]
[3 7]
Domain: Vector space of degree 2 and dimension 2 over Rational Field
User basis matrix:
[ 1 -1]
```

```
[ 2  3]
Codomain: Vector space of degree 2 and dimension 2 over Rational Field
User basis matrix:
[ 2  1]
[-1  4]
sage: psi(u) == 2*x + 5*y
True
sage: psi(v) == 3*x + 7*y
True
```

Functions that act on the domain may be used to compute images of the domain's basis elements, and this mapping can be extended to a unique linear transformation. The function may be a Python function (via `def` or `lambda`) or a Sage symbolic function.

```
sage: def g(x):
...     return vector(QQ, [2*x[0]+x[2], 5*x[1]])
...
sage: phi = linear_transformation(QQ^3, QQ^2, g)
sage: phi
Vector space morphism represented by the matrix:
[2 0]
[0 5]
[1 0]
Domain: Vector space of dimension 3 over Rational Field
Codomain: Vector space of dimension 2 over Rational Field

sage: f = lambda x: vector(QQ, [2*x[0]+x[2], 5*x[1]])
sage: rho = linear_transformation(QQ^3, QQ^2, f)
sage: rho
Vector space morphism represented by the matrix:
[2 0]
[0 5]
[1 0]
Domain: Vector space of dimension 3 over Rational Field
Codomain: Vector space of dimension 2 over Rational Field

sage: x, y, z = var('x y z')
sage: h(x, y, z) = [2*x + z, 5*y]
sage: zeta = linear_transformation(QQ^3, QQ^2, h)
sage: zeta
Vector space morphism represented by the matrix:
[2 0]
[0 5]
[1 0]
Domain: Vector space of dimension 3 over Rational Field
Codomain: Vector space of dimension 2 over Rational Field

sage: phi == rho
True
sage: rho == zeta
True
```

We create a linear transformation relative to non-standard bases, and capture its representation relative to standard bases. With this, we can build functions that create the same linear transformation relative to the nonstandard bases.

```
sage: u = vector(QQ, [1, -1])
sage: v = vector(QQ, [2, 3])
sage: D = (QQ^2).subspace_with_basis([u, v])
```

```
sage: x = vector(QQ, [2, 1])
sage: y = vector(QQ, [-1, 4])
sage: C = (QQ^2).subspace_with_basis([x, y])
sage: A = matrix(QQ, [[2, 5], [3, 7]])
sage: psi = linear_transformation(D, C, A)
sage: rho = psi.restrict_codomain(QQ^2).restrict_domain(QQ^2)
sage: rho.matrix()
[ -4/5  97/5]
[  1/5 -13/5]

sage: f = lambda x: vector(QQ, [(-4/5)*x[0] + (1/5)*x[1], (97/5)*x[0] + (-13/5)*x[1]])
sage: psi = linear_transformation(D, C, f)
sage: psi.matrix()
[2 5]
[3 7]

sage: s, t = var('s t')
sage: h(s, t) = [(-4/5)*s + (1/5)*t, (97/5)*s + (-13/5)*t]
sage: zeta = linear_transformation(D, C, h)
sage: zeta.matrix()
[2 5]
[3 7]
```

Finally, we can give an explicit list of images for the basis elements of the domain.

```
sage: x = polygen(QQ)
sage: F.<a> = NumberField(x^3+x+1)
sage: u = vector(F, [1, a, a^2])
sage: v = vector(F, [a, a^2, 2])
sage: w = u + v
sage: D = F^3
sage: C = F^3
sage: rho = linear_transformation(D, C, [u, v, w])
sage: rho.matrix()
[      1       a     a^2]
[      a     a^2       2]
[  a + 1 a^2 + a a^2 + 2]
sage: C = (F^3).subspace_with_basis([u, v])
sage: D = (F^3).subspace_with_basis([u, v])
sage: psi = linear_transformation(C, D, [u+v, u-v])
sage: psi.matrix()
[ 1  1]
[ 1 -1]
```

TESTS:

We test some bad inputs. First, the wrong things in the wrong places.

```
sage: linear_transformation('junk')
Traceback (most recent call last):
...
TypeError: first argument must be a matrix or a vector space, not junk

sage: linear_transformation(QQ^2, QQ^3, 'stuff')
Traceback (most recent call last):
...
TypeError: third argument must be a matrix, function, or list of images, not stuff

sage: linear_transformation(QQ^2, 'garbage')
```

```
Traceback (most recent call last):
...
TypeError: if first argument is a vector space, then second argument must be a vector space, not

sage: linear_transformation(QQ^2, Integers(7)^2)
Traceback (most recent call last):
...
TypeError: vector spaces must have the same field of scalars, not Rational Field and Ring of int
```

Matrices must be over a field (or a ring that can be promoted to a field), and of the right size.

```
sage: linear_transformation(matrix(Integers(6), [[2, 3],[4, 5]]))
Traceback (most recent call last):
...
TypeError: matrix must have entries from a field, or a ring with a fraction field, not Ring of i

sage: A = matrix(QQ, 3, 4, range(12))
sage: linear_transformation(QQ^4, QQ^4, A)
Traceback (most recent call last):
...
TypeError: domain dimension is incompatible with matrix size

sage: linear_transformation(QQ^3, QQ^3, A, side='right')
Traceback (most recent call last):
...
TypeError: domain dimension is incompatible with matrix size

sage: linear_transformation(QQ^3, QQ^3, A)
Traceback (most recent call last):
...
TypeError: codomain dimension is incompatible with matrix size

sage: linear_transformation(QQ^4, QQ^4, A, side='right')
Traceback (most recent call last):
...
TypeError: codomain dimension is incompatible with matrix size
```

Lists of images can be of the wrong number, or not really elements of the codomain.

```
sage: linear_transformation(QQ^3, QQ^2, [vector(QQ, [1,2])])
Traceback (most recent call last):
...
ValueError: number of images should equal the size of the domain's basis (=3), not 1

sage: C = (QQ^2).subspace_with_basis([vector(QQ, [1,1])])
sage: linear_transformation(QQ^1, C, [vector(QQ, [1,2])])
Traceback (most recent call last):
...
ArithmeticError: some proposed image is not in the codomain, because
element [1, 2] is not in free module
```

Functions may not apply properly to domain elements, or return values outside the codomain.

```
sage: f = lambda x: vector(QQ, [x[0], x[4]])
sage: linear_transformation(QQ^3, QQ^2, f)
Traceback (most recent call last):
...
ValueError: function cannot be applied properly to some basis element because
index out of range
```

```
sage: f = lambda x: vector(QQ, [x[0], x[1]])
sage: C = (QQ^2).span([vector(QQ, [1, 1])])
sage: linear_transformation(QQ^2, C, f)
Traceback (most recent call last):
...
ArithmeticError: some image of the function is not in the codomain, because
element [1, 0] is not in free module
```

A Sage symbolic function can come in a variety of forms that are not representative of a linear transformation.

```
sage: x, y = var('x, y')
sage: f(x, y) = [y, x, y]
sage: linear_transformation(QQ^3, QQ^3, f)
Traceback (most recent call last):
...
ValueError: symbolic function has the wrong number of inputs for domain

sage: linear_transformation(QQ^2, QQ^2, f)
Traceback (most recent call last):
...
ValueError: symbolic function has the wrong number of outputs for codomain

sage: x, y = var('x y')
sage: f(x, y) = [y, x*y]
sage: linear_transformation(QQ^2, QQ^2, f)
Traceback (most recent call last):
...
ValueError: symbolic function must be linear in all the inputs:
unable to convert y to a rational

sage: x, y = var('x y')
sage: f(x, y) = [x, 2*y]
sage: C = (QQ^2).span([vector(QQ, [1, 1])])
sage: linear_transformation(QQ^2, C, f)
Traceback (most recent call last):
...
ArithmeticError: some image of the function is not in the codomain, because
element [1, 0] is not in free module
```

CHAPTER

ELEVEN

HOMSPACES BETWEEN FREE MODULES

EXAMPLES: We create $\mathrm{End}(\mathbf{Z}^2)$ and compute a basis.

```
sage: M = FreeModule(IntegerRing(),2)
sage: E = End(M)
sage: B = E.basis()
sage: len(B)
4
sage: B[0]
Free module morphism defined by the matrix
[1 0]
[0 0]
Domain: Ambient free module of rank 2 over the principal ideal domain ...
Codomain: Ambient free module of rank 2 over the principal ideal domain ...
```

We create $\mathrm{Hom}(\mathbf{Z}^3, \mathbf{Z}^2)$ and compute a basis.

```
sage: V3 = FreeModule(IntegerRing(),3)
sage: V2 = FreeModule(IntegerRing(),2)
sage: H = Hom(V3,V2)
sage: H
Set of Morphisms from Ambient free module of rank 3
over the principal ideal domain Integer Ring to
Ambient free module of rank 2
over the principal ideal domain Integer Ring
in Category of modules with basis over Integer Ring
sage: B = H.basis()
sage: len(B)
6
sage: B[0]
Free module morphism defined by the matrix
[1 0]
[0 0]
[0 0]...
```

TESTS:

```
sage: H = Hom(QQ^2, QQ^1)
sage: loads(dumps(H)) == H
True
```

See trac 5886:

```
sage: V = (ZZ^2).span_of_basis([[1,2],[3,4]])
sage: V.hom([V.0, V.1])
Free module morphism defined by the matrix
```

```
[1 0]
[0 1]...
```

class sage.modules.free_module_homspace.**FreeModuleHomspace**(*X*, *Y*, *category=None*, *check=True*, *base=None*)

Bases: sage.categories.homset.HomsetWithBase

TESTS:

```
sage: X = ZZ['x']; X.rename("X")
sage: Y = ZZ['y']; Y.rename("Y")
sage: class MyHomset(HomsetWithBase):
...       def my_function(self, x):
...           return Y(x[0])
...       def _an_element_(self):
...           return sage.categories.morphism.SetMorphism(self, self.my_function)
...
sage: import __main__; __main__.MyHomset = MyHomset # fakes MyHomset being defined in a Python m
sage: H = MyHomset(X, Y, category=Monoids())
sage: H
Set of Morphisms from X to Y in Category of monoids
sage: H.base()
Integer Ring
sage: TestSuite(H).run()
```

basis()

Return a basis for this space of free module homomorphisms.

OUTPUT:

- tuple

EXAMPLES:

```
sage: H = Hom(ZZ^2, ZZ^1)
sage: H.basis()
(Free module morphism defined by the matrix
[1]
[0]
Domain: Ambient free module of rank 2 over the principal ideal domain ...
Codomain: Ambient free module of rank 1 over the principal ideal domain ..., Free module mor
[0]
[1]
Domain: Ambient free module of rank 2 over the principal ideal domain ...
Codomain: Ambient free module of rank 1 over the principal ideal domain ...)
```

identity()

Return identity morphism in an endomorphism ring.

EXAMPLE:

```
sage: V=FreeModule(ZZ,5)
sage: H=V.Hom(V)
sage: H.identity()
Free module morphism defined by the matrix
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]
Domain: Ambient free module of rank 5 over the principal ideal domain ...
```

```
Codomain: Ambient free module of rank 5 over the principal ideal domain ...
```

zero()

EXAMPLES:

```
sage: E = ZZ^2
sage: F = ZZ^3
sage: H = Hom(E, F)
sage: f = H.zero()
sage: f
Free module morphism defined by the matrix
[0 0 0]
[0 0 0]
Domain: Ambient free module of rank 2 over the principal ideal domain Integer Ring
Codomain: Ambient free module of rank 3 over the principal ideal domain Integer Ring
sage: f(E.an_element())
(0, 0, 0)
sage: f(E.an_element()) == F.zero()
True
```

TESTS:

We check that `H.zero()` is picklable:

```
sage: loads(dumps(f.parent().zero()))
Free module morphism defined by the matrix
[0 0 0]
[0 0 0]
Domain: Ambient free module of rank 2 over the principal ideal domain Integer Ring
Codomain: Ambient free module of rank 3 over the principal ideal domain Integer Ring
```

`sage.modules.free_module_homspace.`**is_FreeModuleHomspace**(*x*)

Return `True` if `x` is a free module homspace.

EXAMPLES:

Notice that every vector space is a free module, but when we construct a set of morphisms between two vector spaces, it is a `VectorSpaceHomspace`, which qualifies as a `FreeModuleHomspace`, since the former is special case of the latter.

sage: H = Hom(ZZ^3, ZZ^2) sage: type(H) <class 'sage.modules.free_module_homspace.FreeModuleHomspace_with_categ
sage: sage.modules.free_module_homspace.is_FreeModuleHomspace(H) True

sage: K = Hom(QQ^3, ZZ^2) sage: type(K) <class 'sage.modules.free_module_homspace.FreeModuleHomspace_with_cate
sage: sage.modules.free_module_homspace.is_FreeModuleHomspace(K) True

sage: L = Hom(ZZ^3, QQ^2) sage: type(L) <class 'sage.modules.free_module_homspace.FreeModuleHomspace_with_categ
sage: sage.modules.free_module_homspace.is_FreeModuleHomspace(L) True

sage: P = Hom(QQ^3, QQ^2) sage: type(P) <class 'sage.modules.vector_space_homspace.VectorSpaceHomspace_with_cate
sage: sage.modules.free_module_homspace.is_FreeModuleHomspace(P) True

sage: sage.modules.free_module_homspace.is_FreeModuleHomspace('junk') False

CHAPTER

TWELVE

MORPHISMS OF FREE MODULES

AUTHORS:

- William Stein: initial version
- Miguel Marco (2010-06-19): added eigenvalues, eigenvectors and minpoly functions

TESTS:

```
sage: V = ZZ^2; f = V.hom([V.1,-2*V.0])
sage: loads(dumps(f))
Free module morphism defined by the matrix
[ 0  1]
[-2  0]
Domain: Ambient free module of rank 2 over the principal ideal domain ...
Codomain: Ambient free module of rank 2 over the principal ideal domain ...
sage: loads(dumps(f)) == f
True
```

class `sage.modules.free_module_morphism.`**`FreeModuleMorphism`**(*parent*, *A*)

Bases: `sage.modules.matrix_morphism.MatrixMorphism`

INPUT:

- `parent` - a homspace in a (sub) category of free modules
- `A` - matrix

EXAMPLES:

```
sage: V = ZZ^3; W = span([[1,2,3],[-1,2,8]], ZZ)
sage: phi = V.hom(matrix(ZZ,3,[1..9]))
sage: type(phi)
<class 'sage.modules.free_module_morphism.FreeModuleMorphism'>
```

`change_ring`(*R*)

Change the ring over which this morphism is defined. This changes the ring of the domain, codomain, and underlying matrix.

EXAMPLES:

```
sage: V0 = span([[0,0,1],[0,2,0]],ZZ); V1 = span([[1/2,0],[0,2]],ZZ); W = span([[1,0],[0,6]]
sage: h = V0.hom([-3*V1.0-3*V1.1, -3*V1.0-3*V1.1])
sage: h.base_ring()
Integer Ring
sage: h
Free module morphism defined by the matrix
[-3 -3]
[-3 -3]...
```

```
sage: h.change_ring(QQ).base_ring()
Rational Field
sage: f = h.change_ring(QQ); f
Vector space morphism represented by the matrix:
[-3 -3]
[-3 -3]
Domain: Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[0 1 0]
[0 0 1]
Codomain: Vector space of degree 2 and dimension 2 over Rational Field
Basis matrix:
[1 0]
[0 1]
sage: f = h.change_ring(GF(7)); f
Vector space morphism represented by the matrix:
[4 4]
[4 4]
Domain: Vector space of degree 3 and dimension 2 over Finite Field of size 7
Basis matrix:
[0 1 0]
[0 0 1]
Codomain: Vector space of degree 2 and dimension 2 over Finite Field of size 7
Basis matrix:
[1 0]
[0 1]
```

eigenspaces (*extend=True*)

Compute a list of subspaces formed by eigenvectors of `self`.

INPUT:

- `extend` – (default: `True`) determines if field extensions should be considered

OUTPUT:

- a list of pairs `(eigenvalue, eigenspace)`

EXAMPLES:

```
sage: V = QQ^3
sage: h = V.hom([[1,0,0],[0,0,1],[0,-1,0]], V)
sage: h.eigenspaces()
[(1,
  Vector space of degree 3 and dimension 1 over Rational Field
  Basis matrix:
  [1 0 0]),
 (-1*I,
  Vector space of degree 3 and dimension 1 over Algebraic Field
  Basis matrix:
  [  0   1 1*I]),
 (1*I,
  Vector space of degree 3 and dimension 1 over Algebraic Field
  Basis matrix:
  [   0    1 -1*I])]

sage: h.eigenspaces(extend=False)
[(1,
  Vector space of degree 3 and dimension 1 over Rational Field
  Basis matrix:
```

```
  [1 0 0])]

sage: h = V.hom([[2,1,0], [0,2,0], [0,0,-1]], V)
sage: h.eigenspaces()
[(-1, Vector space of degree 3 and dimension 1 over Rational Field
  Basis matrix:
  [0 0 1]),
 (2, Vector space of degree 3 and dimension 1 over Rational Field
  Basis matrix:
  [0 1 0])]

sage: h = V.hom([[2,1,0], [0,2,0], [0,0,2]], V)
sage: h.eigenspaces()
[(2, Vector space of degree 3 and dimension 2 over Rational Field
  Basis matrix:
  [0 1 0]
  [0 0 1])]
```

eigenvalues(*extend=True*)

Returns a list with the eigenvalues of the endomorphism of vector spaces.

INPUT:

- `extend` – boolean (default: True) decides if base field extensions should be considered or not.

EXAMPLES:

We compute the eigenvalues of an endomorphism of $\mathbf{Q}^3$:

```
sage: V=QQ^3
sage: H=V.endomorphism_ring()([[1,-1,0],[-1,1,1],[0,3,1]])
sage: H.eigenvalues()
[3, 1, -1]
```

Note the effect of the `extend` option:

```
sage: V=QQ^2
sage: H=V.endomorphism_ring()([[0,-1],[1,0]])
sage: H.eigenvalues()
[-1*I, 1*I]
sage: H.eigenvalues(extend=False)
[]
```

eigenvectors(*extend=True*)

Computes the subspace of eigenvectors of a given eigenvalue.

INPUT:

- `extend` – boolean (default: True) decides if base field extensions should be considered or not.

OUTPUT:

A sequence of tuples. Each tuple contains an eigenvalue, a sequence with a basis of the corresponding subspace of eigenvectors, and the algebraic multiplicity of the eigenvalue.

EXAMPLES:

```
sage: V=(QQ^4).subspace([[0,2,1,4],[1,2,5,0],[1,1,1,1]])
sage: H=(V.Hom(V))(matrix(QQ, [[0,1,0],[-1,0,0],[0,0,3]]))
sage: H.eigenvectors()
[(3, [
(0, 0, 1, -6/7)
```

```
], 1), (-1*I, [
(1, 1*I, 0, -0.571428571428572? + 2.428571428571429?*I)
], 1), (1*I, [
(1, -1*I, 0, -0.571428571428572? - 2.428571428571429?*I)
], 1)]
sage: H.eigenvectors(extend=False)
[(3, [
(0, 0, 1, -6/7)
], 1)]
sage: H1=(V.Hom(V))(matrix(QQ, [[2,1,0],[0,2,0],[0,0,3]]))
sage: H1.eigenvectors()
[(3, [
(0, 0, 1, -6/7)
], 1), (2, [
(0, 1, 0, 17/7)
], 2)]
sage: H1.eigenvectors(extend=False)
[(3, [
(0, 0, 1, -6/7)
], 1), (2, [
(0, 1, 0, 17/7)
], 2)]
```

inverse_image(*V*)

Given a submodule V of the codomain of self, return the inverse image of V under self, i.e., the biggest submodule of the domain of self that maps into V.

EXAMPLES:

We test computing inverse images over a field:

```
sage: V = QQ^3; W = span([[1,2,3],[-1,2,5/3]], QQ)
sage: phi = V.hom(matrix(QQ,3,[1..9]))
sage: phi.rank()
2
sage: I = phi.inverse_image(W); I
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[   1    0    0]
[   0    1 -1/2]
sage: phi(I.0) in W
True
sage: phi(I.1) in W
True
sage: W = phi.image()
sage: phi.inverse_image(W) == V
True
```

We test computing inverse images between two spaces embedded in different ambient spaces.:

```
sage: V0 = span([[0,0,1],[0,2,0]],ZZ); V1 = span([[1/2,0],[0,2]],ZZ); W = span([[1,0],[0,6]]
sage: h = V0.hom([-3*V1.0-3*V1.1, -3*V1.0-3*V1.1])
sage: h.inverse_image(W)
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[0 2 1]
[0 0 2]
sage: h(h.inverse_image(W)).is_submodule(W)
True
sage: h(h.inverse_image(W)).index_in(W)
```

```
+Infinity
sage: h(h.inverse_image(W))
Free module of degree 2 and rank 1 over Integer Ring
Echelon basis matrix:
[ 3 12]
```

We test computing inverse images over the integers:

```
sage: V = QQ^3; W = V.span_of_basis([[2,2,3],[-1,2,5/3]], ZZ)
sage: phi = W.hom([W.0, W.0-W.1])
sage: Z = W.span([2*W.1]); Z
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[    2    -4 -10/3]
sage: Y = phi.inverse_image(Z); Y
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[  6   0 8/3]
sage: phi(Y) == Z
True
```

lift (*x*)

Given an element of the image, return an element of the codomain that maps onto it.

Note that `lift` and `preimage_representative` are equivalent names for this method, with the latter suggesting that the return value is a coset representative of the domain modulo the kernel of the morphism.

EXAMPLE:

```
sage: X = QQ**2
sage: V = X.span([[2, 0], [0, 8]], ZZ)
sage: W = (QQ**1).span([[1/12]], ZZ)
sage: f = V.hom([W([1/3]), W([1/2])], W)
sage: f.lift([1/3])
(8, -16)
sage: f.lift([1/2])
(12, -24)
sage: f.lift([1/6])
(4, -8)
sage: f.lift([1/12])
Traceback (most recent call last):
...
ValueError: element is not in the image
sage: f.lift([1/24])
Traceback (most recent call last):
...
TypeError: element [1/24] is not in free module
```

This works for vector spaces, too:

```
sage: V = VectorSpace(GF(3), 2)
sage: W = VectorSpace(GF(3), 3)
sage: f = V.hom([W.1, W.1 - W.0])
sage: f.lift(W.1)
(1, 0)
sage: f.lift(W.2)
Traceback (most recent call last):
...
ValueError: element is not in the image
```

```
sage: w = W((17, -2, 0))
sage: f(f.lift(w)) == w
True
```

This example illustrates the use of the `preimage_representative` as an equivalent name for this method.

```
sage: V = ZZ^3
sage: W = ZZ^2
sage: w = vector(ZZ, [1,2])
sage: f = V.hom([w, w, w], W)
sage: f.preimage_representative(vector(ZZ, [10, 20]))
(0, 0, 10)
```

minimal_polynomial(*var='x'*)

Computes the minimal polynomial.

`minpoly()` and `minimal_polynomial()` are the same method.

INPUT:

- `var` - string (default: 'x') a variable name

OUTPUT:

polynomial in var - the minimal polynomial of the endomorphism.

EXAMPLES:

Compute the minimal polynomial, and check it.

```
sage: V=GF(7)^3
sage: H=V.Hom(V)([[0,1,2],[-1,0,3],[2,4,1]])
sage: H
Vector space morphism represented by the matrix:
[0 1 2]
[6 0 3]
[2 4 1]
Domain: Vector space of dimension 3 over Finite Field of size 7
Codomain: Vector space of dimension 3 over Finite Field of size 7

sage: H.minpoly()
x^3 + 6*x^2 + 6*x + 1

sage: H.minimal_polynomial()
x^3 + 6*x^2 + 6*x + 1

sage: H^3 + (H^2)*6 + H*6 + 1
Vector space morphism represented by the matrix:
[0 0 0]
[0 0 0]
[0 0 0]
Domain: Vector space of dimension 3 over Finite Field of size 7
Codomain: Vector space of dimension 3 over Finite Field of size 7
```

minpoly(*var='x'*)

Computes the minimal polynomial.

`minpoly()` and `minimal_polynomial()` are the same method.

INPUT:

•var - string (default: 'x') a variable name

OUTPUT:

polynomial in var - the minimal polynomial of the endomorphism.

EXAMPLES:

Compute the minimal polynomial, and check it.

```
sage: V=GF(7)^3
sage: H=V.Hom(V)([[0,1,2],[-1,0,3],[2,4,1]])
sage: H
Vector space morphism represented by the matrix:
[0 1 2]
[6 0 3]
[2 4 1]
Domain: Vector space of dimension 3 over Finite Field of size 7
Codomain: Vector space of dimension 3 over Finite Field of size 7

sage: H.minpoly()
x^3 + 6*x^2 + 6*x + 1

sage: H.minimal_polynomial()
x^3 + 6*x^2 + 6*x + 1

sage: H^3 + (H^2)*6 + H*6 + 1
Vector space morphism represented by the matrix:
[0 0 0]
[0 0 0]
[0 0 0]
Domain: Vector space of dimension 3 over Finite Field of size 7
Codomain: Vector space of dimension 3 over Finite Field of size 7
```

preimage_representative(*x*)

Given an element of the image, return an element of the codomain that maps onto it.

Note that `lift` and `preimage_representative` are equivalent names for this method, with the latter suggesting that the return value is a coset representative of the domain modulo the kernel of the morphism.

EXAMPLE:

```
sage: X = QQ**2
sage: V = X.span([[2, 0], [0, 8]], ZZ)
sage: W = (QQ**1).span([[1/12]], ZZ)
sage: f = V.hom([W([1/3]), W([1/2])], W)
sage: f.lift([1/3])
(8, -16)
sage: f.lift([1/2])
(12, -24)
sage: f.lift([1/6])
(4, -8)
sage: f.lift([1/12])
Traceback (most recent call last):
...
ValueError: element is not in the image
sage: f.lift([1/24])
Traceback (most recent call last):
...
TypeError: element [1/24] is not in free module
```

This works for vector spaces, too:

```
sage: V = VectorSpace(GF(3), 2)
sage: W = VectorSpace(GF(3), 3)
sage: f = V.hom([W.1, W.1 - W.0])
sage: f.lift(W.1)
(1, 0)
sage: f.lift(W.2)
Traceback (most recent call last):
...
ValueError: element is not in the image
sage: w = W((17, -2, 0))
sage: f(f.lift(w)) == w
True
```

This example illustrates the use of the `preimage_representative` as an equivalent name for this method.

```
sage: V = ZZ^3
sage: W = ZZ^2
sage: w = vector(ZZ, [1,2])
sage: f = V.hom([w, w, w], W)
sage: f.preimage_representative(vector(ZZ, [10, 20]))
(0, 0, 10)
```

`sage.modules.free_module_morphism.`**`is_FreeModuleMorphism`**`(x)`

EXAMPLES:

```
sage: V = ZZ^2; f = V.hom([V.1,-2*V.0])
sage: sage.modules.free_module_morphism.is_FreeModuleMorphism(f)
True
sage: sage.modules.free_module_morphism.is_FreeModuleMorphism(0)
False
```

CHAPTER

THIRTEEN

MORPHISMS DEFINED BY A MATRIX

A matrix morphism is a morphism that is defined by multiplication by a matrix. Elements of domain must either have a method `vector()` that returns a vector that the defining matrix can hit from the left, or be coercible into vector space of appropriate dimension.

EXAMPLES:

```
sage: from sage.modules.matrix_morphism import MatrixMorphism, is_MatrixMorphism
sage: V = QQ^3
sage: T = End(V)
sage: M = MatrixSpace(QQ,3)
sage: I = M.identity_matrix()
sage: m = MatrixMorphism(T, I); m
Morphism defined by the matrix
[1 0 0]
[0 1 0]
[0 0 1]
sage: is_MatrixMorphism(m)
True
sage: m.charpoly('x')
x^3 - 3*x^2 + 3*x - 1
sage: m.base_ring()
Rational Field
sage: m.det()
1
sage: m.fcp('x')
(x - 1)^3
sage: m.matrix()
[1 0 0]
[0 1 0]
[0 0 1]
sage: m.rank()
3
sage: m.trace()
3
```

AUTHOR:

- William Stein: initial versions
- David Joyner (2005-12-17): added examples
- William Stein (2005-01-07): added __reduce__
- Craig Citro (2008-03-18): refactored MatrixMorphism class
- Rob Beezer (2011-07-15): additional methods, bug fixes, documentation

class sage.modules.matrix_morphism.**MatrixMorphism**(*parent*, *A*, *copy_matrix=True*)
Bases: sage.modules.matrix_morphism.MatrixMorphism_abstract

A morphism defined by a matrix.

INPUT:

- parent – a homspace
- A – matrix or a MatrixMorphism_abstract instance
- copy_matrix – (default: True) make an immutable copy of the matrix A if it is mutable; if False, then this makes A immutable

is_injective()
Tell whether self is injective.

EXAMPLE:

```
sage: V1 = QQ^2
sage: V2 = QQ^3
sage: phi = V1.hom(Matrix([[1,2,3],[4,5,6]]),V2)
sage: phi.is_injective()
True
sage: psi = V2.hom(Matrix([[1,2],[3,4],[5,6]]),V1)
sage: psi.is_injective()
False
```

AUTHOR:

– Simon King (2010-05)

is_surjective()
Tell whether self is surjective.

EXAMPLES:

```
sage: V1 = QQ^2
sage: V2 = QQ^3
sage: phi = V1.hom(Matrix([[1,2,3],[4,5,6]]), V2)
sage: phi.is_surjective()
False
sage: psi = V2.hom(Matrix([[1,2],[3,4],[5,6]]), V1)
sage: psi.is_surjective()
True
```

An example over a PID that is not $\mathbf{Z}$.

```
sage: R = PolynomialRing(QQ, 'x')
sage: A = R^2
sage: B = R^2
sage: H = A.hom([B([x^2-1, 1]), B([x^2, 1])])
sage: H.image()
Free module of degree 2 and rank 2 over Univariate Polynomial Ring in x over Rational Field
Echelon basis matrix:
[ 1  0]
[ 0 -1]
sage: H.is_surjective()
True
```

This tests if Trac #11552 is fixed.

```
sage: V = ZZ^2
sage: m = matrix(ZZ, [[1,2],[0,2]])
```

```
sage: phi = V.hom(m, V)
sage: phi.lift(vector(ZZ, [0, 1]))
Traceback (most recent call last):
...
ValueError: element is not in the image
sage: phi.is_surjective()
False
```

AUTHORS:

- Simon King (2010-05)
- Rob Beezer (2011-06-28)

matrix (*side='left'*)

Return a matrix that defines this morphism.

INPUT:

- `side` – (default: `'left'`) the side of the matrix where a vector is placed to effect the morphism (function)

OUTPUT:

A matrix which represents the morphism, relative to bases for the domain and codomain. If the modules are provided with user bases, then the representation is relative to these bases.

Internally, Sage represents a matrix morphism with the matrix multiplying a row vector placed to the left of the matrix. If the option `side='right'` is used, then a matrix is returned that acts on a vector to the right of the matrix. These two matrices are just transposes of each other and the difference is just a preference for the style of representation.

EXAMPLES:

```
sage: V = ZZ^2; W = ZZ^3
sage: m = column_matrix([3*V.0 - 5*V.1, 4*V.0 + 2*V.1, V.0 + V.1])
sage: phi = V.hom(m, W)
sage: phi.matrix()
[ 3  4  1]
[-5  2  1]

sage: phi.matrix(side='right')
[ 3 -5]
[ 4  2]
[ 1  1]
```

TESTS:

```
sage: V = ZZ^2
sage: phi = V.hom([3*V.0, 2*V.1])
sage: phi.matrix(side='junk')
Traceback (most recent call last):
...
ValueError: side must be 'left' or 'right', not junk
```

class `sage.modules.matrix_morphism.`**MatrixMorphism_abstract** (*parent*)

Bases: `sage.categories.morphism.Morphism`

INPUT:

- `parent` - a homspace

•A - matrix

EXAMPLES:

```
sage: from sage.modules.matrix_morphism import MatrixMorphism
sage: T = End(ZZ^3)
sage: M = MatrixSpace(ZZ,3)
sage: I = M.identity_matrix()
sage: A = MatrixMorphism(T, I)
sage: loads(A.dumps()) == A
True
```

base_ring()

Return the base ring of self, that is, the ring over which self is given by a matrix.

EXAMPLES:

```
sage: sage.modules.matrix_morphism.MatrixMorphism((ZZ**2).endomorphism_ring(), Matrix(ZZ,2,[
Integer Ring
```

characteristic_polynomial(*var='x'*)

Return the characteristic polynomial of this endomorphism.

`characteristic_polynomial` and `char_poly` are the same method.

INPUT:

- var – variable

EXAMPLES:

```
sage: V = ZZ^2; phi = V.hom([V.0+V.1, 2*V.1])
sage: phi.characteristic_polynomial()
x^2 - 3*x + 2
sage: phi.charpoly()
x^2 - 3*x + 2
sage: phi.matrix().charpoly()
x^2 - 3*x + 2
sage: phi.charpoly('T')
T^2 - 3*T + 2
```

charpoly(*var='x'*)

Return the characteristic polynomial of this endomorphism.

`characteristic_polynomial` and `char_poly` are the same method.

INPUT:

- var – variable

EXAMPLES:

```
sage: V = ZZ^2; phi = V.hom([V.0+V.1, 2*V.1])
sage: phi.characteristic_polynomial()
x^2 - 3*x + 2
sage: phi.charpoly()
x^2 - 3*x + 2
sage: phi.matrix().charpoly()
x^2 - 3*x + 2
sage: phi.charpoly('T')
T^2 - 3*T + 2
```

decomposition(**args, **kwds*)

Return decomposition of this endomorphism, i.e., sequence of subspaces obtained by finding invariant subspaces of self.

See the documentation for self.matrix().decomposition for more details. All inputs to this function are passed onto the matrix one.

EXAMPLES:

```
sage: V = ZZ^2; phi = V.hom([V.0+V.1, 2*V.1])
sage: phi.decomposition()
[
Free module of degree 2 and rank 1 over Integer Ring
Echelon basis matrix:
[0 1],
Free module of degree 2 and rank 1 over Integer Ring
Echelon basis matrix:
[ 1 -1]
]
```

det()

Return the determinant of this endomorphism.

EXAMPLES:

```
sage: V = ZZ^2; phi = V.hom([V.0+V.1, 2*V.1])
sage: phi.det()
2
```

fcp(*var='x'*)

Return the factorization of the characteristic polynomial.

EXAMPLES:

```
sage: V = ZZ^2; phi = V.hom([V.0+V.1, 2*V.1])
sage: phi.fcp()
(x - 2) * (x - 1)
sage: phi.fcp('T')
(T - 2) * (T - 1)
```

image()

Compute the image of this morphism.

EXAMPLES:

```
sage: V = VectorSpace(QQ,3)
sage: phi = V.Hom(V)(matrix(QQ, 3, range(9)))
sage: phi.image()
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1  0 -1]
[ 0  1  2]
sage: hom(GF(7)^3, GF(7)^2, zero_matrix(GF(7), 3, 2)).image()
Vector space of degree 2 and dimension 0 over Finite Field of size 7
Basis matrix:
[]
```

Compute the image of the identity map on a ZZ-submodule:

```
sage: V = (ZZ^2).span([[1,2],[3,4]])
sage: phi = V.Hom(V)(identity_matrix(ZZ,2))
sage: phi(V.0) == V.0
True
sage: phi(V.1) == V.1
```

```
True
sage: phi.image()
Free module of degree 2 and rank 2 over Integer Ring
Echelon basis matrix:
[1 0]
[0 2]
sage: phi.image() == V
True
```

inverse()

Returns the inverse of this matrix morphism, if the inverse exists.

Raises a `ZeroDivisionError` if the inverse does not exist.

EXAMPLES:

An invertible morphism created as a restriction of a non-invertible morphism, and which has an unequal domain and codomain.

```
sage: V = QQ^4
sage: W = QQ^3
sage: m = matrix(QQ, [[2, 0, 3], [-6, 1, 4], [1, 2, -4], [1, 0, 1]])
sage: phi = V.hom(m, W)
sage: rho = phi.restrict_domain(V.span([V.0, V.3]))
sage: zeta = rho.restrict_codomain(W.span([W.0, W.2]))
sage: x = vector(QQ, [2, 0, 0, -7])
sage: y = zeta(x); y
(-3, 0, -1)
sage: inv = zeta.inverse(); inv
Vector space morphism represented by the matrix:
[-1  3]
[ 1 -2]
Domain: Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[1 0 0]
[0 0 1]
Codomain: Vector space of degree 4 and dimension 2 over Rational Field
Basis matrix:
[1 0 0 0]
[0 0 0 1]
sage: inv(y) == x
True
```

An example of an invertible morphism between modules, (rather than between vector spaces).

```
sage: M = ZZ^4
sage: p = matrix(ZZ, [[ 0, -1,  1, -2],
...                   [ 1, -3,  2, -3],
...                   [ 0,  4, -3,  4],
...                   [-2,  8, -4,  3]])
sage: phi = M.hom(p, M)
sage: x = vector(ZZ, [1, -3, 5, -2])
sage: y = phi(x); y
(1, 12, -12, 21)
sage: rho = phi.inverse(); rho
Free module morphism defined by the matrix
[ -5   3  -1   1]
[ -9   4  -3   2]
[-20   8  -7   4]
[ -6   2  -2   1]
```

```
Domain: Ambient free module of rank 4 over the principal ideal domain ...
Codomain: Ambient free module of rank 4 over the principal ideal domain ...
sage: rho(y) == x
True
```

A non-invertible morphism, despite having an appropriate domain and codomain.

```
sage: V = QQ^2
sage: m = matrix(QQ, [[1, 2], [20, 40]])
sage: phi = V.hom(m, V)
sage: phi.is_bijective()
False
sage: phi.inverse()
Traceback (most recent call last):
...
ZeroDivisionError: matrix morphism not invertible
```

The matrix representation of this morphism is invertible over the rationals, but not over the integers, thus the morphism is not invertible as a map between modules. It is easy to notice from the definition that every vector of the image will have a second entry that is an even integer.

```
sage: V = ZZ^2
sage: q = matrix(ZZ, [[1, 2], [3, 4]])
sage: phi = V.hom(q, V)
sage: phi.matrix().change_ring(QQ).inverse()
[  -2    1]
[ 3/2 -1/2]
sage: phi.is_bijective()
False
sage: phi.image()
Free module of degree 2 and rank 2 over Integer Ring
Echelon basis matrix:
[1 0]
[0 2]
sage: phi.lift(vector(ZZ, [1, 1]))
Traceback (most recent call last):
...
ValueError: element is not in the image
sage: phi.inverse()
Traceback (most recent call last):
...
ZeroDivisionError: matrix morphism not invertible
```

The unary invert operator (~, tilde, "wiggle") is synonymous with the `inverse()` method (and a lot easier to type).

```
sage: V = QQ^2
sage: r = matrix(QQ, [[4, 3], [-2, 5]])
sage: phi = V.hom(r, V)
sage: rho = phi.inverse()
sage: zeta = ~phi
sage: rho.is_equal_function(zeta)
True
```

TESTS:

```
sage: V = QQ^2
sage: W = QQ^3
sage: U = W.span([W.0, W.1])
```

```
sage: m = matrix(QQ, [[2, 1], [3, 4]])
sage: phi = V.hom(m, U)
sage: inv = phi.inverse()
sage: (inv*phi).is_identity()
True
sage: (phi*inv).is_identity()
True
```

is_bijective()

Tell whether `self` is bijective.

EXAMPLES:

Two morphisms that are obviously not bijective, simply on considerations of the dimensions. However, each fullfills half of the requirements to be a bijection.

```
sage: V1 = QQ^2
sage: V2 = QQ^3
sage: m = matrix(QQ, [[1, 2, 3], [4, 5, 6]])
sage: phi = V1.hom(m, V2)
sage: phi.is_injective()
True
sage: phi.is_bijective()
False
sage: rho = V2.hom(m.transpose(), V1)
sage: rho.is_surjective()
True
sage: rho.is_bijective()
False
```

We construct a simple bijection between two one-dimensional vector spaces.

```
sage: V1 = QQ^3
sage: V2 = QQ^2
sage: phi = V1.hom(matrix(QQ, [[1, 2], [3, 4], [5, 6]]), V2)
sage: x = vector(QQ, [1, -1, 4])
sage: y = phi(x); y
(18, 22)
sage: rho = phi.restrict_domain(V1.span([x]))
sage: zeta = rho.restrict_codomain(V2.span([y]))
sage: zeta.is_bijective()
True
```

AUTHOR:

- Rob Beezer (2011-06-28)

is_equal_function(*other*)

Determines if two morphisms are equal functions.

INPUT:

- `other` - a morphism to compare with `self`

OUTPUT:

Returns `True` precisely when the two morphisms have equal domains and codomains (as sets) and produce identical output when given the same input. Otherwise returns `False`.

This is useful when `self` and `other` may have different representations.

Sage's default comparison of matrix morphisms requires the domains to have the same bases and the codomains to have the same bases, and then compares the matrix representations. This notion of equality is more permissive (it will return `True` "more often"), but is more correct mathematically.

EXAMPLES:

Three morphisms defined by combinations of different bases for the domain and codomain and different functions. Two are equal, the third is different from both of the others.

```
sage: B = matrix(QQ, [[-3,  5, -4,  2],
...                   [-1,  2, -1,  4],
...                   [ 4, -6,  5, -1],
...                   [-5,  7, -6,  1]])
sage: U = (QQ^4).subspace_with_basis(B.rows())
sage: C = matrix(QQ, [[-1, -6, -4],
...                   [ 3, -5,  6],
...                   [ 1,  2,  3]])
sage: V = (QQ^3).subspace_with_basis(C.rows())
sage: H = Hom(U, V)

sage: D = matrix(QQ, [[-7, -2, -5,  2],
...                   [-5,  1, -4, -8],
...                   [ 1, -1,  1,  4],
...                   [-4, -1, -3,   1]])
sage: X = (QQ^4).subspace_with_basis(D.rows())
sage: E = matrix(QQ, [[ 4, -1,  4],
...                   [ 5, -4, -5],
...                   [-1,  0, -2]])
sage: Y = (QQ^3).subspace_with_basis(E.rows())
sage: K = Hom(X, Y)

sage: f = lambda x: vector(QQ, [x[0]+x[1], 2*x[1]-4*x[2], 5*x[3]])
sage: g = lambda x: vector(QQ, [x[0]-x[2], 2*x[1]-4*x[2], 5*x[3]])

sage: rho = H(f)
sage: phi = K(f)
sage: zeta = H(g)

sage: rho.is_equal_function(phi)
True
sage: phi.is_equal_function(rho)
True
sage: zeta.is_equal_function(rho)
False
sage: phi.is_equal_function(zeta)
False
```

TEST:

```
sage: H = Hom(ZZ^2, ZZ^2)
sage: phi = H(matrix(ZZ, 2, range(4)))
sage: phi.is_equal_function('junk')
Traceback (most recent call last):
...
TypeError: can only compare to a matrix morphism, not junk
```

AUTHOR:

- Rob Beezer (2011-07-15)

is_identity()

Determines if this morphism is an identity function or not.

EXAMPLES:

A homomorphism that cannot possibly be the identity due to an unequal domain and codomain.

```
sage: V = QQ^3
sage: W = QQ^2
sage: m = matrix(QQ, [[1, 2], [3, 4], [5, 6]])
sage: phi = V.hom(m, W)
sage: phi.is_identity()
False
```

A bijection, but not the identity.

```
sage: V = QQ^3
sage: n = matrix(QQ, [[3, 1, -8], [5, -4, 6], [1, 1, -5]])
sage: phi = V.hom(n, V)
sage: phi.is_bijective()
True
sage: phi.is_identity()
False
```

A restriction that is the identity.

```
sage: V = QQ^3
sage: p = matrix(QQ, [[1, 0, 0], [5, 8, 3], [0, 0, 1]])
sage: phi = V.hom(p, V)
sage: rho = phi.restrict(V.span([V.0, V.2]))
sage: rho.is_identity()
True
```

An identity linear transformation that is defined with a domain and codomain with wildly different bases, so that the matrix representation is not simply the identity matrix.

```
sage: A = matrix(QQ, [[1, 1, 0], [2, 3, -4], [2, 4, -7]])
sage: B = matrix(QQ, [[2, 7, -2], [-1, -3, 1], [-1, -6, 2]])
sage: U = (QQ^3).subspace_with_basis(A.rows())
sage: V = (QQ^3).subspace_with_basis(B.rows())
sage: H = Hom(U, V)
sage: id = lambda x: x
sage: phi = H(id)
sage: phi([203, -179, 34])
(203, -179, 34)
sage: phi.matrix()
[  1   0   1]
[ -9 -18  -2]
[-17 -31  -5]
sage: phi.is_identity()
True
```

TEST:

```
sage: V = QQ^10
sage: H = Hom(V, V)
sage: id = H.identity()
sage: id.is_identity()
True
```

AUTHOR:

- Rob Beezer (2011-06-28)

is_zero()

Determines if this morphism is a zero function or not.

EXAMPLES:

A zero morphism created from a function.

```
sage: V = ZZ^5
sage: W = ZZ^3
sage: z = lambda x: zero_vector(ZZ, 3)
sage: phi = V.hom(z, W)
sage: phi.is_zero()
True
```

An image list that just barely makes a non-zero morphism.

```
sage: V = ZZ^4
sage: W = ZZ^6
sage: z = zero_vector(ZZ, 6)
sage: images = [z, z, W.5, z]
sage: phi = V.hom(images, W)
sage: phi.is_zero()
False
```

TEST:

```
sage: V = QQ^10
sage: W = QQ^3
sage: H = Hom(V, W)
sage: rho = H.zero()
sage: rho.is_zero()
True
```

AUTHOR:

- Rob Beezer (2011-07-15)

kernel()

Compute the kernel of this morphism.

EXAMPLES:

```
sage: V = VectorSpace(QQ,3)
sage: id = V.Hom(V)(identity_matrix(QQ,3))
sage: null = V.Hom(V)(0*identity_matrix(QQ,3))
sage: id.kernel()
Vector space of degree 3 and dimension 0 over Rational Field
Basis matrix:
[]
sage: phi = V.Hom(V)(matrix(QQ,3,range(9)))
sage: phi.kernel()
Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[ 1 -2  1]
sage: hom(CC^2, CC^2, matrix(CC, [[1,0], [0,1]])).kernel()
Vector space of degree 2 and dimension 0 over Complex Field with 53 bits of precision
Basis matrix:
[]
```

matrix()

EXAMPLES:

```
sage: V = ZZ^2; phi = V.hom(V.basis())
sage: phi.matrix()
[1 0]
[0 1]
sage: sage.modules.matrix_morphism.MatrixMorphism_abstract.matrix(phi)
Traceback (most recent call last):
...
NotImplementedError: this method must be overridden in the extension class
```

nullity()

Returns the nullity of the matrix representing this morphism, which is the dimension of its kernel.

EXAMPLES:

```
sage: V = ZZ^2; phi = V.hom(V.basis())
sage: phi.nullity()
0
sage: V = ZZ^2; phi = V.hom([V.0, V.0])
sage: phi.nullity()
1
```

rank()

Returns the rank of the matrix representing this morphism.

EXAMPLES:

```
sage: V = ZZ^2; phi = V.hom(V.basis())
sage: phi.rank()
2
sage: V = ZZ^2; phi = V.hom([V.0, V.0])
sage: phi.rank()
1
```

restrict(*sub*)

Restrict this matrix morphism to a subspace sub of the domain.

The codomain and domain of the resulting matrix are both sub.

EXAMPLES:

```
sage: V = ZZ^2; phi = V.hom([3*V.0, 2*V.1])
sage: phi.restrict(V.span([V.0]))
Free module morphism defined by the matrix
[3]
Domain: Free module of degree 2 and rank 1 over Integer Ring
Echelon ...
Codomain: Free module of degree 2 and rank 1 over Integer Ring
Echelon ...

sage: V = (QQ^2).span_of_basis([[1,2],[3,4]])
sage: phi = V.hom([V.0+V.1, 2*V.1])
sage: phi(V.1) == 2*V.1
True
sage: W = span([V.1])
sage: phi(W)
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[  1 4/3]
sage: psi = phi.restrict(W); psi
Vector space morphism represented by the matrix:
[2]
```

```
Domain: Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[  1 4/3]
Codomain: Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[  1 4/3]
sage: psi.domain() == W
True
sage: psi(W.0) == 2*W.0
True
```

restrict_codomain(*sub*)

Restrict this matrix morphism to a subspace sub of the codomain.

The resulting morphism has the same domain as before, but a new codomain.

EXAMPLES:

```
sage: V = ZZ^2; phi = V.hom([4*(V.0+V.1),0])
sage: W = V.span([2*(V.0+V.1)])
sage: phi
Free module morphism defined by the matrix
[4 4]
[0 0]
Domain: Ambient free module of rank 2 over the principal ideal domain ...
Codomain: Ambient free module of rank 2 over the principal ideal domain ...
sage: psi = phi.restrict_codomain(W); psi
Free module morphism defined by the matrix
[2]
[0]
Domain: Ambient free module of rank 2 over the principal ideal domain ...
Codomain: Free module of degree 2 and rank 1 over Integer Ring
Echelon ...
```

An example in which the codomain equals the full ambient space, but with a different basis:

```
sage: V = QQ^2
sage: W = V.span_of_basis([[1,2],[3,4]])
sage: phi = V.hom(matrix(QQ,2,[1,0,2,0]),W)
sage: phi.matrix()
[1 0]
[2 0]
sage: phi(V.0)
(1, 2)
sage: phi(V.1)
(2, 4)
sage: X = V.span([[1,2]]); X
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[1 2]
sage: phi(V.0) in X
True
sage: phi(V.1) in X
True
sage: psi = phi.restrict_codomain(X); psi
Vector space morphism represented by the matrix:
[1]
[2]
Domain: Vector space of dimension 2 over Rational Field
Codomain: Vector space of degree 2 and dimension 1 over Rational Field
```

```
Basis matrix:
[1 2]
sage: psi(V.0)
(1, 2)
sage: psi(V.1)
(2, 4)
sage: psi(V.0).parent() is X
True
```

restrict_domain(*sub*)

Restrict this matrix morphism to a subspace sub of the domain. The subspace sub should have a basis() method and elements of the basis should be coercible into domain.

The resulting morphism has the same codomain as before, but a new domain.

EXAMPLES:

```
sage: V = ZZ^2; phi = V.hom([3*V.0, 2*V.1])
sage: phi.restrict_domain(V.span([V.0]))
Free module morphism defined by the matrix
[3 0]
Domain: Free module of degree 2 and rank 1 over Integer Ring
Echelon ...
Codomain: Ambient free module of rank 2 over the principal ideal domain ...
sage: phi.restrict_domain(V.span([V.1]))
Free module morphism defined by the matrix
[0 2]...
```

trace()

Return the trace of this endomorphism.

EXAMPLES:

```
sage: V = ZZ^2; phi = V.hom([V.0+V.1, 2*V.1])
sage: phi.trace()
3
```

sage.modules.matrix_morphism.**is_MatrixMorphism**(*x*)

Return True if x is a Matrix morphism of free modules.

EXAMPLES:

```
sage: V = ZZ^2; phi = V.hom([3*V.0, 2*V.1])
sage: sage.modules.matrix_morphism.is_MatrixMorphism(phi)
True
sage: sage.modules.matrix_morphism.is_MatrixMorphism(3)
False
```

CHAPTER

FOURTEEN

FINITELY GENERATED MODULES OVER A PID

You can use Sage to compute with finitely generated modules (FGM's) over a principal ideal domain R presented as a quotient V/W, where V and W are free.

NOTE: Currently this is only enabled over R=ZZ, since it has not been tested and debugged over more general PIDs. All algorithms make sense whenever there is a Hermite form implementation. In theory the obstruction to extending the implementation is only that one has to decide how elements print. If you're annoyed that by this, fix things and post a patch!

We represent M=V/W as a pair (V,W) with W contained in V, and we internally represent elements of M non-canonically as elements x of V. We also fix independent generators g[i] for M in V, and when we print out elements of V we print their coordinates with respect to the g[i]; over $\mathbf{Z}$ this is canonical, since each coefficient is reduce modulo the additive order of g[i]. To obtain the vector in V corresponding to x in M, use x.lift().

Morphisms between finitely generated R modules are well supported. You create a homomorphism by simply giving the images of generators of M0 in M1. Given a morphism phi:M0–>M1, you can compute the image of phi, the kernel of phi, and using y=phi.lift(x) you can lift an elements x in M1 to an element y in M0, if such a y exists.

TECHNICAL NOTE: For efficiency, we introduce a notion of optimized representation for quotient modules. The optimized representation of M=V/W is the quotient V'/W' where V' has as basis lifts of the generators g[i] for M. We internally store a morphism from M0=V0/W0 to M1=V1/W1 by giving a morphism from the optimized representation V0' of M0 to V1 that sends W0 into W1.

The following TUTORIAL illustrates several of the above points.

First we create free modules V0 and W0 and the quotient module M0. Notice that everything works fine even though V0 and W0 are not contained inside $\mathbf{Z}^n$, which is extremely convenient.

```
sage: V0 = span([[1/2,0,0],[3/2,2,1],[0,0,1]],ZZ); W0 = V0.span([V0.0+2*V0.1, 9*V0.0+2*V0.1, 4*V0.2])
sage: M0 = V0/W0; M0
Finitely generated module V/W over Integer Ring with invariants (4, 16)
```

The invariants are computed using the Smith normal form algorithm, and determine the structure of this finitely generated module.

You can get the V and W used in constructing the quotient module using V() and W() methods:

```
sage: M0.V()
Free module of degree 3 and rank 3 over Integer Ring
Echelon basis matrix:
[1/2   0   0]
[  0   2   0]
[  0   0   1]
sage: M0.W()
Free module of degree 3 and rank 3 over Integer Ring
Echelon basis matrix:
[1/2   4   0]
```

```
[  0  32   0]
[  0   0   4]
```

We note that the optimized representation of M0, mentioned above in the technical note has a V that need not be equal to V0, in general.

```
sage: M0.optimized()[0].V()
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[0 0 1]
[0 2 0]
```

Create elements of M0 either by coercing in elements of V0, getting generators, or coercing in a list or tuple or coercing in 0. Finally, one can express an element as a linear combination of the smith form generators

```
sage: M0(V0.0)
(0, 14)
sage: M0(V0.0 + W0.0)  # no difference modulo W0
(0, 14)
sage: M0.linear_combination_of_smith_form_gens([3,20])
(3, 4)
sage: 3*M0.0 + 20*M0.1
(3, 4)
```

We make an element of M0 by taking a difference of two generators, and lift it. We also illustrate making an element from a list, which coerces to V0, then take the equivalence class modulo W0.

```
sage: x = M0.0 - M0.1; x
(1, 15)
sage: x.lift()
(0, -2, 1)
sage: M0(vector([1/2,0,0]))
(0, 14)
sage: x.additive_order()
16
```

Similarly, we construct V1 and W1, and the quotient M1, in a completely different 2-dimensional ambient space.

```
sage: V1 = span([[1/2,0],[3/2,2]],ZZ); W1 = V1.span([2*V1.0, 3*V1.1])
sage: M1 = V1/W1; M1
Finitely generated module V/W over Integer Ring with invariants (6)
```

We create the homomorphism from M0 to M1 that sends both generators of M0 to 3 times the generator of M1. This is well defined since 3 times the generator has order 2.

```
sage: f = M0.hom([3*M1.0, 3*M1.0]); f
Morphism from module over Integer Ring with invariants (4, 16) to module with invariants (6,) that se
```

We evaluate the homomorphism on our element x of the domain, and on the first generator of the domain. We also evaluate at an element of V0, which is coerced into M0.

```
sage: f(x)
(0)
sage: f(M0.0)
(3)
sage: f(V0.1)
(3)
```

Here we illustrate lifting an element of the image of f, i.e., finding an element of M0 that maps to a given element of M1:

```
sage: y = f.lift(3*M1.0); y
(0, 13)
sage: f(y)
(3)
```

We compute the kernel of f, i.e., the submodule of elements of M0 that map to 0. Note that the kernel is not explicitly represented as a submodule, but as another quotient V/W where V is contained in V0. You can explicitly coerce elements of the kernel into M0 though.

```
sage: K = f.kernel(); K
Finitely generated module V/W over Integer Ring with invariants (2, 16)
```

```
sage: M0(K.0)
(2, 0)
sage: M0(K.1)
(3, 1)
sage: f(M0(K.0))
(0)
sage: f(M0(K.1))
(0)
```

We compute the image of f.

```
sage: f.image()
Finitely generated module V/W over Integer Ring with invariants (2)
```

Notice how the elements of the image are written as (0) and (1), despite the image being naturally a submodule of M1, which has elements (0), (1), (2), (3), (4), (5). However, below we coerce the element (1) of the image into the codomain, and get (3):

```
sage: list(f.image())
[(0), (1)]
sage: list(M1)
[(0), (1), (2), (3), (4), (5)]
sage: x = f.image().0; x
(1)
sage: M1(x)
(3)
```

TESTS:

```
sage: from sage.modules.fg_pid.fgp_module import FGP_Module
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ)
sage: W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: Q = FGP_Module(V, W); Q
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: Q.linear_combination_of_smith_form_gens([1,3])
(1, 3)
sage: Q(V([1,3,4]))
(0, 11)
sage: Q(W([1,16,0]))
(0, 0)
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],QQ)
sage: W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1])
sage: Q = FGP_Module(V, W); Q
```

```
Finitely generated module V/W over Rational Field with invariants (0)
sage: q = Q.an_element(); q
(1)
sage: q*(1/2)
(1/2)
sage: (1/2)*q
(1/2)
```

AUTHOR:

- William Stein, 2009

sage.modules.fg_pid.fgp_module.**FGP_Module**(*V*, *W*, *check=True*)

INPUT:

- V – a free R-module
- W – a free R-submodule of V
- check – bool (default: True); if True, more checks on correctness are performed; in particular, we check the data types of V and W, and that W is a submodule of V with the same base ring.

OUTPUT:

- the quotient V/W as a finitely generated R-module

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: import sage.modules.fg_pid.fgp_module
sage: Q = sage.modules.fg_pid.fgp_module.FGP_Module(V, W)
sage: type(Q)
<class 'sage.modules.fg_pid.fgp_module.FGP_Module_class_with_category'>
sage: Q is sage.modules.fg_pid.fgp_module.FGP_Module(V, W, check=False)
True
```

class sage.modules.fg_pid.fgp_module.**FGP_Module_class**(*V*, *W*, *check=True*)

Bases: sage.modules.module.Module

A finitely generated module over a PID presented as a quotient V/W.

INPUT:

- V – an R-module
- W – an R-submodule of V
- check – bool (default: True)

EXAMPLES:

```
sage: A = (ZZ^1)/span([[100]], ZZ); A
Finitely generated module V/W over Integer Ring with invariants (100)
sage: A.V()
Ambient free module of rank 1 over the principal ideal domain Integer Ring
sage: A.W()
Free module of degree 1 and rank 1 over Integer Ring
Echelon basis matrix:
[100]

sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 12)
```

```
sage: type(Q)
<class 'sage.modules.fg_pid.fgp_module.FGP_Module_class_with_category'>
```

Element
alias of FGP_Element

V()
If this module was constructed as a quotient V/W, returns V.

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W
sage: Q.V()
Free module of degree 3 and rank 3 over Integer Ring
Echelon basis matrix:
[1/2   0   0]
[  0   1   0]
[  0   0   1]
```

W()
If this module was constructed as a quotient V/W, returns W.

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W
sage: Q.W()
Free module of degree 3 and rank 3 over Integer Ring
Echelon basis matrix:
[1/2   8   0]
[  0  12   0]
[  0   0   4]
```

annihilator()
Return the ideal of the base ring that annihilates self. This is precisely the ideal generated by the LCM of the invariants of self if self is finite, and is 0 otherwise.

EXAMPLES:

```
sage: V = span([[1/2,0,0],[3/2,2,1],[0,0,1]],ZZ); W = V.span([V.0+2*V.1, 9*V.0+2*V.1, 4*V.2]
sage: Q = V/W; Q.annihilator()
Principal ideal (16) of Integer Ring
sage: Q.annihilator().gen()
16

sage: Q = V/V.span([V.0]); Q
Finitely generated module V/W over Integer Ring with invariants (0, 0)
sage: Q.annihilator()
Principal ideal (0) of Integer Ring
```

base_ring()
EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W
sage: Q.base_ring()
Integer Ring
```

cardinality()

Return the cardinality of this module as a set.

EXAMPLES:

```
sage: V = ZZ^2; W = V.span([[1,2],[3,4]]); A = V/W; A
Finitely generated module V/W over Integer Ring with invariants (2)
sage: A.cardinality()
2
sage: V = ZZ^2; W = V.span([[1,2]]); A = V/W; A
Finitely generated module V/W over Integer Ring with invariants (0)
sage: A.cardinality()
+Infinity
sage: V = QQ^2; W = V.span([[1,2]]); A = V/W; A
Vector space quotient V/W of dimension 1 over Rational Field where
V: Vector space of dimension 2 over Rational Field
W: Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[1 2]
sage: A.cardinality()
+Infinity
```

coordinate_vector(*x*, *reduce=False*)

Return coordinates of x with respect to the optimized representation of self.

INPUT:

- x – element of self
- reduce – (default: False); if True, reduce coefficients modulo invariants; this is ignored if the base ring isn't ZZ.

OUTPUT:

The coordinates as a vector. That is, the same type as `self.V()`, but in general with fewer entries.

EXAMPLES:

```
sage: V = span([[1/4,0,0],[3/4,4,2],[0,0,2]],ZZ); W = V.span([4*V.0+12*V.1])
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 0, 0)
sage: Q.coordinate_vector(-Q.0)
(-1, 0, 0)
sage: Q.coordinate_vector(-Q.0, reduce=True)
(3, 0, 0)
```

If x isn't in self, it is coerced in:

```
sage: Q.coordinate_vector(V.0)
(1, 0, -3)
sage: Q.coordinate_vector(Q(V.0))
(1, 0, -3)
```

TESTS:

```
sage: V = span([[1/2,0,0],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: Q.coordinate_vector(Q.0 - Q.1)
(1, -1)

sage: O, X = Q.optimized()
sage: O.V()
Free module of degree 3 and rank 2 over Integer Ring
```

```
User basis matrix:
[0 0 1]
[0 2 0]
sage: phi = Q.hom([Q.0, 4*Q.1])
sage: x = Q(V.0); x
(0, 4)
sage: Q.coordinate_vector(x, reduce=True)
(0, 4)
sage: Q.coordinate_vector(-x, reduce=False)
(0, -4)
sage: x == 4*Q.1
True
sage: x = Q(V.1); x
(0, 1)
sage: Q.coordinate_vector(x)
(0, 1)
sage: x == Q.1
True
sage: x = Q(V.2); x
(1, 0)
sage: Q.coordinate_vector(x)
(1, 0)
sage: x == Q.0
True
```

cover()

If this module was constructed as V/W, returns the cover module V. This is the same as self.V().

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W
sage: Q.V()
Free module of degree 3 and rank 3 over Integer Ring
Echelon basis matrix:
[1/2   0   0]
[  0   1   0]
[  0   0   1]
```

gen(*i*)

Return the i-th generator of self.

INPUT:

- i – integer

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: Q.gen(0)
(1, 0)
sage: Q.gen(1)
(0, 1)
sage: Q.gen(2)
Traceback (most recent call last):
...
ValueError: Generator 2 not defined
sage: Q.gen(-1)
```

```
Traceback (most recent call last):
...
ValueError: Generator -1 not defined
```

gens()

Returns tuple of elements $g_0, ..., g_n$ of self such that the module generated by the gi is isomorphic to the direct sum of R/ei*R, where ei are the invariants of self and R is the base ring.

Note that these are not generally uniquely determined, and depending on how Smith normal form is implemented for the base ring, they may not even be deterministic.

This can safely be overridden in all derived classes.

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W
sage: Q.gens()
((1, 0), (0, 1))
sage: Q.0
(1, 0)
```

has_canonical_map_to(*A*)

Return True if self has a canonical map to A, relative to the given presentation of A. This means that A is a finitely generated quotient module, self.V() is a submodule of A.V() and self.W() is a submodule of A.W(), i.e., that there is a natural map induced by inclusion of the V's. Note that we do *not* require that this natural map be injective; for this use `is_submodule()`.

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: A = Q.submodule((Q.0, Q.0 + 3*Q.1)); A
Finitely generated module V/W over Integer Ring with invariants (4, 4)
sage: A.has_canonical_map_to(Q)
True
sage: Q.has_canonical_map_to(A)
False
```

hom(*im_gens*, *codomain=None*, *check=True*)

Homomorphism defined by giving the images of `self.gens()` in some fixed fg R-module.

Note: We do not assume that the generators given by `self.gens()` are the same as the Smith form generators, since this may not be true for a general derived class.

INPUTS:

- `im_gens` – a list of the images of `self.gens()` in some R-module

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W
sage: phi = Q.hom([3*Q.1, Q.0])
sage: phi
Morphism from module over Integer Ring with invariants (4, 12) to module with invariants (4,
sage: phi(Q.0)
(0, 3)
sage: phi(Q.1)
```

```
(1, 0)
sage: Q.0 == phi(Q.1)
True
```

This example illustrates creating a morphism to a free module. The free module is turned into an FGP module (i.e., quotient V/W with W=0), and the morphism is constructed:

```
sage: V = span([[1/2,0,0],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1])
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (2, 0, 0)
sage: phi = Q.hom([0,V.0,V.1]); phi
Morphism from module over Integer Ring with invariants (2, 0, 0) to module with invariants (
sage: phi.domain()
Finitely generated module V/W over Integer Ring with invariants (2, 0, 0)
sage: phi.codomain()
Finitely generated module V/W over Integer Ring with invariants (0, 0, 0)
sage: phi(Q.0)
(0, 0, 0)
sage: phi(Q.1)
(1, 0, 0)
sage: phi(Q.2) == V.1
True
```

Constructing two zero maps from the zero module:

```
sage: A = (ZZ^2)/(ZZ^2); A
Finitely generated module V/W over Integer Ring with invariants ()
sage: A.hom([])
Morphism from module over Integer Ring with invariants () to module with invariants () that
sage: A.hom([]).codomain() is A
True
sage: B = (ZZ^3)/(ZZ^3)
sage: A.hom([],codomain=B)
Morphism from module over Integer Ring with invariants () to module with invariants () that
sage: phi = A.hom([],codomain=B); phi
Morphism from module over Integer Ring with invariants () to module with invariants () that
sage: phi(A(0))
()
sage: phi(A(0)) == B(0)
True
```

A degenerate case:

```
sage: A = (ZZ^2)/(ZZ^2)
sage: phi = A.hom([]); phi
Morphism from module over Integer Ring with invariants () to module with invariants () that
sage: phi(A(0))
()
```

The code checks that the morphism is valid. In the example below we try to send a generator of order 2 to an element of order 14:

```
sage: V = span([[1/14,3/14],[0,1/2]],ZZ); W = ZZ^2
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (2, 14)
sage: Q.linear_combination_of_smith_form_gens([1,11]).additive_order()
14
sage: f = Q.hom([Q.linear_combination_of_smith_form_gens([1,11]), Q.linear_combination_of_sm
Traceback (most recent call last):
...
```

```
ValueError: phi must send optimized submodule of M.W() into N.W()
```

invariants(*include_ones=False*)

Return the diagonal entries of the smith form of the relative matrix that defines self (see `_relative_matrix()`) padded with zeros, excluding 1's by default. Thus if v is the list of integers returned, then self is abstractly isomorphic to the product of cyclic groups Z/nZ where n is in v.

INPUT:

- `include_ones` – bool (default: False); if True, also include 1's in the output list.

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W
sage: Q.invariants()
(4, 12)
```

An example with 1 and 0 rows:

```
sage: V = ZZ^3; W = V.span([[1,2,0],[0,1,0], [0,2,0]]); Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (0)
sage: Q.invariants()
(0,)
sage: Q.invariants(include_ones=True)
(1, 1, 0)
```

is_finite()

Return True if self is finite and False otherwise.

EXAMPLES:

```
sage: V = span([[1/2,0,0],[3/2,2,1],[0,0,1]],ZZ); W = V.span([V.0+2*V.1, 9*V.0+2*V.1, 4*V.2]
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 16)
sage: Q.is_finite()
True
sage: Q = V/V.zero_submodule(); Q
Finitely generated module V/W over Integer Ring with invariants (0, 0, 0)
sage: Q.is_finite()
False
```

is_submodule(*A*)

Return True if self is a submodule of A. More precisely, this returns True if if `self.V()` is a submodule of `A.V()`, with `self.W()` equal to `A.W()`.

Compare `has_canonical_map_to()`.

EXAMPLES:

```
sage: V = ZZ^2; W = V.span([[1,2]]); W2 = W.scale(2)
sage: A = V/W; B = W/W2
sage: B.is_submodule(A)
False
sage: A = V/W2; B = W/W2
sage: B.is_submodule(A)
True
```

This example illustrates that this command works in a subtle cases.:

```
sage: A = ZZ^1
sage: Q3 = A / A.span([[3]])
sage: Q6 = A / A.span([[6]])
sage: Q6.is_submodule(Q3)
False
sage: Q6.has_canonical_map_to(Q3)
True
sage: Q = A.span([[2]]) / A.span([[6]])
sage: Q.is_submodule(Q6)
True
```

linear_combination_of_smith_form_gens(*x*)

Compute a linear combination of the optimised generators of this module as returned by `smith_form_gens()`.

EXAMPLE:

```
sage: X = ZZ**2 / span([[3,0],[0,2]], ZZ)
sage: X.linear_combination_of_smith_form_gens([1])
(1)
```

ngens()

Return the number of generators of self.

(Note for developers: This is just the length of `gens()`, rather than of the minimal set of generators as returned by `smith_form_gens()`; these are the same in the `FGP_Module_class`, but not necessarily in derived classes.)

EXAMPLES:

```
sage: A = (ZZ**2) / span([[4,0],[0,3]], ZZ)
sage: A.ngens()
1
```

This works (but please don't do it in production code!)

```
sage: A.gens = lambda: [1,2,"Barcelona!"]
sage: A.ngens()
3
```

optimized()

Return a module isomorphic to this one, but with V replaced by a submodule of V such that the generators of self all lift trivially to generators of V. Replace W by the intersection of V and W. This has the advantage that V has small dimension and any homomorphism from self trivially extends to a homomorphism from V.

OUTPUT:

- `Q` – an optimized quotient V0/W0 with V0 a submodule of V such that phi: V0/W0 –> V/W is an isomorphism
- `Z` – matrix such that if x is in self.V() and c gives the coordinates of x in terms of the basis for self.V(), then c*Z is in V0 and c*Z maps to x via phi above.

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W
sage: O, X = Q.optimized(); O
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: O.V()
```

```
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[0 0 1]
[0 1 0]
sage: O.W()
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[ 0 12  0]
[ 0  0  4]
sage: X
[0 4 0]
[0 1 0]
[0 0 1]
sage: OV = O.V()
sage: Q(OV([0,-8,0])) == V.0
True
sage: Q(OV([0,1,0])) == V.1
True
sage: Q(OV([0,0,1])) == V.2
True
```

random_element (**args*, ***kwds*)

Create a random element of self=V/W, by creating a random element of V and reducing it modulo W.

All arguments are passed onto the random_element method of V.

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W
sage: Q.random_element()
(1, 10)
```

relations ()

If this module was constructed as V/W, returns the relations module V. This is the same as self.W().

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W
sage: Q.relations()
Free module of degree 3 and rank 3 over Integer Ring
Echelon basis matrix:
[1/2   8   0]
[  0  12   0]
[  0   0   4]
```

smith_form_gen (*i*)

Return the i-th generator of self. A private name (so we can freely override gen() in derived classes).

INPUT:

- i – integer

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: Q.smith_form_gen(0)
(1, 0)
```

```
sage: Q.smith_form_gen(1)
(0, 1)
```

smith_form_gens()

Return a set of generators for self which are in Smith normal form.

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W
sage: Q.smith_form_gens()
((1, 0), (0, 1))
sage: [x.lift() for x in Q.smith_form_gens()]
[(0, 0, 1), (0, 1, 0)]
```

submodule(*x*)

Return the submodule defined by x.

INPUT:

- x – list, tuple, or FGP module

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: Q.gens()
((1, 0), (0, 1))
```

We create submodules generated by a list or tuple of elements:

```
sage: Q.submodule([Q.0])
Finitely generated module V/W over Integer Ring with invariants (4)
sage: Q.submodule([Q.1])
Finitely generated module V/W over Integer Ring with invariants (12)
sage: Q.submodule((Q.0, Q.0 + 3*Q.1))
Finitely generated module V/W over Integer Ring with invariants (4, 4)
```

A submodule defined by a submodule:

```
sage: A = Q.submodule((Q.0, Q.0 + 3*Q.1)); A
Finitely generated module V/W over Integer Ring with invariants (4, 4)
sage: Q.submodule(A)
Finitely generated module V/W over Integer Ring with invariants (4, 4)
```

Inclusion is checked:

```
sage: A.submodule(Q)
Traceback (most recent call last):
...
ValueError: x.V() must be contained in self's V.
```

sage.modules.fg_pid.fgp_module.**is_FGP_Module**(*x*)

Return true of x is an FGP module, i.e., a finitely generated module over a PID represented as a quotient of finitely generated free modules over a PID.

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: sage.modules.fg_pid.fgp_module.is_FGP_Module(V)
False
```

```
sage: sage.modules.fg_pid.fgp_module.is_FGP_Module(Q)
True
```

sage.modules.fg_pid.fgp_module.**random_fgp_module**(*n*, *R=Integer Ring*, *finite=False*)

Return a random FGP module inside a rank n free module over R.

INPUT:

- n – nonnegative integer
- R – base ring (default: ZZ)
- finite – bool (default: True); if True, make the random module finite.

EXAMPLES:

```
sage: import sage.modules.fg_pid.fgp_module as fgp
sage: fgp.random_fgp_module(4)
Finitely generated module V/W over Integer Ring with invariants (4)
```

sage.modules.fg_pid.fgp_module.**random_fgp_morphism_0**(**args*, ***kwds*)

Construct a random fgp module using random_fgp_module, then construct a random morphism that sends each generator to a random multiple of itself. Inputs are the same as to random_fgp_module.

EXAMPLES:

```
sage: import sage.modules.fg_pid.fgp_module as fgp
sage: fgp.random_fgp_morphism_0(4)
Morphism from module over Integer Ring with invariants (4,) to module with invariants (4,) that
```

sage.modules.fg_pid.fgp_module.**test_morphism_0**(**args*, ***kwds*)

EXAMPLES:

```
sage: import sage.modules.fg_pid.fgp_module as fgp
sage: s = 0  # we set a seed so results clearly and easily reproducible across runs.
sage: set_random_seed(s); v = [fgp.test_morphism_0(1) for _ in range(30)]
sage: set_random_seed(s); v = [fgp.test_morphism_0(2) for _ in range(30)]
sage: set_random_seed(s); v = [fgp.test_morphism_0(3) for _ in range(10)]
sage: set_random_seed(s); v = [fgp.test_morphism_0(i) for i in range(1,20)]
sage: set_random_seed(s); v = [fgp.test_morphism_0(4) for _ in range(50)]    # long time
```

CHAPTER

FIFTEEN

ELEMENTS OF FINITELY GENERATED MODULES OVER A PID

AUTHOR:

- William Stein, 2009

class sage.modules.fg_pid.fgp_element.**FGP_Element** (*parent*, *x*, *check=True*)

Bases: sage.structure.element.ModuleElement

An element of a finitely generated module over a PID.

INPUT:

- parent – parent module M
- x – element of M.V()

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: Q = V/W
sage: x = Q(V.0-V.1); x #indirect doctest
(0, 3)
sage: isinstance(x, sage.modules.fg_pid.fgp_element.FGP_Element)
True
sage: type(x)
<class 'sage.modules.fg_pid.fgp_element.FGP_Module_class_with_category.element_class'>
sage: x is Q(x)
True
sage: x.parent() is Q
True
```

TESTS:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: loads(dumps(Q.0)) == Q.0
True
```

additive_order ()

Return the additive order of this element.

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: Q.0.additive_order()
4
sage: Q.1.additive_order()
12
sage: (Q.0+Q.1).additive_order()
```

```
12
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1])
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (12, 0)
sage: Q.0.additive_order()
12
sage: type(Q.0.additive_order())
<type 'sage.rings.integer.Integer'>
sage: Q.1.additive_order()
+Infinity
```

lift()

Lift self to an element of V, where the parent of self is the quotient module V/W.

EXAMPLES:

```
sage: V = span([[1/2,0,0],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: Q.0
(1, 0)
sage: Q.1
(0, 1)
sage: Q.0.lift()
(0, 0, 1)
sage: Q.1.lift()
(0, 2, 0)
sage: x = Q(V.0); x
(0, 4)
sage: x.lift()
(1/2, 0, 0)
sage: x == 4*Q.1
True
sage: x.lift().parent() == V
True
```

A silly version of the integers modulo 100:

```
sage: A = (ZZ^1)/span([[100]], ZZ); A
Finitely generated module V/W over Integer Ring with invariants (100)
sage: x = A([5]); x
doctest:...: DeprecationWarning: The default behaviour changed!
 If you *really* want a linear combination of smith generators,
 use .linear_combination_of_smith_form_gens.
See http://trac.sagemath.org/16261 for details.
(5)
sage: v = x.lift(); v
(5)
sage: v.parent()
Ambient free module of rank 1 over the principal ideal domain Integer Ring
```

vector()

EXAMPLES:

```
sage: V = span([[1/2,0,0],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: x = Q.0 + 3*Q.1; x
(1, 3)
```

```
sage: x.vector()
(1, 3)
sage: tuple(x)
(1, 3)
sage: list(x)
[1, 3]
sage: x.vector().parent()
Ambient free module of rank 2 over the principal ideal domain Integer Ring
```

CHAPTER

SIXTEEN

MORPHISMS BETWEEN FINITELY GENERATED MODULES OVER A PID

AUTHOR: - William Stein, 2009

sage.modules.fg_pid.fgp_morphism.**FGP_Homset**(*X*, *Y*)

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: Q.Hom(Q)           # indirect doctest
Set of Morphisms from Finitely generated module V/W over Integer Ring with invariants (4, 12) to
sage: True # Q.Hom(Q) is Q.Hom(Q)
True
sage: type(Q.Hom(Q))
<class 'sage.modules.fg_pid.fgp_morphism.FGP_Homset_class_with_category'>
```

class sage.modules.fg_pid.fgp_morphism.**FGP_Homset_class**(*X*, *Y*, *category=None*)

Bases: sage.categories.homset.Homset

Homsets of FGP_Module

TESTS:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: H = Hom(Q,Q); H    # indirect doctest
Set of Morphisms from Finitely generated module V/W over Integer Ring with invariants (4, 12) to
sage: type(H)
<class 'sage.modules.fg_pid.fgp_morphism.FGP_Homset_class_with_category'>
```

Element

alias of FGP_Morphism

class sage.modules.fg_pid.fgp_morphism.**FGP_Morphism**(*parent*, *phi*, *check=True*)

Bases: sage.categories.morphism.Morphism

A morphism between finitely generated modules over a PID.

EXAMPLES:

An endomorphism:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: phi = Q.hom([Q.0+3*Q.1, -Q.1]); phi
Morphism from module over Integer Ring with invariants (4, 12) to module with invariants (4, 12)
sage: phi(Q.0) == Q.0 + 3*Q.1
True
sage: phi(Q.1) == -Q.1
True
```

A morphism between different modules V1/W1 —> V2/W2 in different ambient spaces:

```
sage: V1 = ZZ^2; W1 = V1.span([[1,2],[3,4]]); A1 = V1/W1
sage: V2 = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W2 = V2.span([2*V2.0+4*V2.1, 9*V2.0+12*V2.1,
sage: A1
Finitely generated module V/W over Integer Ring with invariants (2)
sage: A2
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: phi = A1.hom([2*A2.0])
sage: phi(A1.0)
(2, 0)
sage: 2*A2.0
(2, 0)
sage: phi(2*A1.0)
(0, 0)
```

TESTS:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V.2])
sage: phi = Q.hom([Q.0,Q.0 + 2*Q.1])
sage: loads(dumps(phi)) == phi
True
```

im_gens()

Return tuple of the images of the generators of the domain under this morphism.

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: phi = Q.hom([Q.0,Q.0 + 2*Q.1])
sage: phi.im_gens()
((1, 0), (1, 2))
sage: phi.im_gens() is phi.im_gens()
True
```

image()

Compute the image of this homomorphism.

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: Q.hom([Q.0+3*Q.1, -Q.1]).image()
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: Q.hom([3*Q.1, Q.1]).image()
Finitely generated module V/W over Integer Ring with invariants (12)
```

inverse_image(*A*)

Given a submodule A of the codomain of this morphism, return the inverse image of A under this morphism.

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: phi = Q.hom([0, Q.1])
sage: phi.inverse_image(Q.submodule([]))
Finitely generated module V/W over Integer Ring with invariants (4)
sage: phi.kernel()
```

```
Finitely generated module V/W over Integer Ring with invariants (4)
sage: phi.inverse_image(phi.codomain())
Finitely generated module V/W over Integer Ring with invariants (4, 12)

sage: phi.inverse_image(Q.submodule([Q.0]))
Finitely generated module V/W over Integer Ring with invariants (4)
sage: phi.inverse_image(Q.submodule([Q.1]))
Finitely generated module V/W over Integer Ring with invariants (4, 12)

sage: phi.inverse_image(ZZ^3)
Traceback (most recent call last):
...
TypeError: A must be a finitely generated quotient module
sage: phi.inverse_image(ZZ^3 / W.scale(2))
Traceback (most recent call last):
...
ValueError: A must be a submodule of the codomain
```

kernel()

Compute the kernel of this homomorphism.

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q = V/W; Q
Finitely generated module V/W over Integer Ring with invariants (4, 12)
sage: Q.hom([0, Q.1]).kernel()
Finitely generated module V/W over Integer Ring with invariants (4)
sage: A = Q.hom([Q.0, 0]).kernel(); A
Finitely generated module V/W over Integer Ring with invariants (12)
sage: Q.1 in A
True
sage: phi = Q.hom([Q.0-3*Q.1, Q.0+Q.1])
sage: A = phi.kernel(); A
Finitely generated module V/W over Integer Ring with invariants (4)
sage: phi(A)
Finitely generated module V/W over Integer Ring with invariants ()
```

lift(*x*)

Given an element x in the codomain of self, if possible find an element y in the domain such that self(y) == x. Raise a ValueError if no such y exists.

INPUT:

- x – element of the codomain of self.

EXAMPLES:

```
sage: V = span([[1/2,1,1],[3/2,2,1],[0,0,1]],ZZ); W = V.span([2*V.0+4*V.1, 9*V.0+12*V.1, 4*V
sage: Q=V/W; phi = Q.hom([2*Q.0, Q.1])
sage: phi.lift(Q.1)
(0, 1)
sage: phi.lift(Q.0)
Traceback (most recent call last):
...
ValueError: no lift of element to domain
sage: phi.lift(2*Q.0)
(1, 0)
sage: phi.lift(2*Q.0+Q.1)
(1, 1)
```

```
sage: V = span([[5, -1/2]],ZZ); W = span([[20,-2]],ZZ); Q = V/W; phi=Q.hom([2*Q.0])
sage: x = phi.image().0; phi(phi.lift(x)) == x
True
```

CHAPTER

SEVENTEEN

DIAMOND CUTTING IMPLEMENTATION

AUTHORS:

- Jan Poeschko (2012-07-02): initial version

sage.modules.diamond_cutting.**calculate_voronoi_cell**(*basis*, *radius=None*, *verbose=False*)

Calculate the Voronoi cell of the lattice defined by basis

INPUT:

- `basis` – embedded basis matrix of the lattice
- `radius` – radius of basis vectors to consider
- `verbose` – whether to print debug information

OUTPUT:

A :class:`Polyhedron` instance.

EXAMPLES:

```
sage: from sage.modules.diamond_cutting import calculate_voronoi_cell
sage: V = calculate_voronoi_cell(matrix([[1, 0], [0, 1]]))
sage: V.volume()
1
```

sage.modules.diamond_cutting.**diamond_cut**(*V*, *GM*, *C*, *verbose=False*)

Perform diamond cutting on polyhedron `V` with basis matrix `GM` and radius `C`.

INPUT:

- `V` – polyhedron to cut from
- `GM` – half of the basis matrix of the lattice
- `C` – radius to use in cutting algorithm
- `verbose` – (default: `False`) whether to print debug information

OUTPUT:

A :class:`Polyhedron` instance.

EXAMPLES:

```
sage: from sage.modules.diamond_cutting import diamond_cut
sage: V = Polyhedron([[0], [2]])
sage: GM = matrix([2])
sage: V = diamond_cut(V, GM, 4)
sage: V.vertices()
(A vertex at (2), A vertex at (0))
```

sage.modules.diamond_cutting.**jacobi**(*M*)

Compute the upper-triangular part of the Cholesky/Jacobi decomposition of the symmetric matrix M.

Let M be a symmetric $n \times n$-matrix over a field F. Let $m_{i,j}$ denote the (i,j)-th entry of M for any $1 \leq i \leq n$ and $1 \leq j \leq n$. Then, the upper-triangular part computed by this method is the upper-triangular $n \times n$-matrix Q whose (i,j)-th entry $q_{i,j}$ satisfies

$$q_{i,j} = \begin{cases} \frac{1}{q_{i,i}} \left(m_{i,j} - \sum_{r<i} q_{r,r} q_{r,i} q_{r,j} \right) & i < j, \\ a_{i,j} - \sum_{r<i} q_{r,r} q_{r,i}^2 & i = j, \\ 0 & i > j, \end{cases}$$

for all $1 \leq i \leq n$ and $1 \leq j \leq n$. (These equalities determine the entries of Q uniquely by recursion.) This matrix Q is defined for all M in a certain Zariski-dense open subset of the set of all $n \times n$-matrices.

EXAMPLES:

```
sage: from sage.modules.diamond_cutting import jacobi
sage: jacobi(identity_matrix(3) * 4)
[4 0 0]
[0 4 0]
[0 0 4]

sage: def testall(M):
....:      Q = jacobi(M)
....:      for j in range(3):
....:          for i in range(j):
....:              if Q[i,j] * Q[i,i] != M[i,j] - sum(Q[r,i] * Q[r,j] * Q[r,r] for r in range(i)
....:                  return False
....:      for i in range(3):
....:          if Q[i,i] != M[i,i] - sum(Q[r,i] ** 2 * Q[r,r] for r in range(i)):
....:              return False
....:          for j in range(i):
....:              if Q[i,j] != 0:
....:                  return False
....:      return True

sage: M = Matrix(QQ, [[8,1,5], [1,6,0], [5,0,3]])
sage: Q = jacobi(M); Q
[    8   1/8   5/8]
[    0  47/8 -5/47]
[    0     0 -9/47]
sage: testall(M)
True

sage: M = Matrix(QQ, [[3,6,-1,7],[6,9,8,5],[-1,8,2,4],[7,5,4,0]])
sage: testall(M)
True
```

sage.modules.diamond_cutting.**plane_inequality**(*v*)

Return the inequality for points on the same side as the origin with respect to the plane through v normal to v.

EXAMPLES:

```
sage: from sage.modules.diamond_cutting import plane_inequality
sage: ieq = plane_inequality([1, -1]); ieq
[2, -1, 1]
sage: ieq[0] + vector(ieq[1:]) * vector([1, -1])
0
```

CHAPTER

EIGHTEEN

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[Vit1996] E. Viterbo, E. Biglieri. *Computing the Voronoi Cell of a Lattice: The Diamond-Cutting Algorithm*. IEEE Transactions on Information Theory, 1996.

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