Sage Reference Manual: Asymptotic Expansions

Release 7.5

The Sage Development Team

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CHAPTER

ONE

THE ASYMPTOTIC RING

The asymptotic ring, as well as its main documentation is contained in the module

• Asymptotic Ring.

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CHAPTER

TWO

ASYMPTOTIC EXPANSION GENERATORS

Some common asymptotic expansions can be generated in

• Common Asymptotic Expansions.

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CHAPTER

THREE

SUPPLEMENTS

Behind the scenes of working with asymptotic expressions a couple of additional classes and tools turn up. For instance the growth of each summand is managed in growth groups, see below.

3.1 Growth Groups

The growth of a summand of an asymptotic expression is managed in

- (Asymptotic) Growth Groups and
- Cartesian Products of Growth Groups.

3.2 Term Monoids

A summand of an asymptotic expression is basically a term out of the following monoid:

• (Asymptotic) Term Monoids.

3.3 Miscellaneous

Various useful functions and tools are collected in

• Asymptotic Expansions — Miscellaneous.

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ASYMPTOTIC EXPANSIONS — TABLE OF CONTENTS

4.1 Asymptotic Ring

This module provides a ring (called AsymptoticRing) for computations with asymptotic expansions.

4.1.1 (Informal) Definition

An asymptotic expansion is a sum such as

$$5z^3 + 4z^2 + O(z)$$

as $z \to \infty$ or

$$3x^{42}y^2 + 7x^3y^3 + O(x^2) + O(y)$$

as x and y tend to ∞ . It is a truncated series (after a finite number of terms), which approximates a function.

The summands of the asymptotic expansions are partially ordered. In this module these summands are the following:

- Exact terms $c \cdot g$ with a coefficient c and an element g of a growth group (see below).
- O-terms O(g) (see Big O notation; also called *Bachmann–Landau notation*) for a growth group element g (again see below).

See the Wikipedia article on asymptotic expansions for more details. Further examples of such elements can be found

Growth Groups and Elements

The elements of a *growth group* are equipped with a partial order and usually contain a variable. Examples—the order is described below these examples—are

- elements of the form z^q for some integer or rational q (growth groups with description strings z^ZZ or z^QQ),
- elements of the form $\log(z)^q$ for some integer or rational q (growth groups $\log(z)^2$ or $\log(z)^2$ or $\log(z)^2$),
- elements of the form a^z for some rational a (growth group QQ^z), or
- more sophisticated constructions like products $x^r \cdot \log(x)^s \cdot a^y \cdot y^q$ (this corresponds to an element of the growth group $x \cdot QQ \times \log(x) \cdot ZZ \times QQ \cdot y \times y \cdot QQ$).

The order in all these examples is induced by the magnitude of the elements as x, y, or z (independently) tend to ∞ . For elements only using the variable z this means that $g_1 \leq g_2$ if

$$\lim_{z \to \infty} \frac{g_1}{g_2} \le 1.$$

Note: Asymptotic rings where the variable tend to some value distinct from ∞ are not yet implemented.

To find out more about

- · growth groups,
- · on how they are created and
- about the above used descriptions strings

see the top of the module growth group.

Warning: As this code is experimental, a warning is thrown when an asymptotic ring (or an associated structure) is created for the first time in a session (see sage.misc.superseded.experimental).

TESTS:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ')
doctest:...: FutureWarning: This class/method/function is marked as
experimental. It, its functionality or its interface might change
without a formal deprecation.
See http://trac.sagemath.org/17601 for details.
```

4.1.2 Introductory Examples

We start this series of examples by defining two asymptotic rings.

Two Rings

A Univariate Asymptotic Ring

First, we construct the following (very simple) asymptotic ring in the variable z:

```
sage: A.<z> = AsymptoticRing(growth_group='z^QQ', coefficient_ring=ZZ); A
Asymptotic Ring <z^QQ> over Integer Ring
```

A typical element of this ring is

```
sage: A.an_element()
z^(3/2) + O(z^(1/2))
```

This element consists of two summands: the exact term with coefficient 1 and growth $z^{3/2}$ and the O-term $O(z^{1/2})$. Note that the growth of $z^{3/2}$ is larger than the growth of $z^{1/2}$ as $z \to \infty$, thus this expansion cannot be simplified (which would be done automatically, see below).

Elements can be constructed via the generator z and the function O(), for example

```
sage: 4 \times z^2 + O(z)
4 \times z^2 + O(z)
```

A Multivariate Asymptotic Ring

Next, we construct a more sophisticated asymptotic ring in the variables x and y by

Again, we can look at a typical (nontrivial) element:

```
sage: B.an_element() 1/8 \times x^{(3/2)} \times \log(x)^{3} \times (1/8)^{y} \times y^{(3/2)} + O(x^{(1/2)} \times \log(x) \times (1/2)^{y} \times y^{(1/2)}
```

Again, elements can be created using the generators x and y, as well as the function O():

```
sage: log(x)*y/42 + O(1/2^y)
1/42*log(x)*y + O((1/2)^y)
```

Arithmetical Operations

In this section we explain how to perform various arithmetical operations with the elements of the asymptotic rings constructed above.

The Ring Operations Plus and Times

We start our calculations in the ring

```
sage: A
Asymptotic Ring <z^QQ> over Integer Ring
```

Of course, we can perform the usual ring operations + and *:

```
sage: z^2 + 3*z*(1-z)
-2*z^2 + 3*z
sage: (3*z + 2)^3
27*z^3 + 54*z^2 + 36*z + 8
```

In addition to that, special powers—our growth group z^Q allows the exponents to be out of Q—can also be computed:

```
sage: (z^{(5/2)}+z^{(1/7)}) * z^{(-1/5)}
z^{(23/10)} + z^{(-2/35)}
```

The central concepts of computations with asymptotic expansions is that the O-notation can be used. For example, we have

```
sage: z^3 + z^2 + z + O(z^2)
z^3 + O(z^2)
```

where the result is simplified automatically. A more sophisticated example is

```
sage: (z+2*z^2+3*z^3+4*z^4) * (O(z)+z^2)
4*z^6 + O(z^5)
```

Division

The asymptotic expansions support division. For example, we can expand 1/(z-1) to a geometric series:

```
sage: 1 / (z-1)
 z^{(-1)} + z^{(-2)} + z^{(-3)} + z^{(-4)} + ... + z^{(-20)} + O(z^{(-21)})
```

A default precision (parameter default_prec of AsymptoticRing) is predefined. Thus, only the first 20 summands are calculated. However, if we only want the first 5 exact terms, we cut of the rest by using

```
sage: (1 / (z-1)).truncate(5)
z^{(-1)} + z^{(-2)} + z^{(-3)} + z^{(-4)} + z^{(-5)} + O(z^{(-6)})
```

or

```
sage: 1 / (z-1) + O(z^{(-6)})
z^{(-1)} + z^{(-2)} + z^{(-3)} + z^{(-4)} + z^{(-5)} + O(z^{(-6)})
```

Of course, we can work with more complicated expansions as well:

```
sage: (4*z+1) / (z^3+z^2+z+0(z^0))
4*z^(-2) - 3*z^(-3) - z^(-4) + 0(z^(-5))
```

Not all elements are invertible, for instance,

```
sage: 1 / O(z)
Traceback (most recent call last):
...
ZeroDivisionError: Cannot invert O(z).
```

is not invertible, since it includes 0.

Powers, Expontials and Logarithms

```
sage: -\log(1-1/z)
z^{(-1)} + 1/2*z^{(-2)} + 1/3*z^{(-3)} + ... + O(z^{(-21)})
```

as $z \to \infty$.

Similarly, we can apply the exponential function of an asymptotic expansion:

```
sage: \exp(1/z)
1 + z^{(-1)} + 1/2*z^{(-2)} + 1/6*z^{(-3)} + 1/24*z^{(-4)} + ... + O(z^{(-20)})
```

Arbitrary powers work as well; for example, we have

```
sage: (1 + 1/z + O(1/z^5))^(1 + 1/z)

1 + z^(-1) + z^(-2) + 1/2*z^(-3) + 1/3*z^(-4) + O(z^(-5))
```

Note: In the asymptotic ring

```
sage: M.<n> = AsymptoticRing(growth_group='QQ^n * n^QQ', coefficient_ring=ZZ)
```

the operation

```
sage: (1/2)^n
Traceback (most recent call last):
...
ValueError: 1/2 is not in Exact Term Monoid QQ^n * n^QQ
with coefficients in Integer Ring. ...
```

fails, since the rational 1/2 is not contained in M. You can use

```
sage: n.rpow(1/2)
(1/2)^n
```

instead. (See also the examples in ExactTerm.rpow() for a detailed explanation.) Another way is to use a larger coefficient ring:

```
sage: M_QQ.<n> = AsymptoticRing(growth_group='QQ^n * n^QQ', coefficient_ring=QQ)
sage: (1/2)^n
(1/2)^n
```

Multivariate Arithmetic

Now let us move on to arithmetic in the multivariate ring

```
sage: B Asymptotic Ring \langle x^0Q \rangle \times \log(x)^2Z \times QQ^3 \times y^0Q \rangle over Rational Field
```

Todo

write this part

4.1.3 More Examples

The mathematical constant e as a limit

The base of the natural logarithm e satisfies the equation

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n$$

By using asymptotic expansions, we obtain the more precise result

4.1.4 Selected Technical Details

Coercions and Functorial Constructions

The AsymptoticRing fully supports coercion. For example, the coefficient ring is automatically extended when needed:

```
sage: A
Asymptotic Ring <z^QQ> over Integer Ring
sage: (z + 1/2).parent()
Asymptotic Ring <z^QQ> over Rational Field
```

Here, the coefficient ring was extended to allow 1/2 as a coefficient. Another example is

```
sage: C.<c> = AsymptoticRing(growth_group='c^ZZ', coefficient_ring=ZZ['e'])
sage: C.an_element()
e^3*c^3 + O(c)
sage: C.an_element() / 7
1/7*e^3*c^3 + O(c)
```

Here the result's coefficient ring is the newly found

```
sage: (C.an_element() / 7).parent()
Asymptotic Ring <c^ZZ> over
Univariate Polynomial Ring in e over Rational Field
```

Not only the coefficient ring can be extended, but the growth group as well. For example, we can add/multiply elements of the asymptotic rings A and C to get an expansion of new asymptotic ring:

```
sage: r = c*z + c/2 + O(z); r
c*z + 1/2*c + O(z)
sage: r.parent()
Asymptotic Ring <c^ZZ * z^QQ> over
Univariate Polynomial Ring in e over Rational Field
```

Data Structures

The summands of an asymptotic expansion are wrapped growth group elements. This wrapping is done by the term monoid module. However, inside an asymptotic expansion these summands (terms) are stored together with their growth-relationship, i.e., each summand knows its direct predecessors and successors. As a data structure a special poset (namely a mutable poset) is used. We can have a look at this:

```
sage: b = x^3*y + x^2*y + x*y^2 + O(x) + O(y)
sage: print(b.summands.repr_full(reverse=True))
poset(x*y^2, x^3*y, x^2*y, O(x), O(y))
+-- oo
| +-- no successors
| +-- predecessors: x*y^2, x^3*y
```

```
+-- x*y^2
| +-- successors: oo
 +-- predecessors: O(x), O(y)
+-- x^3*y
  +-- successors: oo
 +-- predecessors: x^2*y
+-- x^2*y
  +-- successors: x^3*y
  +-- predecessors: O(x), O(y)
+-- O(x)
  +-- successors: x*y^2, x^2*y
  +-- predecessors: null
+-- O(y)
  +-- successors: x*y^2, x^2*y
   +-- predecessors: null
+-- null
 +-- successors: O(x), O(y)
| +-- no predecessors
```

4.1.5 Various

AUTHORS:

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- Daniel Krenn (2015)
- Clemens Heuberger (2016)

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4.1.6 Classes and Methods

Bases: sage.structure.element.CommutativeAlgebraElement

Class for asymptotic expansions, i.e., the elements of an AsymptoticRing.

INPUT:

- •parent the parent of the asymptotic expansion.
- •summands the summands as a MutablePoset, which represents the underlying structure.
- •simplify a boolean (default: True). It controls automatic simplification (absorption) of the asymptotic expansion.
- •convert a boolean (default: True). If set, then the summands are converted to the asymptotic ring (the parent of this expansion). If not, then the summands are taken as they are. In that case, the caller must ensure that the parent of the terms is set correctly.

EXAMPLES:

There are several ways to create asymptotic expansions; usually this is done by using the corresponding asymptotic rings:

```
sage: R_x.<x> = AsymptoticRing(growth_group='x^QQ', coefficient_ring=QQ); R_x
Asymptotic Ring <x^QQ> over Rational Field
sage: R_y.<y> = AsymptoticRing(growth_group='y^ZZ', coefficient_ring=ZZ); R_y
Asymptotic Ring <y^ZZ> over Integer Ring
```

At this point, x and y are already asymptotic expansions:

The usual ring operations, but allowing rational exponents (growth group x^QQ) can be performed:

```
sage: x^2 + 3*(x - x^2(2/5))
x^2 + 3*x - 3*x^2(2/5)
sage: (3*x^2(1/3) + 2)^3
27*x + 54*x^2(2/3) + 36*x^2(1/3) + 8
```

One of the central ideas behind computing with asymptotic expansions is that the O-notation (see Wikipedia article Big_O_notation) can be used. For example, we have:

```
sage: (x+2*x^2+3*x^3+4*x^4) * (O(x)+x^2)
4*x^6 + O(x^5)
```

In particular, O() can be used to construct the asymptotic expansions. With the help of the summands(), we can also have a look at the inner structure of an asymptotic expansion:

```
sage: expr1 = x + 2*x^2 + 3*x^3 + 4*x^4; expr2 = 0(x) + x^2
sage: print(expr1.summands.repr_full())
poset (x, 2*x^2, 3*x^3, 4*x^4)
+-- null
   +-- no predecessors
   +-- successors: x
+-- x
  +-- predecessors: null
  +-- successors: 2*x^2
+-- 2*x^2
  +-- predecessors: x
  +-- successors: 3*x^3
+-- 3*x^3
  +-- predecessors: 2*x^2
   +-- successors: 4*x^4
+-- 4 * x^4
  +-- predecessors: 3*x^3
   +-- successors: oo
+-- predecessors: 4*x^4
   +-- no successors
sage: print(expr2.summands.repr_full())
poset (O(x), x^2)
+-- null
  +-- no predecessors
  +-- successors: O(x)
+-- O(x)
```

```
+-- predecessors: null
| +-- successors: x^2
+-- x^2
+-- predecessors: O(x)
 +-- successors: oo
+-- 00
+-- predecessors: x^2
+-- no successors
sage: print((expr1 * expr2).summands.repr_full())
poset (O(x^5), 4*x^6)
+-- null
  +-- no predecessors
   +-- successors: O(x^5)
+-- O(x^5)
   +-- predecessors: null
   +-- successors: 4*x^6
+-- 4*x^6
+-- predecessors: O(x^5)
  +-- successors: oo
+-- 00
 +-- predecessors: 4*x^6
+-- no successors
```

In addition to the monomial growth elements from above, we can also compute with logarithmic terms (simply by constructing the appropriate growth group):

```
sage: R_log = AsymptoticRing(growth_group='log(x)^QQ', coefficient_ring=QQ)
sage: lx = R_log(log(SR.var('x')))
sage: (O(lx) + lx^3)^4
log(x)^12 + O(log(x)^10)
```

See also:

(Asymptotic) Growth Groups, (Asymptotic) Term Monoids, mutable_poset.

0 ()

Convert all terms in this asymptotic expansion to *O*-terms.

INPUT:

Nothing.

OUTPUT:

An asymptotic expansion.

EXAMPLES:

See also:

sage.rings.power_series_ring.PowerSeriesRing(),sage.rings.laurent_series_ring.Laurent_series_ring.Laurent_series_ring.

TESTS:

```
sage: AR(0).0()
Traceback (most recent call last):
...
NotImplementedOZero: The error term in the result is O(0)
which means 0 for sufficiently large x.
```

Compute the (rescaled) difference between this asymptotic expansion and the given values.

INPUT:

- •variable an asymptotic expansion or a string.
- •function a callable or symbolic expression giving the comparison values.
- •values a list or iterable of values where the comparison shall be carried out.
- •rescaled (default: True) determines whether the difference is divided by the error term of the asymptotic expansion.
- •ring (default: RIF) the parent into which the difference is converted.

OUTPUT:

A list of pairs containing comparison points and (rescaled) difference values.

EXAMPLES:

```
sage: A.<n> = AsymptoticRing('QQ^n * n^ZZ', SR)
sage: catalan = binomial(2 \times x, x)/(x+1)
sage: expansion = 4^n*(1/sqrt(pi)*n^(-3/2)
         -9/8/sqrt(pi)*n^{-5/2}
         + 145/128/sqrt(pi)*n^{-7/2} + O(n^{-9/2})
sage: expansion.compare_with_values(n, catalan, srange(5, 10))
[(5, 0.5303924444775?),
 (6, 0.5455279498787?),
 (7, 0.556880411050?),
(8, 0.565710587724?),
(9, 0.572775029098?)]
sage: expansion.compare_with_values(n, catalan, [5, 10, 20], rescaled=False)
[(5, 0.3886263699387?), (10, 19.1842458318?), (20, 931314.63637?)]
sage: expansion.compare_with_values(n, catalan, [5, 10, 20], rescaled=False,_
→ring=SR)
[(5, 168/5*sqrt(5)/sqrt(pi) - 42),
 (10, 1178112/125*sqrt(10)/sqrt(pi) - 16796),
 (20, 650486218752/125*sqrt(5)/sqrt(pi) - 6564120420)]
```

Instead of a symbolic expression, a callable function can be specified as well:

```
sage: A.<n> = AsymptoticRing('n^ZZ * log(n)^ZZ', SR)
sage: def H(n):
...: return sum(1/k for k in srange(1, n+1))
sage: H_expansion = (log(n) + euler_gamma + 1/(2*n)
...: - 1/(12*n^2) + O(n^-4))
```

```
sage: H_expansion.compare_with_values(n, H, srange(25, 30)) # rel tol 1e-6
[(25, -0.008326995?),
  (26, -0.008327472?),
  (27, -0.008327898?),
  (28, -0.00832828?),
  (29, -0.00832862?)]
```

See also:

```
plot_comparison()
```

TESTS:

```
sage: A.\langle x, y \rangle = AsymptoticRing('x^2Z^*y^2Z^1, QQ)
sage: expansion = x^2 + O(x) + O(y)
sage: expansion.compare_with_values(y, lambda z: z^2, srange(20, 30))
Traceback (most recent call last):
NotImplementedError: exactly one error term required
sage: expansion = x^2
sage: expansion.compare_with_values(y, lambda z: z^2, srange(20, 30))
Traceback (most recent call last):
NotImplementedError: exactly one error term required
sage: expansion = x^2 + O(x)
sage: expansion.compare_with_values(y, lambda z: z^2, srange(20, 30))
Traceback (most recent call last):
NameError: name 'x' is not defined
sage: expansion.compare_with_values(x, lambda z: z^2, srange(20, 30))
[(20, 0), (21, 0), \ldots, (29, 0)]
sage: expansion.compare_with_values(x, SR('x*y'), srange(20, 30))
Traceback (most recent call last):
NotImplementedError: expression x*y has more than one variable
```

exact_part ()

Return the expansion consisting of all exact terms of this expansion.

INPUT:

Nothing

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: R.<x> = AsymptoticRing('x^QQ * log(x)^QQ', QQ)
sage: (x^2 + O(x)).exact_part()
x^2
sage: (x + log(x)/2 + O(log(x)/x)).exact_part()
x + 1/2*log(x)
```

TESTS:

```
sage: R.<x, y> = AsymptoticRing('x^QQ * y^QQ', QQ)
sage: (x + y + O(1/(x*y))).exact_part()
x + y
```

```
sage: 0(x).exact_part()
0
```

exp (precision=None)

Return the exponential of (i.e., the power of e to) this asymptotic expansion.

INPUT:

•precision – the precision used for truncating the expansion. If None (default value) is used, the default precision of the parent is used.

OUTPUT:

An asymptotic expansion.

Note: The exponential function of this expansion can only be computed exactly if the respective growth element can be constructed in the underlying growth group.

ALGORITHM:

If the corresponding growth can be constructed, return the exact exponential function. Otherwise, if this term is o(1), try to expand the series and truncate according to the given precision.

Todo

As soon as L-terms are implemented, this implementation has to be adapted as well in order to yield correct results.

EXAMPLES:

```
sage: A.<x> = AsymptoticRing('(e^x)^ZZ * x^ZZ * log(x)^ZZ', SR)
sage: exp(x)
e^x
sage: exp(2*x)
(e^x)^2
sage: exp(x + log(x))
e^x*x
```

```
sage: (x^{(-1)}) \cdot \exp(\text{precision} = 7)
1 + x^{(-1)} + \frac{1}{2} \cdot x^{(-2)} + \frac{1}{6} \cdot x^{(-3)} + \dots + o(x^{(-7)})
```

TESTS:

```
sage: A.<x> = AsymptoticRing('(e^x)^ZZ * x^QQ * log(x)^QQ', SR)
sage: exp(log(x))
x
sage: log(exp(x))
x
```

```
sage: exp(x+1)
e*e^x
```

See trac ticket #19521:

```
sage: A.<n> = AsymptoticRing('n^ZZ', SR.subring(no_variables=True))
sage: exp(O(n^(-3))).parent()
Asymptotic Ring <n^ZZ> over Symbolic Constants Subring
```

factorial ()

Return the factorial of this asymptotic expansion.

OUTPUT:

An asymptotic expansion.

EXAMPLES:

Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$:

```
sage: (2*n).factorial() / n.factorial()^2 / (n+1) # long time
1/sqrt(pi)*(e^n)^(2*log(2))*n^(-3/2)
- 9/8/sqrt(pi)*(e^n)^(2*log(2))*n^(-5/2)
+ 145/128/sqrt(pi)*(e^n)^(2*log(2))*n^(-7/2)
+ O((e^n)^(2*log(2))*n^(-9/2))
```

Note that this method substitutes the asymptotic expansion into Stirling's formula. This substitution has to be possible which is not always guaranteed:

See also:

Stirling()

TESTS:

```
sqrt(2)*sqrt(pi)*e^(m*log(m))*(e^m)^(-1)*m^(1/2)
+ 1/12*sqrt(2)*sqrt(pi)*e^(m*log(m))*(e^m)^(-1)*m^(-1/2)
+ 1/288*sqrt(2)*sqrt(pi)*e^(m*log(m))*(e^m)^(-1)*m^(-3/2)
+ O(e^(m*log(m))*(e^m)^(-1)*m^(-5/2))
```

```
sage: A(1/2).factorial()
1/2*sqrt(pi)
sage: _.parent()
Asymptotic Ring <m^ZZ * log(m)^ZZ> over Symbolic Ring
```

```
sage: B.<a, b> = AsymptoticRing('a^ZZ * b^ZZ', QQ, default_prec=3)
sage: b.factorial()
O(e^(b*log(b))*(e^b)^(-1)*b^(1/2))
sage: (a*b).factorial()
Traceback (most recent call last):
...
ValueError: Cannot build the factorial of a*b
since it is not univariate.
```

has_same_summands (other)

Return whether this asymptotic expansion and other have the same summands.

INPUT:

•other - an asymptotic expansion.

OUTPUT:

A boolean.

Note: While for example O(x) == O(x) yields False, these expansions *do* have the same summands and this method returns True.

Moreover, this method uses the coercion model in order to find a common parent for this asymptotic expansion and other.

EXAMPLES:

TESTS:

```
sage: x_ZZ.has_same_summands(None)
False
```

invert (precision=None)

Return the multiplicative inverse of this element.

INPUT:

•precision – the precision used for truncating the expansion. If None (default value) is used, the default precision of the parent is used.

OUTPUT:

An asymptotic expansion.

Warning: Due to truncation of infinite expansions, the element returned by this method might not fulfill $el \star el = 1$.

Todo

As soon as L-terms are implemented, this implementation has to be adapted as well in order to yield correct results.

EXAMPLES:

TESTS:

```
sage: A.<a> = AsymptoticRing(growth_group='a^ZZ', coefficient_ring=ZZ)
sage: (1 / a).parent()
Asymptotic Ring <a^ZZ> over Rational Field
sage: (a / 2).parent()
Asymptotic Ring <a^ZZ> over Rational Field
```

```
sage: ~A(0)
Traceback (most recent call last):
...
ZeroDivisionError: Cannot invert 0 in
Asymptotic Ring <a^ZZ> over Integer Ring.
```

is_exact ()

Return whether all terms of this expansion are exact.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: A.<x> = AsymptoticRing('x^QQ * log(x)^QQ', QQ)
sage: (x^2 + O(x)).is_exact()
False
sage: (x^2 - x).is_exact()
True
```

TESTS:

```
sage: A(0).is_exact()
True
sage: A.one().is_exact()
True
```

is_little_o_of_one()

Return whether this expansion is of order o(1).

INPUT:

Nothing.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: A.<x> = AsymptoticRing('x^ZZ * log(x)^ZZ', QQ)
sage: (x^4 * log(x)^(-2) + x^(-4) * log(x)^2).is_little_o_of_one()
False
sage: (x^(-1) * log(x)^1234 + x^(-2) + O(x^(-3))).is_little_o_of_one()
True
sage: (log(x) - log(x-1)).is_little_o_of_one()
True
```

```
sage: A.<x, y> = AsymptoticRing('x^QQ * y^QQ * log(y)^ZZ', QQ)
sage: (x^(-1/16) * y^32 + x^32 * y^(-1/16)).is_little_o_of_one()
False
sage: (x^(-1) * y^(-3) + x^(-3) * y^(-1)).is_little_o_of_one()
True
sage: (x^(-1) * y / log(y)).is_little_o_of_one()
False
sage: (log(y-1)/log(y) - 1).is_little_o_of_one()
True
```

log (base=None, precision=None)

The logarithm of this asymptotic expansion.

INPUT:

•base - the base of the logarithm. If None (default value) is used, the natural logarithm is taken.

•precision – the precision used for truncating the expansion. If None (default value) is used, the default precision of the parent is used.

OUTPUT:

An asymptotic expansion.

Note: Computing the logarithm of an asymptotic expansion is possible if and only if there is exactly one maximal summand in the expansion.

ALGORITHM:

If the expansion has more than one summand, the asymptotic expansion for log(1 + t) as t tends to 0 is used.

Todo

As soon as *L*-terms are implemented, this implementation has to be adapted as well in order to yield correct results.

EXAMPLES:

TESTS:

```
sage: log(R(1))
0
sage: log(R(0))
Traceback (most recent call last):
...
ArithmeticError: Cannot compute log(0) in
Asymptotic Ring <x^ZZ * log(x)^ZZ> over Rational Field.
sage: C.<s, t> = AsymptoticRing(growth_group='s^ZZ * t^ZZ', coefficient_
ing=QQ)
sage: log(s + t)
Traceback (most recent call last):
...
ValueError: Cannot determine main term of s + t since
there are several maximal elements s, t.
```

map_coefficients (f, new_coefficient_ring=None)

Return the asymptotic expansion obtained by applying f to each coefficient of this asymptotic expansion.

INPUT:

```
•f – a callable. A coefficient c will be mapped to f(c).
•new_coefficient_ring – (default: None) a ring.
```

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: A.<n> = AsymptoticRing(growth_group='n^ZZ', coefficient_ring=ZZ)
sage: a = n^4 + 2*n^3 + 3*n^2 + O(n)
```

```
sage: a.map_coefficients(lambda c: c+1)
2*n^4 + 3*n^3 + 4*n^2 + O(n)
sage: a.map_coefficients(lambda c: c-2)
-n^4 + n^2 + O(n)
```

TESTS:

```
sage: a.map_coefficients(lambda c: 1/c, new_coefficient_ring=QQ)
n^4 + 1/2*n^3 + 1/3*n^2 + O(n)
sage: _.parent()
Asymptotic Ring <n^ZZ> over Rational Field
sage: a.map_coefficients(lambda c: 1/c)
Traceback (most recent call last):
...
ValueError: ... is not a coefficient in
Exact Term Monoid n^ZZ with coefficients in Integer Ring.
```

Plot the (rescaled) difference between this asymptotic expansion and the given values.

INPUT:

- •variable an asymptotic expansion or a string.
- •function a callable or symbolic expression giving the comparison values.
- •values a list or iterable of values where the comparison shall be carried out.
- •rescaled (default: True) determines whether the difference is divided by the error term of the asymptotic expansion.
- •ring (default: RIF) the parent into which the difference is converted.
- •relative_tolerance (default: 0.025). Raise error when relative error exceeds this tolerance.

Other keyword arguments are passed to list_plot().

OUTPUT:

A graphics object.

Note: If rescaled (i.e. divided by the error term), the output should be bounded.

This method is mainly meant to have an easily usable plausability check for asymptotic expansion created in some way.

EXAMPLES:

We want to check the quality of the asymptotic expansion of the harmonic numbers:

```
sage: A.<n> = AsymptoticRing('n^ZZ * log(n)^ZZ', SR)
sage: def H(n):
....: return sum(1/k for k in srange(1, n+1))
sage: H_expansion = (log(n) + euler_gamma + 1/(2*n)
....: - 1/(12*n^2) + O(n^-4))
sage: H_expansion.plot_comparison(n, H, srange(1, 30))
Graphics object consisting of 1 graphics primitive
```

Alternatively, the unscaled (absolute) difference can be plotted as well:

Additional keywords are passed to list_plot():

See also:

```
compare_with_values()
```

TESTS:

```
sage: H_expansion.plot_comparison(n, H, [600])
Traceback (most recent call last):
...
ValueError: Numerical noise is too high, the comparison is inaccurate
sage: H_expansion.plot_comparison(n, H, [600], relative_tolerance=2)
Graphics object consisting of 1 graphics primitive
```

pow (exponent, precision=None)

Calculate the power of this asymptotic expansion to the given exponent.

INPUT:

•exponent - an element.

•precision – the precision used for truncating the expansion. If None (default value) is used, the default precision of the parent is used.

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: Q.<x> = AsymptoticRing(growth_group='x^QQ', coefficient_ring=QQ)
sage: x^{(1/7)}
x^{(1/7)}
sage: (x^{(1/2)} + O(x^{(0)})^{15}
x^{(15/2)} + O(x^{(7)})
```

```
sage: Z.<y> = AsymptoticRing(growth_group='y^ZZ', coefficient_ring=ZZ)
sage: y^(1/7)
y^(1/7)
sage: _.parent()
Asymptotic Ring <y^QQ> over Rational Field
sage: (y^2 + O(y))^(1/2)
y + O(1)
sage: (y^2 + O(y))^(-2)
y^(-4) + O(y^(-5))
sage: (1 + 1/y + O(1/y^3))^pi
1 + pi*y^(-1) + (1/2*pi*(pi - 1))*y^(-2) + O(y^(-3))
```

```
sage: A.<x> = AsymptoticRing('QQ^x * x^SR * log(x)^ZZ', QQ)
sage: x * 2^x
2^x*x
sage: 5^x * 2^x
10^x
sage: 2^log(x)
x^(log(2))
sage: 2^(x + 1/x)
2^x + log(2)*2^x*x^(-1) + 1/2*log(2)^2*2^x*x^(-2) + ... + O(2^x*x^(-20))
sage: _.parent()
Asymptotic Ring <QQ^x * x^SR * log(x)^QQ> over Symbolic Ring
```

TESTS:

See trac ticket #19110:

```
sage: O(x)^(-1)
Traceback (most recent call last):
...
ZeroDivisionError: Cannot take O(x) to exponent -1.
> *previous* ZeroDivisionError: rational division by zero
```

```
sage: B(0)^(-7)
Traceback (most recent call last):
...
ZeroDivisionError: Cannot take 0 to the negative exponent -7.
sage: B(0)^SR.var('a')
Traceback (most recent call last):
...
NotImplementedError: Taking 0 to the exponent a not implemented.
```

Check that trac ticket #19946 is fixed:

```
sage: A.<n> = AsymptoticRing('QQ^n * n^QQ', SR)
sage: e = 2^n; e
2^n
sage: e.parent()
Asymptotic Ring <SR^n * n^SR> over Symbolic Ring
sage: e = A(e); e
2^n
sage: e.parent()
Asymptotic Ring <QQ^n * n^QQ> over Symbolic Ring
```

rpow (base, precision=None)

Return the power of base to this asymptotic expansion.

INPUT:

```
•base - an element or 'e'.
```

•precision – the precision used for truncating the expansion. If None (default value) is used, the default precision of the parent is used.

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: A.<x> = AsymptoticRing('x^ZZ', QQ)
sage: (1/x).rpow('e', precision=5)
1 + x^{(-1)} + 1/2*x^{(-2)} + 1/6*x^{(-3)} + 1/24*x^{(-4)} + 0(x^{(-5)})
```

TESTS:

```
sage: x.rpow(SR.var('y'))
Traceback (most recent call last):
...
ArithmeticError: Cannot construct y^x in Growth Group x^ZZ
> *previous* TypeError: unsupported operand parent(s) for '*':
'Growth Group x^ZZ' and 'Growth Group SR^x'
```

Check that trac ticket #19946 is fixed:

```
sage: A.<n> = AsymptoticRing('QQ^n * n^QQ', SR)
sage: n.rpow(2)
2^n
sage: _.parent()
Asymptotic Ring <QQ^n * n^SR> over Symbolic Ring
```

show ()

Pretty-print this asymptotic expansion.

OUTPUT:

Nothing, the representation is printed directly on the screen.

EXAMPLES:

TESTS:

sqrt (precision=None)

Return the square root of this asymptotic expansion.

INPUT:

•precision – the precision used for truncating the expansion. If None (default value) is used, the default precision of the parent is used.

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: A.<s> = AsymptoticRing(growth_group='s^QQ', coefficient_ring=QQ)
sage: s.sqrt()
s^(1/2)
sage: a = (1 + 1/s).sqrt(precision=6); a
1 + 1/2*s^(-1) - 1/8*s^(-2) + 1/16*s^(-3)
- 5/128*s^(-4) + 7/256*s^(-5) + O(s^(-6))
```

See also:

```
pow(), rpow(), exp().
```

TESTS:

```
sage: P. = PowerSeriesRing(QQ, default_prec=6)
sage: bool(SR(a.exact_part()).subs(s=1/x) -
...: SR((1+p).sqrt().polynomial()).subs(p=x) == 0)
True
```

subs (rules=None, domain=None, **kwds)

Substitute the given rules in this asymptotic expansion.

INPUT:

- •rules a dictionary.
- •kwds keyword arguments will be added to the substitution rules.

•domain - (default: None) a parent. The neutral elements 0 and 1 (rules for the keys '_zero_' and '_one_', see note box below) are taken out of this domain. If None, then this is determined automatically.

OUTPUT:

An object.

Note: The neutral element of the asymptotic ring is replaced by the value to the key '_zero_'; the neutral element of the growth group is replaced by the value to the key '_one_'.

EXAMPLES:

```
sage: (e^x * x^2 + log(x)).subs(x=SR('s'))
s^2*e^s + log(s)
sage: _.parent()
Symbolic Ring
```

```
sage: (x^3 + x + log(x)).subs(x=x+5).truncate(5)
x^3 + 15*x^2 + 76*x + log(x) + 130 + O(x^(-1))
sage: _.parent()
Asymptotic Ring <(e^x)^QQ * x^ZZ * log(x)^ZZ> over Rational Field
```

```
sage: (e^x * x^2 + log(x)).subs(x=2*x)
4*(e^x)^2*x^2 + log(x) + log(2)
sage: _.parent()
Asymptotic Ring <(e^x)^QQ * x^QQ * log(x)^QQ> over Symbolic Ring
```

```
sage: (x^2 + log(x)).subs(x=4*x+2).truncate(5)
16*x^2 + 16*x + log(x) + log(4) + 4 + 1/2*x^(-1) + O(x^(-2))
sage: _.parent()
Asymptotic Ring <(e^x)^QQ * x^ZZ * log(x)^ZZ> over Symbolic Ring
```

```
sage: (e^x * x^2 + log(x)).subs(x=RIF(pi))
229.534211738584?
sage: _.parent()
Real Interval Field with 53 bits of precision
```

See also:

sage.symbolic.expression.Expression.subs()

TESTS:

```
TypeError: Cannot apply the substitution rules {u: 0} on u^(-1) in
Asymptotic Ring <u^QQ * v^QQ * w^QQ> over Rational Field.
> *previous* ZeroDivisionError: Cannot substitute in u^(-1) in
Asymptotic Ring <u^QQ * v^QQ * w^QQ> over Rational Field.
>> *previous* ZeroDivisionError: Cannot substitute in u^(-1) in
Exact Term Monoid u^Q \times v^Q \times w^Q with coefficients in Rational Field.
>...> *previous* ZeroDivisionError: Cannot substitute in u^(-1) in
Growth Group u^QQ * v^QQ * w^QQ.
>...> *previous* ZeroDivisionError: Cannot substitute in u^{-1} in
Growth Group u^QQ.
>...> *previous* ZeroDivisionError: rational division by zero
sage: (1/u).subs({'u': 0, 'v': SR.var('v')})
Traceback (most recent call last):
. . .
TypeError: Cannot apply the substitution rules {u: 0, v: v} on u^(-1) in
Asymptotic Ring <u^QQ * v^QQ * w^QQ> over Rational Field.
> *previous* ZeroDivisionError: Cannot substitute in u^(-1) in
Asymptotic Ring \langle u^QQ * v^QQ * w^QQ \rangle over Rational Field.
>> *previous* ZeroDivisionError: Cannot substitute in u^(-1) in
Exact Term Monoid u^Q = v^Q = w^Q with coefficients in Rational Field.
>...> *previous* ZeroDivisionError: Cannot substitute in u^(-1) in
Growth Group u^Q = v^Q + v^Q = v^Q .
>...> *previous* ZeroDivisionError: Cannot substitute in u^(-1) in
Growth Group u^QQ.
>...> *previous* ZeroDivisionError: rational division by zero
```

```
sage: u.subs({u: 0, 'v': SR.var('v')})
0
sage: v.subs({u: 0, 'v': SR.var('v')})
v
sage: _.parent()
Symbolic Ring
```

```
sage: u.subs({SR.var('u'): -1})
Traceback (most recent call last):
...
TypeError: Cannot substitute u in u since it is neither an
asymptotic expansion nor a string
(but a <type 'sage.symbolic.expression.Expression'>).
```

```
sage: u.subs({u: 1, 'u': 1})
1
sage: u.subs({u: 1}, u=1)
1
sage: u.subs({u: 1, 'u': 2})
Traceback (most recent call last):
...
ValueError: Cannot substitute in u: duplicate key u.
sage: u.subs({u: 1}, u=3)
Traceback (most recent call last):
...
ValueError: Cannot substitute in u: duplicate key u.
```

substitute (rules=None, domain=None, **kwds)

Substitute the given rules in this asymptotic expansion.

INPUT:

- •rules a dictionary.
- •kwds keyword arguments will be added to the substitution rules.
- •domain (default: None) a parent. The neutral elements 0 and 1 (rules for the keys '_zero_' and '_one_', see note box below) are taken out of this domain. If None, then this is determined automatically.

OUTPUT:

An object.

Note: The neutral element of the asymptotic ring is replaced by the value to the key '_zero_'; the neutral element of the growth group is replaced by the value to the key '_one_'.

EXAMPLES:

```
sage: (e^x * x^2 + log(x)).subs(x=SR('s'))
s^2*e^s + log(s)
sage: _.parent()
Symbolic Ring
```

```
sage: (x^3 + x + log(x)).subs(x=x+5).truncate(5)
x^3 + 15*x^2 + 76*x + log(x) + 130 + O(x^(-1))
sage: _.parent()
Asymptotic Ring <(e^x)^QQ * x^ZZ * log(x)^ZZ> over Rational Field
```

```
sage: (x^2 + log(x)).subs(x=4*x+2).truncate(5)
16*x^2 + 16*x + log(x) + log(4) + 4 + 1/2*x^(-1) + O(x^(-2))
sage: _.parent()
Asymptotic Ring <(e^x)^QQ * x^ZZ * log(x)^ZZ> over Symbolic Ring
```

```
sage: (e^x * x^2 + log(x)).subs(x=RIF(pi))
229.534211738584?
sage: _.parent()
Real Interval Field with 53 bits of precision
```

See also:

sage.symbolic.expression.Expression.subs()

TESTS:

```
sage: x.subs({'y': -1})
Traceback (most recent call last):
...
ValueError: Cannot substitute y in x since it is not a generator of
```

```
Asymptotic Ring <(e^x)^Q \times x^Z \times \log(x)^Z > \text{ over Rational Field.}
sage: B.<u, v, w> = AsymptoticRing(growth_group='u^QQ * v^QQ * w^QQ',...
sage: (1/u).subs({'u': 0})
Traceback (most recent call last):
TypeError: Cannot apply the substitution rules {u: 0} on u^(-1) in
Asymptotic Ring <u^QQ * v^QQ * w^QQ> over Rational Field.
> *previous* ZeroDivisionError: Cannot substitute in u^(-1) in
Asymptotic Ring <u^QQ * v^QQ * w^QQ> over Rational Field.
>> *previous* ZeroDivisionError: Cannot substitute in u^(-1) in
Exact Term Monoid u^Q \times v^Q \times w^Q with coefficients in Rational Field.
>...> *previous* ZeroDivisionError: Cannot substitute in u^(-1) in
Growth Group u^Q \times v^Q \times w^Q.
>...> *previous* ZeroDivisionError: Cannot substitute in u^(-1) in
Growth Group u^QQ.
>...> *previous* ZeroDivisionError: rational division by zero
sage: (1/u).subs({'u': 0, 'v': SR.var('v')})
Traceback (most recent call last):
TypeError: Cannot apply the substitution rules {u: 0, v: v} on u^(-1) in
Asymptotic Ring <u^QQ * v^QQ * w^QQ> over Rational Field.
> *previous* ZeroDivisionError: Cannot substitute in u^(-1) in
Asymptotic Ring <u^QQ * v^QQ * w^QQ> over Rational Field.
>> *previous* ZeroDivisionError: Cannot substitute in u^(-1) in
Exact Term Monoid u^Q = v^Q = w^Q 
>...> *previous* ZeroDivisionError: Cannot substitute in u^(-1) in
Growth Group u^Q = v^Q + v^Q = v^Q 
>...> *previous* ZeroDivisionError: Cannot substitute in u^(-1) in
Growth Group u^QQ.
>...> *previous* ZeroDivisionError: rational division by zero
```

```
sage: u.subs({u: 0, 'v': SR.var('v')})
0
sage: v.subs({u: 0, 'v': SR.var('v')})
v
sage: _.parent()
Symbolic Ring
```

```
sage: u.subs({SR.var('u'): -1})
Traceback (most recent call last):
...
TypeError: Cannot substitute u in u since it is neither an
asymptotic expansion nor a string
(but a <type 'sage.symbolic.expression.Expression'>).
```

```
sage: u.subs({u: 1, 'u': 1})
1
sage: u.subs({u: 1}, u=1)
1
sage: u.subs({u: 1, 'u': 2})
Traceback (most recent call last):
...
ValueError: Cannot substitute in u: duplicate key u.
sage: u.subs({u: 1}, u=3)
Traceback (most recent call last):
...
```

```
ValueError: Cannot substitute in u: duplicate key u.
```

summands

The summands of this asymptotic expansion stored in the underlying data structure (a MutablePoset).

EXAMPLES:

```
sage: R.<x> = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ)
sage: expr = 7*x^12 + x^5 + O(x^3)
sage: expr.summands
poset(O(x^3), x^5, 7*x^12)
```

See also:

sage.data_structures.mutable_poset.MutablePoset

symbolic_expression (R=None)

Return this asymptotic expansion as a symbolic expression.

INPUT:

 $\bullet R$ – (a subring of) the symbolic ring or None. The output is will be an element of R. If None, then the symbolic ring is used.

OUTPUT:

A symbolic expression.

EXAMPLES:

```
sage: A.<x, y, z> = AsymptoticRing(growth_group='x^ZZ * y^QQ * log(y)^QQ * QQ^
\rightarrowZ * z^QQ', coefficient_ring=QQ)
sage: SR(A.an_element()) # indirect doctest
1/8*(1/8)^2*x^3*y^(3/2)*z^(3/2)*log(y)^(3/2) +
Order((1/2)^z*x*sqrt(y)*sqrt(z)*sqrt(log(y)))
```

TESTS:

```
sage: a = A.an_element(); a

1/8*x^3*y^(3/2)*log(y)^(3/2)*(1/8)^z*z^(3/2) +

0(x*y^(1/2)*log(y)^(1/2)*(1/2)^z*z^(1/2))

sage: a.symbolic_expression()

1/8*(1/8)^z*x^3*y^(3/2)*z^(3/2)*log(y)^(3/2) +

Order((1/2)^z*x*sqrt(y)*sqrt(z)*sqrt(log(y)))

sage: _.parent()

Symbolic Ring
```

```
sage: from sage.symbolic.ring import SymbolicRing
sage: class MySymbolicRing(SymbolicRing):
....:    pass
sage: mySR = MySymbolicRing()
sage: a.symbolic_expression(mySR).parent() is mySR
True
```

truncate (precision=None)

Truncate this asymptotic expansion.

INPUT:

•precision — a positive integer or None. Number of summands that are kept. If None (default value) is given, then default_prec from the parent is used.

OUTPUT:

An asymptotic expansion.

Note: For example, truncating an asymptotic expansion with precision=20 does not yield an expansion with exactly 20 summands! Rather than that, it keeps the 20 summands with the largest growth, and adds appropriate *O*-Terms.

EXAMPLES:

```
sage: R.<x> = AsymptoticRing('x^ZZ', QQ)
sage: ex = sum(x^k for k in range(5)); ex
x^4 + x^3 + x^2 + x + 1
sage: ex.truncate(precision=2)
x^4 + x^3 + O(x^2)
sage: ex.truncate(precision=0)
O(x^4)
sage: ex.truncate()
x^4 + x^3 + x^2 + x + 1
```

variable_names ()

Return the names of the variables of this asymptotic expansion.

OUTPUT:

A tuple of strings.

EXAMPLES:

```
sage: A.<m, n > = AsymptoticRing('QQ^m * m^QQ * n^ZZ * log(n)^ZZ', QQ)
sage: (4 \times 2^n \times m^4 \times \log(n)).variable_names()
('m', 'n')
sage: (4*2^m*m^4).variable_names()
('m',)
sage: (4*log(n)).variable_names()
('n',)
sage: (4*m^3).variable_names()
('m',)
sage: (4*m^0).variable_names()
sage: (4 \times 2^n \times m^4 + \log(n)).variable_names()
('m', 'n')
sage: (2^m + m^4 + \log(n)).variable_names()
('m', 'n')
sage: (2^m + m^4).variable_names()
('m',)
```

Bases: sage.rings.ring.Algebra, sage.structure.unique_representation.UniqueRepresentation

A ring consisting of asymptotic expansions.

INPUT:

•growth_group - either a partially ordered group (see (Asymptotic) Growth Groups) or a string describing such a growth group (see GrowthGroupFactory).

- •coefficient_ring the ring which contains the coefficients of the expansions.
- •default_prec a positive integer. This is the number of summands that are kept before truncating an infinite series.
- •category the category of the parent can be specified in order to broaden the base structure. It has to be a subcategory of Category of rings. This is also the default category if None is specified.

EXAMPLES:

We begin with the construction of an asymptotic ring in various ways. First, we simply pass a string specifying the underlying growth group:

```
sage: R1_x.<x> = AsymptoticRing(growth_group='x^QQ', coefficient_ring=QQ); R1_x
Asymptotic Ring <x^QQ> over Rational Field
sage: x
x
```

This is equivalent to the following code, which explicitly specifies the underlying growth group:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G_QQ = GrowthGroup('x^QQ')
sage: R2_x.<x> = AsymptoticRing(growth_group=G_QQ, coefficient_ring=QQ); R2_x
Asymptotic Ring <x^QQ> over Rational Field
```

Of course, the coefficient ring of the asymptotic ring and the base ring of the underlying growth group do not need to coincide:

```
sage: R_ZZ_x.<x> = AsymptoticRing(growth_group='x^QQ', coefficient_ring=ZZ); R_ZZ_ \leftrightarrowx
Asymptotic Ring <x^QQ> over Integer Ring
```

Note, we can also create and use logarithmic growth groups:

Other growth groups are available. See Asymptotic Ring for more examples.

Below there are some technical details.

According to the conventions for parents, uniqueness is ensured:

```
sage: R1_x is R2_x
True
```

Furthermore, the coercion framework is also involved. Coercion between two asymptotic rings is possible (given that the underlying growth groups and coefficient rings are chosen appropriately):

```
sage: R1_x.has_coerce_map_from(R_ZZ_x)
True
```

Additionally, for the sake of convenience, the coefficient ring also coerces into the asymptotic ring (representing constant quantities):

```
sage: R1_x.has_coerce_map_from(QQ)
True
```

Element

alias of AsymptoticExpansion

change_parameter (**kwds)

Return an asymptotic ring with a change in one or more of the given parameters.

INPUT:

- •growth_group (default: None) the new growth group.
- •coefficient_ring (default: None) the new coefficient ring.
- •category (default: None) the new category.
- •default_prec (default: None) the new default precision.

OUTPUT:

An asymptotic ring.

EXAMPLES:

```
sage: A = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ)
sage: A.change_parameter(coefficient_ring=QQ)
Asymptotic Ring <x^ZZ> over Rational Field
```

TESTS:

```
sage: A.change_parameter(coefficient_ring=ZZ) is A
True
sage: A.change_parameter(coefficient_ring=None) is A
True
```

coefficient_ring

The coefficient ring of this asymptotic ring.

EXAMPLES:

```
sage: AR = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ)
sage: AR.coefficient_ring
Integer Ring
```

coefficients_of_generating_function (function, singularities, precision=None, return_singular_expansions=False)

Return the asymptotic growth of the coefficients of some generating function by means of Singularity Analysis.

INPUT:

•function – a callable function in one variable.

- •singularities list of dominant singularities of the function.
- •precision (default: None) an integer. If None, then the default precision of the asymptotic ring is used.
- •return_singular_expansions (default: False) a boolean. If set, the singular expansions are also returned.

OUTPUT:

- •If return singular expansions=False: An asymptotic expansion from this ring.
- •If return_singular_expansions=True : A named tuple with components asymptotic_expansion and singular_expansions. The former contains an asymptotic expansion from this ring, the latter is a dictionary which contains the singular expansions around the singularities.

Todo

Make this method more usable by implementing the processing of symbolic expressions.

EXAMPLES:

Catalan numbers:

Unit fractions:

```
sage: def logarithmic(z):
....: return -log(1-z)
sage: B.coefficients_of_generating_function(logarithmic, (1,), precision=5)
n^(-1) + O(n^(-3))
```

Harmonic numbers:

Warning: Once singular expansions around points other than infinity are implemented (trac ticket #20050), the output in the case return_singular_expansions will change to return singular expansions around the singularities.

TESTS:

```
sage: def f(z):
....: return z/(1-z)
sage: B.coefficients_of_generating_function(f, (1,), precision=3)
Traceback (most recent call last):
...
NotImplementedOZero: The error term in the result is O(0)
which means 0 for sufficiently large n.
```

construction ()

Return the construction of this asymptotic ring.

OUTPUT:

A pair whose first entry is an asymptotic ring construction functor and its second entry the coefficient ring.

EXAMPLES:

```
sage: A = AsymptoticRing(growth_group='x^ZZ * QQ^y', coefficient_ring=QQ)
sage: A.construction()
(AsymptoticRing<x^ZZ * QQ^y>, Rational Field)
```

See also:

Asymptotic Ring, Asymptotic Ring, Asymptotic Ring Functor.

create_summand (type, data=None, **kwds)

Create a simple asymptotic expansion consisting of a single summand.

INPUT:

```
•type - 'O' or 'exact'.
```

•data - the element out of which a summand has to be created.

```
•growth - an element of the growth_group().
```

•coefficient - an element of the coefficient_ring().

Note: Either growth and coefficient or data have to be specified.

OUTPUT:

An asymptotic expansion.

Note: This method calls the factory *TermMonoid* with the appropriate arguments.

```
sage: R = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ)
sage: R.create_summand('0', x^2)
```

```
O(x^2)
sage: R.create_summand('exact', growth=x^456, coefficient=123)
123*x^456
sage: R.create_summand('exact', data=12*x^13)
12*x^13
```

TESTS:

```
sage: R.create_summand('exact', data='12*x^13')
12*x^13
sage: R.create_summand('exact', data='x^13 * 12')
12*x^13
sage: R.create_summand('exact', data='x^13')
x^13
sage: R.create_summand('exact', data='12')
12
sage: R.create_summand('exact', data=12)
12
```

```
sage: Z = R.change_parameter(coefficient_ring=Zmod(3))
sage: Z.create_summand('exact', data=42)
0
```

```
sage: AR.<z> = AsymptoticRing('z^QQ', QQ)
sage: AR.create_summand('exact', growth='z^2')
Traceback (most recent call last):
...
TypeError: Cannot create exact term: only 'growth' but
no 'coefficient' specified.
```

default_prec

The default precision of this asymptotic ring.

This is the parameter used to determine how many summands are kept before truncating an infinite series (which occur when inverting asymptotic expansions).

EXAMPLES:

```
sage: AR = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ)
sage: AR.default_prec
20
sage: AR = AsymptoticRing('x^ZZ', ZZ, default_prec=123)
sage: AR.default_prec
123
```

gen (n=0)

Return the n -th generator of this asymptotic ring.

INPUT:

•n – (default: 0) a non-negative integer.

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: R.<x> = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ)
sage: R.gen()
x
```

gens ()

Return a tuple with generators of this asymptotic ring.

INPUT:

Nothing.

OUTPUT:

A tuple of asymptotic expansions.

Note: Generators do not necessarily exist. This depends on the underlying growth group. For example, monomial growth groups have a generator, and exponential growth groups do not.

EXAMPLES:

growth_group

The growth group of this asymptotic ring.

EXAMPLES:

```
sage: AR = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ)
sage: AR.growth_group
Growth Group x^ZZ
```

See also:

(Asymptotic) Growth Groups

ngens ()

Return the number of generators of this asymptotic ring.

INPUT:

Nothing.

OUTPUT:

An integer.

```
sage: AR.<x> = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ)
sage: AR.ngens()
1
```

some_elements ()

Return some elements of this term monoid.

See TestSuite for a typical use case.

INPUT:

Nothing.

OUTPUT:

An iterator.

EXAMPLES:

variable_names ()

Return the names of the variables.

OUTPUT:

A tuple of strings.

EXAMPLES:

```
sage: A = AsymptoticRing(growth_group='x^ZZ * QQ^y', coefficient_ring=QQ)
sage: A.variable_names()
('x', 'y')
```

A construction functor for asymptotic rings.

INPUT:

•growth_group - a partially ordered group (see AsymptoticRing or (Asymptotic) Growth Groups for details).

See also:

Asymptotic Ring, AsymptoticRing, sage.rings.asymptotic.growth_group.AbstractGrowthGroupFunctor, sage.rings.asymptotic.growth_group.ExponentialGrowthGroupFunctor, sage.rings.asymptotic.growth_group.MonomialGrowthGroupFunctor, sage.categories.pushout.ConstructionFunctor.

TESTS:

```
sage: X = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=QQ)
sage: Y = AsymptoticRing(growth_group='y^ZZ', coefficient_ring=QQ)
sage: cm = sage.structure.element.get_coercion_model()
sage: cm.record_exceptions()
sage: cm.common_parent(X, Y)
Asymptotic Ring <x^ZZ * y^ZZ> over Rational Field
sage: sage.structure.element.coercion_traceback() # not tested
```

```
sage: from sage.categories.pushout import pushout
sage: pushout(AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ), QQ)
Asymptotic Ring <x^ZZ> over Rational Field
```

merge (other)

Merge this functor with other if possible.

INPUT:

•other -a functor.

OUTPUT:

A functor or None.

EXAMPLES:

```
sage: X = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=QQ)
sage: Y = AsymptoticRing(growth_group='y^ZZ', coefficient_ring=QQ)
sage: F_X = X.construction()[0]
sage: F_Y = Y.construction()[0]
sage: F_X.merge(F_X)
AsymptoticRing<x^ZZ>
sage: F_X.merge(F_Y)
AsymptoticRing<x^ZZ * y^ZZ>
```

exception sage.rings.asymptotic.asymptotic_ring. NoConvergenceError
Bases: exceptions.RuntimeError

A special RuntimeError which is raised when an algorithm does not converge/stop.

4.2 Common Asymptotic Expansions

Warning: As this code is experimental, a warning is thrown when an asymptotic ring (or an associated structure) is created for the first time in a session (see sage.misc.superseded.experimental).

```
sage: AsymptoticRing(growth_group='z^ZZ * log(z)^QQ', coefficient_ring=ZZ)
doctest:...: FutureWarning: This class/method/function is marked as
experimental. It, its functionality or its interface might change
without a formal deprecation.
See http://trac.sagemath.org/17601 for details.
Asymptotic Ring <z^ZZ * log(z)^QQ> over Integer Ring
```

Asymptotic expansions in SageMath can be built through the asymptotic_expansions object. It contains generators for common asymptotic expressions. For example,

```
sage: asymptotic_expansions.Stirling('n', precision=5)
sqrt(2)*sqrt(pi)*e^(n*log(n))*(e^n)^(-1)*n^(1/2) +

1/12*sqrt(2)*sqrt(pi)*e^(n*log(n))*(e^n)^(-1)*n^(-1/2) +

1/288*sqrt(2)*sqrt(pi)*e^(n*log(n))*(e^n)^(-1)*n^(-3/2) +

O(e^(n*log(n))*(e^n)^(-1)*n^(-5/2))
```

generates the first 5 summands of Stirling's approximation formula for factorials.

To construct an asymptotic expression manually, you can use the class <code>AsymptoticRing</code> . See asymptotic ring for more details and a lot of examples.

Asymptotic Expansions

HarmonicNumber()	harmonic numbers
Stirling()	Stirling's approximation formula for factorials
log_Stirling()	the logarithm of Stirling's approximation formula for factorials
Binomial_kn_over_n()	an asymptotic expansion of the binomial coefficient
SingularityAnalysis()	an asymptotic expansion obtained by singularity analysis

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4.2.1 Classes and Methods

A collection of constructors for several common asymptotic expansions.

A list of all asymptotic expansions in this database is available via tab completion. Type "asymptotic_expansions." and then hit tab to see which expansions are available.

The asymptotic expansions currently in this class include:

- Harmonic Number ()
- •Stirling()
- •log Stirling()

- •Binomial kn over n()
- •SingularityAnalysis()

static Binomial_kn_over_n (var, k, precision=None, skip_constant_factor=False)

Return the asymptotic expansion of the binomial coefficient kn choose n.

INPUT:

- •var a string for the variable name.
- •k a number or symbolic constant.

•precision – (default: None) an integer. If None , then the default precision of the asymptotic ring is used.

•skip_constant_factor — (default: False) a boolean. If set, then the constant factor $\sqrt{k/(2\pi(k-1))}$ is left out. As a consequence, the coefficient ring of the output changes from Symbolic Constants Subring (if False) to Rational Field (if True).

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: asymptotic_expansions.Binomial_kn_over_n('n', k=2, precision=3)
1/sqrt(pi)*4^n*n^(-1/2)
- 1/8/sqrt(pi)*4^n*n^(-3/2)
+ 1/128/sqrt(pi)*4^n*n^(-5/2)
+ O(4^n*n^(-7/2))
sage: _.parent()
Asymptotic Ring <QQ^n * n^QQ> over Symbolic Constants Subring
```

```
sage: asymptotic_expansions.Binomial_kn_over_n('n', k=3, precision=3)
1/2*sqrt(3)/sqrt(pi)*(27/4)^n*n^(-1/2)
- 7/144*sqrt(3)/sqrt(pi)*(27/4)^n*n^(-3/2)
+ 49/20736*sqrt(3)/sqrt(pi)*(27/4)^n*n^(-5/2)
+ O((27/4)^n*n^(-7/2))
```

```
sage: asymptotic_expansions.Binomial_kn_over_n('n', k=7/5, precision=3)
1/2*sqrt(7)/sqrt(pi)*(7/10*7^(2/5)*2^(3/5))^n*n^(-1/2)
- 13/112*sqrt(7)/sqrt(pi)*(7/10*7^(2/5)*2^(3/5))^n*n^(-3/2)
+ 169/12544*sqrt(7)/sqrt(pi)*(7/10*7^(2/5)*2^(3/5))^n*n^(-5/2)
+ O((7/10*7^(2/5)*2^(3/5))^n*n^(-7/2))
sage: _.parent()
Asymptotic Ring <(Symbolic Constants Subring)^n * n^QQ>
over Symbolic Constants Subring
```

```
+ 49/12800*(3125/256)^n*n^(-5/2)
+ O((3125/256)^n*n^(-7/2))
sage: _.parent()
Asymptotic Ring <QQ^n * n^QQ> over Rational Field
sage: asymptotic_expansions.Binomial_kn_over_n(
....: 'n', k=4, precision=1, skip_constant_factor=True)
(256/27)^n*n^(-1/2) + O((256/27)^n*n^(-3/2))
```

```
sage: S = asymptotic_expansions.Stirling('n', precision=5)
sage: n = S.parent().gen()
sage: all( # long time
...: SR(asymptotic_expansions.Binomial_kn_over_n(
...: 'n', k=k, precision=3)).canonicalize_radical() ==
...: SR(S.subs(n=k*n) / (S.subs(n=(k-1)*n) * S)).canonicalize_radical()
...: for k in [2, 3, 4])
True
```

static HarmonicNumber (var, precision=None, skip_constant_summand=False)

Return the asymptotic expansion of a harmonic number.

INPUT:

- •var a string for the variable name.
- •precision (default: None) an integer. If None, then the default precision of the asymptotic ring is used.
- •skip_constant_summand (default: False) a boolean. If set, then the constant summand euler_gamma is left out. As a consequence, the coefficient ring of the output changes from Symbolic Constants Subring (if False) to Rational Field (if True).

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: asymptotic_expansions.HarmonicNumber('n', precision=5)
log(n) + euler_gamma + 1/2*n^{(-1)} - 1/12*n^{(-2)} + 1/120*n^{(-4)} + O(n^{(-6)})
```

```
sage: ex = asymptotic_expansions.HarmonicNumber('n', precision=5)
sage: n = ex.parent().gen()
sage: ex.compare_with_values(n,
                                                        # rel tol 1e-6
           lambda x: sum(1/k \text{ for } k \text{ in srange}(1, x+1)), [5, 10, 20])
[(5, 0.0038125360?), (10, 0.00392733?), (20, 0.0039579?)]
sage: asymptotic_expansions.HarmonicNumber('n')
log(n) + euler_gamma + 1/2*n^(-1) - 1/12*n^(-2) + 1/120*n^(-4)
-1/252*n^{(-6)} + 1/240*n^{(-8)} - 1/132*n^{(-10)}
+691/32760*n^{(-12)} - 1/12*n^{(-14)} + 3617/8160*n^{(-16)}
-43867/14364*n^{(-18)} + 174611/6600*n^{(-20)} - 77683/276*n^{(-22)}
+ 236364091/65520*n^{(-24)} - 657931/12*n^{(-26)}
+ 3392780147/3480*n^(-28) - 1723168255201/85932*n^(-30)
+ 7709321041217/16320*n^(-32)
-151628697551/12*n^{(-34)} + O(n^{(-36)})
sage: _.parent()
Asymptotic Ring < n^2ZZ * log(n)^2ZZ > over Symbolic Constants Subring
```

```
sage: asymptotic_expansions.HarmonicNumber(
         'n', precision=5, skip_constant_summand=True)
\log(n) + 1/2*n^{(-1)} - 1/12*n^{(-2)} + 1/120*n^{(-4)} + O(n^{(-6)})
sage: _.parent()
Asymptotic Ring \langle n^2Z \rangle \times \log(n)^2Z > \text{over Rational Field}
sage: for p in range(5):
....: print(asymptotic_expansions.HarmonicNumber(
. . . . :
               'n', precision=p))
O(log(n))
log(n) + O(1)
log(n) + euler_gamma + O(n^(-1))
log(n) + euler_gamma + 1/2*n^(-1) + O(n^(-2))
log(n) + euler_gamma + 1/2*n^(-1) - 1/12*n^(-2) + O(n^(-4))
sage: asymptotic_expansions.HarmonicNumber('m', precision=5)
log(m) + euler_gamma + 1/2*m^(-1) - 1/12*m^(-2) + 1/120*m^(-4) + O(m^(-6))
```

static SingularityAnalysis (var, zeta=1, alpha=0, beta=0, delta=0, precision=None, normalized=True)

Return the asymptotic expansion of the coefficients of an power series with specified pole and logarithmic singularity.

More precisely, this extracts the n-th coefficient

$$[z^n] \left(\frac{1}{1 - z/\zeta}\right)^{\alpha} \left(\frac{1}{z/\zeta} \log \frac{1}{1 - z/\zeta}\right)^{\beta} \left(\frac{1}{z/\zeta} \log \left(\frac{1}{z/\zeta} \log \frac{1}{1 - z/\zeta}\right)\right)^{\delta}$$

(if normalized=True, the default) or

$$[z^n] \left(\frac{1}{1-z/\zeta}\right)^{\alpha} \left(\log \frac{1}{1-z/\zeta}\right)^{\beta} \left(\log \left(\frac{1}{z/\zeta} \log \frac{1}{1-z/\zeta}\right)\right)^{\delta}$$

(if normalized=False).

INPUT:

- •var -a string for the variable name.
- •zeta (default: 1) the location of the singularity.
- •alpha (default: 0) the pole order of the singularty.
- •beta (default: 0) the order of the logarithmic singularity.
- \bullet delta (default: 0) the order of the log-log singularity. Not yet implemented for delta != 0.
- •precision (default: None) an integer. If None , then the default precision of the asymptotic ring is used.
- •normalized (default: True) a boolean, see above.

OUTPUT:

An asymptotic expansion.

```
sage: asymptotic_expansions.SingularityAnalysis('n', alpha=1)
1
sage: asymptotic_expansions.SingularityAnalysis('n', alpha=2)
n + 1
sage: asymptotic_expansions.SingularityAnalysis('n', alpha=3)
1/2*n^2 + 3/2*n + 1
```

```
sage: _.parent()
Asymptotic Ring <n^ZZ> over Rational Field
```

```
sage: asymptotic_expansions.SingularityAnalysis('n', alpha=-3/2,
. . . . :
         precision=3)
3/4/sqrt(pi)*n^{-5/2}
+ 45/32/sqrt(pi)*n^{(-7/2)}
+ 1155/512/sqrt(pi)*n^(-9/2)
+ O(n^{(-11/2)})
sage: asymptotic_expansions.SingularityAnalysis('n', alpha=-1/2,
        precision=3)
-1/2/sqrt(pi)*n^{(-3/2)}
-3/16/sqrt(pi)*n^{-5/2}
-25/256/sqrt(pi)*n^{-7/2}
+ O(n^{(-9/2)})
sage: asymptotic_expansions.SingularityAnalysis('n', alpha=1/2,
. . . . :
        precision=4)
1/sqrt(pi)*n^{(-1/2)}
-1/8/sqrt(pi)*n^{-3/2}
+ 1/128/sqrt(pi)*n^{-5/2}
+ 5/1024/sqrt(pi)*n^{(-7/2)}
+ O(n^{(-9/2)})
sage: _.parent()
Asymptotic Ring <n^QQ> over Symbolic Constants Subring
```

```
sage: asymptotic_expansions.SingularityAnalysis('n',
...: alpha=1, beta=1/2, precision=4)
log(n)^(1/2) + 1/2*euler_gamma*log(n)^(-1/2)
+ (-1/8*euler_gamma^2 + 1/48*pi^2)*log(n)^(-3/2)
+ (1/16*euler_gamma^3 - 1/32*euler_gamma*pi^2 + 1/8*zeta(3))*log(n)^(-5/2)
```

```
+ O(log(n)^(-7/2))
```

```
sage: ae = asymptotic_expansions.SingularityAnalysis('n',
...: alpha=0, beta=2, precision=14)
sage: n = ae.parent().gen()
sage: ae.subs(n=n-2)
2*n^(-1)*log(n) + 2*euler_gamma*n^(-1) - n^(-2) - 1/6*n^(-3) + O(n^(-5))
```

```
sage: asymptotic_expansions.SingularityAnalysis(
...: 'n', 1, alpha=-1/2, beta=1, precision=2, normalized=False)
-1/2/sqrt(pi)*n^(-3/2)*log(n)
+ (-1/2*(euler_gamma + 2*log(2) - 2)/sqrt(pi))*n^(-3/2)
+ O(n^(-5/2)*log(n))
sage: asymptotic_expansions.SingularityAnalysis(
...: 'n', 1/2, alpha=0, beta=1, precision=3, normalized=False)
2^n*n^(-1) + O(2^n*n^(-2))
```

ALGORITHM:

See [FS2009] together with the errata list.

REFERENCES:

```
sage: ex = asymptotic_expansions.SingularityAnalysis('n', alpha=-1/2,
        precision=4)
sage: n = ex.parent().gen()
sage: coefficients = ((1-x)^{(1/2)}).series(
        x, 21).truncate().coefficients(x, sparse=False)
sage: ex.compare_with_values(n, # rel tol 1e-6
        lambda k: coefficients[k], [5, 10, 20])
[(5, 0.015778873294?), (10, 0.01498952777?), (20, 0.0146264622?)]
sage: asymptotic_expansions.SingularityAnalysis(
        'n', alpha=3, precision=2)
1/2*n^2 + 3/2*n + 0(1)
sage: asymptotic_expansions.SingularityAnalysis(
....: 'n', alpha=3, precision=3)
1/2*n^2 + 3/2*n + 1
sage: asymptotic_expansions.SingularityAnalysis(
         'n', alpha=3, precision=4)
. . . . :
1/2*n^2 + 3/2*n + 1
```

```
sage: asymptotic_expansions.SingularityAnalysis(
...: 'n', alpha=0)
Traceback (most recent call last):
...
NotImplementedOZero: The error term in the result is O(0)
which means 0 for sufficiently large n.
sage: asymptotic_expansions.SingularityAnalysis(
...: 'n', alpha=-1)
Traceback (most recent call last):
...
NotImplementedOZero: The error term in the result is O(0)
which means 0 for sufficiently large n.
```

```
sage: asymptotic_expansions.SingularityAnalysis(
....: 'm', alpha=-1/2, precision=3)
-1/2/sqrt(pi)*m^(-3/2)
- 3/16/sqrt(pi)*m^(-5/2)
- 25/256/sqrt(pi)*m^(-7/2)
+ O(m^(-9/2))
sage: _.parent()
Asymptotic Ring <m^QQ> over Symbolic Constants Subring
```

Location of the singularity:

```
sage: asymptotic_expansions.SingularityAnalysis(
. . . . :
         'n', alpha=1, zeta=2, precision=3)
(1/2)^n
sage: asymptotic_expansions.SingularityAnalysis(
         'n', alpha=1, zeta=1/2, precision=3)
2^n
sage: asymptotic_expansions.SingularityAnalysis(
....: 'n', alpha=1, zeta=CyclotomicField(3).gen(),
         precision=3)
(-zeta3 - 1)^n
sage: asymptotic_expansions.SingularityAnalysis(
....: 'n', alpha=4, zeta=2, precision=3)
1/6*(1/2)^n*n^3 + (1/2)^n*n^2 + 11/6*(1/2)^n*n + O((1/2)^n)
sage: asymptotic_expansions.SingularityAnalysis(
....: 'n', alpha=-1, zeta=2, precision=3)
Traceback (most recent call last):
NotImplementedOZero: The error term in the result is O(0)
which means 0 for sufficiently large n.
sage: asymptotic_expansions.SingularityAnalysis(
....: 'n', alpha=1/2, zeta=2, precision=3)
1/sqrt(pi)*(1/2)^n*n^(-1/2) - 1/8/sqrt(pi)*(1/2)^n*n^(-3/2)
+ 1/128/sqrt(pi)*(1/2)^n*n^(-5/2) + O((1/2)^n*n^(-7/2))
```

The following tests correspond to Table VI.5 in [FS2009].

```
sage: A.<n> = AsymptoticRing('n^QQ * log(n)^QQ', QQ)
sage: asymptotic_expansions.SingularityAnalysis(
....: 'n', 1, alpha=-1/2, beta=1, precision=2,
        normalized=False) * (- sqrt(pi*n^3))
1/2 * \log(n) + 1/2 * euler_gamma + \log(2) - 1 + O(n^{-1}) * \log(n)
sage: asymptotic_expansions.SingularityAnalysis(
....: 'n', 1, alpha=0, beta=1, precision=3,
        normalized=False)
n^{(-1)} + O(n^{(-2)})
sage: asymptotic_expansions.SingularityAnalysis(
....: 'n', 1, alpha=0, beta=2, precision=14,
        normalized=False) * n
. . . . :
2*log(n) + 2*euler_gamma - n^(-1) - 1/6*n^(-2) + O(n^(-4))
sage: (asymptotic_expansions.SingularityAnalysis(
....: 'n', 1, alpha=1/2, beta=1, precision=4,
        normalized=False) * sqrt(pi*n)).\
. . . . :
        map_coefficients(lambda x: x.expand())
log(n) + euler_gamma + 2*log(2) - 1/8*n^(-1)*log(n) +
(-1/8 \times \text{euler\_gamma} - 1/4 \times \log(2)) \times n^{(-1)} + O(n^{(-2)} \times \log(n))
sage: asymptotic_expansions.SingularityAnalysis(
         'n', 1, alpha=1, beta=1, precision=13,
```

```
normalized=False)
log(n) + euler_gamma + 1/2*n^(-1) - 1/12*n^(-2) + 1/120*n^(-4)
+ O(n^{(-6)})
sage: asymptotic_expansions.SingularityAnalysis(
....: 'n', 1, alpha=1, beta=2, precision=4,
         normalized=False)
\log(n)^2 + 2 \cdot \text{euler\_gamma} \cdot \log(n) + \text{euler\_gamma}^2 - \frac{1}{6 \cdot \text{pi}^2}
+ O(n^{(-1)} * log(n))
sage: asymptotic_expansions.SingularityAnalysis(
        'n', 1, alpha=3/2, beta=1, precision=3,
         normalized=False) * sqrt(pi/n)
2*\log(n) + 2*euler_qamma + 4*log(2) - 4 + 3/4*n^(-1)*log(n)
+ O(n^{(-1)})
sage: asymptotic_expansions.SingularityAnalysis(
....: 'n', 1, alpha=2, beta=1, precision=5,
         normalized=False)
n*log(n) + (euler_gamma - 1)*n + log(n) + euler_gamma + 1/2
+ O(n^{(-1)})
sage: asymptotic_expansions.SingularityAnalysis(
         'n', 1, alpha=2, beta=2, precision=4,
        normalized=False) / n
log(n)^2 + (2*euler_gamma - 2)*log(n)
- 2*euler_gamma + euler_gamma^2 - 1/6*pi^2 + 2
+ n^{(-1)} * log(n)^2 + O(n^{(-1)} * log(n))
```

Be aware that the last result does *not* coincide with [FS2009], they do have a different error term.

Checking parameters:

```
sage: asymptotic_expansions.SingularityAnalysis(
....: 'n', 1, 1, 1/2, precision=0, normalized=False)
Traceback (most recent call last):
...
ValueError: beta and delta must be integers
sage: asymptotic_expansions.SingularityAnalysis(
....: 'n', 1, 1, 1, 1/2, normalized=False)
Traceback (most recent call last):
...
ValueError: beta and delta must be integers
```

```
sage: asymptotic_expansions.SingularityAnalysis(
...: 'n', alpha=0, beta=0, delta=1, precision=3)
Traceback (most recent call last):
...
NotImplementedError: not implemented for delta!=0
```

static Stirling (var, precision=None, skip_constant_factor=False)

Return Stirling's approximation formula for factorials.

INPUT:

- •var a string for the variable name.
- •precision (default: None) an integer ≥ 3 . If None, then the default precision of the asymptotic ring is used.
- •skip_constant_factor (default: False) a boolean. If set, then the constant factor $\sqrt{2\pi}$ is left out. As a consequence, the coefficient ring of the output changes from Symbolic Constants Subring (if False) to Rational Field (if True).

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: asymptotic_expansions.Stirling('n', precision=5)
sqrt(2)*sqrt(pi)*e^(n*log(n))*(e^n)^(-1)*n^(1/2) +
1/12*sqrt(2)*sqrt(pi)*e^(n*log(n))*(e^n)^(-1)*n^(-1/2) +
1/288*sqrt(2)*sqrt(pi)*e^(n*log(n))*(e^n)^(-1)*n^(-3/2) +
0(e^(n*log(n))*(e^n)^(-1)*n^(-5/2))
sage: _.parent()
Asymptotic Ring <(e^(n*log(n)))^QQ * (e^n)^QQ * n^QQ * log(n)^QQ>
over Symbolic Constants Subring
```

See also:

```
log Stirling(), factorial().
```

TESTS:

```
sage: expansion = asymptotic_expansions.Stirling('n', precision=5)
sage: n = expansion.parent().gen()
sage: expansion.compare_with_values(n, lambda x: x.factorial(), [5, 10, 20])
\rightarrow# rel tol 1e-6
[(5, 0.00675841118?), (10, 0.0067589306?), (20, 0.006744925?)]
sage: asymptotic_expansions.Stirling('n', precision=5,
                                        skip_constant_factor=True)
e^{(n+\log(n))} (e^n)^{(-1)} n^{(1/2)} +
1/12 *e^{(n)} (n*log(n)) *(e^n)^{(-1)} *n^{(-1/2)} +
1/288 \times e^{(n+\log(n))} \times (e^n)^{(-1)} \times n^{(-3/2)} +
O(e^{(n*\log(n))*(e^n)^{(-1)*n^{(-5/2)}})
sage: _.parent()
Asymptotic Ring <(e^(n*log(n)))^QQ * (e^n)^QQ * n^QQ * log(n)^QQ>
over Rational Field
sage: asymptotic_expansions.Stirling('m', precision=4)
sqrt(2) * sqrt(pi) * e^(m*log(m)) * (e^m)^(-1) * m^(1/2) +
O(e^{(m*\log(m))*(e^m)^{(-1)*m^{(-1/2)}}}
sage: asymptotic_expansions.Stirling('m', precision=3)
O(e^{(m*\log(m))*(e^m)^(-1)*m^(1/2)})
sage: asymptotic_expansions.Stirling('m', precision=2)
Traceback (most recent call last):
ValueError: precision must be at least 3
```

static log_Stirling (var, precision=None, skip_constant_summand=False)

Return the logarithm of Stirling's approximation formula for factorials.

INPUT:

- •var a string for the variable name.
- •precision (default: None) an integer. If None , then the default precision of the asymptotic ring is used.
- •skip_constant_summand (default: False) a boolean. If set, then the constant summand $\log(2\pi)/2$ is left out. As a consequence, the coefficient ring of the output changes from Symbolic Constants Subring (if False) to Rational Field (if True).

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: asymptotic_expansions.log_Stirling('n', precision=7)
n*log(n) - n + 1/2*log(n) + 1/2*log(2*pi) + 1/12*n^(-1)
- 1/360*n^(-3) + 1/1260*n^(-5) + O(n^(-7))
```

See also:

```
Stirling(), factorial().
```

TESTS:

```
sage: expansion = asymptotic_expansions.log_Stirling('n', precision=7)
sage: n = expansion.parent().gen()
sage: expansion.compare_with_values(n, lambda x: x.factorial().log(), [5, 10, ...
→20]) # rel tol le-6
[(5, 0.000564287?), (10, 0.0005870?), (20, 0.0006?)]
sage: asymptotic_expansions.log_Stirling('n')
n*log(n) - n + 1/2*log(n) + 1/2*log(2*pi) + 1/12*n^(-1)
- 1/360*n^(-3) + 1/1260*n^(-5) - 1/1680*n^(-7) + 1/1188*n^(-9)
- 691/360360*n^(-11) + 1/156*n^(-13) - 3617/122400*n^(-15)
+ 43867/244188*n^(-17) - 174611/125400*n^(-19) + 77683/5796*n^(-21)
- 236364091/1506960*n^(-23) + 657931/300*n^(-25)
- 3392780147/93960*n^(-27) + 1723168255201/2492028*n^(-29)
- 7709321041217/505920*n^(-31) + O(n^(-33))
sage: _.parent()
Asymptotic Ring <n^ZZ * log(n)^ZZ> over Symbolic Constants Subring
```

```
sage: asymptotic_expansions.log_Stirling(
        'n', precision=7, skip_constant_summand=True)
n*log(n) - n + 1/2*log(n) + 1/12*n^{(-1)} - 1/360*n^{(-3)} +
1/1260*n^{(-5)} + O(n^{(-7)})
sage: _.parent()
Asymptotic Ring <n^ZZ * log(n)^ZZ> over Rational Field
sage: asymptotic_expansions.log_Stirling(
         'n', precision=0)
O(n*log(n))
sage: asymptotic_expansions.log_Stirling(
....: 'n', precision=1)
n*log(n) + O(n)
sage: asymptotic_expansions.log_Stirling(
....: 'n', precision=2)
n*log(n) - n + O(log(n))
sage: asymptotic_expansions.log_Stirling(
....: 'n', precision=3)
n*log(n) - n + 1/2*log(n) + O(1)
sage: asymptotic_expansions.log_Stirling(
....: 'n', precision=4)
n*log(n) - n + 1/2*log(n) + 1/2*log(2*pi) + O(n^(-1))
sage: asymptotic_expansions.log_Stirling(
        'n', precision=5)
n*log(n) - n + 1/2*log(n) + 1/2*log(2*pi) + 1/12*n^(-1)
+ O(n^{(-3)})
sage: asymptotic_expansions.log_Stirling(
         'm', precision=7, skip_constant_summand=True)
m*log(m) - m + 1/2*log(m) + 1/12*m^(-1) - 1/360*m^(-3) +
1/1260*m^{(-5)} + O(m^{(-7)})
```

sage.rings.asymptotic_expansion_generators.asymptotic_expansions = <class 'sage.rings.</pre>

A collection of several common asymptotic expansions.

This is an instance of AsymptoticExpansionGenerators whose documentation provides more details.

4.3 (Asymptotic) Growth Groups

This module provides support for (asymptotic) growth groups.

Such groups are equipped with a partial order: the elements can be seen as functions, and the behavior as their argument (or arguments) gets large (tend to ∞) is compared.

Growth groups are used for the calculations done in the *asymptotic ring*. There, take a look at the *informal definition*, where examples of growth groups and elements are given as well.

Warning: As this code is experimental, warnings are thrown when a growth group is created for the first time in a session (see sage.misc.superseded.experimental).

TESTS:

```
sage: from sage.rings.asymptotic.growth_group import \
....:     GenericGrowthGroup, GrowthGroup
sage: GenericGrowthGroup(ZZ)
doctest:...: FutureWarning: This class/method/function is marked as
experimental. It, its functionality or its interface might change
without a formal deprecation.
See http://trac.sagemath.org/17601 for details.
Growth Group Generic(ZZ)
```

4.3.1 Description of Growth Groups

Many growth groups can be described by a string, which can also be used to create them. For example, the string $'x^QQ * log(x)^ZZ * QQ^y * y^QQ'$ represents a growth group with the following properties:

- It is a growth group in the two variables x and y.
- Its elements are of the form

$$x^r \cdot \log(x)^s \cdot a^y \cdot y^q$$

for $r \in \mathbf{Q}$, $s \in \mathbf{Z}$, $a \in \mathbf{Q}$ and $q \in \mathbf{Q}$.

- The order is with respect to $x \to \infty$ and $y \to \infty$ independently of each other.
- To compare such elements, they are split into parts belonging to only one variable. In the example above,

$$x^{r_1} \cdot \log(x)^{s_1} \le x^{r_2} \cdot \log(x)^{s_2}$$

if $(r_1, s_1) \le (r_2, s_2)$ lexicographically. This reflects the fact that elements x^r are larger than elements $\log(x)^s$ as $x \to \infty$. The factors belonging to the variable y are compared analogously.

The results of these comparisons are then put together using the product order, i.e., \leq if each component satisfies \leq .

Each description string consists of ordered factors—yes, this means \star is noncommutative—of strings describing "elementary" growth groups (see the examples below). As stated in the example above, these factors are split by their variable; factors with the same variable are grouped. Reading such factors from left to right determines the order: Comparing elements of two factors (growth groups) L and R, then all elements of L are considered to be larger than each element of R.

4.3.2 Creating a Growth Group

For many purposes the factory <code>GrowthGroup</code> (see <code>GrowthGroupFactory</code>) is the most convenient way to generate a growth group.

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
```

Here are some examples:

```
sage: GrowthGroup('z^ZZ')
Growth Group z^ZZ
sage: M = GrowthGroup('z^QQ'); M
Growth Group z^QQ
```

Each of these two generated groups is a MonomialGrowthGroup, whose elements are powers of a fixed symbol (above 'z'). For the order of the elements it is assumed that $z \to \infty$.

Note: Growth groups where the variable tend to some value distinct from ∞ are not yet implemented.

To create elements of M, a generator can be used:

```
sage: z = M.gen()
sage: z^(3/5)
z^(3/5)
```

Strings can also be parsed:

```
sage: M('z^7')
z^7
```

Similarly, we can construct logarithmic factors by:

```
sage: GrowthGroup('log(z)^QQ')
Growth Group log(z)^QQ
```

which again creates a MonomialGrowthGroup. An ExponentialGrowthGroup is generated in the same way. Our factory gives

```
sage: E = GrowthGroup('QQ^z'); E
Growth Group QQ^z
```

and a typical element looks like this:

```
sage: E.an_element()
(1/2)^z
```

More complex groups are created in a similar fashion. For example

```
sage: C = GrowthGroup('QQ^z * z^QQ * log(z)^QQ'); C
Growth Group QQ^z * z^QQ * log(z)^QQ
```

This contains elements of the form

```
sage: C.an_element()
(1/2)^z*z^(1/2)*log(z)^(1/2)
```

The group C itself is a Cartesian product; to be precise a UnivariateProduct. We can see its factors:

```
sage: C.cartesian_factors()
(Growth Group QQ^z, Growth Group z^QQ, Growth Group log(z)^QQ)
```

Multivariate constructions are also possible:

```
sage: GrowthGroup('x^QQ * y^QQ')
Growth Group x^QQ * y^QQ
```

This gives a MultivariateProduct.

Both these Cartesian products are derived from the class <code>GenericProduct</code> . Moreover all growth groups have the abstract base class <code>GenericGrowthGroup</code> in common.

Some Examples

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G_x = GrowthGroup('x^ZZ'); G_x
Growth Group x^ZZ
sage: G_xy = GrowthGroup('x^ZZ * y^ZZ'); G_xy
Growth Group x^ZZ * y^ZZ
sage: G_xy.an_element()
x*y
sage: x = G_xy('x'); y = G_xy('y')
sage: x^2
x^2
sage: elem = x^21*y^21; elem^2
x^42*y^42
```

A monomial growth group itself is totally ordered, all elements are comparable. However, this does **not** hold for Cartesian products:

```
sage: e1 = x^2*y; e2 = x*y^2
sage: e1 <= e2 or e2 <= e1
False</pre>
```

In terms of uniqueness, we have the following behaviour:

```
sage: GrowthGroup('x^ZZ * y^ZZ') is GrowthGroup('y^ZZ * x^ZZ')
True
```

The above is True since the order of the factors does not play a role here; they use different variables. But when using the same variable, it plays a role:

```
sage: GrowthGroup('x^ZZ * \log(x)^ZZ') is GrowthGroup('\log(x)^ZZ * x^ZZ') False
```

In this case the components are ordered lexicographically, which means that in the second growth group, log(x) is assumed to grow faster than x (which is nonsense, mathematically). See CartesianProduct for more details or see *above* for a more extensive description.

Short notation also allows the construction of more complicated growth groups:

```
sage: G = GrowthGroup('QQ^x * x^ZZ * log(x)^QQ * y^QQ')
sage: G.an_element()
(1/2)^x*x*log(x)^(1/2)*y^(1/2)
```

```
sage: x, y = var('x y')
sage: G(2^x * log(x) * y^(1/2)) * G(x^(-5) * 5^x * y^(1/3))
10^x*x^(-5)*log(x)*y^(5/6)
```

AUTHORS:

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- Daniel Krenn (2015)
- Clemens Heuberger (2016)

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4.3.3 Classes and Methods

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('z^QQ').construction()[0] # indirect doctest
MonomialGrowthGroup[z]
```

See also:

```
\begin{tabular}{ll} Asymptotic & \it Ring, & \it Exponential Growth Group Functor & \it , & \it Monomial Growth Group Functor & \it , & \it sage.rings.asymptotic.asymptotic\_ring. Asymptotic Ring Functor & \it , & \it sage.categories.pushout. Construction Functor & \it . & \it sage.categories.pushout. Construction Functor & \it sage.categories.pushout. Construction
```

merge (other)

Merge this functor with other of possible.

INPUT:

•other -a functor.

OUTPUT:

A functor or None.

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: F = GrowthGroup('QQ^t').construction()[0]
sage: G = GrowthGroup('t^QQ').construction()[0]
sage: F.merge(F)
ExponentialGrowthGroup[t]
```

```
sage: F.merge(G) is None
True
```

class sage.rings.asymptotic.growth_group. ExponentialGrowthElement (parent,

raw_element)

Bases: sage.rings.asymptotic.growth_group.GenericGrowthElement

An implementation of exponential growth elements.

INPUT:

- •parent an Exponential Growth Group.
- •raw_element an element from the base ring of the parent.

This raw_element is the base of the created exponential growth element.

An exponential growth element represents a term of the type base variable. The multiplication corresponds to the multiplication of the bases.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: P = GrowthGroup('ZZ^x')
sage: e1 = P(1); e1
1
sage: e2 = P(raw_element=2); e2
2^x
sage: e1 == e2
False
sage: P.le(e1, e2)
True
sage: P.le(e1, P(1)) and P.le(P(1), e2)
```

base

The base of this exponential growth element.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: P = GrowthGroup('ZZ^x')
sage: P(42^x).base
42
```

 ${\bf class} \; {\tt sage.rings.asymptotic.growth_group.} \; {\bf ExponentialGrowthGroup} \; (\; {\it base}, \; {\it var}, \; {\it cate-total} \; {\bf class} \; {\bf class}$

```
Bases: sage.rings.asymptotic.growth_group.GenericGrowthGroup
```

A growth group dealing with expressions involving a fixed variable/symbol as the exponent.

The elements <code>ExponentialGrowthElement</code> of this group represent exponential functions with bases from a fixed base ring; the group law is the multiplication.

INPUT:

•base - one of SageMath's parents, out of which the elements get their data (raw_element).

As exponential expressions are represented by this group, the elements in base are the bases of these exponentials.

```
•var - an object.
```

The string representation of var acts as an exponent of the elements represented by this group.

•category — (default: None) the category of the newly created growth group. It has to be a subcategory of Join of Category of groups and Category of posets. This is also the default category if None is specified.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import ExponentialGrowthGroup
sage: P = ExponentialGrowthGroup(QQ, 'x'); P
Growth Group QQ^x
```

See also:

GenericGrowthGroup

Element

alias of Exponential Growth Element

Magmas

alias of Magmas

Posets

alias of Posets

Sets

alias of Sets

construction()

Return the construction of this growth group.

OUTPUT:

A pair whose first entry is an exponential construction functor and its second entry the base.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('QQ^x').construction()
(ExponentialGrowthGroup[x], Rational Field)
```

gens ()

Return a tuple of all generators of this exponential growth group.

INPUT:

Nothing.

OUTPUT:

An empty tuple.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: E = GrowthGroup('ZZ^x')
sage: E.gens()
()
```

some elements ()

Return some elements of this exponential growth group.

See TestSuite for a typical use case.

INPUT:

Nothing.

OUTPUT:

An iterator.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: tuple(GrowthGroup('QQ^z').some_elements())
((1/2)^z, (-1/2)^z, 2^z, (-2)^z, 1, (-1)^z,
42^z, (2/3)^z, (-2/3)^z, (3/2)^z, (-3/2)^z, ...)
```

class sage.rings.asymptotic.growth_group. ExponentialGrowthGroupFunctor (var)

Bases: sage.rings.asymptotic.growth_group.AbstractGrowthGroupFunctor

A construction functor for exponential growth groups.

INPUT:

•var – a string or list of strings (or anything else *Variable* accepts).

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup,

→ExponentialGrowthGroupFunctor
sage: GrowthGroup('QQ^z').construction()[0]
ExponentialGrowthGroup[z]
```

See also:

Asymptotic Ring, AbstractGrowthGroupFunctor, MonomialGrowthGroupFunctor, sage.rings.asymptotic.asymptotic_ring.AsymptoticRingFunctor, sage.categories.pushout.ConstructionFunctor.

TESTS:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup,_

→ExponentialGrowthGroupFunctor
sage: cm = sage.structure.element.get_coercion_model()
sage: A = GrowthGroup('QQ^x')
sage: B = ExponentialGrowthGroupFunctor('x')(ZZ['t'])
sage: cm.common_parent(A, B)
Growth Group QQ[t]^x
```

class sage.rings.asymptotic.growth_group. GenericGrowthElement (parent,

Bases: sage.structure.element.MultiplicativeGroupElement

A basic implementation of a generic growth element.

Growth elements form a group by multiplication, and (some of) the elements can be compared to each other, i.e., all elements form a poset.

INPUT:

```
•parent - a GenericGrowthGroup.
```

•raw_element - an element from the base of the parent.

factors ()

Return the atomic factors of this growth element. An atomic factor cannot be split further.

INPUT:

Nothing.

OUTPUT:

A tuple of growth elements.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ')
sage: G.an_element().factors()
(x,)
```

is_lt_one()

Return whether this element is less than 1.

INPUT:

Nothing.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ'); x = G(x)
sage: (x^42).is_lt_one() # indirect doctest
False
sage: (x^(-42)).is_lt_one() # indirect doctest
True
```

log (base=None)

Return the logarithm of this element.

INPUT:

•base - the base of the logarithm. If None (default value) is used, the natural logarithm is taken.

OUTPUT:

A growth element.

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ * log(x)^ZZ')
sage: x, = G.gens_monomial()
sage: log(x)  # indirect doctest
log(x)
sage: log(x^5)  # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: When calculating log(x^5) a factor 5 != 1 appeared,
which is not contained in Growth Group x^ZZ * log(x)^ZZ.
```

```
sage: G = GrowthGroup('QQ^x * x^ZZ')
sage: x, = G.gens_monomial()
sage: el = x.rpow(2); el
2^x
sage: log(el) # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: When calculating log(2^x) a factor log(2) != 1
appeared, which is not contained in Growth Group QQ^x * x^ZZ.
sage: log(el, base=2) # indirect doctest
x
```

```
sage: from sage.rings.asymptotic.growth_group import GenericGrowthGroup
sage: x = GenericGrowthGroup(ZZ).an_element()
sage: log(x) # indirect doctest
Traceback (most recent call last):
...
NotImplementedError: Cannot determine logarithmized factorization of
GenericGrowthElement(1) in abstract base class.
```

```
sage: x = GrowthGroup('x^ZZ').an_element()
sage: log(x) # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: Cannot build log(x) since log(x) is not in
Growth Group x^ZZ.
```

```
sage: G = GrowthGroup("(e^x)^QQ * x^ZZ")
sage: x, = G.gens_monomial()
sage: log(exp(x)) # indirect doctest
x
sage: G.one().log() # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: log(1) is zero, which is not contained in
Growth Group (e^x)^QQ * x^ZZ.
```

```
sage: G = GrowthGroup("(e^x)^ZZ * x^ZZ")
sage: x, = G.gens_monomial()
sage: log(exp(x)) # indirect doctest
x
sage: G.one().log() # indirect doctest
Traceback (most recent call last):
```

```
ArithmeticError: log(1) is zero, which is not contained in Growth Group (e^x)^ZZ * x^ZZ.
```

```
sage: G = GrowthGroup('QQ^x * x^ZZ * log(x)^ZZ * y^ZZ * log(y)^ZZ')
sage: x, y = G.gens_monomial()
sage: (x * y).log() # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: Calculating log(x*y) results in a sum,
which is not contained in
Growth Group QQ^x * x^ZZ * log(x)^ZZ * y^ZZ * log(y)^ZZ.
```

log_factor (base=None)

Return the logarithm of the factorization of this element.

INPUT:

•base - the base of the logarithm. If None (default value) is used, the natural logarithm is taken.

OUTPUT:

A tuple of pairs, where the first entry is a growth element and the second a multiplicative coefficient.

ALGORITHM:

This function factors the given element and calculates the logarithm of each of these factors.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('QQ^x * x^ZZ * log(x)^ZZ * y^ZZ * log(y)^ZZ')
sage: x, y = G.gens_monomial()
sage: (x * y).log_factor() # indirect doctest
((log(x), 1), (log(y), 1))
sage: (x^123).log_factor() # indirect doctest
((log(x), 123),)
sage: (G('2^x') * x^2).log_factor(base=2) # indirect doctest
((x, 1), (log(x), 2/log(2)))
```

```
sage: G(1).log_factor()
()
```

```
sage: log(x).log_factor() # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: Cannot build log(log(x)) since log(log(x)) is
not in Growth Group QQ^x * x^ZZ * log(x)^ZZ * y^ZZ * log(y)^ZZ.
```

See also:

```
factors(), log().
```

```
sage: G = GrowthGroup("(e^x)^ZZ * x^ZZ * log(x)^ZZ")
sage: x, = G.gens_monomial()
sage: (exp(x) * x).log_factor() # indirect doctest
((x, 1), (log(x), 1))
```

rpow (base)

Calculate the power of base to this element.

INPUT:

•base - an element.

OUTPUT:

A growth element.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('QQ^x * x^ZZ')
sage: x = G('x')
sage: x.rpow(2) # indirect doctest
2^x
sage: x.rpow(1/2) # indirect doctest
(1/2)^x
```

```
sage: x.rpow(0) # indirect doctest
Traceback (most recent call last):
...
ValueError: 0 is not an allowed base for calculating the power to x.
sage: (x^2).rpow(2) # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: Cannot construct 2^(x^2) in Growth Group QQ^x * x^ZZ
> *previous* TypeError: unsupported operand parent(s) for '*':
'Growth Group QQ^x * x^ZZ' and 'Growth Group ZZ^(x^2)'
```

```
sage: G = GrowthGroup('QQ^(x*log(x)) * x^ZZ * log(x)^ZZ')
sage: x = G('x')
sage: (x * log(x)).rpow(2) # indirect doctest
2^(x*log(x))
```

```
sage: n = GrowthGroup('QQ^n * n^QQ')('n')
sage: n.rpow(2)
2^n
sage: _.parent()
Growth Group QQ^n * n^QQ
```

variable names ()

Return the names of the variables of this growth element.

OUTPUT:

A tuple of strings.

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('m^QQ')
sage: G('m^2').variable_names()
('m',)
sage: G('m^0').variable_names()
()
```

```
sage: G = GrowthGroup('QQ^m')
sage: G('2^m').variable_names()
('m',)
sage: G('1^m').variable_names()
()
```

TESTS:

```
sage: from sage.rings.asymptotic.growth_group import GenericGrowthGroup
sage: G = GenericGrowthGroup(QQ)
sage: G(raw_element=2).variable_names()
Traceback (most recent call last):
...
AttributeError: 'GenericGrowthGroup_with_category.element_class' object
has no attribute 'is_one'
```

A basic implementation for growth groups.

INPUT:

- •base one of SageMath's parents, out of which the elements get their data (raw_element).
- •category (default: None) the category of the newly created growth group. It has to be a subcategory of Join of Category of groups and Category of posets. This is also the default category if None is specified.
- •ignore_variables (default: None) a tuple (or other iterable) of strings. The specified names are not considered as variables.

Note: This class should be derived for concrete implementations.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GenericGrowthGroup
sage: G = GenericGrowthGroup(ZZ); G
Growth Group Generic(ZZ)
```

See also:

MonomialGrowthGroup, ExponentialGrowthGroup

AdditiveMagmas

alias of AdditiveMagmas

Element

alias of GenericGrowthElement

Magmas

alias of Magmas

Posets

alias of Posets

Sets

alias of Sets

gen (n=0)

Return the n-th generator (as a group) of this growth group.

INPUT:

```
•n - default: 0.
```

OUTPUT:

A Monomial Growth Element.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: P = GrowthGroup('x^ZZ')
sage: P.gen()
x
```

```
sage: P = GrowthGroup('QQ^x')
sage: P.gen()
Traceback (most recent call last):
...
IndexError: tuple index out of range
```

gens ()

Return a tuple of all generators of this growth group.

INPUT:

Nothing.

OUTPUT:

A tuple whose entries are growth elements.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: P = GrowthGroup('x^ZZ')
sage: P.gens()
(x,)
sage: GrowthGroup('log(x)^ZZ').gens()
(log(x),)
```

gens monomial ()

Return a tuple containing monomial generators of this growth group.

INPUT:

Nothing.

OUTPUT:

An empty tuple.

Note: A generator is called monomial generator if the variable of the underlying growth group is a valid identifier. For example, x^2Z has x as a monomial generator, while $\log(x)^ZZ$ or $\operatorname{icecream}(x)^ZZ$ do not have monomial generators.

```
sage: from sage.rings.asymptotic.growth_group import GenericGrowthGroup
sage: GenericGrowthGroup(ZZ).gens_monomial()
()
```

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('x^QQ').gens_monomial()
(x,)
sage: GrowthGroup('QQ^x').gens_monomial()
()
```

le (*left*, *right*)

Return whether the growth of left is at most (less than or equal to) the growth of right.

INPUT:

- •left an element.
- •right an element.

OUTPUT:

A boolean.

Note: This function uses the coercion model to find a common parent for the two operands.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ')
sage: x = G.gen()
sage: G.le(x, x^2)
True
sage: G.le(x^2, x)
False
sage: G.le(x^0, 1)
True
```

ngens ()

Return the number of generators (as a group) of this growth group.

INPUT:

Nothing.

OUTPUT:

A Python integer.

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: P = GrowthGroup('x^ZZ')
sage: P.ngens()
1
sage: GrowthGroup('log(x)^ZZ').ngens()
1
```

```
sage: P = GrowthGroup('QQ^x')
sage: P.ngens()
0
```

some_elements ()

Return some elements of this growth group.

See TestSuite for a typical use case.

INPUT:

Nothing.

OUTPUT:

An iterator.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: tuple(GrowthGroup('z^ZZ').some_elements())
(1, z, z^(-1), z^2, z^(-2), z^3, z^(-3),
    z^4, z^(-4), z^5, z^(-5), ...)
sage: tuple(GrowthGroup('z^QQ').some_elements())
(z^(1/2), z^(-1/2), z^2, z^(-2),
    1, z, z^(-1), z^42,
    z^(2/3), z^(-2/3), z^(3/2), z^(-3/2),
    z^(4/5), z^(-4/5), z^(5/4), z^(-5/4), ...)
```

variable names ()

Return the names of the variables of this growth group.

OUTPUT:

A tuple of strings.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GenericGrowthGroup
sage: GenericGrowthGroup(ZZ).variable_names()
()
```

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('x^ZZ').variable_names()
('x',)
sage: GrowthGroup('log(x)^ZZ').variable_names()
('x',)
```

```
sage: GrowthGroup('QQ^x').variable_names()
('x',)
sage: GrowthGroup('QQ^(x*log(x))').variable_names()
('x',)
```

sage.rings.asymptotic.growth_group. **GrowthGroup = <class 'sage.rings.asymptotic.growth_group.GrowthGroup**A factory for growth groups. This is an instance of *GrowthGroupFactory* whose documentation provides more details.

```
class sage.rings.asymptotic.growth_group. GrowthGroupFactory
```

Bases: sage.structure.factory.UniqueFactory

A factory creating asymptotic growth groups.

INPUT:

- •specification a string.
- •keyword arguments are passed on to the growth group constructor. If the keyword ignore_variables is not specified, then ignore_variables=('e',) (to ignore e as a variable name) is used.

OUTPUT:

An asymptotic growth group.

Note: An instance of this factory is available as GrowthGroup.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('x^ZZ')
Growth Group x^ZZ
sage: GrowthGroup('log(x)^QQ')
Growth Group log(x)^QQ
```

This factory can also be used to construct Cartesian products of growth groups:

```
sage: GrowthGroup('x^ZZ * y^ZZ')
Growth Group x^ZZ * y^ZZ
sage: GrowthGroup('x^ZZ * log(x)^ZZ')
Growth Group x^ZZ * log(x)^ZZ
sage: GrowthGroup('x^ZZ * log(x)^ZZ * y^QQ')
Growth Group x^ZZ * log(x)^ZZ * y^QQ
sage: GrowthGroup('QQ^x * x^ZZ * y^QQ * QQ^z')
Growth Group QQ^x * x^ZZ * y^QQ * QQ^z
sage: GrowthGroup('exp(x)^ZZ * x^ZZ')
Growth Group exp(x)^ZZ * x^ZZ')
Growth Group exp(x)^ZZ * x^ZZ'
sage: GrowthGroup('(e^x)^ZZ * x^ZZ')
Growth Group (e^x)^ZZ * x^ZZ')
```

```
sage: G = GrowthGroup('(e^(n*log(n)))^ZZ')
sage: G, G._var_
(Growth Group (e^{(n*\log(n))})^{ZZ}, e^{(n*\log(n))}
sage: G = GrowthGroup('(e^n)^ZZ')
sage: G, G._var_
(Growth Group (e^n)^ZZ, e^n)
sage: G = GrowthGroup('(e^(n*log(n)))^ZZ * (e^n)^ZZ * n^ZZ * log(n)^ZZ')
sage: G, tuple(F._var_ for F in G.cartesian_factors())
(Growth Group (e^{(n \cdot \log(n))})^2Z * (e^n)^2Z * n^2Z * \log(n)^2Z,
(e^{(n*\log(n))}, e^{n}, n, \log(n))
sage: TestSuite(GrowthGroup('x^ZZ')).run(verbose=True) # long time
running ._test_an_element() . . . pass
running ._test_associativity() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_elements() . . .
 Running the test suite of self.an_element()
 running ._test_category() . . . pass
 running ._test_eq() . . . pass
```

```
running ._test_new() . . . pass
 running . test not implemented methods() . . . pass
 running ._test_pickling() . . . pass
 pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_inverse() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_one() . . . pass
running ._test_pickling() . . . pass
running ._test_prod() . . . pass
running ._test_some_elements() . . . pass
```

```
sage: TestSuite(GrowthGroup('QQ^y')).run(verbose=True) # long time
running ._test_an_element() . . . pass
running ._test_associativity() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_elements() . . .
 Running the test suite of self.an_element()
 running ._test_category() . . . pass
 running ._test_eq() . . . pass
 running ._test_new() . . . pass
 running ._test_not_implemented_methods() . . . pass
 running ._test_pickling() . . . pass
 pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_inverse() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_one() . . . pass
running ._test_pickling() . . . pass
running ._test_prod() . . . pass
running ._test_some_elements() . . . pass
```

```
sage: TestSuite(GrowthGroup('x^QQ * log(x)^ZZ')).run(verbose=True) # long time
running ._test_an_element() . . . pass
running ._test_associativity() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_elements() . . .

Running the test suite of self.an_element()
running ._test_category() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
pass
running ._test_elements_eq_reflexive() . . . pass
```

```
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_inverse() . . . pass
running ._test_new() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_one() . . . pass
running ._test_pickling() . . . pass
running ._test_prod() . . . pass
running ._test_some_elements() . . . pass
```

create_key_and_extra_args (specification, **kwds)

Given the arguments and keyword, create a key that uniquely determines this object.

TESTS:

create_object (version, factors, **kwds)

Create an object from the given arguments.

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('as^df') # indirect doctest
Traceback (most recent call last):
ValueError: 'as^df' is not a valid substring of as^df
describing a growth group.
> *previous* ValueError: Cannot create a parent out of 'as'.
>> *previous* SyntaxError: unexpected EOF while parsing (<string>, line 1)
> *and* ValueError: Cannot create a parent out of 'df'.
>> *previous* NameError: name 'df' is not defined
sage: GrowthGroup('x^y^z')
Traceback (most recent call last):
ValueError: 'x^y^z' is an ambigous substring of
a growth group description of 'x^y^z'.
Use parentheses to make it unique.
sage: GrowthGroup('(x^y)^z')
Traceback (most recent call last):
ValueError: '(x^y)^z' is not a valid substring of (x^y)^z
describing a growth group.
> *previous* ValueError: Cannot create a parent out of 'x^y'.
>> *previous* NameError: name 'x' is not defined
> *and* ValueError: Cannot create a parent out of 'z'.
>> *previous* NameError: name 'z' is not defined
sage: GrowthGroup('x^(y^z)')
Traceback (most recent call last):
ValueError: 'x^(y^z)' is not a valid substring of x^(y^z)
```

```
describing a growth group.
> *previous* ValueError: Cannot create a parent out of 'x'.
>> *previous* NameError: name 'x' is not defined
> *and* ValueError: Cannot create a parent out of 'y^z'.
>> *previous* NameError: name 'y' is not defined
```

class sage.rings.asymptotic.growth_group. MonomialGrowthElement (parent,

raw element)

Bases: sage.rings.asymptotic.growth group.GenericGrowthElement

An implementation of monomial growth elements.

INPUT:

```
•parent - a Monomial Growth Group.
```

•raw_element - an element from the base ring of the parent.

This raw_element is the exponent of the created monomial growth element.

A monomial growth element represents a term of the type variable exponent. The multiplication corresponds to the addition of the exponents.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import MonomialGrowthGroup
sage: P = MonomialGrowthGroup(ZZ, 'x')
sage: e1 = P(1); e1
1
sage: e2 = P(raw_element=2); e2
x^2
sage: e1 == e2
False
sage: P.le(e1, e2)
True
sage: P.le(e1, P.gen()) and P.le(P.gen(), e2)
True
```

exponent

The exponent of this growth element.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: P = GrowthGroup('x^ZZ')
sage: P(x^42).exponent
42
```

A growth group dealing with powers of a fixed object/symbol.

The elements MonomialGrowthElement of this group represent powers of a fixed base; the group law is the multiplication, which corresponds to the addition of the exponents of the monomials.

INPUT:

•base - one of SageMath's parents, out of which the elements get their data (raw element).

As monomials are represented by this group, the elements in base are the exponents of these monomials.

```
•var - an object.
```

The string representation of var acts as a base of the monomials represented by this group.

•category — (default: None) the category of the newly created growth group. It has to be a subcategory of Join of Category of groups and Category of posets. This is also the default category if None is specified.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import MonomialGrowthGroup
sage: P = MonomialGrowthGroup(ZZ, 'x'); P
Growth Group x^ZZ
sage: MonomialGrowthGroup(ZZ, log(SR.var('y')))
Growth Group log(y)^ZZ
```

See also:

GenericGrowthGroup

TESTS:

```
sage: L1 = MonomialGrowthGroup(QQ, log(x))
sage: L2 = MonomialGrowthGroup(QQ, 'log(x)')
sage: L1 is L2
True
```

AdditiveMagmas

alias of AdditiveMagmas

Element

alias of Monomial Growth Element

Magmas

alias of Magmas

Posets

alias of Posets

Sets

alias of Sets

construction()

Return the construction of this growth group.

OUTPUT:

A pair whose first entry is a monomial construction functor and its second entry the base.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('x^ZZ').construction()
(MonomialGrowthGroup[x], Integer Ring)
```

gens_logarithmic()

Return a tuple containing logarithmic generators of this growth group.

INPUT:

Nothing.

OUTPUT:

A tuple containing elements of this growth group.

Note: A generator is called logarithmic generator if the variable of the underlying growth group is the logarithm of a valid identifier. For example, x^2Z has no logarithmic generator, while $\log(x)^ZZ$ has $\log(x)$ as logarithmic generator.

TESTS:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('x^ZZ').gens_logarithmic()
()
sage: GrowthGroup('log(x)^QQ').gens_logarithmic()
(log(x),)
```

gens monomial ()

Return a tuple containing monomial generators of this growth group.

INPUT:

Nothing.

OUTPUT:

A tuple containing elements of this growth group.

Note: A generator is called monomial generator if the variable of the underlying growth group is a valid identifier. For example, x^2Z has x as a monomial generator, while $\log(x)^2Z$ or $\operatorname{icecream}(x)^2Z$ do not have monomial generators.

TESTS:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('x^ZZ').gens_monomial()
(x,)
sage: GrowthGroup('log(x)^QQ').gens_monomial()
()
```

class sage.rings.asymptotic.growth_group. MonomialGrowthGroupFunctor (var)

Bases: sage.rings.asymptotic.growth_group.AbstractGrowthGroupFunctor

A construction functor for monomial growth groups.

INPUT:

•var – a string or list of strings (or anything else *Variable* accepts).

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup,

→MonomialGrowthGroupFunctor
sage: GrowthGroup('z^QQ').construction()[0]
MonomialGrowthGroup[z]
```

See also:

Asymptotic Ring, AbstractGrowthGroupFunctor, ExponentialGrowthGroupFunctor, sage.rings.asymptotic.asymptotic_ring.AsymptoticRingFunctor, sage.categories.pushout.ConstructionFunctor.

TESTS:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup,

→MonomialGrowthGroupFunctor
sage: cm = sage.structure.element.get_coercion_model()
sage: A = GrowthGroup('x^QQ')
sage: B = MonomialGrowthGroupFunctor('x')(ZZ['t'])
sage: cm.common_parent(A, B)
Growth Group x^QQ[t]
```

Bases: sage.structure.unique_representation.CachedRepresentation sage.structure.sage_object.SageObject

A class managing the variable of a growth group.

INPUT:

- •var an object whose representation string is used as the variable. It has to be a valid Python identifier. var can also be a tuple (or other iterable) of such objects.
- •repr (default: None) if specified, then this string will be displayed instead of var . Use this to get e.g. $\log(x)^2$: var is then used to specify the variable x.
- •latex_name (default: None) if specified, then this string will be used as LaTeX-representation of var.
- •ignore (default: None) a tuple (or other iterable) of strings which are not variables.

```
sage: from sage.rings.asymptotic.growth_group import Variable
sage: v = Variable('x'); repr(v), v.variable_names()
('x', ('x',))
sage: v = Variable('x1'); repr(v), v.variable_names()
('x1', ('x1',))
sage: v = Variable('x_42'); repr(v), v.variable_names()
('x_42', ('x_42',))
sage: v = Variable(' x'); repr(v), v.variable_names()
('x', ('x',))
sage: v = Variable('x '); repr(v), v.variable_names()
('x', ('x',))
sage: v = Variable(''); repr(v), v.variable_names()
('x', ('x',))
```

```
sage: v = Variable(('x', 'y')); repr(v), v.variable_names()
('x, y', ('x', 'y'))
sage: v = Variable(('x', 'log(y)')); repr(v), v.variable_names()
('x, log(y)', ('x', 'y'))
sage: v = Variable(('x', 'log(x)')); repr(v), v.variable_names()
Traceback (most recent call last):
...
ValueError: Variable names ('x', 'x') are not pairwise distinct.
```

```
sage: v = Variable('log(x)'); repr(v), v.variable_names()
('log(x)', ('x',))
sage: v = Variable('log(log(x))'); repr(v), v.variable_names()
('log(log(x))', ('x',))
```

```
sage: v = Variable('x', repr='log(x)'); repr(v), v.variable_names()
('log(x)', ('x',))
```

```
sage: v = Variable('e^x', ignore=('e',)); repr(v), v.variable_names()
('e^x', ('x',))
```

```
sage: v = Variable('(e^n)', ignore=('e',)); repr(v), v.variable_names()
('e^n', ('n',))
sage: v = Variable('(e^(n*log(n)))', ignore=('e',)); repr(v), v.variable_names()
('e^(n*log(n))', ('n',))
```

static extract variable names (s)

Determine the name of the variable for the given string.

INPUT:

•s -a string.

OUTPUT:

A tuple of strings.

```
sage: from sage.rings.asymptotic.growth_group import Variable
sage: Variable.extract_variable_names('')
()
sage: Variable.extract_variable_names('x')
('x',)
sage: Variable.extract_variable_names('exp(x)')
('x',)
sage: Variable.extract_variable_names('sin(cos(ln(x)))')
('x',)
```

```
sage: Variable.extract_variable_names('log(77w)')
('w',)
sage: Variable.extract_variable_names('log(x')
Traceback (most recent call last):
...
TypeError: Bad function call: log(x !!!
sage: Variable.extract_variable_names('x)')
Traceback (most recent call last):
...
TypeError: Malformed expression: x) !!!
sage: Variable.extract_variable_names('log)x(')
Traceback (most recent call last):
...
TypeError: Malformed expression: log) !!! x(
sage: Variable.extract_variable_names('log(x)+y')
('x', 'y')
sage: Variable.extract_variable_names('icecream(summer)')
('summer',)
```

```
sage: Variable.extract_variable_names('a + b')
('a', 'b')
sage: Variable.extract_variable_names('a+b')
('a', 'b')
sage: Variable.extract_variable_names('a +b')
```

```
('a', 'b')
sage: Variable.extract_variable_names('+a')
('a',)
sage: Variable.extract_variable_names('a+')
Traceback (most recent call last):
...
TypeError: Malformed expression: a+ !!!
sage: Variable.extract_variable_names('b!')
('b',)
sage: Variable.extract_variable_names('-a')
('a',)
sage: Variable.extract_variable_names('a*b')
('a', 'b')
sage: Variable.extract_variable_names('2^q')
('q',)
sage: Variable.extract_variable_names('77')
()
```

```
sage: Variable.extract_variable_names('a + (b + c) + d')
('a', 'b', 'c', 'd')
```

is_monomial()

Return whether this is a monomial variable.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import Variable
sage: Variable('x').is_monomial()
True
sage: Variable('log(x)').is_monomial()
False
```

variable_names ()

Return the names of the variables.

OUTPUT:

A tuple of strings.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import Variable
sage: Variable('x').variable_names()
('x',)
sage: Variable('log(x)').variable_names()
('x',)
```

4.4 Cartesian Products of Growth Groups

See (Asymptotic) Growth Groups for a description.

AUTHORS:

• Benjamin Hackl (2015)

- Daniel Krenn (2015)
- Clemens Heuberger (2016)

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Warning: As this code is experimental, warnings are thrown when a growth group is created for the first time in a session (see sage.misc.superseded.experimental).

TESTS:

```
sage: from sage.rings.asymptotic.growth_group import GenericGrowthGroup, GrowthGroup
sage: GenericGrowthGroup(ZZ)
doctest:...: FutureWarning: This class/method/function is marked as
experimental. It, its functionality or its interface might change
without a formal deprecation.
See http://trac.sagemath.org/17601 for details.
Growth Group Generic(ZZ)
```

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: A = GrowthGroup('QQ^x * x^ZZ'); A
Growth Group QQ^x * x^ZZ
sage: A.construction()
(The cartesian_product functorial construction,
(Growth Group QQ^x, Growth Group x^ZZ))
sage: A.construction()[1][0].construction()
(ExponentialGrowthGroup[x], Rational Field)
sage: A.construction()[1][1].construction()
(MonomialGrowthGroup[x], Integer Ring)
sage: B = GrowthGroup('x^2Z * y^2Z'); B
Growth Group x^{ZZ} * y^{ZZ}
sage: B.construction()
(The cartesian_product functorial construction,
(Growth Group x^ZZ, Growth Group y^ZZ))
sage: C = GrowthGroup('x^ZZ * log(x)^{ZZ} * y^{ZZ'}); C
Growth Group x^2Z * log(x)^2Z * y^2Z
sage: C.construction()
(The cartesian_product functorial construction,
(Growth Group x^2Z * log(x)^2Z, Growth Group y^2Z))
sage: C.construction()[1][0].construction()
(The cartesian_product functorial construction,
(Growth Group x^2Z, Growth Group log(x)^2Z))
sage: C.construction()[1][1].construction()
(MonomialGrowthGroup[y], Integer Ring)
```

```
sage: cm = sage.structure.element.get_coercion_model()
sage: D = GrowthGroup('QQ^x * x^QQ')
sage: cm.common_parent(A, D)
Growth Group QQ^x * x^QQ
sage: E = GrowthGroup('ZZ^x * x^QQ')
sage: cm.record_exceptions() # not tested, see #19411
```

```
sage: cm.common_parent(A, E)
Growth Group QQ^x * x^QQ
sage: for t in cm.exception_stack(): # not tested, see #19411
....: print(t)
```

```
sage: A.an_element()
(1/2)^x*x
sage: tuple(E.an_element())
(1, x^(1/2))
```

4.4.1 Classes and Methods

```
{\bf class}\ {\bf sage.rings.asymptotic.growth\_group\_cartesian.}\ {\bf CartesianProductFactory}\ {\bf Bases:}\ {\bf sage.structure.factory.UniqueFactory}
```

Create various types of Cartesian products depending on its input.

INPUT:

- •growth_groups a tuple (or other iterable) of growth groups.
- •order (default: None) if specified, then this order is taken for comparing two Cartesian product elements. If order is None this is determined automatically.

Note: The Cartesian product of growth groups is again a growth group. In particular, the resulting structure is partially ordered.

The order on the product is determined as follows:

- •Cartesian factors with respect to the same variable are ordered lexicographically. This causes GrowthGroup(' $x^2Z * log(x)^2Z'$) and GrowthGroup(' $log(x)^2Z * x^2Z'$) to produce two different growth groups.
- •Factors over different variables are equipped with the product order (i.e. the comparison is component-wise).

Also, note that the sets of variables of the Cartesian factors have to be either equal or disjoint.

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: A = GrowthGroup('x^ZZ'); A
Growth Group x^ZZ
sage: B = GrowthGroup('log(x)^2Z'); B
Growth Group log(x)^ZZ
sage: C = cartesian_product([A, B]); C # indirect doctest
Growth Group x^2Z * log(x)^2Z
sage: C._le_ == C.le_lex
True
sage: D = GrowthGroup('y^ZZ'); D
Growth Group y^ZZ
sage: E = cartesian_product([A, D]); E # indirect doctest
Growth Group x^{ZZ} * y^{ZZ}
sage: E._le_ == E.le_product
sage: F = cartesian_product([C, D]); F # indirect doctest
Growth Group x^{ZZ} * log(x)^{ZZ} * y^{ZZ}
```

```
sage: F._le_ == F.le_product
True
sage: cartesian_product([A, E]); G # indirect doctest
Traceback (most recent call last):
...
ValueError: The growth groups (Growth Group x^ZZ, Growth Group x^ZZ * y^ZZ)
need to have pairwise disjoint or equal variables.
sage: cartesian_product([A, B, D]) # indirect doctest
Growth Group x^ZZ * log(x)^ZZ * y^ZZ
```

TESTS:

create_key_and_extra_args (growth_groups, category, **kwds)

Given the arguments and keywords, create a key that uniquely determines this object.

TESTS:

create_object (version, args, **kwds)

Create an object from the given arguments.

TESTS:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: cartesian_product([GrowthGroup('x^ZZ')]) # indirect doctest
Growth Group x^ZZ
```

Bases: sage.combinat.posets.cartesian_product.CartesianProductPoset sage.rings.asymptotic.growth_group.GenericGrowthGroup

A Cartesian product of growth groups.

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: P = GrowthGroup('x^QQ')
sage: L = GrowthGroup('log(x)^ZZ')
sage: C = cartesian_product([P, L], order='lex'); C # indirect doctest
Growth Group x^QQ * log(x)^ZZ
```

```
sage: C.an_element()
x^(1/2)*log(x)
```

```
sage: Px = GrowthGroup('x^QQ')
sage: Lx = GrowthGroup('log(x)^ZZ')
sage: Cx = cartesian_product([Px, Lx], order='lex') # indirect doctest
sage: Py = GrowthGroup('y^QQ')
sage: C = cartesian_product([Cx, Py], order='product'); C # indirect doctest
Growth Group x^QQ * log(x)^ZZ * y^QQ
sage: C.an_element()
x^(1/2)*log(x)*y^(1/2)
```

See also:

CartesianProduct, CartesianProductPoset.

class Element

Bases: sage.combinat.posets.cartesian_product.CartesianProductPoset.Element

exp()

The exponential of this element.

INPUT:

Nothing.

OUTPUT:

A growth element.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ * log(x)^ZZ * log(log(x))^ZZ')
sage: x = G('x')
sage: exp(log(x))
x
sage: exp(log(log(x)))
log(x)
```

```
sage: exp(x)
Traceback (most recent call last):
...
ArithmeticError: Cannot construct e^x in
Growth Group x^ZZ * log(x)^ZZ * log(log(x))^ZZ
> *previous* TypeError: unsupported operand parent(s) for '*':
'Growth Group x^ZZ * log(x)^ZZ * log(log(x))^ZZ' and
'Growth Group (e^x)^ZZ'
```

```
sage: E = GrowthGroup("(e^y)^QQ * y^QQ * log(y)^QQ")
sage: y = E('y')
sage: log(exp(y))
y
sage: exp(log(y))
```

factors ()

Return the atomic factors of this growth element. An atomic factor cannot be split further and is not the identity (1).

INPUT:

Nothing.

OUTPUT:

A tuple of growth elements.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ * log(x)^ZZ * y^ZZ')
sage: x, y = G.gens_monomial()
sage: x.factors()
(x,)
sage: f = (x * y).factors(); f
(x, y)
sage: tuple(factor.parent() for factor in f)
(Growth Group x^ZZ, Growth Group y^ZZ)
sage: f = (x * log(x)).factors(); f
(x, log(x))
sage: tuple(factor.parent() for factor in f)
(Growth Group x^ZZ, Growth Group log(x)^ZZ)
```

```
sage: G = GrowthGroup('x^ZZ * log(x)^ZZ * log(log(x))^ZZ * y^QQ')
sage: x, y = G.gens_monomial()
sage: f = (x * log(x) * y).factors(); f
(x, log(x), y)
sage: tuple(factor.parent() for factor in f)
(Growth Group x^ZZ, Growth Group log(x)^ZZ, Growth Group y^QQ)
```

```
sage: G.one().factors()
()
```

is_lt_one()

Return whether this element is less than 1.

INPUT:

Nothing.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ'); x = G(x)
sage: (x^42).is_lt_one() # indirect doctest
False
sage: (x^(-42)).is_lt_one() # indirect doctest
True
```

log (base=None)

Return the logarithm of this element.

INPUT:

•base – the base of the logarithm. If None (default value) is used, the natural logarithm is taken. OUTPUT:

A growth element.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ * log(x)^ZZ')
sage: x, = G.gens_monomial()
sage: log(x)  # indirect doctest
log(x)
sage: log(x^5)  # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: When calculating log(x^5) a factor 5 != 1 appeared,
which is not contained in Growth Group x^ZZ * log(x)^ZZ.
```

```
sage: G = GrowthGroup('QQ^x * x^ZZ')
sage: x, = G.gens_monomial()
sage: el = x.rpow(2); el
2^x
sage: log(el) # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: When calculating log(2^x) a factor log(2) != 1
appeared, which is not contained in Growth Group QQ^x * x^ZZ.
sage: log(el, base=2) # indirect doctest
x
```

```
sage: from sage.rings.asymptotic.growth_group import GenericGrowthGroup
sage: x = GenericGrowthGroup(ZZ).an_element()
sage: log(x) # indirect doctest
Traceback (most recent call last):
...
NotImplementedError: Cannot determine logarithmized factorization of
GenericGrowthElement(1) in abstract base class.
```

```
sage: x = GrowthGroup('x^ZZ').an_element()
sage: log(x) # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: Cannot build log(x) since log(x) is not in
Growth Group x^ZZ.
```

```
sage: G = GrowthGroup("(e^x)^QQ * x^ZZ")
sage: x, = G.gens_monomial()
sage: log(exp(x)) # indirect doctest
x
sage: G.one().log() # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: log(1) is zero, which is not contained in
Growth Group (e^x)^QQ * x^ZZ.
```

```
sage: G = GrowthGroup("(e^x)^ZZ * x^ZZ")
sage: x, = G.gens_monomial()
sage: log(exp(x)) # indirect doctest
x
sage: G.one().log() # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: log(1) is zero, which is not contained in
Growth Group (e^x)^ZZ * x^ZZ.
```

```
sage: G = GrowthGroup('QQ^x * x^ZZ * log(x)^ZZ * y^ZZ * log(y)^ZZ')
sage: x, y = G.gens_monomial()
sage: (x * y).log() # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: Calculating log(x*y) results in a sum,
which is not contained in
Growth Group QQ^x * x^ZZ * log(x)^ZZ * y^ZZ * log(y)^ZZ.
```

log_factor (base=None)

Return the logarithm of the factorization of this element.

INPUT:

•base — the base of the logarithm. If None (default value) is used, the natural logarithm is taken. OUTPUT:

A tuple of pairs, where the first entry is a growth element and the second a multiplicative coefficient.

ALGORITHM:

This function factors the given element and calculates the logarithm of each of these factors. EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('QQ^x * x^ZZ * log(x)^ZZ * y^ZZ * log(y)^ZZ')
sage: x, y = G.gens_monomial()
sage: (x * y).log_factor() # indirect doctest
((log(x), 1), (log(y), 1))
sage: (x^123).log_factor() # indirect doctest
((log(x), 123),)
sage: (G('2^x') * x^2).log_factor(base=2) # indirect doctest
((x, 1), (log(x), 2/log(2)))
```

```
sage: G(1).log_factor()
()
```

```
sage: log(x).log_factor() # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: Cannot build log(log(x)) since log(log(x)) is
not in Growth Group QQ^x * x^ZZ * log(x)^ZZ * y^ZZ * log(y)^ZZ.
```

See also:

```
factors(), log().
```

```
sage: G = GrowthGroup("(e^x)^ZZ * x^ZZ * log(x)^ZZ")
sage: x, = G.gens_monomial()
sage: (exp(x) * x).log_factor() # indirect doctest
((x, 1), (log(x), 1))
```

rpow (base)

Calculate the power of base to this element.

INPUT:

•base - an element.

OUTPUT:

A growth element.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('QQ^x * x^ZZ')
sage: x = G('x')
sage: x.rpow(2) # indirect doctest
2^x
sage: x.rpow(1/2) # indirect doctest
(1/2)^x
```

```
sage: x.rpow(0) # indirect doctest
Traceback (most recent call last):
...
ValueError: 0 is not an allowed base for calculating the power to x.
sage: (x^2).rpow(2) # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: Cannot construct 2^(x^2) in Growth Group QQ^x * x^ZZ
> *previous* TypeError: unsupported operand parent(s) for '*':
'Growth Group QQ^x * x^ZZ' and 'Growth Group ZZ^(x^2)'
```

```
sage: G = GrowthGroup('QQ^(x*log(x)) * x^ZZ * log(x)^ZZ')
sage: x = G('x')
sage: (x * log(x)).rpow(2) # indirect doctest
2^(x*log(x))
```

```
sage: n = GrowthGroup('QQ^n * n^QQ')('n')
sage: n.rpow(2)
2^n
sage: _.parent()
Growth Group QQ^n * n^QQ
```

variable_names ()

Return the names of the variables of this growth element.

OUTPUT:

A tuple of strings.

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('QQ^m * m^QQ * log(n)^ZZ')
sage: G('2^m * m^4 * log(n)').variable_names()
```

```
('m', 'n')
sage: G('2^m * m^4').variable_names()
('m',)
sage: G('log(n)').variable_names()
('n',)
sage: G('m^3').variable_names()
('m',)
sage: G('m^0').variable_names()
()
```

GenericProduct.cartesian_injection (factor, element)

Inject the given element into this Cartesian product at the given factor.

INPUT:

- •factor a growth group (a factor of this Cartesian product).
- •element an element of factor.

OUTPUT:

An element of this Cartesian product.

TESTS:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ * y^QQ')
sage: G.cartesian_injection(G.cartesian_factors()[1], 'y^7')
y^7
```

```
GenericProduct.gens_monomial()
```

Return a tuple containing monomial generators of this growth group.

INPUT:

Nothing.

OUTPUT:

A tuple containing elements of this growth group.

Note: This method calls the <code>gens_monomial()</code> method on the individual factors of this Cartesian product and concatenates the respective outputs.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ * log(x)^ZZ * y^QQ * log(z)^ZZ')
sage: G.gens_monomial()
(x, y)
```

TESTS:

```
sage: all(g.parent() == G for g in G.gens_monomial())
True
```

```
GenericProduct. some_elements ()
```

Return some elements of this Cartesian product of growth groups.

See TestSuite for a typical use case.

INPUT:

Nothing.

OUTPUT:

An iterator.

EXAMPLES:

```
sage: from itertools import islice
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('QQ^y * x^QQ * log(x)^ZZ')
sage: tuple(islice(G.some_elements(), 10))
(x^(1/2)*(1/2)^y,
    x^(-1/2)*log(x)*(-1/2)^y,
    x^2*log(x)^(-1)*2^y,
    x^2*log(x)^2*(-2)^y,
    log(x)^(-2),
    x*log(x)^3*(-1)^y,
    x^(-1)*log(x)^(-3)*42^y,
    x^42*log(x)^4*(2/3)^y,
    x^(2/3)*log(x)^(-4)*(-2/3)^y,
    x^(-2/3)*log(x)^5*(3/2)^y)
```

GenericProduct. variable names ()

Return the names of the variables.

OUTPUT:

A tuple of strings.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('x^ZZ * log(x)^ZZ * y^QQ * log(z)^ZZ').variable_names()
('x', 'y', 'z')
```

Bases: sage.rings.asymptotic.growth_group_cartesian.GenericProduct

A Cartesian product of growth groups with pairwise disjoint (or equal) variable sets.

Note: A multivariate product of growth groups is ordered by means of the product order, i.e. component-wise. This is motivated by the assumption that different variables are considered to be independent (e.g. $x^2ZZ * y^2Z$).

See also:

UnivariateProduct, GenericProduct.

Bases: sage.rings.asymptotic.growth_group_cartesian.GenericProduct

A Cartesian product of growth groups with the same variables.

Note: A univariate product of growth groups is ordered lexicographically. This is motivated by the assumption that univariate growth groups can be ordered in a chain with respect to the growth they model (e.g. $x^2Z \times \log(x)^2Z$); polynomial growth dominates logarithmic growth).

See also:

MultivariateProduct, GenericProduct.

4.5 (Asymptotic) Term Monoids

This module implements asymptotic term monoids. The elements of these monoids are used behind the scenes when performing calculations in an *asymptotic ring*.

The monoids build upon the (asymptotic) growth groups. While growth elements only model the growth of a function as it tends towards infinity (or tends towards another fixed point; see (*Asymptotic*) *Growth Groups* for more details), an asymptotic term additionally specifies its "type" and performs the actual arithmetic operations (multiplication and partial addition/absorption of terms).

Besides an abstract base term GenericTerm, this module implements the following types of terms:

- OTerm O-terms at infinity, see Wikipedia article Big_O_notation.
- TermWithCoefficient abstract base class for asymptotic terms with coefficients.
- Exact Term this class represents a growth element multiplied with some non-zero coefficient from a coefficient ring.

A characteristic property of asymptotic terms is that some terms are able to "absorb" other terms (see absorb()). For instance, $O(x^2)$ is able to absorb O(x) (with result $O(x^2)$), and $3 \cdot x^5$ is able to absorb $-2 \cdot x^5$ (with result x^5). Essentially, absorption can be interpreted as the addition of "compatible" terms (partial addition).

Warning: As this code is experimental, a warning is thrown when a term monoid is created for the first time in a session (see sage.misc.superseded.experimental).

TESTS:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import GenericTermMonoid
sage: G = GrowthGroup('x^ZZ * log(x)^ZZ')
doctest:...: FutureWarning: This class/method/function is marked as
experimental. It, its functionality or its interface might change
without a formal deprecation.
See http://trac.sagemath.org/17601 for details.
```

4.5.1 Absorption of Asymptotic Terms

A characteristic property of asymptotic terms is that some terms are able to "absorb" other terms. This is realized with the method <code>absorb()</code>.

For instance, $O(x^2)$ is able to absorb O(x) (with result $O(x^2)$). This is because the functions bounded by linear growth are bounded by quadratic growth as well. Another example would be that $3x^5$ is able to absorb $-2x^5$ (with result x^5), which simply corresponds to addition.

Essentially, absorption can be interpreted as the addition of "compatible" terms (partial addition).

We want to show step by step which terms can be absorbed by which other terms. We start by defining the necessary term monoids and some terms:

```
sage: from sage.rings.asymptotic.term_monoid import OTermMonoid, ExactTermMonoid
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ'); x = G.gen()
sage: OT = OTermMonoid(growth_group=G, coefficient_ring=QQ)
sage: ET = ExactTermMonoid(growth_group=G, coefficient_ring=QQ)
sage: ot1 = OT(x); ot2 = OT(x^2)
sage: et1 = ET(x^2, 2)
```

• Because of the definition of O-terms (see Wikipedia article Big_O_notation), OTerm are able to absorb all other asymptotic terms with weaker or equal growth. In our implementation, this means that OTerm is able to absorb other OTerm, as well as ExactTerm, as long as the growth of the other term is less than or equal to the growth of this element:

```
sage: ot1, ot2
(O(x), O(x^2))
sage: ot1.can_absorb(ot2), ot2.can_absorb(ot1)
(False, True)
sage: et1
2*x^2
sage: ot1.can_absorb(et1)
False
sage: ot2.can_absorb(et1)
True
```

The result of this absorption always is the dominant (absorbing) OTerm:

```
sage: ot1.absorb(ot1)
O(x)
sage: ot2.absorb(ot1)
O(x^2)
sage: ot2.absorb(et1)
O(x^2)
```

These examples correspond to O(x) + O(x) = O(x), $O(x^2) + O(x) = O(x^2)$, and $O(x^2) + 2x^2 = O(x^2)$.

• Exact Term can only absorb another Exact Term if the growth coincides with the growth of this element:

```
sage: et1.can_absorb(ET(x^2, 5))
True
sage: any(et1.can_absorb(t) for t in [ot1, ot2])
False
```

As mentioned above, absorption directly corresponds to addition in this case:

```
sage: et1.absorb(ET(x^2, 5))
7*x^2
```

When adding two exact terms, they might cancel out. For technical reasons, None is returned in this case:

```
sage: ET(x^2, 5).can_absorb(ET(x^2, -5))
True
sage: ET(x^2, 5).absorb(ET(x^2, -5)) is None
True
```

• The abstract base terms <code>GenericTerm</code> and <code>TermWithCoefficient</code> can neither absorb any other term, nor be absorbed by any other term.

If absorb is called on a term that cannot be absorbed, an Arithmetic Error is raised:

```
sage: ot1.absorb(ot2)
Traceback (most recent call last):
...
ArithmeticError: O(x) cannot absorb O(x^2)
```

This would only work the other way around:

```
sage: ot2.absorb(ot1)
O(x^2)
```

4.5.2 Comparison

The comparison of asymptotic terms with \leq is implemented as follows:

- When comparing t1 <= t2, the coercion framework first tries to find a common parent for both terms. If this fails, False is returned.
- In case the coerced terms do not have a coefficient in their common parent (e.g. OTerm), the growth of the two terms is compared.
- Otherwise, if the coerced terms have a coefficient (e.g. <code>ExactTerm</code>), we compare whether t1 has a growth that is strictly weaker than the growth of t2. If so, we return <code>True</code>. If the terms have equal growth, then we return <code>True</code> if and only if the coefficients coincide as well.

In all other cases, we return False.

Long story short: we consider terms with different coefficients that have equal growth to be incomparable.

4.5.3 Various

Todo

• Implementation of more term types (e.g. L terms, Ω terms, o terms, Θ terms).

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4.5.4 Classes and Methods

Class for asymptotic exact terms. These terms primarily consist of an asymptotic growth element as well as a coefficient specifying the growth of the asymptotic term.

INPUT:

- •parent the parent of the asymptotic term.
- •growth an asymptotic growth element from parent.growth_group.
- •coefficient an element from parent.coefficient ring.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import (ExactTermMonoid, TermMonoid)
sage: G = GrowthGroup('x^ZZ'); x = G.gen()
sage: ET = ExactTermMonoid(G, QQ)
```

Asymptotic exact terms may be multiplied (with the usual rules applying):

```
sage: ET(x^2, 3) * ET(x, -1)
-3*x^3
sage: ET(x^0, 4) * ET(x^5, 2)
8*x^5
```

They may also be multiplied with *O*-terms:

```
sage: OT = TermMonoid('0', G, QQ)
sage: ET(x^2, 42) * OT(x)
O(x^3)
```

Absorption for asymptotic exact terms relates to addition:

```
sage: ET(x^2, 5).can_absorb(ET(x^5, 12))
False
sage: ET(x^2, 5).can_absorb(ET(x^2, 1))
True
sage: ET(x^2, 5).absorb(ET(x^2, 1))
6*x^2
```

Note that, as for technical reasons, 0 is not allowed as a coefficient for an asymptotic term with coefficient. Instead None is returned if two asymptotic exact terms cancel out each other during absorption:

```
sage: ET(x^2, 42).can_absorb(ET(x^2, -42))
True
sage: ET(x^2, 42).absorb(ET(x^2, -42)) is None
True
```

Exact terms can also be created by converting monomials with coefficient from the symbolic ring, or a suitable polynomial or power series ring:

```
sage: x = var('x'); x.parent()
Symbolic Ring
sage: ET(5*x^2)
5*x^2
```

can absorb (other)

Check whether this exact term can absorb other.

INPUT:

•other - an asymptotic term.

OUTPUT:

A boolean.

Note: For *ExactTerm*, absorption corresponds to addition. This means that an exact term can absorb only other exact terms with the same growth.

See the *module description* for a detailed explanation of absorption.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: ET = TermMonoid('exact', GrowthGroup('x^ZZ'), ZZ)
sage: t1 = ET(x^21, 1); t2 = ET(x^21, 2); t3 = ET(x^42, 1)
sage: t1.can_absorb(t2)
True
sage: t2.can_absorb(t1)
True
sage: t1.can_absorb(t3) or t3.can_absorb(t1)
False
```

is_constant()

Return whether this term is an (exact) constant.

INPUT:

Nothing.

OUTPUT:

A boolean.

Note: Only *ExactTerm* with constant growth (1) are constant.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: T = TermMonoid('exact', GrowthGroup('x^ZZ * log(x)^ZZ'), QQ)
sage: T('x * log(x)').is_constant()
False
sage: T('3*x').is_constant()
False
sage: T(1/2).is_constant()
True
sage: T(42).is_constant()
```

is_exact ()

Return whether this term is an exact term.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: T = TermMonoid('exact', GrowthGroup('x^ZZ * log(x)^ZZ'), QQ)
sage: T('x * log(x)').is_exact()
True
sage: T('3 * x^2').is_exact()
True
```

TESTS:

```
sage: T = TermMonoid('O', GrowthGroup('x^ZZ'), QQ)
sage: T('x').is_exact()
False
```

is_little_o_of_one ()

Return whether this exact term is of order o(1).

INPUT:

Nothing.

OUTPUT:

A boolean.

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: T = TermMonoid('exact', GrowthGroup('x^ZZ'), QQ)
sage: T(x).is_little_o_of_one()
False
sage: T(1).is_little_o_of_one()
False
sage: T(x^(-1)).is_little_o_of_one()
True
```

```
sage: T = TermMonoid('exact', GrowthGroup('x^ZZ * y^ZZ'), QQ)
sage: T('x * y^(-1)').is_little_o_of_one()
False
sage: T('x^(-1) * y').is_little_o_of_one()
False
sage: T('x^(-2) * y^(-3)').is_little_o_of_one()
True
```

```
sage: T = TermMonoid('exact', GrowthGroup('x^QQ * log(x)^QQ'), QQ)
sage: T('x * log(x)^2').is_little_o_of_one()
False
sage: T('x^2 * log(x)^(-1234)').is_little_o_of_one()
False
sage: T('x^(-1) * log(x)^4242').is_little_o_of_one()
True
sage: T('x^(-1/100) * log(x)^(1000/7)').is_little_o_of_one()
True
```

log term (base=None)

Determine the logarithm of this exact term.

INPUT:

•base - the base of the logarithm. If None (default value) is used, the natural logarithm is taken.

OUTPUT:

A tuple of terms.

Note: This method returns a tuple with the summands that come from applying the rule $\log(x \cdot y) = \log(x) + \log(y)$.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: T = TermMonoid('exact', GrowthGroup('x^ZZ * log(x)^ZZ'), SR)
sage: T(3*x^2).log_term()
(log(3), 2*log(x))
sage: T(x^1234).log_term()
(1234*log(x),)
sage: T(49*x^7).log_term(base=7)
(log(49)/log(7), 7/log(7)*log(x))
```

See also:

```
OTerm.log_term().
```

rpow (base)

Return the power of base to this exact term.

INPUT:

•base - an element or 'e'.

OUTPUT:

A term.

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: T = TermMonoid('exact', GrowthGroup('QQ^x * x^Z * log(x)^ZZ'), QQ)
sage: T('x').rpow(2)
2^x
sage: T('log(x)').rpow('e')
x
sage: T('42*log(x)').rpow('e')
x^42
sage: T('3*x').rpow(2)
8^x
```

```
sage: T('3*x^2').rpow(2)
Traceback (most recent call last):
...
ArithmeticError: Cannot construct 2^(x^2) in
Growth Group QQ^x * x^ZZ * log(x)^ZZ
> *previous* TypeError: unsupported operand parent(s) for '*':
'Growth Group QQ^x * x^ZZ * log(x)^ZZ' and 'Growth Group ZZ^(x^2)'
```

```
sage: T = TermMonoid('exact', GrowthGroup('QQ^n * n^QQ'), SR)
sage: n = T('n')
sage: n.rpow(2)
2^n
sage: _.parent()
Exact Term Monoid QQ^n * n^SR with coefficients in Symbolic Ring
```

Above, we get QQ^n * n^SR. The reason is the following: Since $n = 1_{SR} \cdot (1_{\mathbf{Q}})^n \cdot n^{1_{\mathbf{Q}}}$, we have

$$2^{n} = (2_{\mathbf{Q}})^{1_{SR} \cdot (1_{\mathbf{Q}})^{n} \cdot n^{1_{\mathbf{Q}}}} = \left((2_{\mathbf{Q}})^{n} \cdot n^{0_{\mathbf{Q}}} \right)^{1_{SR}} = \left((2_{\mathbf{Q}})^{1_{SR}} \right)^{n} \cdot n^{0_{\mathbf{Q}} 1_{SR}} = (2_{\mathbf{Q}})^{n} \cdot n^{0_{SR}}$$

where

```
sage: (QQ(2)^SR(1)).parent(), (QQ(0)*SR(1)).parent()
(Rational Field, Symbolic Ring)
```

was used.

Bases: sage.rings.asymptotic.term_monoid.TermWithCoefficientMonoid

Parent for asymptotic exact terms, implemented in Exact Term.

INPUT:

- •growth_group a growth group.
- •category The category of the parent can be specified in order to broaden the base structure. It has to be a subcategory of Join of Category of monoids and Category of posets. This is also the default category if None is specified.
- •coefficient_ring the ring which contains the coefficients of the elements.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import ExactTermMonoid
sage: G_ZZ = GrowthGroup('x^ZZ'); x_ZZ = G_ZZ.gen()
sage: G_QQ = GrowthGroup('x^QQ'); x_QQ = G_QQ.gen()
sage: ET_ZZ = ExactTermMonoid(G_ZZ, ZZ); ET_ZZ
Exact Term Monoid x^ZZ with coefficients in Integer Ring
sage: ET_QQ = ExactTermMonoid(G_QQ, QQ); ET_QQ
Exact Term Monoid x^QQ with coefficients in Rational Field
sage: ET_QQ.coerce_map_from(ET_ZZ)
Conversion map:
    From: Exact Term Monoid x^ZZ with coefficients in Integer Ring
To: Exact Term Monoid x^QQ with coefficients in Rational Field
```

Exact term monoids can also be created using the term factory:

```
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: TermMonoid('exact', G_ZZ, ZZ) is ET_ZZ
True
sage: TermMonoid('exact', GrowthGroup('x^ZZ'), QQ)
Exact Term Monoid x^ZZ with coefficients in Rational Field
```

Element

alias of ExactTerm

```
class sage.rings.asymptotic.term_monoid. GenericTerm ( parent, growth)
     Bases: sage.structure.element.MultiplicativeGroupElement
```

Base class for asymptotic terms. Mainly the structure and several properties of asymptotic terms are handled here.

INPUT:

- •parent the parent of the asymptotic term.
- •growth an asymptotic growth element.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import GenericTermMonoid
sage: G = GrowthGroup('x^ZZ'); x = G.gen()
sage: T = GenericTermMonoid(G, QQ)
sage: t1 = T(x); t2 = T(x^2); (t1, t2)
(Generic Term with growth x, Generic Term with growth x^2)
sage: t1 * t2
Generic Term with growth x^3
sage: t1.can_absorb(t2)
False
sage: t1.absorb(t2)
Traceback (most recent call last):
...
ArithmeticError: Generic Term with growth x cannot absorb Generic Term with_
Growth x^2
sage: t1.can_absorb(t1)
False
```

absorb (other, check=True)

Absorb the asymptotic term other and return the resulting asymptotic term.

INPUT:

- •other an asymptotic term.
- •check a boolean. If check is True (default), then can_absorb is called before absorption.

OUTPUT:

An asymptotic term or None if a cancellation occurs. If no absorption can be performed, an ArithmeticError is raised.

Note: Setting check to False is meant to be used in cases where the respective comparison is done externally (in order to avoid duplicate checking).

For a more detailed explanation of the absorption of asymptotic terms see the module description.

EXAMPLES:

We want to demonstrate in which cases an asymptotic term is able to absorb another term, as well as explain the output of this operation. We start by defining some parents and elements:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: G_QQ = GrowthGroup('x^QQ'); x = G_QQ.gen()
sage: OT = TermMonoid('O', G_QQ, coefficient_ring=ZZ)
sage: ET = TermMonoid('exact', G_QQ, coefficient_ring=QQ)
sage: ot1 = OT(x); ot2 = OT(x^2)
sage: et1 = ET(x, 100); et2 = ET(x^2, 2)
sage: et3 = ET(x^2, 1); et4 = ET(x^2, -2)
```

O-Terms are able to absorb other O-terms and exact terms with weaker or equal growth.

```
sage: ot1.absorb(ot1)
O(x)
sage: ot1.absorb(et1)
O(x)
sage: ot1.absorb(et1) is ot1
True
```

ExactTerm is able to absorb another ExactTerm if the terms have the same growth. In this case, absorption is nothing else than an addition of the respective coefficients:

```
sage: et2.absorb(et3)
3*x^2
sage: et3.absorb(et2)
3*x^2
sage: et3.absorb(et4)
-x^2
```

Note that, for technical reasons, the coefficient 0 is not allowed, and thus None is returned if two exact terms cancel each other out:

```
sage: et2.absorb(et4)
sage: et4.absorb(et2) is None
True
```

TESTS:

When disabling the check flag, absorb might produce wrong results:

```
sage: et1.absorb(ot2, check=False)
O(x)
```

can_absorb (other)

Check whether this asymptotic term is able to absorb the asymptotic term other.

INPUT:

•other - an asymptotic term.

OUTPUT:

A boolean.

Note: A *GenericTerm* cannot absorb any other term.

See the *module description* for a detailed explanation of absorption.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GenericGrowthGroup
sage: from sage.rings.asymptotic.term_monoid import GenericTermMonoid
sage: G = GenericGrowthGroup(ZZ)
sage: T = GenericTermMonoid(G, QQ)
sage: g1 = G(raw_element=21); g2 = G(raw_element=42)
sage: t1 = T(g1); t2 = T(g2)
sage: t1.can_absorb(t2) # indirect doctest
False
sage: t2.can_absorb(t1) # indirect doctest
False
```

is_constant()

Return whether this term is an (exact) constant.

INPUT:

Nothing.

OUTPUT:

A boolean.

Note: Only *ExactTerm* with constant growth (1) are constant.

EXAMPLES:

```
sage: T = TermMonoid('O', GrowthGroup('x^ZZ'), QQ)
sage: T('x').is_constant()
False
sage: T(1).is_constant()
False
```

is_exact ()

Return whether this term is an exact term.

OUTPUT:

A boolean.

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import GenericTermMonoid
sage: T = GenericTermMonoid(GrowthGroup('x^ZZ * log(x)^ZZ'), QQ)
```

```
sage: T.an_element().is_exact()
False
```

is_little_o_of_one ()

Return whether this generic term is of order o(1).

INPUT:

Nothing.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import (GenericTermMonoid,
→TermWithCoefficientMonoid)
sage: T = GenericTermMonoid(GrowthGroup('x^ZZ'), QQ)
sage: T.an_element().is_little_o_of_one()
Traceback (most recent call last):
NotImplementedError: Cannot check whether Generic Term with growth x is o(1)
in the abstract base class
Generic Term Monoid x^{ZZ} with (implicit) coefficients in Rational Field.
sage: T = TermWithCoefficientMonoid(GrowthGroup('x^ZZ'), QQ)
sage: T.an_element().is_little_o_of_one()
Traceback (most recent call last):
NotImplementedError: Cannot check whether Term with coefficient 1/2 and
⇒growth x
is o(1) in the abstract base class
Generic Term Monoid x^{ZZ} with (implicit) coefficients in Rational Field.
```

log_term (base=None)

Determine the logarithm of this term.

INPUT:

•base - the base of the logarithm. If None (default value) is used, the natural logarithm is taken.

OUTPUT:

A tuple of terms.

Note: This abstract method raises a NotImplementedError. See *ExactTerm* and *OTerm* for a concrete implementation.

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import GenericTermMonoid
sage: T = GenericTermMonoid(GrowthGroup('x^ZZ'), QQ)
sage: T.an_element().log_term()
Traceback (most recent call last):
...
```

```
NotImplementedError: This method is not implemented in this abstract base class.
```

```
sage: from sage.rings.asymptotic.term_monoid import TermWithCoefficientMonoid
sage: T = TermWithCoefficientMonoid(GrowthGroup('x^ZZ'), QQ)
sage: T.an_element().log_term()
Traceback (most recent call last):
...
NotImplementedError: This method is not implemented in
this abstract base class.
```

See also:

```
ExactTerm.log_term(),OTerm.log_term().
```

rpow (base)

Return the power of base to this generic term.

INPUT:

```
•base - an element or 'e'.
```

OUTPUT:

A term.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import GenericTermMonoid
sage: T = GenericTermMonoid(GrowthGroup('x^ZZ * log(x)^ZZ'), QQ)
sage: T.an_element().rpow('e')
Traceback (most recent call last):
...
NotImplementedError: Cannot take e to the exponent
Generic Term with growth x*log(x) in the abstract base class
Generic Term Monoid x^ZZ * log(x)^ZZ with (implicit) coefficients in Rational_
→Field.
```

variable names ()

Return the names of the variables of this term.

OUTPUT:

A tuple of strings.

```
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: T = TermMonoid('exact', 'QQ^m * m^QQ * log(n)^ZZ', QQ)
sage: T('4 * 2^m * m^4 * log(n)').variable_names()
('m', 'n')
sage: T('4 * 2^m * m^4').variable_names()
('m',)
sage: T('4 * log(n)').variable_names()
('n',)
sage: T('4 * m^3').variable_names()
('m',)
sage: T('4 * m^0').variable_names()
```

Parent for generic asymptotic terms.

INPUT:

- •growth_group a growth group (i.e. an instance of GenericGrowthGroup).
- •coefficient_ring a ring which contains the (maybe implicit) coefficients of the elements.
- •category The category of the parent can be specified in order to broaden the base structure. It has to be a subcategory of Join of Category of Monoids and Category of posets. This is also the default category if None is specified.

In this class the base structure for asymptotic term monoids will be handled. These monoids are the parents of asymptotic terms (for example, see <code>GenericTerm</code> or <code>OTerm</code>). Basically, asymptotic terms consist of a <code>growth</code> (which is an asymptotic growth group element, for example <code>MonomialGrowthElement</code>); additional structure and properties are added by the classes inherited from <code>GenericTermMonoid</code>.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import GenericTermMonoid
sage: G_x = GrowthGroup('x^ZZ'); x = G_x.gen()
sage: G_y = GrowthGroup('y^QQ'); y = G_y.gen()
sage: T_x_ZZ = GenericTermMonoid(G_x, QQ)
sage: T_y_QQ = GenericTermMonoid(G_y, QQ)
sage: T_x_ZZ
Generic Term Monoid x^ZZ with (implicit) coefficients in Rational Field
sage: T_y_QQ
Generic Term Monoid y^QQ with (implicit) coefficients in Rational Field
```

Element

alias of GenericTerm

change_parameter (growth_group=None, coefficient_ring=None)

Return a term monoid with a change in one or more of the given parameters.

INPUT:

- •growth_group (default: None) the new growth group.
- •coefficient_ring (default: None) the new coefficient ring.

OUTPUT:

A term monoid.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: E = TermMonoid('exact', GrowthGroup('n^ZZ'), ZZ)
sage: E.change_parameter(coefficient_ring=QQ)
Exact Term Monoid n^ZZ with coefficients in Rational Field
sage: E.change_parameter(growth_group=GrowthGroup('n^QQ'))
Exact Term Monoid n^QQ with coefficients in Integer Ring
```

```
sage: E.change_parameter() is E
True
sage: E.change_parameter(growth_group=None) is E
True
sage: E.change_parameter(coefficient_ring=None) is E
True
sage: E.change_parameter(growth_group=None, coefficient_ring=None) is E
True
```

coefficient ring

The coefficient ring of this term monoid, i.e. the ring where the coefficients are from.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import GenericTermMonoid
sage: GenericTermMonoid(GrowthGroup('x^ZZ'), ZZ).coefficient_ring
Integer Ring
```

growth_group

The growth group underlying this term monoid.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: TermMonoid('exact', GrowthGroup('x^ZZ'), ZZ).growth_group
Growth Group x^ZZ
```

le (*left*, *right*)

Return whether the term left is at most (less than or equal to) the term right.

INPUT:

- •left an element.
- •right an element.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import GenericTermMonoid
sage: G = GrowthGroup('x^ZZ'); x = G.gen()
sage: T = GenericTermMonoid(G, QQ)
sage: t1 = T(x); t2 = T(x^2)
sage: T.le(t1, t2)
True
```

some elements ()

Return some elements of this term monoid.

See TestSuite for a typical use case.

INPUT:

Nothing.

OUTPUT:

An iterator.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: G = GrowthGroup('x^ZZ')
sage: tuple(TermMonoid('O', G, QQ).some_elements())
(O(1), O(x), O(x^{(-1)}), O(x^{2}), O(x^{(-2)}), O(x^{3}), ...)
```

class sage.rings.asymptotic.term_monoid. OTerm (parent, growth)

Bases: sage.rings.asymptotic.term_monoid.GenericTerm

Class for an asymptotic term representing an *O*-term with specified growth. For the mathematical properties of *O*-terms see Wikipedia article Big O Notation.

O-terms can absorb terms of weaker or equal growth.

INPUT:

•parent - the parent of the asymptotic term.

•growth - a growth element.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term monoid import OTermMonoid
sage: G = GrowthGroup('x^ZZ'); x = G.gen()
sage: OT = OTermMonoid(G, QQ)
sage: t1 = OT(x^-7); t2 = OT(x^5); t3 = OT(x^42)
sage: t1, t2, t3
(O(x^{(-7)}), O(x^{5}), O(x^{42}))
sage: t1.can_absorb(t2)
False
sage: t2.can_absorb(t1)
True
sage: t2.absorb(t1)
0(x^5)
sage: t1 \le t2 and t2 \le t3
sage: t3 <= t1
False
```

The conversion of growth elements also works for the creation of *O*-terms:

```
sage: x = SR('x'); x.parent()
Symbolic Ring
sage: OT(x^17)
O(x^17)
```

can_absorb (other)

Check whether this O-term can absorb other.

INPUT:

•other - an asymptotic term.

OUTPUT:

A boolean.

Note: An OTerm can absorb any other asymptotic term with weaker or equal growth.

See the *module description* for a detailed explanation of absorption.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: OT = TermMonoid('O', GrowthGroup('x^ZZ'), QQ)
sage: t1 = OT(x^21); t2 = OT(x^42)
sage: t1.can_absorb(t2)
False
sage: t2.can_absorb(t1)
True
```

is_little_o_of_one ()

Return whether this O-term is of order o(1).

INPUT:

Nothing.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: T = TermMonoid('O', GrowthGroup('x^ZZ'), QQ)
sage: T(x).is_little_o_of_one()
False
sage: T(1).is_little_o_of_one()
False
sage: T(x^(-1)).is_little_o_of_one()
```

```
sage: T = TermMonoid('0', GrowthGroup('x^ZZ * y^ZZ'), QQ)
sage: T('x * y^(-1)').is_little_o_of_one()
False
sage: T('x^(-1) * y').is_little_o_of_one()
False
sage: T('x^(-2) * y^(-3)').is_little_o_of_one()
True
```

```
sage: T = TermMonoid('O', GrowthGroup('x^QQ * log(x)^QQ'), QQ)
sage: T('x * log(x)^2').is_little_o_of_one()
False
sage: T('x^2 * log(x)^(-1234)').is_little_o_of_one()
False
sage: T('x^(-1) * log(x)^4242').is_little_o_of_one()
True
sage: T('x^(-1/100) * log(x)^(1000/7)').is_little_o_of_one()
True
```

log_term (base=None)

Determine the logarithm of this O-term.

INPUT:

•base - the base of the logarithm. If None (default value) is used, the natural logarithm is taken.

OUTPUT:

A tuple of terms.

Note: This method returns a tuple with the summands that come from applying the rule $\log(x \cdot y) = \log(x) + \log(y)$.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: T = TermMonoid('O', GrowthGroup('x^ZZ * log(x)^ZZ'), QQ)
sage: T(x^2).log_term()
(O(log(x)),)
sage: T(x^1234).log_term()
(O(log(x)),)
```

See also:

```
ExactTerm.log_term().
```

rpow (base)

Return the power of base to this O-term.

INPUT:

•base - an element or 'e'.

OUTPUT:

A term.

Note: For OTerm, the powers can only be constructed for exponents O(1) or if base is 1.

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: T = TermMonoid('O', GrowthGroup('x^ZZ * log(x)^ZZ'), QQ)
sage: T(1).rpow('e')
O(1)
sage: T(1).rpow(2)
O(1)
```

```
sage: T.an_element().rpow(1)
1
sage: T('x^2').rpow(1)
1
```

```
sage: T.an_element().rpow('e')
Traceback (most recent call last):
...
ValueError: Cannot take e to the exponent O(x*log(x)) in
O-Term Monoid x^ZZ * log(x)^ZZ with implicit coefficients in Rational Field
sage: T('log(x)').rpow('e')
Traceback (most recent call last):
...
ValueError: Cannot take e to the exponent O(log(x)) in
O-Term Monoid x^ZZ * log(x)^ZZ with implicit coefficients in Rational Field
```

Bases: sage.rings.asymptotic.term_monoid.GenericTermMonoid

Parent for asymptotic big *O*-terms.

INPUT:

- •growth_group a growth group.
- •category The category of the parent can be specified in order to broaden the base structure. It has to be a subcategory of Join of Category of monoids and Category of posets. This is also the default category if None is specified.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import OTermMonoid
sage: G_x_ZZ = GrowthGroup('x^ZZ')
sage: G_y_QQ = GrowthGroup('y^QQ')
sage: OT_x_ZZ = OTermMonoid(G_x_ZZ, QQ); OT_x_ZZ
O-Term Monoid x^ZZ with implicit coefficients in Rational Field
sage: OT_y_QQ = OTermMonoid(G_y_QQ, QQ); OT_y_QQ
O-Term Monoid y^QQ with implicit coefficients in Rational Field
```

O-term monoids can also be created by using the term factory:

```
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: TermMonoid('O', G_x_ZZ, QQ) is OT_x_ZZ
True
sage: TermMonoid('O', GrowthGroup('x^QQ'), QQ)
O-Term Monoid x^QQ with implicit coefficients in Rational Field
```

Element

alias of OTerm

sage.rings.asymptotic.term_monoid. **TermMonoid = <class 'sage.rings.asymptotic.term_monoid.TermMonoidFact**A factory for asymptotic term monoids. This is an instance of *TermMonoidFactory* whose documentation provides more details.

class sage.rings.asymptotic.term_monoid. TermMonoidFactory
 Bases: sage.structure.factory.UniqueFactory

Factory for asymptotic term monoids. It can generate the following term monoids:

- •OTermMonoid,
- •ExactTermMonoid.

Note: An instance of this factory is available as TermMonoid.

INPUT:

- •term_monoid the kind of terms held in the new term monoid. Either a string 'exact' or 'O' (capital letter O), or an existing instance of a term monoid.
- •growth group a growth group or a string describing a growth group.
- •coefficient_ring a ring.
- •asymptotic_ring if specified, then growth_group and coefficient_ring are taken from this asymptotic ring.

OUTPUT:

An asymptotic term monoid.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: G = GrowthGroup('x^ZZ')
sage: TermMonoid('O', G, QQ)
O-Term Monoid x^ZZ with implicit coefficients in Rational Field
sage: TermMonoid('exact', G, ZZ)
Exact Term Monoid x^ZZ with coefficients in Integer Ring
```

```
sage: R = AsymptoticRing(growth_group=G, coefficient_ring=QQ)
sage: TermMonoid('exact', asymptotic_ring=R)
Exact Term Monoid x^ZZ with coefficients in Rational Field
sage: TermMonoid('O', asymptotic_ring=R)
O-Term Monoid x^ZZ with implicit coefficients in Rational Field
sage: TermMonoid('exact', 'QQ^m * m^QQ * log(n)^ZZ', ZZ)
Exact Term Monoid QQ^m * m^QQ * log(n)^ZZ
with coefficients in Integer Ring
```

TESTS:

```
sage: TermMonoid(TermMonoid('O', G, ZZ), asymptotic_ring=R)
O-Term Monoid x^ZZ with implicit coefficients in Rational Field
sage: TermMonoid(TermMonoid('exact', G, ZZ), asymptotic_ring=R)
Exact Term Monoid x^ZZ with coefficients in Rational Field
sage: from sage.rings.asymptotic.term_monoid import GenericTermMonoid
sage: TermMonoid(GenericTermMonoid(G, ZZ), asymptotic_ring=R)
Generic Term Monoid x^ZZ with (implicit) coefficients in Rational Field
```

```
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_one() . . . pass
running ._test_pickling() . . . pass
running ._test_prod() . . . pass
running ._test_some_elements() . . . pass
```

```
sage: TestSuite(TermMonoid('O', GrowthGroup('x^QQ'), ZZ)).run(verbose=True)
\hookrightarrowlong time
running ._test_an_element() . . . pass
running ._test_associativity() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_elements() . . .
 Running the test suite of self.an_element()
 running ._test_category() . . . pass
 running ._test_eq() . . . pass
 running ._test_new() . . . pass
 running ._test_not_implemented_methods() . . . pass
 running ._test_pickling() . . . pass
 pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_one() . . . pass
running ._test_pickling() . . . pass
running ._test_prod() . . . pass
running ._test_some_elements() . . . pass
```

Given the arguments and keyword, create a key that uniquely determines this object.

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: G = GrowthGroup('x^ZZ')
sage: TermMonoid.create_key_and_extra_args('0', G, QQ)
((<class 'sage.rings.asymptotic.term_monoid.OTermMonoid'>,
    Growth Group x^ZZ, Rational Field), {})
sage: TermMonoid.create_key_and_extra_args('exact', G, ZZ)
((<class 'sage.rings.asymptotic.term_monoid.ExactTermMonoid'>,
    Growth Group x^ZZ, Integer Ring), {})
sage: TermMonoid.create_key_and_extra_args('exact', G)
```

```
Traceback (most recent call last):
...
ValueError: A coefficient ring has to be specified
to create a term monoid of type 'exact'
```

TESTS:

```
sage: TermMonoid.create_key_and_extra_args('icecream', G)
Traceback (most recent call last):
...
ValueError: Term specification 'icecream' has to be either
'exact' or 'O' or an instance of an existing term.
sage: TermMonoid.create_key_and_extra_args('O', ZZ)
Traceback (most recent call last):
...
ValueError: Integer Ring has to be an asymptotic growth group
```

create_object (version, key, **kwds)

Create a object from the given arguments.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: G = GrowthGroup('x^ZZ')
sage: TermMonoid('O', G, QQ) # indirect doctest
O-Term Monoid x^ZZ with implicit coefficients in Rational Field
sage: TermMonoid('exact', G, ZZ) # indirect doctest
Exact Term Monoid x^ZZ with coefficients in Integer Ring
```

Bases: sage.rings.asymptotic.term_monoid.GenericTerm

Base class for asymptotic terms possessing a coefficient. For example, <code>ExactTerm</code> directly inherits from this class.

INPUT:

- •parent the parent of the asymptotic term.
- •growth an asymptotic growth element of the parent's growth group.
- •coefficient an element of the parent's coefficient ring.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermWithCoefficientMonoid
sage: G = GrowthGroup('x^ZZ'); x = G.gen()
sage: CT_ZZ = TermWithCoefficientMonoid(G, ZZ)
sage: CT_QQ = TermWithCoefficientMonoid(G, QQ)
sage: CT_ZZ(x^2, 5)
Term with coefficient 5 and growth x^2
sage: CT_QQ(x^3, 3/8)
Term with coefficient 3/8 and growth x^3
```

Bases: sage.rings.asymptotic.term_monoid.GenericTermMonoid

This class implements the base structure for parents of asymptotic terms possessing a coefficient from some coefficient ring. In particular, this is also the parent for <code>TermWithCoefficient</code>.

INPUT:

- •growth_group a growth group.
- •category The category of the parent can be specified in order to broaden the base structure. It has to be a subcategory of Join of Category of monoids and Category of posets. This is also the default category if None is specified.
- •coefficient_ring the ring which contains the coefficients of the elements.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermWithCoefficientMonoid
sage: G_ZZ = GrowthGroup('x^ZZ'); x_ZZ = G_ZZ.gen()
sage: G_QQ = GrowthGroup('x^QQ'); x_QQ = G_QQ.gen()
sage: TC_ZZ = TermWithCoefficientMonoid(G_ZZ, QQ); TC_ZZ
Generic Term Monoid x^ZZ with (implicit) coefficients in Rational Field
sage: TC_QQ = TermWithCoefficientMonoid(G_QQ, QQ); TC_QQ
Generic Term Monoid x^QQ with (implicit) coefficients in Rational Field
sage: TC_ZZ == TC_QQ or TC_ZZ is TC_QQ
False
sage: TC_QQ.coerce_map_from(TC_ZZ)
Conversion map:
    From: Generic Term Monoid x^ZZ with (implicit) coefficients in Rational Field
    To: Generic Term Monoid x^QQ with (implicit) coefficients in Rational Field
```

Element

alias of TermWithCoefficient

some elements ()

Return some elements of this term with coefficient monoid.

See TestSuite for a typical use case.

INPUT:

Nothing.

OUTPUT:

An iterator.

EXAMPLES:

```
sage: from itertools import islice
sage: from sage.rings.asymptotic.term_monoid import TermMonoid
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('z^QQ')
sage: T = TermMonoid('exact', G, ZZ)
sage: tuple(islice(T.some_elements(), 10))
(z^(1/2), z^(-1/2), -z^(1/2), z^2, -z^(-1/2), 2*z^(1/2),
z^(-2), -z^2, 2*z^(-1/2), -2*z^(1/2))
```

exception sage.rings.asymptotic.term_monoid. ZeroCoefficientError
 Bases: exceptions.ValueError

```
sage.rings.asymptotic.term_monoid.absorption (left, right)
```

Let one of the two passed terms absorb the other.

Helper function used by AsymptoticExpansion.

Note: If neither of the terms can absorb the other, an ArithmeticError is raised.

See the *module description* for a detailed explanation of absorption.

INPUT:

- •left an asymptotic term.
- •right an asymptotic term.

OUTPUT:

An asymptotic term or None.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import (TermMonoid, absorption)
sage: T = TermMonoid('O', GrowthGroup('x^ZZ'), ZZ)
sage: absorption(T(x^2), T(x^3))
O(x^3)
sage: absorption(T(x^3), T(x^2))
O(x^3)
```

```
sage: T = TermMonoid('exact', GrowthGroup('x^ZZ'), ZZ)
sage: absorption(T(x^2), T(x^3))
Traceback (most recent call last):
...
ArithmeticError: Absorption between x^2 and x^3 is not possible.
```

sage.rings.asymptotic.term_monoid.can_absorb (left, right)

Return whether one of the two input terms is able to absorb the other.

Helper function used by AsymptoticExpansion.

INPUT:

- •left an asymptotic term.
- •right an asymptotic term.

OUTPUT:

A boolean.

Note: See the *module description* for a detailed explanation of absorption.

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import (TermMonoid, can_absorb)
sage: T = TermMonoid('O', GrowthGroup('x^ZZ'), ZZ)
sage: can_absorb(T(x^2), T(x^3))
True
```

```
sage: can_absorb(T(x^3), T(x^2))
True
```

4.6 Asymptotic Expansions — Miscellaneous

AUTHORS:

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4.6.1 Functions, Classes and Methods

A special NotImplementedError which is raised when the result is O(0) which means 0 for sufficiently large values of the variable.

```
sage.rings.asymptotic.misc. combine\_exceptions ( e, *f)
```

Helper function which combines the messages of the given exceptions.

INPUT:

- •e an exception.
- •*f exceptions.

OUTPUT:

An exception.

```
sage: from sage.rings.asymptotic.misc import combine_exceptions
sage: raise combine_exceptions(ValueError('Outer.'), TypeError('Inner.'))
Traceback (most recent call last):
ValueError: Outer.
> *previous* TypeError: Inner.
sage: raise combine_exceptions(ValueError('Outer.'),
                               TypeError('Inner1.'), TypeError('Inner2.'))
Traceback (most recent call last):
ValueError: Outer.
> *previous* TypeError: Inner1.
> *and* TypeError: Inner2.
sage: raise combine_exceptions(ValueError('Outer.'),
                               combine_exceptions(TypeError('Middle.'),
                                                  TypeError('Inner.')))
Traceback (most recent call last):
ValueError: Outer.
```

```
> *previous* TypeError: Middle.
>> *previous* TypeError: Inner.
```

sage.rings.asymptotic.misc. log_string (element, base=None)

Return a representation of the log of the given element to the given base.

INPUT:

- •element an object.
- •base an object or None.

OUTPUT:

A string.

EXAMPLES:

```
sage: from sage.rings.asymptotic.misc import log_string
sage: log_string(3)
'log(3)'
sage: log_string(3, base=42)
'log(3, base=42)'
```

sage.rings.asymptotic.misc.merge_overlapping (A, B, key=None)

Merge the two overlapping tuples/lists.

INPUT:

- $\bullet A$ a list or tuple (type has to coincide with type of B).
- $\bullet B$ a list or tuple (type has to coincide with type of A).
- •key (default: None) a function. If None, then the identity is used. This key-function applied on an element of the list/tuple is used for comparison. Thus elements with the same key are considered as equal.

OUTPUT:

A pair of lists or tuples (depending on the type of A and B).

Note: Suppose we can decompose the list A = ac and B = cb with lists a, b, c, where c is nonempty. Then $merge_overlapping()$ returns the pair (acb, acb).

Suppose a key-function is specified and $A = ac_A$ and $B = c_B b$, where the list of keys of the elements of c_A equals the list of keys of the elements of c_B . Then $merge_overlapping()$ returns the pair $(ac_A b, ac_B b)$.

After unsuccessfully merging A=ac and B=cb, a merge of A=ca and B=bc is tried.

TESTS:

```
sage: from sage.rings.asymptotic.misc import merge_overlapping
sage: def f(L, s):
....:    return list((ell, s) for ell in L)
sage: key = lambda k: k[0]
sage: merge_overlapping(f([0..3], 'a'), f([5..7], 'b'), key)
Traceback (most recent call last):
...
ValueError: Input does not have an overlap.
sage: merge_overlapping(f([0..2], 'a'), f([4..7], 'b'), key)
Traceback (most recent call last):
...
```

```
ValueError: Input does not have an overlap.
sage: merge_overlapping(f([4..7], 'a'), f([0..2], 'b'), key)
Traceback (most recent call last):
ValueError: Input does not have an overlap.
sage: merge_overlapping(f([0..3], 'a'), f([3..4], 'b'), key)
([(0, 'a'), (1, 'a'), (2, 'a'), (3, 'a'), (4, 'b')],
[(0, 'a'), (1, 'a'), (2, 'a'), (3, 'b'), (4, 'b')])
sage: merge_overlapping(f([3..4], 'a'), f([0..3], 'b'), key)
([(0, 'b'), (1, 'b'), (2, 'b'), (3, 'a'), (4, 'a')],
[(0, 'b'), (1, 'b'), (2, 'b'), (3, 'b'), (4, 'a')])
sage: merge_overlapping(f([0..1], 'a'), f([0..4], 'b'), key)
([(0, 'a'), (1, 'a'), (2, 'b'), (3, 'b'), (4, 'b')],
[(0, 'b'), (1, 'b'), (2, 'b'), (3, 'b'), (4, 'b')])
sage: merge_overlapping(f([0..3], 'a'), f([0..1], 'b'), key)
([(0, 'a'), (1, 'a'), (2, 'a'), (3, 'a')],
[(0, 'b'), (1, 'b'), (2, 'a'), (3, 'a')])
sage: merge_overlapping(f([0..3], 'a'), f([1..3], 'b'), key)
([(0, 'a'), (1, 'a'), (2, 'a'), (3, 'a')],
[(0, 'a'), (1, 'b'), (2, 'b'), (3, 'b')])
sage: merge_overlapping(f([1..3], 'a'), f([0..3], 'b'), key)
([(0, 'b'), (1, 'a'), (2, 'a'), (3, 'a')],
[(0, 'b'), (1, 'b'), (2, 'b'), (3, 'b')])
sage: merge_overlapping(f([0..6], 'a'), f([3..4], 'b'), key)
([(0, 'a'), (1, 'a'), (2, 'a'), (3, 'a'), (4, 'a'), (5, 'a'), (6, 'a')],
[(0, 'a'), (1, 'a'), (2, 'a'), (3, 'b'), (4, 'b'), (5, 'a'), (6, 'a')])
sage: merge_overlapping(f([0..3], 'a'), f([1..2], 'b'), key)
([(0, 'a'), (1, 'a'), (2, 'a'), (3, 'a')],
[(0, 'a'), (1, 'b'), (2, 'b'), (3, 'a')])
sage: merge_overlapping(f([1..2], 'a'), f([0..3], 'b'), key)
([(0, 'b'), (1, 'a'), (2, 'a'), (3, 'b')],
[(0, 'b'), (1, 'b'), (2, 'b'), (3, 'b')])
sage: merge_overlapping(f([1..3], 'a'), f([1..3], 'b'), key)
([(1, 'a'), (2, 'a'), (3, 'a')],
[(1, 'b'), (2, 'b'), (3, 'b')])
```

sage.rings.asymptotic.misc.parent_to_repr_short (P)

Helper method which generates a short(er) representation string out of a parent.

INPUT:

•P – a parent.

OUTPUT:

A string.

```
sage: from sage.rings.asymptotic.misc import parent_to_repr_short
sage: parent_to_repr_short(ZZ)
'ZZ'
sage: parent_to_repr_short(QQ)
'QQ'
sage: parent_to_repr_short(SR)
'SR'
sage: parent_to_repr_short(ZZ['x'])
'ZZ[x]'
sage: parent_to_repr_short(QQ['d, k'])
'QQ[d, k]'
```

```
sage: parent_to_repr_short(QQ['e'])
'QQ[e]'
sage: parent_to_repr_short(SR[['a, r']])
'SR[[a, r]]'
sage: parent_to_repr_short(Zmod(3))
'Ring of integers modulo 3'
sage: parent_to_repr_short(Zmod(3)['g'])
'Univariate Polynomial Ring in g over Ring of integers modulo 3'
```

sage.rings.asymptotic.misc.repr_op (left, op, right=None, latex=False)

Create a string left op right with taking care of parentheses in its operands.

INPUT:

- •left an element.
- •op -a string.
- •right an alement.
- •latex (default: False) a boolean. If set, then LaTeX-output is returned.

OUTPUT:

A string.

EXAMPLES:

```
sage: from sage.rings.asymptotic.misc import repr_op
sage: repr_op('a^b', '^', 'c')
'(a^b)^c'
```

TESTS:

```
sage: repr_op('a-b', '^', 'c')
'(a-b)^c'
sage: repr_op('a+b', '^', 'c')
'(a+b)^c'
```

```
sage: print(repr_op(r'\frac{1}{2}', '^', 'c', latex=True))
\left(\frac{1}{2}\right)^c
```

```
sage.rings.asymptotic.misc.repr_short_to_parent (s)
```

Helper method for the growth group factory, which converts a short representation string to a parent.

INPUT:

•s - a string, short representation of a parent.

OUTPUT:

A parent.

The possible short representations are shown in the examples below.

```
sage: from sage.rings.asymptotic.misc import repr_short_to_parent
sage: repr_short_to_parent('ZZ')
Integer Ring
sage: repr_short_to_parent('QQ')
Rational Field
```

```
sage: repr_short_to_parent('SR')
Symbolic Ring
sage: repr_short_to_parent('NN')
Non negative integer semiring
```

TESTS:

```
sage: repr_short_to_parent('abcdef')
Traceback (most recent call last):
...
ValueError: Cannot create a parent out of 'abcdef'.
> *previous* NameError: name 'abcdef' is not defined
```

sage.rings.asymptotic.misc. **split_str_by_op** (*string*, *op*, *strip_parentheses=True*)

Split the given string into a tuple of substrings arising by splitting by op and taking care of parentheses.

INPUT:

- •string -a string.
- •op a string. This is used by str.split. Thus, if this is None, then any whitespace string is a separator and empty strings are removed from the result.
- •strip_parentheses (default: True) a boolean.

OUTPUT:

A tuple of strings.

TESTS:

```
sage: from sage.rings.asymptotic.misc import split_str_by_op
sage: split_str_by_op('x^ZZ', '*')
('x^ZZ',)
sage: split_str_by_op('log(x)^ZZ * y^QQ', '*')
('log(x)^ZZ', 'y^QQ')
sage: split_str_by_op('log(x)**ZZ * y**QQ', '*')
('log(x)**ZZ', 'y**QQ')
sage: split_str_by_op('a^b * c^d', '*')
Traceback (most recent call last):
...
ValueError: 'a^b * c^d' is invalid since a '*' follows a '*'.
sage: split_str_by_op('a^b * (c*d^e)', '*')
('a^b', 'c*d^e')
```

```
sage: split_str_by_op('(a^b)^c', '^')
('a^b', 'c')
sage: split_str_by_op('a^(b^c)', '^')
('a', 'b^c')
```

```
sage: split_str_by_op('(a) + (b)', op='+', strip_parentheses=True)
('a', 'b')
sage: split_str_by_op('(a) + (b)', op='+', strip_parentheses=False)
('(a)', '(b)')
sage: split_str_by_op(' ( t      ) ', op='+', strip_parentheses=False)
('( t     )',)
```

```
sage: split_str_by_op(' ( t ) ', op=None)
('t',)
```

```
sage: split_str_by_op(' ( t )s', op=None)
('(t)s',)
sage: split_str_by_op(' ( t ) s', op=None)
('t', 's')
```

```
sage: split_str_by_op('(e^(n*log(n)))^SR.subring(no_variables=True)', '*')
('(e^(n*log(n)))^SR.subring(no_variables=True)',)
```

sage.rings.asymptotic.misc. substitute_raise_exception (element, e)

Raise an error describing what went wrong with the substitution.

INPUT:

- •element an element.
- •e an exception which is included in the raised error message.

OUTPUT:

Raise an exception of the same type as e.

TESTS:

```
sage: from sage.rings.asymptotic.misc import substitute_raise_exception
sage: substitute_raise_exception(x, Exception('blub'))
Traceback (most recent call last):
...
Exception: Cannot substitute in x in Symbolic Ring.
> *previous* Exception: blub
```

sage.rings.asymptotic.misc.transform_category (category, subcategory_mapping, axiom mapping, initial category=None)

Transform category to a new category according to the given mappings.

INPUT:

- •category a category.
- •subcategory_mapping a list (or other iterable) of triples (from, to, mandatory), where
 - -from and to are categories and
 - -mandatory is a boolean.
- \bullet axiom_mapping -a list (or other iterable) of triples (from, to, mandatory), where
 - -from and to are strings describing axioms and
 - -mandatory is a boolean.
- •initial_category (default: None) a category. When transforming the given category, this initial_category is used as a starting point of the result. This means the resulting category will be a subcategory of initial_category. If initial_category is None, then the category of objects is used.

OUTPUT:

A category.

Note: Consider a subcategory mapping (from, to, mandatory) . If category is a subcategory of from , then the returned category will be a subcategory of to . Otherwise and if mandatory is set, then an error is raised.

Consider an axiom mapping (from, to, mandatory). If category is has axiom from, then the returned category will have axiom to. Otherwise and if mandatory is set, then an error is raised.

```
sage: from sage.rings.asymptotic.misc import transform_category
sage: from sage.categories.additive_semigroups import AdditiveSemigroups
sage: from sage.categories.additive monoids import AdditiveMonoids
sage: from sage.categories.additive_groups import AdditiveGroups
sage: S = [
....: (Sets(), Sets(), True),
        (Posets(), Posets(), False),
. . . . :
         (AdditiveMagmas(), Magmas(), False)]
. . . . :
sage: A = [
....: ('AdditiveAssociative', 'Associative', False),
        ('AdditiveUnital', 'Unital', False),
. . . . :
        ('AdditiveInverse', 'Inverse', False),
        ('AdditiveCommutative', 'Commutative', False)]
sage: transform category(Objects(), S, A)
Traceback (most recent call last):
ValueError: Category of objects is not
a subcategory of Category of sets.
sage: transform_category(Sets(), S, A)
Category of sets
sage: transform_category(Posets(), S, A)
Category of posets
sage: transform_category(AdditiveSemigroups(), S, A)
Category of semigroups
sage: transform_category(AdditiveMonoids(), S, A)
Category of monoids
sage: transform_category(AdditiveGroups(), S, A)
Category of groups
sage: transform_category(AdditiveGroups().AdditiveCommutative(), S, A)
Category of commutative groups
```

```
sage: transform_category(AdditiveGroups().AdditiveCommutative(), S, A,
...: initial_category=Posets())
Join of Category of commutative groups
    and Category of posets
```

```
sage: transform_category(ZZ.category(), S, A)
Category of commutative groups
sage: transform_category(QQ.category(), S, A)
Category of commutative groups
sage: transform_category(SR.category(), S, A)
Category of commutative groups
sage: transform_category(Fields(), S, A)
Category of commutative groups
sage: transform_category(ZZ['t'].category(), S, A)
Category of commutative groups
```

```
sage: A[-1] = ('Commutative', 'AdditiveCommutative', True)
sage: transform_category(Groups(), S, A)
Traceback (most recent call last):
...
ValueError: Category of groups does not have
```

```
axiom Commutative.
```

sage.rings.asymptotic.misc.underlying_class (P)

Return the underlying class (class without the attached categories) of the given instance.

OUTPUT:

A class.

EXAMPLES:

```
sage: from sage.rings.asymptotic.misc import underlying_class
sage: type(QQ)
<class 'sage.rings.rational_field.RationalField_with_category'>
sage: underlying_class(QQ)
<class 'sage.rings.rational_field.RationalField'>
```

4.7 Asymptotics of Multivariate Generating Series

Let $F(x) = \sum_{\nu \in \mathbb{N}^d} F_{\nu} x^{\nu}$ be a multivariate power series with complex coefficients that converges in a neighborhood of the origin. Assume that F = G/H for some functions G and H holomorphic in a neighborhood of the origin. Assume also that H is a polynomial.

This computes asymptotics for the coefficients $F_{r\alpha}$ as $r \to \infty$ with $r\alpha \in \mathbf{N}^d$ for α in a permissible subset of d-tuples of positive reals. More specifically, it computes arbitrary terms of the asymptotic expansion for $F_{r\alpha}$ when the asymptotics are controlled by a strictly minimal multiple point of the algebraic variety H = 0.

The algorithms and formulas implemented here come from [RaWi2008a] and [RaWi2012]. For a general reference take a look in the book [PeWi2013].

4.7.1 Introductory Examples

A univariate smooth point example:

```
sage: R.<x> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: H = (x - 1/2)^3
sage: Hfac = H.factor()
sage: G = -1/(x + 3)/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F
(-1/(x + 3), [(x - 1/2, 3)])
sage: alpha = [1]
sage: decomp = F.asymptotic_decomposition(alpha)
sage: decomp
(0, []) +
(-1/2*(x^2 + 6*x + 9)*r^2/(x^5 + 9*x^4 + 27*x^3 + 27*x^2)
-1/2*(5*x^2 + 24*x + 27)*r/(x^5 + 9*x^4 + 27*x^3 + 27*x^2)
-3*(x^2 + 3*x + 3)/(x^5 + 9*x^4 + 27*x^3 + 27*x^2)
[(x - 1/2, 1)])
sage: F1 = decomp[1]
sage: p = \{x: 1/2\}
```

```
sage: asy = F1.asymptotics(p, alpha, 3)
sage: asy
(8/343*(49*r^2 + 161*r + 114)*2^r, 2, 8/7*r^2 + 184/49*r + 912/343)
sage: F.relative_error(asy[0], alpha, [1, 2, 4, 8, 16], asy[1])
[((1,), 7.55555556, [7.556851312], [-0.0001714971672]),
((2,), 14.74074074, [14.74052478], [0.00001465051901]),
((4,), 35.96502058, [35.96501458], [1.667911934e-7]),
((8,), 105.8425656, [105.8425656], [4.399565380e-11]),
((16,), 355.3119534, [355.3119534], [0.00000000000])]
```

Another smooth point example (Example 5.4 of [RaWi2008a]):

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: q = 1/2
sage: qq = q.denominator()
sage: H = 1 - q*x + q*x*y - x^2*y
sage: Hfac = H.factor()
sage: G = (1 - q*x)/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: alpha = list(qq*vector([2, 1 - q]))
sage: alpha
[4, 1]
sage: I = F.smooth_critical_ideal(alpha)
sage: I
Ideal (v^2 - 2*v + 1, x + 1/4*v - 5/4) of
Multivariate Polynomial Ring in x, y over Rational Field
sage: s = solve([SR(z) for z in I.gens()],
               [SR(z) for z in R.gens()], solution_dict=true)
sage: s == [\{SR(x): 1, SR(y): 1\}]
True
sage: p = s[0]
sage: asy = F.asymptotics(p, alpha, 1, verbose=True)
Creating auxiliary functions...
Computing derivatives of auxiallary functions...
Computing derivatives of more auxiliary functions...
Computing second order differential operator actions...
sage: asy
(1/24*2^{(2/3)}*(sqrt(3) + 4/(sqrt(3) + I) + I)*qamma(1/3)/(pi*r^{(1/3)}),
1,
1/24*2^{(2/3)}*(sqrt(3) + 4/(sqrt(3) + I) + I)*gamma(1/3)/(pi*r^{(1/3)}))
sage: r = SR('r')
sage: tuple((a*r^(1/3)).full_simplify() / r^(1/3) for a in asy) # make nicer_
⇒coefficients
(1/12*sqrt(3)*2^(2/3)*gamma(1/3)/(pi*r^(1/3)),
1/12*sqrt(3)*2^(2/3)*qamma(1/3)/(pi*r^(1/3))
sage: F.relative_error(asy[0], alpha, [1, 2, 4, 8, 16], asy[1])
[((4, 1), 0.1875000000, [0.1953794675...], [-0.042023826...]),
 ((8, 2), 0.1523437500, [0.1550727862...], [-0.017913673...]),
((16, 4), 0.1221771240, [0.1230813519...], [-0.0074009592...]),
 ((32, 8), 0.09739671811, [0.09768973377...], [-0.0030084757...]),
 ((64, 16), 0.07744253816, [0.07753639308...], [-0.0012119297...]))
```

A multiple point example (Example 6.5 of [RaWi2012]):

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
```

```
sage: H = (1 - 2*x - y)**2 * (1 - x - 2*y)**2
sage: Hfac = H.factor()
sage: G = 1/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F
(1, [(x + 2*y - 1, 2), (2*x + y - 1, 2)])
sage: I = F.singular_ideal()
sage: I
Ideal (x - 1/3, y - 1/3) of
Multivariate Polynomial Ring in x, y over Rational Field
sage: p = \{x: 1/3, y: 1/3\}
sage: F.is_convenient_multiple_point(p)
(True, 'convenient in variables [x, y]')
sage: alpha = (var('a'), var('b'))
sage: decomp = F.asymptotic_decomposition(alpha); decomp
(0, []) +
(-1/9*(2*b^2*x^2 - 5*a*b*x*y + 2*a^2*y^2)*r^2/(x^2*y^2)
 -1/9*(6*b*x^2 - 5*(a + b)*x*y + 6*a*y^2)*r/(x^2*y^2)
 -1/9*(4*x^2 - 5*x*y + 4*y^2)/(x^2*y^2),
[(x + 2*y - 1, 1), (2*x + y - 1, 1)])
sage: F1 = decomp[1]
sage: F1.asymptotics(p, alpha, 2)
(-3*((2*a^2 - 5*a*b + 2*b^2)*r^2 + (a + b)*r + 3)*((1/3)^(-a)*(1/3)^(-b))^r
(1/3)^{(-a)}*(1/3)^{(-b)}, -3*(2*a^2 - 5*a*b + 2*b^2)*r^2 - 3*(a + b)*r - 9)
sage: alpha = [4, 3]
sage: decomp = F.asymptotic_decomposition(alpha)
sage: F1 = decomp[1]
sage: asy = F1.asymptotics(p, alpha, 2)
sage: asy
(3*(10*r^2 - 7*r - 3)*2187^r, 2187, 30*r^2 - 21*r - 9)
sage: F.relative_error(asy[0], alpha, [1, 2, 4, 8], asy[1])
[((4, 3), 30.72702332, [0.000000000], [1.00000000]),
((8, 6), 111.9315678, [69.00000000], [0.3835519207]),
((16, 12), 442.7813138, [387.0000000], [0.1259793763]),
((32, 24), 1799.879232, [1743.000000], [0.03160169385])]
```

TESTS:

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: H = (1 - 2*x - y) * (1 - x - 2*y)
sage: G = 1
sage: Hfac = H.factor()
sage: G = G / Hfac.unit()
sage: F = FFPD(G, Hfac); F
(1, [(x + 2*y - 1, 1), (2*x + y - 1, 1)])
sage: p = {x: 1, y: 1}
sage: alpha = [1, 1]
sage: F.asymptotics(p, alpha, 1)
(1/3, 1, 1/3)
```

```
sage: R.<x,y,t> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: H = (1 - y) * (1 + x^2) * (1 - t*(1 + x^2 + x*y^2))
sage: G = (1 + x) * (1 + x^2 - x*y^2)
sage: Hfac = H.factor()
sage: G = G / Hfac.unit()
sage: F = FFPD(G, Hfac); F
```

```
(-x^2*y^2 + x^3 - x*y^2 + x^2 + x + 1,
  [(y - 1, 1), (x^2 + 1, 1), (x*y^2*t + x^2*t + t - 1, 1)])
sage: p = {x: 1, y: 1, t: 1/3}
sage: alpha = [1, 1, 1]
sage: F.asymptotics_multiple(p, alpha, 1, var('r')) # not tested - see #19989
```

4.7.2 Various

AUTHORS:

- Alexander Raichev (2008)
- Daniel Krenn (2014, 2016)

4.7.3 Classes and Methods

class sage.rings.asymptotic.asymptotics_multivariate_generating_functions. FractionWithFactor

Bases: sage.structure.element.RingElement

This element represents a fraction with a factored polynomial denominator. See also its parent FractionWithFactoredDenominatorRing for details.

Represents a fraction with factored polynomial denominator (FFPD) $p/(q_1^{e_1}\cdots q_n^{e_n})$ by storing the parts p and $[(q_1,e_1),\ldots,(q_n,e_n)]$. Here q_1,\ldots,q_n are elements of a 0- or multi-variate factorial polynomial ring R, q_1,\ldots,q_n are distinct irreducible elements of R, e_1,\ldots,e_n are positive integers, and p is a function of the indeterminates of R (e.g., a Sage symbolic expression). An element r with no polynomial denominator is represented as (r,[]).

INPUT:

- •numerator an element p; this can be of any ring from which parent's base has coercion in
- •denominator_factored a list of the form $[(q_1, e_1), \ldots, (q_n, e_n)]$, where the q_1, \ldots, q_n are distinct irreducible elements of R and the e_i are positive integers
- •reduce (optional) if True, then represent $p/(q_1^{e_1}\cdots q_n^{e_n})$ in lowest terms, otherwise this won't attempt to divide p by any of the q_i

OUTPUT:

An element representing the rational expression $p/(q_1^{e_1}\cdots q_n^{e_n})$.

```
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: df = [x, 1], [y, 1], [x*y+1, 1]
sage: f = FFPD(x, df)
sage: f
(1, [(y, 1), (x*y + 1, 1)])
sage: ff = FFPD(x, df, reduce=False)
sage: ff
(x, [(y, 1), (x, 1), (x*y + 1, 1)])
sage: f = FFPD(x + y, [(x + y, 1)])
sage: f
(1, [])
```

```
sage: R.<x> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = 5*x^3 + 1/x + 1/(x-1) + 1/(3*x^2 + 1)
sage: FFPD(f)
(5*x^7 - 5*x^6 + 5/3*x^5 - 5/3*x^4 + 2*x^3 - 2/3*x^2 + 1/3*x - 1/3,
[(x - 1, 1), (x, 1), (x^2 + 1/3, 1)])
```

```
sage: R. \langle x, y \rangle = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: f = 2*y/(5*(x^3 - 1)*(y + 1))
sage: FFPD(f)
(2/5*y, [(y + 1, 1), (x - 1, 1), (x^2 + x + 1, 1)])
sage: p = 1/x^2
sage: q = 3*x**2*y
sage: qs = q.factor()
sage: f = FFPD(p/qs.unit(), qs)
sage: f
(1/3/x^2, [(y, 1), (x, 2)])
sage: f = FFPD(\cos(x) *x*y^2, [(x, 2), (y, 1)])
sage: f
(x*y^2*cos(x), [(y, 1), (x, 2)])
sage: G = \exp(x + y)
sage: H = (1 - 2*x - y) * (1 - x - 2*y)
sage: a = FFPD(G/H)
sage: a
(e^{(x + y)}, [(x + 2*y - 1, 1), (2*x + y - 1, 1)])
sage: a.denominator_ring
Multivariate Polynomial Ring in x, y over Rational Field
sage: b = FFPD(G, H.factor())
sage: b
(e^{(x + y)}, [(x + 2*y - 1, 1), (2*x + y - 1, 1)])
sage: b.denominator_ring
Multivariate Polynomial Ring in x, y over Rational Field
```

Singular throws a 'not implemented' error when trying to factor in a multivariate polynomial ring over an inexact field:

```
sage: R.<x,y> = PolynomialRing(CC)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = (x + 1) / (x*y*(x*y + 1)^2)
sage: FFPD(f)
```

```
Traceback (most recent call last):
...

TypeError: Singular error:
   ? not implemented
   ? error occurred in or before STDIN line ...:
   `def sage...=factorize(sage...);`
```

AUTHORS:

- •Alexander Raichev (2012-07-26)
- •Daniel Krenn (2014-12-01)

algebraic_dependence_certificate ()

Return the algebraic dependence certificate of self.

The algebraic dependence certificate is the ideal J of annihilating polynomials for the set of polynomials [q^e for (q,e) in self.denominator_factored()], which could be the zero ideal. The ideal J lies in a polynomial ring over the field self.denominator_ring.base_ring() that has $m = len(self.denominator_factored())$ indeterminates.

OUTPUT:

An ideal.

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating

→functions import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = 1/(x^2 * (x*y + 1) * y^3)
sage: ff = FFPD(f)
sage: J = ff.algebraic_dependence_certificate(); J
Ideal (1 - 6*T2 + 15*T2^2 - 20*T2^3 + 15*T2^4 - T0^2*T1^3 -
6*T2^5 + T2^6) of Multivariate Polynomial Ring in
T0, T1, T2 over Rational Field
sage: g = J.gens()[0]
sage: df = ff.denominator_factored()
sage: g(*(q**e for q, e in df)) == 0
True
```

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: G = exp(x + y)
sage: H = x^2 * (x*y + 1) * y^3
sage: ff = FFPD(G, H.factor())
sage: J = ff.algebraic_dependence_certificate(); J
Ideal (1 - 6*T2 + 15*T2^2 - 20*T2^3 + 15*T2^4 - T0^2*T1^3 -
6*T2^5 + T2^6) of Multivariate Polynomial Ring in
T0, T1, T2 over Rational Field
sage: g = J.gens()[0]
sage: df = ff.denominator_factored()
sage: g(*(q**e for q, e in df)) == 0
True
```

```
sage: f = 1/(x^3 * y^2)
sage: J = FFPD(f).algebraic_dependence_certificate()
```

```
sage: J
Ideal (0) of Multivariate Polynomial Ring in TO, T1 over Rational Field
```

```
sage: f = sin(1)/(x^3 * y^2)
sage: J = FFPD(f).algebraic_dependence_certificate()
sage: J
Ideal (0) of Multivariate Polynomial Ring in TO, T1 over Rational Field
```

algebraic_dependence_decomposition (whole_and_parts=True)

Return an algebraic dependence decomposition of self.

Let f = p/q where q lies in a d-variate polynomial ring K[X] for some field K. Let $q_1^{e_1} \cdots q_n^{e_n}$ be the unique factorization of q in K[X] into irreducible factors and let V_i be the algebraic variety $\{x \in L^d \mid q_i(x) = 0\}$ of q_i over the algebraic closure L of K. By [Raic2012], f can be written as

$$(*) \quad \sum_{A} \frac{p_A}{\prod_{i \in A} q_i^{b_i}},$$

where the b_i are positive integers, each p_A is a products of p and an element in K[X], and the sum is taken over all subsets $A \subseteq \{1, \ldots, m\}$ such that $|A| \le d$ and $\{q_i \mid i \in A\}$ is algebraically independent.

We call (*) an algebraic dependence decomposition of f. Algebraic dependence decompositions are not unique.

The algorithm used comes from [Raic2012].

OUTPUT:

An instance of FractionWithFactoredDenominatorSum.

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_
→ functions import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = 1/(x^2 * (x*y + 1) * y^3)
sage: ff = FFPD(f)
sage: decomp = ff.algebraic_dependence_decomposition()
sage: decomp
(0, []) + (-x, [(x*y + 1, 1)]) +
(x^2*y^2 - x*y + 1, [(y, 3), (x, 2)])
sage: decomp.sum().guotient() == f
True
sage: for r in decomp:
      J = r.algebraic_dependence_certificate()
         J is None or J == J.ring().ideal() # The zero ideal
True
True
True
```

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: G = sin(x)
sage: H = x^2 * (x*y + 1) * y^3
sage: f = FFPD(G, H.factor())
sage: decomp = f.algebraic_dependence_decomposition()
sage: decomp
(0, []) + (x^4*y^3*sin(x), [(x*y + 1, 1)]) +
```

asymptotic_decomposition (alpha, asy_var=None)

Return the asymptotic decomposition of self.

The asymptotic decomposition of F is a sum that has the same asymptotic expansion as f in the direction alpha but each summand has a denominator factorization of the form $[(q_1, 1), \ldots, (q_n, 1)]$, where n is at most the dimension() of F.

INPUT:

- •alpha a *d*-tuple of positive integers or symbolic variables
- •asy_var (default: None) a symbolic variable with respect to which to compute asymptotics; if None is given, we set asy_var = var('r')

OUTPUT:

An instance of FractionWithFactoredDenominatorSum.

The output results from a Leinartas decomposition followed by a cohomology decomposition.

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: H = (1 - 2*x -y)*(1 - x -2*y)**2
sage: Hfac = H.factor()
sage: G = 1/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: alpha = var('a, b')
sage: F.asymptotic_decomposition(alpha)
(0, []) +
(1/3*(2*b*x - a*y)*r/(x*y) + 1/3*(2*x - y)/(x*y),
  [(x + 2*y - 1, 1), (2*x + y - 1, 1)])
```

asymptotics (p, alpha, N, asy_var=None, numerical=0, verbose=False)

Return the asymptotics in the given direction.

This function returns the first N terms (some of which could be zero) of the asymptotic expansion of the Maclaurin ray coefficients $F_{r\alpha}$ of the function F represented by self as $r \to \infty$, where r is asy_var and alpha is a tuple of positive integers of length d which is self.dimension(). Assume that

- $\bullet F$ is holomorphic in a neighborhood of the origin;
- •the unique factorization of the denominator H of F in the local algebraic ring at p equals its unique factorization in the local analytic ring at p;
- •the unique factorization of H in the local algebraic ring at p has at most d irreducible factors, none of which are repeated (one can reduce to this case via $asymptotic_decomposition()$);
- $\bullet p$ is a convenient strictly minimal smooth or multiple point with all nonzero coordinates that is critical and nondegenerate for alpha.

The algorithms used here come from [RaWi2008a] and [RaWi2012].

INPUT:

- •p a dictionary with keys that can be coerced to equal self.denominator_ring.gens()
- •alpha a tuple of length self.dimension() of positive integers or, if p is a smooth point, possibly of symbolic variables
- •N a positive integer
- •asy_var (default: None) a symbolic variable for the asymptotic expansion; if none is given, then var('r') will be assigned
- •numerical (default: 0) a natural number; if numerical is greater than 0, then return a numerical approximation of $F_{r\alpha}$ with numerical digits of precision; otherwise return exact values
- •verbose (default: False) print the current state of the algorithm

OUTPUT:

The tuple (asy, exp_scale, subexp_part). Here asy is the sum of the first N terms (some of which might be 0) of the asymptotic expansion of $F_{r\alpha}$ as $r \to \infty$; exp_scale**r is the exponential factor of asy; subexp_part is the subexponential factor of asy.

EXAMPLES:

A smooth point example:

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: H = (1 - x - y - x*y)**2
sage: Hfac = H.factor()
sage: G = 1/Hfac.unit()
sage: F = FFPD(G, Hfac); print(F)
(1, [(x*y + x + y - 1, 2)])
sage: alpha = [4, 3]
sage: decomp = F.asymptotic_decomposition(alpha); decomp
(0, []) + (-3/2*r*(y + 1)/y - 1/2*(y + 1)/y, [(x*y + x + y - 1, 1)])
sage: F1 = decomp[1]
sage: p = \{y: 1/3, x: 1/2\}
sage: asy = F1.asymptotics(p, alpha, 2, verbose=True)
Creating auxiliary functions...
Computing derivatives of auxiallary functions...
Computing derivatives of more auxiliary functions...
```

```
Computing second order differential operator actions...

sage: asy

(1/6000*(3600*sqrt(5)*sqrt(3)*sqrt(2)*sqrt(r)/sqrt(pi)

+ 463*sqrt(5)*sqrt(3)*sqrt(2)/(sqrt(pi)*sqrt(r)))*432^r,

432,

3/5*sqrt(5)*sqrt(3)*sqrt(2)*sqrt(r)/sqrt(pi)

+ 463/6000*sqrt(5)*sqrt(3)*sqrt(2)/(sqrt(pi)*sqrt(r)))

sage: F.relative_error(asy[0], alpha, [1, 2, 4, 8, 16], asy[1])

[((4, 3), 2.083333333, [2.092576110], [-0.0044365330...]),

((8, 6), 2.787374614, [2.790732875], [-0.0012048112...]),

((16, 12), 3.826259447, [3.827462310], [-0.0003143703...]),

((32, 24), 5.328112821, [5.328540787], [-0.0000803222...]),

((64, 48), 7.475927885, [7.476079664], [-0.0000203023...])]
```

A multiple point example:

```
sage: R.<x,y,z>= PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: H = (4 - 2 \times x - y - z) \times 2 \times (4 - x - 2 \times y - z)
sage: Hfac = H.factor()
sage: G = 16/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F
(-16, [(x + 2*y + z - 4, 1), (2*x + y + z - 4, 2)])
sage: alpha = [3, 3, 2]
sage: decomp = F.asymptotic_decomposition(alpha); decomp
(0, []) +
(16*r*(4*y - 3*z)/(y*z) + 16*(2*y - z)/(y*z),
[(x + 2*y + z - 4, 1), (2*x + y + z - 4, 1)])
sage: F1 = decomp[1]
sage: p = \{x: 1, y: 1, z: 1\}
sage: asy = F1.asymptotics(p, alpha, 2, verbose=True) # long time
Creating auxiliary functions...
Computing derivatives of auxiliary functions...
Computing derivatives of more auxiliary functions...
Computing second-order differential operator actions...
sage: asy # long time
(4/3*sqrt(3)*sqrt(r)/sqrt(pi) + 47/216*sqrt(3)/(sqrt(pi)*sqrt(r)),
1, 4/3*sqrt(3)*sqrt(r)/sqrt(pi) + 47/216*sqrt(3)/(sqrt(pi)*sqrt(r)))
sage: F.relative_error(asy[0], alpha, [1, 2, 4, 8], asy[1]) # long time
[((3, 3, 2), 0.9812164307, [1.515572606], [-0.54458543...]),
((6, 6, 4), 1.576181132, [1.992989399], [-0.26444185...]),
 ((12, 12, 8), 2.485286378, [2.712196351], [-0.091301338...]),
 ((24, 24, 16), 3.700576827, [3.760447895], [-0.016178847...]))
```

asymptotics_multiple (p, alpha, N, asy_var , coordinate=None, numerical=0, verbose=False) Return the asymptotics in the given direction of a multiple point nondegenerate for alpha.

This is the same as <code>asymptotics()</code>, but only in the case of a convenient multiple point nondegenerate for alpha. Assume also that <code>self.dimension >= 2</code> and that the <code>p.values()</code> are not symbolic variables.

The formulas used for computing the asymptotic expansion are Theorem 3.4 and Theorem 3.7 of [RaWi2012].

INPUT:

•p - a dictionary with keys that can be coerced to equal self.denominator_ring.gens()

- •alpha a tuple of length d = self.dimension() of positive integers or, if p is a smooth point, possibly of symbolic variables
- •N a positive integer
- •asy_var -(optional; default: None) a symbolic variable; the variable of the asymptotic expansion, if none is given, var('r') will be assigned
- •coordinate (optional; default: None) an integer in $\{0, \ldots, d-1\}$ indicating a convenient coordinate to base the asymptotic calculations on; if None is assigned, then choose coordinate=d-1
- •numerical (optional; default: 0) a natural number; if numerical is greater than 0, then return a numerical approximation of the Maclaurin ray coefficients of self with numerical digits of precision; otherwise return exact values
- •verbose (default: False) print the current state of the algorithm

OUTPUT:

The asymptotic expansion.

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_
→ functions import FractionWithFactoredDenominatorRing
sage: R.<x,y,z>= PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: H = (4 - 2*x - y - z)*(4 - x -2*y - z)
sage: Hfac = H.factor()
sage: G = 16/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F
(16, [(x + 2*y + z - 4, 1), (2*x + y + z - 4, 1)])
sage: p = \{x: 1, y: 1, z: 1\}
sage: alpha = [3, 3, 2]
sage: F.asymptotics_multiple(p, alpha, 2, var('r'), verbose=True) # long time
Creating auxiliary functions...
Computing derivatives of auxiliary functions...
Computing derivatives of more auxiliary functions...
Computing second-order differential operator actions...
(4/3*sqrt(3)/(sqrt(pi)*sqrt(r)) - 25/216*sqrt(3)/(sqrt(pi)*r^(3/2)),
1,
4/3*sqrt(3)/(sqrt(pi)*sqrt(r)) - 25/216*sqrt(3)/(sqrt(pi)*r^(3/2)))
sage: H = (1 - x*(1 + y))*(1 - z*x**2*(1 + 2*y))
sage: Hfac = H.factor()
sage: G = 1/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F
(1, [(x*y + x - 1, 1), (2*x^2*y*z + x^2*z - 1, 1)])
sage: p = \{x: 1/2, z: 4/3, y: 1\}
sage: alpha = [8, 3, 3]
sage: F.asymptotics_multiple(p, alpha, 2, var('r'), coordinate=1,...
→verbose=True) # long time
Creating auxiliary functions...
Computing derivatives of auxiliary functions...
Computing derivatives of more auxiliary functions...
Computing second-order differential operator actions...
(1/172872*108^r*(24696*sqrt(7)*sqrt(3)/(sqrt(pi)*sqrt(r))
  -1231*sqrt(7)*sqrt(3)/(sqrt(pi)*r^(3/2))),
108,
```

```
1/7*sqrt(7)*sqrt(3)/(sqrt(pi)*sqrt(r))
- 1231/172872*sqrt(7)*sqrt(3)/(sqrt(pi)*r^(3/2)))
```

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: H = (1 - 2*x - y) * (1 - x - 2*y)
sage: Hfac = H.factor()
sage: G = exp(x + y)/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F
(e^(x + y), [(x + 2*y - 1, 1), (2*x + y - 1, 1)])
sage: p = {x: 1/3, y: 1/3}
sage: alpha = (var('a'), var('b'))
sage: F.asymptotics_multiple(p, alpha, 2, var('r')) # long time
(3*((1/3)^(-a)*(1/3)^(-b))^r*e^((2/3), (1/3)^(-a)*(1/3)^(-b), 3*e^((2/3))
```

asymptotics_smooth (p, alpha, N, asy_var, coordinate=None, numerical=0, verbose=False) Return the asymptotics in the given direction of a smooth point.

This is the same as asymptotics (), but only in the case of a convenient smooth point.

The formulas used for computing the asymptotic expansions are Theorems 3.2 and 3.3 [RaWi2008a] with the exponent of H equal to 1. Theorem 3.2 is a specialization of Theorem 3.4 of [RaWi2012] with n = 1.

INPUT:

- •p a dictionary with keys that can be coerced to equal self.denominator_ring.gens()
- •alpha a tuple of length d = self.dimension() of positive integers or, if p is a smooth point, possibly of symbolic variables
- •N a positive integer
- •asy_var -(optional; default: None) a symbolic variable; the variable of the asymptotic expansion, if none is given, var('r') will be assigned
- •coordinate (optional; default: None) an integer in $\{0,\ldots,d-1\}$ indicating a convenient coordinate to base the asymptotic calculations on; if None is assigned, then choose coordinate=d-1
- •numerical (optional; default: 0) a natural number; if numerical is greater than 0, then return a numerical approximation of the Maclaurin ray coefficients of self with numerical digits of precision; otherwise return exact values
- •verbose (default: False) print the current state of the algorithm

OUTPUT:

The asymptotic expansion.

```
sage: p = {x: 2/3}
sage: asy = F.asymptotics_smooth(p, alpha, 3, asy_var=var('r'))
sage: asy
(1/2*(9/4)^r, 9/4, 1/2)
```

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: H = 1-x-y-x*y
sage: Hfac = H.factor()
sage: G = 1/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: alpha = [3, 2]
sage: p = \{y: 1/2*sqrt(13) - 3/2, x: 1/3*sqrt(13) - 2/3\}
sage: F.asymptotics_smooth(p, alpha, 2, var('r'), numerical=3, verbose=True)
Creating auxiliary functions...
Computing derivatives of auxiallary functions...
Computing derivatives of more auxiliary functions...
Computing second order differential operator actions...
(71.2^r*(0.369/sqrt(r) - 0.018.../r^(3/2)), 71.2, 0.369/sqrt(r) - 0.018.../r^(3/2))
\hookrightarrow (3/2))
sage: q = 1/2
sage: qq = q.denominator()
sage: H = 1 - q*x + q*x*y - x^2*y
sage: Hfac = H.factor()
sage: G = (1 - q*x)/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: alpha = list(qq*vector([2, 1 - q]))
sage: alpha
[4, 1]
sage: p = \{x: 1, y: 1\}
sage: F.asymptotics_smooth(p, alpha, 5, var('r'), verbose=True) # not tested_
\hookrightarrow (140 seconds)
Creating auxiliary functions...
Computing derivatives of auxiallary functions...
Computing derivatives of more auxiliary functions...
Computing second order differential operator actions...
(1/12*sqrt(3)*2^(2/3)*gamma(1/3)/(pi*r^(1/3))
 -1/96*sqrt(3)*2^(1/3)*gamma(2/3)/(pi*r^(5/3)),
1,
 1/12*sqrt(3)*2^(2/3)*gamma(1/3)/(pi*r^(1/3))
  -1/96*sqrt(3)*2^(1/3)*qamma(2/3)/(pi*r^(5/3))
```

cohomology_decomposition ()

Return the cohomology decomposition of self.

Let $p/(q_1^{e_1}\cdots q_n^{e_n})$ be the fraction represented by self and let $K[x_1,\ldots,x_d]$ be the polynomial ring in which the q_i lie. Assume that $n\leq d$ and that the gradients of the q_i are linearly independent at all points in the intersection $V_1\cap\ldots\cap V_n$ of the algebraic varieties $V_i=\{x\in L^d\mid q_i(x)=0\}$, where L is the algebraic closure of the field K. Return a FractionWithFactoredDenominatorSum f such that the differential form $fdx_1\wedge\cdots\wedge dx_d$ is de Rham cohomologous to the differential form $p/(q_1^{e_1}\cdots q_n^{e_n})dx_1\wedge\cdots\wedge dx_d$ and such that the denominator of each summand of f contains no repeated irreducible factors.

The algorithm used here comes from the proof of Theorem 17.4 of [AiYu1983].

OUTPUT:

An instance of FractionWithFactoredDenominatorSum.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_
→ functions import FractionWithFactoredDenominatorRing
sage: R.<x> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = 1/(x^2 + x + 1)^3
sage: decomp = FFPD(f).cohomology_decomposition()
sage: decomp
(0, []) + (2/3, [(x^2 + x + 1, 1)])
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: FFPD(1, [(x, 1), (y, 2)]).cohomology_decomposition()
(0, [])
sage: p = 1
sage: qs = [(x*y - 1, 1), (x**2 + y**2 - 1, 2)]
sage: f = FFPD(p, qs)
sage: f.cohomology_decomposition()
(0, []) + (4/3*x*y + 4/3, [(x^2 + y^2 - 1, 1)]) +
(1/3, [(x*y - 1, 1), (x^2 + y^2 - 1, 1)])
```

critical_cone (p, coordinate=None)

Return the critical cone of the convenient multiple point p .

INPUT:

 \bullet p — a dictionary with keys that can be coerced to equal self.denominator_ring.gens() and values in a field

•coordinate - (optional; default: None) a natural number

OUTPUT:

A list of vectors.

This list of vectors generate the critical cone of p and the cone itself, which is None if the values of p don't lie in \mathbf{Q} . Divide logarithmic gradients by their component coordinate entries. If coordinate = None, then search from d-1 down to 0 for the first index j such that for all i we have self.log_grads()[i][j] != 0 and set coordinate = j.

EXAMPLES:

denominator()

Return the denominator of self.

OUTPUT:

The denominator (i.e., the product of the factored denominator).

EXAMPLES:

denominator_factored ()

Return the factorization in self.denominator_ring of the denominator of self but without the unit part.

OUTPUT:

The factored denominator as a list of tuple (f, m), where f is a factor and m its multiplicity.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_
    →functions import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: H = (1 - x - y - x*y)**2*(1-x)
sage: Hfac = H.factor()
sage: G = exp(y)/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F.denominator_factored()
[(x - 1, 1), (x*y + x + y - 1, 2)]
```

denominator_ring

Return the ring of the denominator.

OUTPUT:

A ring.

```
sage: F.denominator_ring
Multivariate Polynomial Ring in x, y over Rational Field
```

dimension ()

Return the number of indeterminates of self.denominator_ring.

OUTPUT:

An integer.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_
    →functions import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: H = (1 - x - y - x*y)**2*(1-x)
sage: Hfac = H.factor()
sage: G = exp(y)/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F.dimension()
```

grads(p)

Return a list of the gradients of the polynomials [q for (q,e) in self.denominator_factored()] evaluated at p.

INPUT:

•p - (optional; default: None) a dictionary whose keys are the generators of self.denominator_ring

OUTPUT:

A list.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_
→ functions import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: p = exp(x)
sage: df = [(x^3 + 3*y^2, 5), (x*y, 2), (y, 1)]
sage: f = FFPD(p, df)
sage: f
(e^x, [(y, 1), (x*y, 2), (x^3 + 3*y^2, 5)])
sage: R.gens()
(x, y)
sage: p = None
sage: f.grads(p)
[(0, 1), (y, x), (3*x^2, 6*y)]
sage: p = {x: sqrt(2), y: var('a')}
sage: f.grads(p)
[(0, 1), (a, sqrt(2)), (6, 6*a)]
```

is_convenient_multiple_point (p)

Tests if p is a convenient multiple point of self.

In case p is a convenient multiple point, verdict = True and comment is a string stating which variables it's convenient to use. In case p is not, verdict = False and comment is a string explaining why p fails to be a convenient multiple point.

See [RaWi2012] for more details.

INPUT:

•p - a dictionary with keys that can be coerced to equal self.denominator_ring.gens()

OUTPUT:

A pair (verdict, comment).

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating

import FractionWithFactoredDenominatorRing
sage: R. ⟨x, y, z⟩ = PolynomialRing (QQ)
sage: FFPD = FractionWithFactoredDenominatorRing (R)
sage: H = (1 - x*(1 + y)) * (1 - z*x**2*(1 + 2*y))
sage: df = H.factor()
sage: G = 1 / df.unit()
sage: F = FFPD(G, df)
sage: p1 = {x: 1/2, y: 1, z: 4/3}
sage: p2 = {x: 1, y: 2, z: 1/2}
sage: F.is_convenient_multiple_point(p1)
(True, 'convenient in variables [x, y]')
sage: F.is_convenient_multiple_point(p2)
(False, 'not a singular point')
```

leinartas decomposition ()

Return a Leinartas decomposition of self.

Let f = p/q where q lies in a d-variate polynomial ring K[X] for some field K. Let $q_1^{e_1} \cdots q_n^{e_n}$ be the unique factorization of q in K[X] into irreducible factors and let V_i be the algebraic variety $\{x \in L^d \mid q_i(x) = 0\}$ of q_i over the algebraic closure L of K. By [Raic2012], f can be written as

$$(*) \quad \sum_{A} \frac{p_A}{\prod_{i \in A} q_i^{b_i}},$$

where the b_i are positive integers, each p_A is a product of p and an element of K[X], and the sum is taken over all subsets $A \subseteq \{1, \dots, m\}$ such that

```
1.|A| < d,
```

$$2.\bigcap_{i\in A}T_i\neq\emptyset$$
, and

 $3.\{q_i \mid i \in A\}$ is algebraically independent.

In particular, any rational expression in d variables can be represented as a sum of rational expressions whose denominators each contain at most d distinct irreducible factors.

We call (*) a *Leinartas decomposition* of f. Leinartas decompositions are not unique.

The algorithm used comes from [Raic2012].

OUTPUT:

An instance of FractionWithFactoredDenominatorSum.

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = 1/x + 1/y + 1/(x*y + 1)
sage: decomp = FFPD(f).leinartas_decomposition()
sage: decomp
(0, []) + (1, [(x*y + 1, 1)]) + (x + y, [(y, 1), (x, 1)])
sage: decomp.sum().quotient() == f
True
sage: def check_decomp(r):
...: L = r.nullstellensatz_certificate()
...: J = r.algebraic_dependence_certificate()
...: return L is None and (J is None or J == J.ring().ideal())
sage: all(check_decomp(r) for r in decomp)
True
```

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: f = \sin(x)/x + 1/y + 1/(x*y + 1)
sage: G = f.numerator()
sage: H = R(f.denominator())
sage: ff = FFPD(G, H.factor())
sage: decomp = ff.leinartas_decomposition()
sage: decomp
(0, []) +
(-(x*y^2*sin(x) + x^2*y + x*y + y*sin(x) + x)*y, [(y, 1)]) +
((x*y^2*sin(x) + x^2*y + x*y + y*sin(x) + x)*x*y, [(x*y + 1, 1)]) +
(x*y^2*sin(x) + x^2*y + x*y + y*sin(x) + x, [(y, 1), (x, 1)])
sage: bool(decomp.sum().quotient() == f)
True
sage: all(check_decomp(r) for r in decomp)
True
```

```
sage: R.<x,y,z>= PolynomialRing(GF(2, 'a'))
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = 1/(x * y * z * (x*y + z))
sage: decomp = FFPD(f).leinartas_decomposition()
sage: decomp
(0, []) + (1, [(z, 2), (x*y + z, 1)]) +
(1, [(z, 2), (y, 1), (x, 1)])
sage: decomp.sum().quotient() == f
True
```

$log_grads(p)$

Return a list of the logarithmic gradients of the polynomials $[q \text{ for } (q,e) \text{ in self.denominator}_factored()]$ evalutated at p.

The logarithmic gradient of a function f at point p is the vector $(x_1\partial_1 f(x), \dots, x_d\partial_d f(x))$ evaluated at p. INPUT:

•p - (optional; default: None) a dictionary whose keys are the generators of self.denominator_ring

OUTPUT:

A list.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_
→ functions import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: p = exp(x)
sage: df = [(x^3 + 3*y^2, 5), (x*y, 2), (y, 1)]
sage: f = FFPD(p, df)
sage: f
(e^x, [(y, 1), (x*y, 2), (x^3 + 3*y^2, 5)])
sage: R.gens()
(x, y)
sage: p = None
sage: f.log_grads(p)
[(0, y), (x*y, x*y), (3*x^3, 6*y^2)]
sage: p = \{x: sqrt(2), y: var('a')\}
sage: f.log_grads(p)
[(0, a), (sqrt(2)*a, sqrt(2)*a), (6*sqrt(2), 6*a^2)]
```

maclaurin_coefficients (multi_indices, numerical=0)

Return the Maclaurin coefficients of self with given multi_indices.

INPUT:

•multi_indices - a list of tuples of positive integers, where each tuple has length self.dimension()

•numerical – (optional; default: 0) a natural number; if positive, return numerical approximations of coefficients with numerical digits of accuracy

OUTPUT:

A dictionary whose value of the key nu are the Maclaurin coefficient of index nu of self.

Note: Uses iterated univariate Maclaurin expansions. Slow.

```
(-1/3, [(x - 2/3, 1)])
sage: F.maclaurin_coefficients([(2*k,) for k in range(6)])
{(0,): 1/2,
   (2,): 9/8,
   (4,): 81/32,
   (6,): 729/128,
   (8,): 6561/512,
   (10,): 59049/2048}
```

```
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: H = (4 - 2*x - y - z) * (4 - x - 2*y - z)
sage: Hfac = H.factor()
sage: G = 16 / Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: alpha = vector([3, 3, 2])
sage: interval = [1, 2, 4]
sage: S = [r*alpha for r in interval]
sage: F.maclaurin_coefficients(S, numerical=10)
{(3, 3, 2): 0.7849731445,
    (6, 6, 4): 0.7005249476,
    (12, 12, 8): 0.5847732654}
```

nullstellensatz_certificate()

Return a Nullstellensatz certificate of self if it exists.

Let $[(q_1,e_1),\ldots,(q_n,e_n)]$ be the denominator factorization of self. The Nullstellensatz certificate is a list of polynomials h_1,\ldots,h_m in self.denominator_ring that satisfies $h_1q_1+\cdots+h_mq_n=1$ if it exists.

Note: Only works for multivariate base rings.

OUTPUT:

A list of polynomials or None if no Nullstellensatz certificate exists.

```
sage: f = 1/(x*y)
sage: L = FFPD(f).nullstellensatz_certificate()
sage: L is None
True
```

nullstellensatz decomposition ()

Return a Nullstellensatz decomposition of self.

Let f = p/q where q lies in a d-variate polynomial ring K[X] for some field K and $d \ge 1$. Let $q_1^{e_1} \cdots q_n^{e_n}$ be the unique factorization of q in K[X] into irreducible factors and let V_i be the algebraic variety $\{x \in L^d \mid q_i(x) = 0\}$ of q_i over the algebraic closure L of K. By [Raic2012], f can be written as

$$(*) \quad \sum_{A} \frac{p_A}{\prod_{i \in A} q_i^{e_i}},$$

where the p_A are products of p and elements in K[X] and the sum is taken over all subsets $A \subseteq \{1, \dots, m\}$ such that $\bigcap_{i \in A} T_i \neq \emptyset$.

We call (*) a Nullstellensatz decomposition of f. Nullstellensatz decompositions are not unique.

The algorithm used comes from [Raic2012].

Note: Recursive. Only works for multivariate self.

OUTPUT:

An instance of FractionWithFactoredDenominatorSum.

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: G = sin(y)
sage: H = x*(x*y + 1)
sage: f = FFPD(G, H.factor())
sage: decomp = f.nullstellensatz_decomposition()
sage: decomp
(0, []) + (sin(y), [(x, 1)]) + (-y*sin(y), [(x*y + 1, 1)])
sage: bool(decomp.sum().quotient() == G/H)
True
sage: [r.nullstellensatz_certificate() is None for r in decomp]
[True, True, True]
```

numerator()

Return the numerator of self.

OUTPUT:

The numerator.

numerator_ring

Return the ring of the numerator.

OUTPUT:

A ring.

EXAMPLES:

quotient ()

Convert self into a quotient.

OUTPUT:

An element.

```
relative error (approx, alpha, interval, exp scale=1, digits=10)
```

Return the relative error between the values of the Maclaurin coefficients of self with multi-indices r alpha for r in interval and the values of the functions (of the variable r) in approx.

INPUT:

- •approx an individual or list of symbolic expressions in one variable
- •alpha a list of positive integers of length self.denominator_ring.ngens()
- •interval a list of positive integers
- •exp_scale (optional; default: 1) a number

OUTPUT:

A list of tuples with properties described below.

This outputs a list whose entries are a tuple (r*alpha, a_r, b_r, err_r) for r in interval. Here r*alpha is a tuple; a_r is the r*alpha (multi-index) coefficient of the Maclaurin series for self divided by $\exp_scale**r$; b_r is a list of the values of the functions in approx evaluated at r and divided by $\exp_scale**m$; err_r is the list of relative errors (a_r -f)/a_r for f in b_r. All outputs are decimal approximations.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics multivariate generating
→ functions import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: H = 1 - x - y - x * y
sage: Hfac = H.factor()
sage: G = 1 / Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: alpha = [1, 1]
sage: r = var('r')
sage: a1 = (0.573/sqrt(r))*5.83^r
sage: a2 = (0.573/sqrt(r) - 0.0674/r^(3/2))*5.83^r
sage: es = 5.83
sage: F.relative_error([a1, a2], alpha, [1, 2, 4, 8], es) # long time
[((1, 1), 0.5145797599,
  [0.5730000000, 0.5056000000], [-0.1135300000, 0.01745066667]),
 ((2, 2), 0.3824778089,
 [0.4051721856, 0.3813426871], [-0.05933514614, 0.002967810973]),
 ((4, 4), 0.2778630595,
  [0.2865000000, 0.2780750000], [-0.03108344267, -0.0007627515584]),
 ((8, 8), 0.1991088276,
  [0.2025860928, 0.1996074055], [-0.01746414394, -0.002504047242])]
```

singular_ideal ()

Return the singular ideal of self.

Let R be the ring of self and H its denominator. Let H_{red} be the reduction (square-free part) of H. Return the ideal in R generated by H_{red} and its partial derivatives. If the coefficient field of R is algebraically closed, then the output is the ideal of the singular locus (which is a variety) of the variety of H.

OUTPUT:

An ideal.

smooth_critical_ideal (alpha)

Return the smooth critical ideal of self.

Let R be the ring of self and H its denominator. Return the ideal in R of smooth critical points of the variety of H for the direction alpha. If the variety V of H has no smooth points, then return the ideal in R of V.

See [RaWi2012] for more details.

INPUT:

•alpha — a tuple of positive integers and/or symbolic entries of length self.denominator_ring.ngens()

OUTPUT:

An ideal.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics multivariate generating
→ functions import FractionWithFactoredDenominatorRing
sage: R. < x, y > = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: H = (1 - x - y - x*y)^2
sage: Hfac = H.factor()
sage: G = 1/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: alpha = var('a1, a2')
sage: F.smooth_critical_ideal(alpha)
Ideal (y^2 + 2*a1/a2*y - 1, x + ((-a2)/a1)*y + (-a1 + a2)/a1) of
Multivariate Polynomial Ring in \mathbf{x}, \mathbf{y} over Fraction Field of
Multivariate Polynomial Ring in al, a2 over Rational Field
sage: H = (1-x-y-x*y)^2
sage: Hfac = H.factor()
sage: G = 1/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: alpha = [7/3, var('a')]
sage: F.smooth_critical_ideal(alpha)
Ideal (y^2 + 14/(3*a)*y - 1, x + (-3/7*a)*y + 3/7*a - 1) of
Multivariate Polynomial Ring in \mathbf{x}, \mathbf{y} over Fraction Field of
Univariate Polynomial Ring in a over Rational Field
```

univariate_decomposition ()

Return the usual univariate partial fraction decomposition of self.

Assume that the numerator of self lies in the same univariate factorial polynomial ring as the factors of

the denominator.

Let f = p/q be a rational expression where p and q lie in a univariate factorial polynomial ring R. Let $q_1^{e_1} \cdots q_n^{e_n}$ be the unique factorization of q in R into irreducible factors. Then f can be written uniquely as:

$$(*) \quad p_0 + \sum_{i=1}^m \frac{p_i}{q_i^{e_i}},$$

for some $p_i \in R$. We call (*) the usual partial fraction decomposition of f.

Note: This partial fraction decomposition can be computed using partial_fraction() or partial_fraction_decomposition() as well. However, here we use the already obtained/cached factorization of the denominator. This gives a speed up for non-small instances.

OUTPUT:

An instance of FractionWithFactoredDenominatorSum.

EXAMPLES:

One variable:

```
sage: R.<x> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = 5*x^3 + 1/x + 1/(x-1) + 1/(3*x^2 + 1)
sage: f
(15*x^7 - 15*x^6 + 5*x^5 - 5*x^4 + 6*x^3 - 2*x^2 + x - 1)/(3*x^4 - 3*x^3 + x^2 - x)
sage: decomp = FFPD(f).univariate_decomposition()
sage: decomp
(5*x^3, []) + (1, [(x - 1, 1)]) + (1, [(x, 1)]) + (1/3, [(x^2 + 1/3, 1)])
sage: decomp.sum().quotient() == f
True
```

One variable with numerator in symbolic ring:

```
sage: R.<x> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: f = 5*x^3 + 1/x + 1/(x-1) + exp(x)/(3*x^2 + 1)
sage: f
(5*x^5 - 5*x^4 + 2*x - 1)/(x^2 - x) + e^x/(3*x^2 + 1)
sage: decomp = FFPD(f).univariate_decomposition()
sage: decomp
(0, []) +
(15/4*x^7 - 15/4*x^6 + 5/4*x^5 - 5/4*x^4 + 3/2*x^3 + 1/4*x^2*e^x - 3/4*x^2 - 1/4*x*e^x + 1/2*x - 1/4, [(x - 1, 1)]) +
(-15*x^7 + 15*x^6 - 5*x^5 + 5*x^4 - 6*x^3 - x^2*e^x + 3*x^2 + x*e^x - 2*x + 1, [(x, 1)]) +
(1/4*(15*x^7 - 15*x^6 + 5*x^5 - 5*x^4 + 6*x^3 + x^2*e^x - 3*x^2 - x*e^x + 2*x - 1)*(3*x - 1), [(x^2 + 1/3, 1)])
```

One variable over a finite field:

```
sage: R.<x> = PolynomialRing(GF(2))
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = 5*x^3 + 1/x + 1/(x-1) + 1/(3*x^2 + 1)
sage: f
(x^6 + x^4 + 1)/(x^3 + x)
sage: decomp = FFPD(f).univariate_decomposition()
sage: decomp
(x^3, []) + (1, [(x, 1)]) + (x, [(x + 1, 2)])
sage: decomp.sum().quotient() == f
True
```

One variable over an inexact field:

```
sage: R.<x> = PolynomialRing(CC)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = 5*x^3 + 1/x + 1/(x-1) + 1/(3*x^2 + 1)
sage: f
(15.0000000000000 \times x^7 - 15.00000000000 \times x^6 + 5.000000000000 \times x^5)
-5.0000000000000000x^4 + 6.0000000000000x^3
-3.0000000000000000 * x^3 + x^2 - x
sage: decomp = FFPD(f).univariate_decomposition()
sage: decomp
(5.0000000000000000 *x^3, []) +
(1.000000000000000, [(x - 1.0000000000000, 1)]) +
(-0.288675134594813*I, [(x - 0.577350269189626*I, 1)]) +
(1.000000000000000, [(x, 1)]) +
(0.288675134594813*I, [(x + 0.577350269189626*I, 1)])
sage: decomp.sum().quotient() == f # Rounding error coming
False
```

TESTS:

```
sage: R.<x> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: f = exp(x) / (x^2-x)
sage: f
e^x/(x^2 - x)
sage: FFPD(f).univariate_decomposition()
(0, []) + (e^x, [(x - 1, 1)]) + (-e^x, [(x, 1)])
```

AUTHORS:

- •Robert Bradshaw (2007-05-31)
- •Alexander Raichev (2012-06-25)
- •Daniel Krenn (2014-12-01)

class sage.rings.asymptotic.asymptotics_multivariate_generating_functions. FractionWithFactor

Bases: sage.structure.unique_representation.UniqueRepresentation

```
sage.rings.ring.Ring
```

This is the ring of fractions with factored denominator.

INPUT:

- •denominator_ring the base ring (a polynomial ring)
- •numerator ring (optional) the numerator ring; the default is the denominator ring
- •category (default: Rings) the category

See also:

FractionWithFactoredDenominator, asymptotics_multivariate_generating_functions

EXAMPLES:

AUTHORS:

•Daniel Krenn (2014-12-01)

Element

alias of FractionWithFactoredDenominator

base_ring()

Returns the base ring.

OUTPUT:

A ring.

EXAMPLES:

rename_keyword

alias of rename_keyword

A list representing the sum of FractionWithFactoredDenominator objects with distinct denominator factorizations.

AUTHORS:

- •Alexander Raichev (2012-06-25)
- •Daniel Krenn (2014-12-01)

denominator_ring

Return the polynomial ring of the denominators of self.

OUTPUT:

A ring or None if the list is empty.

EXAMPLES:

sum ()

Return the sum of the elements in self.

OUTPUT:

An instance of FractionWithFactoredDenominator.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_
functions import FractionWithFactoredDenominatorRing,...
\hookrightarrowFractionWithFactoredDenominatorSum
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: df = (x, 1), (y, 1), (x*y + 1, 1)
sage: f = FFPD(2, df)
sage: g = FFPD(2*x*y, df)
sage: FractionWithFactoredDenominatorSum([f, g])
(2, [(y, 1), (x, 1), (x*y + 1, 1)]) + (2, [(x*y + 1, 1)])
sage: FractionWithFactoredDenominatorSum([f, q]).sum()
(2, [(y, 1), (x, 1)])
sage: f = FFPD(cos(x), [(x, 2)])
sage: g = FFPD(cos(y), [(x, 1), (y, 2)])
sage: FractionWithFactoredDenominatorSum([f, g])
(\cos(x), [(x, 2)]) + (\cos(y), [(y, 2), (x, 1)])
sage: FractionWithFactoredDenominatorSum([f, g]).sum()
(y^2*\cos(x) + x*\cos(y), [(y, 2), (x, 2)])
```

whole_and_parts ()

Rewrite self as a sum of a (possibly zero) polynomial followed by reduced rational expressions.

OUTPUT:

An instance of FractionWithFactoredDenominatorSum.

Only useful for multivariate decompositions.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics multivariate generating
→ functions import FractionWithFactoredDenominatorRing,
→ FractionWithFactoredDenominatorSum
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: f = x**2 + 3*y + 1/x + 1/y
sage: f = FFPD(f); f
(x^3*y + 3*x*y^2 + x + y, [(y, 1), (x, 1)])
sage: FractionWithFactoredDenominatorSum([f]).whole_and_parts()
(x^2 + 3*y, []) + (x + y, [(y, 1), (x, 1)])
sage: f = cos(x) **2 + 3*y + 1/x + 1/y; f
\cos(x)^2 + 3*y + 1/x + 1/y
sage: G = f.numerator()
sage: H = R(f.denominator())
sage: f = FFPD(G, H.factor()); f
(x*y*cos(x)^2 + 3*x*y^2 + x + y, [(y, 1), (x, 1)])
sage: FractionWithFactoredDenominatorSum([f]).whole_and_parts()
(0, []) + (x*y*cos(x)^2 + 3*x*y^2 + x + y, [(y, 1), (x, 1)])
```

sage.rings.asymptotics_multivariate_generating_functions. $coerce_point$ (R, p)

Coerce the keys of the dictionary p into the ring R.

Warning: This method assumes that it is possible.

EXAMPLES:

```
sage.rings.asymptotic.asymptotics_multivariate_generating_functions. \mbox{diff\_all} (f, V, n, end-ing=[], sub=None, sub_final=None, zero_order=0, rekey=None)
```

Return a dictionary of representative mixed partial derivatives of f from order 1 up to order n with respect to the variables in V.

The default is to key the dictionary by all nondecreasing sequences in V of length 1 up to length n.

INPUT:

- •f an individual or list of C^{n+1} functions
- •V a list of variables occurring in f
- •n a natural number
- •ending a list of variables in V
- •sub an individual or list of dictionaries
- •sub_final an individual or list of dictionaries
- •rekey a callable symbolic function in V or list thereof
- •zero_order a natural number

OUTPUT:

The dictionary {s 1:deriv 1, ..., sr:deriv r}.

Here s_1 , ..., s_r is a listing of all nondecreasing sequences of length 1 up to length n over the alphabet V, where w>v in X if and only if str(w)>str(v), and $deriv_j$ is the derivative of f with respect to the derivative sequence s_j and simplified with respect to the substitutions in sub and evaluated at sub_final . Moreover, all derivatives with respect to sequences of length less than $zero_order$ (derivatives of order less than $zero_order$) will be made zero.

If rekey is nonempty, then s_1, \ldots, s_r will be replaced by the symbolic derivatives of the functions in rekey.

If ending is nonempty, then every derivative sequence s_j will be suffixed by ending.

EXAMPLES:

```
24*x

sage: a = {x:1}
sage: dd = diff_all(f, [x], 3, sub=d1, rekey=f, sub_final=a)
sage: dd[diff(f, x, 3)]
24
```

```
sage: X = var('x, y, z')
sage: f = function('f')(*X)
sage: dd = diff_all(f, X, 2, ending=[y, y, y])
sage: dd[(z, y, y, y)]
diff(f(x, y, z), y, y, y, z)
```

```
sage: g = function('g')(*X)
sage: dd = diff_all([f, g], X, 2)
sage: dd[(0, y, z)]
diff(f(x, y, z), y, z)

sage: dd[(1, z, z)]
diff(g(x, y, z), z, z)

sage: f = exp(x*y*z)
sage: f = function('ff')(*X)
sage: dd = diff_all(f, X, 2, rekey=ff)
sage: dd[diff(ff, x, z)]
x*y^2*z*e^(x*y*z) + y*e^(x*y*z)
```

```
sage.rings.asymptotics_multivariate_generating_functions. \mbox{diff_op} ( A, B, AB\_derivs, V, M, r, N)
```

Return the derivatives $DD^{(l+k)}(A[j]B^l)$ evaluated at a point p for various natural numbers j, k, l which depend on r and N.

Here DD is a specific second-order linear differential operator that depends on M, A is a list of symbolic functions, B is symbolic function, and AB_derivs contains all the derivatives of A and B evaluated at p that are necessary for the computation.

INPUT:

- •A a single or length r list of symbolic functions in the variables V
- $\bullet B a$ symbolic function in the variables V.
- •AB_derivs a dictionary whose keys are the (symbolic) derivatives of A[0],...,A[r-1] up to order $2 \times N-2$ and the (symbolic) derivatives of B up to order $2 \times N$; the values of the dictionary are complex numbers that are the keys evaluated at a common point p
- •V the variables of the A[j] and B
- •M a symmetric $l \times l$ matrix, where l is the length of V
- •r, N natural numbers

OUTPUT:

A dictionary.

The output is a dictionary whose keys are natural number tuples of the form (j,k,l), where $l \leq 2k, j \leq r-1$, and $j+k \leq N-1$, and whose values are $DD^{(l+k)}(A[j]B^{l})$ evaluated at a point p, where DD is the linear second-order differential operator $-\sum_{i=0}^{l-1}\sum_{j=0}^{l-1}M[i][j]\partial^{2}/(\partial V[j]\partial V[i])$.

Note: For internal use by $FractionWithFactoredDenominator.asymptotics_smooth()$ and $FractionWithFactoredDenominator.asymptotics_multiple()$.

EXAMPLES:

 $\verb|sage.rings.asymptotics_multivariate_generating_functions. \verb|diff_op_simple|| (A, and all of the context of$

B,
AB_deriv
x,
v,

a, *N*)

Return $DD^{(ek+vl)}(AB^l)$ evaluated at a point p for various natural numbers e, k, l that depend on v and N.

Here DD is a specific linear differential operator that depends on a and v, A and B are symbolic functions, and AB_derivs contains all the derivatives of A and B evaluated at p that are necessary for the computation.

Note: For internal use by the function FractionWithFactoredDenominator.asymptotics_smooth()

INPUT:

- •A, B Symbolic functions in the variable x
- •AB_derivs a dictionary whose keys are the (symbolic) derivatives of A up to order 2 * N if v is even or N if v is odd and the (symbolic) derivatives of B up to order 2 * N + v if v is even or N + v if v is odd; the values of the dictionary are complex numbers that are the keys evaluated at a common point p
- •x a symbolic variable
- •a −a complex number
- •v, N natural numbers

OUTPUT:

A dictionary.

The output is a dictionary whose keys are natural number pairs of the form (k, l), where k < N and $l \le 2k$ and whose values are $DD^(ek + vl)(AB^l)$ evaluated at a point p. Here e = 2 if v is even, e = 1 if v is odd, and DD is the linear differential operator $(a^{-1/v}d/dt)$ if v is even and $(|a|^{-1/v}i\mathrm{sgn}(a)d/dt)$ if v is odd.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions_
import diff_op_simple
sage: A = function('A') (x)
sage: B = function('B') (x)
sage: AB_derivs = {}
sage: sorted(diff_op_simple(A, B, AB_derivs, x, 3, 2, 2).items())
[((0, 0), A(x)),
  ((1, 0), 1/2*I*2^(2/3)*diff(A(x), x)),
  ((1, 1),
  1/4*2^(2/3)*(B(x)*diff(A(x), x, x, x, x) + 4*diff(A(x), x, x, x)*diff(B(x), x),
  →+ 6*diff(A(x), x, x)*diff(B(x), x, x, x) + 4*diff(A(x), x)*diff(B(x), x, x, x, x) +
  →A(x)*diff(B(x), x, x, x, x))]
```

sage.rings.asymptotics_multivariate_generating_functions. $diff_prod$ (f_derivs_r), u_r)

u,
g,
X,
interval,
end,
uderivs,
atc)

Take various derivatives of the equation f = ug, evaluate them at a point c, and solve for the derivatives of u.

INPUT:

- \bullet f_derivs a dictionary whose keys are all tuples of the form s + end, where s is a sequence of variables from X whose length lies in interval, and whose values are the derivatives of a function f evaluated at c
- •u a callable symbolic function
- •q an expression or callable symbolic function
- •X a list of symbolic variables
- •interval a list of positive integers Call the first and last values n and nn, respectively
- •end a possibly empty list of repetitions of the variable z, where z is the last element of X
- •uderivs a dictionary whose keys are the symbolic derivatives of order 0 to order n-1 of u evaluated at c and whose values are the corresponding derivatives evaluated at c
- •atc a dictionary whose keys are the keys of c and all the symbolic derivatives of order 0 to order nn of g evaluated c and whose values are the corresponding derivatives evaluated at c

OUTPUT:

A dictionary whose keys are the derivatives of ${\bf u}$ up to order nn and whose values are those derivatives evaluated at c.

This function works by differentiating the equation f=ug with respect to the variable sequence s+end, for all tuples s of X of lengths in interval, evaluating at the point c, and solving for the remaining derivatives of u. This function assumes that u never appears in the differentiations of f=ug after evaluating at c.

Note: For internal use by FractionWithFactoredDenominator.asymptotics_multiple().

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions_
import diff_prod
sage: u = function('u')(x)
sage: g = function('g')(x)
sage: fd = {(x,):1, (x, x):1}
sage: ud = {u(x=2): 1}
sage: atc = {x: 2, g(x=2): 3, diff(g, x)(x=2): 5}
sage: atc[diff(g, x, x)(x=2)] = 7
sage: dd = diff_prod(fd, u, g, [x], [1, 2], [], ud, atc)
sage: dd[diff(u, x, 2)(x=2)]
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```

Given a list s of tuples of natural numbers, return the list of elements of V with indices the elements of the elements of s.

INPUT:

- •∨ a list
- •s a list of tuples of natural numbers in the interval range (len (V))

OUTPUT:

The tuple ([V[tt] for tt in sorted(t)]), where t is the list of elements of the elements of s.

EXAMPLES:

Note: This function is for internal use by diff_op().

Return [vv/v[coordinate]] for vv in v] where coordinate is the last index of v if not specified otherwise.

INPUT:

- •v a vector
- •coordinate (optional; default: None) an index for v

EXAMPLES:

sage.rings.asymptotic.asymptotics_multivariate_generating_functions. $permutation_sign$ (s, u)

This function returns the sign of the permutation on $1, \ldots, len(u)$ that is induced by the sublist s of u.

Note: For internal use by $FractionWithFactoredDenominator.cohomology_decomposition()$

INPUT:

- •s a sublist of u
- •u a list

OUTPUT:

The sign of the permutation obtained by taking indices within u of the list s + sc, where sc is u with the elements of s removed.

EXAMPLES:

```
sage.rings.asymptotic.asymptotics_multivariate_generating_functions. {\bf subs\_all} ( f, sub, sim- plify=False)
```

Return the items of f substituted by the dictionaries of sub in order of their appearance in sub.

INPUT:

- •f an individual or list of symbolic expressions or dictionaries
- •sub an individual or list of dictionaries
- \bullet simplify -(default: False) boolean; set to True to simplify the result

OUTPUT:

The items of f substituted by the dictionaries of sub in order of their appearance in sub. The subs() command is used. If simplify is True, then simplify() is used after substitution.

EXAMPLES:

```
sage: var('x, y')
(x, y)
sage: a = {'foo': x**2 + y**2, 'bar': x - y}
sage: b = {x: 1 , y: 2}
sage: subs_all(a, b)
{'bar': -1, 'foo': 5}
```



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