Sage Reference Manual: Basic Structures

Release 7.6

The Sage Development Team

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SAGE OBJECTS

1.1 Abstract base class for Sage objects

class `sage.structure.sage_object.` **SageObject**
Bases: `object`

Base class for all (user-visible) objects in Sage

Every object that can end up being returned to the user should inherit from *SageObject* .

_ascii_art_ `( )`
Return an ASCII art representation.

To implement multi-line ASCII art output in a derived class you must override this method. Unlike `_repr_()` , which is sometimes used for the hash key, the output of *_ascii_art_()* may depend on settings and is allowed to change during runtime.

OUTPUT:

An `AsciiArt` object, see `sage.typeset.ascii_art` for details.

EXAMPLES:

You can use the `ascii_art()` function to get the ASCII art representation of any object in Sage:

```
sage: ascii_art(integral(exp(x+x^2)/(x+1), x))
  /
 |
 |    2
 |   x  + x
 | e
 | ------- dx
 |   x + 1
 |
 /
```

Alternatively, you can use the `%display ascii_art/simple` magic to switch all output to ASCII art and back:

```
sage: from sage.repl.interpreter import get_test_shell
sage: shell = get_test_shell()
sage: shell.run_cell('tab = StandardTableaux(3)[2]; tab')
[[1, 2], [3]]
sage: shell.run_cell('%display ascii_art')
sage: shell.run_cell('tab')
1  2
3
```

```
sage: shell.run_cell('Tableaux.options(ascii_art="table", convention="French")
↪')
sage: shell.run_cell('tab')
+---+
| 3 |
+---+---+
| 1 | 2 |
+---+---+
sage: shell.run_cell('%display plain')
sage: shell.run_cell('Tableaux.options._reset()')
sage: shell.quit()
```

_cache_key ()

Return a hashable key which identifies this objects for caching. The output must be hashable itself, or a tuple of objects which are hashable or define a `_cache_key`.

This method will only be called if the object itself is not hashable.

Some immutable objects (such as p-adic numbers) cannot implement a reasonable hash function because their `==` operator has been modified to return `True` for objects which might behave differently in some computations:

```
sage: K.<a> = Qq(9)
sage: b = a + O(3)
sage: c = a + 3
sage: b
a + O(3)
sage: c
a + 3 + O(3^20)
sage: b == c
True
sage: b == a
True
sage: c == a
False
```

If such objects defined a non-trivial hash function, this would break caching in many places. However, such objects should still be usable in caches. This can be achieved by defining an appropriate `_cache_key`:

```
sage: hash(b)
Traceback (most recent call last):
...
TypeError: unhashable type: 'sage.rings.padics.qadic_flint_CR.
↪qAdicCappedRelativeElement'
sage: @cached_method
....: def f(x): return x==a
sage: f(b)
True
sage: f(c) # if b and c were hashable, this would return True
False

sage: b._cache_key()
(..., ((0, 1),), 0, 1)
sage: c._cache_key()
(..., ((0, 1), (1,)), 0, 20)
```

An implementation must make sure that for elements `a` and `b`, if `a != b`, then also `a._cache_key()` `!= b._cache_key()`. In practice this means that the `_cache_key` should always include the parent

as its first argument:

```
sage: S.<a> = Qq(4)
sage: d = a + O(2)
sage: b._cache_key() == d._cache_key() # this would be True if the parents
↪were not included
False
```

category ( )

db (*name*, *compress=True*)
Dumps self into the Sage database. Use db(name) by itself to reload.

The database directory is `$HOME/.sage/db`

dump (*filename*, *compress=True*)
Same as self.save(filename, compress)

dumps (*compress=True*)
Dump `self` to a string `s`, which can later be reconstituted as `self` using `loads(s)`.

There is an optional boolean argument `compress` which defaults to `True`.

EXAMPLES:

```
sage: O=SageObject(); O.dumps()
'x\x9ck`J.NLO\xd5+.)*M.)-\x02\xb2\x80\xdc\xf8\xfc\xa4\xac\xd4\xe4\x12\xae`
↪\xdb\x1f\xc2,d\xd4l,d\xd2\x03\x00\xb7X\x10\xf1'
sage: O.dumps(compress=False)
'\x80\x02csage.structure.sage_object\nSageObject\nq\x01)\x81q\x02.'
```

parent ( )
Return the type of `self` to support the coercion framework.

EXAMPLES:

```
sage: t = log(sqrt(2) - 1) + log(sqrt(2) + 1); t
log(sqrt(2) + 1) + log(sqrt(2) - 1)
sage: u = t.maxima_methods()
sage: u.parent()
<class 'sage.symbolic.maxima_wrapper.MaximaWrapper'>
```

rename (*x=None*)
Change self so it prints as x, where x is a string.

Note: This is *only* supported for Python classes that derive from SageObject.

EXAMPLES:

```
sage: x = PolynomialRing(QQ, 'x', sparse=True).gen()
sage: g = x^3 + x - 5
sage: g
x^3 + x - 5
sage: g.rename('a polynomial')
sage: g
a polynomial
sage: g + x
x^3 + 2*x - 5
sage: h = g^100
```

```
sage: str(h)[:20]
'x^300 + 100*x^298 - '
sage: h.rename('x^300 + ...')
sage: h
x^300 + ...
```

Real numbers are not Python classes, so rename is not supported:

```
sage: a = 3.14
sage: type(a)
<type 'sage.rings.real_mpfr.RealLiteral'>
sage: a.rename('pi')
Traceback (most recent call last):
...
NotImplementedError: object does not support renaming: 3.14000000000000
```

Note: The reason C-extension types are not supported by default is if they were then every single one would have to carry around an extra attribute, which would be slower and waste a lot of memory.

To support them for a specific class, add a `cdef public __custom_name` attribute.

---

reset_name ()
> Remove the custom name of an object.
>
> EXAMPLES:

```
sage: P.<x> = QQ[]
sage: P
Univariate Polynomial Ring in x over Rational Field
sage: P.rename('A polynomial ring')
sage: P
A polynomial ring
sage: P.reset_name()
sage: P
Univariate Polynomial Ring in x over Rational Field
```

save (*filename=None*, *compress=True*)
> Save self to the given filename.
>
> EXAMPLES:

```
sage: f = x^3 + 5
sage: f.save(os.path.join(SAGE_TMP, 'file'))
sage: load(os.path.join(SAGE_TMP, 'file.sobj'))
x^3 + 5
```

sage.structure.sage_object. **dumps** (*obj*, *compress=True*)
> Dump obj to a string s. To recover obj, use `loads(s)`.
>
> See also:
>
> *dumps()*
>
> EXAMPLES:

```
sage: a = 2/3
sage: s = dumps(a)
sage: len(s)
```

```
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sage: loads(s)
2/3
```

`sage.structure.sage_object.` **load** (*compress=True*, *verbose=True*, **filename*)

> Load Sage object from the file with name filename, which will have an `.sobj` extension added if it doesn't have one. Or, if the input is a filename ending in `.py` , `.pyx` , `.sage` , `.spyx` , `.f` , `.f90` or `.m` , load that file into the current running session.
>
> Loaded files are not loaded into their own namespace, i.e., this is much more like Python's `execfile` than Python's `import` .
>
> This function also loads a `.sobj` file over a network by specifying the full URL. (Setting `verbose = False` suppresses the loading progress indicator.)
>
> Finally, if you give multiple positional input arguments, then all of those files are loaded, or all of the objects are loaded and a list of the corresponding loaded objects is returned.
>
> EXAMPLES:

```
sage: u = 'http://sage.math.washington.edu/home/was/db/test.sobj'
sage: s = load(u)                                              # optional -
↪internet
Attempting to load remote file: http://sage.math.washington.edu/home/was/db/test.
↪sobj
Loading: [.]
sage: s                                                        # optional -
↪internet
'hello SAGE'
```

> We test loading a file or multiple files or even mixing loading files and objects:

```
sage: t = tmp_filename(ext='.py')
sage: open(t,'w').write("print('hello world')")
sage: load(t)
hello world
sage: load(t,t)
hello world
hello world
sage: t2 = tmp_filename(); save(2/3,t2)
sage: load(t,t,t2)
hello world
hello world
[None, None, 2/3]
```

> Files with a `.sage` extension are preparsed. Also note that we can access global variables:

```
sage: t = tmp_filename(ext=".sage")
sage: with open(t, 'w') as f:
....:     f.write("a += Mod(2/3, 11)")  # This evaluates to Mod(8, 11)
sage: a = -1
sage: load(t)
sage: a
7
```

> We can load Fortran files:

```
sage: code = '        subroutine hello\n        print *, "Hello World!"\n        end␣
→subroutine hello\n'
sage: t = tmp_filename(ext=".F")
sage: open(t, 'w').write(code)
sage: load(t)
sage: hello
<fortran object>
```

sage.structure.sage_object. **loads** (*s*, *compress=True*)

Recover an object x that has been dumped to a string s using `s = dumps(x)`.

See also:

dumps()

EXAMPLES:

```
sage: a = matrix(2, [1,2,3,-4/3])
sage: s = dumps(a)
sage: loads(s)
[   1    2]
[   3 -4/3]
```

If compress is True (the default), it will try to decompress the data with zlib and with bz2 (in turn); if neither succeeds, it will assume the data is actually uncompressed. If compress=False is explicitly specified, then no decompression is attempted.

```
sage: v = [1..10]
sage: loads(dumps(v, compress=False)) == v
True
sage: loads(dumps(v, compress=False), compress=True) == v
True
sage: loads(dumps(v, compress=True), compress=False)
Traceback (most recent call last):
...
UnpicklingError: invalid load key, 'x'.
```

sage.structure.sage_object. **make_None** (**args*, ***kwds*)

Do nothing and return `None`. Used for overriding pickles when that pickle is no longer needed.

EXAMPLES:

```
sage: from sage.structure.sage_object import make_None
sage: print(make_None(42, pi, foo='bar'))
None
```

sage.structure.sage_object. **picklejar** (*obj*, *dir=None*)

Create pickled sobj of obj in dir, with name the absolute value of the hash of the pickle of obj. This is used in conjunction with *unpickle_all()*.

To use this to test the whole Sage library right now, set the environment variable SAGE_PICKLE_JAR, which will make it so dumps will by default call picklejar with the default dir. Once you do that and doctest Sage, you'll find that the SAGE_ROOT /tmp/ contains a bunch of pickled objects along with corresponding txt descriptions of them. Use the *unpickle_all()* to see if they unpickle later.

INPUT:

- •obj – a pickleable object

- •dir – a string or None; if None then dir defaults to SAGE_ROOT/tmp/pickle_jar

EXAMPLES:

```
sage: dir = tmp_dir()
sage: sage.structure.sage_object.picklejar(1, dir)
sage: sage.structure.sage_object.picklejar('test', dir)
sage: len(os.listdir(dir))   # Two entries (sobj and txt) for each object
4
```

sage.structure.sage_object.**register_unpickle_override** (*module*, *name*, *callable*, *call_name=None*)

Python pickles include the module and class name of classes. This means that rearranging the Sage source can invalidate old pickles. To keep the old pickles working, you can call register_unpickle_override with an old module name and class name, and the Python callable (function, class with __call__ method, etc.) to use for unpickling. (If this callable is a value in some module, you can specify the module name and class name, for the benefit of `explain_pickle()` when called with `in_current_sage=True`).)

EXAMPLES:

Imagine that there used to be an `old_integer` module and old pickles essentially trying to do the following:

```
sage: unpickle_global('sage.rings.old_integer', 'OldInteger')
Traceback (most recent call last):
...
ImportError: cannot import OldInteger from sage.rings.old_integer, call register_
→unpickle_override('sage.rings.old_integer', 'OldInteger', ...) to fix this
```

After following the advice from the error message, unpickling works:

```
sage: from sage.structure.sage_object import register_unpickle_override
sage: register_unpickle_override('sage.rings.old_integer', 'OldInteger', Integer)
sage: unpickle_global('sage.rings.old_integer', 'OldInteger')
<type 'sage.rings.integer.Integer'>
```

In many cases, unpickling problems for old pickles can be resolved with a simple call to `register_unpickle_override`, as in the example above and in many of the `sage` source files. However, if the underlying data structure has changed significantly then unpickling may fail and it will be necessary to explicitly implement unpickling methods for the associated objects. The python pickle protocol is described in detail on the web and, in particular, in the python pickling documentation. For example, the following excerpt from this documentation shows that the unpickling of classes is controlled by their __setstate__() method.

```
object.__setstate__(state)

    Upon unpickling, if the class also defines the method :meth:`__setstate__`,␣
→it is
    called with the unpickled state. If there is no :meth:`__setstate__` method,
    the pickled state must be a dictionary and its items are assigned to the new
    instance's dictionary. If a class defines both :meth:`getstate__` and
    :meth:`__setstate__`, the state object needn't be a dictionary and these␣
→methods
    can do what they want.
```

By implementing a __setstate__() method for a class it should be possible to fix any unpickling problems for the class. As an example of what needs to be done, we show how to unpickle a `CombinatorialObject` object using a class which also inherits from *Element* . This exact problem often arises when refactoring old code into the element framework. First we create a pickle to play with:

```
sage: from sage.structure.element import Element
sage: class SourPickle(CombinatorialObject): pass
sage: class SweetPickle(CombinatorialObject,Element): pass
sage: import __main__
sage: __main__.SourPickle=SourPickle
sage: __main__.SweetPickle=SweetPickle   # a hack to allow us to pickle command
→line classes
sage: gherkin = dumps( SourPickle([1,2,3]) )
```

Using *register_unpickle_override()* we try to sweeten our pickle, but we are unable to eat it:

```
sage: from sage.structure.sage_object import register_unpickle_override
sage: register_unpickle_override('__main__','SourPickle',SweetPickle)
sage: loads( gherkin )
Traceback (most recent call last):
...
KeyError: 0
```

The problem is that the `SweetPickle` has inherited a `__setstate__()` method from *Element* which is not compatible with unpickling for `CombinatorialObject`. We can fix this by explicitly defining a new `__setstate__()` method:

```
sage: class SweeterPickle(CombinatorialObject,Element):
....:     def __setstate__(self, state):
....:         if isinstance(state, dict):        # a pickle from
→CombinatorialObject is just its instance dictionary
....:             self._set_parent(Tableaux())   # this is a fudge: we need an
→appropriate parent here
....:             self.__dict__ = state
....:         else:
....:             self._set_parent(state[0])
....:             self.__dict__ = state[1]
sage: __main__.SweeterPickle = SweeterPickle
sage: register_unpickle_override('__main__','SourPickle',SweeterPickle)
sage: loads( gherkin )
[1, 2, 3]
sage: loads(dumps( SweeterPickle([1,2,3]) ))   # check that pickles work for
→SweeterPickle
[1, 2, 3]
```

The `state` passed to `__setstate__()` will usually be something like the instance dictionary of the pickled object, however, with some older classes such as `CombinatorialObject` it will be a tuple. In general, the `state` can be any python object. Sage provides a special tool, `explain_pickle()`, which can help in figuring out the contents of an old pickle. Here is a second example.

```
sage: class A(object):
....:     def __init__(self,value):
....:         self.original_attribute = value
....:     def __repr__(self):
....:         return 'A(%s)'%self.original_attribute
sage: class B(object):
....:     def __init__(self,value):
....:         self.new_attribute = value
....:     def __setstate__(self,state):
....:         try:
....:             self.new_attribute = state['new_attribute']
....:         except KeyError:      # an old pickle
```

```
....:             self.new_attribute = state['original_attribute']
....:     def __repr__(self):
....:         return 'B(%s)'%self.new_attribute
sage: import __main__
sage: __main__.A=A; __main__.B=B  # a hack to allow us to pickle command line
↪classes
sage: A(10)
A(10)
sage: loads( dumps(A(10)) )
A(10)
sage: sage.misc.explain_pickle.explain_pickle( dumps(A(10)) )
pg_A = unpickle_global('__main__', 'A')
si = unpickle_newobj(pg_A, ())
pg_make_integer = unpickle_global('sage.rings.integer', 'make_integer')
unpickle_build(si, {'original_attribute':pg_make_integer('a')})
si
sage: from sage.structure.sage_object import register_unpickle_override
sage: register_unpickle_override('__main__', 'A', B)
sage: loads( dumps(A(10)) )
B(10)
sage: loads( dumps(B(10)) )
B(10)
```

Pickling for python classes and extension classes, such as cython, is different – again this is discussed in the python pickling documentation. For the unpickling of extension classes you need to write a `__reduce__()` method which typically returns a tuple `(f,args,...)` such that `f(*args)` returns (a copy of) the original object. The following code snippet is the `__reduce__()` method from `sage.rings.integer.Integer`.

```python
def __reduce__(self):
    'Including the documentation properly causes a doc-test failure so we include
↪it as a comment:'
    #* '''
    #* This is used when pickling integers.
    #*
    #* EXAMPLES::
    #*
    #*     sage: n = 5
    #*     sage: t = n.__reduce__(); t
    #*     (<built-in function make_integer>, ('5',))
    #*     sage: t[0](*t[1])
    #*     5
    #*     sage: loads(dumps(n)) == n
    #*     True
    #* '''
    # This single line below took me HOURS to figure out.
    # It is the *trick* needed to pickle Cython extension types.
    # The trick is that you must put a pure Python function
    # as the first argument, and that function must return
    # the result of unpickling with the argument in the second
    # tuple as input. All kinds of problems happen
    # if we don't do this.
    return sage.rings.integer.make_integer, (self.str(32),)
```

sage.structure.sage_object.**rich_to_bool** (*op*, *c*)

Return the corresponding `True` or `False` value for a rich comparison, given the result of an ordinary comparison.

INPUT:

- `op` – a rich comparison operation (e.g. `Py_EQ`)

- `c` – the result of an ordinary comparison: -1, 0 or 1.

OUTPUT: 1 or 0 (corresponding to `True` and `False`)

See also:

`rich_to_bool_sgn` if `c` could be outside the [-1, 1] range.

EXAMPLES:

```
sage: from sage.structure.sage_object import (rich_to_bool,
....:     op_EQ, op_NE, op_LT, op_LE, op_GT, op_GE)
sage: for op in (op_LT, op_LE, op_EQ, op_NE, op_GT, op_GE):
....:     for c in (-1,0,1):
....:         print(rich_to_bool(op, c))
True False False
True True False
False True False
True False True
False False True
False True True
```

Indirect tests using integers:

```
sage: 0 < 5, 5 < 5, 5 < -8
(True, False, False)
sage: 0 <= 5, 5 <= 5, 5 <= -8
(True, True, False)
sage: 0 >= 5, 5 >= 5, 5 >= -8
(False, True, True)
sage: 0 > 5, 5 > 5, 5 > -8
(False, False, True)
sage: 0 == 5, 5 == 5, 5 == -8
(False, True, False)
sage: 0 != 5, 5 != 5, 5 != -8
(True, False, True)
```

`sage.structure.sage_object.` **`rich_to_bool_sgn`** (*op*, *c*)

Same as `rich_to_bool` , but allow any $c < 0$ and $c > 0$ instead of only -1 and 1.

Note: This is in particular needed for `mpz_cmp()` .

`sage.structure.sage_object.` **`richcmp_not_equal`** (*x*, *y*, *op*)

Like `richcmp(x,y,op)` but assuming that x is not equal to y.

INPUT:

- `op` – a rich comparison operation (e.g. `Py_EQ`)

OUTPUT:

If `op` is not `op_EQ` or `op_NE` , the result of `richcmp(x,y,op)` . If `op` is `op_EQ` , return `False` . If `op` is `op_NE` , return `True` .

This is useful to compare lazily two objects A and B according to 2 (or more) different parameters, say width and height for example. One could use:

```
return richcmp((A.width(), A.height()), (B.width(), B.height()), op)
```

but this will compute both width and height in all cases, even if A.width() and B.width() are enough to decide the comparison.

Instead one can do:

```
wA = A.width()
wB = B.width()
if wA != wB:
    return richcmp_not_equal(wA, wB, op)
return richcmp(A.height(), B.height(), op)
```

The difference with `richcmp` is that `richcmp_not_equal` assumes that its arguments are not equal, which is excluding the case where the comparison cannot be decided so far, without knowing the rest of the parameters.

EXAMPLES:

```
sage: from sage.structure.sage_object import (richcmp_not_equal,
....:     op_EQ, op_NE, op_LT, op_LE, op_GT, op_GE)
sage: for op in (op_LT, op_LE, op_EQ, op_NE, op_GT, op_GE):
....:     print(richcmp_not_equal(3, 4, op))
True
True
False
True
False
False
sage: for op in (op_LT, op_LE, op_EQ, op_NE, op_GT, op_GE):
....:     print(richcmp_not_equal(5, 4, op))
False
False
False
True
True
True
```

`sage.structure.sage_object.` **save** (*obj*, *filename=None*, *compress=True*, ***kwds*)
  Save `obj` to the file with name `filename`, which will have an `.sobj` extension added if it doesn't have one and if `obj` doesn't have its own `save()` method, like e.g. Python tuples.

  For image objects and the like (which have their own `save()` method), you may have to specify a specific extension, e.g. `.png`, if you don't want the object to be saved as a Sage object (or likewise, if `filename` could be interpreted as already having some extension).

> **Warning:** This will *replace* the contents of the file if it already exists.

  EXAMPLES:

```
sage: a = matrix(2, [1,2,3,-5/2])
sage: objfile = os.path.join(SAGE_TMP, 'test.sobj')
sage: objfile_short = os.path.join(SAGE_TMP, 'test')
sage: save(a, objfile)
sage: load(objfile_short)
[   1    2]
[   3 -5/2]
sage: E = EllipticCurve([-1,0])
```

```
sage: P = plot(E)
sage: save(P, objfile_short)     # saves the plot to "test.sobj"
sage: save(P, filename=os.path.join(SAGE_TMP, "sage.png"), xmin=-2)
sage: save(P, os.path.join(SAGE_TMP, "filename.with.some.wrong.ext"))
Traceback (most recent call last):
...
ValueError: allowed file extensions for images are '.eps', '.pdf', '.pgf', '.png
→', '.ps', '.sobj', '.svg'!
sage: print(load(objfile))
Graphics object consisting of 2 graphics primitives
sage: save("A python string", os.path.join(SAGE_TMP, 'test'))
sage: load(objfile)
'A python string'
sage: load(objfile_short)
'A python string'
```

`sage.structure.sage_object.` **unpickle_all** (*dir=None*, *debug=False*, *run_test_suite=False*)

Unpickle all sobj's in the given directory, reporting failures as they occur. Also printed the number of successes and failure.

INPUT:

- `dir` – a string; the name of a directory (or of a .tar.bz2 file that decompresses to a directory) full of pickles. (default: the standard pickle jar)

- `debug` – a boolean (default: False) whether to report a stacktrace in case of failure

- `run_test_suite` – a boolean (default: False) whether to run `TestSuite(x).run()` on the un-pickled objects

EXAMPLES:

```
sage: dir = tmp_dir()
sage: sage.structure.sage_object.picklejar('hello', dir)
sage: sage.structure.sage_object.unpickle_all(dir)
Successfully unpickled 1 objects.
Failed to unpickle 0 objects.
```

When run with no arguments *unpickle_all()* tests that all of the "standard" pickles stored in the pickle_jar at `SAGE_SHARE/sage/ext/pickle_jar/pickle_jar.tar.bz2` can be unpickled.

```
sage: sage.structure.sage_object.unpickle_all()     # (4s on sage.math, 2011)
doctest:... DeprecationWarning: ...
See http://trac.sagemath.org/... for details.
Successfully unpickled ... objects.
Failed to unpickle 0 objects.
```

Check that unpickling a second time works (see trac ticket #5838):

```
sage: sage.structure.sage_object.unpickle_all()
Successfully unpickled ... objects.
Failed to unpickle 0 objects.
```

When it is not possible to unpickle a pickle in the pickle_jar then *unpickle_all()* prints the following error message which warns against removing pickles from the pickle_jar and directs the user towards *register_unpickle_override()*. The following code intentionally breaks a pickle to illustrate this:

```
sage: from sage.structure.sage_object import register_unpickle_override, unpickle_
↪all, unpickle_global
sage: class A(CombinatorialObject,sage.structure.element.Element):
....:     pass  # to break a pickle
sage: tableau_unpickler=unpickle_global('sage.combinat.tableau','Tableau_class')
sage: register_unpickle_override('sage.combinat.tableau','Tableau_class',A) #␣
↪breaking the pickle
sage: unpickle_all()  # todo: not tested
...
Failed:
_class__sage_combinat_crystals_affine_AffineCrystalFromClassicalAndPromotion_with_
↪category_element_class__.sobj
_class__sage_combinat_crystals_tensor_product_CrystalOfTableaux_with_category_
↪element_class__.sobj
_class__sage_combinat_crystals_tensor_product_
↪TensorProductOfCrystalsWithGenerators_with_category__.sobj
_class__sage_combinat_tableau_Tableau_class__.sobj
----------------------------------------------------------------
** This error is probably due to an old pickle failing to unpickle.
** See sage.structure.sage_object.register_unpickle_override for
** how to override the default unpickling methods for (old) pickles.
** NOTE: pickles should never be removed from the pickle_jar!
----------------------------------------------------------------
Successfully unpickled 583 objects.
Failed to unpickle 4 objects.
sage: register_unpickle_override('sage.combinat.tableau','Tableau_class',tableau_
↪unpickler) # restore to default
```

Todo

Create a custom-made `SourPickle` for the last example.

If you want to find *lots* of little issues in Sage then try the following:

```
sage: sage.structure.sage_object.unpickle_all(run_test_suite = True) # todo: not␣
↪tested
```

This runs `TestSuite` tests on all objects in the Sage pickle jar. Some of those objects seem to unpickle properly, but do not pass the tests because their internal data structure is messed up. In most cases though it is just that their source file misses a TestSuite call, and therefore some misfeatures went unnoticed (typically Parents not implementing the `an_element` method).

Note: Every so often the standard pickle jar should be updated by running the doctest suite with the environment variable `SAGE_PICKLE_JAR` set, then copying the files from `SAGE_ROOT/tmp/pickle_jar*` into the standard pickle jar.

> **Warning:** Sage's pickle jar helps to ensure backward compatibility in sage. Pickles should **only** be removed from the pickle jar after the corresponding objects have been properly deprecated. Any proposal to remove pickles from the pickle jar should first be discussed on sage-devel.

sage.structure.sage_object. **unpickle_global** (*module*, *name*)
  Given a module name and a name within that module (typically a class name), retrieve the corresponding object.

This normally just looks up the name in the module, but it can be overridden by register_unpickle_override. This is used in the Sage unpickling mechanism, so if the Sage source code organization changes, register_unpickle_override can allow old pickles to continue to work.

EXAMPLES:

```
sage: from sage.structure.sage_object import unpickle_override, register_unpickle_
↪override
sage: unpickle_global('sage.rings.integer', 'Integer')
<type 'sage.rings.integer.Integer'>
```

Now we horribly break the pickling system:

```
sage: register_unpickle_override('sage.rings.integer', 'Integer', Rational, call_
↪name=('sage.rings.rational', 'Rational'))
sage: unpickle_global('sage.rings.integer', 'Integer')
<type 'sage.rings.rational.Rational'>
```

and we reach into the internals and put it back:

```
sage: del unpickle_override[('sage.rings.integer', 'Integer')]
sage: unpickle_global('sage.rings.integer', 'Integer')
<type 'sage.rings.integer.Integer'>
```

A meaningful error message with resolution instructions is displayed for old pickles that accidentally got broken because a class or entire module was moved or renamed:

```
sage: unpickle_global('sage.all', 'some_old_class')
Traceback (most recent call last):
...
ImportError: cannot import some_old_class from sage.all, call
register_unpickle_override('sage.all', 'some_old_class', ...)
to fix this

sage: unpickle_global('sage.some_old_module', 'some_old_class')
Traceback (most recent call last):
...
ImportError: cannot import some_old_class from sage.some_old_module, call
register_unpickle_override('sage.some_old_module', 'some_old_class', ...)
to fix this
```

1.2 Base class for objects of a category

CLASS HIERARCHY:

- *SageObject*

  - **CategoryObject**

    * *Parent*

Many category objects in Sage are equipped with generators, which are usually special elements of the object. For example, the polynomial ring $\mathbf{Z}[x, y, z]$ is generated by x, y, and z. In Sage the `i` th generator of an object `X` is obtained using the notation `X.gen(i)`. From the Sage interactive prompt, the shorthand notation `X.i` is also allowed.

The following examples illustrate these functions in the context of multivariate polynomial rings and free modules.

EXAMPLES:

```
sage: R = PolynomialRing(ZZ, 3, 'x')
sage: R.ngens()
3
sage: R.gen(0)
x0
sage: R.gens()
(x0, x1, x2)
sage: R.variable_names()
('x0', 'x1', 'x2')
```

This example illustrates generators for a free module over **Z**.

```
sage: M = FreeModule(ZZ, 4)
sage: M
Ambient free module of rank 4 over the principal ideal domain Integer Ring
sage: M.ngens()
4
sage: M.gen(0)
(1, 0, 0, 0)
sage: M.gens()
((1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1))
```

class sage.structure.category_object. **CategoryObject**

    Bases: *sage.structure.sage_object.SageObject*

    An object in some category.

    Hom (*codomain*, *cat=None*)

        Return the homspace Hom(self,codomain,cat) of all homomorphisms from self to codomain in the category cat. The default category is determined by self.category() and codomain.category().

        EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ, 2)
sage: R.Hom(QQ)
Set of Homomorphisms from Multivariate Polynomial Ring in x, y over Rational␣
↪Field to Rational Field
```

        Homspaces are defined for very general Sage objects, even elements of familiar rings.

```
sage: n = 5; Hom(n,7)
Set of Morphisms from 5 to 7 in Category of elements of Integer Ring
sage: z=(2/3); Hom(z,8/1)
Set of Morphisms from 2/3 to 8 in Category of elements of Rational Field
```

        This example illustrates the optional third argument:

```
sage: QQ.Hom(ZZ, Sets())
Set of Morphisms from Rational Field to Integer Ring in Category of sets
```

    base ()

    base_ring ()

        Return the base ring of self.

        INPUT:

- `self` – an object over a base ring; typically a module

EXAMPLES:

```
sage: from sage.modules.module import Module
sage: Module(ZZ).base_ring()
Integer Ring

sage: F = FreeModule(ZZ,3)
sage: F.base_ring()
Integer Ring
sage: F.__class__.base_ring
<method 'base_ring' of 'sage.structure.category_object.CategoryObject'
 ↪objects>
```

Note that the coordinates of the elements of a module can lie in a bigger ring, the `coordinate_ring`:

```
sage: M = (ZZ^2) * (1/2)
sage: v = M([1/2, 0])
sage: v.base_ring()
Integer Ring
sage: parent(v[0])
Rational Field
sage: v.coordinate_ring()
Rational Field
```

More examples:

```
sage: F = FreeAlgebra(QQ, 'x')
sage: F.base_ring()
Rational Field
sage: F.__class__.base_ring
<method 'base_ring' of 'sage.structure.category_object.CategoryObject'
 ↪objects>

sage: E = CombinatorialFreeModule(ZZ, [1,2,3])
sage: F = CombinatorialFreeModule(ZZ, [2,3,4])
sage: H = Hom(E, F)
sage: H.base_ring()
Integer Ring
sage: H.__class__.base_ring
<method 'base_ring' of 'sage.structure.category_object.CategoryObject'
 ↪objects>
```

Todo

Move this method elsewhere (typically in the Modules category) so as not to pollute the namespace of all category objects.

`categories()`
Return the categories of `self`.

EXAMPLES:

```
sage: ZZ.categories()
[Join of Category of euclidean domains
     and Category of infinite enumerated sets
     and Category of metric spaces,
```

```
Category of euclidean domains,
Category of principal ideal domains,
Category of unique factorization domains,
Category of gcd domains,
Category of integral domains,
Category of domains,
Category of commutative rings, ...
Category of monoids, ...,
Category of commutative additive groups, ...,
Category of sets, ...,
Category of objects]
```

category ( )

gens_dict ( )

Return a dictionary whose entries are {name:variable,...} , where name stands for the variable names of this object (as strings) and variable stands for the corresponding defining generators (as elements of this object).

EXAMPLES:

```
sage: B.<a,b,c,d> = BooleanPolynomialRing()
sage: B.gens_dict()
{'a': a, 'b': b, 'c': c, 'd': d}
```

gens_dict_recursive ( )

Return the dictionary of generators of self and its base rings.

OUTPUT:

•a dictionary with string names of generators as keys and generators of self and its base rings as values.

EXAMPLES:

```
sage: R = QQ['x,y']['z,w']
sage: sorted(R.gens_dict_recursive().items())
[('w', w), ('x', x), ('y', y), ('z', z)]
```

has_base (*category=None*)

inject_variables (*scope=None*, *verbose=True*)

Inject the generators of self with their names into the namespace of the Python code from which this function is called. Thus, e.g., if the generators of self are labeled 'a', 'b', and 'c', then after calling this method the variables a, b, and c in the current scope will be set equal to the generators of self.

NOTE: If Foo is a constructor for a Sage object with generators, and Foo is defined in Cython, then it would typically call inject_variables() on the object it creates. E.g., PolynomialRing(QQ,'y') does this so that the variable y is the generator of the polynomial ring.

latex_name ( )

latex_variable_names ( )

Returns the list of variable names suitable for latex output.

All _SOMETHING substrings are replaced by _{SOMETHING} recursively so that subscripts of subscripts work.

EXAMPLES:

```
sage: R, x = PolynomialRing(QQ, 'x', 12).objgens()
sage: x
(x0, x1, x2, x3, x4, x5, x6, x7, x8, x9, x10, x11)
sage: R.latex_variable_names ()
['x_{0}', 'x_{1}', 'x_{2}', 'x_{3}', 'x_{4}', 'x_{5}', 'x_{6}', 'x_{7}', 'x_
↪{8}', 'x_{9}', 'x_{10}', 'x_{11}']
sage: f = x[0]^3 + 15/3 * x[1]^10
sage: print(latex(f))
5 x_{1}^{10} + x_{0}^{3}
```

normalize_names (*ngens*, *names*)

objgen ( )
Return the tuple (self,self.gen()).

EXAMPLES:

```
sage: R, x = PolynomialRing(QQ,'x').objgen()
sage: R
Univariate Polynomial Ring in x over Rational Field
sage: x
x
```

objgens ( )
Return the tuple (self,self.gens()).

EXAMPLES:

```
sage: R = PolynomialRing(QQ, 3, 'x'); R
Multivariate Polynomial Ring in x0, x1, x2 over Rational Field
sage: R.objgens()
(Multivariate Polynomial Ring in x0, x1, x2 over Rational Field, (x0, x1, x2))
```

variable_name ( )
Return the first variable name.

OUTPUT: a string

EXAMPLES:

```
sage: R.<z,y,a42> = ZZ[]
sage: R.variable_name()
'z'
sage: R.<x> = InfinitePolynomialRing(ZZ)
sage: R.variable_name()
'x'
```

variable_names ( )
Return the list of variable names corresponding to the generators.

OUTPUT: a tuple of strings

EXAMPLES:

```
sage: R.<z,y,a42> = QQ[]
sage: R.variable_names()
('z', 'y', 'a42')
sage: S = R.quotient_ring(z+y)
sage: S.variable_names()
('zbar', 'ybar', 'a42bar')
```

```
sage: T.<x> = InfinitePolynomialRing(ZZ)
sage: T.variable_names()
('x',)
```

sage.structure.category_object. **certify_names** (*names*)

Check that `names` are valid variable names.

INPUT:

- `names` – an iterable with strings representing variable names

OUTPUT: `True` (for efficiency of the Cython call)

EXAMPLES:

```
sage: from sage.structure.category_object import certify_names as cn
sage: cn(["a", "b", "c"])
1
sage: cn("abc")
1
sage: cn([])
1
sage: cn([""])
Traceback (most recent call last):
...
ValueError: variable name must be nonempty
sage: cn(["_foo"])
Traceback (most recent call last):
...
ValueError: variable name '_foo' does not start with a letter
sage: cn(["x'"])
Traceback (most recent call last):
...
ValueError: variable name "x'" is not alphanumeric
sage: cn(["a", "b", "b"])
Traceback (most recent call last):
...
ValueError: variable name 'b' appears more than once
```

sage.structure.category_object. **check_default_category** (*default_category*, *category*)

sage.structure.category_object. **guess_category** (*obj*)

sage.structure.category_object. **normalize_names** (*ngens*, *names*)

Return a tuple of strings of variable names of length ngens given the input names.

INPUT:

- `ngens` – integer: number of generators. The value `ngens=-1` means that the number of generators is unknown a priori.

- `names` – any of the following:

  - a tuple or list of strings, such as (`'x'`,`'y'`)

  - a comma-separated string, such as `x,y`

  - a string prefix, such as 'alpha'

  - a string of single character names, such as 'xyz'

OUTPUT: a tuple of `ngens` strings to be used as variable names.

EXAMPLES:

```
sage: from sage.structure.category_object import normalize_names as nn
sage: nn(0, "")
()
sage: nn(0, [])
()
sage: nn(0, None)
()
sage: nn(1, 'a')
('a',)
sage: nn(2, 'z_z')
('z_z0', 'z_z1')
sage: nn(3, 'x, y, z')
('x', 'y', 'z')
sage: nn(2, 'ab')
('a', 'b')
sage: nn(2, 'x0')
('x00', 'x01')
sage: nn(3, (' a ', ' bb ', ' ccc '))
('a', 'bb', 'ccc')
sage: nn(4, ['a1', 'a2', 'b1', 'b11'])
('a1', 'a2', 'b1', 'b11')
```

Arguments are converted to strings:

```
sage: nn(1, u'a')
('a',)
sage: var('alpha')
alpha
sage: nn(2, alpha)
('alpha0', 'alpha1')
sage: nn(1, [alpha])
('alpha',)
```

With an unknown number of generators:

```
sage: nn(-1, 'a')
('a',)
sage: nn(-1, 'x, y, z')
('x', 'y', 'z')
```

Test errors:

```
sage: nn(3, ["x", "y"])
Traceback (most recent call last):
...
IndexError: the number of names must equal the number of generators
sage: nn(None, "a")
Traceback (most recent call last):
...
TypeError: 'NoneType' object cannot be interpreted as an index
sage: nn(1, "")
Traceback (most recent call last):
...
ValueError: variable name must be nonempty
sage: nn(1, "foo@")
Traceback (most recent call last):
...
```

```
ValueError: variable name 'foo@' is not alphanumeric
sage: nn(2, "_foo")
Traceback (most recent call last):
...
ValueError: variable name '_foo0' does not start with a letter
sage: nn(1, 3/2)
Traceback (most recent call last):
...
ValueError: variable name '3/2' is not alphanumeric
```

PARENTS

2.1 Parents

2.1.1 Base class for parent objects

CLASS HIERARCHY:

```
SageObject
    CategoryObject
        Parent
```

A simple example of registering coercions:

```
sage: class A_class(Parent):
....:    def __init__(self, name):
....:        Parent.__init__(self, name=name)
....:        self._populate_coercion_lists_()
....:        self.rename(name)
....:    #
....:    def category(self):
....:        return Sets()
....:    #
....:    def _element_constructor_(self, i):
....:        assert(isinstance(i, (int, Integer)))
....:        return ElementWrapper(self, i)
....:
sage: A = A_class("A")
sage: B = A_class("B")
sage: C = A_class("C")

sage: def f(a):
....:    return B(a.value+1)
....:
sage: class MyMorphism(Morphism):
....:    def __init__(self, domain, codomain):
....:        Morphism.__init__(self, Hom(domain, codomain))
....:    #
....:    def _call_(self, x):
....:        return self.codomain()(x.value)
....:
sage: f = MyMorphism(A,B)
sage: f
    Generic morphism:
      From: A
```

```
       To:   B
sage: B.register_coercion(f)
sage: C.register_coercion(MyMorphism(B,C))
sage: A(A(1)) == A(1)
True
sage: B(A(1)) == B(1)
True
sage: C(A(1)) == C(1)
True


sage: A(B(1))
Traceback (most recent call last):
...
AssertionError
```

When implementing an element of a ring, one would typically provide the element class with _rmul_ and/or _lmul_ methods for the action of a base ring, and with _mul_ for the ring multiplication. However, prior to trac ticket #14249, it would have been necessary to additionally define a method _an_element_() for the parent. But now, the following example works:

```
sage: from sage.structure.element import RingElement
sage: class MyElement(RingElement):
....:       def __init__(self, parent, x, y):
....:           RingElement.__init__(self, parent)
....:       def _mul_(self, other):
....:           return self
....:       def _rmul_(self, other):
....:           return self
....:       def _lmul_(self, other):
....:           return self
sage: class MyParent(Parent):
....:       Element = MyElement
```

Now, we define

```
sage: P = MyParent(base=ZZ, category=Rings())
sage: a = P(1,2)
sage: a*a is a
True
sage: a*2 is a
True
sage: 2*a is a
True
```

```
sage: sage.structure.parent.normalize_names(5, 'x')
doctest:...: DeprecationWarning:
Importing normalize_names from here is deprecated. If you need to use it, please␣
↪import it directly from sage.structure.category_object
See http://trac.sagemath.org/19675 for details.
('x0', 'x1', 'x2', 'x3', 'x4')
sage: sage.structure.parent.normalize_names(2, ['x','y'])
('x', 'y')
```

class sage.structure.parent. **EltPair**
    Bases: object

    short_repr ()

---

class sage.structure.parent. **Parent**

    Bases: *sage.structure.category_object.CategoryObject*

    Base class for all parents.

    Parents are the Sage/mathematical analogues of container objects in computer science.

    INPUT:

    • base – An algebraic structure considered to be the "base" of this parent (e.g. the base field for a vector space).

    • category – a category or list/tuple of categories. The category in which this parent lies (or list or tuple thereof). Since categories support more general super-categories, this should be the most specific category possible. If category is a list or tuple, a JoinCategory is created out of them. If category is not specified, the category will be guessed (see *CategoryObject*), but won't be used to inherit parent's or element's code from this category.

    • element_constructor – A class or function that creates elements of this Parent given appropriate input (can also be filled in later with *_populate_coercion_lists_()*)

    • names – Names of generators.

    • normalize – Whether to standardize the names (remove punctuation, etc)

    • facade – a parent, or tuple thereof, or True

    If facade is specified, then Sets().Facade() is added to the categories of the parent. Furthermore, if facade is not True , the internal attribute _facade_for is set accordingly for use by Sets.Facade.ParentMethods.facade_for() .

    Internal invariants:

    • self._element_init_pass_parent == guess_pass_parent(self,self._element_constructor) Ensures that *__call__()* passes down the parent properly to _element_constructor() . See trac ticket #5979.

    Todo

    Eventually, category should be Sets by default.

    __call__ (*x=0*, **args*, ***kwds*)

        This is the generic call method for all parents.

        When called, it will find a map based on the Parent (or type) of x. If a coercion exists, it will always be chosen. This map will then be called (with the arguments and keywords if any).

        By default this will dispatch as quickly as possible to _element_constructor_() though faster pathways are possible if so desired.

    _populate_coercion_lists_ (*coerce_list=[]*, *action_list=[]*, *convert_list=[]*, *embedding=None*, *convert_method_name=None*, *element_constructor=None*, *init_no_parent=None*, *unpickling=False*)

        This function allows one to specify coercions, actions, conversions and embeddings involving this parent.

        IT SHOULD ONLY BE CALLED DURING THE __INIT__ method, often at the end.

        INPUT:

        • coerce_list – a list of coercion Morphisms to self and parents with canonical coercions to self

        • action_list – a list of actions on and by self

- **convert_list** – **a list of conversion Maps to self and** parents with conversions to self

- embedding – a single Morphism from self

- convert_method_name – a name to look for that other elements can implement to create elements of self (e.g. _integer_)

- element_constructor – A callable object used by the __call__ method to construct new elements. Typically the element class or a bound method (defaults to self._element_constructor_).

- init_no_parent – if True omit passing self in as the first argument of element_constructor for conversion. This is useful if parents are unique, or element_constructor is a bound method (this latter case can be detected automatically).

__mul__ (*x*)

This is a multiplication method that more or less directly calls another attribute _mul_ (single underscore). This is because __mul__ can not be implemented via inheritance from the parent methods of the category, but _mul_ can be inherited. This is, e.g., used when creating twosided ideals of matrix algebras. See trac ticket #7797.

EXAMPLES:

```
sage: MS = MatrixSpace(QQ,2,2)
```

This matrix space is in fact an algebra, and in particular it is a ring, from the point of view of categories:

```
sage: MS.category()
Category of infinite algebras over (quotient fields and metric spaces)
sage: MS in Rings()
True
```

However, its class does not inherit from the base class `Ring` :

```
sage: isinstance(MS,Ring)
False
```

Its _mul_ method is inherited from the category, and can be used to create a left or right ideal:

```
sage: MS._mul_.__module__
'sage.categories.rings'
sage: MS*MS.1        # indirect doctest
Left Ideal
(
  [0 1]
  [0 0]
)
 of Full MatrixSpace of 2 by 2 dense matrices over Rational Field
sage: MS*[MS.1,2]
Left Ideal
(
  [0 1]
  [0 0],

  [2 0]
  [0 2]
)
 of Full MatrixSpace of 2 by 2 dense matrices over Rational Field
sage: MS.1*MS
Right Ideal
(
```

```
  [0 1]
  [0 0]
)
 of Full MatrixSpace of 2 by 2 dense matrices over Rational Field
sage: [MS.1,2]*MS
Right Ideal
(
  [0 1]
  [0 0],

  [2 0]
  [0 2]
)
 of Full MatrixSpace of 2 by 2 dense matrices over Rational Field
```

__contains__ (*x*)

> True if there is an element of self that is equal to x under ==, or if x is already an element of self. Also, True in other cases involving the Symbolic Ring, which is handled specially.
>
> For many structures we test this by using *__call__()* and then testing equality between x and the result.
>
> The Symbolic Ring is treated differently because it is ultra-permissive about letting other rings coerce in, but ultra-strict about doing comparisons.
>
> EXAMPLES:

```
sage: 2 in Integers(7)
True
sage: 2 in ZZ
True
sage: Integers(7)(3) in ZZ
True
sage: 3/1 in ZZ
True
sage: 5 in QQ
True
sage: I in RR
False
sage: SR(2) in ZZ
True
sage: RIF(1, 2) in RIF
True
sage: pi in RIF # there is no element of RIF equal to pi
False
sage: sqrt(2) in CC
True
sage: pi in RR
True
sage: pi in CC
True
sage: pi in RDF
True
sage: pi in CDF
True
```

> Note that we have

```
sage: 3/2 in RIF
True
```

because `3/2` has an exact representation in `RIF` (i.e. can be represented as an interval that contains exactly one value):

```
sage: RIF(3/2).is_exact()
True
```

On the other hand, we have

```
sage: 2/3 in RIF
False
```

because `2/3` has no exact representation in `RIF`. Since `RIF(2/3)` is a nontrivial interval, it can not be equal to anything (not even itself):

```
sage: RIF(2/3).is_exact()
False
sage: RIF(2/3).endpoints()
(0.666666666666666, 0.666666666666667)
sage: RIF(2/3) == RIF(2/3)
False
```

_coerce_map_from_ (*S*)

Override this method to specify coercions beyond those specified in coerce_list.

If no such coercion exists, return None or False. Otherwise, it may return either an actual Map to use for the coercion, a callable (in which case it will be wrapped in a Map), or True (in which case a generic map will be provided).

_convert_map_from_ (*S*)

Override this method to provide additional conversions beyond those given in convert_list.

This function is called after coercions are attempted. If there is a coercion morphism in the opposite direction, one should consider adding a section method to that.

This MUST return a Map from S to self, or None. If None is returned then a generic map will be provided.

_get_action_ (*S*, *op*, *self_on_left*)

Override this method to provide an action of self on S or S on self beyond what was specified in action_list.

This must return an action which accepts an element of self and an element of S (in the order specified by self_on_left).

_an_element_ ( )

Returns an element of self. Want it in sufficient generality that poorly-written functions won't work when they're not supposed to. This is cached so doesn't have to be super fast.

EXAMPLES:

```
sage: QQ._an_element_()
1/2
sage: ZZ['x,y,z']._an_element_()
x
```

_repr_option (*key*)

Metadata about the _repr_() output.

INPUT:

- `key` – string. A key for different metadata informations that can be inquired about.

Valid `key` arguments are:

- `'ascii_art'` : The `_repr_()` output is multi-line ascii art and each line must be printed starting at the same column, or the meaning is lost.

- `'element_ascii_art'` : same but for the output of the elements. Used in `sage.repl.display.formatter`.

- `'element_is_atomic'` : the elements print atomically, that is, parenthesis are not required when *printing* out any of $x - y$, $x + y$, x^y and x/y.

OUTPUT:

Boolean.

EXAMPLES:

```
sage: ZZ._repr_option('ascii_art')
False
sage: MatrixSpace(ZZ, 2)._repr_option('element_ascii_art')
True
```

_init_category_ (*category*)
    Initialize the category framework

    Most parents initialize their category upon construction, and this is the recommended behavior. For example, this happens when the constructor calls `Parent.__init__()` directly or indirectly. However, some parents defer this for performance reasons. For example, `sage.matrix.matrix_space.MatrixSpace` does not.

    EXAMPLES:

```
sage: P = Parent()
sage: P.category()
Category of sets
sage: class MyParent(Parent):
....:     def __init__(self):
....:         self._init_category_(Groups())
sage: MyParent().category()
Category of groups
```

Hom (*codomain*, *category=None*)
    Return the homspace `Hom(self,codomain,category)` .

    INPUT:

- `codomain` – a parent

- `category` – a category or `None` (default: `None`) If `None` , the meet of the category of `self` and `codomain` is used.

    OUTPUT:

    The homspace of all homomorphisms from `self` to `codomain` in the category `category` .

    See also:

    `Hom()`

    EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ, 2)
sage: R.Hom(QQ)
Set of Homomorphisms from Multivariate Polynomial Ring in x, y over Rational␣
 ↪Field to Rational Field
```

Homspaces are defined for very general Sage objects, even elements of familiar rings:

```
sage: n = 5; Hom(n,7)
Set of Morphisms from 5 to 7 in Category of elements of Integer Ring
sage: z=(2/3); Hom(z,8/1)
Set of Morphisms from 2/3 to 8 in Category of elements of Rational Field
```

This example illustrates the optional third argument:

```
sage: QQ.Hom(ZZ, Sets())
Set of Morphisms from Rational Field to Integer Ring in Category of sets
```

A parent may specify how to construct certain homsets by implementing a method
`_Hom_`(codomain,category). See :func:`~sage.categories.homset.Hom()`
for details.

an_element ( )

Returns a (preferably typical) element of this parent.

This is used both for illustration and testing purposes. If the set `self` is empty, *an_element()* raises
the exception `EmptySetError`.

This calls *_an_element_()* (which see), and caches the result. Parent are thus encouraged to override
_an_element_().

EXAMPLES:

```
sage: CDF.an_element()
1.0*I
sage: ZZ[['t']].an_element()
t
```

In case the set is empty, an `EmptySetError` is raised:

```
sage: Set([]).an_element()
Traceback (most recent call last):
...
EmptySetError
```

category ( )

EXAMPLES:

```
sage: P = Parent()
sage: P.category()
Category of sets
sage: class MyParent(Parent):
....:     def __init__(self): pass
sage: MyParent().category()
Category of sets
```

coerce (*x*)

Return x as an element of self, if and only if there is a canonical coercion from the parent of x to self.

EXAMPLES:

```
sage: QQ.coerce(ZZ(2))
2
sage: ZZ.coerce(QQ(2))
Traceback (most recent call last):
...
TypeError: no canonical coercion from Rational Field to Integer Ring
```

We make an exception for zero:

```
sage: V = GF(7)^7
sage: V.coerce(0)
(0, 0, 0, 0, 0, 0, 0)
```

coerce_embedding ( )

> Return the embedding of self into some other parent, if such a parent exists.
>
> This does not mean that there are no coercion maps from self into other fields, this is simply a specific morphism specified out of self and usually denotes a special relationship (e.g. sub-objects, choice of completion, etc.)
>
> EXAMPLES:

```
sage: K.<a>=NumberField(x^3+x^2+1,embedding=1)
sage: K.coerce_embedding()
Generic morphism:
  From: Number Field in a with defining polynomial x^3 + x^2 + 1
  To:   Real Lazy Field
  Defn: a -> -1.465571231876768?
sage: K.<a>=NumberField(x^3+x^2+1,embedding=CC.gen())
sage: K.coerce_embedding()
Generic morphism:
  From: Number Field in a with defining polynomial x^3 + x^2 + 1
  To:   Complex Lazy Field
  Defn: a -> 0.2327856159383841? + 0.7925519925154479?*I
```

coerce_map_from (*S*)

> Return a Map object to coerce from S to self if one exists, or None if no such coercion exists.
>
> EXAMPLES:
>
> By trac ticket #12313, a special kind of weak key dictionary is used to store coercion and conversion maps, namely MonoDict. In that way, a memory leak was fixed that would occur in the following test:

```
sage: import gc
sage: _ = gc.collect()
sage: K = GF(1<<55,'t')
sage: for i in range(50):
....:    a = K.random_element()
....:    E = EllipticCurve(j=a)
....:    b = K.has_coerce_map_from(E)
sage: _ = gc.collect()
sage: len([x for x in gc.get_objects() if isinstance(x,type(E))])
1
```

construction ( )

> Returns a pair (functor, parent) such that functor(parent) return self. If this ring does not have a functorial construction, return None.
>
> EXAMPLES:

```
sage: QQ.construction()
(FractionField, Integer Ring)
sage: f, R = QQ['x'].construction()
sage: f
Poly[x]
sage: R
Rational Field
sage: f(R)
Univariate Polynomial Ring in x over Rational Field
```

convert_map_from (S)

> This function returns a `Map` from S to $self$, which may or may not succeed on all inputs. If a coercion map from S to self exists, then the it will be returned. If a coercion from $self$ to S exists, then it will attempt to return a section of that map.

> Under the new coercion model, this is the fastest way to convert elements of S to elements of $self$ (short of manually constructing the elements) and is used by *__call__()* .

> EXAMPLES:

```
sage: m = ZZ.convert_map_from(QQ)
sage: m
Generic map:
  From: Rational Field
  To:   Integer Ring
sage: m(-35/7)
-5
sage: parent(m(-35/7))
Integer Ring
```

element_class ( )

> The (default) class for the elements of this parent

> FIXME's and design issues:

> • If self.Element is "trivial enough", should we optimize it away with: self.element_class = dynamic_class("%s.element_class"%self.__class__.__name__, (category.element_class,), self.Element)

> • This should lookup for Element classes in all super classes

get_action (S, *op=None*, *self_on_left=True*, *self_el=None*, *S_el=None*)

> Returns an action of self on S or S on self.

> To provide additional actions, override *_get_action_()* .

has_coerce_map_from (S)

> Return True if there is a natural map from S to self. Otherwise, return False.

> EXAMPLES:

```
sage: RDF.has_coerce_map_from(QQ)
True
sage: RDF.has_coerce_map_from(QQ['x'])
False
sage: RDF['x'].has_coerce_map_from(QQ['x'])
True
sage: RDF['x,y'].has_coerce_map_from(QQ['x'])
True
```

hom (*im_gens*, *codomain=None*, *check=None*)

> Return the unique homomorphism from self to codomain that sends `self.gens()` to the entries of `im_gens` . Raises a TypeError if there is no such homomorphism.
>
> INPUT:
>
> > •`im_gens` – the images in the codomain of the generators of this object under the homomorphism
> >
> > •`codomain` – the codomain of the homomorphism
> >
> > •`check` – whether to verify that the images of generators extend to define a map (using only canonical coercions).
>
> OUTPUT:
>
> A homomorphism self –> codomain
>
> ---
>
> **Note:** As a shortcut, one can also give an object X instead of `im_gens` , in which case return the (if it exists) natural map to X.
>
> ---
>
> EXAMPLES:
>
> Polynomial Ring: We first illustrate construction of a few homomorphisms involving a polynomial ring:
>
> ```
> sage: R.<x> = PolynomialRing(ZZ)
> sage: f = R.hom([5], QQ)
> sage: f(x^2 - 19)
> 6
>
> sage: R.<x> = PolynomialRing(QQ)
> sage: f = R.hom([5], GF(7))
> Traceback (most recent call last):
> ...
> TypeError: images do not define a valid homomorphism
>
> sage: R.<x> = PolynomialRing(GF(7))
> sage: f = R.hom([3], GF(49,'a'))
> sage: f
> Ring morphism:
>   From: Univariate Polynomial Ring in x over Finite Field of size 7
>   To:   Finite Field in a of size 7^2
>   Defn: x |--> 3
> sage: f(x+6)
> 2
> sage: f(x^2+1)
> 3
> ```
>
> Natural morphism:
>
> ```
> sage: f = ZZ.hom(GF(5))
> sage: f(7)
> 2
> sage: f
> Ring Coercion morphism:
>   From: Integer Ring
>   To:   Finite Field of size 5
> ```
>
> There might not be a natural morphism, in which case a `TypeError` is raised:

```
sage: QQ.hom(ZZ)
Traceback (most recent call last):
...
TypeError: Natural coercion morphism from Rational Field to Integer Ring not␣
↪defined.
```

is_atomic_repr ( )

The old way to signal atomic string reps.

True if the elements have atomic string representations, in the sense that if they print at s, then -s means the negative of s. For example, integers are atomic but polynomials are not.

EXAMPLES:

```
sage: Parent().is_atomic_repr()
doctest:...: DeprecationWarning: Use _repr_option to return metadata about␣
↪string rep
See http://trac.sagemath.org/14040 for details.
False
```

is_coercion_cached (*domain*)

is_conversion_cached (*domain*)

is_exact ( )

Test whether the ring is exact.

Note: This defaults to true, so even if it does return `True` you have no guarantee (unless the ring has properly overloaded this).

OUTPUT:

Return True if elements of this ring are represented exactly, i.e., there is no precision loss when doing arithmetic.

EXAMPLES:

```
sage: QQ.is_exact()
True
sage: ZZ.is_exact()
True
sage: Qp(7).is_exact()
False
sage: Zp(7, type='capped-abs').is_exact()
False
```

register_action (*action*)

Update the coercion model to use `action` to act on self.

`action` should be of type `sage.categories.action.Action`.

EXAMPLES:

```
sage: import sage.categories.action
sage: import operator

sage: class SymmetricGroupAction(sage.categories.action.Action):
....:     "Act on a multivariate polynomial ring by permuting the generators."
....:     def __init__(self, G, M, is_left=True):
```

```
....:          sage.categories.action.Action.__init__(self, G, M, is_left,
→operator.mul)
....:
....:      def _call_(self, g, a):
....:          if not self.is_left():
....:              g, a = a, g
....:          D = {}
....:          for k, v in a.dict().items():
....:              nk = [0]*len(k)
....:              for i in range(len(k)):
....:                  nk[g(i+1)-1] = k[i]
....:              D[tuple(nk)] = v
....:          return a.parent()(D)

sage: R.<x, y, z> = QQ['x, y, z']
sage: G = SymmetricGroup(3)
sage: act = SymmetricGroupAction(G, R)
sage: t = x + 2*y + 3*z

sage: act(G((1, 2)), t)
2*x + y + 3*z
sage: act(G((2, 3)), t)
x + 3*y + 2*z
sage: act(G((1, 2, 3)), t)
3*x + y + 2*z
```

This should fail, since we haven't registered the left action:

```
sage: G((1,2)) * t
Traceback (most recent call last):
...
TypeError: ...
```

Now let's make it work:

```
sage: R._unset_coercions_used()
sage: R.register_action(act)
sage: G((1, 2)) * t
2*x + y + 3*z
```

register_coercion (*mor*)

Update the coercion model to use $mor : P \to$ self to coerce from a parent `P` into `self`.

For safety, an error is raised if another coercion has already been registered or discovered between `P` and `self`.

EXAMPLES:

```
sage: K.<a> = ZZ['a']
sage: L.<b> = ZZ['b']
sage: L_into_K = L.hom([-a]) # non-trivial automorphism
sage: K.register_coercion(L_into_K)

sage: K(0) + b
-a
sage: a + b
0
sage: K(b) # check that convert calls coerce first; normally this is just a
```

```
-a

sage: L(0) + a in K # this goes through the coercion mechanism of K
True
sage: L(a) in L # this still goes through the convert mechanism of L
True

sage: K.register_coercion(L_into_K)
Traceback (most recent call last):
...
AssertionError: coercion from Univariate Polynomial Ring in b over Integer␣
→Ring to Univariate Polynomial Ring in a over Integer Ring already␣
→registered or discovered
```

register_conversion (*mor*)

Update the coercion model to use mor : $P \to$ self to convert from `P` into `self` .

EXAMPLES:

```
sage: K.<a> = ZZ['a']
sage: M.<c> = ZZ['c']
sage: M_into_K = M.hom([a]) # trivial automorphism
sage: K._unset_coercions_used()
sage: K.register_conversion(M_into_K)

sage: K(c)
a
sage: K(0) + c
Traceback (most recent call last):
...
TypeError: ...
```

register_embedding (*embedding*)

Add embedding to coercion model.

This method updates the coercion model to use embedding : self $\to P$ to embed `self` into the parent `P` .

There can only be one embedding registered; it can only be registered once; and it must be registered before using this parent in the coercion model.

EXAMPLES:

```
sage: S3 = AlternatingGroup(3)
sage: G = SL(3, QQ)
sage: p = S3[2]; p.matrix()
[0 0 1]
[1 0 0]
[0 1 0]
```

In general one can't mix matrices and permutations:

```
sage: G(p)
Traceback (most recent call last):
...
TypeError: entries must be coercible to a list or integer
sage: phi = S3.hom(lambda p: G(p.matrix()), codomain = G)
sage: phi(p)
[0 0 1]
[1 0 0]
```

```
[0 1 0]
sage: S3._unset_coercions_used()
sage: S3.register_embedding(phi)
```

By trac ticket #14711, coerce maps should be copied when using outside of the coercion system:

```
sage: phi = copy(S3.coerce_embedding()); phi
Generic morphism:
  From: Alternating group of order 3!/2 as a permutation group
  To:   Special Linear Group of degree 3 over Rational Field
sage: phi(p)
[0 0 1]
[1 0 0]
[0 1 0]
```

This does not work since matrix groups are still old-style parents (see trac ticket #14014):

```
sage: G(p)                                          # todo: not implemented
```

Though one can have a permutation act on the rows of a matrix:

```
sage: G(1) * p
[0 0 1]
[1 0 0]
[0 1 0]
```

Some more advanced examples:

```
sage: x = QQ['x'].0
sage: t = abs(ZZ.random_element(10^6))
sage: K = NumberField(x^2 + 2*3*7*11, "a"+str(t))
sage: a = K.gen()
sage: K_into_MS = K.hom([a.matrix()])
sage: K._unset_coercions_used()
sage: K.register_embedding(K_into_MS)

sage: L = NumberField(x^2 + 2*3*7*11*19*31, "b"+str(abs(ZZ.random_element(10^
→6))))
sage: b = L.gen()
sage: L_into_MS = L.hom([b.matrix()])
sage: L._unset_coercions_used()
sage: L.register_embedding(L_into_MS)

sage: K.coerce_embedding()(a)
[   0    1]
[-462    0]
sage: L.coerce_embedding()(b)
[      0       1]
[-272118       0]

sage: a.matrix() * b.matrix()
[-272118        0]
[      0     -462]
sage: a.matrix() * b.matrix()
[-272118        0]
[      0     -462]
```

sage.structure.parent.**Set_PythonType**(*theType*)

Return the (unique) Parent that represents the set of Python objects of a specified type.

EXAMPLES:

```
sage: from sage.structure.parent import Set_PythonType
sage: Set_PythonType(list)
Set of Python objects of type 'list'
sage: Set_PythonType(list) is Set_PythonType(list)
True
sage: S = Set_PythonType(tuple)
sage: S([1,2,3])
(1, 2, 3)

S is a parent which models the set of all lists:
sage: S.category()
Category of sets
```

EXAMPLES:

```
sage: R = sage.structure.parent.Set_PythonType(int)
sage: S = sage.structure.parent.Set_PythonType(float)
sage: Hom(R, S)
Set of Morphisms from Set of Python objects of type 'int' to Set of Python
→objects of type 'float' in Category of sets
```

class sage.structure.parent. **Set_PythonType_class**

Bases: *sage.structure.parent.Set_generic*

The set of Python objects of a given type.

EXAMPLES:

```
sage: S = sage.structure.parent.Set_PythonType(int)
sage: S
Set of Python objects of type 'int'
sage: int('1') in S
True
sage: Integer('1') in S
False

sage: sage.structure.parent.Set_PythonType(2)
Traceback (most recent call last):
...
TypeError: must be initialized with a type, not 2
```

cardinality ()

EXAMPLES:

```
sage: S = sage.structure.parent.Set_PythonType(bool)
sage: S.cardinality()
2
sage: S = sage.structure.parent.Set_PythonType(int)
sage: S.cardinality()
4294967296                         # 32-bit
18446744073709551616               # 64-bit
sage: S = sage.structure.parent.Set_PythonType(float)
sage: S.cardinality()
18437736874454810627
sage: S = sage.structure.parent.Set_PythonType(long)
```

```
sage: S.cardinality()
+Infinity
```

object ( )
    EXAMPLES:

```
sage: S = sage.structure.parent.Set_PythonType(tuple)
sage: S.object()
<... 'tuple'>
```

class sage.structure.parent. **Set_generic**
    Bases: *sage.structure.parent.Parent*

    Abstract base class for sets.

    object ( )
        Return the underlying object of self .

        EXAMPLES:

```
sage: Set(QQ).object()
Rational Field
```

sage.structure.parent. **is_Parent** (*x*)
    Return True if x is a parent object, i.e., derives from sage.structure.parent.Parent and False otherwise.

    EXAMPLES:

```
sage: from sage.structure.parent import is_Parent
sage: is_Parent(2/3)
False
sage: is_Parent(ZZ)
True
sage: is_Parent(Primes())
True
```

2.1.2 Indexed Generators

class sage.structure.indexed_generators. **IndexedGenerators** (*indices,   prefix='x',*
                                                **kwds)
    Bases: object

    Abstract base class for parents whose elements consist of generators indexed by an arbitrary set.

    Options controlling the printing of elements:

        •prefix – string, prefix used for printing elements of this module (optional, default 'x'). With the default, a monomial indexed by 'a' would be printed as x['a'] .

        •latex_prefix – string or None , prefix used in the LATEX representation of elements (optional, default None). If this is anything except the empty string, it prints the index as a subscript. If this is None, it uses the setting for prefix , so if prefix is set to "B", then a monomial indexed by 'a' would be printed as B_{a} . If this is the empty string, then don't print monomials as subscripts: the monomial indexed by 'a' would be printed as a , or as [a] if latex_bracket is True.

        •bracket – None , bool, string, or list or tuple of strings (optional, default None): if None , use the value of the attribute self._repr_option_bracket , which has default value True . (self._repr_option_bracket is available for backwards compatibility. Users should set bracket instead. If bracket is set to anything except None , it overrides the value of

self._repr_option_bracket .) If `False` , do not include brackets when printing elements: a monomial indexed by 'a' would be printed as `B'a'` , and a monomial indexed by (1,2,3) would be printed as `B(1,2,3)` . If True, use "[" and "]" as brackets. If it is one of "[", "(", or "{", use it and its partner as brackets. If it is any other string, use it as both brackets. If it is a list or tuple of strings, use the first entry as the left bracket and the second entry as the right bracket.

- `latex_bracket` – bool, string, or list or tuple of strings (optional, default False): if `False` , do not include brackets in the LaTeX representation of elements. This option is only relevant if `latex_prefix` is the empty string; otherwise, brackets are not used regardless. If `True` , use "left[" and "right]" as brackets. If this is one of "[", "(", "\{", "|", or "||", use it and its partner, prepended with "left" and "right", as brackets. If this is any other string, use it as both brackets. If this is a list or tuple of strings, use the first entry as the left bracket and the second entry as the right bracket.

- `scalar_mult` – string to use for scalar multiplication in the print representation (optional, default "*")

- `latex_scalar_mult` – string or `None` (default: `None`), string to use for scalar multiplication in the latex representation. If None, use the empty string if `scalar_mult` is set to "*", otherwise use the value of `scalar_mult` .

- `tensor_symbol` – string or `None` (default: `None`), string to use for tensor product in the print representation. If `None` , use `sage.categories.tensor.symbol` .

- `generator_cmp` – deprecated

- `sorting_key` – a key function (default: `lambda x:   x`), to use for sorting elements in the output of elements

- `sorting_reverse` – bool (default: `False`), if `True` sort elements in reverse order in the output of elements

- `string_quotes` – bool (default: `True`), if `True` then display string indices with quotes

---

Note: These print options may also be accessed and modified using the *`print_options()`* method, after the parent has been defined.

EXAMPLES:

We demonstrate a variety of the input options:

```
sage: from sage.structure.indexed_generators import IndexedGenerators
sage: I = IndexedGenerators(ZZ, prefix='A')
sage: I._repr_generator(2)
'A[2]'
sage: I._latex_generator(2)
'A_{2}'

sage: I = IndexedGenerators(ZZ, bracket='(')
sage: I._repr_generator(2)
'x(2)'
sage: I._latex_generator(2)
'x_{2}'

sage: I = IndexedGenerators(ZZ, prefix="", latex_bracket='(')
sage: I._repr_generator(2)
'[2]'
sage: I._latex_generator(2)
\left( 2 \right)

sage: I = IndexedGenerators(ZZ, bracket=['|', '>'])
```

```
sage: I._repr_generator(2)
'x|2>'
```

indices ( )

> Return the indices of `self`.

> EXAMPLES:

```
sage: F = CombinatorialFreeModule(QQ, ['a', 'b', 'c'])
sage: F.indices()
{'a', 'b', 'c'}
```

prefix ( )

> Return the prefix used when displaying elements of self.

> EXAMPLES:

```
sage: F = CombinatorialFreeModule(QQ, ['a', 'b', 'c'])
sage: F.prefix()
'B'
```

```
sage: X = SchubertPolynomialRing(QQ)
sage: X.prefix()
'X'
```

print_options (***kwds*)

> Return the current print options, or set an option.

> INPUT: all of the input is optional; if present, it should be in the form of keyword pairs, such as `latex_bracket='('` . The allowable keywords are:

> - `prefix`
> - `latex_prefix`
> - `bracket`
> - `latex_bracket`
> - `scalar_mult`
> - `latex_scalar_mult`
> - `tensor_symbol`
> - `string_quotes`
> - `sorting_key`
> - `sorting_reverse`

> See the documentation for *IndexedGenerators* for descriptions of the effects of setting each of these options.

> OUTPUT: if the user provides any input, set the appropriate option(s) and return nothing. Otherwise, return the dictionary of settings for print and LaTeX representations.

> EXAMPLES:

```
sage: F = CombinatorialFreeModule(ZZ, [1,2,3], prefix='x')
sage: F.print_options()
{...'prefix': 'x'...}
sage: F.print_options(bracket='(')
```

```
sage: F.print_options()
{...'bracket': '('...}
```

2.1.3 Precision management for non-exact objects

class sage.structure.nonexact. **Nonexact** (*prec=20*)

    default_prec ( )
        Return the default precision for self. Use set_default_prec to set the default precision.

    set_default_prec (*prec*)

2.1.4 Global options

The *GlobalOptions* class provides a generic mechanism for setting and accessing **global** options for parents in one or several related classes, typically for customizing the representation of their elements. This class will eventually also support setting options on a parent by parent basis.

These options should be "attached" to one or more classes as an options method. For example, see sage.combinat.partition.Partitions.options() and sage.combinat.tableau.Tableaux.options().

See also:

For better examples of *GlobalOptions* in action see sage.combinat.partition.Partitions.options() and sage.combinat.tableau.Tableaux.options().

Construction of options classes

The general setup for creating a set of global options is:

```
MyOptions=GlobalOptions('option name',
    module='sage.some_module.some_file',
    options_class='name_of_class_controlled_by_options',
    doc='Nice options',
    first_option=dict(default='default value',
                      description='Changes the functionality of _repr_',
                      values=dict(with_bells='causes _repr_ to print with bells',
                                  with_whistles='causes _repr_ to print with whistles
↪',
                                  ...),
                      aliases=dict(bells='option1', whistles='option2', ...),
    second_option=dict(...),
    third_option=dict(),
    end_doc='end of options documentation'
)
```

The options constructed by *GlobalOptions* have to be explicitly associated to the class that they control using the following arguments:

- name= – A descriptive name for the options class

- module – The sage module containing the options class (optional)

- option_class – The name of the options class. This is optional and defaults to name if not explicitly set.

It is only possible to pickle an instance of a *GlobalOptions* if the corresponding module is specified *and* if the options are explicitly attached to the corresponding class as a *options* method.

Each option is specified as a dictionary which describes the possible values for the option and its documentation. The possible entries in this dictionary are:

- `alias` – Allows for several option values to do the same thing.

- `alt_name` – An alternative name for this option.

- `checker` – A validation function which returns whether a user supplied value is valid or not. This is typically useful for large lists of legal values such as `NN`.

- `default` – Gives the default value for the option.

- `description` – A one line description of the option.

- `link_to` – Links this option to another one in another set of global options. This is used for example to allow `Partitions` and `Tableaux` to share the same `convention` option.

- `setter` – A function which is called **after** the value of the option is changed.

- `values` – A dictionary assigning each valid value for the option to a short description of what it does.

- `case_sensitive` – (Default: `True`) `True` or `False` depending on whether the values of the option are case sensitive.

For each option, either a complete list of possible values, via `values`, or a validation function, via `checker`, must be given. The values can be quite arbitrary, including user-defined functions which customize the default behaviour of the classes such as the output of `_repr_` or `latex()`. See *Dispatchers* below, and `_dispatcher()`, for more information.

The documentation for the options is automatically constructed by combining the description of each option with a header and footer which are given by the following optional, but recommended, arguments:

- `doc` – The top half of the documentation which appears before the automatically generated list of options and their possible values.

- `end_doc` – The second half of the documentation which appears after the list of options and their values.

The basic structure for defining a *GlobalOptions* class is best illustrated by an example:

```
sage: from sage.structure.global_options import GlobalOptions
sage: class Menu(object):
....:     options = GlobalOptions('menu',
....:         doc='Fancy documentation\n'+'-'*19, end_doc='The END!',
....:         entree=dict(default='soup',
....:                     description='The first course of a meal',
....:                     values=dict(soup='soup of the day', bread='oven baked'),
....:                     alias=dict(rye='bread')),
....:         appetizer=dict(alt_name='entree'),
....:         main=dict(default='pizza', description='Main meal',
....:                   values=dict(pizza='thick crust', pasta='penne arrabiata'),
....:                   case_sensitive=False),
....:         dessert=dict(default='espresso', description='Dessert',
....:                      values=dict(espresso='life begins again',
....:                                  cake='waist begins again',
....:                                  cream='fluffy, white stuff')),
....:         tip=dict(default=10, description='Reward for good service',
....:         checker=lambda tip: tip in range(0,20))
....:     )
sage: Menu.options
Current options for menu
```

```
 - dessert: espresso
 - entree:  soup
 - main:    pizza
 - tip:     10
```

I all of the examples above the options are contructed with single call to *GlobalOptions* , however, it is also possible to construct the options dynamically using the `GlobalOptions._add_to_options()` methods.

For more details see *GlobalOptions* .

Accessing and setting option values

All options and their values, when they are strings, are forced to be lower case. The values of an options class can be set and accessed by calling the class or by treating the class as an array.

Continuing the example from *Construction of options classes*:

```
sage: Menu.options
Current options for menu
  - dessert: espresso
  - entree:  soup
  - main:    pizza
  - tip:     10
sage: Menu.options.dessert
espresso
sage: Menu.options.dessert = 'cake'
sage: Menu.options.dessert
cake
```

Note that, provided there is no ambiguity, options and their values can be abbreviated:

```
sage: Menu.options('d')
'cake'
sage: Menu.options('m','t',des='esp', ent='sou')  # get and set several values at once
['pizza', 10]
sage: Menu.options(t=15);
sage: Menu.options('tip')
15
sage: Menu.options.tip
15
sage: Menu.options(e='s', m='Pi'); Menu.options()
Current options for menu
  - dessert: cake
  - entree:  soup
  - main:    pizza
  - tip:     15
sage: Menu.options(m='P')
Traceback (most recent call last):
...
ValueError: P is not a valid value for main in the options for menu
```

Setter functions

Each option of a *GlobalOptions* can be equipped with an optional setter function which is called **after** the value of the option is changed. In the following example, setting the option 'add' changes the state of the class by setting

an attribute in this class using a `classmethod()`. Note that the options object is inserted after the creation of the class in order to access the `classmethod()` as `A.setter`:

```
sage: from sage.structure.global_options import GlobalOptions
sage: class A(SageObject):
....:     state = 0
....:     @classmethod
....:     def setter(cls, option, val):
....:         cls.state += int(val)
...
sage: A.options = GlobalOptions('A',
....:                 add=dict(default=1,
....:                     checker=lambda v: int(v)>0,
....:                     description='An option with a setter',
....:                     setter=A.setter))
sage: A.options
Current options for A
- add: 1
sage: a = A(); a.state
1
sage: a.options()
Current options for A
- add: 1
sage: a.options(add=4)
sage: a.state
5
sage: a.options()
Current options for A
- add: 4
```

Another alternative is to construct the options class inside the `__init__` method of the class `A`.

Documentation for options

The documentation for a *GlobalOptions* is automatically generated from the supplied options. For example, the generated documentation for the options `menu` defined in *Construction of options classes* is the following:

```
Fancy documentation
-------------------

OPTIONS:

- ``appetizer`` -- alternative name for ``entree``

- ``dessert`` -- (default: ``espresso``)
  Dessert

  - ``cake``     -- waist begins again
  - ``cream``    -- fluffy, white stuff
  - ``espresso`` -- life begins again

- ``entree`` -- (default: ``soup``)
  The first course of a meal

  - ``bread`` -- oven baked
  - ``rye``   -- alias for bread
  - ``soup``  -- soup of the day
```

```
- ``main`` -- (default: ``pizza``)
  Main meal

  - ``pasta`` -- penne arrabiata
  - ``pizza`` -- thick crust

- tip -- (default: 10)
  Reward for good service

The END!

See :class:`~sage.structure.global_options.GlobalOptions` for more features of these␣
→options.
```

In addition, help on each option, and its list of possible values, can be obtained by (trying to) set the option equal to '?':

```
sage: Menu.options.dessert?                      #not tested
- ``dessert`` -- (default: ``espresso``)
  Dessert
  - ``cake``     -- waist begins again
  - ``cream``    -- fluffy, white stuff
  - ``espresso`` -- life begins again
Current value: espresso
```

Dispatchers

The whole idea of a *GlobalOptions* class is that the options change the default behaviour of the associated classes. This can be done either by simply checking what the current value of the relevant option is. Another possibility is to use the options class as a dispatcher to associated methods. To use the dispatcher feature of a *GlobalOptions* class it is necessary to implement separate methods for each value of the option where the naming convention for these methods is that they start with a common prefix and finish with the value of the option.

If the value of a dispatchable option is set equal to a (user defined) function then this function is called instead of a class method.

For example, the options `MyOptions` can be used to dispatch the `_repr_` method of the associated class `MyClass` as follows:

```python
class MyClass(...):
    def _repr_(self):
        return self.options._dispatch(self,'_repr_','first_option')
    def _repr_with_bells(self):
        print('Bell!')
    def _repr_with_whistles(self):
        print('Whistles!')
GlobalOptions(MyClass,
    ...
)
```

In this example, `first_option` is an option of `MyOptions` which takes values `bells`, `whistles`, and so on. Note that it is necessary to make `self`, which is an instance of `MyClass`, an argument of the dispatcher because `_dispatch()` is a method of *GlobalOptions* and not a method of `MyClass`. Apart from `MyOptions`, as it is a method of this class, the arguments are the attached class (here `MyClass`), the prefix of the method of `MyClass`

being dispatched, the option of `MyOptions` which controls the dispatching. All other arguments are passed through to the corresponding methods of `MyClass`. In general, a dispatcher is invoked as:

```
self.options._dispatch(self, dispatch_to, option, *args, **kargs)
```

Usually this will result in the method `dispatch_to + '_' + MyOptions(options)` of `self` being called with arguments `*args` and `**kargs` (if `dispatch_to[-1] == '_'` then the method `dispatch_to + MyOptions(options)` is called).

If `MyOptions(options)` is itself a function then the dispatcher will call this function instead. In this way, it is possible to allow the user to customise the default behaviour of this method. See `_dispatch()` for an example of how this can be achieved.

The dispatching capabilities of *GlobalOptions* allows options to be applied automatically without needing to parse different values of the option (the cost is that there must be a method for each value). The dispatching capabilities can also be used to make one option control several methods:

```python
def __le__(self, other):
    return self.options._dispatch(self, '_le_','cmp', other)
def __ge__(self, other):
    return self.options._dispatch(self, '_ge_','cmp', other)
def _le_option_a(self, other):
    return ...
def _ge_option_a(self, other):
    return ...
def _le_option_b(self, other):
    return ...
def _ge_option_b(self, other):
    return ...
```

See `_dispatch()` for more details.

Doc testing

All of the options and their effects should be doc-tested. However, in order not to break other tests, all options should be returned to their default state at the end of each test. To make this easier, every *GlobalOptions* class has a `_reset()` method for doing exactly this.

Tests

Options classes can only be pickled if they are the options for some standard sage class. In this case the class is specified using the arguments to *GlobalOptions*. For example `options()` is defined as:

```python
class Partitions(UniqueRepresentation, Parent):
    ....
    options = GlobalOptions(
        name = 'Partitions',
        module='sage.combinat.partition',
    ...
```

Here is an example to test the pickling of a *GlobalOptions* instance:

```
sage: class Menu(object):
....:     options = GlobalOptions(Menu, doc='Fancy documentation\n'+'-'*19, end_doc=
→'The END!',
....:         entree=dict(default='soup',
```

```
....:                    description='The first course of a meal',
....:                    values=dict(soup='soup of the day', bread='oven baked'),
....:                    alias=dict(rye='bread')),
....:        appetizer=dict(alt_name='entree'),
....:        main=dict(default='pizza', description='Main meal',
....:                    values=dict(pizza='thick crust', pasta='penne arrabiata'),
....:                    case_sensitive=False),
....:        dessert=dict(default='espresso', description='Dessert',
....:                      values=dict(espresso='life begins again',
....:                                    cake='waist begins again',
....:                                    cream='fluffy, white stuff')),
....:        tip=dict(default=10, description='Reward for good service',
....:        checker=lambda tip: tip in range(0,20))
....:    )
sage: TestSuite(Menu.options).run( skip = '_test_pickling' ) # not attached to a
→class => can't pickle
sage: TestSuite(Partitions.options).run()
```

AUTHORS:

- Andrew Mathas (2013): initial version

- **Andrew Mathas (2016): overhaul making the options attributes, enabling** pickling and attaching the options to a class.

class sage.structure.global_options. **GlobalOptions** (*name=''*,   *module=''*,   *option_class=''*, *doc=''*, *end_doc=''*, ***options*)

Bases: object

The *GlobalOptions* class is a generic class for setting and accessing global options for sage objects. It takes as inputs a name for the collection of options and a dictionary of dictionaries which specifies the individual options. The allowed/expected keys in the dictionary are the following:

INPUT:

- name – specifies a name for the options class (required)

- module – gives the module that contains the associated options class

- option_class – gives the name of the associated module class (default: name)

- doc – initial documentation string

- end_doc – final documentation string

- <options>=dict(...) – dictionary specifying an option

The options are specified by keyword arguments with their values being a dictionary which describes the option. The allowed/expected keys in the dictionary are:

- alias – defines alias/synonym for option values

- alt_name – alternative name for an option

- checker – a function for checking whether a particular value for the option is valid

- default – the default value of the option

- description – documentation string

- link_to – links to an option for this set of options to an option in another *GlobalOptions*

- setter – a function (class method) which is called whenever this option changes

- •values – a dictionary of the legal values for this option (this automatically defines the corresponding
  `checker`); this dictionary gives the possible options, as keys, together with a brief description of them

- •case_sensitive – (default: `True`) `True` or `False` depending on whether the values of the option
  are case sensitive

Options and their values can be abbreviated provided that this abbreviation is a prefix of a unique option.

Calling the options with no arguments results in the list of current options being printed.

EXAMPLES:

```
sage: from sage.structure.global_options import GlobalOptions
sage: class Menu(object):
....:     options = GlobalOptions('menu', doc='Fancy documentation\n'+'-'*19, end_
→doc='End of Fancy documentation',
....:         entree=dict(default='soup',
....:                     description='The first course of a meal',
....:                     values=dict(soup='soup of the day', bread='oven baked'),
....:                     alias=dict(rye='bread')),
....:         appetizer=dict(alt_name='entree'),
....:         main=dict(default='pizza', description='Main meal',
....:                   values=dict(pizza='thick crust', pasta='penne arrabiata'),
....:                   case_sensitive=False),
....:         dessert=dict(default='espresso', description='Dessert',
....:                     values=dict(espresso='life begins again',
....:                                 cake='waist begins again',
....:                                 cream='fluffy white stuff')),
....:         tip=dict(default=10, description='Reward for good service',
....:                 checker=lambda tip: tip in range(0,20))
....:     )
sage: Menu.options
Current options for menu
  - dessert: espresso
  - entree:  soup
  - main:    pizza
  - tip:     10
sage: Menu.options(entree='s')          # unambiguous abbreviations are allowed
sage: Menu.options(t=15);
sage: (Menu.options['tip'], Menu.options('t'))
(15, 15)
sage: Menu.options()
Current options for menu
  - dessert: espresso
  - entree:  soup
  - main:    pizza
  - tip:     15
sage: Menu.options._reset(); Menu.options()
Current options for menu
  - dessert: espresso
  - entree:  soup
  - main:    pizza
  - tip:     10
sage: Menu.options.tip=40
Traceback (most recent call last):
...
ValueError: 40 is not a valid value for tip in the options for menu
sage: Menu.options(m='p')               # ambiguous abbreviations are not allowed
Traceback (most recent call last):
...
```

```
ValueError: p is not a valid value for main in the options for menu
```

The documentation for the options class is automatically generated from the information which specifies the options:

```
Fancy documentation
-------------------

OPTIONS:

- dessert:  (default: espresso)
  Dessert

  - ``cake``     -- waist begins again
  - ``cream``    -- fluffy white stuff
  - ``espresso`` -- life begins again

- entree:  (default: soup)
  The first course of a meal

  - ``bread`` -- oven baked
  - ``rye``   -- alias for bread
  - ``soup``  -- soup of the day

- main:  (default: pizza)
  Main meal

  - ``pasta`` -- penne arrabiata
  - ``pizza`` -- thick crust

- tip:  (default: 10)
  Reward for good service

End of Fancy documentation

See :class:`~sage.structure.global_options.GlobalOptions` for more features of␣
↪these options.
```

The possible values for an individual option can be obtained by (trying to) set it equal to '?':

```
sage: Menu.options(des='?')
- ``dessert`` -- (default: ``espresso``)
  Dessert

  - ``cake``     -- waist begins again
  - ``cream``    -- fluffy white stuff
  - ``espresso`` -- life begins again

Current value: espresso
```

default_value (*args*, ***kwds*)
    Deprecated: Use _default_value() instead. See trac ticket #18555 for details.

dispatch (*args*, ***kwds*)
    Deprecated: Use _dispatch() instead. See trac ticket #18555 for details.

reset (*args*, ***kwds*)
    Deprecated: Use _reset() instead. See trac ticket #18555 for details.

class `sage.structure.global_options.` **Option** (*options*, *name*)

> Bases: `object`

> An option.

> Each option for an options class is an instance of this class which implements the magic that allows the options to the attributes of the options class that can be looked up, set and called.

> By way of example, this class implements the following functionality.

> EXAMPLES:

```
sage: Partitions.options.display
list
sage: Partitions.options.display='compact'
sage: Partitions.options.display('list')
sage: Partitions.options._reset()
```

2.2 Old-Style Parents (Deprecated)

2.2.1 Base class for old-style parent objects

CLASS HIERARCHY:

SageObject

> **Parent**

> > **ParentWithBase** ParentWithGens

This came up in some subtle bug once.

```
sage: gp(2) + gap(3)
5
```

class `sage.structure.parent_old.` **Parent**

> Bases: *sage.structure.parent.Parent*

> Parents are the SAGE/mathematical analogues of container objects in computer science.

> **coerce_map_from_c** (*S*)

> > EXAMPLES:

> > Check to make sure that we handle coerce maps from Python native types correctly:

> > ```
> > sage: QQ['q,t'].coerce_map_from(int)
> > Composite map:
> >   From: Set of Python objects of type 'int'
> >   To:   Multivariate Polynomial Ring in q, t over Rational Field
> >   Defn:   Native morphism:
> >           From: Set of Python objects of type 'int'
> >           To:   Rational Field
> >         then
> >           Polynomial base injection morphism:
> >           From: Rational Field
> >           To:   Multivariate Polynomial Ring in q, t over Rational Field
> > ```

> **get_action_c** (*S*, *op*, *self_on_left*)

> **get_action_impl** (*S*, *op*, *self_on_left*)

has_coerce_map_from_c (S)
>   Return True if there is a natural map from S to self. Otherwise, return False.

2.2.2 Base class for old-style parent objects with a base ring

class sage.structure.parent_base. **ParentWithBase**
>   Bases: *sage.structure.parent_old.Parent*

>   This class is being deprecated, see parent.Parent for the new model.

>   **base_extend** (X)

sage.structure.parent_base. **is_ParentWithBase** (x)
>   Return True if x is a parent object with base.

2.2.3 Base class for old-style parent objects with generators

---

Note: This   class   is   being   deprecated,   see   sage.structure.parent.Parent   and
sage.structure.category_object.CategoryObject for the new model.

Many parent objects in Sage are equipped with generators, which are special elements of the object. For example, the
polynomial ring $\mathbf{Z}[x, y, z]$ is generated by x, y, and z. In Sage the ith generator of an object X is obtained using the
notation X.gen(i) . From the Sage interactive prompt, the shorthand notation X.i is also allowed.

REQUIRED: A class that derives from ParentWithGens *must* define the ngens() and gen(i) methods.

OPTIONAL: It is also good if they define gens() to return all gens, but this is not necessary.

The gens function returns a tuple of all generators, the ngens function returns the number of generators.

The _assign_names functions is for internal use only, and is called when objects are created to set the generator
names. It can only be called once.

The following examples illustrate these functions in the context of multivariate polynomial rings and free modules.

EXAMPLES:

```
sage: R = PolynomialRing(ZZ, 3, 'x')
sage: R.ngens()
3
sage: R.gen(0)
x0
sage: R.gens()
(x0, x1, x2)
sage: R.variable_names()
('x0', 'x1', 'x2')
```

This example illustrates generators for a free module over **Z**.

```
sage: M = FreeModule(ZZ, 4)
sage: M
Ambient free module of rank 4 over the principal ideal domain Integer Ring
sage: M.ngens()
4
sage: M.gen(0)
(1, 0, 0, 0)
```

```
sage: M.gens()
((1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1))
```

class sage.structure.parent_gens. **ParentWithGens**

    Bases: *sage.structure.parent_base.ParentWithBase*

    EXAMPLES:

```
sage: class MyParent(ParentWithGens):
....:     def ngens(self): return 3
sage: P = MyParent(base = QQ, names = 'a,b,c', normalize = True, category =
↪Groups())
sage: P.category()
Category of groups
sage: P._names
('a', 'b', 'c')
```

    gen (*i=0*)

    gens ( )

        Return a tuple whose entries are the generators for this object, in order.

    hom (*im_gens*, *codomain=None*, *check=True*)

        Return the unique homomorphism from self to codomain that sends self.gens() to the entries of im_gens . Raises a TypeError if there is no such homomorphism.

        INPUT:

            •im_gens - the images in the codomain of the generators of this object under the homomorphism

            •codomain - the codomain of the homomorphism

            •check - whether to verify that the images of generators extend to define a map (using only canonical coercions).

        OUTPUT:

            •a homomorphism self –> codomain

        Note: As a shortcut, one can also give an object X instead of im_gens , in which case return the (if it exists) natural map to X.

        EXAMPLES: Polynomial Ring We first illustrate construction of a few homomorphisms involving a polynomial ring.

```
sage: R.<x> = PolynomialRing(ZZ)
sage: f = R.hom([5], QQ)
sage: f(x^2 - 19)
6

sage: R.<x> = PolynomialRing(QQ)
sage: f = R.hom([5], GF(7))
Traceback (most recent call last):
...
TypeError: images do not define a valid homomorphism

sage: R.<x> = PolynomialRing(GF(7))
sage: f = R.hom([3], GF(49,'a'))
sage: f
```

```
Ring morphism:
  From: Univariate Polynomial Ring in x over Finite Field of size 7
  To:   Finite Field in a of size 7^2
  Defn: x |--> 3
sage: f(x+6)
2
sage: f(x^2+1)
3
```

EXAMPLES: Natural morphism

```
sage: f = ZZ.hom(GF(5))
sage: f(7)
2
sage: f
Ring Coercion morphism:
  From: Integer Ring
  To:   Finite Field of size 5
```

There might not be a natural morphism, in which case a TypeError exception is raised.

```
sage: QQ.hom(ZZ)
Traceback (most recent call last):
...
TypeError: Natural coercion morphism from Rational Field to Integer Ring not␣
↪defined.
```

ngens()

class sage.structure.parent_gens. **localvars**

Bases: object

Context manager for safely temporarily changing the variables names of an object with generators.

Objects with named generators are globally unique in Sage. Sometimes, though, it is very useful to be able to temporarily display the generators differently. The new Python with statement and the localvars context manager make this easy and safe (and fun!)

Suppose X is any object with generators. Write

```
with localvars(X, names[, latex_names] [,normalize=False]):
    some code
    ...
```

and the indented code will be run as if the names in X are changed to the new names. If you give normalize=True, then the names are assumed to be a tuple of the correct number of strings.

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ,2)
sage: with localvars(R, 'z,w'):
....:     print(x^3 + y^3 - x*y)
z^3 + w^3 - z*w
```

Note: I wrote this because it was needed to print elements of the quotient of a ring R by an ideal I using the print function for elements of R. See the code in quotient_ring_element.pyx.

AUTHOR:

•William Stein (2006-10-31)

2.2.4 Pure python code for abstract base class for objects with generators

sage.structure.gens_py. **abelian_iterator** (M)

sage.structure.gens_py. **multiplicative_iterator** (M)

ELEMENTS

3.1 Elements

AUTHORS:

- David Harvey (2006-10-16): changed CommutativeAlgebraElement to derive from CommutativeRingElement instead of AlgebraElement

- David Harvey (2006-10-29): implementation and documentation of new arithmetic architecture

- William Stein (2006-11): arithmetic architecture – pushing it through to completion.

- Gonzalo Tornaria (2007-06): recursive base extend for coercion – lots of tests

- Robert Bradshaw (2007-2010): arithmetic operators and coercion

- Maarten Derickx (2010-07): added architecture for is_square and sqrt

- Jeroen Demeyer (2016-08): moved all coercion to the base class *Element* , see trac ticket #20767

3.1.1 The Abstract Element Class Hierarchy

This is the abstract class hierarchy, i.e., these are all abstract base classes.

```
SageObject
    Element
        ModuleElement
            RingElement
                CommutativeRingElement
                    IntegralDomainElement
                        DedekindDomainElement
                            PrincipalIdealDomainElement
                                EuclideanDomainElement
                    FieldElement
                    CommutativeAlgebraElement
                AlgebraElement   (note -- can't derive from module, since no multiple
→inheritance)
                    Matrix
                InfinityElement
            AdditiveGroupElement
            Vector

        MonoidElement
            MultiplicativeGroupElement
    ElementWithCachedMethod
```

3.1.2 How to Define a New Element Class

Elements typically define a method _new_c , e.g.,

```
cdef _new_c(self, defining data):
    cdef FreeModuleElement_generic_dense x
    x = FreeModuleElement_generic_dense.__new__(FreeModuleElement_generic_dense)
    x._parent = self._parent
    x._entries = v
```

that creates a new sibling very quickly from defining data with assumed properties.

Arithmetic for Elements

Sage has a special system for handling arithmetic operations on Sage elements (that is instances of *Element*), in particular to manage uniformly mixed arithmetic operations using the coercion model . We describe here the rules that must be followed by both arithmetic implementers and callers.

A quick summary for the impatient

To implement addition for any *Element* subclass, override the def _add_(self,other) method instead of the usual Python __add__ special method. Within _add_(self,other) , you may assume that self and other have the same parent.

If the implementation is generic across all elements in a given category C, then this method can be put in C.ElementMethods .

When writing *Cython* code, _add_ should be a cpdef method: cpdef _add_(self,other) .

When doing arithmetic with two elements having different parents, the coercion model is responsible for "coercing" them to a common parent and performing arithmetic on the coerced elements.

Arithmetic in more detail

The aims of this system are to provide (1) an efficient calling protocol from both Python and Cython, (2) uniform coercion semantics across Sage, (3) ease of use, (4) readability of code.

We will take addition as an example; all other operators are similar. There are two relevant functions, with differing names (single vs. double underscores).

- **def Element.__add__(left, right)**

  This function is called by Python or Cython when the binary "+" operator is encountered. It assumes that at least one of its arguments is an *Element* .

  It has a fast pathway to deal with the most common case where both arguments have the same parent. Otherwise, it uses the coercion model to work out how to make them have the same parent. The coercion model then adds the coerced elements (technically, it calls operator.add). Note that the result of coercion is not required to be a Sage *Element* , it could be a plain Python type.

  Note that, although this function is declared as def , it doesn't have the usual overheads associated with Python functions (either for the caller or for __add__ itself). This is because Python has optimised calling protocols for such special functions.

- **def Element._add_(self, other)**

  This is the function that you should override to implement addition in a subclass of *Element* .

> The two arguments to this function are guaranteed to have the **same parent**, but not necessarily the same Python type.
>
> When implementing `_add_` in a Cython extension type, use `cpdef _add_` instead of `def _add_`.
>
> In Cython code, if you want to add two elements and you know that their parents are identical, you are encouraged to call this function directly, instead of using `x + y`. This only works if Cython knows that the left argument is an `Element`. You can always cast explicitly: `(<Element>x)._add_(y)` to force this. In plain Python, `x + y` is always the fastest way to add two elements because the special method `__add__` is optimized unlike the normal method `_add_`.

The difference in the names of the arguments (`left`, `right` versus `self`, `other`) is intentional: `self` is guaranteed to be an instance of the class in which the method is defined. In Cython, we know that at least one of `left` or `right` is an instance of the class but we do not know a priori which one.

For addition and multiplication (not for other operators), there is a fast path for operations with a Python `int` (which corresponds to a C `long`). Implement `cdef _add_long(self, long n)` or `cdef _mul_long(self, long n)` with optimized code for `self + n` or `self * n`. These are assumed to be commutative, so they are also called for `n + self` or `n * self`. From Cython code, you can also call `_add_long` or `_mul_long` directly.

Examples

We need some *Parent* to work with:

```
sage: from sage.structure.parent import Parent
sage: class ExampleParent(Parent):
....:     def __init__(self, name, **kwds):
....:         Parent.__init__(self, **kwds)
....:         self.rename(name)
```

We start with a very basic example of a Python class implementing `_add_`:

```
sage: from sage.structure.element import Element
sage: class MyElement(Element):
....:     def _add_(self, other):
....:         return 42
sage: p = ExampleParent("Some parent")
sage: x = MyElement(p)
sage: x + x
42
```

When two different parents are involved, this no longer works since there is no coercion:

```
sage: q = ExampleParent("Other parent")
sage: y = MyElement(q)
sage: x + y
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for +: 'Some parent' and 'Other parent'
```

If `_add_` is not defined, an error message is raised, referring to the parents:

```
sage: x = Element(p)
sage: x._add_(x)
Traceback (most recent call last):
...
AttributeError: 'sage.structure.element.Element' object has no attribute '_add_'
```

```
sage: x + x
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for +: 'Some parent' and 'Some parent'
sage: y = Element(q)
sage: x + y
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for +: 'Some parent' and 'Other parent'
```

We can also implement arithmetic generically in categories:

```
sage: class MyCategory(Category):
....:     def super_categories(self):
....:         return [Sets()]
....:     class ElementMethods:
....:         def _add_(self, other):
....:             return 42
sage: p = ExampleParent("Parent in my category", category=MyCategory())
sage: x = Element(p)
sage: x + x
42
```

Implementation details

Implementing the above features actually takes a bit of magic. Casual callers and implementers can safely ignore it, but here are the details for the curious.

To achieve fast arithmetic, it is critical to have a fast path in Cython to call the _add_ method of a Cython object. So we would like to declare _add_ as a cpdef method of class *Element* . Remember however that the abstract classes coming from categories come after *Element* in the method resolution order (or fake method resolution order in case of a Cython class). Hence any generic implementation of _add_ in such an abstract class would in principle be shadowed by Element._add_ . This is worked around by defining Element._add_ as a cdef instead of a cpdef method. Concrete implementations in subclasses should be cpdef or def methods.

Let us now see what happens upon evaluating x + y when x and y are instances of a class that does not implement _add_ but where _add_ is implemented in the category. First, x.__add__(y) is called, where __add__ is implemented in *Element* . Assuming that x and y have the same parent, a Cython call to x._add_(y) will be done. The latter is implemented to trigger a Python level call to x._add_(y) which will succeed as desired.

In case that Python code calls x._add_(y) directly, Element._add_ will be invisible, and the method lookup will continue down the MRO and find the _add_ method in the category.

class sage.structure.element.**AdditiveGroupElement**
    Bases: *sage.structure.element.ModuleElement*

    Generic element of an additive group.

    order ()
        Return additive order of element

class sage.structure.element.**AlgebraElement**
    Bases: *sage.structure.element.RingElement*

class sage.structure.element.**CoercionModel**
    Bases: object

    Most basic coercion scheme. If it doesn't already match, throw an error.

bin_op (*x*, *y*, *op*)

canonical_coercion (*x*, *y*)

richcmp (*x*, *y*, *op*)

class sage.structure.element. **CommutativeAlgebraElement**
 Bases: *sage.structure.element.CommutativeRingElement*

class sage.structure.element. **CommutativeRingElement**
 Bases: *sage.structure.element.RingElement*

 Base class for elements of commutative rings.

 divides (*x*)
  Return `True` if `self` divides x.

  EXAMPLES:

```
sage: P.<x> = PolynomialRing(QQ)
sage: x.divides(x^2)
True
sage: x.divides(x^2+2)
False
sage: (x^2+2).divides(x)
False
sage: P.<x> = PolynomialRing(ZZ)
sage: x.divides(x^2)
True
sage: x.divides(x^2+2)
False
sage: (x^2+2).divides(x)
False
```

  trac ticket #5347 has been fixed:

```
sage: K = GF(7)
sage: K(3).divides(1)
True
sage: K(3).divides(K(1))
True
```

```
sage: R = Integers(128)
sage: R(0).divides(1)
False
sage: R(0).divides(0)
True
sage: R(0).divides(R(0))
True
sage: R(1).divides(0)
True
sage: R(121).divides(R(120))
True
sage: R(120).divides(R(121))
Traceback (most recent call last):
...
ArithmeticError: reduction modulo 120 not defined
```

  If `x` has different parent than `self`, they are first coerced to a common parent if possible. If this coercion
  fails, it returns a TypeError. This fixes trac ticket #5759.

```
sage: Zmod(2)(0).divides(Zmod(2)(0))
True
sage: Zmod(2)(0).divides(Zmod(2)(1))
False
sage: Zmod(5)(1).divides(Zmod(2)(1))
Traceback (most recent call last):
...
TypeError: no common canonical parent for objects with parents: 'Ring of␣
↪integers modulo 5' and 'Ring of integers modulo 2'
sage: Zmod(35)(4).divides(Zmod(7)(1))
True
sage: Zmod(35)(7).divides(Zmod(7)(1))
False
```

inverse_mod (*I*)

Return an inverse of `self` modulo the ideal *I*, if defined, i.e., if *I* and `self` together generate the unit ideal.

is_square (*root=False*)

Return whether or not the ring element `self` is a square.

If the optional argument root is `True`, then also return the square root (or `None`, if it is not a square).

INPUT:

- `root` - whether or not to also return a square root (default: `False`)

OUTPUT:

- `bool` – whether or not a square

- `object` – (optional) an actual square root if found, and `None` otherwise.

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ)
sage: f = 12*(x+1)^2 * (x+3)^2
sage: f.is_square()
False
sage: f.is_square(root=True)
(False, None)
sage: h = f/3
sage: h.is_square()
True
sage: h.is_square(root=True)
(True, 2*x^2 + 8*x + 6)
```

Note: This is the is_square implementation for general commutative ring elements. It's implementation is to raise a NotImplementedError. The function definition is here to show what functionality is expected and provide a general framework.

---

mod (*I*)

Return a representative for `self` modulo the ideal I (or the ideal generated by the elements of I if I is not an ideal.)

EXAMPLES: Integers Reduction of 5 modulo an ideal:

```
sage: n = 5
sage: n.mod(3*ZZ)
2
```

Reduction of 5 modulo the ideal generated by 3:

```
sage: n.mod(3)
2
```

Reduction of 5 modulo the ideal generated by 15 and 6, which is (3).

```
sage: n.mod([15,6])
2
```

EXAMPLES: Univariate polynomials

```
sage: R.<x> = PolynomialRing(QQ)
sage: f = x^3 + x + 1
sage: f.mod(x + 1)
-1
```

Reduction for $\mathbf{Z}[x]$:

```
sage: R.<x> = PolynomialRing(ZZ)
sage: f = x^3 + x + 1
sage: f.mod(x + 1)
-1
```

When little is implemented about a given ring, then mod may return simply return f.

EXAMPLES: Multivariate polynomials We reduce a polynomial in two variables modulo a polynomial and an ideal:

```
sage: R.<x,y,z> = PolynomialRing(QQ, 3)
sage: (x^2 + y^2 + z^2).mod(x+y+z)
2*y^2 + 2*y*z + 2*z^2
```

Notice above that x is eliminated. In the next example, both y and z are eliminated:

```
sage: (x^2 + y^2 + z^2).mod( (x - y, y - z) )
3*z^2
sage: f = (x^2 + y^2 + z^2)^2; f
x^4 + 2*x^2*y^2 + y^4 + 2*x^2*z^2 + 2*y^2*z^2 + z^4
sage: f.mod( (x - y, y - z) )
9*z^4
```

In this example y is eliminated:

```
sage: (x^2 + y^2 + z^2).mod( (x^3, y - z) )
x^2 + 2*z^2
```

sqrt (*extend=True*, *all=False*, *name=None*)
  It computes the square root.

  INPUT:

  •`extend` - Whether to make a ring extension containing a square root if `self` is not a square (default: `True`)

- `all` - Whether to return a list of all square roots or just a square root (default: False)

- `name` - Required when `extend=True` and `self` is not a square. This will be the name of the generator extension.

OUTPUT:

- if `all=False` it returns a square root. (throws an error if `extend=False` and `self` is not a square)

- if `all=True` it returns a list of all the square roots (could be empty if `extend=False` and `self` is not a square)

ALGORITHM:

It uses `is_square(root=true)` for the hard part of the work, the rest is just wrapper code.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: (x^2).sqrt()
x
sage: f=x^2-4*x+4; f.sqrt(all=True)
[x - 2, -x + 2]
sage: sqrtx=x.sqrt(name="y"); sqrtx
y
sage: sqrtx^2
x
sage: x.sqrt(all=true,name="y")
[y, -y]
sage: x.sqrt(extend=False,all=True)
[]
sage: x.sqrt()
Traceback (most recent call last):
...
TypeError: Polynomial is not a square. You must specify the name of the
→square root when using the default extend = True
sage: x.sqrt(extend=False)
Traceback (most recent call last):
...
ValueError: trying to take square root of non-square x with extend = False
```

class sage.structure.element. **DedekindDomainElement**
    Bases: *sage.structure.element.IntegralDomainElement*

class sage.structure.element. **Element**
    Bases: *sage.structure.sage_object.SageObject*

Generic element of a structure. All other types of elements (*RingElement*, *ModuleElement*, etc) derive from this type.

Subtypes must either call `__init__()` to set `_parent`, or may set `_parent` themselves if that would be more efficient.

cmp (*left*, *right*)
    Default three-way comparison method which only checks for a Python class defining `__cmp__`.

richcmp (*left*, *right*, *op*)
    Default implementation of rich comparisons for elements with equal parents.

    It tries to see if `_cmp_` is implemented. Otherwise it does a comparison by id for `==` and `!=`. Calling this default method with `<`, `<=`, `>` or `>=` will raise a `NotImplementedError`.

EXAMPLES:

```
sage: from sage.structure.parent import Parent
sage: from sage.structure.element import Element
sage: P = Parent()
sage: e1 = Element(P); e2 = Element(P)
sage: e1 == e1    # indirect doctest
True
sage: e1 == e2    # indirect doctest
False
sage: e1 < e2     # indirect doctest
Traceback (most recent call last):
...
NotImplementedError: comparison not implemented for <type 'sage.structure.
→element.Element'>
```

__add__ (*left*, *right*)

Top-level addition operator for `Element` invoking the coercion model.

See *Arithmetic for Elements*.

EXAMPLES:

```
sage: from sage.structure.element import Element
sage: class MyElement(Element):
....:     def _add_(self, other):
....:         return 42
sage: e = MyElement(Parent())
sage: e + e
42
```

__sub__ (*left*, *right*)

Top-level subtraction operator for `Element` invoking the coercion model.

See *Arithmetic for Elements*.

EXAMPLES:

```
sage: from sage.structure.element import Element
sage: class MyElement(Element):
....:     def _sub_(self, other):
....:         return 42
sage: e = MyElement(Parent())
sage: e - e
42
```

__neg__ ( )

Top-level negation operator for `Element` .

EXAMPLES:

```
sage: from sage.structure.element import Element
sage: class MyElement(Element):
....:     def _neg_(self):
....:         return 42
sage: e = MyElement(Parent())
sage: -e
42
```

__mul__ (*left*, *right*)

> Top-level multiplication operator for `Element` invoking the coercion model.
>
> See *Arithmetic for Elements*.
>
> EXAMPLES:

```
sage: from sage.structure.element import Element
sage: class MyElement(Element):
....:     def _mul_(self, other):
....:         return 42
sage: e = MyElement(Parent())
sage: e * e
42
```

```
sage: A = AlgebrasWithBasis(QQ).example(); A
An example of an algebra with basis: the free algebra
on the generators ('a', 'b', 'c') over Rational Field
sage: x = A.an_element()
sage: x
B[word: ] + 2*B[word: a] + 3*B[word: b] + B[word: bab]
sage: x.__mul__(x)
B[word: ] + 4*B[word: a] + 4*B[word: aa] + 6*B[word: ab]
+ 2*B[word: abab] + 6*B[word: b] + 6*B[word: ba]
+ 2*B[word: bab] + 2*B[word: baba] + 3*B[word: babb]
+ B[word: babbab] + 9*B[word: bb] + 3*B[word: bbab]
```

__div__ (*left*, *right*)

> Top-level division operator for `Element` invoking the coercion model.
>
> See *Arithmetic for Elements*.
>
> EXAMPLES:

```
sage: 2 / 3
2/3
sage: pi / 3
1/3*pi
sage: K.<i> = NumberField(x^2+1)
sage: 2 / K.ideal(i+1)
Fractional ideal (-i + 1)
```

```
sage: from sage.structure.element import Element
sage: class MyElement(Element):
....:     def _div_(self, other):
....:         return 42
sage: e = MyElement(Parent())
sage: e / e
42
```

__truediv__ (*left*, *right*)

> Top-level true division operator for `Element` invoking the coercion model.
>
> See *Arithmetic for Elements*.
>
> EXAMPLES:

```
sage: operator.truediv(2, 3)
2/3
sage: operator.truediv(pi, 3)
```

```
1/3*pi
sage: K.<i> = NumberField(x^2+1)
sage: operator.truediv(2, K.ideal(i+1))
Fractional ideal (-i + 1)
```

```
sage: from sage.structure.element import Element
sage: class MyElement(Element):
....:     def _div_(self, other):
....:         return 42
sage: e = MyElement(Parent())
sage: operator.truediv(e, e)
42
```

__floordiv__ (*left*, *right*)

Top-level floor division operator for *Element* invoking the coercion model.

See *Arithmetic for Elements*.

EXAMPLES:

```
sage: 7 // 3
2
sage: 7 // int(3)
2
sage: int(7) // 3
2
```

```
sage: from sage.structure.element import Element
sage: class MyElement(Element):
....:     def _floordiv_(self, other):
....:         return 42
sage: e = MyElement(Parent())
sage: e // e
42
```

__mod__ (*left*, *right*)

Top-level modulo operator for *Element* invoking the coercion model.

See *Arithmetic for Elements*.

EXAMPLES:

```
sage: 7 % 3
1
sage: 7 % int(3)
1
sage: int(7) % 3
1
```

```
sage: from sage.structure.element import Element
sage: class MyElement(Element):
....:     def _mod_(self, other):
....:         return 42
sage: e = MyElement(Parent())
sage: e % e
42
```

N (**args*, ***kwds*)
> Deprecated: Use *n()* instead. See trac ticket #13055 for details.

base_extend (*R*)

base_ring ( )
> Return the base ring of this element's parent (if that makes sense).

category ( )

is_zero ( )
> Return `True` if `self` equals `self.parent()(0)`.
>
> The default implementation is to fall back to `not self.__nonzero__`.
>
> > **Warning:** Do not re-implement this method in your subclass but implement `__nonzero__` instead.

n (*prec=None*, *digits=None*, *algorithm=None*)
> Alias for *numerical_approx()*.
>
> EXAMPLES:

```
sage: (2/3).n()
0.666666666666667
```

numerical_approx (*prec=None*, *digits=None*, *algorithm=None*)
> Return a numerical approximation of `self` with `prec` bits (or decimal `digits`) of precision.
>
> No guarantee is made about the accuracy of the result.
>
> INPUT:
>
> > •`prec` – precision in bits
> >
> > •`digits` – precision in decimal digits (only used if `prec` is not given)
> >
> > •`algorithm` – which algorithm to use to compute this approximation (the accepted algorithms depend on the object)
>
> If neither `prec` nor `digits` is given, the default precision is 53 bits (roughly 16 digits).
>
> EXAMPLES:

```
sage: (2/3).numerical_approx()
0.666666666666667
sage: pi.n(digits=10)
3.141592654
sage: pi.n(prec=20)
3.1416
```

parent (*x=None*)
> Return the parent of this element; or, if the optional argument x is supplied, the result of coercing x into the parent of this element.

subs (*in_dict=None*, ***kwds*)
> Substitutes given generators with given values while not touching other generators. This is a generic wrapper around `__call__`. The syntax is meant to be compatible with the corresponding method for symbolic expressions.
>
> INPUT:
>
> > •`in_dict` - (optional) dictionary of inputs

•**kwds** - named parameters

OUTPUT:

•new object if substitution is possible, otherwise self.

EXAMPLES:

```
sage: x, y = PolynomialRing(ZZ,2,'xy').gens()
sage: f = x^2 + y + x^2*y^2 + 5
sage: f((5,y))
25*y^2 + y + 30
sage: f.subs({x:5})
25*y^2 + y + 30
sage: f.subs(x=5)
25*y^2 + y + 30
sage: (1/f).subs(x=5)
1/(25*y^2 + y + 30)
sage: Integer(5).subs(x=4)
5
```

substitute (*in_dict=None*, ***kwds*)
　　This is an alias for self.subs().

INPUT:

•**in_dict** - (optional) dictionary of inputs

•**kwds** - named parameters

OUTPUT:

•new object if substitution is possible, otherwise self.

EXAMPLES:

```
sage: x, y = PolynomialRing(ZZ,2,'xy').gens()
sage: f = x^2 + y + x^2*y^2 + 5
sage: f((5,y))
25*y^2 + y + 30
sage: f.substitute({x:5})
25*y^2 + y + 30
sage: f.substitute(x=5)
25*y^2 + y + 30
sage: (1/f).substitute(x=5)
1/(25*y^2 + y + 30)
sage: Integer(5).substitute(x=4)
5
```

class sage.structure.element. **ElementWithCachedMethod**
　　Bases: *sage.structure.element.Element*

An element class that fully supports cached methods.

NOTE:

The cached_method decorator provides a convenient way to automatically cache the result of a computation. Since trac ticket #11115, the cached method decorator applied to a method without optional arguments is faster than a hand-written cache in Python, and a cached method without any arguments (except self) is actually faster than a Python method that does nothing more but to return 1 . A cached method can also be inherited from the parent or element class of a category.

However, this holds true only if attribute assignment is supported. If you write an extension class in Cython that does not accept attribute assignment then a cached method inherited from the category will be slower (for *Parent*) or the cache would even break (for *Element*).

This class should be used if you write an element class, can not provide it with attribute assignment, but want that it inherits a cached method from the category. Under these conditions, your class should inherit from this class rather than *Element* . Then, the cache will work, but certainly slower than with attribute assignment. Lazy attributes work as well.

EXAMPLES:

We define three element extension classes. The first inherits from *Element* , the second from this class, and the third simply is a Python class. We also define a parent class and, in Python, a category whose element and parent classes define cached methods.

```
sage: cython_code = ["from sage.structure.element cimport Element,␣
↪ElementWithCachedMethod",
....:     "cdef class MyBrokenElement(Element):",
....:     "    cdef public object x",
....:     "    def __init__(self,P,x):",
....:     "        self.x=x",
....:     "        Element.__init__(self,P)",
....:     "    def __neg__(self):",
....:     "        return MyBrokenElement(self.parent(),-self.x)",
....:     "    def _repr_(self):",
....:     "        return '<%s>'%self.x",
....:     "    def __hash__(self):",
....:     "        return hash(self.x)",
....:     "    cpdef int _cmp_(left, right) except -2:",
....:     "        return cmp(left.x,right.x)",
....:     "    def raw_test(self):",
....:     "        return -self",
....:     "cdef class MyElement(ElementWithCachedMethod):",
....:     "    cdef public object x",
....:     "    def __init__(self,P,x):",
....:     "        self.x=x",
....:     "        Element.__init__(self,P)",
....:     "    def __neg__(self):",
....:     "        return MyElement(self.parent(),-self.x)",
....:     "    def _repr_(self):",
....:     "        return '<%s>'%self.x",
....:     "    def __hash__(self):",
....:     "        return hash(self.x)",
....:     "    cpdef int _cmp_(left, right) except -2:",
....:     "        return cmp(left.x,right.x)",
....:     "    def raw_test(self):",
....:     "        return -self",
....:     "class MyPythonElement(MyBrokenElement): pass",
....:     "from sage.structure.parent cimport Parent",
....:     "cdef class MyParent(Parent):",
....:     "    Element = MyElement"]
sage: cython('\n'.join(cython_code))
sage: cython_code = ["from sage.all import cached_method, cached_in_parent_
↪method, Category, Objects",
....:     "class MyCategory(Category):",
....:     "    @cached_method",
....:     "    def super_categories(self):",
....:     "        return [Objects()]",
....:     "    class ElementMethods:",
```

```
....:    "        @cached_method",
....:    "        def element_cache_test(self):",
....:    "            return -self",
....:    "        @cached_in_parent_method",
....:    "        def element_via_parent_test(self):",
....:    "            return -self",
....:    "    class ParentMethods:",
....:    "        @cached_method",
....:    "        def one(self):",
....:    "            return self.element_class(self,1)",
....:    "        @cached_method",
....:    "        def invert(self, x):",
....:    "            return -x"]
sage: cython('\n'.join(cython_code))
sage: C = MyCategory()
sage: P = MyParent(category=C)
sage: ebroken = MyBrokenElement(P,5)
sage: e = MyElement(P,5)
```

The cached methods inherited by `MyElement` works:

```
sage: e.element_cache_test()
<-5>
sage: e.element_cache_test() is e.element_cache_test()
True
sage: e.element_via_parent_test()
<-5>
sage: e.element_via_parent_test() is e.element_via_parent_test()
True
```

The other element class can only inherit a `cached_in_parent_method`, since the cache is stored in the parent. In fact, equal elements share the cache, even if they are of different types:

```
sage: e == ebroken
True
sage: type(e) == type(ebroken)
False
sage: ebroken.element_via_parent_test() is e.element_via_parent_test()
True
```

However, the cache of the other inherited method breaks, although the method as such works:

```
sage: ebroken.element_cache_test()
<-5>
sage: ebroken.element_cache_test() is ebroken.element_cache_test()
False
```

Since `e` and `ebroken` share the cache, when we empty it for one element it is empty for the other as well:

```
sage: b = ebroken.element_via_parent_test()
sage: e.element_via_parent_test.clear_cache()
sage: b is ebroken.element_via_parent_test()
False
```

Note that the cache only breaks for elements that do no allow attribute assignment. A Python version of `MyBrokenElement` therefore allows for cached methods:

```
sage: epython = MyPythonElement(P,5)
sage: epython.element_cache_test()
<-5>
sage: epython.element_cache_test() is epython.element_cache_test()
True
```

class sage.structure.element. **EuclideanDomainElement**

    Bases: *sage.structure.element.PrincipalIdealDomainElement*

    degree ( )

    leading_coefficient ( )

    quo_rem (*other*)

class sage.structure.element. **FieldElement**

    Bases: *sage.structure.element.CommutativeRingElement*

    divides (*other*)

        Check whether self divides other, for field elements.

        Since this is a field, all values divide all other values, except that zero does not divide any non-zero values.

        EXAMPLES:

```
sage: K.<rt3> = QQ[sqrt(3)]
sage: K(0).divides(rt3)
False
sage: rt3.divides(K(17))
True
sage: K(0).divides(K(0))
True
sage: rt3.divides(K(0))
True
```

    is_unit ( )

        Return True if self is a unit in its parent ring.

        EXAMPLES:

```
sage: a = 2/3; a.is_unit()
True
```

        On the other hand, 2 is not a unit, since its parent is **Z**.

```
sage: a = 2; a.is_unit()
False
sage: parent(a)
Integer Ring
```

        However, a is a unit when viewed as an element of QQ:

```
sage: a = QQ(2); a.is_unit()
True
```

    quo_rem (*right*)

        Return the quotient and remainder obtained by dividing self by right . Since this element lives in a field, the remainder is always zero and the quotient is self/right .

class sage.structure.element. **InfinityElement**

    Bases: *sage.structure.element.RingElement*

class sage.structure.element.**IntegralDomainElement**
 Bases: *sage.structure.element.CommutativeRingElement*

 is_nilpotent ()

class sage.structure.element.**Matrix**
 Bases: *sage.structure.element.ModuleElement*

class sage.structure.element.**ModuleElement**
 Bases: *sage.structure.element.Element*

 Generic element of a module.

 additive_order ()
  Return the additive order of self.

 order ()
  Return the additive order of self.

class sage.structure.element.**MonoidElement**
 Bases: *sage.structure.element.Element*

 Generic element of a monoid.

 multiplicative_order ()
  Return the multiplicative order of self.

 order ()
  Return the multiplicative order of self.

 powers (*n*)
  Return the list $[x^0, x^1, \ldots, x^{n-1}]$.

  EXAMPLES:

```
sage: G = SymmetricGroup(4)
sage: g = G([2, 3, 4, 1])
sage: g.powers(4)
[(), (1,2,3,4), (1,3)(2,4), (1,4,3,2)]
```

class sage.structure.element.**MultiplicativeGroupElement**
 Bases: *sage.structure.element.MonoidElement*

 Generic element of a multiplicative group.

 order ()
  Return the multiplicative order of self.

class sage.structure.element.**PrincipalIdealDomainElement**
 Bases: *sage.structure.element.DedekindDomainElement*

 lcm (*right*)
  Return the least common multiple of self and right.

class sage.structure.element.**RingElement**
 Bases: *sage.structure.element.ModuleElement*

 abs ()
  Return the absolute value of self. (This just calls the __abs__ method, so it is equivalent to the abs() built-in function.)

  EXAMPLES:

```
sage: RR(-1).abs()
1.00000000000000
sage: ZZ(-1).abs()
1
sage: CC(I).abs()
1.00000000000000
sage: Mod(-15, 37).abs()
Traceback (most recent call last):
...
ArithmeticError: absolute valued not defined on integers modulo n.
```

additive_order ( )
> Return the additive order of `self`.

is_nilpotent ( )
> Return `True` if `self` is nilpotent, i.e., some power of `self` is 0.

is_one ( )

is_prime ( )
> Is `self` a prime element?

> A *prime* element is a non-zero, non-unit element p such that, whenever p divides ab for some a and b, then p divides a or p divides b.

> EXAMPLES:

> For polynomial rings, prime is the same as irreducible:

```
sage: R.<x,y> = QQ[]
sage: x.is_prime()
True
sage: (x^2 + y^3).is_prime()
True
sage: (x^2 - y^2).is_prime()
False
sage: R(0).is_prime()
False
sage: R(2).is_prime()
False
```

> For the Gaussian integers:

```
sage: K.<i> = QuadraticField(-1)
sage: ZI = K.ring_of_integers()
sage: ZI(3).is_prime()
True
sage: ZI(5).is_prime()
False
sage: ZI(2+i).is_prime()
True
sage: ZI(0).is_prime()
False
sage: ZI(1).is_prime()
False
```

> In fields, an element is never prime:

```
sage: RR(0).is_prime()
False
```

```
sage: RR(2).is_prime()
False
```

For integers, prime numbers are redefined to be positive:

```
sage: RingElement.is_prime(-2)
True
sage: Integer.is_prime(-2)
False
```

multiplicative_order ( )
    Return the multiplicative order of `self`, if `self` is a unit, or raise `ArithmeticError` otherwise.

powers (*n*)
    Return the list $[x^0, x^1, \ldots, x^{n-1}]$.

    EXAMPLES:

```
sage: 5.powers(3)
[1, 5, 25]
```

class sage.structure.element. **Vector**
    Bases: *sage.structure.element.ModuleElement*

sage.structure.element. **bin_op** (*x, y, op*)

sage.structure.element. **canonical_coercion** (*x, y*)
    canonical_coercion(x,y) is what is called before doing an arithmetic operation between x and y. It
    returns a pair (z,w) such that z is got from x and w from y via canonical coercion and the parents of z and
    w are identical.

    EXAMPLES:

```
sage: A = Matrix([[0, 1], [1, 0]])
sage: canonical_coercion(A, 1)
(
[0 1]  [1 0]
[1 0], [0 1]
)
```

sage.structure.element. **coerce_binop** (*method*)
    Decorator for a binary operator method for applying coercion to the arguments before calling the method.

    Consider a parent class in the category framework, S, whose element class expose a method $binop$. If a and b are
    elements of S, then $a.binop(b)$ behaves as expected. If a and b are not elements of S, but rather have a common
    parent T whose element class also exposes $binop$, we would rather expect $a.binop(b)$ to compute $aa.binop(bb)$,
    where $aa = T(a)$ and $bb = T(b)$. This decorator ensures that behaviour without having to otherwise modify
    the implementation of $binop$ on the element class of A.

    Since coercion will be attempted on the arguments of the decorated method, a $TypeError$ will be thrown if
    there is no common parent between the elements. An $AttributeError$ or $NotImplementedError$ or similar
    will be thrown if there is a common parent of the arguments, but its element class does not implement a method
    of the same name as the decorated method.

    EXAMPLES:

    Sparse polynomial rings uses $@coerce_binop$ on gcd:

```
sage: S.<x> = PolynomialRing(ZZ,sparse=True)
sage: f = x^2
```

```
sage: g = x
sage: f.gcd(g)    #indirect doctest
x
sage: T = PolynomialRing(QQ, name='x', sparse=True)
sage: h = 1/2*T(x)
sage: u = f.gcd(h); u   #indirect doctest
x
sage: u.parent() == T
True
```

Another real example:

```
sage: R1=QQ['x,y']
sage: R2=QQ['x,y,z']
sage: f=R1(1)
sage: g=R1(2)
sage: h=R2(1)
sage: f.gcd(g)
1
sage: f.gcd(g,algorithm='modular')
1
sage: f.gcd(h)
1
sage: f.gcd(h,algorithm='modular')
1
sage: h.gcd(f)
1
sage: h.gcd(f,'modular')
1
```

We demonstrate a small class using $@coerce_binop$ on a method:

```
sage: from sage.structure.element import coerce_binop
sage: class MyRational(Rational):
....:     def __init__(self,value):
....:         self.v = value
....:     @coerce_binop
....:     def test_add(self, other, keyword='z'):
....:         return (self.v, other, keyword)
```

Calls func directly if the two arguments have the same parent:

```
sage: x = MyRational(1)
sage: x.test_add(1/2)
(1, 1/2, 'z')
sage: x.test_add(1/2, keyword=3)
(1, 1/2, 3)
```

Passes through coercion and does a method lookup if the left operand is not the same. If the common parent's element class does not have a method of the same name, an exception is raised:

```
sage: x.test_add(2)
(1, 2, 'z')
sage: x.test_add(2, keyword=3)
(1, 2, 3)
sage: x.test_add(CC(2))
Traceback (most recent call last):
```

```
...
AttributeError: 'sage.rings.complex_number.ComplexNumber' object has no attribute
↪'test_add'
```

sage.structure.element. **coercion_traceback** (*dump=True*)

This function is very helpful in debugging coercion errors. It prints the tracebacks of all the errors caught in the coercion detection. Note that failure is cached, so some errors may be omitted the second time around (as it remembers not to retry failed paths for speed reasons.

For performance and caching reasons, exception recording must be explicitly enabled before using this function.

EXAMPLES:

```
sage: cm = sage.structure.element.get_coercion_model()
sage: cm.record_exceptions()
sage: 1 + 1/5
6/5
sage: coercion_traceback()  # Should be empty, as all went well.
sage: 1/5 + GF(5).gen()
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for +: 'Rational Field' and 'Finite
↪Field of size 5'
sage: coercion_traceback()
Traceback (most recent call last):
...
TypeError: no common canonical parent for objects with parents: 'Rational Field'
↪and 'Finite Field of size 5'
```

sage.structure.element. **generic_power** (*a*, *n*, *one=None*)

Computes a^n, where n is an integer, and a is an object which supports multiplication. Optionally an additional argument, which is used in the case that n == 0 :

- one - the "unit" element, returned directly (can be anything)

If this is not supplied, int(1) is returned.

EXAMPLES:

```
sage: from sage.structure.element import generic_power
sage: generic_power(int(12),int(0))
1
sage: generic_power(int(0),int(100))
0
sage: generic_power(Integer(10),Integer(0))
1
sage: generic_power(Integer(0),Integer(23))
0
sage: sum([generic_power(2,i) for i in range(17)]) #test all 4-bit combinations
131071
sage: F = Zmod(5)
sage: a = generic_power(F(2), 5); a
2
sage: a.parent() is F
True
sage: a = generic_power(F(1), 2)
sage: a.parent() is F
True
```

```
sage: generic_power(int(5), 0)
1
```

sage.structure.element. **get_coercion_model** ( )
>   Return the global coercion model.

>   EXAMPLES:

```
sage: import sage.structure.element as e
sage: cm = e.get_coercion_model()
sage: cm
<sage.structure.coerce.CoercionModel_cache_maps object at ...>
```

sage.structure.element. **have_same_parent** (*left*, *right*)
>   Return `True` if and only if `left` and `right` have the same parent.

>   > **Warning:** This function assumes that at least one of the arguments is a Sage *Element* . When in doubt, use the slower `parent(left) is parent(right)` instead.

>   EXAMPLES:

```
sage: from sage.structure.element import have_same_parent
sage: have_same_parent(1, 3)
True
sage: have_same_parent(1, 1/2)
False
sage: have_same_parent(gap(1), gap(1/2))
True
```

>   These have different types but the same parent:

```
sage: a = RLF(2)
sage: b = exp(a)
sage: type(a)
<type 'sage.rings.real_lazy.LazyWrapper'>
sage: type(b)
<type 'sage.rings.real_lazy.LazyNamedUnop'>
sage: have_same_parent(a, b)
True
```

sage.structure.element. **is_AdditiveGroupElement** (*x*)
>   Return `True` if x is of type AdditiveGroupElement.

sage.structure.element. **is_AlgebraElement** (*x*)
>   Return `True` if x is of type AlgebraElement.

sage.structure.element. **is_CommutativeAlgebraElement** (*x*)
>   Return `True` if x is of type CommutativeAlgebraElement.

sage.structure.element. **is_CommutativeRingElement** (*x*)
>   Return `True` if x is of type CommutativeRingElement.

sage.structure.element. **is_DedekindDomainElement** (*x*)
>   Return `True` if x is of type DedekindDomainElement.

sage.structure.element. **is_Element** (*x*)
>   Return `True` if x is of type Element.

EXAMPLES:

```
sage: from sage.structure.element import is_Element
sage: is_Element(2/3)
True
sage: is_Element(QQ^3)
False
```

sage.structure.element.**is_EuclideanDomainElement**(*x*)
    Return `True` if x is of type EuclideanDomainElement.

sage.structure.element.**is_FieldElement**(*x*)
    Return `True` if x is of type FieldElement.

sage.structure.element.**is_InfinityElement**(*x*)
    Return `True` if x is of type InfinityElement.

sage.structure.element.**is_IntegralDomainElement**(*x*)
    Return `True` if x is of type IntegralDomainElement.

sage.structure.element.**is_Matrix**(*x*)

sage.structure.element.**is_ModuleElement**(*x*)
    Return `True` if x is of type ModuleElement.

    This is even faster than using isinstance inline.

    EXAMPLES:

```
sage: from sage.structure.element import is_ModuleElement
sage: is_ModuleElement(2/3)
True
sage: is_ModuleElement((QQ^3).0)
True
sage: is_ModuleElement('a')
False
```

sage.structure.element.**is_MonoidElement**(*x*)
    Return `True` if x is of type MonoidElement.

sage.structure.element.**is_MultiplicativeGroupElement**(*x*)
    Return `True` if x is of type MultiplicativeGroupElement.

sage.structure.element.**is_PrincipalIdealDomainElement**(*x*)
    Return `True` if x is of type PrincipalIdealDomainElement.

sage.structure.element.**is_RingElement**(*x*)
    Return `True` if x is of type RingElement.

sage.structure.element.**is_Vector**(*x*)

sage.structure.element.**make_element**(*_class*, *_dict*, *parent*)
    This function is only here to support old pickles.

    Pickling functionality is moved to Element.{__getstate__,__setstate__} functions.

sage.structure.element.**parent**(*x*)
    Return the parent of the element x .

    Usually, this means the mathematical object of which x is an element.

    INPUT:

        •x – an element

OUTPUT:

- •If `x` is a Sage *Element* , return `x.parent()`.

- •If `x` has a `parent` method and `x` does not have an `__int__` or `__float__` method, return `x.parent()`.

- •Otherwise, return `type(x)`.

See also:

Parents, Conversion and Coercion Section in the Sage Tutorial

EXAMPLES:

```
sage: a = 42
sage: parent(a)
Integer Ring
sage: b = 42/1
sage: parent(b)
Rational Field
sage: c = 42.0
sage: parent(c)
Real Field with 53 bits of precision
```

Some more complicated examples:

```
sage: x = Partition([3,2,1,1,1])
sage: parent(x)
Partitions
sage: v = vector(RDF, [1,2,3])
sage: parent(v)
Vector space of dimension 3 over Real Double Field
```

The following are not considered to be elements, so the type is returned:

```
sage: d = int(42)   # Python int
sage: parent(d)
<... 'int'>
sage: L = list(range(10))
sage: parent(L)
<... 'list'>
```

sage.structure.element. **set_coercion_model** (*cm*)

3.2 Element Wrapper

Wrapping Sage or Python objects as Sage elements.

AUTHORS:

- • Nicolas Thiery (2008-2010): Initial version

- • Travis Scrimshaw (2013-05-04): Cythonized version

class sage.structure.element_wrapper. **DummyParent** (*name*)

Bases: *sage.structure.unique_representation.UniqueRepresentation* , *sage.structure.parent.Parent*

A class for creating dummy parents for testing ElementWrapper

class sage.structure.element_wrapper. **ElementWrapper**

> Bases: *sage.structure.element.Element*

> A class for wrapping Sage or Python objects as Sage elements.

> EXAMPLES:

```
sage: from sage.structure.element_wrapper import DummyParent
sage: parent = DummyParent("A parent")
sage: o = ElementWrapper(parent, "bla"); o
'bla'
sage: isinstance(o, sage.structure.element.Element)
True
sage: o.parent()
A parent
sage: o.value
'bla'
```

> Note that o is not *an instance of* str , but rather *contains a* str . Therefore, o does not inherit the string methods. On the other hand, it is provided with reasonable default implementations for equality testing, hashing, etc.

> The typical use case of ElementWrapper is for trivially constructing new element classes from pre-existing Sage or Python classes, with a containment relation. Here we construct the tropical monoid of integers endowed with min as multiplication. There, it is desirable *not* to inherit the factor method from Integer :

```
sage: class MinMonoid(Parent):
....:     def _repr_(self):
....:         return "The min monoid"
....:
sage: M = MinMonoid()
sage: class MinMonoidElement(ElementWrapper):
....:     wrapped_class = Integer
....:
....:     def __mul__(self, other):
....:         return MinMonoidElement(self.parent(), min(self.value, other.value))
sage: x = MinMonoidElement(M, 5); x
5
sage: x.parent()
The min monoid
sage: x.value
5
sage: y = MinMonoidElement(M, 3)
sage: x * y
3
```

> This example was voluntarily kept to a bare minimum. See the examples in the categories (e.g. Semigroups().example()) for several full featured applications.

> > **Warning:** Versions before trac ticket #14519 had parent as the second argument and the value as the first.

> **value**

class sage.structure.element_wrapper. **ElementWrapperCheckWrappedClass**

> Bases: *sage.structure.element_wrapper.ElementWrapper*

> An *element wrapper* such that comparison operations are done against subclasses of wrapped_class .

class sage.structure.element_wrapper. **ElementWrapperTester**

    Bases: *sage.structure.element_wrapper.ElementWrapper*

    Test class for the default __copy() method of subclasses of *ElementWrapper*.

    append (*x*)

3.3 Elements, Array and Lists With Clone Protocol

This module defines several classes which are subclasses of *Element* and which roughly implement the "prototype" design pattern (see [Prototype_pattern], [GHJV1994]). Those classes are intended to be used to model *mathematical* objects, which are by essence immutable. However, in many occasions, one wants to construct the data-structure encoding of a new mathematical object by small modifications of the data structure encoding some already built object. For the resulting data-structure to correctly encode the mathematical object, some structural invariants must hold. One problem is that, in many cases, during the modification process, there is no possibility but to break the invariants.

For example, in a list modeling a permutation of $\{1, \ldots, n\}$, the integers must be distinct. A very common operation is to take a permutation to make a copy with some small modifications, like exchanging two consecutive values in the list or cycling some values. Though the result is clearly a permutations there's no way to avoid breaking the permutations invariants at some point during the modifications.

The main purpose of this module is to define the class

    • *ClonableElement* as an abstract super class,

and its subclasses:

    • *ClonableArray* for arrays (lists of fixed length) of objects;

    • *ClonableList* for (resizable) lists of objects;

    • *NormalizedClonableList* for lists of objects with a normalization method;

    • *ClonableIntArray* for arrays of int.

See also:

The following parents from *sage.structure.list_clone_demo* demonstrate how to use them:

    • IncreasingArrays() (see *IncreasingArray* and the parent class *IncreasingArrays*)

    • IncreasingLists() (see *IncreasingList* and the parent class *IncreasingLists*)

    • SortedLists() (see *SortedList* and the parent class *SortedLists*)

    • IncreasingIntArrays() (see *IncreasingIntArray* and the parent class *IncreasingIntArrays*)

EXAMPLES:

We now demonstrate how *IncreasingArray* works, creating an instance el through its parent IncreasingArrays():

```
sage: from sage.structure.list_clone_demo import IncreasingArrays
sage: P = IncreasingArrays()
sage: P([1, 4 ,8])
[1, 4, 8]
```

If one tries to create this way a list which in not increasing, an error is raised:

```
sage: IncreasingArrays()([5, 4 ,8])
Traceback (most recent call last):
...
ValueError: array is not increasing
```

Once created modifying `el` is forbidden:

```
sage: el = P([1, 4 ,8])
sage: el[1] = 3
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead.
```

However, you can modify a temporarily mutable clone:

```
sage: with el.clone() as elc:
....:      elc[1] = 3
sage: [el, elc]
[[1, 4, 8], [1, 3, 8]]
```

We check that the original and the modified copy now are in a proper immutable state:

```
sage: el.is_immutable(), elc.is_immutable()
(True, True)
sage: elc[1] = 5
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead.
```

You can break the property that the list is increasing during the modification:

```
sage: with el.clone() as elc2:
....:      elc2[1] = 12
....:      print(elc2)
....:      elc2[2] = 25
[1, 12, 8]
sage: elc2
[1, 12, 25]
```

But this property must be restored at the end of the `with` block; otherwise an error is raised:

```
sage: with elc2.clone() as el3:
....:      el3[1] = 100
Traceback (most recent call last):
...
ValueError: array is not increasing
```

Finally, as an alternative to the `with` syntax one can use:

```
sage: el4 = copy(elc2)
sage: el4[1] = 10
sage: el4.set_immutable()
sage: el4.check()
```

REFERENCES:

- [Prototype_pattern]

- [GHJV1994]

AUTHORS:

- Florent Hivert (2010-03): initial revision

class sage.structure.list_clone. **ClonableArray**

    Bases: *sage.structure.list_clone.ClonableElement*

    Array with clone protocol

    The class of objects which are *Element* behave as arrays (i.e. lists of fixed length) and implement the clone protocol. See *ClonableElement* for details about clone protocol.

    INPUT:

- parent – a *Parent*

- lst – a list

- check – a boolean specifying if the invariant must be checked using method *check()* .

- immutable – a boolean telling wether the created element is immutable (defaults to True)

    See also:

    IncreasingArray for an example of usage.

    EXAMPLES:

```
sage: from sage.structure.list_clone_demo import IncreasingArrays
sage: IA = IncreasingArrays()
sage: ia1 = IA([1, 4, 6]); ia1
[1, 4, 6]
sage: with ia1.clone() as ia2:
....:       ia2[1] = 5
sage: ia2
[1, 5, 6]
sage: with ia1.clone() as ia2:
....:       ia2[1] = 7
Traceback (most recent call last):
...
ValueError: array is not increasing
```

    check ( )

        Check that self fulfill the invariants

        This is an abstract method. Subclasses are supposed to overload check .

        EXAMPLES:

```
sage: from sage.structure.list_clone import ClonableArray
sage: ClonableArray(Parent(), [1,2,3]) # indirect doctest
Traceback (most recent call last):
...
NotImplementedError: this should never be called, please overload the check
→method
sage: from sage.structure.list_clone_demo import IncreasingArrays
sage: el = IncreasingArrays()([1,2,4]) # indirect doctest
```

    count (*key*)

        Returns number of i 's for which s[i] == key

        EXAMPLES:

```
sage: from sage.structure.list_clone_demo import IncreasingArrays
sage: c = IncreasingArrays()([1,2,2,4])
sage: c.count(1)
1
sage: c.count(2)
2
sage: c.count(3)
0
```

index (*x*, *start=None*, *stop=None*)

Returns the smallest `k` such that `s[k] == x` and `i <= k < j`

EXAMPLES:

```
sage: from sage.structure.list_clone_demo import IncreasingArrays
sage: c = IncreasingArrays()([1,2,4])
sage: c.index(1)
0
sage: c.index(4)
2
sage: c.index(5)
Traceback (most recent call last):
...
ValueError: 5 is not in list
```

class sage.structure.list_clone. **ClonableElement**

Bases: *sage.structure.element.Element*

Abstract class for elements with clone protocol

This class is a subclass of *Element* and implements the "prototype" design pattern (see [Prototype_pattern], [GHJV1994]). The role of this class is:

• to manage copy and mutability and hashing of elements

• to ensure that at the end of a piece of code an object is restored in a meaningful mathematical state.

A class `C` inheriting from *ClonableElement* must implement the following two methods

• `obj.__copy__()` – returns a fresh copy of obj

• `obj.check()` – returns nothing, raise an exception if `obj` doesn't satisfy the data structure invariants

and ensure to call `obj._require_mutable()` at the beginning of any modifying method.

Additionally, one can also implement

• `obj._hash_()` – return the hash value of `obj`.

Then, given an instance `obj` of `C`, the following sequences of instructions ensures that the invariants of `new_obj` are properly restored at the end:

```
with obj.clone() as new_obj:
    ...
    # lot of invariant breaking modifications on new_obj
    ...
# invariants are ensured to be fulfilled
```

The following equivalent sequence of instructions can be used if speed is needed, in particular in Cython code:

```
new_obj = obj.__copy__()
...
# lot of invariant breaking modifications on new_obj
...
new_obj.set_immutable()
new_obj.check()
# invariants are ensured to be fulfilled
```

Finally, if the class implements the _hash_ method, then *ClonableElement* ensures that the hash value can only be computed on an immutable object. It furthermore caches it so that it is only computed once.

> **Warning:** for the hash caching mechanism to work correctly, the hash value cannot be 0.

EXAMPLES:

The following code shows a minimal example of usage of *ClonableElement* . We implement a class or pairs (x, y) such that $x < y$:

```
sage: from sage.structure.list_clone import ClonableElement
sage: class IntPair(ClonableElement):
....:         def __init__(self, parent, x, y):
....:             ClonableElement.__init__(self, parent=parent)
....:             self._x = x
....:             self._y = y
....:             self.set_immutable()
....:             self.check()
....:         def _repr_(self):
....:             return "(x=%s, y=%s)"%(self._x, self._y)
....:         def check(self):
....:             if self._x >= self._y:
....:                 raise ValueError("Incorrectly ordered pair")
....:         def get_x(self): return self._x
....:         def get_y(self): return self._y
....:         def set_x(self, v): self._require_mutable(); self._x = v
....:         def set_y(self, v): self._require_mutable(); self._y = v
```

> **Note:** we don't need to define __copy__ since it is properly inherited from *Element* .

We now demonstrate the behavior. Let's create an IntPair :

```
sage: myParent = Parent()
sage: el = IntPair(myParent, 1, 3); el
(x=1, y=3)
sage: el.get_x()
1
```

Modifying it is forbidden:

```
sage: el.set_x(4)
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead.
```

However, you can modify a mutable copy:

```
sage: with el.clone() as el1:
....:        el1.set_x(2)
sage: [el, el1]
[(x=1, y=3), (x=2, y=3)]
```

We check that the original and the modified copy are in a proper immutable state:

```
sage: el.is_immutable(), el1.is_immutable()
(True, True)
sage: el1.set_x(4)
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead.
```

A modification which doesn't restore the invariant $x < y$ at the end is illegal and raise an exception:

```
sage: with el.clone() as elc2:
....:        elc2.set_x(5)
Traceback (most recent call last):
...
ValueError: Incorrectly ordered pair
```

clone (*check=True*)
   Returns a clone that is mutable copy of `self`.

   INPUT:

   • `check` – a boolean indicating if `self.check()` must be called after modifications.

   EXAMPLES:

```
sage: from sage.structure.list_clone_demo import IncreasingArrays
sage: el = IncreasingArrays()([1,2,3])
sage: with el.clone() as el1:
....:        el1[2] = 5
sage: el1
[1, 2, 5]
```

is_immutable ( )
   Returns `True` if `self` is immutable (can not be changed) and `False` if it is not.

   To make `self` immutable use `self.set_immutable()`.

   EXAMPLES:

```
sage: from sage.structure.list_clone_demo import IncreasingArrays
sage: el = IncreasingArrays()([1,2,3])
sage: el.is_immutable()
True
sage: copy(el).is_immutable()
False
sage: with el.clone() as el1:
....:        print([el.is_immutable(), el1.is_immutable()])
[True, False]
```

is_mutable ( )
   Returns `True` if `self` is mutable (can be changed) and `False` if it is not.

   To make this object immutable use `self.set_immutable()`.

EXAMPLES:

```
sage: from sage.structure.list_clone_demo import IncreasingArrays
sage: el = IncreasingArrays()([1,2,3])
sage: el.is_mutable()
False
sage: copy(el).is_mutable()
True
sage: with el.clone() as el1:
....:         print([el.is_mutable(), el1.is_mutable()])
[False, True]
```

set_immutable ()

Makes `self` immutable, so it can never again be changed.

EXAMPLES:

```
sage: from sage.structure.list_clone_demo import IncreasingArrays
sage: el = IncreasingArrays()([1,2,3])
sage: el1 = copy(el); el1.is_mutable()
True
sage: el1.set_immutable();  el1.is_mutable()
False
sage: el1[2] = 4
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead.
```

class sage.structure.list_clone. **ClonableIntArray**

Bases: *sage.structure.list_clone.ClonableElement*

Array of int with clone protocol

The class of objects which are *Element* behave as list of int and implement the clone protocol. See *ClonableElement* for details about clone protocol.

INPUT:

  • `parent` – a *Parent*

  • `lst` – a list

  • `check` – a boolean specifying if the invariant must be checked using method *check()*

  • `immutable` – a boolean telling wether the created element is immutable (defaults to `True`)

See also:

IncreasingIntArray for an example of usage.

check ()

Check that `self` fulfill the invariants

This is an abstract method. Subclasses are supposed to overload `check` .

EXAMPLES:

```
sage: from sage.structure.list_clone import ClonableArray
sage: ClonableArray(Parent(), [1,2,3]) # indirect doctest
Traceback (most recent call last):
...
NotImplementedError: this should never be called, please overload the check␣
→method
```

```
sage: from sage.structure.list_clone_demo import IncreasingIntArrays
sage: el = IncreasingIntArrays()([1,2,4]) # indirect doctest
```

index (*item*)
> EXAMPLES:

```
sage: from sage.structure.list_clone_demo import IncreasingIntArrays
sage: c = IncreasingIntArrays()([1,2,4])
sage: c.index(1)
0
sage: c.index(4)
2
sage: c.index(5)
Traceback (most recent call last):
...
ValueError: list.index(x): x not in list
```

list ( )
> Convert self into a Python list.

> EXAMPLES:

```
sage: from sage.structure.list_clone_demo import IncreasingIntArrays
sage: I = IncreasingIntArrays()(range(5))
sage: I == list(range(5))
False
sage: I.list() == list(range(5))
True
sage: I = IncreasingIntArrays()(range(1000))
sage: I.list() == list(range(1000))
True
```

class sage.structure.list_clone. **ClonableList**
> Bases: *sage.structure.list_clone.ClonableArray*

> List with clone protocol

> The class of objects which are *Element* behave as lists and implement the clone protocol. See *ClonableElement* for details about clone protocol.

> **See also:**

> *IncreasingList* for an example of usage.

> **append** (*el*)
>> Appends el to self

>> INPUT: el – any object

>> EXAMPLES:

```
sage: from sage.structure.list_clone_demo import IncreasingLists
sage: el = IncreasingLists()([1])
sage: el.append(3)
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead.
sage: with el.clone() as elc:
....:         elc.append(4)
....:         elc.append(6)
```

```
sage: elc
[1, 4, 6]
sage: with el.clone() as elc:
....:        elc.append(4)
....:        elc.append(2)
Traceback (most recent call last):
...
ValueError: array is not increasing
```

extend (*it*)

Extends `self` by the content of the iterable `it`

INPUT: `it` – any iterable

EXAMPLES:

```
sage: from sage.structure.list_clone_demo import IncreasingLists
sage: el = IncreasingLists()([1, 4, 5, 8, 9])
sage: el.extend((10,11))
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead.

sage: with el.clone() as elc:
....:        elc.extend((10,11))
sage: elc
[1, 4, 5, 8, 9, 10, 11]

sage: el2 = IncreasingLists()([15, 16])
sage: with el.clone() as elc:
....:        elc.extend(el2)
sage: elc
[1, 4, 5, 8, 9, 15, 16]

sage: with el.clone() as elc:
....:        elc.extend((6,7))
Traceback (most recent call last):
...
ValueError: array is not increasing
```

insert (*index*, *el*)

Inserts `el` in `self` at position `index`

INPUT:

- `el` – any object

- `index` – any int

EXAMPLES:

```
sage: from sage.structure.list_clone_demo import IncreasingLists
sage: el = IncreasingLists()([1, 4, 5, 8, 9])
sage: el.insert(3, 6)
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead.
sage: with el.clone() as elc:
....:        elc.insert(3, 6)
sage: elc
```

```
[1, 4, 5, 6, 8, 9]
sage: with el.clone() as elc:
....:       elc.insert(2, 6)
Traceback (most recent call last):
...
ValueError: array is not increasing
```

pop (*index=-1*)

Remove `self[index]` from `self` and returns it

INPUT: `index` - any int, default to -1

EXAMPLES:

```
sage: from sage.structure.list_clone_demo import IncreasingLists
sage: el = IncreasingLists()([1, 4, 5, 8, 9])
sage: el.pop()
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead.
sage: with el.clone() as elc:
....:       print(elc.pop())
9
sage: elc
[1, 4, 5, 8]
sage: with el.clone() as elc:
....:       print(elc.pop(2))
5
sage: elc
[1, 4, 8, 9]
```

remove (*el*)

Remove the first occurence of `el` from `self`

INPUT: `el` - any object

EXAMPLES:

```
sage: from sage.structure.list_clone_demo import IncreasingLists
sage: el = IncreasingLists()([1, 4, 5, 8, 9])
sage: el.remove(4)
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead.
sage: with el.clone() as elc:
....:       elc.remove(4)
sage: elc
[1, 5, 8, 9]
sage: with el.clone() as elc:
....:       elc.remove(10)
Traceback (most recent call last):
...
ValueError: list.remove(x): x not in list
```

class `sage.structure.list_clone.` **NormalizedClonableList**

Bases: *`sage.structure.list_clone.ClonableList`*

List with clone protocol and normal form

This is a subclass of *ClonableList* which call a method *normalize()* at creation and after any modification of its instance.

See also:

SortedList for an example of usage.

EXAMPLES:

We construct a sorted list through its parent:

```
sage: from sage.structure.list_clone_demo import SortedLists
sage: SL = SortedLists()
sage: sl1 = SL([4,2,6,1]); sl1
[1, 2, 4, 6]
```

Normalization is also performed atfer modification:

```
sage: with sl1.clone() as sl2:
....:        sl2[1] = 12
sage: sl2
[1, 4, 6, 12]
```

normalize ( )
Normalize `self`

This is an abstract method. Subclasses are supposed to overload *normalize()* . The call `self.normalize()` is supposed to

- call `self._require_mutable()` to check that `self` is in a proper mutable state

- modify `self` to put it in a normal form

EXAMPLES:

```
sage: from sage.structure.list_clone_demo import SortedList, SortedLists
sage: l = SortedList(SortedLists(), [2,3,2], False, False)
sage: l
[2, 2, 3]
sage: l.check()
Traceback (most recent call last):
...
ValueError: list is not strictly increasing
```

3.4 Elements, Array and Lists With Clone Protocol, demonstration classes

This module demonstrate the usage of the various classes defined in *list_clone*

class sage.structure.list_clone_demo. **IncreasingArray**
Bases: *sage.structure.list_clone.ClonableArray*

A small extension class for testing *ClonableArray* .

check ( )
Check that `self` is increasing.

EXAMPLES:

```
sage: from sage.structure.list_clone_demo import IncreasingArrays
sage: IncreasingArrays()([1,2,3]) # indirect doctest
[1, 2, 3]
sage: IncreasingArrays()([3,2,1]) # indirect doctest
Traceback (most recent call last):
...
ValueError: array is not increasing
```

class sage.structure.list_clone_demo. **IncreasingArrays**

> Bases: *sage.structure.unique_representation.UniqueRepresentation* , *sage.structure.parent.Parent*
>
> A small (incomplete) parent for testing *ClonableArray*
>
> **Element**
> > alias of *IncreasingArray*

class sage.structure.list_clone_demo. **IncreasingIntArray**

> Bases: *sage.structure.list_clone.ClonableIntArray*
>
> A small extension class for testing *ClonableIntArray* .
>
> **check** ( )
> > Check that self is increasing.
> >
> > EXAMPLES:
> >
> > ```
> > sage: from sage.structure.list_clone_demo import IncreasingIntArrays
> > sage: IncreasingIntArrays()([1,2,3]) # indirect doctest
> > [1, 2, 3]
> > sage: IncreasingIntArrays()([3,2,1]) # indirect doctest
> > Traceback (most recent call last):
> > ...
> > ValueError: array is not increasing
> > ```

class sage.structure.list_clone_demo. **IncreasingIntArrays**

> Bases: *sage.structure.list_clone_demo.IncreasingArrays*
>
> A small (incomplete) parent for testing *ClonableIntArray*
>
> **Element**
> > alias of *IncreasingIntArray*

class sage.structure.list_clone_demo. **IncreasingList**

> Bases: *sage.structure.list_clone.ClonableList*
>
> A small extension class for testing *ClonableList*
>
> **check** ( )
> > Check that self is increasing
> >
> > EXAMPLES:
> >
> > ```
> > sage: from sage.structure.list_clone_demo import IncreasingLists
> > sage: IncreasingLists()([1,2,3]) # indirect doctest
> > [1, 2, 3]
> > sage: IncreasingLists()([3,2,1]) # indirect doctest
> > Traceback (most recent call last):
> > ...
> > ValueError: array is not increasing
> > ```

class sage.structure.list_clone_demo. **IncreasingLists**

    Bases: *sage.structure.list_clone_demo.IncreasingArrays*

    A small (incomplete) parent for testing *ClonableList*

    Element

        alias of *IncreasingList*

class sage.structure.list_clone_demo. **SortedList**

    Bases: *sage.structure.list_clone.NormalizedClonableList*

    A small extension class for testing *NormalizedClonableList* .

    check ( )

        Check that self is strictly increasing

        EXAMPLES:

```
sage: from sage.structure.list_clone_demo import SortedList, SortedLists
sage: SortedLists()([1,2,3]) # indirect doctest
[1, 2, 3]
sage: SortedLists()([3,2,2]) # indirect doctest
Traceback (most recent call last):
...
ValueError: list is not strictly increasing
```

    normalize ( )

        Normalize self

        Sort the list stored in self .

        EXAMPLES:

```
sage: from sage.structure.list_clone_demo import SortedList, SortedLists
sage: l = SortedList(SortedLists(), [3,1,2], False, False)
sage: l            # indirect doctest
[1, 2, 3]
sage: l[1] = 5; l
[1, 5, 3]
sage: l.normalize(); l
[1, 3, 5]
```

class sage.structure.list_clone_demo. **SortedLists**

    Bases: *sage.structure.list_clone_demo.IncreasingLists*

    A small (incomplete) parent for testing *NormalizedClonableList*

    Element

        alias of *SortedList*

"MATHEMATICAL" DATA STRUCTURES

4.1 Formal sums

AUTHORS:

- David Harvey (2006-09-20): changed FormalSum not to derive from "list" anymore, because that breaks new Element interface

- Nick Alexander (2006-12-06): added test cases.

- William Stein (2006, 2009): wrote the first version in 2006, documented it in 2009.

- Volker Braun (2010-07-19): new-style coercions, documentation added. FormalSums now derives from UniqueRepresentation.

FUNCTIONS:

- **FormalSums(ring)** **– create the module of formal finite sums with** coefficients in the given ring.

- FormalSum(list of pairs (coeff,number)) – create a formal sum

EXAMPLES:

```
sage: A = FormalSum([(1, 2/3)]); A
2/3
sage: B = FormalSum([(3, 1/5)]); B
3*1/5
sage: -B
-3*1/5
sage: A + B
3*1/5 + 2/3
sage: A - B
-3*1/5 + 2/3
sage: B*3
9*1/5
sage: 2*A
2*2/3
sage: list(2*A + A)
[(3, 2/3)]
```

class sage.structure.formal_sum. **FormalSum** (*x*, *parent=None*, *check=True*, *reduce=True*)

Bases: *sage.structure.element.ModuleElement*

A formal sum over a ring.

reduce ( )
EXAMPLES:

```
sage: a = FormalSum([(-2,3), (2,3)], reduce=False); a
-2*3 + 2*3
sage: a.reduce()
sage: a
0
```

class sage.structure.formal_sum. **FormalSums**

> Bases: *sage.structure.unique_representation.UniqueRepresentation*   ,
> sage.modules.module.Module

> The R-module of finite formal sums with coefficients in some ring R.

> EXAMPLES:

```
sage: FormalSums()
Abelian Group of all Formal Finite Sums over Integer Ring
sage: FormalSums(ZZ)
Abelian Group of all Formal Finite Sums over Integer Ring
sage: FormalSums(GF(7))
Abelian Group of all Formal Finite Sums over Finite Field of size 7
sage: FormalSums(ZZ[sqrt(2)])
Abelian Group of all Formal Finite Sums over Order in Number Field in sqrt2 with␣
↪defining polynomial x^2 - 2
sage: FormalSums(GF(9,'a'))
Abelian Group of all Formal Finite Sums over Finite Field in a of size 3^2
```

> **Element**
> > alias of *FormalSum*

> **base_extend** (*R*)
> > EXAMPLES:

```
sage: F7 = FormalSums(ZZ).base_extend(GF(7)); F7
Abelian Group of all Formal Finite Sums over Finite Field of size 7
```

> > The following tests against a bug that was fixed at trac ticket #18795:

```
sage: isinstance(F7, F7.category().parent_class)
True
```

4.2 Factorizations

The *Factorization* class provides a structure for holding quite general lists of objects with integer multiplicities. These may hold the results of an arithmetic or algebraic factorization, where the objects may be primes or irreducible polynomials and the multiplicities are the (non-zero) exponents in the factorization. For other types of examples, see below.

Factorization class objects contain a list , so can be printed nicely and be manipulated like a list of prime-exponent pairs, or easily turned into a plain list. For example, we factor the integer -45:

```
sage: F = factor(-45)
```

This returns an object of type *Factorization* :

```
sage: type(F)
<class 'sage.structure.factorization_integer.IntegerFactorization'>
```

It prints in a nice factored form:

```
sage: F
-1 * 3^2 * 5
```

There is an underlying list representation, which ignores the unit part:

```
sage: list(F)
[(3, 2), (5, 1)]
```

A *Factorization* is not actually a list:

```
sage: isinstance(F, list)
False
```

However, we can access the *Factorization* F itself as if it were a list:

```
sage: F[0]
(3, 2)
sage: F[1]
(5, 1)
```

To get at the unit part, use the *Factorization.unit()* function:

```
sage: F.unit()
-1
```

All factorizations are immutable, up to ordering with `sort()` and simplifying with `simplify()`. Thus if you write a function that returns a cached version of a factorization, you do not have to return a copy.

```
sage: F = factor(-12); F
-1 * 2^2 * 3
sage: F[0] = (5,4)
Traceback (most recent call last):
...
TypeError: 'Factorization' object does not support item assignment
```

EXAMPLES:

This more complicated example involving polynomials also illustrates that the unit part is not discarded from factorizations:

```
sage: x = QQ['x'].0
sage: f = -5*(x-2)*(x-3)
sage: f
-5*x^2 + 25*x - 30
sage: F = f.factor(); F
(-5) * (x - 3) * (x - 2)
sage: F.unit()
-5
sage: F.value()
-5*x^2 + 25*x - 30
```

The underlying list is the list of pairs (p_i, e_i), where each p_i is a 'prime' and each e_i is an integer. The unit part is discarded by the list:

```
sage: list(F)
[(x - 3, 1), (x - 2, 1)]
```

```
sage: len(F)
2
sage: F[1]
(x - 2, 1)
```

In the ring $\mathbf{Z}[x]$, the integer -5 is not a unit, so the factorization has three factors:

```
sage: x = ZZ['x'].0
sage: f = -5*(x-2)*(x-3)
sage: f
-5*x^2 + 25*x - 30
sage: F = f.factor(); F
(-1) * 5 * (x - 3) * (x - 2)
sage: F.universe()
Univariate Polynomial Ring in x over Integer Ring
sage: F.unit()
-1
sage: list(F)
[(5, 1), (x - 3, 1), (x - 2, 1)]
sage: F.value()
-5*x^2 + 25*x - 30
sage: len(F)
3
```

On the other hand, -1 is a unit in $\mathbf{Z}$, so it is included in the unit:

```
sage: x = ZZ['x'].0
sage: f = -1*(x-2)*(x-3)
sage: F = f.factor(); F
(-1) * (x - 3) * (x - 2)
sage: F.unit()
-1
sage: list(F)
[(x - 3, 1), (x - 2, 1)]
```

Factorizations can involve fairly abstract mathematical objects:

```
sage: F = ModularSymbols(11,4).factorization()
sage: F
(Modular Symbols subspace of dimension 2 of Modular Symbols space of dimension 6 for
→Gamma_0(11) of weight 4 with sign 0 over Rational Field) *
(Modular Symbols subspace of dimension 2 of Modular Symbols space of dimension 6 for
→Gamma_0(11) of weight 4 with sign 0 over Rational Field) *
(Modular Symbols subspace of dimension 2 of Modular Symbols space of dimension 6 for
→Gamma_0(11) of weight 4 with sign 0 over Rational Field)
sage: type(F)
<class 'sage.structure.factorization.Factorization'>


sage: K.<a> = NumberField(x^2 + 3); K
Number Field in a with defining polynomial x^2 + 3
sage: f = K.factor(15); f
(Fractional ideal (-a))^2 * (Fractional ideal (5))
sage: f.universe()
Monoid of ideals of Number Field in a with defining polynomial x^2 + 3
sage: f.unit()
Fractional ideal (1)
sage: g=K.factor(9); g
```

```
(Fractional ideal (-a))^4
sage: f.lcm(g)
(Fractional ideal (-a))^4 * (Fractional ideal (5))
sage: f.gcd(g)
(Fractional ideal (-a))^2
sage: f.is_integral()
True
```

AUTHORS:

- William Stein (2006-01-22): added unit part as suggested by David Kohel.

- William Stein (2008-01-17): wrote much of the documentation and fixed a couple of bugs.

- Nick Alexander (2008-01-19): added support for non-commuting factors.

- John Cremona (2008-08-22): added division, lcm, gcd, is_integral and universe functions

class sage.structure.factorization. **Factorization** (*x*, *unit=None*, *cr=False*, *sort=True*, *simplify=True*)

Bases: *sage.structure.sage_object.SageObject*

A formal factorization of an object.

EXAMPLES:

```
sage: N = 2006
sage: F = N.factor(); F
2 * 17 * 59
sage: F.unit()
1
sage: F = factor(-2006); F
-1 * 2 * 17 * 59
sage: F.unit()
-1
sage: loads(F.dumps()) == F
True
sage: F = Factorization([(x,1/3)])
Traceback (most recent call last):
...
TypeError: no conversion of this rational to integer
```

base_change (*U*)

Return the factorization self, with its factors (including the unit part) coerced into the universe *U*.

EXAMPLES:

```
sage: F = factor(2006)
sage: F.universe()
Integer Ring
sage: P.<x> = ZZ[]
sage: F.base_change(P).universe()
Univariate Polynomial Ring in x over Integer Ring
```

This method will return a TypeError if the coercion is not possible:

```
sage: g = x^2 - 1
sage: F = factor(g); F
(x - 1) * (x + 1)
sage: F.universe()
Univariate Polynomial Ring in x over Integer Ring
```

```
sage: F.base_change(ZZ)
Traceback (most recent call last):
...
TypeError: Impossible to coerce the factors of (x - 1) * (x + 1) into Integer
↪Ring
```

expand ( )

Return the product of the factors in the factorization, multiplied out.

EXAMPLES:

```
sage: F = factor(-2006); F
-1 * 2 * 17 * 59
sage: F.value()
-2006

sage: R.<x,y> = FreeAlgebra(ZZ, 2)
sage: F = Factorization([(x,3), (y, 2), (x,1)]); F
x^3 * y^2 * x
sage: F.value()
x^3*y^2*x
```

gcd (*other*)

Return the gcd of two factorizations.

If the two factorizations have different universes, this method will attempt to find a common universe for the gcd. A TypeError is raised if this is impossible.

EXAMPLES:

```
sage: factor(-30).gcd(factor(-160))
2 * 5
sage: factor(gcd(-30,160))
2 * 5

sage: R.<x> = ZZ[]
sage: (factor(-20).gcd(factor(5*x+10))).universe()
Univariate Polynomial Ring in x over Integer Ring
```

is_commutative ( )

Return True if my factors commute.

EXAMPLES:

```
sage: F = factor(2006)
sage: F.is_commutative()
True
sage: K = QuadraticField(23, 'a')
sage: F = K.factor(13)
sage: F.is_commutative()
True
sage: R.<x,y,z> = FreeAlgebra(QQ, 3)
sage: F = Factorization([(z, 2)], 3)
sage: F.is_commutative()
False
sage: (F*F^-1).is_commutative()
False
```

is_integral ( )
> Return True iff all exponents of this Factorization are non-negative.
>
> EXAMPLES:

```
sage: F = factor(-10); F
-1 * 2 * 5
sage: F.is_integral()
True

sage: F = factor(-10) / factor(16); F
-1 * 2^-3 * 5
sage: F.is_integral()
False
```

lcm (*other*)
> Return the lcm of two factorizations.
>
> If the two factorizations have different universes, this method will attempt to find a common universe for the lcm. A TypeError is raised if this is impossible.
>
> EXAMPLES:

```
sage: factor(-10).lcm(factor(-16))
2^4 * 5
sage: factor(lcm(-10,16))
2^4 * 5

sage: R.<x> = ZZ[]
sage: (factor(-20).lcm(factor(5*x+10))).universe()
Univariate Polynomial Ring in x over Integer Ring
```

prod ( )
> Return the product of the factors in the factorization, multiplied out.
>
> EXAMPLES:

```
sage: F = factor(-2006); F
-1 * 2 * 17 * 59
sage: F.value()
-2006

sage: R.<x,y> = FreeAlgebra(ZZ, 2)
sage: F = Factorization([(x,3), (y, 2), (x,1)]); F
x^3 * y^2 * x
sage: F.value()
x^3*y^2*x
```

radical ( )
> Return the factorization of the radical of the value of self.
>
> First, check that all exponents in the factorization are positive, raise ValueError otherwise. If all exponents are positive, return self with all exponents set to 1 and with the unit set to 1.
>
> EXAMPLES:

```
sage: F = factor(-100); F
-1 * 2^2 * 5^2
sage: F.radical()
2 * 5
```

```
sage: factor(1/2).radical()
Traceback (most recent call last):
...
ValueError: All exponents in the factorization must be positive.
```

radical_value ( )

> Return the product of the prime factors in self.

> First, check that all exponents in the factorization are positive, raise ValueError otherwise. If all exponents are positive, return the product of the prime factors in self. This should be functionally equivalent to self.radical().value()

> EXAMPLES:

```
sage: F = factor(-100); F
-1 * 2^2 * 5^2
sage: F.radical_value()
10
sage: factor(1/2).radical_value()
Traceback (most recent call last):
...
ValueError: All exponents in the factorization must be positive.
```

simplify ( )

> Combine adjacent products as much as possible.

sort (*_cmp=None*, *key=None*)

> Sort the factors in this factorization.

> INPUT:

>> •_cmp - (default: None) comparison function (deprecated)

>> •key - (default: None) comparison key

> OUTPUT:

>> •changes this factorization to be sorted (inplace)

> If _cmp is None , we use a comparison key.

> If key is None , we determine the comparison key as follows:

> If the prime in the first factor has a dimension method, then we sort based first on *dimension* then on the exponent.

> If there is no dimension method, we next attempt to sort based on a degree method, in which case, we sort based first on *degree*, then exponent to break ties when two factors have the same degree, and if those match break ties based on the actual prime itself.

> Otherwise, we sort according to the prime itself.

> EXAMPLES:

> We create a factored polynomial:

```
sage: x = polygen(QQ,'x')
sage: F = factor(x^3 + 1); F
(x + 1) * (x^2 - x + 1)
```

> We sort it by decreasing degree:

```
sage: F.sort(key=lambda x:(-x[0].degree(), x))
sage: F
(x^2 - x + 1) * (x + 1)
```

unit ( )
   Return the unit part of this factorization.

   EXAMPLES:

   We create a polynomial over the real double field and factor it:

```
sage: x = polygen(RDF, 'x')
sage: F = factor(-2*x^2 - 1); F
(-2.0) * (x^2 + 0.5000000000000001)
```

   Note that the unit part of the factorization is -2.0:

```
sage: F.unit()
-2.0

sage: F = factor(-2006); F
-1 * 2 * 17 * 59
sage: F.unit()
-1
```

universe ( )
   Return the parent structure of my factors.

   Note: This used to be called `base_ring`, but the universe of a factorization need not be a ring.

   EXAMPLES:

```
sage: F = factor(2006)
sage: F.universe()
Integer Ring

sage: R.<x,y,z> = FreeAlgebra(QQ, 3)
sage: F = Factorization([(z, 2)], 3)
sage: (F*F^-1).universe()
Free Algebra on 3 generators (x, y, z) over Rational Field

sage: F = ModularSymbols(11,4).factorization()
sage: F.universe()
```

value ( )
   Return the product of the factors in the factorization, multiplied out.

   EXAMPLES:

```
sage: F = factor(-2006); F
-1 * 2 * 17 * 59
sage: F.value()
-2006

sage: R.<x,y> = FreeAlgebra(ZZ, 2)
sage: F = Factorization([(x,3), (y, 2), (x,1)]); F
x^3 * y^2 * x
```

```
sage: F.value()
x^3*y^2*x
```

4.3 IntegerFactorization objects

class sage.structure.factorization_integer. **IntegerFactorization** (*x*, *unit=None*, *cr=False*, *sort=True*, *simplify=True*, *unsafe=False*)

Bases: *sage.structure.factorization.Factorization*

A lightweight class for an IntegerFactorization object, inheriting from the more general Factorization class.

In the Factorization class the user has to create a list containing the factorization data, which is then passed to the actual Factorization object upon initialization.

However, for the typical use of integer factorization via the Integer.factor() method in sage.rings.integer this is noticeably too much overhead, slowing down the factorization of integers of up to about 40 bits by a factor of around 10. Moreover, the initialization done in the Factorization class is typically unnecessary: the caller can guarantee that the list contains pairs of an Integer and an int, as well as that the list is sorted.

AUTHOR:

- Sebastian Pancratz (2010-01-10)

4.4 Finite Homogenous Sequences

A mutable sequence of elements with a common guaranteed category, which can be set immutable.

Sequence derives from list, so has all the functionality of lists and can be used wherever lists are used. When a sequence is created without explicitly given the common universe of the elements, the constructor coerces the first and second element to some *canonical* common parent, if possible, then the second and third, etc. If this is possible, it then coerces everything into the canonical parent at the end. (Note that canonical coercion is very restrictive.) The sequence then has a function universe() which returns either the common canonical parent (if the coercion succeeded), or the category of all objects (Objects()). So if you have a list v and type:

```
sage: v = [1, 2/3, 5]
sage: w = Sequence(v)
sage: w.universe()
Rational Field
```

then since w.universe() is $\mathbf{Q}$, you're guaranteed that all elements of w are rationals:

```
sage: v[0].parent()
Integer Ring
sage: w[0].parent()
Rational Field
```

If you do assignment to w this property of being rationals is guaranteed to be preserved:

```
sage: w[0] = 2
sage: w[0].parent()
Rational Field
sage: w[0] = 'hi'
Traceback (most recent call last):
...
TypeError: unable to convert 'hi' to a rational
```

However, if you do `w = Sequence(v)` and the resulting universe is `Objects()`, the elements are not guaranteed to have any special parent. This is what should happen, e.g., with finite field elements of different characteristics:

```
sage: v = Sequence([GF(3)(1), GF(7)(1)])
sage: v.universe()
Category of objects
```

You can make a list immutable with `v.freeze()`. Assignment is never again allowed on an immutable list.

Creation of a sequence involves making a copy of the input list, and substantial coercions. It can be greatly sped up by explicitly specifying the universe of the sequence:

```
sage: v = Sequence(range(10000), universe=ZZ)
```

`sage.structure.sequence.` **Sequence** (*x*,   *universe=None*,   *check=True*,   *immutable=False*,
                                        cr=False, *cr_str=None*, *use_sage_types=False*)
A mutable list of elements with a common guaranteed universe, which can be set immutable.

A universe is either an object that supports coercion (e.g., a parent), or a category.

INPUT:

- `x` - a list or tuple instance

- `universe` - (default: None) the universe of elements; if None determined using canonical coercions and the entire list of elements. If list is empty, is category Objects() of all objects.

- `check` – (default: True) whether to coerce the elements of x into the universe

- `immutable` - (default: True) whether or not this sequence is immutable

- `cr` - (default: False) if True, then print a carriage return after each comma when printing this sequence.

- `cr_str` - (default: False) if True, then print a carriage return after each comma when calling `str()` on this sequence.

- **use_sage_types – (default: False) if True, coerce the** built-in Python numerical types int, long, float, complex to the corresponding Sage types (this makes functions like vector() more flexible)

OUTPUT:

- a sequence

EXAMPLES:

```
sage: v = Sequence(range(10))
sage: v.universe()
<... 'int'>
sage: v
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
```

We can request that the built-in Python numerical types be coerced to Sage objects:

```
sage: v = Sequence(range(10), use_sage_types=True)
sage: v.universe()
Integer Ring
sage: v
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
```

You can also use seq for "Sequence", which is identical to using Sequence:

```
sage: v = seq([1,2,1/1]); v
[1, 2, 1]
sage: v.universe()
Rational Field
```

Note that assignment coerces if possible,:

```
sage: v = Sequence(range(10), ZZ)
sage: a = QQ(5)
sage: v[3] = a
sage: parent(v[3])
Integer Ring
sage: parent(a)
Rational Field
sage: v[3] = 2/3
Traceback (most recent call last):
...
TypeError: no conversion of this rational to integer
```

Sequences can be used absolutely anywhere lists or tuples can be used:

```
sage: isinstance(v, list)
True
```

Sequence can be immutable, so entries can't be changed:

```
sage: v = Sequence([1,2,3], immutable=True)
sage: v.is_immutable()
True
sage: v[0] = 5
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead.
```

Only immutable sequences are hashable (unlike Python lists), though the hashing is potentially slow, since it first involves conversion of the sequence to a tuple, and returning the hash of that.:

```
sage: v = Sequence(range(10), ZZ, immutable=True)
sage: hash(v)
1591723448               # 32-bit
-4181190870548101704     # 64-bit
```

If you really know what you are doing, you can circumvent the type checking (for an efficiency gain):

```
sage: list.__setitem__(v, int(1), 2/3)        # bad circumvention
sage: v
[0, 2/3, 2, 3, 4, 5, 6, 7, 8, 9]
sage: list.__setitem__(v, int(1), int(2))     # not so bad circumvention
```

You can make a sequence with a new universe from an old sequence.:

```
sage: w = Sequence(v, QQ)
sage: w
[0, 2, 2, 3, 4, 5, 6, 7, 8, 9]
sage: w.universe()
Rational Field
sage: w[1] = 2/3
sage: w
[0, 2/3, 2, 3, 4, 5, 6, 7, 8, 9]
```

The default universe for any sequence, if no compatible parent structure can be found, is the universe of all Sage objects.

This example illustrates how every element of a list is taken into account when constructing a sequence.:

```
sage: v = Sequence([1,7,6,GF(5)(3)]); v
[1, 2, 1, 3]
sage: v.universe()
Finite Field of size 5
```

class sage.structure.sequence. **Sequence_generic** (*x*, *universe=None*, *check=True*, *immutable=False*, *cr=False*, *cr_str=None*, *use_sage_types=False*)

Bases: *sage.structure.sage_object.SageObject* , list

A mutable list of elements with a common guaranteed universe, which can be set immutable.

A universe is either an object that supports coercion (e.g., a parent), or a category.

INPUT:

   •x - a list or tuple instance

   •universe - (default: None) the universe of elements; if None determined using canonical coercions and the entire list of elements. If list is empty, is category Objects() of all objects.

   •check – (default: True) whether to coerce the elements of x into the universe

   •immutable - (default: True) whether or not this sequence is immutable

   •cr - (default: False) if True, then print a carriage return after each comma when printing this sequence.

   •**use_sage_types – (default: False) if True, coerce the** built-in Python numerical types int, long, float, complex to the corresponding Sage types (this makes functions like vector() more flexible)

OUTPUT:

   •a sequence

EXAMPLES:

```
sage: v = Sequence(range(10))
sage: v.universe()
<... 'int'>
sage: v
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
```

We can request that the built-in Python numerical types be coerced to Sage objects:

```
sage: v = Sequence(range(10), use_sage_types=True)
sage: v.universe()
Integer Ring
```

```
sage: v
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
```

You can also use seq for "Sequence", which is identical to using Sequence:

```
sage: v = seq([1,2,1/1]); v
[1, 2, 1]
sage: v.universe()
Rational Field
```

Note that assignment coerces if possible,

```
sage: v = Sequence(range(10), ZZ)
sage: a = QQ(5)
sage: v[3] = a
sage: parent(v[3])
Integer Ring
sage: parent(a)
Rational Field
sage: v[3] = 2/3
Traceback (most recent call last):
...
TypeError: no conversion of this rational to integer
```

Sequences can be used absolutely anywhere lists or tuples can be used:

```
sage: isinstance(v, list)
True
```

Sequence can be immutable, so entries can't be changed:

```
sage: v = Sequence([1,2,3], immutable=True)
sage: v.is_immutable()
True
sage: v[0] = 5
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead.
```

Only immutable sequences are hashable (unlike Python lists), though the hashing is potentially slow, since it first involves conversion of the sequence to a tuple, and returning the hash of that.

```
sage: v = Sequence(range(10), ZZ, immutable=True)
sage: hash(v)
1591723448                 # 32-bit
-4181190870548101704    # 64-bit
```

If you really know what you are doing, you can circumvent the type checking (for an efficiency gain):

```
sage: list.__setitem__(v, int(1), 2/3)         # bad circumvention
sage: v
[0, 2/3, 2, 3, 4, 5, 6, 7, 8, 9]
sage: list.__setitem__(v, int(1), int(2))      # not so bad circumvention
```

You can make a sequence with a new universe from an old sequence.

```
sage: w = Sequence(v, QQ)
sage: w
[0, 2, 2, 3, 4, 5, 6, 7, 8, 9]
sage: w.universe()
Rational Field
sage: w[1] = 2/3
sage: w
[0, 2/3, 2, 3, 4, 5, 6, 7, 8, 9]
```

The default universe for any sequence, if no compatible parent structure can be found, is the universe of all Sage objects.

This example illustrates how every element of a list is taken into account when constructing a sequence.

```
sage: v = Sequence([1,7,6,GF(5)(3)]); v
[1, 2, 1, 3]
sage: v.universe()
Finite Field of size 5
```

append (*x*)
    EXAMPLES:

```
sage: v = Sequence([1,2,3,4], immutable=True)
sage: v.append(34)
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead.
sage: v = Sequence([1/3,2,3,4])
sage: v.append(4)
sage: type(v[4])
<type 'sage.rings.rational.Rational'>
```

extend (*iterable*)
    Extend list by appending elements from the iterable.

    EXAMPLES:

```
sage: B = Sequence([1,2,3])
sage: B.extend(range(4))
sage: B
[1, 2, 3, 0, 1, 2, 3]
```

insert (*index*, *object*)
    Insert object before index.

    EXAMPLES:

```
sage: B = Sequence([1,2,3])
sage: B.insert(10, 5)
sage: B
[1, 2, 3, 5]
```

is_immutable ( )
    Return True if this object is immutable (can not be changed) and False if it is not.

    To make this object immutable use *set_immutable()* .

    EXAMPLES:

```
sage: v = Sequence([1,2,3,4/5])
sage: v[0] = 5
sage: v
[5, 2, 3, 4/5]
sage: v.is_immutable()
False
sage: v.set_immutable()
sage: v.is_immutable()
True
```

is_mutable ()
> EXAMPLES:

```
sage: a = Sequence([1,2/3,-2/5])
sage: a.is_mutable()
True
sage: a[0] = 100
sage: type(a[0])
<type 'sage.rings.rational.Rational'>
sage: a.set_immutable()
sage: a[0] = 50
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead.
sage: a.is_mutable()
False
```

pop (*index=-1*)
> Remove and return item at index (default last)

> EXAMPLES:

```
sage: B = Sequence([1,2,3])
sage: B.pop(1)
2
sage: B
[1, 3]
```

remove (*value*)
> Remove first occurrence of value

> EXAMPLES:

```
sage: B = Sequence([1,2,3])
sage: B.remove(2)
sage: B
[1, 3]
```

reverse ()
> Reverse the elements of self, in place.

> EXAMPLES:

```
sage: B = Sequence([1,2,3])
sage: B.reverse(); B
[3, 2, 1]
```

set_immutable ()
> Make this object immutable, so it can never again be changed.

EXAMPLES:

```
sage: v = Sequence([1,2,3,4/5])
sage: v[0] = 5
sage: v
[5, 2, 3, 4/5]
sage: v.set_immutable()
sage: v[3] = 7
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead.
```

sort (*cmp=None*, *key=None*, *reverse=False*)

Sort this list *IN PLACE*.

INPUT:

- •key - see Python `list sort`

- •reverse - see Python `list sort`

- •cmp - see Python `list sort` (deprecated)

Because `cmp` is not allowed in Python3, it must be avoided.

EXAMPLES:

```
sage: B = Sequence([3,2,1/5])
sage: B.sort()
sage: B
[1/5, 2, 3]
sage: B.sort(reverse=True); B
[3, 2, 1/5]
```

universe ( )

EXAMPLES:

```
sage: Sequence([1,2/3,-2/5]).universe()
Rational Field
sage: Sequence([1,2/3,'-2/5']).universe()
Category of objects
```

sage.structure.sequence. **seq** (*x*, *universe=None*, *check=True*, *immutable=False*, *cr=False*, *cr_str=None*, *use_sage_types=False*)

A mutable list of elements with a common guaranteed universe, which can be set immutable.

A universe is either an object that supports coercion (e.g., a parent), or a category.

INPUT:

- •x - a list or tuple instance

- •universe - (default: None) the universe of elements; if None determined using canonical coercions and the entire list of elements. If list is empty, is category Objects() of all objects.

- •check – (default: True) whether to coerce the elements of x into the universe

- •immutable - (default: True) whether or not this sequence is immutable

- •cr - (default: False) if True, then print a carriage return after each comma when printing this sequence.

- •cr_str - (default: False) if True, then print a carriage return after each comma when calling `str()` on this sequence.

- •**use_sage_types** – (default: False) **if True, coerce the** built-in Python numerical types int, long, float, complex to the corresponding Sage types (this makes functions like vector() more flexible)

OUTPUT:

- •a sequence

EXAMPLES:

```
sage: v = Sequence(range(10))
sage: v.universe()
<... 'int'>
sage: v
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
```

We can request that the built-in Python numerical types be coerced to Sage objects:

```
sage: v = Sequence(range(10), use_sage_types=True)
sage: v.universe()
Integer Ring
sage: v
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
```

You can also use seq for "Sequence", which is identical to using Sequence:

```
sage: v = seq([1,2,1/1]); v
[1, 2, 1]
sage: v.universe()
Rational Field
```

Note that assignment coerces if possible,:

```
sage: v = Sequence(range(10), ZZ)
sage: a = QQ(5)
sage: v[3] = a
sage: parent(v[3])
Integer Ring
sage: parent(a)
Rational Field
sage: v[3] = 2/3
Traceback (most recent call last):
...
TypeError: no conversion of this rational to integer
```

Sequences can be used absolutely anywhere lists or tuples can be used:

```
sage: isinstance(v, list)
True
```

Sequence can be immutable, so entries can't be changed:

```
sage: v = Sequence([1,2,3], immutable=True)
sage: v.is_immutable()
True
sage: v[0] = 5
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead.
```

Only immutable sequences are hashable (unlike Python lists), though the hashing is potentially slow, since it first involves conversion of the sequence to a tuple, and returning the hash of that.:

```
sage: v = Sequence(range(10), ZZ, immutable=True)
sage: hash(v)
1591723448              # 32-bit
-4181190870548101704    # 64-bit
```

If you really know what you are doing, you can circumvent the type checking (for an efficiency gain):

```
sage: list.__setitem__(v, int(1), 2/3)      # bad circumvention
sage: v
[0, 2/3, 2, 3, 4, 5, 6, 7, 8, 9]
sage: list.__setitem__(v, int(1), int(2))   # not so bad circumvention
```

You can make a sequence with a new universe from an old sequence.:

```
sage: w = Sequence(v, QQ)
sage: w
[0, 2, 2, 3, 4, 5, 6, 7, 8, 9]
sage: w.universe()
Rational Field
sage: w[1] = 2/3
sage: w
[0, 2/3, 2, 3, 4, 5, 6, 7, 8, 9]
```

The default universe for any sequence, if no compatible parent structure can be found, is the universe of all Sage objects.

This example illustrates how every element of a list is taken into account when constructing a sequence.:

```
sage: v = Sequence([1,7,6,GF(5)(3)]); v
[1, 2, 1, 3]
sage: v.universe()
Finite Field of size 5
```

4.5 Cartesian products

AUTHORS:

- Nicolas Thiery (2010-03): initial version

class sage.sets.cartesian_product. **CartesianProduct** (*sets*, *category*, *flatten=False*)

Bases: *sage.structure.unique_representation.UniqueRepresentation* , *sage.structure.parent.Parent*

A class implementing a raw data structure for Cartesian products of sets (and elements thereof). See cartesian_product for how to construct full fledged Cartesian products.

EXAMPLES:

```
sage: G = cartesian_product([GF(5), Permutations(10)])
sage: G.cartesian_factors()
(Finite Field of size 5, Standard permutations of 10)
sage: G.cardinality()
18144000
sage: G.random_element()     # random
```

```
(1, [4, 7, 6, 5, 10, 1, 3, 2, 8, 9])
sage: G.category()
Join of Category of finite monoids
    and Category of Cartesian products of monoids
    and Category of Cartesian products of finite enumerated sets
```

_cartesian_product_of_elements (*elements*)

Return the Cartesian product of the given `elements`.

This implements `Sets.CartesianProducts.ParentMethods._cartesian_product_of_elements()`. INPUT:

- `elements` – an iterable (e.g. tuple, list) with one element of each Cartesian factor of `self`

> **Warning:** This is meant as a fast low-level method. In particular, no coercion is attempted. When coercion or sanity checks are desirable, please use instead `self(elements)` or `self._element_constructor(elements)`.

EXAMPLES:

```
sage: S1 = Sets().example()
sage: S2 = InfiniteEnumeratedSets().example()
sage: C = cartesian_product([S2, S1, S2])
sage: C._cartesian_product_of_elements([S2.an_element(), S1.an_element(), S2.
↪an_element()])
(42, 47, 42)
```

class **Element**

Bases: *sage.structure.element_wrapper.ElementWrapperCheckWrappedClass*

cartesian_factors ( )

Return the tuple of elements that compose this element.

EXAMPLES:

```
sage: A = cartesian_product([ZZ, RR])
sage: A((1, 1.23)).cartesian_factors()
(1, 1.23000000000000)
sage: type(_)
<... 'tuple'>
```

cartesian_projection (*i*)

Return the projection of `self` on the i-th factor of the Cartesian product, as per `Sets.CartesianProducts.ElementMethods.cartesian_projection()`.

INPUT:

- `i` – the index of a factor of the Cartesian product

EXAMPLES:

```
sage: C = Sets().CartesianProducts().example(); C
The Cartesian product of (Set of prime numbers (basic implementation), An␣
↪example of an infinite enumerated set: the non negative integers, An␣
↪example of a finite enumerated set: {1,2,3})
sage: x = C.an_element(); x
(47, 42, 1)
sage: x.cartesian_projection(1)
42
```

```
sage: x.summand_projection(1)
doctest:...: DeprecationWarning: summand_projection is deprecated. Please
→use cartesian_projection instead.
See http://trac.sagemath.org/10963 for details.
42
```

CartesianProduct.**an_element**()

> EXAMPLES:

```
sage: C = Sets().CartesianProducts().example(); C
The Cartesian product of (Set of prime numbers (basic implementation),
 An example of an infinite enumerated set: the non negative integers,
 An example of a finite enumerated set: {1,2,3})
sage: C.an_element()
(47, 42, 1)
```

CartesianProduct.**cartesian_factors**()

> Return the Cartesian factors of `self`.
>
> **See also:**
>
> > `Sets.CartesianProducts.ParentMethods.cartesian_factors()`.
>
> EXAMPLES:

```
sage: cartesian_product([QQ, ZZ, ZZ]).cartesian_factors()
(Rational Field, Integer Ring, Integer Ring)
```

CartesianProduct.**cartesian_projection**(*i*)

> Return the natural projection onto the i-th Cartesian factor of `self` as per `Sets.CartesianProducts.ParentMethods.cartesian_projection()`.
>
> INPUT:
>
> > • `i` – the index of a Cartesian factor of `self`
>
> EXAMPLES:

```
sage: C = Sets().CartesianProducts().example(); C
The Cartesian product of (Set of prime numbers (basic implementation), An
→example of an infinite enumerated set: the non negative integers, An
→example of a finite enumerated set: {1,2,3})
sage: x = C.an_element(); x
(47, 42, 1)
sage: pi = C.cartesian_projection(1)
sage: pi(x)
42

sage: C.cartesian_projection('hey')
Traceback (most recent call last):
...
ValueError: i (=hey) must be in {0, 1, 2}
```

CartesianProduct.**construction**()

> Return the construction functor and its arguments for this Cartesian product.
>
> OUTPUT:
>
> A pair whose first entry is a Cartesian product functor and its second entry is a list of the Cartesian factors.

EXAMPLES:

```
sage: cartesian_product([ZZ, QQ]).construction()
(The cartesian_product functorial construction,
 (Integer Ring, Rational Field))
```

CartesianProduct. **summand_projection** (*args*, ***kwds*)

Deprecated: Use *cartesian_projection()* instead. See trac ticket #10963 for details.

4.6 Families

A Family is an associative container which models a family $(f_i)_{i \in I}$. Then, `f[i]` returns the element of the family indexed by `i`. Whenever available, set and combinatorial class operations (counting, iteration, listing) on the family are induced from those of the index set. Families should be created through the *Family()* function.

AUTHORS:

- Nicolas Thiery (2008-02): initial release

- Florent Hivert (2008-04): various fixes, cleanups and improvements.

class sage.sets.family. **AbstractFamily**

Bases: *sage.structure.parent.Parent*

The abstract class for family

Any family belongs to a class which inherits from *AbstractFamily* .

hidden_keys ( )

Returns the hidden keys of the family, if any.

EXAMPLES:

```
sage: f = Family({3: 'a', 4: 'b', 7: 'd'})
sage: f.hidden_keys()
[]
```

inverse_family ( )

Returns the inverse family, with keys and values exchanged. This presumes that there are no duplicate values in `self` .

This default implementation is not lazy and therefore will only work with not too big finite families. It is also cached for the same reason:

```
sage: Family({3: 'a', 4: 'b', 7: 'd'}).inverse_family()
Finite family {'a': 3, 'b': 4, 'd': 7}

sage: Family((3,4,7)).inverse_family()
Finite family {3: 0, 4: 1, 7: 2}
```

map (*f*, *name=None*)

Returns the family $(f(\mathtt{self}[i]))_{i \in I}$, where I is the index set of self.

Todo

good name?

EXAMPLES:

```
sage: f = Family({3: 'a', 4: 'b', 7: 'd'})
sage: g = f.map(lambda x: x+'1')
sage: list(g)
['a1', 'b1', 'd1']
```

zip (*f*, *other*, *name=None*)

Given two families with same index set I (and same hidden keys if relevant), returns the family $(f(self[i], other[i]))_{i \in I}$

Todo

generalize to any number of families and merge with map?

EXAMPLES:

```
sage: f = Family({3: 'a', 4: 'b', 7: 'd'})
sage: g = Family({3: '1', 4: '2', 7: '3'})
sage: h = f.zip(lambda x,y: x+y, g)
sage: list(h)
['a1', 'b2', 'd3']
```

class sage.sets.family. **EnumeratedFamily** (*enumset*)

Bases: *sage.sets.family.LazyFamily*

EnumeratedFamily turns an enumerated set c into a family indexed by the set $\{0, \ldots, |c| - 1\}$.

Instances should be created via the *Family()* factory. See its documentation for examples and tests.

cardinality ( )

Return the number of elements in self.

EXAMPLES:

```
sage: from sage.sets.family import EnumeratedFamily
sage: f = EnumeratedFamily(Permutations(3))
sage: f.cardinality()
6

sage: from sage.categories.examples.infinite_enumerated_sets import␣
→NonNegativeIntegers
sage: f = Family(NonNegativeIntegers())
sage: f.cardinality()
+Infinity
```

keys ( )

Returns self's keys.

EXAMPLES:

```
sage: from sage.sets.family import EnumeratedFamily
sage: f = EnumeratedFamily(Permutations(3))
sage: f.keys()
Standard permutations of 3

sage: from sage.categories.examples.infinite_enumerated_sets import␣
→NonNegativeIntegers
sage: f = Family(NonNegativeIntegers())
```

```
sage: f.keys()
An example of an infinite enumerated set: the non negative integers
```

sage.sets.family. **Family** (*indices*, *function=None*, *hidden_keys=[]*, *hidden_function=None*, *lazy=False*, *name=None*)

A Family is an associative container which models a family $(f_i)_{i \in I}$. Then, `f[i]` returns the element of the family indexed by i. Whenever available, set and combinatorial class operations (counting, iteration, listing) on the family are induced from those of the index set.

There are several available implementations (classes) for different usages; Family serves as a factory, and will create instances of the appropriate classes depending on its arguments.

INPUT:

> •`indices` – the indices for the family
>
> •`function` – (optional) the function f applied to all visible indices; the default is the identity function
>
> •`hidden_keys` – (optional) a list of hidden indices that can be accessed through `my_family[i]`
>
> •`hidden_function` – (optional) a function for the hidden indices
>
> •`lazy` – boolean (default: `False`); whether the family is lazily created or not; if the indices are infinite, then this is automatically made `True`
>
> •`name` – (optional) the name of the function; only used when the family is lazily created via a function

EXAMPLES:

In its simplest form, a list $l = [l_0, l_1, \ldots, l_\ell]$ or a tuple by itself is considered as the family $(l_i)_{i \in I}$ where I is the set $\{0, \ldots, \ell\}$ where ℓ is `len(l)` -1. So `Family(l)` returns the corresponding family:

```
sage: f = Family([1,2,3])
sage: f
Family (1, 2, 3)
sage: f = Family((1,2,3))
sage: f
Family (1, 2, 3)
```

Instead of a list you can as well pass any iterable object:

```
sage: f = Family(2*i+1 for i in [1,2,3]);
sage: f
Family (3, 5, 7)
```

A family can also be constructed from a dictionary `t` . The resulting family is very close to `t` , except that the elements of the family are the values of `t` . Here, we define the family $(f_i)_{i \in \{3,4,7\}}$ with $f_3 = a$, $f_4 = b$, and $f_7 = d$:

```
sage: f = Family({3: 'a', 4: 'b', 7: 'd'})
sage: f
Finite family {3: 'a', 4: 'b', 7: 'd'}
sage: f[7]
'd'
sage: len(f)
3
sage: list(f)
['a', 'b', 'd']
sage: [ x for x in f ]
['a', 'b', 'd']
sage: f.keys()
```

```
[3, 4, 7]
sage: 'b' in f
True
sage: 'e' in f
False
```

A family can also be constructed by its index set I and a function f, as in $(f(i))_{i \in I}$:

```
sage: f = Family([3,4,7], lambda i: 2*i)
sage: f
Finite family {3: 6, 4: 8, 7: 14}
sage: f.keys()
[3, 4, 7]
sage: f[7]
14
sage: list(f)
[6, 8, 14]
sage: [x for x in f]
[6, 8, 14]
sage: len(f)
3
```

By default, all images are computed right away, and stored in an internal dictionary:

```
sage: f = Family((3,4,7), lambda i: 2*i)
sage: f
Finite family {3: 6, 4: 8, 7: 14}
```

Note that this requires all the elements of the list to be hashable. One can ask instead for the images $f(i)$ to be computed lazily, when needed:

```
sage: f = Family([3,4,7], lambda i: 2*i, lazy=True)
sage: f
Lazy family (<lambda>(i))_{i in [3, 4, 7]}
sage: f[7]
14
sage: list(f)
[6, 8, 14]
sage: [x for x in f]
[6, 8, 14]
```

This allows in particular for modeling infinite families:

```
sage: f = Family(ZZ, lambda i: 2*i, lazy=True)
sage: f
Lazy family (<lambda>(i))_{i in Integer Ring}
sage: f.keys()
Integer Ring
sage: f[1]
2
sage: f[-5]
-10
sage: i = iter(f)
sage: next(i), next(i), next(i), next(i), next(i)
(0, 2, -2, 4, -4)
```

Note that the `lazy` keyword parameter is only needed to force laziness. Usually it is automatically set to a correct default value (ie: `False` for finite data structures and `True` for enumerated sets:

```
sage: f == Family(ZZ, lambda i: 2*i)
True
```

Beware that for those kind of families len(f) is not supposed to work. As a replacement, use the .cardinality()
method:

```
sage: f = Family(Permutations(3), attrcall("to_lehmer_code"))
sage: list(f)
[[0, 0, 0], [0, 1, 0], [1, 0, 0], [1, 1, 0], [2, 0, 0], [2, 1, 0]]
sage: f.cardinality()
6
```

Caveat: Only certain families with lazy behavior can be pickled. In particular, only functions that work with
Sage's pickle_function and unpickle_function (in sage.misc.fpickle) will correctly unpickle. The following two
work:

```
sage: f = Family(Permutations(3), lambda p: p.to_lehmer_code()); f
Lazy family (<lambda>(i))_{i in Standard permutations of 3}
sage: f == loads(dumps(f))
True

sage: f = Family(Permutations(3), attrcall("to_lehmer_code")); f
Lazy family (i.to_lehmer_code())_{i in Standard permutations of 3}
sage: f == loads(dumps(f))
True
```

But this one does not:

```
sage: def plus_n(n): return lambda x: x+n
sage: f = Family([1,2,3], plus_n(3), lazy=True); f
Lazy family (<lambda>(i))_{i in [1, 2, 3]}
sage: f == loads(dumps(f))
Traceback (most recent call last):
...
ValueError: Cannot pickle code objects from closures
```

Finally, it can occasionally be useful to add some hidden elements in a family, which are accessible as f[i],
but do not appear in the keys or the container operations:

```
sage: f = Family([3,4,7], lambda i: 2*i, hidden_keys=[2])
sage: f
Finite family {3: 6, 4: 8, 7: 14}
sage: f.keys()
[3, 4, 7]
sage: f.hidden_keys()
[2]
sage: f[7]
14
sage: f[2]
4
sage: list(f)
[6, 8, 14]
sage: [x for x in f]
[6, 8, 14]
sage: len(f)
3
```

The following example illustrates when the function is actually called:

```
sage: def compute_value(i):
....:       print('computing 2*'+str(i))
....:       return 2*i
sage: f = Family([3,4,7], compute_value, hidden_keys=[2])
computing 2*3
computing 2*4
computing 2*7
sage: f
Finite family {3: 6, 4: 8, 7: 14}
sage: f.keys()
[3, 4, 7]
sage: f.hidden_keys()
[2]
sage: f[7]
14
sage: f[2]
computing 2*2
4
sage: f[2]
4
sage: list(f)
[6, 8, 14]
sage: [x for x in f]
[6, 8, 14]
sage: len(f)
3
```

Here is a close variant where the function for the hidden keys is different from that for the other keys:

```
sage: f = Family([3,4,7], lambda i: 2*i, hidden_keys=[2], hidden_function =
→lambda i: 3*i)
sage: f
Finite family {3: 6, 4: 8, 7: 14}
sage: f.keys()
[3, 4, 7]
sage: f.hidden_keys()
[2]
sage: f[7]
14
sage: f[2]
6
sage: list(f)
[6, 8, 14]
sage: [x for x in f]
[6, 8, 14]
sage: len(f)
3
```

Family accept finite and infinite EnumeratedSets as input:

```
sage: f = Family(FiniteEnumeratedSet([1,2,3]))
sage: f
Family (1, 2, 3)
sage: from sage.categories.examples.infinite_enumerated_sets import
→NonNegativeIntegers
sage: f = Family(NonNegativeIntegers())
sage: f
Family (An example of an infinite enumerated set: the non negative integers)
```

```
sage: f = Family(FiniteEnumeratedSet([3,4,7]), lambda i: 2*i)
sage: f
Finite family {3: 6, 4: 8, 7: 14}
sage: f.keys()
{3, 4, 7}
sage: f[7]
14
sage: list(f)
[6, 8, 14]
sage: [x for x in f]
[6, 8, 14]
sage: len(f)
3
```

```
sage: f = Family({1:'a', 2:'b', 3:'c'}, lazy=True)
Traceback (most recent call last):
ValueError: lazy keyword only makes sense together with function keyword !
```

```
sage: f = Family(list(range(1,27)), lambda i: chr(i+96))
sage: f
    Finite family {1: 'a', 2: 'b', 3: 'c', 4: 'd', 5: 'e', 6: 'f', 7: 'g', 8: 'h
↪', 9: 'i', 10: 'j', 11: 'k', 12: 'l', 13: 'm', 14: 'n', 15: 'o', 16: 'p', 17: 'q
↪', 18: 'r', 19: 's', 20: 't', 21: 'u', 22: 'v', 23: 'w', 24: 'x', 25: 'y', 26:
↪'z'}
sage: f[2]
'b'
```

The factory `Family` is supposed to be idempotent. We test this feature here:

```
sage: from sage.sets.family import FiniteFamily, LazyFamily, TrivialFamily
sage: f = FiniteFamily({3: 'a', 4: 'b', 7: 'd'})
sage: g = Family(f)
sage: f == g
True

sage: f = Family([3,4,7], lambda i: 2*i, hidden_keys=[2])
sage: g = Family(f)
sage: f == g
True

sage: f = LazyFamily([3,4,7], lambda i: 2*i)
sage: g = Family(f)
sage: f == g
True

sage: f = TrivialFamily([3,4,7])
sage: g = Family(f)
sage: f == g
True
```

The family should keep the order of the keys:

```
sage: f = Family(["c", "a", "b"], lambda x: x+x)
sage: list(f)
['cc', 'aa', 'bb']
```

Only the hidden function is applied to the hidden keys:

```
sage: f = lambda x : 2*x
sage: h_f = lambda x:x%2
sage: F = Family([1,2,3,4],function = f, hidden_keys=[5],hidden_function=h_f)
sage: F[5]
1
```

class sage.sets.family. **FiniteFamily** (*dictionary*, *keys=None*)

Bases: *sage.sets.family.AbstractFamily*

A *FiniteFamily* is an associative container which models a finite family $(f_i)_{i \in I}$. Its elements f_i are therefore its values. Instances should be created via the *Family()* factory. See its documentation for examples and tests.

EXAMPLES:

We define the family $(f_i)_{i \in \{3,4,7\}}$ with $f_3 = a$, $f_4 = b$, and $f_7 = d$:

```
sage: from sage.sets.family import FiniteFamily
sage: f = FiniteFamily({3: 'a', 4: 'b', 7: 'd'})
```

Individual elements are accessible as in a usual dictionary:

```
sage: f[7]
'd'
```

And the other usual dictionary operations are also available:

```
sage: len(f)
3
sage: f.keys()
[3, 4, 7]
```

However f behaves as a container for the f_i's:

```
sage: list(f)
['a', 'b', 'd']
sage: [ x for x in f ]
['a', 'b', 'd']
```

The order of the elements can be specified using the keys optional argument:

```
sage: f = FiniteFamily({"a": "aa", "b": "bb", "c" : "cc" }, keys = ["c", "a", "b
→"])
sage: list(f)
['cc', 'aa', 'bb']
```

cardinality ( )

Returns the number of elements in self.

EXAMPLES:

```
sage: from sage.sets.family import FiniteFamily
sage: f = FiniteFamily({3: 'a', 4: 'b', 7: 'd'})
sage: f.cardinality()
3
```

has_key (*k*)

Returns whether k is a key of self

EXAMPLES:

```
sage: Family({"a":1, "b":2, "c":3}).has_key("a")
True
sage: Family({"a":1, "b":2, "c":3}).has_key("d")
False
```

keys ( )
    Returns the index set of this family

    EXAMPLES:

```
sage: f = Family(["c", "a", "b"], lambda x: x+x)
sage: f.keys()
['c', 'a', 'b']
```

values ( )
    Returns the elements of this family

    EXAMPLES:

```
sage: f = Family(["c", "a", "b"], lambda x: x+x)
sage: f.values()
['cc', 'aa', 'bb']
```

class sage.sets.family. **FiniteFamilyWithHiddenKeys** (*dictionary*, *hidden_keys*, *hidden_function*)
    Bases: *sage.sets.family.FiniteFamily*

    A close variant of *FiniteFamily* where the family contains some hidden keys whose corresponding values are computed lazily (and remembered). Instances should be created via the *Family()* factory. See its documentation for examples and tests.

    Caveat: Only instances of this class whose functions are compatible with sage.misc.fpickle can be pickled.

hidden_keys ( )
    Returns self's hidden keys.

    EXAMPLES:

```
sage: f = Family([3,4,7], lambda i: 2*i, hidden_keys=[2])
sage: f.hidden_keys()
[2]
```

class sage.sets.family. **LazyFamily** (*set*, *function*, *name=None*)
    Bases: *sage.sets.family.AbstractFamily*

    A LazyFamily(I, f) is an associative container which models the (possibly infinite) family $(f(i))_{i \in I}$.

    Instances should be created via the *Family()* factory. See its documentation for examples and tests.

cardinality ( )
    Return the number of elements in self.

    EXAMPLES:

```
sage: from sage.sets.family import LazyFamily
sage: f = LazyFamily([3,4,7], lambda i: 2*i)
sage: f.cardinality()
3
```

```
sage: from sage.categories.examples.infinite_enumerated_sets import
→NonNegativeIntegers
sage: l = LazyFamily(NonNegativeIntegers(), lambda i: 2*i)
sage: l.cardinality()
+Infinity
```

> **keys** ( )
>> Returns self's keys.
>>
>> EXAMPLES:

```
sage: from sage.sets.family import LazyFamily
sage: f = LazyFamily([3,4,7], lambda i: 2*i)
sage: f.keys()
[3, 4, 7]
```

class sage.sets.family. **TrivialFamily** (*enumeration*)
> Bases: *sage.sets.family.AbstractFamily*
>
> *TrivialFamily* turns a list/tuple c into a family indexed by the set $\{0, \dots, |c| - 1\}$.
>
> Instances should be created via the *Family()* factory. See its documentation for examples and tests.
>
> **cardinality** ( )
>> Return the number of elements in self.
>>
>> EXAMPLES:

```
sage: from sage.sets.family import TrivialFamily
sage: f = TrivialFamily([3,4,7])
sage: f.cardinality()
3
```

> **keys** ( )
>> Returns self's keys.
>>
>> EXAMPLES:

```
sage: from sage.sets.family import TrivialFamily
sage: f = TrivialFamily([3,4,7])
sage: f.keys()
[0, 1, 2]
```

4.7 Sets

AUTHORS:

- William Stein (2005) - first version

- William Stein (2006-02-16) - large number of documentation and examples; improved code

- Mike Hansen (2007-3-25) - added differences and symmetric differences; fixed operators

- Florent Hivert (2010-06-17) - Adapted to categories

- Nicolas M. Thiery (2011-03-15) - Added subset and superset methods

- Julian Rueth (2013-04-09) - Collected common code in *Set_object_binary* , fixed __hash__ .

sage.sets.set. **Set** (*X=frozenset([])*)

> Create the underlying set of X .
>
> If X is a list, tuple, Python set, or X.is_finite() is True , this returns a wrapper around Python's enumerated immutable frozenset type with extra functionality. Otherwise it returns a more formal wrapper.
>
> If you need the functionality of mutable sets, use Python's builtin set type.
>
> EXAMPLES:

```
sage: X = Set(GF(9,'a'))
sage: X
{0, 1, 2, a, a + 1, a + 2, 2*a, 2*a + 1, 2*a + 2}
sage: type(X)
<class 'sage.sets.set.Set_object_enumerated_with_category'>
sage: Y = X.union(Set(QQ))
sage: Y
Set-theoretic union of {0, 1, 2, a, a + 1, a + 2, 2*a, 2*a + 1, 2*a + 2} and Set␣
↪of elements of Rational Field
sage: type(Y)
<class 'sage.sets.set.Set_object_union_with_category'>
```

> Usually sets can be used as dictionary keys.

```
sage: d={Set([2*I,1+I]):10}
sage: d                       # key is randomly ordered
{{I + 1, 2*I}: 10}
sage: d[Set([1+I,2*I])]
10
sage: d[Set((1+I,2*I))]
10
```

> The original object is often forgotten.

```
sage: v = [1,2,3]
sage: X = Set(v)
sage: X
{1, 2, 3}
sage: v.append(5)
sage: X
{1, 2, 3}
sage: 5 in X
False
```

> Set also accepts iterators, but be careful to only give *finite* sets.

```
sage: list(Set(iter([1, 2, 3, 4, 5])))
[1, 2, 3, 4, 5]
```

> We can also create sets from different types:

```
sage: sorted(Set([Sequence([3,1], immutable=True), 5, QQ, Partition([3,1,1])]),␣
↪key=str)
[5, Rational Field, [3, 1, 1], [3, 1]]
```

> However each of the objects must be hashable:

```
sage: Set([QQ, [3, 1], 5])
Traceback (most recent call last):
```

```
...
TypeError: unhashable type: 'list'
```

class sage.sets.set. **Set_object** (*X*)

 Bases: *sage.structure.parent.Set_generic*

 A set attached to an almost arbitrary object.

 EXAMPLES:

```
sage: K = GF(19)
sage: Set(K)
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18}
sage: S = Set(K)

sage: latex(S)
\left\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\right\}
sage: TestSuite(S).run()

sage: latex(Set(ZZ))
\Bold{Z}
```

 an_element ( )

  Return the first element of self returned by __iter__()

  If self is empty, the exception EmptySetError is raised instead.

  This provides a generic implementation of the method _an_element_() for all parents in EnumeratedSets .

  EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example(); C
An example of a finite enumerated set: {1,2,3}
sage: C.an_element() # indirect doctest
1
sage: S = Set([])
sage: S.an_element()
Traceback (most recent call last):
...
EmptySetError
```

 cardinality ( )

  Return the cardinality of this set, which is either an integer or Infinity .

  EXAMPLES:

```
sage: Set(ZZ).cardinality()
+Infinity
sage: Primes().cardinality()
+Infinity
sage: Set(GF(5)).cardinality()
5
sage: Set(GF(5^2,'a')).cardinality()
25
```

 difference (*X*)

  Return the set difference self -X .

  EXAMPLES:

```
sage: X = Set(ZZ).difference(Primes())
sage: 4 in X
True
sage: 3 in X
False

sage: 4/1 in X
True

sage: X = Set(GF(9,'b')).difference(Set(GF(27,'c')))
sage: X
{0, 1, 2, b, b + 1, b + 2, 2*b, 2*b + 1, 2*b + 2}

sage: X = Set(GF(9,'b')).difference(Set(GF(27,'b')))
sage: X
{0, 1, 2, b, b + 1, b + 2, 2*b, 2*b + 1, 2*b + 2}
```

intersection (*X*)

Return the intersection of `self` and `X`.

EXAMPLES:

```
sage: X = Set(ZZ).intersection(Primes())
sage: 4 in X
False
sage: 3 in X
True

sage: 2/1 in X
True

sage: X = Set(GF(9,'b')).intersection(Set(GF(27,'c')))
sage: X
{}

sage: X = Set(GF(9,'b')).intersection(Set(GF(27,'b')))
sage: X
{}
```

is_empty ( )

Return boolean representing emptiness of the set.

OUTPUT:

True if the set is empty, false if otherwise.

EXAMPLES:

```
sage: Set([]).is_empty()
True
sage: Set([0]).is_empty()
False
sage: Set([1..100]).is_empty()
False
sage: Set(SymmetricGroup(2).list()).is_empty()
False
sage: Set(ZZ).is_empty()
False
```

is_finite ( )
>   Return `True` if `self` is finite.

>   EXAMPLES:

```
sage: Set(QQ).is_finite()
False
sage: Set(GF(250037)).is_finite()
True
sage: Set(Integers(2^1000000)).is_finite()
True
sage: Set([1,'a',ZZ]).is_finite()
True
```

object ( )
>   Return underlying object.

>   EXAMPLES:

```
sage: X = Set(QQ)
sage: X.object()
Rational Field
sage: X = Primes()
sage: X.object()
Set of all prime numbers: 2, 3, 5, 7, ...
```

subsets (*size=None*)
>   Return the `Subsets` object representing the subsets of a set. If size is specified, return the subsets of that size.

>   EXAMPLES:

```
sage: X = Set([1,2,3])
sage: list(X.subsets())
[{}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}]
sage: list(X.subsets(2))
[{1, 2}, {1, 3}, {2, 3}]
```

symmetric_difference (*X*)
>   Returns the symmetric difference of `self` and `X`.

>   EXAMPLES:

```
sage: X = Set([1,2,3]).symmetric_difference(Set([3,4]))
sage: X
{1, 2, 4}
```

union (*X*)
>   Return the union of `self` and `X`.

>   EXAMPLES:

```
sage: Set(QQ).union(Set(ZZ))
Set-theoretic union of Set of elements of Rational Field and Set of elements
→of Integer Ring
sage: Set(QQ) + Set(ZZ)
Set-theoretic union of Set of elements of Rational Field and Set of elements
→of Integer Ring
sage: X = Set(QQ).union(Set(GF(3))); X
Set-theoretic union of Set of elements of Rational Field and {0, 1, 2}
```

```
sage: 2/3 in X
True
sage: GF(3)(2) in X
True
sage: GF(5)(2) in X
False
sage: Set(GF(7)) + Set(GF(3))
{0, 1, 2, 3, 4, 5, 6, 1, 2, 0}
```

class sage.sets.set. **Set_object_binary** (*X, Y, op, latex_op*)
    Bases: *sage.sets.set.Set_object*

    An abstract common base class for sets defined by a binary operation (ex.
    Set_object_union , *Set_object_intersection* , *Set_object_difference* , and
    Set_object_symmetric_difference).

    INPUT:

        •X , Y – sets, the operands to op

        •op – a string describing the binary operation

        •latex_op – a string used for rendering this object in LaTeX

    EXAMPLES:

```
sage: X = Set(QQ^2)
sage: Y = Set(ZZ)
sage: from sage.sets.set import Set_object_binary
sage: S = Set_object_binary(X, Y, "union", "\\cup"); S
Set-theoretic union of Set of elements of Vector space of dimension 2
 over Rational Field and Set of elements of Integer Ring
```

class sage.sets.set. **Set_object_difference** (*X, Y*)
    Bases: *sage.sets.set.Set_object_binary*

    Formal difference of two sets.

    is_finite ( )
        Return whether this set is finite.

        EXAMPLES:

```
sage: X = Set(range(10))
sage: Y = Set(range(-10,5))
sage: Z = Set(QQ)
sage: X.difference(Y).is_finite()
True
sage: X.difference(Z).is_finite()
True
sage: Z.difference(X).is_finite()
False
sage: Z.difference(Set(ZZ)).is_finite()
Traceback (most recent call last):
...
NotImplementedError
```

class sage.sets.set. **Set_object_enumerated** (*X*)
    Bases: *sage.sets.set.Set_object*

    A finite enumerated set.

cardinality ( )
>   Return the cardinality of `self`.
>
>   EXAMPLES:

```
sage: Set([1,1]).cardinality()
1
```

difference (*other*)
>   Return the set difference `self -other`.
>
>   EXAMPLES:

```
sage: X = Set([1,2,3,4])
sage: Y = Set([1,2])
sage: X.difference(Y)
{3, 4}
sage: Z = Set(ZZ)
sage: W = Set([2.5, 4, 5, 6])
sage: W.difference(Z)
{2.50000000000000}
```

frozenset ( )
>   Return the Python frozenset object associated to this set, which is an immutable set (hence hashable).
>
>   EXAMPLES:

```
sage: X = Set(GF(8,'c'))
sage: X
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
sage: s = X.set(); s
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
sage: hash(s)
Traceback (most recent call last):
...
TypeError: unhashable type: 'set'
sage: s = X.frozenset(); s
frozenset({0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1})
sage: hash(s)
-1390224788            # 32-bit
 561411537695332972    # 64-bit
sage: type(s)
<... 'frozenset'>
```

intersection (*other*)
>   Return the intersection of `self` and `other`.
>
>   EXAMPLES:

```
sage: X = Set(GF(8,'c'))
sage: Y = Set([GF(8,'c').0, 1, 2, 3])
sage: X.intersection(Y)
{1, c}
```

is_finite ( )
>   Return `True` as this is a finite set.
>
>   EXAMPLES:

```
sage: Set(GF(19)).is_finite()
True
```

issubset (*other*)

Return whether `self` is a subset of `other`.

INPUT:

•`other` – a finite Set

EXAMPLES:

```
sage: X = Set([1,3,5])
sage: Y = Set([0,1,2,3,5,7])
sage: X.issubset(Y)
True
sage: Y.issubset(X)
False
sage: X.issubset(X)
True
```

issuperset (*other*)

Return whether `self` is a superset of `other`.

INPUT:

•`other` – a finite Set

EXAMPLES:

```
sage: X = Set([1,3,5])
sage: Y = Set([0,1,2,3,5])
sage: X.issuperset(Y)
False
sage: Y.issuperset(X)
True
sage: X.issuperset(X)
True
```

list ( )

Return the elements of `self`, as a list.

EXAMPLES:

```
sage: X = Set(GF(8,'c'))
sage: X
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
sage: X.list()
[0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1]
sage: type(X.list())
<... 'list'>
```

Todo

FIXME: What should be the order of the result? That of `self.object()`? Or the order given by `set(self.object())`? Note that `__getitem__()` is currently implemented in term of this list method, which is really inefficient ...

---

random_element ( )
> Return a random element in this set.
>
> EXAMPLES:

```
sage: Set([1,2,3]).random_element() # random
2
```

set ( )
> Return the Python set object associated to this set.
>
> Python has a notion of finite set, and often Sage sets have an associated Python set. This function returns that set.
>
> EXAMPLES:

```
sage: X = Set(GF(8,'c'))
sage: X
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
sage: X.set()
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
sage: type(X.set())
<... 'set'>
sage: type(X)
<class 'sage.sets.set.Set_object_enumerated_with_category'>
```

symmetric_difference (*other*)
> Return the symmetric difference of `self` and `other`.
>
> EXAMPLES:

```
sage: X = Set([1,2,3,4])
sage: Y = Set([1,2])
sage: X.symmetric_difference(Y)
{3, 4}
sage: Z = Set(ZZ)
sage: W = Set([2.5, 4, 5, 6])
sage: U = W.symmetric_difference(Z)
sage: 2.5 in U
True
sage: 4 in U
False
sage: V = Z.symmetric_difference(W)
sage: V == U
True
sage: 2.5 in V
True
sage: 6 in V
False
```

union (*other*)
> Return the union of `self` and `other`.
>
> EXAMPLES:

```
sage: X = Set(GF(8,'c'))
sage: Y = Set([GF(8,'c').0, 1, 2, 3])
sage: X
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
sage: Y
```

```
{1, c, 3, 2}
sage: X.union(Y)
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1, 2, 3}
```

class sage.sets.set. **Set_object_intersection** (*X*, *Y*)

Bases: *sage.sets.set.Set_object_binary*

Formal intersection of two sets.

is_finite ( )

Return whether this set is finite.

EXAMPLES:

```
sage: X = Set(IntegerRange(100))
sage: Y = Set(ZZ)
sage: X.intersection(Y).is_finite()
True
sage: Y.intersection(X).is_finite()
True
sage: Y.intersection(Set(QQ)).is_finite()
Traceback (most recent call last):
...
NotImplementedError
```

class sage.sets.set. **Set_object_symmetric_difference** (*X*, *Y*)

Bases: *sage.sets.set.Set_object_binary*

Formal symmetric difference of two sets.

is_finite ( )

Return whether this set is finite.

EXAMPLES:

```
sage: X = Set(range(10))
sage: Y = Set(range(-10,5))
sage: Z = Set(QQ)
sage: X.symmetric_difference(Y).is_finite()
True
sage: X.symmetric_difference(Z).is_finite()
False
sage: Z.symmetric_difference(X).is_finite()
False
sage: Z.symmetric_difference(Set(ZZ)).is_finite()
Traceback (most recent call last):
...
NotImplementedError
```

class sage.sets.set. **Set_object_union** (*X*, *Y*)

Bases: *sage.sets.set.Set_object_binary*

A formal union of two sets.

cardinality ( )

Return the cardinality of this set.

EXAMPLES:

```
sage: X = Set(GF(3)).union(Set(GF(2)))
sage: X
```

```
{0, 1, 2, 0, 1}
sage: X.cardinality()
5

sage: X = Set(GF(3)).union(Set(ZZ))
sage: X.cardinality()
+Infinity
```

   is_finite ( )
        Return whether this set is finite.

        EXAMPLES:

```
sage: X = Set(range(10))
sage: Y = Set(range(-10,0))
sage: Z = Set(Primes())
sage: X.union(Y).is_finite()
True
sage: X.union(Z).is_finite()
False
```

sage.sets.set. **is_Set** (*x*)
   Returns `True` if `x` is a Sage *Set_object* (not to be confused with a Python set).

   EXAMPLES:

```
sage: from sage.sets.set import is_Set
sage: is_Set([1,2,3])
False
sage: is_Set(set([1,2,3]))
False
sage: is_Set(Set([1,2,3]))
True
sage: is_Set(Set(QQ))
True
sage: is_Set(Primes())
True
```

4.8 Disjoint-set data structure

The main entry point is *DisjointSet()* which chooses the appropriate type to return. For more on the data structure, see *DisjointSet()* .

AUTHORS:

   • Sébastien Labbé (2008) - Initial version.

   • Sébastien Labbé (2009-11-24) - Pickling support

   • Sébastien Labbé (2010-01) - Inclusion into sage (trac ticket #6775).

EXAMPLES:

Disjoint set of integers from 0 to n −1 :

```
sage: s = DisjointSet(6)
sage: s
{{0}, {1}, {2}, {3}, {4}, {5}}
```

```
sage: s.union(2, 4)
sage: s.union(1, 3)
sage: s.union(5, 1)
sage: s
{{0}, {1, 3, 5}, {2, 4}}
sage: s.find(3)
1
sage: s.find(5)
1
sage: list(map(s.find, range(6)))
[0, 1, 2, 1, 2, 1]
```

Disjoint set of hashables objects:

```
sage: d = DisjointSet('abcde')
sage: d
{{'a'}, {'b'}, {'c'}, {'d'}, {'e'}}
sage: d.union('a','b')
sage: d.union('b','c')
sage: d.union('c','d')
sage: d
{{'a', 'b', 'c', 'd'}, {'e'}}
sage: d.find('c')
'a'
```

sage.sets.disjoint_set. **DisjointSet** (*arg*)

> Constructs a disjoint set where each element of arg is in its own set. If arg is an integer, then the disjoint set
> returned is made of the integers from 0 to arg −1 .

> A disjoint-set data structure (sometimes called union-find data structure) is a data structure that keeps track of a
> partitioning of a set into a number of separate, nonoverlapping sets. It performs two operations:

> > • *find()* – Determine which set a particular element is in.

> > • *union()* – Combine or merge two sets into a single set.

> REFERENCES:

> > • Wikipedia article Disjoint-set_data_structure

> INPUT:

> > • arg – non negative integer or an iterable of hashable objects.

> EXAMPLES:

> From a non-negative integer:

```
sage: DisjointSet(5)
{{0}, {1}, {2}, {3}, {4}}
```

> From an iterable:

```
sage: DisjointSet('abcde')
{{'a'}, {'b'}, {'c'}, {'d'}, {'e'}}
sage: DisjointSet(range(6))
{{0}, {1}, {2}, {3}, {4}, {5}}
sage: DisjointSet(['yi',45,'cheval'])
{{'cheval'}, {'yi'}, {45}}
```

class sage.sets.disjoint_set. **DisjointSet_class**

 Bases: *sage.structure.sage_object.SageObject*

 Common class and methods for *DisjointSet_of_integers* and *DisjointSet_of_hashables* .

 cardinality ()

  Return the number of elements in self , *not* the number of subsets.

  EXAMPLES:

```
sage: d = DisjointSet(5)
sage: d.cardinality()
5
sage: d.union(2, 4)
sage: d.cardinality()
5
sage: d = DisjointSet(range(5))
sage: d.cardinality()
5
sage: d.union(2, 4)
sage: d.cardinality()
5
```

 number_of_subsets ()

  Return the number of subsets in self .

  EXAMPLES:

```
sage: d = DisjointSet(5)
sage: d.number_of_subsets()
5
sage: d.union(2, 4)
sage: d.number_of_subsets()
4
sage: d = DisjointSet(range(5))
sage: d.number_of_subsets()
5
sage: d.union(2, 4)
sage: d.number_of_subsets()
4
```

class sage.sets.disjoint_set. **DisjointSet_of_hashables**

 Bases: *sage.sets.disjoint_set.DisjointSet_class*

 Disjoint set of hashables.

 EXAMPLES:

```
sage: d = DisjointSet('abcde')
sage: d
{{'a'}, {'b'}, {'c'}, {'d'}, {'e'}}
sage: d.union('a', 'c')
sage: d
{{'a', 'c'}, {'b'}, {'d'}, {'e'}}
sage: d.find('a')
'a'
```

```
sage: a.union('a','c')
sage: a == loads(dumps(a))
True
```

element_to_root_dict ( )
>   Return the dictionary where the keys are the elements of `self` and the values are their representative inside a list.
>
>   EXAMPLES:

```
sage: d = DisjointSet(range(5))
sage: d.union(2,3)
sage: d.union(4,1)
sage: e = d.element_to_root_dict(); e
{0: 0, 1: 4, 2: 2, 3: 2, 4: 4}
sage: WordMorphism(e)
WordMorphism: 0->0, 1->4, 2->2, 3->2, 4->4
```

find (*e*)
>   Return the representative of the set that `e` currently belongs to.
>
>   INPUT:
>
>   •e – element in `self`
>
>   EXAMPLES:

```
sage: e = DisjointSet(range(5))
sage: e.union(4,2)
sage: e
{{0}, {1}, {2, 4}, {3}}
sage: e.find(2)
4
sage: e.find(4)
4
sage: e.union(1,3)
sage: e
{{0}, {1, 3}, {2, 4}}
sage: e.find(1)
1
sage: e.find(3)
1
sage: e.union(3,2)
sage: e
{{0}, {1, 2, 3, 4}}
sage: [e.find(i) for i in range(5)]
[0, 1, 1, 1, 1]
sage: e.find(5)
Traceback (most recent call last):
...
KeyError: 5
```

root_to_elements_dict ( )
>   Return the dictionary where the keys are the roots of `self` and the values are the elements in the same set.
>
>   EXAMPLES:

```
sage: d = DisjointSet(range(5))
sage: d.union(2,3)
sage: d.union(4,1)
sage: e = d.root_to_elements_dict(); e
{0: [0], 2: [2, 3], 4: [1, 4]}
```

to_digraph ()

Return the current digraph of `self` where (a, b) is an oriented edge if b is the parent of a.

EXAMPLES:

```
sage: d = DisjointSet(range(5))
sage: d.union(2,3)
sage: d.union(4,1)
sage: d.union(3,4)
sage: d
{{0}, {1, 2, 3, 4}}
sage: g = d.to_digraph(); g
Looped digraph on 5 vertices
sage: g.edges()
[(0, 0, None), (1, 2, None), (2, 2, None), (3, 2, None), (4, 2, None)]
```

The result depends on the ordering of the union:

```
sage: d = DisjointSet(range(5))
sage: d.union(1,2)
sage: d.union(1,3)
sage: d.union(1,4)
sage: d
{{0}, {1, 2, 3, 4}}
sage: d.to_digraph().edges()
[(0, 0, None), (1, 1, None), (2, 1, None), (3, 1, None), (4, 1, None)]
```

union (e, f)

Combine the set of `e` and the set of `f` into one.

All elements in those two sets will share the same representative that can be gotten using find.

INPUT:

- `e` - element in `self`

- `f` - element in `self`

EXAMPLES:

```
sage: e = DisjointSet('abcde')
sage: e
{{'a'}, {'b'}, {'c'}, {'d'}, {'e'}}
sage: e.union('a','b')
sage: e
{{'a', 'b'}, {'c'}, {'d'}, {'e'}}
sage: e.union('c','e')
sage: e
{{'a', 'b'}, {'c', 'e'}, {'d'}}
sage: e.union('b','e')
sage: e
{{'a', 'b', 'c', 'e'}, {'d'}}
```

class sage.sets.disjoint_set. **DisjointSet_of_integers**

Bases: *sage.sets.disjoint_set.DisjointSet_class*

Disjoint set of integers from `0` to `n-1`.

EXAMPLES:

```
sage: d = DisjointSet(5)
sage: d
{{0}, {1}, {2}, {3}, {4}}
sage: d.union(2,4)
sage: d.union(0,2)
sage: d
{{0, 2, 4}, {1}, {3}}
sage: d.find(2)
2
```

```
sage: a.union(3,4)
sage: a == loads(dumps(a))
True
```

element_to_root_dict ( )
> Return the dictionary where the keys are the elements of `self` and the values are their representative inside a list.

> EXAMPLES:

```
sage: d = DisjointSet(5)
sage: d.union(2,3)
sage: d.union(4,1)
sage: e = d.element_to_root_dict(); e
{0: 0, 1: 4, 2: 2, 3: 2, 4: 4}
sage: WordMorphism(e)
WordMorphism: 0->0, 1->4, 2->2, 3->2, 4->4
```

find (*i*)
> Return the representative of the set that `i` currently belongs to.

> INPUT:

> • `i` – element in `self`

> EXAMPLES:

```
sage: e = DisjointSet(5)
sage: e.union(4,2)
sage: e
{{0}, {1}, {2, 4}, {3}}
sage: e.find(2)
4
sage: e.find(4)
4
sage: e.union(1,3)
sage: e
{{0}, {1, 3}, {2, 4}}
sage: e.find(1)
1
sage: e.find(3)
1
sage: e.union(3,2)
sage: e
{{0}, {1, 2, 3, 4}}
sage: [e.find(i) for i in range(5)]
[0, 1, 1, 1, 1]
sage: e.find(5)
Traceback (most recent call last):
```

```
...
ValueError: i(=5) must be between 0 and 4
```

root_to_elements_dict ( )
    Return the dictionary where the keys are the roots of self and the values are the elements in the same
    set as the root.

    EXAMPLES:

```
sage: d = DisjointSet(5)
sage: d.root_to_elements_dict()
{0: [0], 1: [1], 2: [2], 3: [3], 4: [4]}
sage: d.union(2,3)
sage: d.root_to_elements_dict()
{0: [0], 1: [1], 2: [2, 3], 4: [4]}
sage: d.union(3,0)
sage: d.root_to_elements_dict()
{1: [1], 2: [0, 2, 3], 4: [4]}
sage: d
{{0, 2, 3}, {1}, {4}}
```

to_digraph ( )
    Return the current digraph of self where (a, b) is an oriented edge if b is the parent of a.

    EXAMPLES:

```
sage: d = DisjointSet(5)
sage: d.union(2,3)
sage: d.union(4,1)
sage: d.union(3,4)
sage: d
{{0}, {1, 2, 3, 4}}
sage: g = d.to_digraph(); g
Looped digraph on 5 vertices
sage: g.edges()
[(0, 0, None), (1, 2, None), (2, 2, None), (3, 2, None), (4, 2, None)]
```

The result depends on the ordering of the union:

```
sage: d = DisjointSet(5)
sage: d.union(1,2)
sage: d.union(1,3)
sage: d.union(1,4)
sage: d
{{0}, {1, 2, 3, 4}}
sage: d.to_digraph().edges()
[(0, 0, None), (1, 1, None), (2, 1, None), (3, 1, None), (4, 1, None)]
```

union (i, j)
    Combine the set of i and the set of j into one.

    All elements in those two sets will share the same representative that can be gotten using find.

    INPUT:

        • i - element in self

        • j - element in self

    EXAMPLES:

```
sage: d = DisjointSet(5)
sage: d
{{0}, {1}, {2}, {3}, {4}}
sage: d.union(0,1)
sage: d
{{0, 1}, {2}, {3}, {4}}
sage: d.union(2,4)
sage: d
{{0, 1}, {2, 4}, {3}}
sage: d.union(1,4)
sage: d
{{0, 1, 2, 4}, {3}}
sage: d.union(1,5)
Traceback (most recent call last):
...
ValueError: j(=5) must be between 0 and 4
```

4.9 Disjoint union of enumerated sets

AUTHORS:

- Florent Hivert (2009-07/09): initial implementation.

- Florent Hivert (2010-03): classcall related stuff.

- Florent Hivert (2010-12): fixed facade element construction.

class sage.sets.disjoint_union_enumerated_sets.**DisjointUnionEnumeratedSets** (*family*,
facade=True,
keepkey=False,
category=None)

Bases: *sage.structure.unique_representation.UniqueRepresentation* ,
sage.structure.parent.Parent

A class for disjoint unions of enumerated sets.

INPUT:

- `family` – a list (or iterable or family) of enumerated sets

- `keepkey` – a boolean

- `facade` – a boolean

This models the enumerated set obtained by concatenating together the specified ordered sets. The later are supposed to be pairwise disjoint; otherwise, a multiset is created.

The argument `family` can be a list, a tuple, a dictionary, or a family. If it is not a family it is first converted into a family (see *sage.sets.family.Family()*).

Experimental options:

By default, there is no way to tell from which set of the union an element is generated. The option `keepkey=True` keeps track of those by returning pairs `(key,el)` where `key` is the index of the set to which `el` belongs. When this option is specified, the enumerated sets need not be disjoint anymore.

With the option `facade=False` the elements are wrapped in an object whose parent is the disjoint union itself. The wrapped object can then be recovered using the `value` attribute.

The two options can be combined.

The names of those options is imperfect, and subject to change in future versions. Feedback welcome.

EXAMPLES:

The input can be a list or a tuple of FiniteEnumeratedSets:

```
sage: U1 = DisjointUnionEnumeratedSets((
....:        FiniteEnumeratedSet([1,2,3]),
....:        FiniteEnumeratedSet([4,5,6])))
sage: U1
Disjoint union of Family ({1, 2, 3}, {4, 5, 6})
sage: U1.list()
[1, 2, 3, 4, 5, 6]
sage: U1.cardinality()
6
```

The input can also be a dictionary:

```
sage: U2 = DisjointUnionEnumeratedSets({1: FiniteEnumeratedSet([1,2,3]),
....:                                   2: FiniteEnumeratedSet([4,5,6])})
sage: U2
Disjoint union of Finite family {1: {1, 2, 3}, 2: {4, 5, 6}}
sage: U2.list()
[1, 2, 3, 4, 5, 6]
sage: U2.cardinality()
6
```

However in that case the enumeration order is not specified.

In general the input can be any family:

```
sage: U3 = DisjointUnionEnumeratedSets(
....:      Family([2,3,4], Permutations, lazy=True))
sage: U3
Disjoint union of Lazy family (<class 'sage.combinat.permutation.Permutations'>
↪(i))_{i in [2, 3, 4]}
sage: U3.cardinality()
32
sage: it = iter(U3)
sage: [next(it), next(it), next(it), next(it), next(it), next(it)]
[[1, 2], [2, 1], [1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1]]
sage: U3.unrank(18)
[2, 4, 1, 3]
```

This allows for infinite unions:

```
sage: U4 = DisjointUnionEnumeratedSets(
....:      Family(NonNegativeIntegers(), Permutations))
sage: U4
Disjoint union of Lazy family (<class 'sage.combinat.permutation.Permutations'>
↪(i))_{i in Non negative integers}
sage: U4.cardinality()
+Infinity
sage: it = iter(U4)
sage: [next(it), next(it), next(it), next(it), next(it), next(it)]
```

```
[[], [1], [1, 2], [2, 1], [1, 2, 3], [1, 3, 2]]
sage: U4.unrank(18)
[2, 3, 1, 4]
```

> **Warning:** Beware that some of the operations assume in that case that infinitely many of the enumerated sets are non empty.

Experimental options

We demonstrate the `keepkey` option:

```
sage: Ukeep = DisjointUnionEnumeratedSets(
....:             Family(list(range(4)), Permutations), keepkey=True)
sage: it = iter(Ukeep)
sage: [next(it) for i in range(6)]
[(0, []), (1, [1]), (2, [1, 2]), (2, [2, 1]), (3, [1, 2, 3]), (3, [1, 3, 2])]
sage: type(next(it)[1])
<class 'sage.combinat.permutation.StandardPermutations_n_with_category.element_
↪class'>
```

We now demonstrate the `facade` option:

```
sage: UNoFacade = DisjointUnionEnumeratedSets(
....:             Family(list(range(4)), Permutations), facade=False)
sage: it = iter(UNoFacade)
sage: [next(it) for i in range(6)]
[[], [1], [1, 2], [2, 1], [1, 2, 3], [1, 3, 2]]
sage: el = next(it); el
[2, 1, 3]
sage: type(el)
<type 'sage.structure.element_wrapper.ElementWrapper'>
sage: el.parent() == UNoFacade
True
sage: elv = el.value; elv
[2, 1, 3]
sage: type(elv)
<class 'sage.combinat.permutation.StandardPermutations_n_with_category.element_
↪class'>
```

The elements `el` of the disjoint union are simple wrapped elements. So to access the methods, you need to do `el.value`:

```
sage: el[0]
Traceback (most recent call last):
...
TypeError: 'sage.structure.element_wrapper.ElementWrapper' object
 has no attribute '__getitem__'
sage: el.value[0]
2
```

Possible extensions: the current enumeration order is not suitable for unions of infinite enumerated sets (except possibly for the last one). One could add options to specify alternative enumeration orders (anti-diagonal, round robin, ...) to handle this case.

Inheriting from `DisjointUnionEnumeratedSets`

There are two different use cases for inheriting from *DisjointUnionEnumeratedSets* : writing a parent which happens to be a disjoint union of some known parents, or writing generic disjoint unions for some particular classes of sage.categories.enumerated_sets.EnumeratedSets .

- In the first use case, the input of the __init__ method is most likely different from that of *DisjointUnionEnumeratedSets* . Then, one simply writes the __init__ method as usual:

```
sage: class MyUnion(DisjointUnionEnumeratedSets):
....:    def __init__(self):
....:        DisjointUnionEnumeratedSets.__init__(self,
....:            Family([1,2], Permutations))
sage: pp = MyUnion()
sage: pp.list()
[[1], [1, 2], [2, 1]]
```

In case the __init__() method takes optional arguments, or does some normalization on them, a specific method __classcall_private__ is required (see the documentation of *UniqueRepresentation*).

- In the second use case, the input of the __init__ method is the same as that of *DisjointUnionEnumeratedSets* ; one therefore wants to inherit the __classcall_private__() method as well, which can be achieved as follows:

```
sage: class UnionOfSpecialSets(DisjointUnionEnumeratedSets):
....:     __classcall_private__ = staticmethod(DisjointUnionEnumeratedSets.__
→classcall_private__)
sage: psp = UnionOfSpecialSets(Family([1,2], Permutations))
sage: psp.list()
[[1], [1, 2], [2, 1]]
```

Element()

an_element()
    Return an element of this disjoint union, as per Sets.ParentMethods.an_element() .

    EXAMPLES:

```
sage: U4 = DisjointUnionEnumeratedSets(
....:         Family([3, 5, 7], Permutations))
sage: U4.an_element()
[1, 2, 3]
```

cardinality()
    Returns the cardinality of this disjoint union.

    EXAMPLES:

    For finite disjoint unions, the cardinality is computed by summing the cardinalities of the enumerated sets:

```
sage: U = DisjointUnionEnumeratedSets(Family([0,1,2,3], Permutations))
sage: U.cardinality()
10
```

    For infinite disjoint unions, this makes the assumption that the result is infinite:

```
sage: U = DisjointUnionEnumeratedSets(
....:         Family(NonNegativeIntegers(), Permutations))
```

```
sage: U.cardinality()
+Infinity
```

> **Warning:** As pointed out in the main documentation, it is possible to construct examples where this
> is incorrect:
>
> ```
> sage: U = DisjointUnionEnumeratedSets(
> ....:         Family(NonNegativeIntegers(), lambda x: []))
> sage: U.cardinality()  # Should be 0!
> +Infinity
> ```

4.10 Enumerated set from iterator

EXAMPLES:

We build a set from the iterator `graphs` that returns a canonical representative for each isomorphism class of graphs:

```
sage: from sage.sets.set_from_iterator import EnumeratedSetFromIterator
sage: E = EnumeratedSetFromIterator(
....:    graphs,
....:    name = "Graphs",
....:    category = InfiniteEnumeratedSets(),
....:    cache = True)
sage: E
Graphs
sage: E.unrank(0)
Graph on 0 vertices
sage: E.unrank(4)
Graph on 3 vertices
sage: E.cardinality()
+Infinity
sage: E.category()
Category of facade infinite enumerated sets
```

The module also provides decorator for functions and methods:

```
sage: from sage.sets.set_from_iterator import set_from_function
sage: @set_from_function
....: def f(n): return xsrange(n)
sage: f(3)
{0, 1, 2}
sage: f(5)
{0, 1, 2, 3, 4}
sage: f(100)
{0, 1, 2, 3, 4, ...}

sage: from sage.sets.set_from_iterator import set_from_method
sage: class A:
....:    @set_from_method
....:    def f(self,n):
....:        return xsrange(n)
sage: a = A()
sage: a.f(3)
{0, 1, 2}
```

```
sage: f(100)
{0, 1, 2, 3, 4, ...}
```

class sage.sets.set_from_iterator. **Decorator**

>   Abstract class that manage documentation and sources of the wrapped object.
>
>   The method needs to be stored in the attribute self.f

class sage.sets.set_from_iterator. **DummyExampleForPicklingTest**

>   Class example to test pickling with the decorator *set_from_method* .

> > **Warning:** This class is intended to be used in doctest only.

>   EXAMPLES:

```
sage: from sage.sets.set_from_iterator import DummyExampleForPicklingTest
sage: DummyExampleForPicklingTest().f()
{10, 11, 12, 13, 14, ...}
```

>   **f** ( )
>
> >   Returns the set between self.start and self.stop .
> >
> >   EXAMPLES:

```
sage: from sage.sets.set_from_iterator import DummyExampleForPicklingTest
sage: d = DummyExampleForPicklingTest()
sage: d.f()
{10, 11, 12, 13, 14, ...}
sage: d.start = 4
sage: d.stop = 200
sage: d.f()
{4, 5, 6, 7, 8, ...}
```

class sage.sets.set_from_iterator. **EnumeratedSetFromIterator** (*f*, *args=None*, *kwds=None*, *name=None*, *category=None*, *cache=False*)

>   Bases: *sage.structure.parent.Parent*
>
>   A class for enumerated set built from an iterator.
>
>   INPUT:
>
> > - f – a function that returns an iterable from which the set is built from
> >
> > - args – tuple – arguments to be sent to the function f
> >
> > - kwds – dictionary – keywords to be sent to the function f
> >
> > - name – an optional name for the set
> >
> > - category – (default: None) an optional category for that enumerated set. If you know that your iterator will stop after a finite number of steps you should set it as FiniteEnumeratedSets , conversely if you know that your iterator will run over and over you should set it as InfiniteEnumeratedSets .
> >
> > - cache – boolean (default: False) – Whether or not use a cache mechanism for the iterator. If True , then the function f is called only once.
>
>   EXAMPLES:

```
sage: from sage.sets.set_from_iterator import EnumeratedSetFromIterator
sage: E = EnumeratedSetFromIterator(graphs, args = (7,))
sage: E
{Graph on 7 vertices, Graph on 7 vertices, Graph on 7 vertices, Graph on 7
↪vertices, Graph on 7 vertices, ...}
sage: E.category()
Category of facade enumerated sets
```

The same example with a cache and a custom name:

```
sage: E = EnumeratedSetFromIterator(
....:     graphs,
....:     args = (8,),
....:     category = FiniteEnumeratedSets(),
....:     name = "Graphs with 8 vertices",
....:     cache = True)
sage: E
Graphs with 8 vertices
sage: E.unrank(3)
Graph on 8 vertices
sage: E.category()
Category of facade finite enumerated sets
```

Note: In order to make the TestSuite works, the elements of the set should have parents.

---

clear_cache ()
    Clear the cache.

    EXAMPLES:

```
sage: from itertools import count
sage: from sage.sets.set_from_iterator import EnumeratedSetFromIterator
sage: E = EnumeratedSetFromIterator(count, args=(1,), cache=True)
sage: e1 = E._cache
sage: e1
lazy list [1, 2, 3, ...]
sage: E.clear_cache()
sage: E._cache
lazy list [1, 2, 3, ...]
sage: e1 is E._cache
False
```

is_parent_of (*x*)
    Test whether x is in self.

    If the set is infinite, only the answer True should be expected in finite time.

    EXAMPLES:

```
sage: from sage.sets.set_from_iterator import EnumeratedSetFromIterator
sage: P = Partitions(12,min_part=2,max_part=5)
sage: E = EnumeratedSetFromIterator(P.__iter__)
sage: P([5,5,2]) in E
True
```

unrank (*i*)
    Returns the element at position i.

EXAMPLES:

```
sage: from sage.sets.set_from_iterator import EnumeratedSetFromIterator
sage: E = EnumeratedSetFromIterator(graphs, args=(8,), cache=True)
sage: F = EnumeratedSetFromIterator(graphs, args=(8,), cache=False)
sage: E.unrank(2)
Graph on 8 vertices
sage: E.unrank(2) == F.unrank(2)
True
```

class sage.sets.set_from_iterator. **EnumeratedSetFromIterator_function_decorator** (*f=None*,
name=None,
***op-
tions*)

Bases: *sage.sets.set_from_iterator.Decorator*

Decorator for *EnumeratedSetFromIterator* .

Name could be string or a function (args, kwds) -> string .

> **Warning:** If you are going to use this with the decorator cached_function , you must place the
> cached_function first. See the example below.

EXAMPLES:

```
sage: from sage.sets.set_from_iterator import set_from_function
sage: from six.moves import range
sage: @set_from_function
....: def f(n):
....:     for i in range(n):
....:         yield i**2 + i + 1
sage: f(3)
{1, 3, 7}
sage: f(100)
{1, 3, 7, 13, 21, ...}
```

To avoid ambiguity, it is always better to use it with a call which provides optional global initialization for the
call to *EnumeratedSetFromIterator* :

```
sage: @set_from_function(category=InfiniteEnumeratedSets())
....: def Fibonacci():
....:     a = 1; b = 2
....:     while True:
....:         yield a
....:         a, b = b, a + b
sage: F = Fibonacci()
sage: F
{1, 2, 3, 5, 8, ...}
sage: F.cardinality()
+Infinity
```

A simple example with many options:

```
sage: @set_from_function(
....:     name = "From %(m)d to %(n)d",
....:     category = FiniteEnumeratedSets())
....: def f(m, n): return xsrange(m,n+1)
```

```
sage: E = f(3,10); E
From 3 to 10
sage: E.list()
[3, 4, 5, 6, 7, 8, 9, 10]
sage: E = f(1,100); E
From 1 to 100
sage: E.cardinality()
100
sage: f(n=100,m=1) == E
True
```

An example which mixes together `set_from_function` and `cached_method`:

```
sage: @cached_function
....: @set_from_function(
....:   name = "Graphs on %(n)d vertices",
....:   category = FiniteEnumeratedSets(),
....:   cache = True)
....: def Graphs(n): return graphs(n)
sage: Graphs(10)
Graphs on 10 vertices
sage: Graphs(10).unrank(0)
Graph on 10 vertices
sage: Graphs(10) is Graphs(10)
True
```

The `cached_function` must go first:

```
sage: @set_from_function(
....:   name = "Graphs on %(n)d vertices",
....:   category = FiniteEnumeratedSets(),
....:   cache = True)
....: @cached_function
....: def Graphs(n): return graphs(n)
sage: Graphs(10)
Graphs on 10 vertices
sage: Graphs(10).unrank(0)
Graph on 10 vertices
sage: Graphs(10) is Graphs(10)
False
```

class sage.sets.set_from_iterator. **EnumeratedSetFromIterator_method_caller** (*inst*, *f*, *name=None*, ***options*)

Bases: *sage.sets.set_from_iterator.Decorator*

Caller for decorated method in class.

INPUT:

- `inst` – an instance of a class

- `f` – a method of a class of `inst` (and not of the instance itself)

- `name` – optional – either a string (which may contains substitution rules from argument or a function args,kwds -> string.

- `options` – any option accepted by *EnumeratedSetFromIterator*

class sage.sets.set_from_iterator. **EnumeratedSetFromIterator_method_decorator** (*f=None*,
***op-
tions*)

> Bases: `object`
>
> Decorator for enumerated set built from a method.
>
> INPUT:
>
> > •`f` – Optional function from which are built the enumerated sets at each call
> >
> > •`name` – Optional string (which may contains substitution rules from argument) or a function
> > `(args,kwds) -> string`.
> >
> > •any option accepted by *EnumeratedSetFromIterator* .
>
> EXAMPLES:

```
sage: from sage.sets.set_from_iterator import set_from_method
sage: class A():
....:  def n(self): return 12
....:  @set_from_method
....:  def f(self): return xsrange(self.n())
sage: a = A()
sage: print(a.f.__class__)
sage.sets.set_from_iterator.EnumeratedSetFromIterator_method_caller
sage: a.f()
{0, 1, 2, 3, 4, ...}
sage: A.f(a)
{0, 1, 2, 3, 4, ...}
```

> A more complicated example with a parametrized name:

```
sage: class B():
....:  @set_from_method(
....:      name = "Graphs(%(n)d)",
....:      category = FiniteEnumeratedSets())
....:  def graphs(self, n): return graphs(n)
sage: b = B()
sage: G3 = b.graphs(3)
sage: G3
Graphs(3)
sage: G3.cardinality()
4
sage: G3.category()
Category of facade finite enumerated sets
sage: B.graphs(b,3)
Graphs(3)
```

> And a last example with a name parametrized by a function:

```
sage: class D():
....:  def __init__(self, name): self.name = str(name)
....:  def __str__(self): return self.name
....:  @set_from_method(
....:      name = lambda self,n: str(self)*n,
....:      category = FiniteEnumeratedSets())
....:  def subset(self, n):
....:      return xsrange(n)
sage: d = D('a')
sage: E = d.subset(3); E
```

```
aaa
sage: E.list()
[0, 1, 2]
sage: F = d.subset(n=10); F
aaaaaaaaaa
sage: F.list()
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
```

Todo

It is not yet possible to use `set_from_method` in conjunction with `cached_method`.

---

sage.sets.set_from_iterator. **set_from_function**
    alias of *EnumeratedSetFromIterator_function_decorator*

sage.sets.set_from_iterator. **set_from_method**
    alias of *EnumeratedSetFromIterator_method_decorator*

4.11 Finite Enumerated Sets

class sage.sets.finite_enumerated_set. **FiniteEnumeratedSet** (*elements*)
    Bases: *sage.structure.unique_representation.UniqueRepresentation*   ,
    sage.structure.parent.Parent

    A class for finite enumerated set.

    Returns the finite enumerated set with elements in `elements` where `element` is any (finite) iterable object.

    The main purpose is to provide a variant of `list` or `tuple`, which is a parent with an interface consistent with `EnumeratedSets` and has unique representation. The list of the elements is expanded in memory.

    EXAMPLES:

```
sage: S = FiniteEnumeratedSet([1, 2, 3])
sage: S
{1, 2, 3}
sage: S.list()
[1, 2, 3]
sage: S.cardinality()
3
sage: S.random_element()
1
sage: S.first()
1
sage: S.category()
Category of facade finite enumerated sets
sage: TestSuite(S).run()
```

Note that being and enumerated set, the result depends on the order:

```
sage: S1 = FiniteEnumeratedSet((1, 2, 3))
sage: S1
{1, 2, 3}
sage: S1.list()
[1, 2, 3]
sage: S1 == S
```

```
True
sage: S2 = FiniteEnumeratedSet((2, 1, 3))
sage: S2 == S
False
```

As an abuse, repeated entries in `elements` are allowed to model multisets:

```
sage: S1 = FiniteEnumeratedSet((1, 2, 1, 2, 2, 3))
sage: S1
{1, 2, 1, 2, 2, 3}
```

Finaly the elements are not aware of their parent:

```
sage: S.first().parent()
Integer Ring
```

an_element ( )

cardinality ( )

first ( )

> Return the first element of the enumeration or raise an EmptySetError if the set is empty.

> EXAMPLES:

> ```
> sage: S = FiniteEnumeratedSet('abc')
> sage: S.first()
> 'a'
> ```

index (*x*)

> Returns the index of `x` in this finite enumerated set.

> EXAMPLES:

> ```
> sage: S = FiniteEnumeratedSet(['a','b','c'])
> sage: S.index('b')
> 1
> ```

is_parent_of (*x*)

last ( )

> Returns the last element of the iteration or raise an EmptySetError if the set is empty.

> EXAMPLES:

> ```
> sage: S = FiniteEnumeratedSet([0,'a',1.23, 'd'])
> sage: S.last()
> 'd'
> ```

list ( )

random_element ( )

> Return a random element.

> EXAMPLES:

> ```
> sage: S = FiniteEnumeratedSet('abc')
> sage: S.random_element()    # random
> 'b'
> ```

rank (*x*)

    Returns the index of `x` in this finite enumerated set.

    EXAMPLES:

```
sage: S = FiniteEnumeratedSet(['a','b','c'])
sage: S.index('b')
1
```

unrank (*i*)

    Return the element at position `i` .

    EXAMPLES:

```
sage: S = FiniteEnumeratedSet([1,'a',-51])
sage: S[0], S[1], S[2]
(1, 'a', -51)
sage: S[3]
Traceback (most recent call last):
...
IndexError: tuple index out of range
sage: S[-1], S[-2], S[-3]
(-51, 'a', 1)
sage: S[-4]
Traceback (most recent call last):
...
IndexError: list index out of range
```

4.12 Recursively enumerated set

A set S is called recursively enumerable if there is an algorithm that enumerates the members of S. We consider here the recursively enumerated sets that are described by some `seeds` and a successor function `successors` . The successor function may have some structure (symmetric, graded, forest) or not. The elements of a set having a symmetric, graded or forest structure can be enumerated uniquely without keeping all of them in memory. Many kinds of iterators are provided in this module: depth first search, breadth first search or elements of given depth.

See Wikipedia article Recursively_enumerable_set.

See documentation of *RecursivelyEnumeratedSet()* below for the description of the inputs.

AUTHORS:

    • Sebastien Labbe, April 2014, at Sage Days 57, Cernay-la-ville

EXAMPLES:

4.12.1 Forest structure

The set of words over the alphabet $\{a, b\}$ can be generated from the empty word by appending letter a or b as a successor function. This set has a forest structure:

```
sage: seeds = ['']
sage: succ = lambda w: [w+'a', w+'b']
sage: C = RecursivelyEnumeratedSet(seeds, succ, structure='forest')
sage: C
An enumerated set with a forest structure
```

Depth first search iterator:

```
sage: it = C.depth_first_search_iterator()
sage: [next(it) for _ in range(6)]
['', 'a', 'aa', 'aaa', 'aaaa', 'aaaaa']
```

Breadth first search iterator:

```
sage: it = C.breadth_first_search_iterator()
sage: [next(it) for _ in range(6)]
['', 'a', 'b', 'aa', 'ab', 'ba']
```

4.12.2 Symmetric structure

The origin `(0,0)` as seed and the upper, lower, left and right lattice point as successor function. This function is symmetric:

```
sage: succ = lambda a: [(a[0]-1,a[1]), (a[0],a[1]-1), (a[0]+1,a[1]), (a[0],a[1]+1)]
sage: seeds = [(0,0)]
sage: C = RecursivelyEnumeratedSet(seeds, succ, structure='symmetric', enumeration=
→'depth')
sage: C
A recursively enumerated set with a symmetric structure (depth first search)
```

In this case, depth first search is the default enumeration for iteration:

```
sage: it_depth = iter(C)
sage: [next(it_depth) for _ in range(10)]
[(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (0, 9)]
```

Breadth first search:

```
sage: it_breadth = C.breadth_first_search_iterator()
sage: [next(it_breadth) for _ in range(10)]
[(0, 0), (0, 1), (0, -1), (1, 0), (-1, 0), (-1, 1), (-2, 0), (0, 2), (2, 0), (-1, -1)]
```

Levels (elements of given depth):

```
sage: sorted(C.graded_component(0))
[(0, 0)]
sage: sorted(C.graded_component(1))
[(-1, 0), (0, -1), (0, 1), (1, 0)]
sage: sorted(C.graded_component(2))
[(-2, 0), (-1, -1), (-1, 1), (0, -2), (0, 2), (1, -1), (1, 1), (2, 0)]
```

4.12.3 Graded structure

Identity permutation as seed and `permutohedron_succ` as successor function:

```
sage: succ = attrcall("permutohedron_succ")
sage: seed = [Permutation([1..5])]
sage: R = RecursivelyEnumeratedSet(seed, succ, structure='graded')
sage: R
A recursively enumerated set with a graded structure (breadth first search)
```

Depth first search iterator:

```
sage: it_depth = R.depth_first_search_iterator()
sage: [next(it_depth) for _ in range(5)]
[[1, 2, 3, 4, 5],
 [1, 2, 3, 5, 4],
 [1, 2, 5, 3, 4],
 [1, 2, 5, 4, 3],
 [1, 5, 2, 4, 3]]
```

Breadth first search iterator:

```
sage: it_breadth = R.breadth_first_search_iterator()
sage: [next(it_breadth) for _ in range(5)]
[[1, 2, 3, 4, 5],
 [1, 3, 2, 4, 5],
 [1, 2, 4, 3, 5],
 [2, 1, 3, 4, 5],
 [1, 2, 3, 5, 4]]
```

Elements of given depth iterator:

```
sage: list(R.elements_of_depth_iterator(9))
[[5, 3, 4, 2, 1], [4, 5, 3, 2, 1], [5, 4, 2, 3, 1], [5, 4, 3, 1, 2]]
sage: list(R.elements_of_depth_iterator(10))
[[5, 4, 3, 2, 1]]
```

Graded components (set of elements of the same depth):

```
sage: sorted(R.graded_component(0))
[[1, 2, 3, 4, 5]]
sage: sorted(R.graded_component(1))
[[1, 2, 3, 5, 4], [1, 2, 4, 3, 5], [1, 3, 2, 4, 5], [2, 1, 3, 4, 5]]
sage: sorted(R.graded_component(9))
[[4, 5, 3, 2, 1], [5, 3, 4, 2, 1], [5, 4, 2, 3, 1], [5, 4, 3, 1, 2]]
sage: sorted(R.graded_component(10))
[[5, 4, 3, 2, 1]]
```

4.12.4 No hypothesis on the structure

By "no hypothesis" is meant neither a forest, neither symmetric neither graded, it may have other structure like not containing oriented cycle but this does not help for enumeration.

In this example, the seed is 0 and the successor function is either +2 or +3 . This is the set of non negative linear combinations of 2 and 3:

```
sage: succ = lambda a:[a+2,a+3]
sage: C = RecursivelyEnumeratedSet([0], succ)
sage: C
A recursively enumerated set (breadth first search)
```

Breadth first search:

```
sage: it = C.breadth_first_search_iterator()
sage: [next(it) for _ in range(10)]
[0, 2, 3, 4, 5, 6, 8, 9, 7, 10]
```

Depth first search:

```
sage: it = C.depth_first_search_iterator()
sage: [next(it) for _ in range(10)]
[0, 3, 6, 9, 12, 15, 18, 21, 24, 27]
```

sage.sets.recursively_enumerated_set. **RecursivelyEnumeratedSet** (*seeds*, *successors*, *structure=None*, *enumeration=None*, *max_depth=None*, *post_process=None*, *facade=None*, *category=None*)

Return a recursively enumerated set.

A set S is called recursively enumerable if there is an algorithm that enumerates the members of S. We consider here the recursively enumerated set that are described by some `seeds` and a successor function `successors`.

Let U be a set and `successors` $: U \to 2^U$ be a successor function associating to each element of U a subset of U. Let `seeds` be a subset of U. Let $S \subseteq U$ be the set of elements of U that can be reached from a seed by applying recursively the `successors` function. This class provides different kinds of iterators (breadth first, depth first, elements of given depth, etc.) for the elements of S.

See Wikipedia article Recursively_enumerable_set.

INPUT:

- `seeds` – list (or iterable) of hashable objects

- `successors` – function (or callable) returning a list (or iterable) of hashable objects

- `structure` – string (optional, default: `None`), structure of the set, possible values are:

    - `None` – nothing is known about the structure of the set.

    - `'forest'` – if the `successors` function generates a *forest*, that is, each element can be reached uniquely from a seed.

    - `'graded'` – if the `successors` function is *graded*, that is, all paths from a seed to a given element have equal length.

    - `'symmetric'` – if the relation is *symmetric*, that is, `y in successors(x)` if and only if `x in successors(y)`

- `enumeration` – `'depth'` , `'breadth'` , `'naive'` or `None` (optional, default: `None`). The default enumeration for the `__iter__` function.

- `max_depth` – integer (optional, default: `float("inf")`), limit the search to a certain depth, currently works only for breadth first search

- `post_process` – (optional, default: `None`), for forest only

- `facade` – (optional, default: `None`)

- `category` – (optional, default: `None`)

EXAMPLES:

A recursive set with no other information:

```
sage: f = lambda a: [a+3, a+5]
sage: C = RecursivelyEnumeratedSet([0], f)
sage: C
A recursively enumerated set (breadth first search)
sage: it = iter(C)
sage: [next(it) for _ in range(10)]
[0, 3, 5, 8, 10, 6, 9, 11, 13, 15]
```

A recursive set with a forest structure:

```
sage: f = lambda a: [2*a,2*a+1]
sage: C = RecursivelyEnumeratedSet([1], f, structure='forest')
sage: C
An enumerated set with a forest structure
sage: it = C.depth_first_search_iterator()
sage: [next(it) for _ in range(7)]
[1, 2, 4, 8, 16, 32, 64]
sage: it = C.breadth_first_search_iterator()
sage: [next(it) for _ in range(7)]
[1, 2, 3, 4, 5, 6, 7]
```

A recursive set given by a symmetric relation:

```
sage: f = lambda a: [a-1,a+1]
sage: C = RecursivelyEnumeratedSet([10, 15], f, structure='symmetric')
sage: C
A recursively enumerated set with a symmetric structure (breadth first search)
sage: it = iter(C)
sage: [next(it) for _ in range(7)]
[10, 15, 16, 9, 11, 14, 8]
```

A recursive set given by a graded relation:

```
sage: f = lambda a: [a+1, a+I]
sage: C = RecursivelyEnumeratedSet([0], f, structure='graded')
sage: C
A recursively enumerated set with a graded structure (breadth first search)
sage: it = iter(C)
sage: [next(it) for _ in range(7)]
[0, 1, I, I + 1, 2, 2*I, I + 2]
```

> **Warning:** If you do not set the good structure, you might obtain bad results, like elements generated twice:
>
> ```
> sage: f = lambda a: [a-1,a+1]
> sage: C = RecursivelyEnumeratedSet([0], f, structure='graded')
> sage: it = iter(C)
> sage: [next(it) for _ in range(7)]
> [0, 1, -1, 0, 2, -2, 1]
> ```

```
sage: C = RecursivelyEnumeratedSet((1, 2, 3), factor)
sage: C.successors
<function factor at ...>
sage: C._seeds
(1, 2, 3)
```

class sage.sets.recursively_enumerated_set. **RecursivelyEnumeratedSet_generic**
    Bases: *sage.structure.parent.Parent*

A generic recursively enumerated set.

For more information, see *RecursivelyEnumeratedSet()* .

EXAMPLES:

```
sage: f = lambda a:[a+1]
```

Different structure for the sets:

```
sage: RecursivelyEnumeratedSet([0], f, structure=None)
A recursively enumerated set (breadth first search)
sage: RecursivelyEnumeratedSet([0], f, structure='graded')
A recursively enumerated set with a graded structure (breadth first search)
sage: RecursivelyEnumeratedSet([0], f, structure='symmetric')
A recursively enumerated set with a symmetric structure (breadth first search)
sage: RecursivelyEnumeratedSet([0], f, structure='forest')
An enumerated set with a forest structure
```

Different default enumeration algorithms:

```
sage: RecursivelyEnumeratedSet([0], f, enumeration='breadth')
A recursively enumerated set (breadth first search)
sage: RecursivelyEnumeratedSet([0], f, enumeration='naive')
A recursively enumerated set (naive search)
sage: RecursivelyEnumeratedSet([0], f, enumeration='depth')
A recursively enumerated set (depth first search)
```

breadth_first_search_iterator (*max_depth=None*)
    Iterate on the elements of self (breadth first).

    This code remembers every element generated.

    INPUT:

        •max_depth – (Default: None) specifies the maximal depth to which elements are computed; if
            None , the value of self._max_depth is used

    EXAMPLES:

    ```
    sage: f = lambda a: [a+3, a+5]
    sage: C = RecursivelyEnumeratedSet([0], f)
    sage: it = C.breadth_first_search_iterator()
    sage: [next(it) for _ in range(10)]
    [0, 3, 5, 8, 10, 6, 9, 11, 13, 15]
    ```

depth_first_search_iterator ( )
    Iterate on the elements of self (depth first).

    This code remembers every elements generated.

    See Wikipedia article Depth-first_search.

    EXAMPLES:

    ```
    sage: f = lambda a: [a+3, a+5]
    sage: C = RecursivelyEnumeratedSet([0], f)
    sage: it = C.depth_first_search_iterator()
    ```

```
sage: [next(it) for _ in range(10)]
[0, 5, 10, 15, 20, 25, 30, 35, 40, 45]
```

elements_of_depth_iterator (*depth*)

Iterate over the elements of `self` of given depth.

An element of depth n can be obtained applying n times the successor function to a seed.

INPUT:

  •`depth` – integer

OUTPUT:

An iterator.

EXAMPLES:

```
sage: f = lambda a: [a-1, a+1]
sage: S = RecursivelyEnumeratedSet([5, 10], f, structure='symmetric')
sage: it = S.elements_of_depth_iterator(2)
sage: sorted(it)
[3, 7, 8, 12]
```

graded_component (*depth*)

Return the graded component of given depth.

This method caches each lower graded component.

A graded component is a set of elements of the same depth where the depth of an element is its minimal distance to a root.

It is currently implemented only for graded or symmetric structure.

INPUT:

  •`depth` – integer

OUTPUT:

A set.

EXAMPLES:

```
sage: f = lambda a: [a+3, a+5]
sage: C = RecursivelyEnumeratedSet([0], f)
sage: C.graded_component(0)
Traceback (most recent call last):
...
NotImplementedError: graded_component_iterator method currently implemented␣
→only for graded or symmetric structure
```

graded_component_iterator ( )

Iterate over the graded components of `self` .

A graded component is a set of elements of the same depth.

It is currently implemented only for graded or symmetric structure.

OUTPUT:

An iterator of sets.

EXAMPLES:

```
sage: f = lambda a: [a+3, a+5]
sage: C = RecursivelyEnumeratedSet([0], f)
sage: it = C.graded_component_iterator()     # todo: not implemented
```

naive_search_iterator ( )
    Iterate on the elements of `self` (in no particular order).

    This code remembers every elements generated.

seeds ( )
    Return an iterable over the seeds of `self`.

    EXAMPLES:

```
sage: R = RecursivelyEnumeratedSet([1], lambda x: [x+1, x-1])
sage: R.seeds()
[1]
```

successors

to_digraph (*max_depth=None*, *loops=True*, *multiedges=True*)
    Return the directed graph of the recursively enumerated set.

    INPUT:

        •`max_depth` – (default: `None`) specifies the maximal depth for which outgoing edges of elements
         are computed; if `None`, the value of `self._max_depth` is used

        •`loops` – (default: `True`) option for the digraph

        •`multiedges` – (default: `True`) option of the digraph

    OUTPUT:

    A directed graph

> **Warning:** If the set is infinite, this will loop forever unless `max_depth` is finite.

    EXAMPLES:

```
sage: child = lambda i: [(i+3) % 10, (i+8) % 10]
sage: R = RecursivelyEnumeratedSet([0], child)
sage: R.to_digraph()
Looped multi-digraph on 10 vertices
```

Digraph of an recursively enumerated set with a symmetric structure of infinite cardinality using
`max_depth` argument:

```
sage: succ = lambda a: [(a[0]-1,a[1]), (a[0],a[1]-1), (a[0]+1,a[1]),
↪(a[0],a[1]+1)]
sage: seeds = [(0,0)]
sage: C = RecursivelyEnumeratedSet(seeds, succ, structure='symmetric')
sage: C.to_digraph(max_depth=4)
Looped multi-digraph on 41 vertices
```

The `max_depth` argument can be given at the creation of the set:

```
sage: C = RecursivelyEnumeratedSet(seeds, succ, structure='symmetric', max_
→depth=3)
sage: C.to_digraph()
Looped multi-digraph on 25 vertices
```

Digraph of an recursively enumerated set with a graded structure:

```
sage: f = lambda a: [a+1, a+I]
sage: C = RecursivelyEnumeratedSet([0], f, structure='graded')
sage: C.to_digraph(max_depth=4)
Looped multi-digraph on 21 vertices
```

class sage.sets.recursively_enumerated_set. **RecursivelyEnumeratedSet_graded**
    Bases: *sage.sets.recursively_enumerated_set.RecursivelyEnumeratedSet_generic*

Generic tool for constructing ideals of a graded relation.

INPUT:

- seeds – list (or iterable) of hashable objects

- successors – function (or callable) returning a list (or iterable)

- enumeration – 'depth', 'breadth' or None (default: None)

- max_depth – integer (default: float("inf"))

EXAMPLES:

```
sage: f = lambda a: [(a[0]+1,a[1]), (a[0],a[1]+1)]
sage: C = RecursivelyEnumeratedSet([(0,0)], f, structure='graded', max_depth=3)
sage: C
A recursively enumerated set with a graded structure (breadth first
search) with max_depth=3
sage: sorted(C)
[(0, 0), (0, 1), (0, 2), (0, 3), (1, 0),
 (1, 1), (1, 2), (2, 0), (2, 1), (3, 0)]
```

breadth_first_search_iterator (*max_depth=None*)
    Iterate on the elements of self (breadth first).

This iterator make use of the graded structure by remembering only the elements of the current depth.

INPUT:

- max_depth – (Default: None) Specifies the maximal depth to which elements are computed. If None, the value of self._max_depth is used.

EXAMPLES:

```
sage: f = lambda a: [(a[0]+1,a[1]), (a[0],a[1]+1)]
sage: C = RecursivelyEnumeratedSet([(0,0)], f, structure='graded')
sage: it = C.breadth_first_search_iterator(max_depth=3)
sage: list(it)
[(0, 0), (0, 1), (1, 0), (2, 0), (1, 1),
 (0, 2), (3, 0), (1, 2), (0, 3), (2, 1)]
```

graded_component (*depth*)
    Return the graded component of given depth.

This method caches each lower graded component. See *graded_component_iterator()* to generate each graded component without caching the previous ones.

A graded component is a set of elements of the same depth where the depth of an element is its minimal distance to a root.

INPUT:

> •`depth` – integer

OUTPUT:

A set.

EXAMPLES:

```
sage: f = lambda a: [a+1, a+I]
sage: C = RecursivelyEnumeratedSet([0], f, structure='graded')
sage: for i in range(5): sorted(C.graded_component(i))
[0]
[I, 1]
[2*I, I + 1, 2]
[3*I, 2*I + 1, I + 2, 3]
[4*I, 3*I + 1, 2*I + 2, I + 3, 4]
```

graded_component_iterator ()
> Iterate over the graded components of `self`.

> A graded component is a set of elements of the same depth.

> The algorithm remembers only the current graded component generated since the structure is graded.

> OUTPUT:

> An iterator of sets.

> EXAMPLES:

```
sage: f = lambda a: [(a[0]+1,a[1]), (a[0],a[1]+1)]
sage: C = RecursivelyEnumeratedSet([(0,0)], f, structure='graded', max_
↪depth=3)
sage: it = C.graded_component_iterator()
sage: for _ in range(4): sorted(next(it))
[(0, 0)]
[(0, 1), (1, 0)]
[(0, 2), (1, 1), (2, 0)]
[(0, 3), (1, 2), (2, 1), (3, 0)]
```

class sage.sets.recursively_enumerated_set. **RecursivelyEnumeratedSet_symmetric**
> Bases: *sage.sets.recursively_enumerated_set.RecursivelyEnumeratedSet_generic*

Generic tool for constructing ideals of a symmetric relation.

INPUT:

> •`seeds` – list (or iterable) of hashable objects

> •`successors` – function (or callable) returning a list (or iterable)

> •`enumeration` – `'depth'`, `'breadth'` or `None` (default: `None`)

> •`max_depth` – integer (default: `float("inf")`)

EXAMPLES:

```
sage: f = lambda a: [a-1,a+1]
sage: C = RecursivelyEnumeratedSet([0], f, structure='symmetric')
sage: C
A recursively enumerated set with a symmetric structure (breadth first search)
sage: it = iter(C)
sage: [next(it) for _ in range(7)]
[0, 1, -1, 2, -2, 3, -3]
```

breadth_first_search_iterator (*max_depth=None*)

Iterate on the elements of `self` (breadth first).

This code remembers only elements needed by the graded component iterator to generate the next graded component.

This method is the default breadth first search iterator when the structure is symmetric or graded.

INPUT:

• `max_depth` – (Default: `None`) specifies the maximal depth to which elements are computed; if `None` , the value of `self._max_depth` is used

---

Note: Calling next in this iterator will be either quite slow or very fast since it generates the whole graded component before yielding the elements of each graded component.

EXAMPLES:

```
sage: f = lambda a: [(a[0]+1,a[1]), (a[0],a[1]+1)]
sage: C = RecursivelyEnumeratedSet([(0,0)], f, structure='graded')
sage: it = C._breadth_first_search_iterator_from_graded_component_
↪iterator(max_depth=3)
sage: list(it)
[(0, 0), (0, 1), (1, 0), (2, 0), (1, 1), (0, 2)]
```

This iterator is used by default for symmetric structure:

```
sage: f = lambda a: [a-1,a+1]
sage: S = RecursivelyEnumeratedSet([10], f, structure='symmetric')
sage: it = iter(S)
sage: [next(it) for _ in range(7)]
[10, 9, 11, 8, 12, 13, 7]
```

graded_component (*depth*)

Return the graded component of given depth.

This method caches each lower graded component. See *graded_component_iterator()* to generate each graded component without caching the previous ones.

A graded component is a set of elements of the same depth where the depth of an element is its minimal distance to a root.

INPUT:

• `depth` – integer

OUTPUT:

A set.

EXAMPLES:

```
sage: f = lambda a: [a-1,a+1]
sage: C = RecursivelyEnumeratedSet([10, 15], f, structure='symmetric')
sage: for i in range(5): sorted(C.graded_component(i))
[10, 15]
[9, 11, 14, 16]
[8, 12, 13, 17]
[7, 18]
[6, 19]
```

graded_component_iterator ()

> Iterate over the graded components of self .
>
> A graded component is a set of elements of the same depth.
>
> The enumeration remembers only the last two graded components generated since the structure is symmetric.
>
> OUTPUT:
>
> An iterator of sets.
>
> EXAMPLES:

```
sage: f = lambda a: [a-1, a+1]
sage: S = RecursivelyEnumeratedSet([10], f, structure='symmetric')
sage: it = S.graded_component_iterator()
sage: [sorted(next(it)) for _ in range(5)]
[[10], [9, 11], [8, 12], [7, 13], [6, 14]]
```

> Starting with two generators:

```
sage: f = lambda a: [a-1, a+1]
sage: S = RecursivelyEnumeratedSet([5, 10], f, structure='symmetric')
sage: it = S.graded_component_iterator()
sage: [sorted(next(it)) for _ in range(5)]
[[5, 10], [4, 6, 9, 11], [3, 7, 8, 12], [2, 13], [1, 14]]
```

> Gaussian integers:

```
sage: f = lambda a: [a+1, a+I]
sage: S = RecursivelyEnumeratedSet([0], f, structure='symmetric')
sage: it = S.graded_component_iterator()
sage: [sorted(next(it)) for _ in range(7)]
[[0],
 [I, 1],
 [2*I, I + 1, 2],
 [3*I, 2*I + 1, I + 2, 3],
 [4*I, 3*I + 1, 2*I + 2, I + 3, 4],
 [5*I, 4*I + 1, 3*I + 2, 2*I + 3, I + 4, 5],
 [6*I, 5*I + 1, 4*I + 2, 3*I + 3, 2*I + 4, I + 5, 6]]
```

4.13 Maps between finite sets

This module implements parents modeling the set of all maps between two finite sets. At the user level, any such parent should be constructed using the factory class *FiniteSetMaps* which properly selects which of its subclasses to use.

AUTHORS:

- Florent Hivert

class sage.sets.finite_set_maps. **FiniteSetEndoMaps_N** (*n*, *action*, *category=None*)

　　Bases: *sage.sets.finite_set_maps.FiniteSetMaps_MN*

　　The sets of all maps from $\{1, 2, \dots, n\}$ to itself

　　Users should use the factory class *FiniteSetMaps* to create instances of this class.

　　INPUT:

　　　　•n – an integer.

　　　　•category – the category in which the sets of maps is constructed. It must be a sub-category of
　　　　Monoids().Finite() and EnumeratedSets().Finite() which is the default value.

　　Element

　　　　alias of FiniteSetEndoMap_N

　　an_element ( )

　　　　Returns a map in self

　　　　EXAMPLES:

```
sage: M = FiniteSetMaps(4)
sage: M.an_element()
[3, 2, 1, 0]
```

　　one ( )

　　　　EXAMPLES:

```
sage: M = FiniteSetMaps(4)
sage: M.one()
[0, 1, 2, 3]
```

class sage.sets.finite_set_maps. **FiniteSetEndoMaps_Set** (*domain*, 　　 *action*, 　　 *category=None*)

　　Bases: *sage.sets.finite_set_maps.FiniteSetMaps_Set* , *sage.sets.finite_set_maps.FiniteSetEn*

　　The sets of all maps from a set to itself

　　Users should use the factory class *FiniteSetMaps* to create instances of this class.

　　INPUT:

　　　　•domain – an object in the category FiniteSets().

　　　　•category – the category in which the sets of maps is constructed. It must be a sub-category of
　　　　Monoids().Finite() and EnumeratedSets().Finite() which is the default value.

　　Element

　　　　alias of FiniteSetEndoMap_Set

class sage.sets.finite_set_maps. **FiniteSetMaps**

　　Bases: 　　　　*sage.structure.unique_representation.UniqueRepresentation* 　　　,
　　sage.structure.parent.Parent

　　Maps between finite sets

　　Constructs the set of all maps between two sets. The sets can be given using any of the three following ways:

　　　　1.an object in the category Sets().

2.a finite iterable. In this case, an object of the class *FiniteEnumeratedSet* is constructed from the iterable.

3.an integer n designing the set $\{0, 1, \ldots, n-1\}$. In this case an object of the class *IntegerRange* is constructed.

INPUT:

- •domain – a set, finite iterable, or integer.

- •codomain – a set, finite iterable, integer, or None (default). In this last case, the maps are endo-maps of the domain.

- •action – "left" (default) or "right" . The side where the maps act on the domain. This is used in particular to define the meaning of the product (composition) of two maps.

- •category – the category in which the sets of maps is constructed. By default, this is FiniteMonoids() if the domain and codomain coincide, and FiniteEnumeratedSets() otherwise.

OUTPUT:

an instance of a subclass of *FiniteSetMaps* modeling the set of all maps between domain and codomain .

EXAMPLES:

We construct the set M of all maps from $\{a, b\}$ to $\{3, 4, 5\}$:

```
sage: M = FiniteSetMaps(["a", "b"], [3, 4, 5]); M
Maps from {'a', 'b'} to {3, 4, 5}
sage: M.cardinality()
9
sage: M.domain()
{'a', 'b'}
sage: M.codomain()
{3, 4, 5}
sage: for f in M: print(f)
map: a -> 3, b -> 3
map: a -> 3, b -> 4
map: a -> 3, b -> 5
map: a -> 4, b -> 3
map: a -> 4, b -> 4
map: a -> 4, b -> 5
map: a -> 5, b -> 3
map: a -> 5, b -> 4
map: a -> 5, b -> 5
```

Elements can be constructed from functions and dictionaries:

```
sage: M(lambda c: ord(c)-94)
map: a -> 3, b -> 4

sage: M.from_dict({'a':3, 'b':5})
map: a -> 3, b -> 5
```

If the domain is equal to the codomain, then maps can be composed:

```
sage: M = FiniteSetMaps([1, 2, 3])
sage: f = M.from_dict({1:2, 2:1, 3:3}); f
map: 1 -> 2, 2 -> 1, 3 -> 3
sage: g = M.from_dict({1:2, 2:3, 3:1}); g
```

```
map: 1 -> 2, 2 -> 3, 3 -> 1

sage: f * g
map: 1 -> 1, 2 -> 3, 3 -> 2
```

This makes M into a monoid:

```
sage: M.category()
Category of finite enumerated monoids
sage: M.one()
map: 1 -> 1, 2 -> 2, 3 -> 3
```

By default, composition is from right to left, which corresponds to an action on the left. If one specifies `action` to right, then the composition is from left to right:

```
sage: M = FiniteSetMaps([1, 2, 3], action = 'right')
sage: f = M.from_dict({1:2, 2:1, 3:3})
sage: g = M.from_dict({1:2, 2:3, 3:1})
sage: f * g
map: 1 -> 3, 2 -> 2, 3 -> 1
```

If the domains and codomains are both of the form $\{0, \dots\}$, then one can use the shortcut:

```
sage: M = FiniteSetMaps(2,3); M
Maps from {0, 1} to {0, 1, 2}
sage: M.cardinality()
9
```

For a compact notation, the elements are then printed as lists $[f(i), i = 0, \dots]$:

```
sage: list(M)
[[0, 0], [0, 1], [0, 2], [1, 0], [1, 1], [1, 2], [2, 0], [2, 1], [2, 2]]
```

> **cardinality** ( )
>> The cardinality of `self`
>>
>> EXAMPLES:
>>
>> ```
>> sage: FiniteSetMaps(4, 3).cardinality()
>> 81
>> ```

class sage.sets.finite_set_maps. **FiniteSetMaps_MN** (*m*, *n*, *category=None*)

> Bases: *sage.sets.finite_set_maps.FiniteSetMaps*
>
> The set of all maps from $\{1, 2, \dots, m\}$ to $\{1, 2, \dots, n\}$.
>
> Users should use the factory class *FiniteSetMaps* to create instances of this class.
>
> INPUT:
>
>> • m , n – integers
>>
>> • category – the category in which the sets of maps is constructed. It must be a sub-category of `EnumeratedSets().Finite()` which is the default value.
>
> **Element**
>> alias of `FiniteSetMap_MN`
>
> **an_element** ( )
>> Returns a map in `self`

EXAMPLES:

```
sage: M = FiniteSetMaps(4, 2)
sage: M.an_element()
[0, 0, 0, 0]

sage: M = FiniteSetMaps(0, 0)
sage: M.an_element()
[]
```

An exception `EmptySetError` is raised if this set is empty, that is if the codomain is empty and the domain is not.

> sage: M = FiniteSetMaps(4, 0) sage: M.cardinality() 0 sage: M.an_element() Traceback (most recent call last): ... EmptySetError

codomain ( )
> The codomain of `self`

> EXAMPLES:

```
sage: FiniteSetMaps(3,2).codomain()
{0, 1}
```

domain ( )
> The domain of `self`

> EXAMPLES:

```
sage: FiniteSetMaps(3,2).domain()
{0, 1, 2}
```

class sage.sets.finite_set_maps. **FiniteSetMaps_Set** (*domain*, *codomain*, *category=None*)
> Bases: *sage.sets.finite_set_maps.FiniteSetMaps_MN*

> The sets of all maps between two sets

> Users should use the factory class *FiniteSetMaps* to create instances of this class.

> INPUT:

> • `domain` – an object in the category `FiniteSets()`.

> • `codomain` – an object in the category `FiniteSets()`.

> • `category` – the category in which the sets of maps is constructed. It must be a sub-category of `EnumeratedSets().Finite()` which is the default value.

Element
> alias of `FiniteSetMap_Set`

codomain ( )
> The codomain of `self`

> EXAMPLES:

```
sage: FiniteSetMaps(["a", "b"], [3, 4, 5]).codomain()
{3, 4, 5}
```

domain ( )
> The domain of `self`

> EXAMPLES:

```
sage: FiniteSetMaps(["a", "b"], [3, 4, 5]).domain()
{'a', 'b'}
```

from_dict (d)
    Create a map from a dictionary

    EXAMPLES:

```
sage: M = FiniteSetMaps(["a", "b"], [3, 4, 5])
sage: M.from_dict({"a": 4, "b": 3})
map: a -> 4, b -> 3
```

4.14 Data structures for maps between finite sets

This module implements several fast Cython data structures for maps between two finite set. Those classes are not intended to be used directly. Instead, such a map should be constructed via its parent, using the class *FiniteSetMaps*.

EXAMPLES:

To create a map between two sets, one first creates the set of such maps:

```
sage: M = FiniteSetMaps(["a", "b"], [3, 4, 5])
```

The map can then be constructed either from a function:

```
sage: f1 = M(lambda c: ord(c)-94); f1
map: a -> 3, b -> 4
```

or from a dictionary:

```
sage: f2 = M.from_dict({'a':3, 'b':4}); f2
map: a -> 3, b -> 4
```

The two created maps are equal:

```
sage: f1 == f2
True
```

Internally, maps are represented as the list of the ranks of the images `f(x)` in the co-domain, in the order of the domain:

```
sage: list(f2)
[0, 1]
```

A third fast way to create a map it to use such a list. it should be kept for internal use:

```
sage: f3 = M._from_list_([0, 1]); f3
map: a -> 3, b -> 4
sage: f1 == f3
True
```

AUTHORS:

- Florent Hivert

class sage.sets.finite_set_map_cy. **FiniteSetEndoMap_N**

> Bases: *sage.sets.finite_set_map_cy.FiniteSetMap_MN*

> Maps from range(n) to itself.

> **See also:**

> *FiniteSetMap_MN* for assumptions on the parent

class sage.sets.finite_set_map_cy. **FiniteSetEndoMap_Set**

> Bases: *sage.sets.finite_set_map_cy.FiniteSetMap_Set*

> Maps from a set to itself

> **See also:**

> *FiniteSetMap_Set* for assumptions on the parent

class sage.sets.finite_set_map_cy. **FiniteSetMap_MN**

> Bases: *sage.structure.list_clone.ClonableIntArray*

> Data structure for maps from range(m) to range(n).

> We assume that the parent given as argument is such that:

>> •m is stored in self.parent()._m

>> •n is stored in self.parent()._n

>> •the domain is in self.parent().domain()

>> •the codomain is in self.parent().codomain()

> **check** ()

>> Performs checks on self

>> Check that self is a proper function and then calls parent.check_element(self) where parent is the parent of self.

> **codomain** ()

>> Returns the codomain of self

>> EXAMPLES:

>> ```
sage: FiniteSetMaps(4, 3)([1, 0, 2, 1]).codomain()
{0, 1, 2}
```

> **domain** ()

>> Returns the domain of self

>> EXAMPLES:

>> ```
sage: FiniteSetMaps(4, 3)([1, 0, 2, 1]).domain()
{0, 1, 2, 3}
```

> **fibers** ()

>> Returns the fibers of self

>> OUTPUT:

>>> a dictionary d such that d[y] is the set of all x in domain such that f(x) = y

>> EXAMPLES:

```
sage: FiniteSetMaps(4, 3)([1, 0, 2, 1]).fibers()
{0: {1}, 1: {0, 3}, 2: {2}}
sage: F = FiniteSetMaps(["a", "b", "c"])
sage: F.from_dict({"a": "b", "b": "a", "c": "b"}).fibers()
{'a': {'b'}, 'b': {'a', 'c'}}
```

getimage (*i*)

Returns the image of `i` by `self`

INPUT:

   • `i` – any object.

Note: if you need speed, please use instead `_getimage()`

EXAMPLES:

```
sage: fs = FiniteSetMaps(4, 3)([1, 0, 2, 1])
sage: fs.getimage(0), fs.getimage(1), fs.getimage(2), fs.getimage(3)
(1, 0, 2, 1)
```

image_set ( )

Returns the image set of `self`

EXAMPLES:

```
sage: FiniteSetMaps(4, 3)([1, 0, 2, 1]).image_set()
{0, 1, 2}
sage: FiniteSetMaps(4, 3)([1, 0, 0, 1]).image_set()
{0, 1}
```

items ( )

The items of `self`

Return the list of the ordered pairs (`x`, `self(x)`)

EXAMPLES:

```
sage: FiniteSetMaps(4, 3)([1, 0, 2, 1]).items()
[(0, 1), (1, 0), (2, 2), (3, 1)]
```

setimage (*i*, *j*)

Set the image of `i` as `j` in `self`

Warning: `self` must be mutable; otherwise an exception is raised.

INPUT:

   • `i`, `j` – two `object` 's

OUTPUT: `None`

Note: if you need speed, please use instead `_setimage()`

EXAMPLES:

```
sage: fs = FiniteSetMaps(4, 3)([1, 0, 2, 1])
sage: fs2 = copy(fs)
sage: fs2.setimage(2, 1)
sage: fs2
[1, 0, 1, 1]
sage: with fs.clone() as fs3:
....:     fs3.setimage(0, 2)
....:     fs3.setimage(1, 2)
sage: fs3
[2, 2, 2, 1]
```

class sage.sets.finite_set_map_cy. **FiniteSetMap_Set**

Bases: *sage.sets.finite_set_map_cy.FiniteSetMap_MN*

Data structure for maps

We assume that the parent given as argument is such that:

> •the domain is in `parent.domain()`
>
> •the codomain is in `parent.codomain()`
>
> •`parent._m` contains the cardinality of the domain
>
> •`parent._n` contains the cardinality of the codomain
>
> •`parent._unrank_domain` and `parent._rank_domain` is a pair of reciprocal rank and unrank functions beween the domain and `range(parent._m)`.
>
> •`parent._unrank_codomain` and `parent._rank_codomain` is a pair of reciprocal rank and unrank functions beween the codomain and `range(parent._n)`.

classmethod from_dict (*t*, *parent*, *d*)

Creates a `FiniteSetMap` from a dictionary

> **Warning:** no check is performed !

---

classmethod from_list (*t*, *parent*, *lst*)

Creates a `FiniteSetMap` from a list

> **Warning:** no check is performed !

---

getimage (*i*)

Returns the image of `i` by `self`

INPUT:

> •i – an `int`

EXAMPLES:

```
sage: F = FiniteSetMaps(["a", "b", "c", "d"], ["u", "v", "w"])
sage: fs = F._from_list_([1, 0, 2, 1])
sage: list(map(fs.getimage, ["a", "b", "c", "d"]))
['v', 'u', 'w', 'v']
```

image_set ( )

Returns the image set of `self`

EXAMPLES:

```
sage: F = FiniteSetMaps(["a", "b", "c"])
sage: F.from_dict({"a": "b", "b": "a", "c": "b"}).image_set()
{'a', 'b'}
sage: F = FiniteSetMaps(["a", "b", "c"])
sage: F(lambda x: "c").image_set()
{'c'}
```

items ()

The items of `self`

Return the list of the couple `(x, self(x))`

EXAMPLES:

```
sage: F = FiniteSetMaps(["a", "b", "c"])
sage: F.from_dict({"a": "b", "b": "a", "c": "b"}).items()
[('a', 'b'), ('b', 'a'), ('c', 'b')]
```

setimage (*i*, *j*)

Set the image of `i` as `j` in `self`

> **Warning:** `self` must be mutable otherwise an exception is raised.

INPUT:

- `i` , `j` – two `object`'s

OUTPUT: `None`

EXAMPLES:

```
sage: F = FiniteSetMaps(["a", "b", "c", "d"], ["u", "v", "w"])
sage: fs = F(lambda x: "v")
sage: fs2 = copy(fs)
sage: fs2.setimage("a", "w")
sage: fs2
map: a -> w, b -> v, c -> v, d -> v
sage: with fs.clone() as fs3:
....:     fs3.setimage("a", "u")
....:     fs3.setimage("c", "w")
sage: fs3
map: a -> u, b -> v, c -> w, d -> v
```

sage.sets.finite_set_map_cy. **FiniteSetMap_Set_from_dict** (*t*, *parent*, *d*)

Creates a `FiniteSetMap` from a dictionary

> **Warning:** no check is performed !

sage.sets.finite_set_map_cy. **FiniteSetMap_Set_from_list** (*t*, *parent*, *lst*)

Creates a `FiniteSetMap` from a list

> **Warning:** no check is performed !

`sage.sets.finite_set_map_cy.` **`fibers`** (*f*, *domain*)

Returns the fibers of the function `f` on the finite set `domain`

INPUT:

- `f` – a function or callable

- `domain` – a finite iterable

OUTPUT:

- a dictionary `d` such that `d[y]` is the set of all `x` in `domain` such that `f(x) = y`

EXAMPLES:

```
sage: from sage.sets.finite_set_map_cy import fibers, fibers_args
sage: fibers(lambda x: 1, [])
{}
sage: fibers(lambda x: x^2, [-1, 2, -3, 1, 3, 4])
{1: {1, -1}, 4: {2}, 9: {3, -3}, 16: {4}}
sage: fibers(lambda x: 1,    [-1, 2, -3, 1, 3, 4])
{1: {1, 2, 3, 4, -3, -1}}
sage: fibers(lambda x: 1, [1,1,1])
{1: {1}}
```

See also:

fibers_args() if one needs to pass extra arguments to `f` .

`sage.sets.finite_set_map_cy.` **`fibers_args`** (*f*, *domain*, **args*, ***opts*)

Returns the fibers of the function `f` on the finite set `domain`

It is the same as *fibers()* except that one can pass extra argument for `f` (with a small overhead)

EXAMPLES:

```
sage: from sage.sets.finite_set_map_cy import fibers_args
sage: fibers_args(operator.pow, [-1, 2, -3, 1, 3, 4], 2)
{1: {1, -1}, 4: {2}, 9: {3, -3}, 16: {4}}
```

4.15 Totally Ordered Finite Sets

AUTHORS:

- Stepan Starosta (2012): Initial version

class `sage.sets.totally_ordered_finite_set.` **`TotallyOrderedFiniteSet`** (*elements*, *facade=True*)

Bases: *sage.sets.finite_enumerated_set.FiniteEnumeratedSet*

Totally ordered finite set.

This is a finite enumerated set assuming that the elements are ordered based upon their rank (i.e. their position in the set).

INPUT:

- `elements` – A list of elements in the set

- `facade` – (default: `True`) if `True` , a facade is used; it should be set to `False` if the elements do not inherit from *Element* or if you want a funny order. See examples for more details.

See also:

FiniteEnumeratedSet

EXAMPLES:

```
sage: S = TotallyOrderedFiniteSet([1,2,3])
sage: S
{1, 2, 3}
sage: S.cardinality()
3
```

By default, totally ordered finite set behaves as a facade:

```
sage: S(1).parent()
Integer Ring
```

It makes comparison fails when it is not the standard order:

```
sage: T1 = TotallyOrderedFiniteSet([3,2,5,1])
sage: T1(3) < T1(1)
False
sage: T2 = TotallyOrderedFiniteSet([3,var('x')])
sage: T2(3) < T2(var('x'))
3 < x
```

To make the above example work, you should set the argument facade to `False` in the constructor. In that case, the elements of the set have a dedicated class:

```
sage: A = TotallyOrderedFiniteSet([3,2,0,'a',7,(0,0),1], facade=False)
sage: A
{3, 2, 0, 'a', 7, (0, 0), 1}
sage: x = A.an_element()
sage: x
3
sage: x.parent()
{3, 2, 0, 'a', 7, (0, 0), 1}
sage: A(3) < A(2)
True
sage: A('a') < A(7)
True
sage: A(3) > A(2)
False
sage: A(1) < A(3)
False
sage: A(3) == A(3)
True
```

But then, the equality comparison is always False with elements outside of the set:

```
sage: A(1) == 1
False
sage: 1 == A(1)
False
sage: 'a' == A('a')
False
sage: A('a') == 'a'
False
```

and comparisons are comparisons of types:

```
sage: for e in [1,'a',(0, 0)]:
....:     f = A(e)
....:     l = (e == f,
....:           cmp(e,f) == cmp(type(e),type(f)),
....:           cmp(f,e) == cmp(type(f),type(e)))
....:     print(l)
(False, True, True)
(False, True, True)
(False, True, True)
```

This behavior of comparison is the same as the one of `Element`.

Since trac ticket #16280, totally ordered sets support elements that do not inherit from `sage.structure.element.Element`, whether they are facade or not:

```
sage: S = TotallyOrderedFiniteSet(['a','b'])
sage: S('a')
'a'
sage: S = TotallyOrderedFiniteSet(['a','b'], facade = False)
sage: S('a')
'a'
```

Multiple elements are automatically deleted:

```
sage: TotallyOrderedFiniteSet([1,1,2,1,2,2,5,4])
{1, 2, 5, 4}
```

Element
> alias of `TotallyOrderedFiniteSetElement`

le (*x*, *y*)
> Return `True` if $x \le y$ for the order of `self`.
>
> EXAMPLES:
>
> ```
> sage: T = TotallyOrderedFiniteSet([1,3,2], facade=False)
> sage: T1, T3, T2 = T.list()
> sage: T.le(T1,T3)
> True
> sage: T.le(T3,T2)
> True
> ```

class sage.sets.totally_ordered_finite_set. **TotallyOrderedFiniteSetElement** (*parent*, *data*)

> Bases: `sage.structure.element.Element`

Element of a finite totally ordered set.

EXAMPLES:

```
sage: S = TotallyOrderedFiniteSet([2,7], facade=False)
sage: x = S(2)
sage: print(x)
2
sage: x.parent()
{2, 7}
```

SETS

5.1 Integer Range

AUTHORS:

- Nicolas Borie (2010-03): First release.

- Florent Hivert (2010-03): Added a class factory + cardinality method.

- Vincent Delecroix (2012-02): add methods rank/unrank, make it complient with Python int.

class sage.sets.integer_range. **IntegerRange**

Bases: *sage.structure.unique_representation.UniqueRepresentation* , *sage.structure.parent.Parent*

The class of Integer ranges

Returns an enumerated set containing an arithmetic progression of integers.

INPUT:

- begin – an integer, Infinity or -Infinity

- end – an integer, Infinity or -Infinity

- step – a non zero integer (default to 1)

- middle_point – an integer inside the set (default to None)

OUTPUT:

A parent in the category FiniteEnumeratedSets() or InfiniteEnumeratedSets() depending on the arguments defining self .

IntegerRange(i,j) returns the set of $\{i, i+1, i+2, \dots, j-1\}$. start (!) defaults to 0. When step is given, it specifies the increment. The default increment is 1. IntegerRange allows begin and end to be infinite.

IntegerRange is designed to have similar interface Python range. However, whereas range accept and returns Python int , IntegerRange deals with Integer .

If middle_point is given, then the elements are generated starting from it, in a alternating way: $\{m, m+1, m-2, m+2, m-2\dots\}$.

EXAMPLES:

```
sage: list(IntegerRange(5))
[0, 1, 2, 3, 4]
sage: list(IntegerRange(2,5))
[2, 3, 4]
```

```
sage: I = IntegerRange(2,100,5); I
{2, 7, ..., 97}
sage: list(I)
[2, 7, 12, 17, 22, 27, 32, 37, 42, 47, 52, 57, 62, 67, 72, 77, 82, 87, 92, 97]
sage: I.category()
Category of facade finite enumerated sets
sage: I[1].parent()
Integer Ring
```

When `begin` and `end` are both finite, `IntegerRange(begin,end,step)` is the set whose list of elements is equivalent to the python construction `range(begin,end,step)`:

```
sage: list(IntegerRange(4,105,3)) == list(range(4,105,3))
True
sage: list(IntegerRange(-54,13,12)) == list(range(-54,13,12))
True
```

Except for the type of the numbers:

```
sage: type(IntegerRange(-54,13,12)[0]), type(list(range(-54,13,12))[0])
(<type 'sage.rings.integer.Integer'>, <... 'int'>)
```

When `begin` is finite and `end` is +Infinity, `self` is the infinite arithmetic progression starting from the `begin` by step `step`:

```
sage: I = IntegerRange(54,Infinity,3); I
{54, 57, ...}
sage: I.category()
Category of facade infinite enumerated sets
sage: p = iter(I)
sage: (next(p), next(p), next(p), next(p), next(p), next(p))
(54, 57, 60, 63, 66, 69)

sage: I = IntegerRange(54,-Infinity,-3); I
{54, 51, ...}
sage: I.category()
Category of facade infinite enumerated sets
sage: p = iter(I)
sage: (next(p), next(p), next(p), next(p), next(p), next(p))
(54, 51, 48, 45, 42, 39)
```

When `begin` and `end` are both infinite, you will have to specify the extra argument `middle_point`. `self` is then defined by a point and a progression/regression setting by `step`. The enumeration is done this way: (let us call m the `middle_point`) $\{m, m + step, m - step, m + 2step, m - 2step, m + 3step, \dots\}$:

```
sage: I = IntegerRange(-Infinity,Infinity,37,-12); I
Integer progression containing -12 with increment 37 and bounded with -Infinity␣
↪and +Infinity
sage: I.category()
Category of facade infinite enumerated sets
sage: -12 in I
True
sage: -15 in I
False
sage: p = iter(I)
sage: (next(p), next(p), next(p), next(p), next(p), next(p), next(p), next(p))
(-12, 25, -49, 62, -86, 99, -123, 136)
```

It is also possible to use the argument `middle_point` for other cases, finite or infinite. The set will be the same as if you didn't give this extra argument but the enumeration will begin with this `middle_point`:

```
sage: I = IntegerRange(123,-12,-14); I
{123, 109, ..., -3}
sage: list(I)
[123, 109, 95, 81, 67, 53, 39, 25, 11, -3]
sage: J = IntegerRange(123,-12,-14,25); J
Integer progression containing 25 with increment -14 and bounded with 123 and -12
sage: list(J)
[25, 11, 39, -3, 53, 67, 81, 95, 109, 123]
```

Remember that, like for range, if you define a non empty set, `begin` is supposed to be included and `end` is supposed to be excluded. In the same way, when you define a set with a `middle_point`, the `begin` bound will be supposed to be included and the `end` bound supposed to be excluded:

```
sage: I = IntegerRange(-100,100,10,0)
sage: J = list(range(-100,100,10))
sage: 100 in I
False
sage: 100 in J
False
sage: -100 in I
True
sage: -100 in J
True
sage: list(I)
[0, 10, -10, 20, -20, 30, -30, 40, -40, 50, -50, 60, -60, 70, -70, 80, -80, 90, -
→90, -100]
```

Note: The input is normalized so that:

```
sage: IntegerRange(1, 6, 2) is IntegerRange(1, 7, 2)
True
sage: IntegerRange(1, 8, 3) is IntegerRange(1, 10, 3)
True
```

---

element_class
> alias of `Integer`

class sage.sets.integer_range.**IntegerRangeEmpty** (*elements*)
> Bases: *sage.sets.integer_range.IntegerRange*, *sage.sets.finite_enumerated_set.FiniteEnumera*

> A singleton class for empty integer ranges

> See *IntegerRange* for more details.

class sage.sets.integer_range.**IntegerRangeFinite** (*begin*, *end*, *step=1*)
> Bases: *sage.sets.integer_range.IntegerRange*

> The class of finite enumerated sets of integers defined by finite arithmetic progressions

> See *IntegerRange* for more details.

cardinality ( )
> Return the cardinality of `self`

> EXAMPLES:

```
sage: IntegerRange(123,12,-4).cardinality()
28
sage: IntegerRange(-57,12,8).cardinality()
9
sage: IntegerRange(123,12,4).cardinality()
0
```

rank (*x*)

 EXAMPLES:

```
sage: I = IntegerRange(-57,36,8)
sage: I.rank(23)
10
sage: I.unrank(10)
23
sage: I.rank(22)
Traceback (most recent call last):
...
IndexError: 22 not in self
sage: I.rank(87)
Traceback (most recent call last):
...
IndexError: 87 not in self
```

unrank (*i*)

 Return the i-th element of this integer range.

 EXAMPLES:

```
sage: I = IntegerRange(1,13,5)
sage: I[0], I[1], I[2]
(1, 6, 11)
sage: I[3]
Traceback (most recent call last):
...
IndexError: out of range
sage: I[-1]
11
sage: I[-4]
Traceback (most recent call last):
...
IndexError: out of range

sage: I = IntegerRange(13,1,-1)
sage: l = I.list()
sage: [I[i] for i in range(I.cardinality())] == l
True
sage: l.reverse()
sage: [I[i] for i in range(-1,-I.cardinality()-1,-1)] == l
True
```

class sage.sets.integer_range. **IntegerRangeFromMiddle** (*begin*,   *end*,   *step=1*,   *middle_point=1*)

 Bases: *sage.sets.integer_range.IntegerRange*

 The class of finite or infinite enumerated sets defined with an inside point, a progression and two limits.

 See *IntegerRange* for more details.

next (*elt*)

> Return the next element of elt in self.

> EXAMPLES:

```
sage: from sage.sets.integer_range import IntegerRangeFromMiddle
sage: I = IntegerRangeFromMiddle(-100,100,10,0)
sage: (I.next(0), I.next(10), I.next(-10), I.next(20), I.next(-100))
(10, -10, 20, -20, None)
sage: I = IntegerRangeFromMiddle(-Infinity,Infinity,10,0)
sage: (I.next(0), I.next(10), I.next(-10), I.next(20), I.next(-100))
(10, -10, 20, -20, 110)
sage: I.next(1)
Traceback (most recent call last):
...
LookupError: 1 not in Integer progression containing 0 with increment 10 and
→bounded with -Infinity and +Infinity
```

class sage.sets.integer_range. **IntegerRangeInfinite** (*begin*, *step=1*)

> Bases: *sage.sets.integer_range.IntegerRange*

> The class of infinite enumerated sets of integers defined by infinite arithmetic progressions.

> See *IntegerRange* for more details.

> **rank** (*x*)

> > EXAMPLES:

```
sage: I = IntegerRange(-57,Infinity,8)
sage: I.rank(23)
10
sage: I.unrank(10)
23
sage: I.rank(22)
Traceback (most recent call last):
...
IndexError: 22 not in self
```

unrank (*i*)

> Returns the i -th element of self.

> EXAMPLES:

```
sage: I = IntegerRange(-8,Infinity,3)
sage: I.unrank(1)
-5
```

5.2 Positive Integers

class sage.sets.positive_integers. **PositiveIntegers**

> Bases: *sage.sets.integer_range.IntegerRangeInfinite*

> The enumerated set of positive integers. To fix the ideas, we mean $\{1, 2, 3, 4, 5, \ldots\}$.

> This class implements the set of positive integers, as an enumerated set (see InfiniteEnumeratedSets).

> This set is an integer range set. The construction is therefore done by IntegerRange (see *IntegerRange*).

> EXAMPLES:

```
sage: PP = PositiveIntegers()
sage: PP
Positive integers
sage: PP.cardinality()
+Infinity
sage: TestSuite(PP).run()
sage: PP.list()
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
sage: it = iter(PP)
sage: (next(it), next(it), next(it), next(it), next(it))
(1, 2, 3, 4, 5)
sage: PP.first()
1
```

an_element ( )
> Returns an element of `self`.

> EXAMPLES:

```
sage: PositiveIntegers().an_element()
42
```

5.3 Non Negative Integers

class `sage.sets.non_negative_integers.` **NonNegativeIntegers** (*category=None*)
> Bases: *sage.structure.unique_representation.UniqueRepresentation* ,
> *sage.structure.parent.Parent*

> The enumerated set of non negative integers.

> This class implements the set of non negative integers, as an enumerated set (see
> `InfiniteEnumeratedSets`).

> EXAMPLES:

```
sage: NN = NonNegativeIntegers()
sage: NN
Non negative integers
sage: NN.cardinality()
+Infinity
sage: TestSuite(NN).run()
sage: NN.list()
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
sage: NN.element_class
<type 'sage.rings.integer.Integer'>
sage: it = iter(NN)
sage: [next(it), next(it), next(it), next(it), next(it)]
[0, 1, 2, 3, 4]
sage: NN.first()
0
```

> Currently, this is just a "facade" parent; namely its elements are plain Sage `Integers` with `Integer Ring` as parent:

```
sage: x = NN(15); type(x)
<type 'sage.rings.integer.Integer'>
sage: x.parent()
Integer Ring
sage: x+3
18
```

In a later version, there will be an option to specify whether the elements should have `Integer Ring` or `Non negative integers` as parent:

```
sage: NN = NonNegativeIntegers(facade = False)    # todo: not implemented
sage: x = NN(5)                                    # todo: not implemented
sage: x.parent()                                   # todo: not implemented
Non negative integers
```

This runs generic sanity checks on `NN` :

```
sage: TestSuite(NN).run()
```

TODO: do not use `NN` any more in the doctests for `NonNegativeIntegers` .

Element
> alias of `Integer`

an_element ( )
> EXAMPLES:

```
sage: NonNegativeIntegers().an_element()
42
```

from_integer
> alias of `Integer`

next (*o*)
> EXAMPLES:

```
sage: NN = NonNegativeIntegers()
sage: NN.next(3)
4
```

some_elements ( )
> EXAMPLES:

```
sage: NonNegativeIntegers().some_elements()
[0, 1, 3, 42]
```

unrank (*rnk*)
> EXAMPLES:

```
sage: NN = NonNegativeIntegers()
sage: NN.unrank(100)
100
```

5.4 The set of prime numbers

AUTHORS:

- William Stein (2005): original version

- Florent Hivert (2009-11): adapted to the category framework. The following methods were removed:

  - cardinality, __len__, __iter__: provided by EnumeratedSets

  - __cmp__(self, other): __eq__ is provided by UniqueRepresentation and seems to do as good a job (all test pass)

class sage.sets.primes. **Primes** (*proof*)

    Bases: *sage.structure.parent.Set_generic* , *sage.structure.unique_representation.UniqueRepr*

    The set of prime numbers.

    EXAMPLES:

```
sage: P = Primes(); P
Set of all prime numbers: 2, 3, 5, 7, ...
```

    We show various operations on the set of prime numbers:

```
sage: P.cardinality()
+Infinity
sage: R = Primes()
sage: P == R
True
sage: 5 in P
True
sage: 100 in P
False

sage: len(P)
Traceback (most recent call last):
...
NotImplementedError: infinite set
```

    first ( )

        Return the first prime number.

        EXAMPLES:

```
sage: P = Primes()
sage: P.first()
2
```

    next (*pr*)

        Return the next prime number.

        EXAMPLES:

```
sage: P = Primes()
sage: P.next(5)
7
```

    unrank (*n*)

        Return the n-th prime number.

        EXAMPLES:

```
sage: P = Primes()
sage: P.unrank(0)
2
sage: P.unrank(5)
13
sage: P.unrank(42)
191
```

5.5 Subsets of the Real Line

This module contains subsets of the real line that can be constructed as the union of a finite set of open and closed intervals.

EXAMPLES:

```
sage: RealSet(0,1)
(0, 1)
sage: RealSet((0,1), [2,3])
(0, 1) + [2, 3]
sage: RealSet(-oo, oo)
(-oo, +oo)
```

Brackets must be balanced in Python, so the naive notation for half-open intervals does not work:

```
sage: RealSet([0,1))
Traceback (most recent call last):
...
SyntaxError: invalid syntax
```

Instead, you can use the following construction functions:

```
sage: RealSet.open_closed(0,1)
(0, 1]
sage: RealSet.closed_open(0,1)
[0, 1)
sage: RealSet.point(1/2)
{1/2}
sage: RealSet.unbounded_below_open(0)
(-oo, 0)
sage: RealSet.unbounded_below_closed(0)
(-oo, 0]
sage: RealSet.unbounded_above_open(1)
(1, +oo)
sage: RealSet.unbounded_above_closed(1)
[1, +oo)
```

AUTHORS:

- Laurent Claessens (2010-12-10): Interval and ContinuousSet, posted to sage-devel at http://www.mail-archive.com/sage-support@googlegroups.com/msg21326.html.

- Ares Ribo (2011-10-24): Extended the previous work defining the class RealSet.

- Jordi Saludes (2011-12-10): Documentation and file reorganization.

- Volker Braun (2013-06-22): Rewrite

class sage.sets.real_set. **InternalRealInterval** (*lower*, *lower_closed*, *upper*, *upper_closed*, *check=True*)

    Bases: *sage.structure.unique_representation.UniqueRepresentation* , *sage.structure.parent.Parent*

    A real interval.

    You are not supposed to create RealInterval objects yourself. Always use *RealSet* instead.

    INPUT:

- lower – real or minus infinity; the lower bound of the interval.

- lower_closed – boolean; whether the interval is closed at the lower bound

- upper – real or (plus) infinity; the upper bound of the interval

- upper_closed – boolean; whether the interval is closed at the upper bound

- check – boolean; whether to check the other arguments for validity

    closure ( )

        Return the closure

        OUTPUT:

        The closure as a new RealInterval

        EXAMPLES:

```
sage: RealSet.open(0,1)[0].closure()
[0, 1]
sage: RealSet.open(-oo,1)[0].closure()
(-oo, 1]
sage: RealSet.open(0, oo)[0].closure()
[0, +oo)
```

    contains (*x*)

        Return whether x is contained in the interval

        INPUT:

- x – a real number.

        OUTPUT:

        Boolean.

        EXAMPLES:

```
sage: i = RealSet.open_closed(0,2)[0]; i
(0, 2]
sage: i.contains(0)
False
sage: i.contains(1)
True
sage: i.contains(2)
True
```

    convex_hull (*other*)

        Return the convex hull of the two intervals

        OUTPUT:

        The convex hull as a new RealInterval .

EXAMPLES:

```
sage: I1 = RealSet.open(0, 1)[0];  I1
(0, 1)
sage: I2 = RealSet.closed(1, 2)[0];  I2
[1, 2]
sage: I1.convex_hull(I2)
(0, 2]
sage: I2.convex_hull(I1)
(0, 2]
sage: I1.convex_hull(I2.interior())
(0, 2)
sage: I1.closure().convex_hull(I2.interior())
[0, 2)
sage: I1.closure().convex_hull(I2)
[0, 2]
sage: I3 = RealSet.closed(1/2, 3/2)[0]; I3
[1/2, 3/2]
sage: I1.convex_hull(I3)
(0, 3/2]
```

element_class
    alias of `LazyFieldElement`

interior ( )
    Return the interior

    OUTPUT:

    The interior as a new `RealInterval`

    EXAMPLES:

```
sage: RealSet.closed(0, 1)[0].interior()
(0, 1)
sage: RealSet.open_closed(-oo, 1)[0].interior()
(-oo, 1)
sage: RealSet.closed_open(0, oo)[0].interior()
(0, +oo)
```

intersection (*other*)
    Return the intersection of the two intervals

    INPUT:

    •`other` – a `RealInterval`

    OUTPUT:

    The intersection as a new `RealInterval`

    EXAMPLES:

```
sage: I1 = RealSet.open(0, 2)[0];  I1
(0, 2)
sage: I2 = RealSet.closed(1, 3)[0];  I2
[1, 3]
sage: I1.intersection(I2)
[1, 2)
sage: I2.intersection(I1)
[1, 2)
sage: I1.closure().intersection(I2.interior())
```

```
(1, 2]
sage: I2.interior().intersection(I1.closure())
(1, 2]

sage: I3 = RealSet.closed(10, 11)[0];  I3
[10, 11]
sage: I1.intersection(I3)
(0, 0)
sage: I3.intersection(I1)
(0, 0)
```

is_connected (*other*)

Test whether two intervals are connected

OUTPUT:

Boolean. Whether the set-theoretic union of the two intervals has a single connected component.

EXAMPLES:

```
sage: I1 = RealSet.open(0, 1)[0];  I1
(0, 1)
sage: I2 = RealSet.closed(1, 2)[0];  I2
[1, 2]
sage: I1.is_connected(I2)
True
sage: I1.is_connected(I2.interior())
False
sage: I1.closure().is_connected(I2.interior())
True
sage: I2.is_connected(I1)
True
sage: I2.interior().is_connected(I1)
False
sage: I2.closure().is_connected(I1.interior())
True
sage: I3 = RealSet.closed(1/2, 3/2)[0]; I3
[1/2, 3/2]
sage: I1.is_connected(I3)
True
sage: I3.is_connected(I1)
True
```

is_empty ( )

Return whether the interval is empty

The normalized form of [*RealSet*](#) has all intervals non-empty, so this method usually returns False .

OUTPUT:

Boolean.

EXAMPLES:

```
sage: I = RealSet(0, 1)[0]
sage: I.is_empty()
False
```

is_point ( )

Return whether the interval consists of a single point

OUTPUT:

Boolean.

EXAMPLES:

```
sage: I = RealSet(0, 1)[0]
sage: I.is_point()
False
```

lower ()
> Return the lower bound

> OUTPUT:

> The lower bound as it was originally specified.

> EXAMPLES:

```
sage: I = RealSet(0, 1)[0]
sage: I.lower()
0
sage: I.upper()
1
```

lower_closed ()
> Return whether the interval is open at the lower bound

> OUTPUT:

> Boolean.

> EXAMPLES:

```
sage: I = RealSet.open_closed(0, 1)[0];  I
(0, 1]
sage: I.lower_closed()
False
sage: I.lower_open()
True
sage: I.upper_closed()
True
sage: I.upper_open()
False
```

lower_open ()
> Return whether the interval is closed at the upper bound

> OUTPUT:

> Boolean.

> EXAMPLES:

```
sage: I = RealSet.open_closed(0, 1)[0];  I
(0, 1]
sage: I.lower_closed()
False
sage: I.lower_open()
True
sage: I.upper_closed()
True
```

```
sage: I.upper_open()
False
```

upper ( )
>   Return the upper bound
>
>   OUTPUT:
>
>   The upper bound as it was originally specified.
>
>   EXAMPLES:

```
sage: I = RealSet(0, 1)[0]
sage: I.lower()
0
sage: I.upper()
1
```

upper_closed ( )
>   Return whether the interval is closed at the lower bound
>
>   OUTPUT:
>
>   Boolean.
>
>   EXAMPLES:

```
sage: I = RealSet.open_closed(0, 1)[0];  I
(0, 1]
sage: I.lower_closed()
False
sage: I.lower_open()
True
sage: I.upper_closed()
True
sage: I.upper_open()
False
```

upper_open ( )
>   Return whether the interval is closed at the upper bound
>
>   OUTPUT:
>
>   Boolean.
>
>   EXAMPLES:

```
sage: I = RealSet.open_closed(0, 1)[0];  I
(0, 1]
sage: I.lower_closed()
False
sage: I.lower_open()
True
sage: I.upper_closed()
True
sage: I.upper_open()
False
```

class sage.sets.real_set. **RealSet** (*intervals*)
>   Bases: *sage.structure.unique_representation.UniqueRepresentation*   ,
>   *sage.structure.parent.Parent*

A subset of the real line

INPUT:

Arguments defining a real set. Possibilities are either two real numbers to construct an open set or a list/tuple/iterable of intervals. The individual intervals can be specified by either a `RealInterval`, a tuple of two real numbers (constructing an open interval), or a list of two number (constructing a closed interval).

EXAMPLES:

```
sage: RealSet(0,1)      # open set from two numbers
(0, 1)
sage: i = RealSet(0,1)[0]
sage: RealSet(i)        # interval
(0, 1)
sage: RealSet(i, (3,4))     # tuple of two numbers = open set
(0, 1) + (3, 4)
sage: RealSet(i, [3,4])     # list of two numbers = closed set
(0, 1) + [3, 4]
```

an_element ( )
> Return a point of the set
>
> OUTPUT:
>
> A real number. `ValueError` if the set is empty.
>
> EXAMPLES:
>
> ```
> sage: RealSet.open_closed(0, 1).an_element()
> 1
> sage: RealSet(0, 1).an_element()
> 1/2
> ```

static are_pairwise_disjoint (**real_set_collection*)
> Test whether sets are pairwise disjoint
>
> INPUT:
>
> > • **real_set_collection* – a list/tuple/iterable of *RealSet* .
>
> OUTPUT:
>
> Boolean.
>
> EXAMPLES:
>
> ```
> sage: s1 = RealSet((0, 1), (2, 3))
> sage: s2 = RealSet((1, 2))
> sage: s3 = RealSet.point(3)
> sage: RealSet.are_pairwise_disjoint(s1, s2, s3)
> True
> sage: RealSet.are_pairwise_disjoint(s1, s2, s3, [10,10])
> True
> sage: RealSet.are_pairwise_disjoint(s1, s2, s3, [-1, 1/2])
> False
> ```

cardinality ( )
> Return the cardinality of the subset of the real line.
>
> OUTPUT:
>
> Integer or infinity. The size of a discrete set is the number of points; the size of a real interval is Infinity.

EXAMPLES:

```
sage: RealSet([0, 0], [1, 1], [3, 3]).cardinality()
3
sage: RealSet(0,3).cardinality()
+Infinity
```

static **closed** (*lower*, *upper*)

Construct a closed interval

INPUT:

- lower , upper – two real numbers or infinity. They will be sorted if necessary.

OUTPUT:

A new *RealSet* .

EXAMPLES:

```
sage: RealSet.closed(1, 0)
[0, 1]
```

static **closed_open** (*lower*, *upper*)

Construct an half-open interval

INPUT:

- lower , upper – two real numbers or infinity. They will be sorted if necessary.

OUTPUT:

A new *RealSet* that is closed at the lower bound and open an the upper bound.

EXAMPLES:

```
sage: RealSet.closed_open(1, 0)
[0, 1)
```

complement ( )

Return the complement

OUTPUT:

The set-theoretic complement as a new *RealSet* .

EXAMPLES:

```
sage: RealSet(0,1).complement()
(-oo, 0] + [1, +oo)

sage: s1 = RealSet(0,2) + RealSet.unbounded_above_closed(10);  s1
(0, 2) + [10, +oo)
sage: s1.complement()
(-oo, 0] + [2, 10)

sage: s2 = RealSet(1,3) + RealSet.unbounded_below_closed(-10);  s2
(-oo, -10] + (1, 3)
sage: s2.complement()
(-10, 1] + [3, +oo)
```

contains (*x*)

Return whether x is contained in the set

INPUT:

- x – a real number.

OUTPUT:

Boolean.

EXAMPLES:

```
sage: s = RealSet(0,2) + RealSet.unbounded_above_closed(10);  s
(0, 2) + [10, +oo)
sage: s.contains(1)
True
sage: s.contains(0)
False
sage: 10 in s    # syntactic sugar
True
```

difference (**other*)

Return `self` with `other` subtracted

INPUT:

- `other` – a *RealSet* or data that defines one.

OUTPUT:

The set-theoretic difference of `self` with `other` removed as a new *RealSet* .

EXAMPLES:

```
sage: s1 = RealSet(0,2) + RealSet.unbounded_above_closed(10);  s1
(0, 2) + [10, +oo)
sage: s2 = RealSet(1,3) + RealSet.unbounded_below_closed(-10);  s2
(-oo, -10] + (1, 3)
sage: s1.difference(s2)
(0, 1] + [10, +oo)
sage: s1 - s2    # syntactic sugar
(0, 1] + [10, +oo)
sage: s2.difference(s1)
(-oo, -10] + [2, 3)
sage: s2 - s1    # syntactic sugar
(-oo, -10] + [2, 3)
sage: s1.difference(1,11)
(0, 1] + [11, +oo)
```

get_interval (*i*)

Return the `i` -th connected component.

Note that the intervals representing the real set are always normalized, see *normalize()* .

INPUT:

- `i` – integer.

OUTPUT:

The i-th connected component as a `RealInterval` .

EXAMPLES:

```
sage: s = RealSet(RealSet.open_closed(0,1), RealSet.closed_open(2,3))
sage: s.get_interval(0)
(0, 1]
sage: s[0]     # shorthand
(0, 1]
sage: s.get_interval(1)
[2, 3)
sage: s[0] == s.get_interval(0)
True
```

inf ( )

Return the infimum

OUTPUT:

A real number or infinity.

EXAMPLES:

```
sage: s1 = RealSet(0,2) + RealSet.unbounded_above_closed(10);  s1
(0, 2) + [10, +oo)
sage: s1.inf()
0

sage: s2 = RealSet(1,3) + RealSet.unbounded_below_closed(-10);  s2
(-oo, -10] + (1, 3)
sage: s2.inf()
-Infinity
```

intersection (*other*)

Return the intersection of the two sets

INPUT:

•other − a *RealSet* or data that defines one.

OUTPUT:

The set-theoretic intersection as a new *RealSet* .

EXAMPLES:

```
sage: s1 = RealSet(0,2) + RealSet.unbounded_above_closed(10);  s1
(0, 2) + [10, +oo)
sage: s2 = RealSet(1,3) + RealSet.unbounded_below_closed(-10);  s2
(-oo, -10] + (1, 3)
sage: s1.intersection(s2)
(1, 2)
sage: s1 & s2     # syntactic sugar
(1, 2)

sage: s1 = RealSet((0, 1), (2, 3));  s1
(0, 1) + (2, 3)
sage: s2 = RealSet([0, 1], [2, 3]);  s2
[0, 1] + [2, 3]
sage: s3 = RealSet([1, 2]);  s3
[1, 2]
sage: s1.intersection(s2)
(0, 1) + (2, 3)
sage: s1.intersection(s3)
{}
```

```
sage: s2.intersection(s3)
{1} + {2}
```

is_disjoint_from (**other*)
    Test whether the two sets are disjoint

    INPUT:

    •other – a *RealSet* or data defining one.

    OUTPUT:

    Boolean.

    EXAMPLES:

```
sage: s1 = RealSet((0, 1), (2, 3));  s1
(0, 1) + (2, 3)
sage: s2 = RealSet([1, 2]);  s2
[1, 2]
sage: s1.is_disjoint_from(s2)
True
sage: s1.is_disjoint_from([1, 2])
True
```

is_empty ( )
    Return whether the set is empty

    EXAMPLES:

```
sage: RealSet(0, 1).is_empty()
False
sage: RealSet(0, 0).is_empty()
True
```

is_included_in (**other*)
    Tests interval inclusion

    INPUT:

    •*args – a *RealSet* or something that defines one.

    OUTPUT:

    Boolean.

    EXAMPLES:

```
sage: I = RealSet((1,2))
sage: J = RealSet((1,3))
sage: K = RealSet((2,3))
sage: I.is_included_in(J)
True
sage: J.is_included_in(K)
False
```

n_components ( )
    Return the number of connected components

    See also *get_interval()*

    EXAMPLES:

```
sage: s = RealSet(RealSet.open_closed(0,1), RealSet.closed_open(2,3))
sage: s.n_components()
2
```

static **normalize** (*intervals*)

Bring a collection of intervals into canonical form

INPUT:

- intervals – a list/tuple/iterable of intervals.

OUTPUT:

A tuple of intervals such that

- they are sorted in ascending order (by lower bound)

- there is a gap between each interval

- all intervals are non-empty

EXAMPLES:

```
sage: i1 = RealSet((0, 1))[0]
sage: i2 = RealSet([1, 2])[0]
sage: i3 = RealSet((2, 3))[0]
sage: RealSet.normalize([i1, i2, i3])
((0, 3),)

sage: RealSet((0, 1), [1, 2], (2, 3))
(0, 3)
sage: RealSet((0, 1), (1, 2), (2, 3))
(0, 1) + (1, 2) + (2, 3)
sage: RealSet([0, 1], [2, 3])
[0, 1] + [2, 3]
sage: RealSet((0, 2), (1, 3))
(0, 3)
sage: RealSet(0,0)
{}
```

static **open** (*lower*, *upper*)

Construct an open interval

INPUT:

- lower , upper – two real numbers or infinity. They will be sorted if necessary.

OUTPUT:

A new *RealSet* .

EXAMPLES:

```
sage: RealSet.open(1, 0)
(0, 1)
```

static **open_closed** (*lower*, *upper*)

Construct a half-open interval

INPUT:

- lower , upper – two real numbers or infinity. They will be sorted if necessary.

OUTPUT:

A new [`RealSet`](#) that is open at the lower bound and closed at the upper bound.

EXAMPLES:

```
sage: RealSet.open_closed(1, 0)
(0, 1]
```

static **point** (*p*)

Construct an interval containing a single point

INPUT:

- •p – a real number.

OUTPUT:

A new [`RealSet`](#) .

EXAMPLES:

```
sage: RealSet.open(1, 0)
(0, 1)
```

sup ( )

Return the supremum

OUTPUT:

A real number or infinity.

EXAMPLES:

```
sage: s1 = RealSet(0,2) + RealSet.unbounded_above_closed(10);  s1
(0, 2) + [10, +oo)
sage: s1.sup()
+Infinity

sage: s2 = RealSet(1,3) + RealSet.unbounded_below_closed(-10);  s2
(-oo, -10] + (1, 3)
sage: s2.sup()
3
```

static **unbounded_above_closed** (*bound*)

Construct a semi-infinite interval

INPUT:

- •bound – a real number.

OUTPUT:

A new [`RealSet`](#) from the bound (including) to plus infinity.

EXAMPLES:

```
sage: RealSet.unbounded_above_closed(1)
[1, +oo)
```

static **unbounded_above_open** (*bound*)

Construct a semi-infinite interval

INPUT:

> •bound – a real number.

> OUTPUT:

> A new *RealSet* from the bound (excluding) to plus infinity.

> EXAMPLES:

```
sage: RealSet.unbounded_above_open(1)
(1, +oo)
```

static unbounded_below_closed (*bound*)

> Construct a semi-infinite interval

> INPUT:

>> •bound – a real number.

> OUTPUT:

> A new *RealSet* from minus infinity to the bound (including).

> EXAMPLES:

```
sage: RealSet.unbounded_below_closed(1)
(-oo, 1]
```

static unbounded_below_open (*bound*)

> Construct a semi-infinite interval

> INPUT:

>> •bound – a real number.

> OUTPUT:

> A new *RealSet* from minus infinity to the bound (excluding).

> EXAMPLES:

```
sage: RealSet.unbounded_below_open(1)
(-oo, 1)
```

union (**other*)

> Return the union of the two sets

> INPUT:

>> •other – a *RealSet* or data that defines one.

> OUTPUT:

> The set-theoretic union as a new *RealSet* .

> EXAMPLES:

```
sage: s1 = RealSet(0,2)
sage: s2 = RealSet(1,3)
sage: s1.union(s2)
(0, 3)
sage: s1.union(1,3)
(0, 3)
sage: s1 | s2    # syntactic sugar
(0, 3)
```

```
sage: s1 + s2     # syntactic sugar
(0, 3)
```

5.6 Set factories

A *set factory* F is a device for constructing some `Parent` P that models subsets of a big set S. Typically, each such parent is constructed as the subset of S of all elements satisfying a certain collection of constraints *cons*. In such a hierarchy of subsets, one needs an easy and flexible control on how elements are constructed. For example, one may want to construct the elements of P in some subclass of the class of the elements of S. On other occasions, one also often needs P to be a facade parent, whose elements are represented as elements of S (see `FacadeSets`).

The role of a set factory is twofold:

- *Manage a database* of constructors for the different parents $P = F(cons)$ depending on the various kinds of constraints *cons*. Note: currently there is no real support for that. We are gathering use cases before fixing the interface.

- Ensure that the elements $e = P(...)$ created by the different parents follows a consistent policy concerning their *class and parent*.

Basic usage: constructing parents through a factory

The file `sage.structure.set_factories_example` shows an example of a `SetFactory` together with typical implementation. Note that the written code is intentionally kept minimal, many things and in particular several iterators could be written in a more efficient way.

Consider the set S of couples (x, y) with x and y in $I := \{0, 1, 2, 3, 4\}$. We represent an element of S as a 2-elements tuple, wrapped in a class `XYPair` deriving from `ElementWrapper`. You can create a `XYPair` with any `Parent`:

```
sage: from sage.structure.set_factories import *
sage: from sage.structure.set_factories_example import *
sage: p = XYPair(Parent(), (0,1)); p
(0, 1)
```

Now, given $(a, b) \in S$ we want to consider the following subsets of S

$$S_a := \{(x, y) \in S \mid x = a\},$$
$$S^b := \{(x, y) \in S \mid y = b\},$$
$$S_a^b := \{(x, y) \in S \mid x = a, y = b\}.$$

The constraints considered here are admittedly trivial. In a realistic example, there would be much more of them. And for some sets of constraints no good enumeration algorithms would be known.

In Sage, those sets are constructed by passing the constraints to the factory. We first create the set with no constraints at all:

```
sage: XYPairs
Factory for XY pairs
sage: S = XYPairs(); S.list()
[(0, 0), (1, 0), ..., (4, 0), (0, 1), (1, 1), ..., (3, 4), (4, 4)]
sage: S.cardinality()
25
```

Let us construct S_2, S^3 and S_2^3:

```
sage: Sx2 = XYPairs(x=2); Sx2.list()
[(2, 0), (2, 1), (2, 2), (2, 3), (2, 4)]
sage: Sy3 = XYPairs(y=3); Sy3.list()
[(0, 3), (1, 3), (2, 3), (3, 3), (4, 3)]
sage: S23 = XYPairs(x=2, y=3); S23.list()
[(2, 3)]
```

Set factories provide an alternative way to build subsets of an already constructed set: each set constructed by a factory has a method *subset()* which accept new constraints. Sets constructed by the factory or the *subset()* methods are identical:

```
sage: Sx2s = S.subset(x=2); Sx2 is Sx2s
True
sage: Sx2.subset(y=3) is S23
True
```

It is not possible to change an already given constraint:

```
sage: S23.subset(y=5)
Traceback (most recent call last):
...
ValueError: Duplicate value for constraints 'y': was 3 now 5
```

Constructing custom elements: policies

We now come to the point of factories: constructing custom elements. The writer of *XYPairs()* decided that, by default, the parents `Sx2`, `Sy3` and `S23` are facade for parent `S`. This means that each element constructed by those subsets behaves as if they where directly constructed by `S` itself:

```
sage: Sx2.an_element().parent()
AllPairs
sage: el = Sx2.an_element()
sage: el.parent() is S
True
sage: type(el) is S.element_class
True
```

This is not always desirable. The device which decides how to construct an element is called a *policy* (see *SetFactoryPolicy*). Each factory should have a default policy. Here is the policy of *XYPairs()*:

```
sage: XYPairs._default_policy
Set factory policy for <class 'sage.structure.set_factories_example.XYPair'> with␣
↪parent AllPairs[=Factory for XY pairs(())]
```

This means that with the current policy, the parent builds elements with class `XYPair` and parent `AllPairs` which is itself constructed by calling the factory *XYPairs()* with constraints `()`. There is a lot of flexibility to change that. We now illustrate how to make a few different choices.

1 - In a first use case, we want to add some methods to the constructed elements. As illustration, we add here a new method `sum` which returns $x + y$. We therefore create a new class for the elements which inherits from *XYPair*:

```
sage: class NewXYPair(XYPair):
....:     def sum(self):
....:         return sum(self.value)
```

Here is an instance of this class (with a dummy parent):

```
sage: el = NewXYPair(Parent(), (2,3))
sage: el.sum()
5
```

We now want to have subsets generating those new elements while still having a single real parent (the one with no constraint) for each element. The corresponding policy is called *TopMostParentPolicy* . It takes three parameters:

- the factory;
- the parameters for void constraint;
- the class used for elements.

Calling the factory with this policy returns a new set which builds its elements with the new policy:

```
sage: new_policy = TopMostParentPolicy(XYPairs, (), NewXYPair)
sage: NewS = XYPairs(policy=new_policy)
sage: el = NewS.an_element(); el
(0, 0)
sage: el.sum()
0
sage: el.parent() is NewS
True
sage: isinstance(el, NewXYPair)
True
```

Newly constructed subsets inherit the policy:

```
sage: NewS2 = NewS.subset(x=2)
sage: el2 = NewS2.an_element(); el2
(2, 0)
sage: el2.sum()
2
sage: el2.parent() is NewS
True
```

2 - In a second use case, we want the elements to remember which parent created them. The corresponding policy is called *SelfParentPolicy* . It takes only two parameters:

- the factory;
- the class used for elements.

Here is an example:

```
sage: selfpolicy = SelfParentPolicy(XYPairs, NewXYPair)
sage: SelfS = XYPairs(policy=selfpolicy)
sage: el = SelfS.an_element();
sage: el.parent() is SelfS
True
```

Now all subsets are the parent of the elements that they create:

```
sage: SelfS2 = SelfS.subset(x=2)
sage: el2 = SelfS2.an_element()
sage: el2.parent() is NewS
False
```

```
sage: el2.parent() is SelfS2
True
```

3 - Finaly, a common use case is to construct simple python object which are not Sage `sage.structure.Element`. As an example, we show how to build a parent `TupleS` which construct pairs as tuple. The corresponding policy is called *BareFunctionPolicy* . It takes two parameters:

- the factory;

- the function called to construct the elements.

Here is how to do it:

```
sage: cons = lambda t, check: tuple(t) # ignore the check parameter
sage: tuplepolicy = BareFunctionPolicy(XYPairs, cons)
sage: P = XYPairs(x=2, policy=tuplepolicy)
sage: P.list()
[(2, 0), (2, 1), (2, 2), (2, 3), (2, 4)]
sage: el = P.an_element()
sage: type(el)
<... 'tuple'>
```

Here are the currently implemented policies:

- *FacadeParentPolicy* : reuse an existing parent together with its element_class

- *TopMostParentPolicy* : use a parent created by the factory itself and provide a class `Element` for it. In this case, we need to specify the set of constraints which build this parent taking the ownership of all elements and the class which will be used for the `Element` .

- *SelfParentPolicy* : provide systematically `Element` and element_class and ensure that the parent is `self` .

- *BareFunctionPolicy* : instead of calling a class constructor element are passed to a function provided to the policy.

Todo

Generalize *TopMostParentPolicy* to be able to have several topmost parents.

Technicalities: how policies inform parents

Parents built from factories should inherit from *ParentWithSetFactory* . This class provide a few methods related to factories and policies. The `__init__` method of *ParentWithSetFactory* must be provided with the set of constraints and the policy. A parent built from a factory must create elements through a call to the method `_element_constructor_` . The current way in which policies inform parents how to builds their elements is set by a few attributes. So the class must accept attribute adding. The precise details of which attributes are set may be subject to change in the future.

How to write a set factory

See also:

set_factories_example for an example of a factory.

Here are the specifications:

A parent built from a factory should

- *inherit* from *ParentWithSetFactory* . It should accept a `policy` argument and pass it verbatim to the `__init__` method of *ParentWithSetFactory* together with the set of constraints;

- *create its elements* through calls to the method `_element_constructor_`;

- *define a method* *ParentWithSetFactory.check_element* which checks if a built element indeed belongs to it. The method should accept an extra keyword parameter called `check` specifying which level of check should be performed. It will only be called when `bool(check)` evaluates to `True` .

The constructor of the elements of a parent from a factory should:

- receive the parent as first mandatory argument;

- accept an extra optional keyword parameter called `check` which is meant to tell if the input must be checked or not. The precise meaning of `check` is intentionally left vague. The only intent is that if `bool(check)` evaluates to `False` , no check is performed at all.

A factory should

- *define a method* `__call__` which is responsible for calling the appropriate parent constructor given the constraints;

- *define a method* overloading *SetFactory.add_constraints()* which is responsible of computing the union of two sets of constraints;

- *optionally* define a method or an attribute `_default_policy` passed to the *ParentWithSetFactory* if no policy is given to the factory.

Todo

There is currently no support for dealing with sets of constraints. The set factory and the parents must cooperate to consistently handle them. More support, together with a generic mechanism to select the appropriate parent class from the constraints, will be added as soon as we have gathered sufficiently enough use-cases.

AUTHORS:

- Florent Hivert (2011-2012): initial revision

class sage.structure.set_factories. **BareFunctionPolicy** (*factory*, *constructor*)

Bases: *sage.structure.set_factories.SetFactoryPolicy*

Policy where element are contructed using a bare function.

INPUT:

- `factory` – an instance of *SetFactory*

- `contructor` – a function

Given a factory `F` and a function `c` , returns a policy for parent `P` creating element using the function `f` .

EXAMPLES:

```
sage: from sage.structure.set_factories import BareFunctionPolicy
sage: from sage.structure.set_factories_example import XYPairs
sage: cons = lambda t, check: tuple(t) # ignore the check parameter
sage: tuplepolicy = BareFunctionPolicy(XYPairs, cons)
sage: P = XYPairs(x=2, policy=tuplepolicy)
sage: el = P.an_element()
sage: type(el)
<... 'tuple'>
```

element_constructor_attributes (*constraints*)

> Return the element constructor attributes as per *SetFactoryPolicy.element_constructor_attributes()*
>
> .
>
> INPUT:
>
> > •constraints – a bunch of constraints

class sage.structure.set_factories. **FacadeParentPolicy** (*factory*, *parent*)

> Bases: *sage.structure.set_factories.SetFactoryPolicy*

Policy for facade parent.

INPUT:

> •factory – an instance of *SetFactory*
>
> •parent – an instance of *Parent*

Given a factory F and a class E , returns a policy for parent P creating elements as if they were created by parent .

EXAMPLES:

```
sage: from sage.structure.set_factories import SelfParentPolicy,
↪FacadeParentPolicy
sage: from sage.structure.set_factories_example import XYPairs, XYPair
```

We create a custom standard parent P :

```
sage: selfpolicy = SelfParentPolicy(XYPairs, XYPair)
sage: P = XYPairs(x=2, policy=selfpolicy)
sage: pol = FacadeParentPolicy(XYPairs, P)
sage: P2 = XYPairs(x=2, y=3, policy=pol)
sage: el = P2.an_element()
sage: el.parent() is P
True
sage: type(el) is P.element_class
True
```

If parent is itself a facade parent, then transitivity is correctly applied:

```
sage: P =  XYPairs()
sage: P2 = XYPairs(x=2)
sage: P2.category()
Category of facade finite enumerated sets
sage: pol = FacadeParentPolicy(XYPairs, P)
sage: P23 = XYPairs(x=2, y=3, policy=pol)
sage: el = P2.an_element()
sage: el.parent() is P
True
sage: type(el) is P.element_class
True
```

element_constructor_attributes (*constraints*)

> Return the element constructor attributes as per *SetFactoryPolicy.element_constructor_attributes()*
>
> .
>
> INPUT:
>
> > •constraints – a bunch of constraints

class sage.structure.set_factories. **ParentWithSetFactory** (*constraints*, *policy*, *category=None*)

Bases: *sage.structure.parent.Parent*

Abstract class for parent belonging to a set factory.

INPUT:

- constraints – a set of constraints

- policy – the policy for element construction

- category – the category of the parent (default to None)

Depending on the constraints and the policy, initialize the parent in a proper category to set up element construction.

EXAMPLES:

```
sage: from sage.structure.set_factories_example import XYPairs, PairsX_
sage: P = PairsX_(3, XYPairs._default_policy)
sage: P is XYPairs(3)
True
sage: P.category()
Category of facade finite enumerated sets
```

check_element (*x*, *check*)

Check that x verifies the constraints of self .

INPUT:

- x – an instance of self.element_class .

- check – the level of checking to be performed (usually a boolean).

This method may assume that x was properly constructed by self or a possible super-set of self for which self is a facade. It should return nothing if x verifies the constraints and raise a ValueError explaining which constraints x fails otherwise.

The method should accept an extra parameter check specifying which level of check should be performed. It will only be called when bool(check) evaluates to True .

Todo

Should we always call check element and let it decide which check has to be performed ?

EXAMPLES:

```
sage: from sage.structure.set_factories_example import XYPairs
sage: S = XYPairs()
sage: el = S((2,3))
sage: S.check_element(el, True)
sage: XYPairs(x=2).check_element(el, True)
sage: XYPairs(x=3).check_element(el, True)
Traceback (most recent call last):
...
ValueError: Wrong first coordinate
sage: XYPairs(y=4).check_element(el, True)
Traceback (most recent call last):
...
ValueError: Wrong second coordinate
```

constraints ()
>   Return the constraints defining `self`.

>   **Note:** Currently there is no specification on how constraints are passed as arguments.

>   EXAMPLES:

```
sage: from sage.structure.set_factories_example import XYPairs
sage: XYPairs().constraints()
()
sage: XYPairs(x=3).constraints()
(3, None)
sage: XYPairs(y=2).constraints()
(None, 2)
```

facade_policy ()
>   Return the policy for parent facade for `self`.

>   EXAMPLES:

```
sage: from sage.structure.set_factories import SelfParentPolicy
sage: from sage.structure.set_factories_example import XYPairs, XYPair
```

>   We create a custom standard parent `P`:

```
sage: selfpolicy = SelfParentPolicy(XYPairs, XYPair)
sage: P = XYPairs(x=2, policy=selfpolicy)
sage: P.facade_policy()
Set factory policy for facade parent {(2, b) | b in range(5)}
```

>   Now passing `P.facade_policy()` creates parent which are facade for `P`:

```
sage: P3 = XYPairs(x=2, y=3, policy=P.facade_policy())
sage: P3.facade_for() == (P,)
True
sage: el = P3.an_element()
sage: el.parent() is P
True
```

factory ()
>   Return the factory which built `self`.

>   EXAMPLES:

```
sage: from sage.structure.set_factories_example import XYPairs
sage: XYPairs().factory() is XYPairs
True
sage: XYPairs(x=3).factory() is XYPairs
True
```

policy ()
>   Return the policy used when `self` was created.

>   EXAMPLES:

```
sage: from sage.structure.set_factories_example import XYPairs
sage: XYPairs().policy()
Set factory policy for <class 'sage.structure.set_factories_example.XYPair'>␣
→with parent AllPairs[=Factory for XY pairs(())]
```

```
sage: XYPairs(x=3).policy()
Set factory policy for <class 'sage.structure.set_factories_example.XYPair'>␣
→with parent AllPairs[=Factory for XY pairs(())]
```

subset (**args*, ***options*)

> Return a subset of `self` by adding more constraints.

> ---
> **Note:** Currently there is no specification on how constraints are passed as arguments.
> ---

> EXAMPLES:

> ```
> sage: from sage.structure.set_factories_example import XYPairs
> sage: S = XYPairs()
> sage: S3 = S.subset(x=3)
> sage: S3.list()
> [(3, 0), (3, 1), (3, 2), (3, 3), (3, 4)]
> ```

class sage.structure.set_factories. **SelfParentPolicy** (*factory*, *Element*)

> Bases: *sage.structure.set_factories.SetFactoryPolicy*

Policy where each parent is a standard parent.

INPUT:

> •`factory` – an instance of *SetFactory*

> •`Element` – a subclass of *Element*

Given a factory `F` and a class `E`, returns a policy for parent `P` creating elements in class `E` and parent `P` itself.

EXAMPLES:

```
sage: from sage.structure.set_factories import SelfParentPolicy
sage: from sage.structure.set_factories_example import XYPairs, XYPair, Pairs_Y
sage: pol = SelfParentPolicy(XYPairs, XYPair)
sage: S = Pairs_Y(3, pol)
sage: el = S.an_element()
sage: el.parent() is S
True

sage: class Foo(XYPair): pass
sage: pol = SelfParentPolicy(XYPairs, Foo)
sage: S = Pairs_Y(3, pol)
sage: el = S.an_element()
sage: isinstance(el, Foo)
True
```

element_constructor_attributes (*constraints*)

> Return the element constructor attributes as per *SetFactoryPolicy.element_constructor_attributes()*

> INPUT:

> > •`constraints` – a bunch of constraints

class sage.structure.set_factories. **SetFactory**

> Bases: *sage.structure.unique_representation.UniqueRepresentation* , *sage.structure.sage_object.SageObject*

This class is currently just a stub that we will be using to add more structures on factories.

---

add_constraints (*cons*, **args*, ***opts*)
    Add constraints to the set of constraints *cons*.

    Should return a set of constraints.

    Note: Currently there is no specification on how constraints are passed as arguments.

    EXAMPLES:

```
sage: from sage.structure.set_factories_example import XYPairs
sage: XYPairs.add_constraints((3,),((None, 2), {}))
(3, 2)

sage: XYPairs.add_constraints((3,),((None, None), {'y': 2}))
(3, 2)
```

class sage.structure.set_factories. **SetFactoryPolicy** (*factory*)
    Bases: *sage.structure.unique_representation.UniqueRepresentation*   ,
    sage.structure.sage_object.SageObject

    Abstract base class for policies.

    A policy is a device which is passed to a parent inheriting from *ParentWithSetFactory* in order to set-up
    the element construction framework.

    INPUT:

        •factory – a *SetFactory*

    Warning: This class is a base class for policies, one should not try to create instances.

---

    element_constructor_attributes (*constraints*)
        Element constructor attributes.

        INPUT:

            •constraints – a bunch of constraints

        Should return the attributes that are prerequisite for element construction. This is co-
        ordinated with ParentWithSetFactory._element_constructor_() . Currently two
        standard attributes are provided in *facade_element_constructor_attributes()* and
        self_element_constructor_attributes() . You should return the one needed depending
        on the given constraints.

        EXAMPLES:

```
sage: from sage.structure.set_factories_example import XYPairs, XYPair
sage: pol = XYPairs._default_policy
sage: pol.element_constructor_attributes(())
{'Element': <class 'sage.structure.set_factories_example.XYPair'>,
 '_parent_for': 'self'}
sage: pol.element_constructor_attributes((1))
{'_facade_for': AllPairs,
 '_parent_for': AllPairs,
 'element_class': <class 'sage.structure.set_factories_example.AllPairs_with_
→category.element_class'>}
```

facade_element_constructor_attributes (*parent*)
    Element Constructor Attributes for facade parent.

    The list of attributes which must be set during the init of a facade parent with factory.

    INPUT:

        •`parent` – the actual parent for the elements

    EXAMPLES:

```
sage: from sage.structure.set_factories_example import XYPairs, XYPair
sage: pol = XYPairs._default_policy
sage: pol.facade_element_constructor_attributes(XYPairs())
{'_facade_for': AllPairs,
 '_parent_for': AllPairs,
 'element_class': <class 'sage.structure.set_factories_example.AllPairs_with_
→category.element_class'>}
```

factory ( )
    Return the factory for `self`.

    EXAMPLES:

```
sage: from sage.structure.set_factories import SetFactoryPolicy,␣
→SelfParentPolicy
sage: from sage.structure.set_factories_example import XYPairs, XYPair
sage: XYPairs._default_policy.factory()
Factory for XY pairs
sage: XYPairs._default_policy.factory() is XYPairs
True
```

self_element_constructor_attributes (*Element*)
    Element Constructor Attributes for non facade parent.

    The list of attributes which must be set during the init of a non facade parent with factory.

    INPUT:

        •`Element` – the class used for the elements

    EXAMPLES:

```
sage: from sage.structure.set_factories_example import XYPairs, XYPair
sage: pol = XYPairs._default_policy
sage: pol.self_element_constructor_attributes(XYPair)
{'Element': <class 'sage.structure.set_factories_example.XYPair'>,
 '_parent_for': 'self'}
```

class sage.structure.set_factories. **TopMostParentPolicy** (*factory*, *top_constraints*, *Element*)

    Bases: *sage.structure.set_factories.SetFactoryPolicy*

Policy where the parent of the elements is the topmost parent.

INPUT:

    •`factory` – an instance of *SetFactory*

    •`top_constraints` – the empty set of constraints.

    •`Element` – a subclass of *Element*

Given a factory `F` and a class `E`, returns a policy for parent `P` creating element in class `E` and parent factory(*top_constraints,policy).

EXAMPLES:

```
sage: from sage.structure.set_factories_example import XYPairs, XYPair
sage: P = XYPairs(); P.policy()
Set factory policy for <class 'sage.structure.set_factories_example.XYPair'> with
↪parent AllPairs[=Factory for XY pairs(())]
```

element_constructor_attributes (*constraints*)
> Return the element constructor attributes as per *SetFactoryPolicy.element_constructor_attributes()*
> .

> INPUT:

> • `constraints` – a bunch of constraints

5.7 An example of set factory

The goal of this module is to exemplify the use of set factories. Note that the code is intentionally kept minimal; many things and in particular several iterators could be written in a more efficient way.

See also:

set_factories for an introduction to set factories, their specifications, and examples of their use and implementation based on this module.

We describe here a factory used to construct the set S of couples (x, y) with x and y in $I := \{0, 1, 2, 3, 4\}$, together with the following subsets, where $(a, b) \in S$

$$S_a := \{(x, y) \in S \mid x = a\},$$
$$S^b := \{(x, y) \in S \mid y = b\},$$
$$S_a^b := \{(x, y) \in S \mid x = a, y = b\}.$$

class sage.structure.set_factories_example. **AllPairs** (*policy*)
> Bases: *sage.structure.set_factories.ParentWithSetFactory* ,
> *sage.sets.disjoint_union_enumerated_sets.DisjointUnionEnumeratedSets*

> This parent shows how one can use set factories together with *DisjointUnionEnumeratedSets* .

> It is constructed as the disjoint union (*DisjointUnionEnumeratedSets*) of *Pairs_Y* parents:

$$S := \bigcup_{i=0,1,\dots,4} S^y$$

> **Warning:** When writing a parent `P` as a disjoint union of a family of parents `P_i`, the parents `P_i` must be constructed as facade parents for `P`. As a consequence, it should be passed `P.facade_policy()` as policy argument. See the source code of *pairs_y()* for an example.

> **check_element** (*el*, *check*)

pairs_y (*letter*)

> Construct the parent for the disjoint union
>
> Construct a parent in `Pairs_Y` as a facade parent for `self`.
>
> This is an internal function which should be hidden from the user (typically under the name `_pairs_y`. We put it here for documentation.

class `sage.structure.set_factories_example.` **PairsX_** (*x*, *policy*)

> Bases: `sage.structure.set_factories.ParentWithSetFactory` , `sage.structure.unique_representation.UniqueRepresentation`
>
> The set of pairs $(x, 0), (x, 1), ..., (x, 4)$.
>
> **an_element** ( )
>
> **check_element** (*el*, *check*)

class `sage.structure.set_factories_example.` **Pairs_Y** (*y*, *policy*)

> Bases: `sage.structure.set_factories.ParentWithSetFactory` , `sage.sets.disjoint_union_enumerated_sets.DisjointUnionEnumeratedSets`
>
> The set of pairs $(0, y), (1, y), ..., (4, y)$.
>
> It is constructed as the disjoint union (`DisjointUnionEnumeratedSets`) of `SingletonPair` parents:
>
> $$S^y := \bigcup_{i=0,1,...,4} S_i^y$$
>
> See also:
>
> `AllPairs` for how to properly construct `DisjointUnionEnumeratedSets` using `ParentWithSetFactory`.
>
> **an_element** ( )
>
> **check_element** (*el*, *check*)
>
> **single_pair** (*letter*)
>
> > Construct the singleton pair parent
> >
> > Construct a singleton pair for (`self.y, letter`) as a facade parent for `self`.
> >
> > See also:
> >
> > `AllPairs` for how to properly construct `DisjointUnionEnumeratedSets` using `ParentWithSetFactory`.

class `sage.structure.set_factories_example.` **SingletonPair** (*x*, *y*, *policy*)

> Bases: `sage.structure.set_factories.ParentWithSetFactory` , `sage.structure.unique_representation.UniqueRepresentation`
>
> **check_element** (*el*, *check*)

class `sage.structure.set_factories_example.` **XYPair** (*parent*, *value*, *check=True*)

> Bases: `sage.structure.element_wrapper.ElementWrapper`
>
> A class for Elements (x, y) with x and y in $\{0, 1, 2, 3, 4\}$.
>
> EXAMPLES:

```
sage: from sage.structure.set_factories_example import XYPair
sage: p = XYPair(Parent(), (0,1)); p
(0, 1)
sage: p = XYPair(Parent(), (0,8))
```

```
Traceback (most recent call last):
...
ValueError: numbers must be in range(5)
```

sage.structure.set_factories_example.**XYPairs** (*x=None*, *y=None*, *policy=None*)

Construct the subset from constraints.

Consider the set S of couples (x, y) with x and y in $I := \{0, 1, 2, 3, 4\}$. Returns the subsets of element of S satisfying some constraints.

INPUT:

- •x=a – where a is an integer (default to None).

- •y=b – where b is an integer (default to None).

- •policy – the policy passed to the created set.

See also:

set_factories.SetFactoryPolicy

EXAMPLES:

Let us first create the set factory:

```
sage: from sage.structure.set_factories_example import XYPairsFactory
sage: XYPairs = XYPairsFactory()
```

One can then use the set factory to construct a set:

```
sage: P = XYPairs(); P.list()
[(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (0, 1), (1, 1), (2, 1), (3, 1), (4, 1),␣
↪(0, 2), (1, 2), (2, 2), (3, 2), (4, 2), (0, 3), (1, 3), (2, 3), (3, 3), (4, 3),␣
↪(0, 4), (1, 4), (2, 4), (3, 4), (4, 4)]
```

Note: This function is actually the __call__ method of *XYPairsFactory* .

---

class sage.structure.set_factories_example.**XYPairsFactory**

Bases: *sage.structure.set_factories.SetFactory*

An example of set factory, for sets of pairs of integers.

See also:

set_factories for an introduction to set factories.

add_constraints (*cons*, *args_opts*)

Add constraints to the set cons as per *SetFactory.add_constraints* .

This is a crude implementation for the sake of the demonstration which should not be taken as an example.

EXAMPLES:

```
sage: from sage.structure.set_factories_example import XYPairs
sage: XYPairs.add_constraints((3,None), ((2,), {}))
Traceback (most recent call last):
...
ValueError: Duplicate value for constraints 'x': was 3 now 2
sage: XYPairs.add_constraints((), ((2,), {}))
(2, None)
```

```
sage: XYPairs.add_constraints((), ((2,), {'y':3}))
(2, 3)
```

USE OF HEURISTIC AND PROBABILISTIC ALGORITHMS

6.1 Global proof preferences

class sage.structure.proof.proof. **WithProof** (*subsystem*, *t*)

Use WithProof to temparily set the value of one of the proof systems for a block of code, with a guarantee that it will be set back to how it was before after the block is done, even if there is an error.

EXAMPLES:

```
sage: proof.arithmetic(True)
sage: with proof.WithProof('arithmetic',False):    # this would hang "forever" if
→attempted with proof=True
....:        print((10^1000 + 453).is_prime())
....:        print(1/0)
Traceback (most recent call last):
...
ZeroDivisionError: rational division by zero
sage: proof.arithmetic()
True
```

sage.structure.proof.proof. **get_flag** (*t=None*, *subsystem=None*)

Used for easily determining the correct proof flag to use.

EXAMPLES:

```
sage: from sage.structure.proof.proof import get_flag
sage: get_flag(False)
False
sage: get_flag(True)
True
sage: get_flag()
True
sage: proof.all(False)
sage: get_flag()
False
```

6.2 Whether or not computations are provably correct by default

UTILITIES

7.1 Unique Representation

Abstract classes for cached and unique representation behavior.

See also:

sage.structure.factory.UniqueFactory

AUTHORS:

- Nicolas M. Thiery (2008): Original version.

- Simon A. King (2013-02): Separate cached and unique representation.

- Simon A. King (2013-08): Extended documentation.

7.1.1 What is a cached representation?

Instances of a class have a *cached representation behavior* when several instances constructed with the same arguments share the same memory representation. For example, calling twice:

```
sage: G = SymmetricGroup(6)
sage: H = SymmetricGroup(6)
```

to create the symmetric group on six elements gives back the same object:

```
sage: G is H
True
```

This is a standard design pattern. Besides saving memory, it allows for sharing cached data (say representation theoretical information about a group). And of course a look-up in the cache is faster than the creation of a new object.

Implementing a cached representation

Sage provides two standard ways to create a cached representation: *CachedRepresentation* and *UniqueFactory*. Note that, in spite of its name, *UniqueFactory* does not ensure *unique* representation behaviour, which will be explained below.

Using `CachedRepresentation`

It is often very easy to use *CachedRepresentation* : One simply writes a Python class and adds *CachedRepresentation* to the list of base classes. If one does so, then the arguments used to create an instance of this class will by default also be used as keys for the cache:

```
sage: from sage.structure.unique_representation import CachedRepresentation
sage: class C(CachedRepresentation):
....:     def __init__(self, a, b=0):
....:         self.a = a
....:         self.b = b
....:     def __repr__(self):
....:         return "C(%s, %s)"%(self.a, self.b)
....:
sage: a = C(1)
sage: a is C(1)
True
```

In addition, pickling just works, provided that Python is able to look up the class. Hence, in the following two lines, we explicitly put the class into the __main__ module. This is needed in doctests, but not in an interactive session:

```
sage: import __main__
sage: __main__.C = C
sage: loads(dumps(a)) is a
True
```

Often, this very easy approach is sufficient for applications. However, there are some pitfalls. Since the arguments are used for caching, all arguments must be hashable, i.e., must be valid as dictionary keys:

```
sage: C((1,2))
C((1, 2), 0)
sage: C([1,2])
Traceback (most recent call last):
...
TypeError: unhashable type: 'list'
```

In addition, equivalent ways of providing the arguments are *not* automatically normalised when forming the cache key, and hence different but equivalent arguments may yield distinct instances:

```
sage: C(1) is C(1,0)
False
sage: C(1) is C(a=1)
False
sage: repr(C(1)) == repr(C(a=1))
True
```

It should also be noted that the arguments are compared by equality, not by identity. This is often desired, but can imply subtle problems. For example, since `C(1)` already is in the cache, and since the unit elements in different finite fields are all equal to the integer one, we find:

```
sage: GF(5)(1) == 1 == GF(3)(1)
True
sage: C(1) is C(GF(3)(1)) is C(GF(5)(1))
True
```

But `C(2)` is not in the cache, and the number two is not equal in different finite fields (i. e., `GF(5)(2) == GF(3)(2)` returns as `False`), even though it is equal to the number two in the ring of integers (`GF(5)(2)`

`== 2 == GF(3)(2)` returns as `True`; equality is not transitive when comparing elements of *distinct* algebraic structures!!). Hence, we have:

```
sage: GF(5)(2) == GF(3)(2)
False
sage: C(GF(3)(2)) is C(GF(5)(2))
False
```

Normalising the arguments

CachedRepresentation uses the metaclass `ClasscallMetaclass`. Its `__classcall__` method is a `WeakCachedFunction`. This function creates an instance of the given class using the given arguments, unless it finds the result in the cache. This has the following implications:

- The arguments must be valid dictionary keys (i.e., they must be hashable; see above).

- It is a weak cache, hence, if the user does not keep a reference to the resulting instance, then it may be removed from the cache during garbage collection.

- It is possible to preprocess the input arguments by implementing a `__classcall__` or a `__classcall_private__` method, but in order to benefit from caching, `CachedRepresentation.__classcall__()` should at some point be called.

Note: For technical reasons, it is needed that `__classcall__` respectively `__classcall_private__` are "static methods", i.e., they are callable objects that do not bind to an instance or class. For example, a `cached_function` can be used here, because it is callable, but does not bind to an instance or class, because it has no `__get__()` method. A usual Python function, however, has a `__get__()` method and would thus under normal circumstances bind to an instance or class, and thus the instance or class would be passed to the function as the first argument. To prevent a callable object from being bound to the instance or class, one can prepend the `@staticmethod` decorator to the definition; see `staticmethod`.

For more on Python's `__get__()` method, see: http://docs.python.org/2/howto/descriptor.html

---

Warning: If there is preprocessing, then the preprocessed arguments passed to `CachedRepresentation.__classcall__()` must be invariant under the preprocessing. That is to say, preprocessing the input arguments twice must have the same effect as preprocessing the input arguments only once. That is to say, the preprocessing must be idempotent.

The reason for this warning lies in the way pickling is implemented. If the preprocessed arguments are passed to `CachedRepresentation.__classcall__()`, then the resulting instance will store the *preprocessed* arguments in some attribute, and will use them for pickling. If the pickle is unpickled, then preprocessing is applied to the preprocessed arguments—and this second round of preprocessing must not change the arguments further, since otherwise a different instance would be created.

We illustrate the warning by an example. Imagine that one has instances that are created with an integer-valued argument, but only depend on the *square* of the argument. It would be a mistake to square the given argument during preprocessing:

```
sage: class WrongUsage(CachedRepresentation):
....:     @staticmethod
....:     def __classcall__(cls, n):
....:         return super(WrongUsage,cls).__classcall__(cls, n^2)
....:     def __init__(self, n):
```

```
....:            self.n = n
....:        def __repr__(self):
....:            return "Something(%d)"%self.n
....:
sage: import __main__
sage: __main__.WrongUsage = WrongUsage # This is only needed in doctests
sage: w = WrongUsage(3); w
Something(9)
sage: w._reduction
(<class '__main__.WrongUsage'>, (9,), {})
```

Indeed, the reduction data are obtained from the preprocessed argument. By consequence, if the resulting instance is pickled and unpickled, the argument gets squared *again*:

```
sage: loads(dumps(w))
Something(81)
```

Instead, the preprocessing should only take the absolute value of the given argument, while the squaring should happen inside of the __init__ method, where it won't mess with the cache:

```
sage: class BetterUsage(CachedRepresentation):
....:        @staticmethod
....:        def __classcall__(cls, n):
....:            return super(BetterUsage, cls).__classcall__(cls, abs(n))
....:        def __init__(self, n):
....:            self.n = n^2
....:        def __repr__(self):
....:            return "SomethingElse(%d)"%self.n
....:
sage: __main__.BetterUsage = BetterUsage # This is only needed in doctests
sage: b = BetterUsage(3); b
SomethingElse(9)
sage: loads(dumps(b)) is b
True
sage: b is BetterUsage(-3)
True
```

In our next example, we create a cached representation class C that returns an instance of a sub-class C1 or C2 depending on the given arguments. This is implemented in a static __classcall_private__ method of C, letting it choose the sub-class according to the given arguments. Since a __classcall_private__ method will be ignored on sub-classes, the caching of *CachedRepresentation* is available to both C1 and C2. But for illustration, we overload the static __classcall__ method on C2, doing some argument preprocessing. We also create a sub-class C2b of C2, demonstrating that the __classcall__ method is used on the sub-class (in contrast to a __classcall_private__ method!).

```
sage: class C(CachedRepresentation):
....:        @staticmethod
....:        def __classcall_private__(cls, n, implementation=0):
....:            if not implementation:
....:                return C.__classcall__(cls, n)
....:            if implementation==1:
....:                return C1(n)
....:            if implementation>1:
....:                return C2(n,implementation)
....:        def __init__(self, n):
....:            self.n = n
....:        def __repr__(self):
```

```
....:           return "C(%d, 0)"%self.n
....:
sage: class C1(C):
....:       def __repr__(self):
....:           return "C1(%d)"%self.n
....:
sage: class C2(C):
....:       @staticmethod
....:       def __classcall__(cls, n, implementation=0):
....:           if implementation:
....:               return super(C2, cls).__classcall__(cls, (n,)*implementation)
....:           return super(C2, cls).__classcall__(cls, n)
....:       def __init__(self, t):
....:           self.t = t
....:       def __repr__(self):
....:           return "C2(%s)"%repr(self.t)
....:
sage: class C2b(C2):
....:       def __repr__(self):
....:           return "C2b(%s)"%repr(self.t)
....:
sage: __main__.C2 = C2        # not needed in an interactive session
sage: __main__.C2b = C2b
```

In the above example, `C` drops the argument `implementation` if it evaluates to `False`, and since the cached `__classcall__` is called in this case, we have:

```
sage: C(1)
C(1, 0)
sage: C(1) is C(1,0)
True
sage: C(1) is C(1,0) is C(1,None) is C(1,[])
True
```

(Note that we were able to bypass the issue of arguments having to be hashable by catching the empty list `[]` during preprocessing in the `__classcall_private__` method. Similarly, unhashable arguments can be made hashable – e. g., lists normalized to tuples – in the `__classcall_private__` method before they are further delegated to `__classcall__`. See `TCrystal` for an example.)

If we call `C1` directly or if we provide `implementation=1` to `C`, we obtain an instance of `C1`. Since it uses the `__classcall__` method inherited from *CachedRepresentation*, the resulting instances are cached:

```
sage: C1(2)
C1(2)
sage: C(2, implementation=1)
C1(2)
sage: C(2, implementation=1) is C1(2)
True
```

The class `C2` preprocesses the input arguments. Instances can, again, be obtained directly or by calling `C`:

```
sage: C(1, implementation=3)
C2((1, 1, 1))
sage: C(1, implementation=3) is C2(1,3)
True
```

The argument preprocessing of `C2` is inherited by `C2b`, since `__classcall__` and not `__classcall_private__` is used. Pickling works, since the preprocessing of arguments is idempotent:

```
sage: c2b = C2b(2,3); c2b
C2b((2, 2, 2))
sage: loads(dumps(c2b)) is c2b
True
```

Using `UniqueFactory`

For creating a cached representation using a factory, one has to

- create a class *separately* from the factory. This class **must** inherit from `object`. Its instances **must** allow attribute assignment.

- write a method `create_key` (or `create_key_and_extra_args`) that creates the cache key from the given arguments.

- write a method `create_object` that creates an instance of the class from a given cache key.

- create an instance of the factory with a name that allows to conclude where it is defined.

An example:

```
sage: class C(object):
....:     def __init__(self, t):
....:         self.t = t
....:     def __repr__(self):
....:         return "C%s"%repr(self.t)
....:
sage: from sage.structure.factory import UniqueFactory
sage: class MyFactory(UniqueFactory):
....:     def create_key(self, n, m=None):
....:         if isinstance(n, (tuple,list)) and m is None:
....:             return tuple(n)
....:         return (n,)*m
....:     def create_object(self, version, key, **extra_args):
....:         # We ignore version and extra_args
....:         return C(key)
....:
```

Now, we define an instance of the factory, stating that it can be found under the name `"F"` in the __main__ module. By consequence, pickling works:

```
sage: F = MyFactory("__main__.F")
sage: __main__.F = F                    # not needed in an interactive session
sage: loads(dumps(F)) is F
True
```

We can now create *cached* instances of `C` by calling the factory. The cache only takes into account the key computed with the method `create_key` that we provided. Hence, different given arguments may result in the same instance. Note that, again, the cache is weak, hence, the instance might be removed from the cache during garbage collection, unless an external reference is preserved.

```
sage: a = F(1, 2); a
C(1, 1)
sage: a is F((1,1))
True
```

If the class of the returned instances is a sub-class of `object`, and **if** the resulting instance allows attribute assignment, then pickling of the resulting instances is automatically provided for, and respects the cache.

```
sage: loads(dumps(a)) is a
True
```

This is because an attribute is stored that explains how the instance was created:

```
sage: a._factory_data
(<class '__main__.MyFactory'>, (...), (1, 1), {})
```

Note: If a class is used that does not inherit from `object` then unique pickling is *not* provided.

Caching is only available if the factory is called. If an instance of the class is directly created, then the cache is not used:

```
sage: C((1,1))
C(1, 1)
sage: C((1,1)) is a
False
```

Comparing the two ways of implementing a cached representation

In this sub-section, we discuss advantages and disadvantages of the two ways of implementing a cached representation, depending on the type of application.

Simplicity and transparency

In many cases, turning a class into a cached representation requires nothing more than adding `CachedRepresentation` to the list of base classes of this class. This is, of course, a very easy and convenient way. Writing a factory would involve a lot more work.

If preprocessing of the arguments is needed, then we have seen how to do this by a `__classcall_private__` or `__classcall__` method. But these are double underscore methods and hence, for example, invisible in the automatically created reference manual. Moreover, preprocessing *and* caching are implemented in the same method, which might be confusing. In a unique factory, these two tasks are cleanly implemented in two separate methods. With a factory, it is possible to create the resulting instance by arguments that are different from the key used for caching. This is significantly restricted with CachedRepresentation due to the requirement that argument preprocessing be idempotent.

Hence, if advanced preprocessing is needed, then `UniqueFactory` might be easier and more transparent to use than `CachedRepresentation`.

Class inheritance

Using `CachedRepresentation` has the advantage that one has a class and creates cached instances of this class by the usual Python syntax:

```
sage: G = SymmetricGroup(6)
sage: issubclass(SymmetricGroup, sage.structure.unique_representation.
→CachedRepresentation)
True
```

```
sage: isinstance(G, SymmetricGroup)
True
```

In contrast, a factory is just a callable object that returns something that has absolutely nothing to do with the factory, and may in fact return instances of quite different classes:

```
sage: isinstance(GF, sage.structure.factory.UniqueFactory)
True
sage: K5 = GF(5)
sage: type(K5)
<class 'sage.rings.finite_rings.finite_field_prime_modn.FiniteField_prime_modn_with_
↪category'>
sage: K25 = GF(25, 'x')
sage: type(K25)
<class 'sage.rings.finite_rings.finite_field_givaro.FiniteField_givaro_with_category'>
sage: Kp = GF(next_prime_power(1000000)^2, 'x')
sage: type(Kp)
<class 'sage.rings.finite_rings.finite_field_pari_ffelt.FiniteField_pari_ffelt_with_
↪category'>
```

This can be confusing to the user. Namely, the user might determine the class of an instance and try to create further instances by calling the class rather than the factory—which is a mistake since it works around the cache (and also since the class might be more restrictive than the factory – i. e., the type of `K5` in the above doctest cannot be called on a prime power which is not a prime). This mistake can more easily be avoided by using *CachedRepresentation*.

We have seen above that one can easily create new cached-representation classes by subclassing an existing cached-representation class, even making use of an existing argument preprocess. This would be much more complicated with a factory. Namely, one would need to rewrite old factories making them aware of the new classes, and/or write new factories for the new classes.

Python versus extension classes

CachedRepresentation uses a metaclass, namely `ClasscallMetaclass`. Hence, it can currently not be a Cython extension class. Moreover, it is supposed to be used by providing it as a base class. But in typical applications, one also has another base class, say, *Parent*. Hence, one would like to create a class with at least two base classes, which is currently impossible in Cython extension classes.

In other words, when using *CachedRepresentation*, one must work with Python classes. These can be defined in Cython code (`.pyx` files) and can thus benefit from Cython's speed inside of their methods, but they must not be `cdef class` and can thus not use `cdef` attributes or methods.

Such restrictions do not exist when using a factory. However, if attribute assignment does not work, then the automatic pickling provided by *UniqueFactory* will not be available.

7.1.2 What is a unique representation?

Instances of a class have a *unique instance behavior* when instances of this class evaluate equal if and only if they are identical. Sage provides the base class `WithEqualityById`, which provides comparison by identity and a hash that is determined by the memory address of the instance. Both the equality test and the hash are implemented in Cython and are very fast, even when one has a Python class inheriting from `WithEqualityById`.

In many applications, one wants to combine unique instance and cached representation behaviour. This is called *unique representation* behaviour. We have seen above that symmetric groups have a *cached* representation behaviour.

However, they do not show the *unique* representation behaviour, since they are equal to groups created in a totally different way, namely to subgroups:

```
sage: G = SymmetricGroup(6)
sage: G3 = G.subgroup([G((1,2,3,4,5,6)),G((1,2))])
sage: G is G3
False
sage: type(G) == type(G3)
False
sage: G == G3
True
```

The unique representation behaviour can conveniently be implemented with a class that inherits from *UniqueRepresentation* : By adding *UniqueRepresentation* to the base classes, the class will simultaneously inherit from *CachedRepresentation* and from WithEqualityById .

For example, a symmetric function algebra is uniquely determined by the base ring. Thus, it is reasonable to use *UniqueRepresentation* in this case:

```
sage: isinstance(SymmetricFunctions(CC), SymmetricFunctions)
True
sage: issubclass(SymmetricFunctions, UniqueRepresentation)
True
```

UniqueRepresentation differs from *CachedRepresentation* only by adding WithEqualityById as a base class. Hence, the above examples of argument preprocessing work for *UniqueRepresentation* as well.

Note that a cached representation created with *UniqueFactory* does *not* automatically provide unique representation behaviour, in spite of its name! Hence, for unique representation behaviour, one has to implement hash and equality test accordingly, for example by inheriting from WithEqualityById .

class sage.structure.unique_representation. **CachedRepresentation**

    Bases: object

    Classes derived from CachedRepresentation inherit a weak cache for their instances.

    Note: If this class is used as a base class, then instances are (weakly) cached, according to the arguments used to create the instance. Pickling is provided, of course by using the cache.

    Note: Using this class, one can have arbitrary hash and comparison. Hence, *unique* representation behaviour is *not* provided.

    See also:

    UniqueRepresentation , *unique_representation*

    EXAMPLES:

    Providing a class with a weak cache for the instances is easy: Just inherit from *CachedRepresentation* :

```
sage: from sage.structure.unique_representation import CachedRepresentation
sage: class MyClass(CachedRepresentation):
....:     # all the rest as usual
....:     pass
```

    We start with a simple class whose constructor takes a single value as argument (TODO: find a more meaningful example):

```
sage: class MyClass(CachedRepresentation):
....:        def __init__(self, value):
....:            self.value = value
....:        def __cmp__(self, other):
....:            c = cmp(type(self),type(other))
....:            if c: return c
....:            return cmp(self.value, other.value)
```

Two coexisting instances of `MyClass` created with the same argument data are guaranteed to share the same identity. Since trac ticket #12215, this is only the case if there is some strong reference to the returned instance, since otherwise it may be garbage collected:

```
sage: x = MyClass(1)
sage: y = MyClass(1)
sage: x is y                    # There is a strong reference
True
sage: z = MyClass(2)
sage: x is z
False
```

In particular, modifying any one of them modifies the other (reference effect):

```
sage: x.value = 3
sage: x.value, y.value
(3, 3)
sage: y.value = 1
sage: x.value, y.value
(1, 1)
```

The arguments can consist of any combination of positional or keyword arguments, as taken by a usual `__init__` function. However, all values passed in should be hashable:

```
sage: MyClass(value = [1,2,3])
Traceback (most recent call last):
...
TypeError: unhashable type: 'list'
```

Argument preprocessing

Sometimes, one wants to do some preprocessing on the arguments, to put them in some canonical form. The following example illustrates how to achieve this; it takes as argument any iterable, and canonicalizes it into a tuple (which is hashable!):

```
sage: class MyClass2(CachedRepresentation):
....:        @staticmethod
....:        def __classcall__(cls, iterable):
....:            t = tuple(iterable)
....:            return super(MyClass2, cls).__classcall__(cls, t)
....:
....:        def __init__(self, value):
....:            self.value = value
....:
sage: x = MyClass2([1,2,3])
sage: y = MyClass2(tuple([1,2,3]))
sage: z = MyClass2(i for i in [1,2,3])
sage: x.value
```

```
(1, 2, 3)
sage: x is y, y is z
(True, True)
```

A similar situation arises when the constructor accepts default values for some of its parameters. Alas, the obvious implementation does not work:

```
sage: class MyClass3(CachedRepresentation):
....:     def __init__(self, value = 3):
....:         self.value = value
....:
sage: MyClass3(3) is MyClass3()
False
```

Instead, one should do:

```
sage: class MyClass3(UniqueRepresentation):
....:     @staticmethod
....:     def __classcall__(cls, value = 3):
....:         return super(MyClass3, cls).__classcall__(cls, value)
....:
....:     def __init__(self, value):
....:         self.value = value
....:
sage: MyClass3(3) is MyClass3()
True
```

A bit of explanation is in order. First, the call `MyClass2([1,2,3])` triggers a call to `MyClass2.__classcall__(MyClass2,[1,2,3])`. This is an extension of the standard Python behavior, needed by *CachedRepresentation*, and implemented by the `ClasscallMetaclass`. Then, `MyClass2.__classcall__` does the desired transformations on the arguments. Finally, it uses `super` to call the default implementation of `__classcall__` provided by *CachedRepresentation*. This one in turn handles the caching and, if needed, constructs and initializes a new object in the class using `__new__` and `__init__` as usual.

Constraints:

• `__classcall__()` is a staticmethod (like, implicitly, `__new__`)

• the preprocessing on the arguments should be idempotent. That is, if `MyClass2.__classcall__(<arguments>)` calls `CachedRepresentation.__classcall__(<preproce` , then `MyClass2.__classcall__(<preprocessed_arguments>)` should also result in a call to `CachedRepresentation.__classcall__(<preprocessed_arguments>)`.

• `MyClass2.__classcall__` should return the result of `CachedRepresentation.__classcall__()` without modifying it.

Other than that `MyClass2.__classcall__` may play any tricks, like acting as a factory and returning objects from other classes.

> **Warning:** It is possible, but strongly discouraged, to let the `__classcall__` method of a class `C` return objects that are not instances of `C`. Of course, instances of a *subclass* of `C` are fine. Compare the examples in *unique_representation*.

We illustrate what is meant by an "idempotent" preprocessing. Imagine that one has instances that are created with an integer-valued argument, but only depend on the *square* of the argument. It would be a mistake to square the given argument during preprocessing:

```
sage: class WrongUsage(CachedRepresentation):
....:     @staticmethod
....:     def __classcall__(cls, n):
....:         return super(WrongUsage,cls).__classcall__(cls, n^2)
....:     def __init__(self, n):
....:         self.n = n
....:     def __repr__(self):
....:         return "Something(%d)"%self.n
....:
sage: import __main__
sage: __main__.WrongUsage = WrongUsage # This is only needed in doctests
sage: w = WrongUsage(3); w
Something(9)
sage: w._reduction
(<class '__main__.WrongUsage'>, (9,), {})
```

Indeed, the reduction data are obtained from the preprocessed arguments. By consequence, if the resulting instance is pickled and unpickled, the argument gets squared *again*:

```
sage: loads(dumps(w))
Something(81)
```

Instead, the preprocessing should only take the absolute value of the given argument, while the squaring should happen inside of the __init__ method, where it won't mess with the cache:

```
sage: class BetterUsage(CachedRepresentation):
....:     @staticmethod
....:     def __classcall__(cls, n):
....:         return super(BetterUsage, cls).__classcall__(cls, abs(n))
....:     def __init__(self, n):
....:         self.n = n^2
....:     def __repr__(self):
....:         return "SomethingElse(%d)"%self.n
....:
sage: __main__.BetterUsage = BetterUsage # This is only needed in doctests
sage: b = BetterUsage(3); b
SomethingElse(9)
sage: loads(dumps(b)) is b
True
sage: b is BetterUsage(-3)
True
```

Cached representation and mutability

CachedRepresentation is primarily intended for implementing objects which are (at least semantically) immutable. This is in particular assumed by the default implementations of copy and deepcopy :

```
sage: copy(x) is x
True
sage: from copy import deepcopy
sage: deepcopy(x) is x
True
```

However, in contrast to *UniqueRepresentation*, using *CachedRepresentation* allows for a comparison that is not by identity:

```
sage: t = MyClass(3)
sage: z = MyClass(2)
sage: t.value = 2
```

Now `t` and `z` are non-identical, but equal:

```
sage: t.value == z.value
True
sage: t == z
True
sage: t is z
False
```

More on cached representation and identity

CachedRepresentation is implemented by means of a cache. This cache uses weak references. Hence, when all other references to, say, `MyClass(1)` have been deleted, the instance is actually deleted from memory. A later call to `MyClass(1)` reconstructs the instance from scratch.

```
sage: class SomeClass(UniqueRepresentation):
....:     def __init__(self, i):
....:         print("creating new instance for argument %s" % i)
....:         self.i = i
....:     def __del__(self):
....:         print("deleting instance for argument %s" % self.i)
....:
sage: O = SomeClass(1)
creating new instance for argument 1
sage: O is SomeClass(1)
True
sage: O is SomeClass(2)
creating new instance for argument 2
deleting instance for argument 2
False
sage: del O
deleting instance for argument 1
sage: O = SomeClass(1)
creating new instance for argument 1
sage: del O
deleting instance for argument 1
```

Cached representation and pickling

The default Python pickling implementation (by reconstructing an object from its class and dictionary, see "The pickle protocol" in the Python Library Reference) does not preserve cached representation, as Python has no chance to know whether and where the same object already exists.

CachedRepresentation tries to ensure appropriate pickling by implementing a `__reduce__` method returning the arguments passed to the constructor:

```
sage: import __main__              # Fake MyClass being defined in a python module
sage: __main__.MyClass = MyClass
sage: x = MyClass(1)
```

```
sage: loads(dumps(x)) is x
True
```

CachedRepresentation uses the __reduce__ pickle protocol rather than __getnewargs__ because the latter does not handle keyword arguments:

```
sage: x = MyClass(value = 1)
sage: x.__reduce__()
(<function unreduce at ...>, (<class '__main__.MyClass'>, (), {'value': 1}))
sage: x is loads(dumps(x))
True
```

Note: The default implementation of __reduce__ in *CachedRepresentation* requires to store the constructor's arguments in the instance dictionary upon construction:

```
sage: x.__dict__
{'_reduction': (<class '__main__.MyClass'>, (), {'value': 1}), 'value': 1}
```

It is often easy in a derived subclass to reconstruct the constructor's arguments from the instance data structure. When this is the case, __reduce__ should be overridden; automagically the arguments won't be stored anymore:

```
sage: class MyClass3(UniqueRepresentation):
....:     def __init__(self, value):
....:         self.value = value
....:
....:     def __reduce__(self):
....:         return (MyClass3, (self.value,))
....:
sage: import __main__; __main__.MyClass3 = MyClass3  # Fake MyClass3 being␣
→defined in a python module
sage: x = MyClass3(1)
sage: loads(dumps(x)) is x
True
sage: x.__dict__
{'value': 1}
```

Migrating classes to `CachedRepresentation` and unpickling

We check that, when migrating a class to *CachedRepresentation*, older pickles can still be reasonably unpickled. Let us create a (new style) class, and pickle one of its instances:

```
sage: class MyClass4(object):
....:     def __init__(self, value):
....:         self.value = value
....:
sage: import __main__; __main__.MyClass4 = MyClass4  # Fake MyClass4 being␣
→defined in a python module
sage: pickle = dumps(MyClass4(1))
```

It can be unpickled:

```
sage: y = loads(pickle)
sage: y.value
1
```

Now, we upgrade the class to derive from `UniqueRepresentation`, which inherits from `CachedRepresentation`:

```
sage: class MyClass4(UniqueRepresentation, object):
....:     def __init__(self, value):
....:         self.value = value
sage: import __main__; __main__.MyClass4 = MyClass4  # Fake MyClass4 being
→defined in a python module
sage: __main__.MyClass4 = MyClass4
```

The pickle can still be unpickled:

```
sage: y = loads(pickle)
sage: y.value
1
```

Note however that, for the reasons explained above, unique representation is not guaranteed in this case:

```
sage: y is MyClass4(1)
False
```

Todo

Illustrate how this can be fixed on a case by case basis.

Now, we redo the same test for a class deriving from SageObject:

```
sage: class MyClass4(SageObject):
....:     def __init__(self, value):
....:         self.value = value
sage: import __main__; __main__.MyClass4 = MyClass4  # Fake MyClass4 being
→defined in a python module
sage: pickle = dumps(MyClass4(1))

sage: class MyClass4(UniqueRepresentation, SageObject):
....:     def __init__(self, value):
....:         self.value = value
sage: __main__.MyClass4 = MyClass4
sage: y = loads(pickle)
sage: y.value
1
```

Caveat: unpickling instances of a formerly old-style class is not supported yet by default:

```
sage: class MyClass4:
....:     def __init__(self, value):
....:         self.value = value
sage: import __main__; __main__.MyClass4 = MyClass4  # Fake MyClass4 being
→defined in a python module
sage: pickle = dumps(MyClass4(1))

sage: class MyClass4(UniqueRepresentation, SageObject):
```

```
....:        def __init__(self, value):
....:            self.value = value
sage: __main__.MyClass4 = MyClass4
sage: y = loads(pickle)   # todo: not implemented
sage: y.value             # todo: not implemented
1
```

Rationale for the current implementation

CachedRepresentation and derived classes use the `ClasscallMetaclass` of the standard Python type. The following example explains why.

We define a variant of `MyClass` where the calls to `__init__` are traced:

```
sage: class MyClass(CachedRepresentation):
....:        def __init__(self, value):
....:            print("initializing object")
....:            self.value = value
....:
```

Let us create an object twice:

```
sage: x = MyClass(1)
initializing object
sage: z = MyClass(1)
```

As desired the `__init__` method was only called the first time, which is an important feature.

As far as we can tell, this is not achievable while just using `__new__` and `__init__` (as defined by type; see Section Basic Customization in the Python Reference Manual). Indeed, `__init__` is called systematically on the result of `__new__` whenever the result is an instance of the class.

Another difficulty is that argument preprocessing (as in the example above) cannot be handled by `__new__`, since the unprocessed arguments will be passed down to `__init__`.

class `sage.structure.unique_representation.`**UniqueRepresentation**

Bases: *sage.structure.unique_representation.CachedRepresentation* , `sage.misc.fast_methods.WithEqualityById`

Classes derived from UniqueRepresentation inherit a unique representation behavior for their instances.

See also:

unique_representation

EXAMPLES:

The short story: to construct a class whose instances have a unique representation behavior one just has to do:

```
sage: class MyClass(UniqueRepresentation):
....:        # all the rest as usual
....:        pass
```

Everything below is for the curious or for advanced usage.

What is unique representation?

Instances of a class have a *unique representation behavior* when instances evaluate equal if and only if they are identical (i.e., share the same memory representation), if and only if they were created using equal arguments. For example, calling twice:

```
sage: f = SymmetricFunctions(QQ)
sage: g = SymmetricFunctions(QQ)
```

to create the symmetric function algebra over **Q** actually gives back the same object:

```
sage: f == g
True
sage: f is g
True
```

This is a standard design pattern. It allows for sharing cached data (say representation theoretical information about a group) as well as for very fast hashing and equality testing. This behaviour is typically desirable for parents and categories. It can also be useful for intensive computations where one wants to cache all the operations on a small set of elements (say the multiplication table of a small group), and access this cache as quickly as possible.

UniqueRepresentation is very easy to use: a class just needs to derive from it, or make sure some of its super classes does. Also, it groups together the class and the factory in a single gadget:

```
sage: isinstance(SymmetricFunctions(CC), SymmetricFunctions)
True
sage: issubclass(SymmetricFunctions, UniqueRepresentation)
True
```

This nice behaviour is not available when one just uses a factory:

```
sage: isinstance(GF(7), GF)
Traceback (most recent call last):
...
TypeError: isinstance() arg 2 must be a class, type, or tuple of classes and types
sage: isinstance(GF, sage.structure.factory.UniqueFactory)
True
```

In addition, *UniqueFactory* only provides the *cached* representation behaviour, but not the *unique* representation behaviour—the examples in *unique_representation* explain this difference.

On the other hand, the *UniqueRepresentation* class is more intrusive, as it imposes a behavior (and a metaclass) on all the subclasses. In particular, the unique representation behaviour is imposed on *all* subclasses (unless the `__classcall__` method is overloaded and not called in the subclass, which is not recommended). Its implementation is also more technical, which leads to some subtleties.

EXAMPLES:

We start with a simple class whose constructor takes a single value as argument. This pattern is similar to what is done in `sage.combinat.sf.sf.SymmetricFunctions`:

```
sage: class MyClass(UniqueRepresentation):
....:     def __init__(self, value):
....:         self.value = value
....:     def __cmp__(self, other):
....:         c = cmp(type(self),type(other))
....:         if c: return c
```

```
....:           print("custom cmp")
....:           return cmp(self.value, other.value)
....:
```

Two coexisting instances of `MyClass` created with the same argument data are guaranteed to share the same identity. Since trac ticket #12215, this is only the case if there is some strong reference to the returned instance, since otherwise it may be garbage collected:

```
sage: x = MyClass(1)
sage: y = MyClass(1)
sage: x is y               # There is a strong reference
True
sage: z = MyClass(2)
sage: x is z
False
```

In particular, modifying any one of them modifies the other (reference effect):

```
sage: x.value = 3
sage: x.value, y.value
(3, 3)
sage: y.value = 1
sage: x.value, y.value
(1, 1)
```

Rich comparison by identity is used when possible (hence, for == , for != , and for identical arguments in the case of < , <= , >= and >), which is as fast as it can get. Only if identity is not enough to decide the answer of a comparison, the custom comparison is called:

```
sage: x == y
True
sage: z = MyClass(2)
sage: x == z, x is z
(False, False)
sage: x <= x
True
sage: x != z
True
sage: x <= z
custom cmp
True
sage: x > z
custom cmp
False
```

A hash function equivalent to `object.__hash__()` is used, which is compatible with comparison by identity. However this means that the hash function may change in between Sage sessions, or even within the same Sage session.

```
sage: hash(x) == object.__hash__(x)
True
```

> **Warning:** It is possible to inherit from `UniqueRepresentation` and then overload comparison in a way that destroys the unique representation property. We strongly recommend against it! You should use `CachedRepresentation` instead.

Mixing super types and super classes

`sage.structure.unique_representation.` **unreduce** (*cls*, *args*, *keywords*)

Calls a class on the given arguments:

```
sage: sage.structure.unique_representation.unreduce(Integer, (1,), {})
1
```

Todo

should reuse something preexisting ...

7.2 Factory for cached representations

See also:

sage.structure.unique_representation

Using a *UniqueFactory* is one way of implementing a *cached representation behaviour*. In spite of its name, using a *UniqueFactory* is not enough to ensure the *unique representation behaviour*. See *unique_representation* for a detailed explanation.

With a *UniqueFactory* , one can preprocess the given arguments. There is special support for specifying a subset of the arguments that serve as the unique key, so that still *all* given arguments are used to create a new instance, but only the specified subset is used to look up in the cache. Typically, this is used to construct objects that accept an optional `check=[True|False]` argument, but whose result should be unique regardless of said optional argument. (This use case should be handled with care, though: Any checking which isn't done in the `create_key` or `create_key_and_extra_args` method will be done only when a new object is generated, but not when a cached object is retrieved from cache. Consequently, if the factory is once called with `check=False` , a subsequent call with `check=True` cannot be expected to perform all checks unless these checks are all in the `create_key` or `create_key_and_extra_args` method.)

For a class derived from *CachedRepresentation* , argument preprocessing can be obtained by providing a custom static `__classcall__` or `__classcall_private__` method, but this seems less transparent. When argument preprocessing is not needed or the preprocess is not very sophisticated, then generally *CachedRepresentation* is much easier to use than a factory.

AUTHORS:

- Robert Bradshaw (2008): initial version.

- Simon King (2013): extended documentation.

- Julian Rueth (2014-05-09): use `_cache_key` if parameters are unhashable

class `sage.structure.factory.` **UniqueFactory**

Bases: *sage.structure.sage_object.SageObject*

This class is intended to make it easy to cache objects.

It is based on the idea that the object is uniquely defined by a set of defining data (the key). There is also the possibility of some non-defining data (extra args) which will be used in initial creation, but not affect the caching.

> **Warning:** This class only provides *cached representation behaviour*. Hence, using *`UniqueFactory`* , it
> is still possible to create distinct objects that evaluate equal. Unique representation behaviour can be added,
> for example, by additionally inheriting from `sage.misc.fast_methods.WithEqualityById` .

The objects created are cached (using weakrefs) based on their key and returned directly rather than re-created
if requested again. Pickling is taken care of by the factory, and will return the same object for the same version
of Sage, and distinct (but hopefully equal) objects for different versions of Sage.

> **Warning:** The objects returned by a *`UniqueFactory`* must be instances of new style classes (hence,
> they must be instances of `object`) that must not only allow a weak reference, but must accept general
> attribute assignment. Otherwise, pickling won't work.

USAGE:

A *unique factory* provides a way to create objects from parameters (the type of these objects can depend on
the parameters, and is often determined only at runtime) and to cache them by a certain key derived from these
parameters, so that when the factory is being called again with the same parameters (or just with parameters
which yield the same key), the object is being returned from cache rather than constructed anew.

An implementation of a unique factory consists of a factory class and an instance of this factory class.

The factory class has to be a class inheriting from `UniqueFactory` . Typically it only needs to implement
`create_key()` (a method that creates a key from the given parameters, under which key the object will be
stored in the cache) and *`create_object()`* (a method that returns the actual object from the key). Some-
times, one would also implement *`create_key_and_extra_args()`* (this differs from *`create_key()`*
in allowing to also create some additional arguments from the given parameters, which arguments then get
passed to *`create_object()`* and thus can have an effect on the initial creation of the object, but do *not*
affect the key) or *`other_keys()`* . Other methods are not supposed to be overloaded.

The factory class itself cannot be called to create objects. Instead, an instance of the factory class has to be
created first. For technical reasons, this instance has to be provided with a name that allows Sage to find
its definition. Specifically, the name of the factory instance (or the full path to it, if it is not in the global
namespace) has to be passed to the factory class as a string variable. So, if our factory class has been called
`A` and is located in `sage/spam/battletoads.py` , then we need to define an instance (say, `B`) of `A`
by writing `B = A("sage.spam.battletoads.B")` (or `B = A("B")` if this `B` will be imported into
global namespace). This instance can then be used to create objects (by calling `B(*parameters)`).

Notice that the objects created by the factory don't inherit from the factory class. They *do* know about the
factory that created them (this information, along with the keys under which this factory caches them, is stored
in the `_factory_data` attributes of the objects), but not via inheritance.

EXAMPLES:

The below examples are rather artificial and illustrate particular aspects. For
a "real-life" usage case of `UniqueFactory` , see the finite field factory in
`sage.rings.finite_rings.finite_field_constructor` .

In many cases, a factory class is implemented by providing the two methods *`create_key()`* and
`create_object()` . In our example, we want to demonstrate how to use "extra arguments" to choose a
specific implementation, with preference given to an instance found in the cache, even if its implementation
is different. Hence, we implement *`create_key_and_extra_args()`* rather than *`create_key()`* ,
putting the chosen implementation into the extra arguments. Then, in the *`create_object()`* method, we
create and return instances of the specified implementation.

```
sage: from sage.structure.factory import UniqueFactory
sage: class MyFactory(UniqueFactory):
....:     def create_key_and_extra_args(self, *args, **kwds):
....:         return args, {'impl':kwds.get('impl', None)}
....:     def create_object(self, version, key, **extra_args):
....:         impl = extra_args['impl']
....:         if impl=='C':
....:             return C(*key)
....:         if impl=='D':
....:             return D(*key)
....:         return E(*key)
....:
```

Now we can create a factory instance. It is supposed to be found under the name `"F"` in the `"__main__"` module. Note that in an interactive session, `F` would automatically be in the `__main__` module. Hence, the second and third of the following four lines are only needed in doctests.

```
sage: F = MyFactory("__main__.F")
sage: import __main__
sage: __main__.F = F
sage: loads(dumps(F)) is F
True
```

Now we create three classes `C`, `D` and `E`. The first is a Cython extension-type class that does not allow weak references nor attribute assignment. The second is a Python class that is not derived from `object`. The third allows attribute assignment and is derived from `object`.

```
sage: cython("cdef class C: pass")
sage: class D:
....:     def __init__(self, *args):
....:         self.t = args
....:     def __repr__(self):
....:         return "D%s"%repr(self.t)
....:
sage: class E(D, object): pass
```

Again, being in a doctest, we need to put the class `D` into the `__main__` module, so that Python can find it:

```
sage: import __main__
sage: __main__.D = D
```

It is impossible to create an instance of `C` with our factory, since it does not allow weak references:

```
sage: F(1, impl='C')
Traceback (most recent call last):
...
TypeError: cannot create weak reference to '....C' object
```

Let us try again, with a Cython class that does allow weak references. Now, creation of an instance using the factory works:

```
sage: cython('''cdef class C:
....:     cdef __weakref__
....: ''')
....:
sage: c = F(1, impl='C')
sage: isinstance(c, C)
```

```
True
```

The cache is used when calling the factory again—even if it is suggested to use a different implementation. This is because the implementation is only considered an "extra argument" that does not count for the key.

```
sage: c is F(1, impl='C') is F(1, impl="D") is F(1)
True
```

However, pickling and unpickling does not use the cache. This is because the factory has tried to assign an attribute to the instance that provides information on the key used to create the instance, but failed:

```
sage: loads(dumps(c)) is c
False
sage: hasattr(c, '_factory_data')
False
```

We have already seen that our factory will only take the requested implementation into account if the arguments used as key have not been used yet. So, we use other arguments to create an instance of class D :

```
sage: d = F(2, impl='D')
sage: isinstance(d, D)
True
```

The factory only knows about the pickling protocol used by new style classes. Hence, again, pickling and unpickling fails to use the cache, even though the "factory data" are now available:

```
sage: loads(dumps(d)) is d
False
sage: d._factory_data
(<class '__main__.MyFactory'>, (...), (2,), {'impl': 'D'})
```

Only when we have a new style class that can be weak referenced and allows for attribute assignment, everything works:

```
sage: e = F(3)
sage: isinstance(e, E)
True
sage: loads(dumps(e)) is e
True
sage: e._factory_data
(<class '__main__.MyFactory'>, (...), (3,), {'impl': None})
```

create_key (*args*, **kwds*)

    Given the parameters (arguments and keywords), create a key that uniquely determines this object.

    EXAMPLES:

```
sage: from sage.structure.test_factory import test_factory
sage: test_factory.create_key(1, 2, key=5)
(1, 2)
```

create_key_and_extra_args (*args*, **kwds*)

    Return a tuple containing the key (uniquely defining data) and any extra arguments (empty by default).

    Defaults to *create_key()* .

    EXAMPLES:

```
sage: from sage.structure.test_factory import test_factory
sage: test_factory.create_key_and_extra_args(1, 2, key=5)
((1, 2), {})
sage: GF.create_key_and_extra_args(3, foo='value')
((3, ('x',), None, 'modn', "{'foo': 'value'}", 3, 1, True), {'foo': 'value'})
```

create_object (*version*, *key*, ***extra_args*)

Create the object from the key and extra arguments. This is only called if the object was not found in the cache.

EXAMPLES:

```
sage: from sage.structure.test_factory import test_factory
sage: test_factory.create_object(0, (1,2,3))
Making object (1, 2, 3)
<sage.structure.test_factory.A instance at ...>
sage: test_factory('a')
Making object ('a',)
<sage.structure.test_factory.A instance at ...>
sage: test_factory('a') # NOT called again
<sage.structure.test_factory.A instance at ...>
```

get_object (*version*, *key*, *extra_args*)

Returns the object corresponding to `key`, creating it with `extra_args` if necessary (for example, it isn't in the cache or it is unpickling from an older version of Sage).

EXAMPLES:

```
sage: from sage.structure.test_factory import test_factory
sage: a = test_factory.get_object(3.0, 'a', {}); a
Making object a
<sage.structure.test_factory.A instance at ...>
sage: test_factory.get_object(3.0, 'a', {}) is test_factory.get_object(3.0, 'a
→', {})
True
sage: test_factory.get_object(3.0, 'a', {}) is test_factory.get_object(3.1, 'a
→', {})
Making object a
False
sage: test_factory.get_object(3.0, 'a', {}) is test_factory.get_object(3.0, 'b
→', {})
Making object b
False
```

get_version (*sage_version*)

This is provided to allow more or less granular control over pickle versioning. Objects pickled in the same version of Sage will unpickle to the same rather than simply equal objects. This can provide significant gains as arithmetic must be performed on objects with identical parents. However, if there has been an incompatible change (e.g. in element representation) we want the version number to change so coercion is forced between the two parents.

Defaults to the Sage version that is passed in, but coarser granularity can be provided.

EXAMPLES:

```
sage: from sage.structure.test_factory import test_factory
sage: test_factory.get_version((3,1,0))
(3, 1, 0)
```

other_keys (*key*, *obj*)

Sometimes during object creation, certain defaults are chosen which may result in a new (more specific) key. This allows the more specific key to be regarded as equivalent to the original key returned by `create_key()` for the purpose of lookup in the cache, and is used for pickling.

EXAMPLES:

The `GF` factory used to have a custom `other_keys()` method, but this was removed in trac ticket #16934:

```
sage: key, _ = GF.create_key_and_extra_args(27, 'k'); key
(27, ('k',), x^3 + 2*x + 1, 'givaro', '{}', 3, 3, True)
sage: K = GF.create_object(0, key); K
Finite Field in k of size 3^3
sage: GF.other_keys(key, K)
[]

sage: K = GF(7^40, 'a')
sage: loads(dumps(K)) is K
True
```

reduce_data (*obj*)

The results of this function can be returned from __reduce__() . This is here so the factory internals can change without having to re-write __reduce__() methods that use it.

EXAMPLES:

```
sage: V = FreeModule(ZZ, 5)
sage: factory, data = FreeModule.reduce_data(V)
sage: factory(*data)
Ambient free module of rank 5 over the principal ideal domain Integer Ring
sage: factory(*data) is V
True

sage: from sage.structure.test_factory import test_factory
sage: a = test_factory(1, 2)
Making object (1, 2)
sage: test_factory.reduce_data(a)
(<built-in function generic_factory_unpickle>,
 (<class 'sage.structure.test_factory.UniqueFactoryTester'>,
  (...),
  (1, 2),
  {}))
```

Note that the ellipsis (...) here stands for the Sage version.

sage.structure.factory. **generic_factory_reduce** (*self*, *proto*)

Used to provide a __reduce__ method if one does not already exist.

EXAMPLES:

```
sage: V = QQ^6
sage: sage.structure.factory.generic_factory_reduce(V, 1) == V.__reduce_ex__(1)
True
```

sage.structure.factory. **generic_factory_unpickle** (*factory*, **args*)

Method used for unpickling the object.

The unpickling mechanism needs a plain Python function to call. It takes a factory as the first argument, passes the rest of the arguments onto the factory's `UniqueFactory.get_object()` method.

EXAMPLES:

```
sage: V = FreeModule(ZZ, 5)
sage: func, data = FreeModule.reduce_data(V)
sage: func is sage.structure.factory.generic_factory_unpickle
True
sage: sage.structure.factory.generic_factory_unpickle(*data) is V
True
```

sage.structure.factory. **lookup_global** (*name*)

Used in unpickling the factory itself.

EXAMPLES:

```
sage: from sage.structure.factory import lookup_global
sage: lookup_global('ZZ')
Integer Ring
sage: lookup_global('sage.rings.all.ZZ')
Integer Ring
```

sage.structure.factory. **register_factory_unpickle** (*name*, *callable*)

Register a callable to handle the unpickling from an old *UniqueFactory* object.

UniqueFactory pickles use a global name through *generic_factory_unpickle()* , so the usual *register_unpickle_override()* cannot be used here.

See also:

generic_factory_unpickle()

7.3 Dynamic classes

Why dynamic classes?

The short answer:

- Multiple inheritance is a powerful tool for constructing new classes by combining preexisting building blocks.

- There is a combinatorial explosion in the number of potentially useful classes that can be produced this way.

- The implementation of standard mathematical constructions calls for producing such combinations automatically.

- Dynamic classes, i.e. classes created on the fly by the Python interpreter, are a natural mean to achieve this.

The long answer:

Say we want to construct a new class `MyPermutation` for permutations in a given set S (in Sage, S will be modelled by a parent, but we won't discuss this point here). First, we have to choose a data structure for the permutations, typically among the following:

- Stored by cycle type

- Stored by code

- Stored in list notation - C arrays of short ints (for small permutations) - python lists of ints (for huge permutations) - ...

- Stored by reduced word

- Stored as a function

• ...

Luckily, the Sage library provides (or will provide) classes implementing each of those data structures. Those classes all share a common interface (or possibly a common abstract base class). So we can just derive our class from the chosen one:

```
class MyPermutation(PermutationCycleType):
    ...
```

Then we may want to further choose a specific memory behavior (unique representation, copy-on-write) which (hopefuly) can again be achieved by inheritance:

```
class MyPermutation(UniqueRepresentation, PermutationCycleType):
    ...
```

Finaly, we may want to endow the permutations in S with further operations coming from the (algebraic) structure of S:

• group operations

• or just monoid operations (for a subset of permutations not stable by inverse)

• poset operations (for left/right/Bruhat order)

• word operations (searching for substrings, patterns, ...)

Or any combination thereof. Now, our class typically looks like:

```
class MyPermutation(UniqueRepresentation, PermutationCycleType, PosetElement,␣
→GroupElement):
    ...
```

Note the combinatorial explosion in the potential number of classes which can be created this way.

In practice, such classes will be used in mathematical constructions like:

```
SymmetricGroup(5).subset(... TODO: find a good example in the context above ...)
```

In such a construction, the structure of the result, and therefore the operations on its elements can only be determined at execution time. Let us take another standard construction:

```
A = cartesian_product( B, C )
```

Depending on the structure of B and C, and possibly on further options passed down by the user, A may be:

• an enumerated set

• a group

• an algebra

• a poset

• ...

Or any combination thereof.

Hardcoding classes for all potential combinations would be at best tedious. Furthermore, this would require a cumbersome mechanism to lookup the appropriate class depending on the desired combination.

Instead, one may use the ability of Python to create new classes dynamically:

```
type("class name", tuple of base classes, dictionary of methods)
```

This paradigm is powerful, but there are some technicalities to address. The purpose of this library is to standardize its use within Sage, and in particular to ensure that the constructed classes are reused whenever possible (unique representation), and can be pickled.

Combining dynamic classes and Cython classes

Cython classes cannot inherit from a dynamic class (there might be some partial support for this in the future). On the other hand, such an inheritance can be partially emulated using __getattr__() . See `sage.categories.examples.semigroups_cython` for an example.

class sage.structure.dynamic_class. **DynamicClasscallMetaclass**
> Bases: *sage.structure.dynamic_class.DynamicMetaclass* ,
> sage.misc.classcall_metaclass.ClasscallMetaclass

class sage.structure.dynamic_class. **DynamicInheritComparisonClasscallMetaclass**
> Bases: *sage.structure.dynamic_class.DynamicMetaclass* ,
> sage.misc.inherit_comparison.InheritComparisonClasscallMetaclass

class sage.structure.dynamic_class. **DynamicInheritComparisonMetaclass**
> Bases: *sage.structure.dynamic_class.DynamicMetaclass* ,
> sage.misc.inherit_comparison.InheritComparisonMetaclass

class sage.structure.dynamic_class. **DynamicMetaclass**
> Bases: type

> A metaclass implementing an appropriate reduce-by-construction method

sage.structure.dynamic_class. **M**
> alias of *DynamicInheritComparisonClasscallMetaclass*

class sage.structure.dynamic_class. **TestClass**
> A class used for checking that introspection works

> **bla** ( )
> > bla ...

sage.structure.dynamic_class. **dynamic_class** (*name*, *bases*, *cls=None*, *reduction=None*, *doccls=None*, *prepend_cls_bases=True*, *cache=True*)

> INPUT:

> > • name – a string

> > • bases – a tuple of classes

> > • cls – a class or None

> > • reduction – a tuple or None

> > • doccls – a class or None

> > • prepend_cls_bases – a boolean (default: True)

> > • cache – a boolean or "ignore_reduction" (default: True)

> Constructs dynamically a new class C with name name , and bases bases . If cls is provided, then its methods will be inserted into C , and its bases will be prepended to bases (unless prepend_cls_bases is False).

The module, documentation and source instrospection is taken from `doccls`, or `cls` if `doccls` is `None`, or `bases[0]` if both are `None` (therefore `bases` should be non empty if cls\` is \`\`None).

The constructed class can safely be pickled (assuming the arguments themselves can).

Unless `cache` is `False`, the result is cached, ensuring unique representation of dynamic classes.

See *sage.structure.dynamic_class* for a discussion of the dynamic classes paradigm, and its relevance to Sage.

EXAMPLES:

To setup the stage, we create a class Foo with some methods, cached methods, and lazy attributes, and a class Bar:

```
sage: from sage.misc.lazy_attribute import lazy_attribute
sage: from sage.misc.cachefunc import cached_function
sage: from sage.structure.dynamic_class import dynamic_class
sage: class Foo(object):
....:     "The Foo class"
....:     def __init__(self, x):
....:         self._x = x
....:     @cached_method
....:     def f(self):
....:         return self._x^2
....:     def g(self):
....:         return self._x^2
....:     @lazy_attribute
....:     def x(self):
....:         return self._x
sage: class Bar:
....:     def bar(self):
....:         return self._x^2
```

We now create a class FooBar which is a copy of Foo, except that it also inherits from Bar:

```
sage: FooBar = dynamic_class("FooBar", (Bar,), Foo)
sage: x = FooBar(3)
sage: x.f()
9
sage: x.f() is x.f()
True
sage: x.x
3
sage: x.bar()
9
sage: FooBar.__name__
'FooBar'
sage: FooBar.__module__
'__main__'

sage: Foo.__bases__
(<type 'object'>,)
sage: FooBar.__bases__
(<type 'object'>, <class __main__.Bar at ...>)
sage: Foo.mro()
[<class '__main__.Foo'>, <type 'object'>]
sage: FooBar.mro()
[<class '__main__.FooBar'>, <type 'object'>, <class __main__.Bar at ...>]
```

Pickling

Dynamic classes are pickled by construction. Namely, upon unpickling, the class will be reconstructed by recalling dynamic_class with the same arguments:

```
sage: type(FooBar).__reduce__(FooBar)
(<function dynamic_class at ...>, ('FooBar', (<class __main__.Bar at ...>,),
↪<class '__main__.Foo'>, None, None))
```

Technically, this is achieved by using a metaclass, since the Python pickling protocol for classes is to pickle by name:

```
sage: type(FooBar)
<class 'sage.structure.dynamic_class.DynamicMetaclass'>
```

The following (meaningless) example illustrates how to customize the result of the reduction:

```
sage: BarFoo = dynamic_class("BarFoo", (Foo,), Bar, reduction = (str, (3,)))
sage: type(BarFoo).__reduce__(BarFoo)
(<... 'str'>, (3,))
sage: loads(dumps(BarFoo))
'3'
```

Caching

By default, the built class is cached:

```
sage: dynamic_class("FooBar", (Bar,), Foo) is FooBar
True
sage: dynamic_class("FooBar", (Bar,), Foo, cache=True) is FooBar
True
```

and the result depends on the reduction:

```
sage: dynamic_class("BarFoo", (Foo,), Bar, reduction = (str, (3,))) is BarFoo
True
sage: dynamic_class("BarFoo", (Foo,), Bar, reduction = (str, (2,))) is BarFoo
False
```

With `cache=False`, a new class is created each time:

```
sage: FooBar1 = dynamic_class("FooBar", (Bar,), Foo, cache=False); FooBar1
<class '__main__.FooBar'>
sage: FooBar2 = dynamic_class("FooBar", (Bar,), Foo, cache=False); FooBar2
<class '__main__.FooBar'>
sage: FooBar1 is FooBar
False
sage: FooBar2 is FooBar1
False
```

With `cache="ignore_reduction"`, the class does not depend on the reduction:

```
sage: BarFoo = dynamic_class("BarFoo", (Foo,), Bar, reduction = (str, (3,)),
↪cache="ignore_reduction")
sage: dynamic_class("BarFoo", (Foo,), Bar, reduction = (str, (2,)), cache="ignore_
↪reduction") is BarFoo
```

```
True
```

In particular, the reduction used is that provided upon creating the first class:

```
sage: dynamic_class("BarFoo", (Foo,), Bar, reduction = (str, (2,)), cache="ignore_
↪reduction")._reduction
(<... 'str'>, (3,))
```

> **Warning:** The behaviour upon creating several dynamic classes from the same data but with different values
> for `cache` option is currently left unspecified. In other words, for a given application, it is recommended
> to consistently use the same value for that option.

`sage.structure.dynamic_class.` **`dynamic_class_internal`** (*name*, *bases*, *cls=None*, *reduction=None*, *doccls=None*, *prepend_cls_bases=True*)

   See sage.structure.dynamic_class.dynamic_class? for indirect doctests.

7.4 Mutability Cython Implementation

class `sage.structure.mutability.` **`Mutability`**

   Bases: `object`

   `is_immutable` ( )

   Return True if this object is immutable (can not be changed) and False if it is not.

   To make this object immutable use self.set_immutable().

   EXAMPLES:

```
sage: v = Sequence([1,2,3,4/5])
sage: v[0] = 5
sage: v
[5, 2, 3, 4/5]
sage: v.is_immutable()
False
sage: v.set_immutable()
sage: v.is_immutable()
True
```

   `is_mutable` ( )

   `set_immutable` ( )

   Make this object immutable, so it can never again be changed.

   EXAMPLES:

```
sage: v = Sequence([1,2,3,4/5])
sage: v[0] = 5
sage: v
[5, 2, 3, 4/5]
sage: v.set_immutable()
sage: v[3] = 7
Traceback (most recent call last):
...
ValueError: object is immutable; please change a copy instead.
```

sage.structure.mutability. **require_immutable** (*f*)

A decorator that requires mutability for a method to be called.

EXAMPLES:

```
sage: from sage.structure.mutability import require_mutable, require_immutable
sage: class A:
....:  def __init__(self, val):
....:      self._m = val
....:  @require_mutable
....:  def change(self, new_val):
....:      'change self'
....:      self._m = new_val
....:  @require_immutable
....:  def __hash__(self):
....:      'implement hash'
....:      return hash(self._m)
sage: a = A(5)
sage: a.change(6)
sage: hash(a)    # indirect doctest
Traceback (most recent call last):
...
ValueError: <type 'instance'> instance is mutable, <function __hash__ at ...>
→must not be called
sage: a._is_immutable = True
sage: hash(a)
6
sage: a.change(7)
Traceback (most recent call last):
...
ValueError: <type 'instance'> instance is immutable, <function change at ...>
→must not be called
sage: from sage.misc.sageinspect import sage_getdoc
sage: print(sage_getdoc(a.__hash__))
implement hash
```

AUTHORS:

- Simon King <simon.king@uni-jena.de>

sage.structure.mutability. **require_mutable** (*f*)

A decorator that requires mutability for a method to be called.

EXAMPLES:

```
sage: from sage.structure.mutability import require_mutable, require_immutable
sage: class A:
....:  def __init__(self, val):
....:      self._m = val
....:  @require_mutable
....:  def change(self, new_val):
....:      'change self'
....:      self._m = new_val
....:  @require_immutable
....:  def __hash__(self):
....:      'implement hash'
....:      return hash(self._m)
sage: a = A(5)
sage: a.change(6)
sage: hash(a)
```

```
Traceback (most recent call last):
...
ValueError: <type 'instance'> instance is mutable, <function __hash__ at ...>
→must not be called
sage: a._is_immutable = True
sage: hash(a)
6
sage: a.change(7)    # indirect doctest
Traceback (most recent call last):
...
ValueError: <type 'instance'> instance is immutable, <function change at ...>
→must not be called
sage: from sage.misc.sageinspect import sage_getdoc
sage: print(sage_getdoc(a.change))
change self
```

AUTHORS:

- Simon King <simon.king@uni-jena.de>

INTERNALS

8.1 Debug options for the `sage.structure` modules

EXAMPLES:

```
sage: from sage.structure.debug_options import debug
sage: debug.bad_parent_warnings
False
sage: debug.unique_parent_warnings
False
sage: debug.refine_category_hash_check
True
```

class sage.structure.debug_options. **DebugOptions_class**

    Bases: `object`

    bad_parent_warnings

    refine_category_hash_check

    unique_parent_warnings

8.2 Performance Test for Clone Protocol

see *sage.structure.list_clone.ClonableArray*

EXAMPLES:

```
sage: from sage.structure.list_clone_timings import *
sage: cmd =["",
....:     "e.__copy__()",
....:     "copy(e)",
....:     "e.clone()",
....:     "e.__class__(e.parent(), e._get_list())",
....:     "e.__class__(e.parent(), e[:])",
....:     "e.check()",
....:     "",
....:     "add1_internal(e)",
....:     "add1_immutable(e)",
....:     "add1_mutable(e)",
....:     "add1_with(e)",
....:     "",
....:     "cy_add1_internal(e)",
....:     "cy_add1_immutable(e)",
```

```
....:       "cy_add1_mutable(e)",
....:       "cy_add1_with(e)"]
```

Various timings using a Cython class:

```
sage: size = 5
sage: e = IncreasingArrays()(range(size))
sage: # random
....: for p in cmd:
....:     print("{0:36} : ".format(p), end=""); timeit(p)
                                       :
e.__copy__()                          :  625 loops, best of 3: 446 ns per loop
copy(e)                               :  625 loops, best of 3: 1.94 μs per loop
e.clone()                             :  625 loops, best of 3: 736 ns per loop
e.__class__(e.parent(), e._get_list()) :  625 loops, best of 3: 1.34 μs per loop
e.__class__(e.parent(), e[:])         :  625 loops, best of 3: 1.35 μs per loop
e.check()                             :  625 loops, best of 3: 342 ns per loop
                                       :
add1_internal(e)                      :  625 loops, best of 3: 3.53 μs per loop
add1_immutable(e)                     :  625 loops, best of 3: 3.72 μs per loop
add1_mutable(e)                       :  625 loops, best of 3: 3.42 μs per loop
add1_with(e)                          :  625 loops, best of 3: 4.05 μs per loop
                                       :
cy_add1_internal(e)                   :  625 loops, best of 3: 752 ns per loop
cy_add1_immutable(e)                  :  625 loops, best of 3: 1.28 μs per loop
cy_add1_mutable(e)                    :  625 loops, best of 3: 861 ns per loop
cy_add1_with(e)                       :  625 loops, best of 3: 1.51 μs per loop
```

Various timings using a Python class:

```
sage: e = IncreasingArraysPy()(range(size))
sage: # random
....: for p in cmd: print("{0:36} : ".format(p), end=""); timeit(p)
                                       :
e.__copy__()                          :  625 loops, best of 3: 869 ns per loop
copy(e)                               :  625 loops, best of 3: 2.13 μs per loop
e.clone()                             :  625 loops, best of 3: 1.86 μs per loop
e.__class__(e.parent(), e._get_list()) :  625 loops, best of 3: 7.52 μs per loop
e.__class__(e.parent(), e[:])         :  625 loops, best of 3: 7.27 μs per loop
e.check()                             :  625 loops, best of 3: 4.02 μs per loop
                                       :
add1_internal(e)                      :  625 loops, best of 3: 9.34 μs per loop
add1_immutable(e)                     :  625 loops, best of 3: 9.91 μs per loop
add1_mutable(e)                       :  625 loops, best of 3: 12.6 μs per loop
add1_with(e)                          :  625 loops, best of 3: 15.9 μs per loop
                                       :
cy_add1_internal(e)                   :  625 loops, best of 3: 7.13 μs per loop
cy_add1_immutable(e)                  :  625 loops, best of 3: 6.95 μs per loop
cy_add1_mutable(e)                    :  625 loops, best of 3: 14.1 μs per loop
cy_add1_with(e)                       :  625 loops, best of 3: 17.5 μs per loop
```

class sage.structure.list_clone_timings.**IncreasingArraysPy**

    Bases: *sage.structure.list_clone_demo.IncreasingArrays*

    class Element

        Bases: *sage.structure.list_clone.ClonableArray*

        A small class for testing ClonableArray : Increasing Lists

check ( )
   Check that `self` is increasing.

   EXAMPLES:

```
sage: from sage.structure.list_clone_timings import IncreasingArraysPy
sage: IncreasingArraysPy()([1,2,3]) # indirect doctest
[1, 2, 3]
sage: IncreasingArraysPy()([3,2,1]) # indirect doctest
Traceback (most recent call last):
...
ValueError: Lists is not increasing
```

sage.structure.list_clone_timings. **add1_immutable** (*bla*)

sage.structure.list_clone_timings. **add1_internal** (*bla*)

sage.structure.list_clone_timings. **add1_mutable** (*bla*)

sage.structure.list_clone_timings. **add1_with** (*bla*)

8.3 Cython Functions for Timing Clone Protocol

sage.structure.list_clone_timings_cy. **cy_add1_immutable** (*bla*)

sage.structure.list_clone_timings_cy. **cy_add1_internal** (*bla*)

sage.structure.list_clone_timings_cy. **cy_add1_mutable** (*bla*)

sage.structure.list_clone_timings_cy. **cy_add1_with** (*bla*)

8.4 Miscellaneous utilities

class sage.structure.misc. **AttributeErrorMessage**
   Bases: object

   Tries to emulate the standard Python `AttributeError` message.

Note: The typical fate of an attribute error is being caught. Hence, under normal circumstances, nobody will ever see the error message. The idea for this class is to provide an object that is fast to create and whose string representation is an attribute error's message. That string representation is only created if someone wants to see it.

   EXAMPLES:

```
sage: 1.bla  #indirect doctest
Traceback (most recent call last):
...
AttributeError: 'sage.rings.integer.Integer' object has no attribute 'bla'
sage: QQ[x].gen().bla
Traceback (most recent call last):
...
AttributeError: 'sage.rings.polynomial.polynomial_rational_flint.Polynomial_
→rational_flint' object has no attribute 'bla'
sage: from sage.structure.misc import AttributeErrorMessage
```

```
sage: AttributeErrorMessage(int(1),'bla')
'int' object has no attribute 'bla'
```

AUTHOR:

- Simon King (2011-05-21)

cls

name

sage.structure.misc. **dir_with_other_class** (*self*, *cls*)

Emulates `dir(self)` , as if self was also an instance `cls` , right after `caller_class` in the method resolution order (`self.__class__.mro()`)

EXAMPLES:

```
sage: class A(object):
....:     a = 1
....:     b = 2
....:     c = 3
sage: class B(object):
....:     b = 2
....:     c = 3
....:     d = 4
sage: x = A()
sage: x.c = 1; x.e = 1
sage: from sage.structure.misc import dir_with_other_class
sage: dir_with_other_class(x, B)
[..., 'a', 'b', 'c', 'd', 'e']
```

Check that objects without dicts are well handled:

```
sage: cython("cdef class A:\n    cdef public int a")
sage: cython("cdef class B:\n    cdef public int b")
sage: x = A()
sage: x.a = 1
sage: hasattr(x,'__dict__')
False
sage: dir_with_other_class(x, B)
[..., 'a', 'b']
```

sage.structure.misc. **getattr_from_other_class** (*self*, *cls*, *name*)

Emulate `getattr(self,name)` , as if `self` was an instance of `cls` .

INPUT:

- `self` – some object

- `cls` – a new-style class

- `name` – a string

If self is an instance of cls, raises an `AttributeError` , to avoid a double lookup. This function is intended to be called from __getattr__, and so should not be called if name is an attribute of self.

EXAMPLES:

```
sage: from sage.structure.misc import getattr_from_other_class
sage: class A(object):
....:         def inc(self):
```

```
....:            return self + 1
....:
....:        @staticmethod
....:        def greeting():
....:            print("Hello World!")
....:
....:        @lazy_attribute
....:        def lazy_attribute(self):
....:            return repr(self)
sage: getattr_from_other_class(1, A, "inc")
<bound method A.inc of 1>
sage: getattr_from_other_class(1, A, "inc")()
2
```

Static methods work:

```
sage: getattr_from_other_class(1, A, "greeting")()
Hello World!
```

Caveat: lazy attributes work with extension types only if they allow attribute assignment or have a public attribute __cached_methods of type <dict>. This condition is satisfied, e.g., by any class that is derived from *Parent*:

```
sage: getattr_from_other_class(1, A, "lazy_attribute")
Traceback (most recent call last):
...
AttributeError: 'sage.rings.integer.Integer' object has no attribute 'lazy_
↪attribute'
```

The integer ring is a parent, so, lazy attributes work:

```
sage: getattr_from_other_class(ZZ, A, "lazy_attribute")
'Integer Ring'
sage: getattr_from_other_class(PolynomialRing(QQ, name='x', sparse=True).one(),␣
↪A, "lazy_attribute")
'1'
sage: getattr_from_other_class(PolynomialRing(QQ, name='x', implementation="FLINT
↪").one(), A, "lazy_attribute")
Traceback (most recent call last):
...
AttributeError: 'sage.rings.polynomial.polynomial_rational_flint.Polynomial_
↪rational_flint'
object has no attribute 'lazy_attribute'
```

In general, descriptors are not yet well supported, because they often do not accept to be cheated with the type of their instance:

```
sage: A.__weakref__.__get__(1)
Traceback (most recent call last):
...
TypeError: descriptor '__weakref__' for 'A' objects doesn't apply
to 'sage.rings.integer.Integer' object
```

When this occurs, an `AttributeError` is raised:

```
sage: getattr_from_other_class(1, A, "__weakref__")
Traceback (most recent call last):
```

```
...
AttributeError: 'sage.rings.integer.Integer' object has no attribute '__weakref__'
```

This was caught by trac ticket #8296 for which we do a couple more tests:

```
sage: "__weakref__" in dir(A)
True
sage: "__weakref__" in dir(1)
True
sage: 1.__weakref__
Traceback (most recent call last):
...
AttributeError: 'sage.rings.integer.Integer' object has no attribute '__weakref__'

sage: n = 1
sage: ip = get_ipython()                    # not tested: only works in interactive
→shell
sage: ip.magic_psearch('n.N')               # not tested: only works in interactive
→shell
n.N
sage: ip.magic_psearch('n.__weakref__')     # not tested: only works in interactive
→shell
```

Caveat: When __call__ is not defined for instances, using `A.__call__` yields the method __call__ of the class. We use a workaround but there is no guarantee for robustness.

> sage: getattr_from_other_class(1, A, "__call__") Traceback (most recent call last): ... AttributeError: 'sage.rings.integer.Integer' object has no attribute '__call__'

sage.structure.misc. **is_extension_type** (*cls*)

> INPUT:

> > •cls: a class

> Tests whether cls is an extension type (int, list, cython compiled classes, ...)

> EXAMPLES:

```
sage: from sage.structure.parent import is_extension_type
sage: is_extension_type(int)
True
sage: is_extension_type(list)
True
sage: is_extension_type(ZZ.__class__)
True
sage: is_extension_type(QQ.__class__)
False
```

8.5 Test of the `factory` module

class sage.structure.test_factory. **A**

class sage.structure.test_factory. **UniqueFactoryTester**

> Bases: *sage.structure.factory.UniqueFactory*

> **create_key** (*args*, ** *kwds*)
> > EXAMPLES:

```
sage: from sage.structure.test_factory import UniqueFactoryTester
sage: test_factory = UniqueFactoryTester('foo')
sage: test_factory.create_key(1, 2, 3)
(1, 2, 3)
```

create_object (*version*, *key*, ***extra_args*)

EXAMPLES:

```
sage: from sage.structure.test_factory import UniqueFactoryTester
sage: test_factory = UniqueFactoryTester('foo')
sage: test_factory.create_object('version', key=(1, 2, 4))
Making object (1, 2, 4)
<sage.structure.test_factory.A instance at ...>
```

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