

---

# **PREP Tutorials**

***Release 7.3***

**Rob Beezer, Karl-Dieter Crisman, and Jason Grout**

**Aug 05, 2016**



<b>1</b>	<b>Logging on to a Sage Server and Creating a Worksheet</b>	<b>3</b>
1.1	Creating an Account . . . . .	3
1.2	Two Usage Scenarios . . . . .	9
<b>2</b>	<b>Introductory Sage Tutorial</b>	<b>13</b>
2.1	Evaluating Sage Commands . . . . .	13
2.2	Functions in Sage . . . . .	15
2.3	Help inside Sage . . . . .	18
2.4	Annotating with Sage . . . . .	20
<b>3</b>	<b>Tutorial for Symbolics and Plotting</b>	<b>23</b>
3.1	Symbolic Expressions . . . . .	23
3.2	Basic 2D Plotting . . . . .	26
3.3	Basic 3D Plotting . . . . .	28
<b>4</b>	<b>Tutorial for Calculus</b>	<b>31</b>
4.1	Calculus 1 . . . . .	31
4.2	Calculus 2 . . . . .	34
4.3	Calculus 3 . . . . .	37
4.4	‘Exam’ . . . . .	39
<b>5</b>	<b>Sage Introductory Programming Tutorial</b>	<b>41</b>
5.1	Methods and Dot Notation . . . . .	41
5.2	Lists, Loops, and Set Builders . . . . .	43
5.3	Defining Functions . . . . .	46
5.4	Gotchas from names and copies . . . . .	47
5.5	Appendix: Advanced Introductory Topics . . . . .	49
<b>6</b>	<b>Tutorial for Advanced 2d Plotting</b>	<b>51</b>
6.1	Graphing Functions and Plotting Curves . . . . .	51
6.2	Plotting Data . . . . .	55
6.3	Contour-type Plots . . . . .	56
6.4	Miscellaneous Plot Information . . . . .	58
<b>7</b>	<b>PREP Quickstart Tutorials</b>	<b>61</b>
7.1	Sage Quickstart for Abstract Algebra . . . . .	61
7.2	Sage Quickstart for Differential Equations . . . . .	66
7.3	Sage Quickstart for Graph Theory and Discrete Mathematics . . . . .	68
7.4	Sage Quickstart for Linear Algebra . . . . .	73
7.5	Sage Quickstart for Multivariable Calculus . . . . .	80
7.6	Sage Quickstart for Numerical Analysis . . . . .	84

7.7	Sage Quickstart for Number Theory . . . . .	88
7.8	Sage Quickstart for Statistics . . . . .	91
7.9	Sage Interact Quickstart . . . . .	95

This is a set of tutorials developed for the MAA PREP workshops “Sage: Using Open-Source Mathematics Software with Undergraduates” (funding provided by the National Science Foundation under grant DUE 0817071) in the summers of 2010-2012. It is licensed under the Creative Commons Attribution-ShareAlike 3.0 license ([CC BY-SA](#)).

The original audience for these tutorials was mathematics faculty at undergraduate institutions with little or no experience with programming or computer mathematics software. Because the computer experience required is quite minimal, this should be useful to anyone coming with little such experience to Sage.

Although any mathematics the reader is not familiar with can simply be skipped at first, we do assume throughout that the reader is familiar with the concept of a function and different kinds of numbers. We also make liberal use of basic examples from calculus, linear algebra, and other areas. In the *Quickstart tutorials*, we assume familiarity with the topics at the level of a student who has just completed a course in the subject, or of a faculty member who is about to teach it.

Contents:



## LOGGING ON TO A SAGE SERVER AND CREATING A WORKSHEET

This [Sage](#) worksheet is from a series of tutorials developed for the MAA PREP Workshop “Sage: Using Open-Source Mathematics Software with Undergraduates” (funding provided by NSF DUE 0817071). It is licensed under the Creative Commons Attribution-ShareAlike 3.0 license ([CC BY-SA](#)).

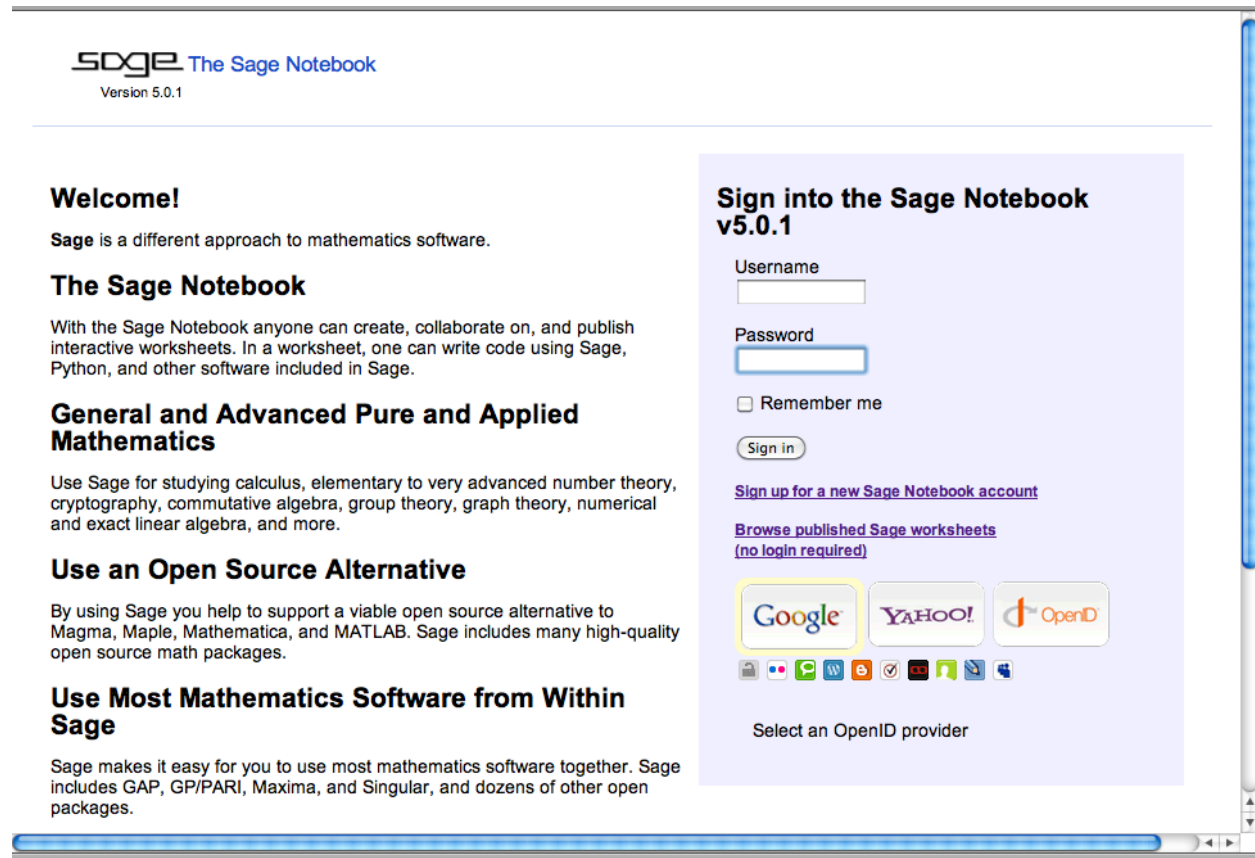
This document describes how to get into a Sage worksheet in the first place, including:

- *logging in* for the first time
- *editing a copy* of a worksheet someone sent you
- *making your own worksheet* from scratch

If you already feel comfortable with this process, or at least comfortable enough to see how to actually use Sage, the main content of the tutorials begins with *the introductory tutorial*.

### 1.1 Creating an Account

When coming to a Sage server for the first time, it will look something like this.



You can create an account in two ways.

- Use an *OpenID* account to verify your account
- Create a login in the *standard way* by creating a username and password.

### 1.1.1 OpenID

With many public Sage servers, you can use an OpenID such as Google, Yahoo!, and so forth to create an account.



**SAGE** The Sage Notebook  
Version 5.0.1

### Welcome!

Sage is a different approach to mathematics software.

### The Sage Notebook

With the Sage Notebook anyone can create, collaborate on, and publish interactive worksheets. In a worksheet, one can write code using Sage, Python, and other software included in Sage.

### General and Advanced Pure and Applied Mathematics

Use Sage for studying calculus, elementary to very advanced number theory, cryptography, commutative algebra, group theory, graph theory, numerical and exact linear algebra, and more.

### Use an Open Source Alternative

By using Sage you help to support a viable open source alternative to Magma, Maple, Mathematica, and MATLAB. Sage includes many high-quality open source math packages.

### Use Most Mathematics Software from Within Sage

Sage makes it easy for you to use most mathematics software together. Sage includes GAP, GP/PARI, Maxima, and Singular, and dozens of other open packages.

### Sign into the Sage Notebook v5.0.1

Choose your favorite OpenID provider




Username

Password

☐ Remember me

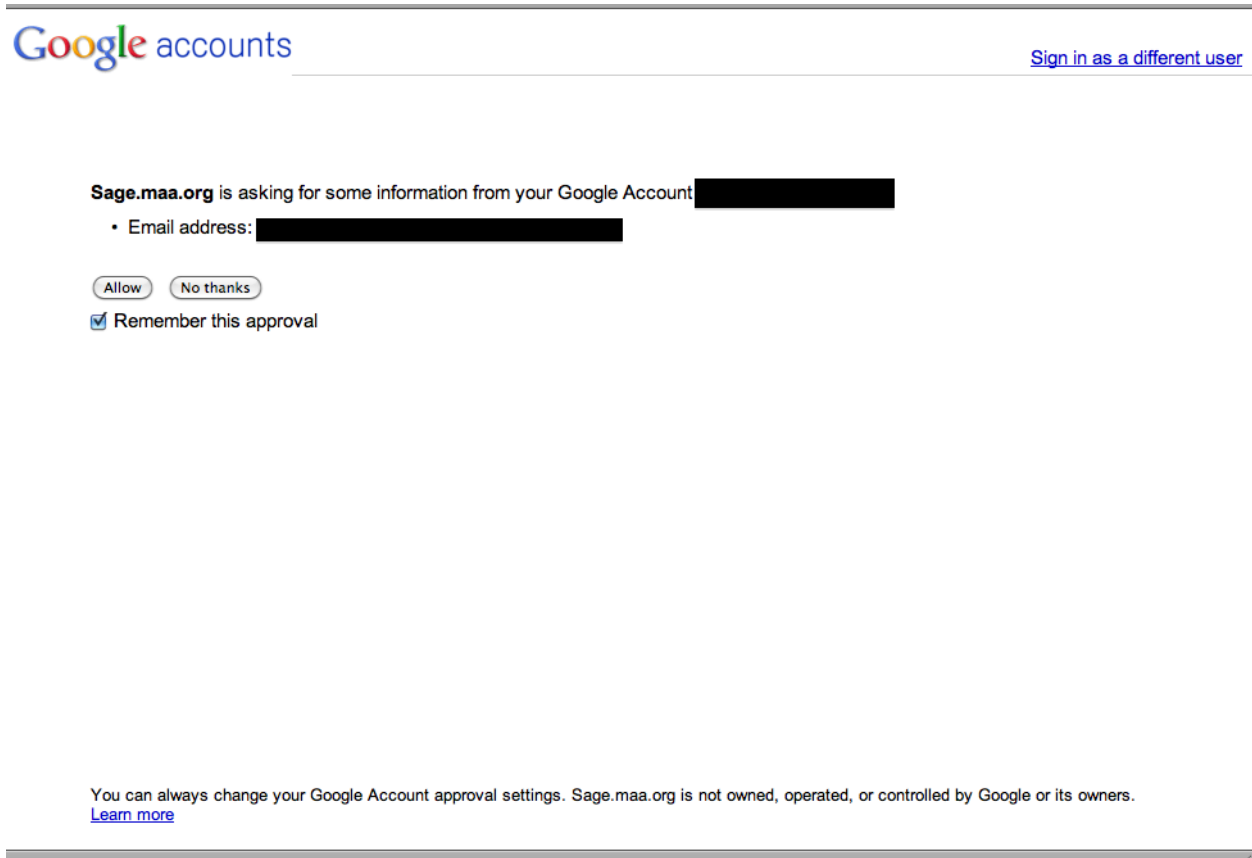
[Sign up for a new Sage Notebook account](#)

[Browse published Sage worksheets \(no login required\)](#)

Select an OpenID provider

To create an account, just make sure you are logged in with your verification website, and then click the correct logo of the many on the lower right. Then you should come to a page like this.



From there, you should be taken directly to your new notebook, ready to make your *first worksheet*.

### 1.1.2 Standard Account Creation

The normal way to create an account is quite straightforward as well.

**Sage** The Sage Notebook  
Version 5.0.1

**Welcome!**  
Sage is a different approach to mathematics software.

**The Sage Notebook**  
With the Sage Notebook anyone can create, collaborate on, and publish interactive worksheets. In a worksheet, one can write code using Sage, Python, and other software included in Sage.

**General and Advanced Pure and Applied Mathematics**  
Use Sage for studying calculus, elementary to very advanced number theory, cryptography, commutative algebra, group theory, graph theory, numerical and exact linear algebra, and more.

**Use an Open Source Alternative**  
By using Sage you help to support a viable open source alternative to Magma, Maple, Mathematica, and MATLAB. Sage includes many high-quality open source math packages.

**Use Most Mathematics Software from Within Sage**  
Sage makes it easy for you to use most mathematics software together. Sage includes GAP, GP/PARI, Maxima, and Singular, and dozens of other open packages.

**Sign into the Sage Notebook v5.0.1**

Username

Password

☐ Remember me

[Sign up for a new Sage Notebook account](#)

[Browse published Sage worksheets \(no login required\)](#)

Google YAHOO! OpenID

Select an OpenID provider

Just click on the relevant link, and you'll be taken to a page where you create a new username and password.

## Sign up for a Sage Notebook account

---

### 1. Create a username

Your username must start with a letter and be between 3 and 64 characters long. You may only use letters, numbers, underscores, and dots.

### 2. Create a good password

Your password must have at least 4 characters. Your password can not contain your username or spaces.

Because the passwords are not encrypted when being sent to the server, do not use a password you use elsewhere.


### 3. Re-type your password

### 4. Answer a challenge

What is the magic word?

In this example, there is a “magic word”; there could be a different security question as well. In that case, you’ll have already been given the information if you’re authorized to be on that server.

In this scenario, you’ll be taken back to the main login page, where you’ll need to put in your new login information.



**The Sage Notebook**  
Version 5.0.1

---

**Congratulations Aaron\_Aardvark! You can now sign into the Sage Notebook.**

**Welcome!**

Sage is a different approach to mathematics software.

**The Sage Notebook**

With the Sage Notebook anyone can create, collaborate on, and publish interactive worksheets. In a worksheet, one can write code using Sage, Python, and other software included in Sage.

**General and Advanced Pure and Applied Mathematics**

Use Sage for studying calculus, elementary to very advanced number theory, cryptography, commutative algebra, group theory, graph theory, numerical and exact linear algebra, and more.

**Use an Open Source Alternative**

By using Sage you help to support a viable open source alternative to Magma, Maple, Mathematica, and MATLAB. Sage includes many high-quality open source math packages.

**Use Most Mathematics Software from Within Sage**

Sage makes it easy for you to use most mathematics software together. Sage includes GAP, GP/PARI, Maxima, and Singular, and dozens of other open packages.

**Use a Mainstream Programming Language**

You work with Sage using the highly regarded scripting language Python. You can write programs that combine serious mathematics with anything else.

**Sign into the Sage Notebook v5.0.1**

Username

Password

☐ Remember me

[Sign up for a new Sage Notebook account](#)

[Browse published Sage worksheets \(no login required\)](#)

Then you'll be sent to a new notebook, ready to make your *first worksheet*.


## 1.2 Two Usage Scenarios

There are two main scenarios when starting with Sage on a server.

- You are going to a Sage server, and just want to start trying some mathematics. We cover this situation *first*.
- Someone has given you a link to a published tutorial or other worksheet (perhaps one similar to this!) and you would like to try out the mathematics there, using your own editable copy of the worksheet. We cover this less common situation *below*.

### 1.2.1 Starting a New Worksheet from Scratch

Sage on a server functions via individual documents called *worksheets*. If you are sent to one and you want to make a *live copy*, that is one thing, but usually you will start your Sage session with just an empty notebook, with no worksheets yet in it.



The Sage Notebook  
Version 5.0.1

Aaron\_Aardvark |
[Home](#) |
[Published](#) |
[Log](#) |
[Settings](#) |
[Help](#) |
[Report a Problem](#) |
[Sign out](#)

[New Worksheet](#)
[Upload](#)
[Download All Active](#)

Archive

Delete


Stop

Download

Current Folder: [Active](#) [Archived](#) [Trash](#)

<input type="checkbox"/>	Active Worksheets	Owner / Collaborators	Last Edited
Welcome to Sage! You can <a href="#">create a new worksheet</a> , view <a href="#">published worksheets</a> , or read the <a href="#">documentation</a> .			
<< previous <span>1</span> next >>			

There are a few things you can do here, but usually you'll want to start a new worksheet.



The Sage Notebook  
Version 5.0.1

Aaron\_Aardvark |
[Home](#) |
[Published](#) |
[Log](#) |
[Settings](#) |
[Help](#) |
[Report a Problem](#) |
[Sign out](#)

[New Worksheet](#)
[Upload](#)
[Download All Active](#)

Archive

Delete

Stop

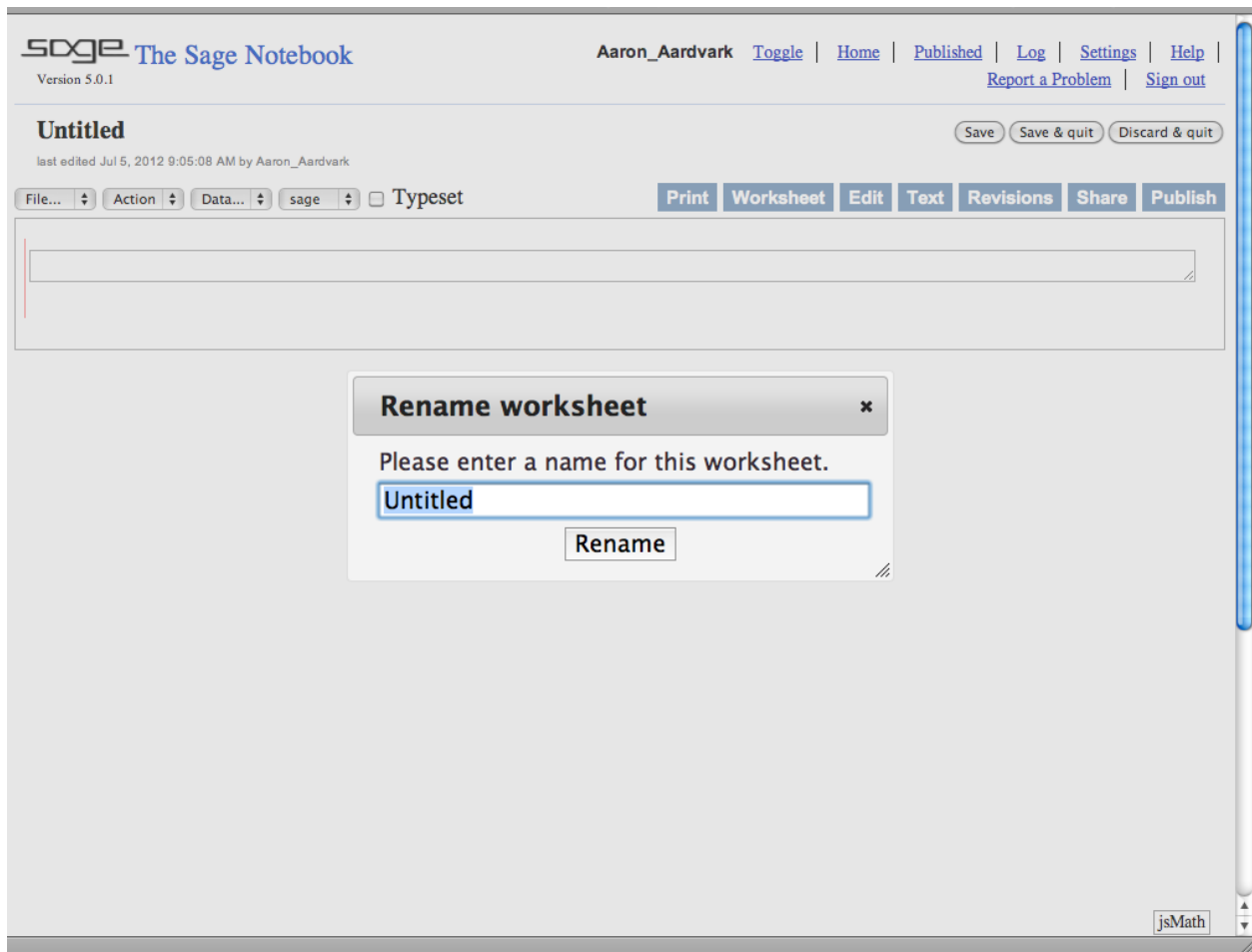
Download

Current Folder: [Active](#) [Archived](#) [Trash](#)

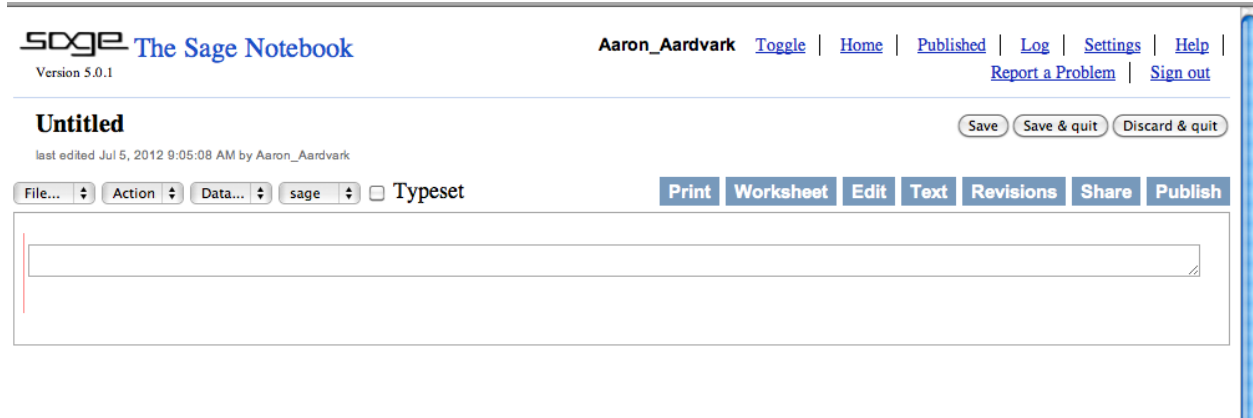
<input type="checkbox"/>	Active Worksheets	Owner / Collaborators	Last Edited
Welcome to Sage! You can <a href="#">create a new worksheet</a> , view <a href="#">published worksheets</a> , or read the <a href="#">documentation</a> .			
<< previous <span>1</span> next >>			

Click here for your first worksheet!

Once you've done this, it should look something like this:



You can leave the name, or call it whatever you like. Then you should see your first “cell”, the rectangle in this picture.



But at this point you are ready to go on *evaluate Sage commands*!

## 1.2.2 Getting a Live Copy of a Worksheet

Occasionally, you’ll get started with Sage by someone giving you a link to a *published worksheet* that someone else has created. In order to do math on it, you’ll need your *own* copy of the worksheet on the server.

If you are logged in and have your own copy, it should look like this at the top:

Except, of course, *your* username will appear! If you already have a live copy, you're all set and should start trying it out, possibly referring to the *first tutorial* for tips.

More likely, you'll need to follow a few steps.

- Take another look at the top of the screen. Does it look like this?

If you already have an account on the server, log in; otherwise, you may want to review how to *get an account*.

- Once you have an account and are logged in, you'll need to go back to your original link for the published worksheet. In either event, the worksheet should now look like this.

- Now just click 'Edit a copy' so that it looks like this!

Now you're ready to learn how to actually *evaluate those Sage commands*! Good luck.



## INTRODUCTORY SAGE TUTORIAL

This [Sage](#) document is the first in a series of tutorials developed for the MAA PREP Workshop “Sage: Using Open-Source Mathematics Software with Undergraduates” (funding provided by NSF DUE 0817071). It is licensed under the Creative Commons Attribution-ShareAlike 3.0 license ([CC BY-SA](#)).

If you are unsure how to log on to a Sage server or to create a new worksheet, you might find the [prelude on logging in](#) helpful. Otherwise, you can continue with this tutorial, which has the following sections:

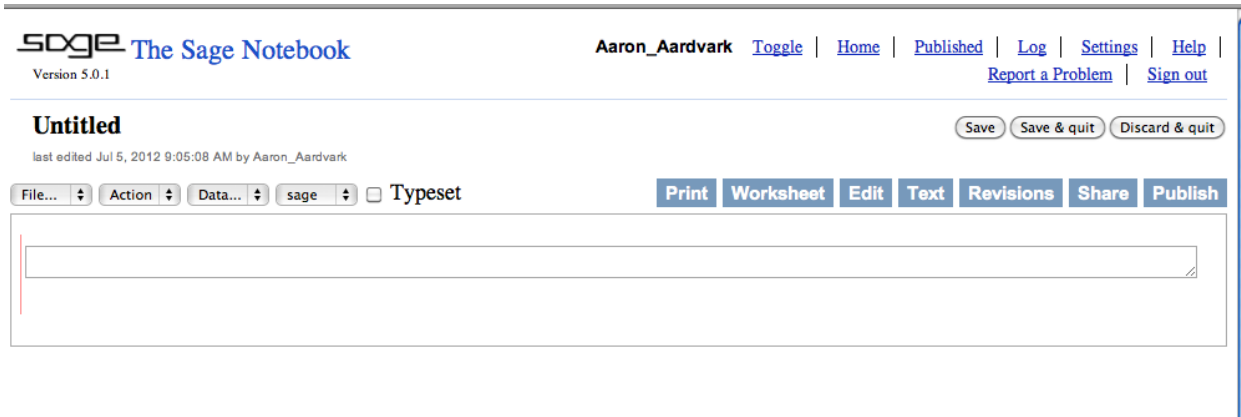
- [Evaluating Sage Commands](#)
- [Functions in Sage](#)
- [Help inside Sage](#)
- [Annotating with Sage](#)

This tutorial only introduces the most basic level of functionality. Later tutorials address topics such as calculus, advanced plotting, and a wide variety of specific mathematical topics.

### 2.1 Evaluating Sage Commands

#### Or, How do I get Sage to do some math?

In any Sage worksheet, there are little boxes called *input cells* or *code cells*. They should be about the width of your browser.



Evaluating the content of an input cell is very easy.

- First, click inside the cell so that the cell is active (i.e., has a bright blue border).

**Untitled**

last edited Jul 5, 2012 9:05:08 AM by Aaron\_Aardvark

File... Action Data... sage ☐ Typeset Print W

[evaluate](#)

- Then, just below the cell on the left, an “evaluate” link appears; clicking this link evaluates the cell.

**Untitled**

last edited Jul 5, 2012 9:05:08 AM by Aaron\_Aardvark

File... Action Data... sage ☐ Typeset Print W

[evaluate](#)

Click here to evaluate  
(or press Shift-Enter)

If you are using the live version of this documentation, try evaluating the following cell.

**sage:** 2+2

4

Sage prints out its response just below the cell (that’s the 4 above, so Sage confirms that  $2 + 2 = 4$ ). Note also that Sage has automatically made the next cell active after you evaluated your first cell.

You can also evaluate a cell using a keyboard shortcut.

- If a cell isn’t active (such as below, in the live documentation), click in it.
- Then hold down the Shift key while you press the Enter key.

We call this “Shift-Enter”. Try doing Shift-Enter with this cell.

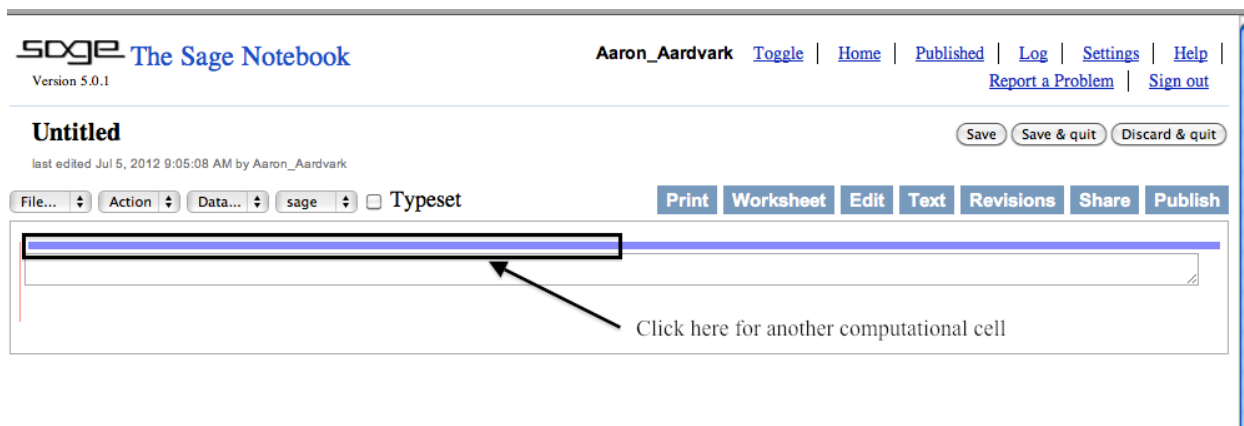
```
sage: factor(2012)
2^2 * 503
```

An input cell isn't much use if it can only do one thing, so you can edit a cell and evaluate it again. Just click inside, and then make any changes you wish by typing as usual.

Try changing the number 2012 above to 2011 and evaluate the cell to find its factorization (surprised?); then try your own favorite number.

To do more math, we'll need to be able to create new input cells. This is also easy.

- Move your cursor over the space above or below another cell.
- A blue horizontal line as wide as the browser should appear.
- Click on the line to insert a new cell.



If for some reason you need to remove or delete an input cell, just delete all the text inside of it, and then press backspace in the now-empty cell.

Try creating a few new input cells below, doing some arithmetic in those cells, and then deleting one of the input cells.

```
'Do some arithmetic in me, and make some other cells after me!'
```

## 2.2 Functions in Sage

To start out, let's explore how to define and use functions in Sage.

For a typical mathematical function, it's pretty straightforward to define it. Below, we define a function.

$$f(x) = x^2$$

```
sage: f(x)=x^2
```

Since all we wanted was to create the function  $f(x)$ , Sage just does this and doesn't print anything out back to us.

We can check the definition by asking Sage what  $f(x)$  is:

```
sage: f(x)
x^2
```

If we just ask Sage what  $f$  is (as opposed to  $f(x)$ ), Sage prints out the standard mathematical notation for a function that maps a variable  $x$  to the value  $x^2$  (with the “maps to” arrow  $\mapsto$  as `|-->`).

```
sage: f
x |--> x^2
```

We can evaluate  $f$  at various values.

```
sage: f(3)
9
```

```
sage: f(3.1)
9.610000000000000
```

```
sage: f(31/10)
961/100
```

Notice that the output type changes depending on whether the input had a decimal; we’ll see that again below.

Naturally, we are not restricted to  $x$  as a variable. In the next cell, we define the function  $g(y) = 2y - 1$ .

```
sage: g(y)=2*y-1
```

However, we need to make sure we do define a function if we use a new variable. In the next cell, we see what happens if we try to use a random input by itself.

```
sage: z^2
Traceback (most recent call last):
...
NameError: name 'z' is not defined
```

This is explained in some detail in following tutorials. At this point, it suffices to know using the function notation (like  $g(y)$ ) tells Sage you are serious about  $y$  being a variable.

One can also do this with the `var('z')` notation below.

```
sage: var('z')
z
sage: z^2
z^2
```

This also demonstrates that we can put several commands in one cell, each on a separate line. The output of the last command (if any) is printed as the output of the cell.

Sage knows various common mathematical constants, like  $\pi$  (`pi`) and  $e$ .

```
sage: f(pi)
pi^2
```

```
sage: f(e^-1)
e^(-2)
```

In order to see a numeric approximation for an expression, just type the expression inside the parentheses of `N()`.

```
sage: N(f(pi))
9.86960440108936
```

Another option, often more useful in practice, is having the expression immediately followed by `.n()` (note the dot).

```
sage: f(pi).n()
9.86960440108936
```

For now, we won't go in great depth explaining the reasons behind this syntax, which may be new to you. For those who are interested, Sage often uses this type of syntax (known as “object-oriented”) because...

- Sage uses the Python programming language, which uses this syntax, ‘under the hood’, and
- Because it makes it easier to distinguish among
- The mathematical object,
- The thing you are doing to it, and
- Any ancillary arguments.

For example, the following numerically evaluates  $(n)$  the constant  $\pi$  (`pi`) to twenty digits (`digits=20`).

```
sage: pi.n(digits=20)
3.1415926535897932385
```

Sage has lots of common mathematical functions built in, like  $\sqrt{x}$  (`sqrt(x)`) and  $\ln(x)$  (`ln(x)` or `log(x)`).

```
sage: log(3)
log(3)
```

Notice that there is no reason to numerically evaluate  $\log(3)$ , so Sage keeps it symbolic. The same is true in the next cell -  $2\log(3) = \log(9)$ , but there isn't any reason to do that; after all, depending on what you want,  $\log(9)$  may be simpler or less simple than you need.

```
sage: log(3)+log(3)
2*log(3)
```

```
sage: log(3).n()
1.09861228866811
```

Notice again that Sage tries to respect the type of input as much as possible; adding the decimal tells Sage that we have approximate input and want a more approximate answer. (Full details are a little too complicated for this introduction.)

```
sage: log(3.)
1.09861228866811
```

```
sage: sqrt(2)
sqrt(2)
```

If we want this to look nicer, we can use the `show` command. We'll see more of this sort of thing below.

```
sage: show(sqrt(2))
```

$$\sqrt{2}$$

```
sage: sqrt(2).n()
1.41421356237310
```

Do you remember what  $f$  does?

```
sage: f(sqrt(2))
2
```

We can also plot functions easily.

```
sage: plot(f, (x,-3,3))  
Graphics object consisting of 1 graphics primitive
```

In another tutorial, we will go more in depth with plotting. Here, note that the preferred syntax has the variable and endpoints for the plotting domain in parentheses, separated by commas.

If you are feeling bold, plot the `sqrt` function in the next cell between 0 and 100.

## 2.3 Help inside Sage

There are various ways to get help for doing things in Sage. Here are several common ways to get help as you are working in a Sage worksheet.

### 2.3.1 Documentation

Sage includes extensive documentation covering thousands of functions, with many examples, tutorials, and other helps.

- One way to access these is to click the “Help” link at the top right of any worksheet, then click your preferred option at the top of the help page.
- They are also available any time online at the [Sage website](#), which has many other links, like video introductions.
- The [Quick Reference cards](#) are another useful tool once you get more familiar with Sage.

Our main focus in this tutorial, though, is help you can immediately access from within a worksheet, where you don’t have to do *any* of those things.

### 2.3.2 Tab completion

The most useful help available in the notebook is “tab completion”. The idea is that even if you aren’t one hundred percent sure of the name of a command, the first few letters should still be enough to help find it. Here’s an example.

- Suppose you want to do a specific type of plot - maybe a slope field plot - but aren’t quite sure what will do it.
- Still, it seems reasonable that the command might start with `pl`.
- Then one can type `pl` in an input cell, and then press the tab key to see all the commands that start with the letters `pl`.

Try tabbing after the `pl` in the following cell to see all the commands that start with the letters `pl`. You should see that `plot_slope_field` is one of them.

```
sage: pl
```

To pick one, just click on it; to stop viewing them, press the Escape/esc key.

You can also use this to see what you can do to an expression or mathematical object.

- Assuming your expression has a name, type it;
- Then type a period after it,
- Then press tab.

You will see a list pop up of all the things you can do to the expression.

To try this, evaluate the following cell, just to make sure  $f$  is defined.

```
sage: f(x)=x^2
```

Now put your cursor after the period and press your tab key.

```
sage: f.
```

Again, Escape should remove the list.

One of the things in that list above was `integrate`. Let's try it.

```
sage: f.integrate(x)
x |--> 1/3*x^3
```

### 2.3.3 Finding documentation

#### Or, Why all the question marks?

In the previous example, you might have wondered why I needed to put `f.integrate(x)` rather than just `f.integrate()`, by analogy with `sqrt(2).n()`.

To find out, there is another help tool one can use from right inside the notebook. Almost all documentation in Sage has extensive examples that can illustrate how to use the function.

- As with tab completion, type the expression, period, and the name of the function.
- Then type a question mark.
- Press tab *or* evaluate to see the documentation.

To see how this help works, move your cursor after the question mark below and press tab.

```
sage: f.integrate?
```

The examples illustrate that the syntax requires `f.integrate(x)` and not just `f.integrate()`. (After all, the latter could be ambiguous if several variables had already been defined).

To stop viewing the documentation after pressing tab, you can press the Escape key, just like with the completion of options.

If you would like the documentation to be visible longer-term, you can *evaluate* a command with the question mark (like below) to access the documentation, rather than just tabbing. Then it will stay there until you remove the input cell.

```
sage: binomial?
```

Try this with another function!

### 2.3.4 Finding the source

There is one more source of help you may find useful in the long run, though perhaps not immediately.

- One can use *two* question marks after a function name to pull up the documentation *and* the source code for the function.

- Again, to see this help, you can either evaluate a cell like below, or just move your cursor after the question mark and press tab.

The ability to see the code (the underlying instructions to the computer) is one of Sage’s great strengths. You can see *all* the code to *everything*.

This means:

- You can see what Sage is doing.
- Your curious students can see what is going on.
- And if you find a better way to do something, then you can see how to change it!

```
sage: binomial??
```

## 2.4 Annotating with Sage

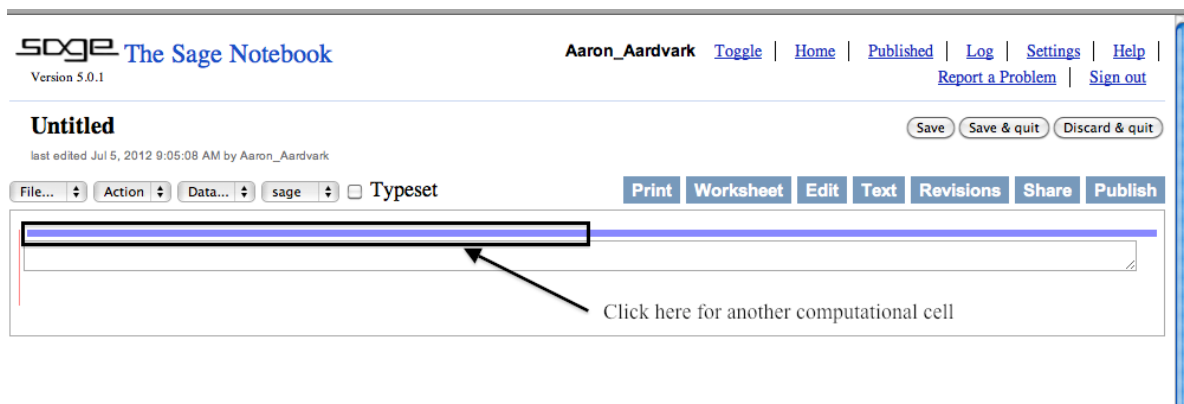
Whether one uses Sage in the classroom or in research, it is usually helpful to describe to the reader what is being done, such as in the description you are now reading.

Thanks to the [mini-word processor TinyMCE](#) and a TeX rendering engine called [MathJax](#), you can type much more in Sage than just Sage commands. This math-aware setup makes Sage perfect for annotating computations.

To use the word processor, we create a *text cell* (as opposed to a *input cell* that contains Sage commands that Sage evaluates).

To create a text cell, do the following.

- First, move the cursor between two input cells, until the thin blue line appears.
- Then hold the Shift key and click on the thin blue line.



So to create an input cell, one merely clicks, but one “Shift-Click”’s to create a text cell. Here is what your text cell will look like.



last edited Jul 5, 2012 9:05:08 AM by Aaron\_Aardvark

21

- One just types things like:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left( \frac{1}{1-p^{-s}} \right)$$

in the word processor.

- Whether this shows up as nicely as possible depends on what fonts you have in your browser, but it should be legible.
- More realistically, we might type `$f(x)=x^2$` so that we remember that  $f(x) = x^2$  in this worksheet.

Here is a simpler example.

```
sage: f(x) = x^2
sage: f(9)
81
```

If  $f(x) = x^2$ , then  $f(9) = 81$ .

It is simple to edit a text cell; simply double-click on the text.

If you are in the live version of this tutorial, try double-clicking on this text to edit this text cell (or any text cell) to see how we typed the mathematics!

Of course, one can do much more, since Sage can execute arbitrary commands in the [Python](#) programming language, as well as output nicely formatted HTML, and so on. If you have enough programming experience to do things like this, go for it!

```
sage: html("Sage is <a style='text-decoration:line-through'>somewhat</a> <b>really</b>  
→ cool! <p style='color:red'>(It even does HTML.)</p>")
```

This concludes the introductory tutorial. Our hope is that now you can try finding and using simple commands and functions in Sage. Remember, help is as close as the notebook, or at [the Sage website](#).

## TUTORIAL FOR SYMBOLICS AND PLOTTING

This Sage document is one of the tutorials developed for the MAA PREP Workshop “Sage: Using Open-Source Mathematics Software with Undergraduates” (funding provided by NSF DUE 0817071). It is licensed under the Creative Commons Attribution-ShareAlike 3.0 license (CC BY-SA).

This tutorial has the following sections:

- *Symbolic Expressions*
- *Basic 2D Plotting*
- *Basic 3D Plotting*

It assumes that one is familiar with the absolute basics of functions and evaluation in Sage. We provide a (very) brief refresher.

1. Make sure the syntax below for defining a function and getting a value makes sense.
2. Then evaluate the cell by clicking the “evaluate” link, or by pressing Shift-Enter (hold down Shift while pressing the Enter key).

```
sage: f(x)=x^3+1
sage: f(2)
9
```

### 3.1 Symbolic Expressions

In the first tutorial, we defined *functions* using notation similar to that one would use in (say) a calculus course.

There is a useful variant on this - defining *expressions* involving variables. This will give us the opportunity to point out several important, and sometimes subtle, things.

In the cell below, we define an expression  $FV$  which is the future value of an investment of \$100, compounded continuously. We then substitute in values for  $r$  and  $t$  which calculate the future value for  $t = 5$  years and  $r = 5\%$  nominal interest.

```
sage: var('r,t')
(r, t)
sage: FV=100*e^(r*t)
```

```
sage: FV(r=.05,t=5)
128.402541668774
```

The previous cells point out several things to remember when working with symbolic expressions. Some are fairly standard.

- An asterisk (`*`) signifies multiplication. This should be how you always do multiplication.
- Although it is possible to allow implicit multiplication, this can easily lead to ambiguity.
- We can access the most important constants; for instance,  $e$  stands for the constant 2.71828.... Likewise,  $\pi$  (or  $\pi$ ) and  $I$  (think complex numbers) are also defined.
- Of course, if you redefine  $e$  to be something else, all bets are off!

However, two others may be unfamiliar, especially if you have not used much mathematical software before.

- You must tell Sage what the variables are before using them in a symbolic expression.
- We did that above by typing `var('r,t')`.
- This is automatically done with the  $f(x)$  notation, but without that it is necessary, so that Sage knows one intends  $t$  (for instance) is a symbolic variable and not a number or something else.
- If you then wish to substitute some values into the expression, you must explicitly tell Sage which variables are being assigned which values.
- For instance, above we used `FV(r=.05,t=5)` to indicate the precise values of  $r$  and  $t$ .

Notice that when we define a function, we don't need to specify which variable has which value. In the function defined below, we have already specified an order.

```
sage: FV2(r,t)=100*e^(r*t)
sage: FV2(.05,5)
128.402541668774
```

In this case it is clear that  $r$  is first and  $t$  is second.

But with `FV=100*e^(r*t)`, there is no particular reason  $r$  or  $t$  should be first.

```
sage: FV(r=.05,t=5); FV(t=5,r=.05)
128.402541668774
128.402541668774
```

This is why we receive a deprecation error message when we try to do `FV` without explicitly mentioning the variables.

```
sage: FV(5,.05)
doctest:...: DeprecationWarning: Substitution using function-call syntax and unnamed
↳arguments is deprecated and will be removed from a future release of Sage; you can
↳use named arguments instead, like EXPR(x=..., y=...)
See http://trac.sagemath.org/5930 for details.
128.402541668774
```

In this case, the outcome is the same, since  $rt = tr$ ! Of course, in most expressions, one would not be so lucky, as the following example indicates.

```
sage: y = var('y')
sage: G = x*y^2
sage: G(1,2); G(2,1)
4
2
```

Also remember that when we don't use function notation, we'll need to define our variables.

One of the great things we can do with expressions is manipulate them. Let's make a typical expression.

```
sage: z = (x+1)^3
```

In the cells below, you'll notice something new: the character `#`. In Sage (and in [Python](#)), anything on a single line after the number/pound sign (the [octothorpe](#)) is ignored. We say that `#` is a comment character. We use it below to mention alternative ways to do the same thing.

```
sage: expand(z) # or z.expand()
x^3 + 3*x^2 + 3*x + 1
```

```
sage: y = expand(z)
sage: y.factor() # or factor(y)
(x + 1)^3
```

In the previous cell, we *assigned* the expression which is the expansion of  $z$  to the variable  $y$  with the first line. After that, anything we want to do to the expansion of  $z$  can be done by doing it to  $y$ .

There are more commands like this as well. Notice that  $z$  will no longer be  $(x + 1)^3$  after this cell is evaluated, since we've assigned  $z$  to a (much more complex) expression.

```
sage: z = ((x - 1)^(3/2) - (x + 1)*sqrt(x - 1))/sqrt((x - 1)*(x + 1))
sage: z.simplify_full()
-2*sqrt(x - 1)/sqrt(x^2 - 1)
```

This is a good place for a few reminders of basic help.

- You can see various methods for simplifying an expression by using tab completion. Put your cursor at the end of the next cell (after the `simplify`) and press tab to see lots of different methods.
- Also remember that you can use the question mark (e.g., `z.simplify_rational?`) to get help about a particular method.

```
sage: z.simplify
<built-in method simplify of sage.symbolic.expression.Expression object at ...>
```

Finally, recall that you can get nicely typeset versions of the output in several ways.

- One option is to click the 'Typeset' button at the top.
- Another - which does not require scrolling! - is to use the `show` command.

```
sage: show(z.simplify_rational())
```

$$-\frac{2\sqrt{x-1}}{\sqrt{x^2-1}}$$

Another Sage command that is useful in this context is `solve`.

Here, we solve the simple equation  $x^2 = -1$ .

```
sage: solve(x^2==-1,x) # solve x^2==-1 for x
[x == -I, x == I]
```

- In the `solve` command, one types an equals sign in the equation as *two* equal signs.
- This is because the single equals sign means assignment to a variable, as we've done above, so Sage (along with Python) uses the double equals sign for symbolic equality.
- We also include the variable we'd like to solve for after the comma.

It's also possible to solve more than one expression simultaneously.

```
sage: solve([x^2==1, x^3==1], x)
[[x == 1]]
```

## 3.2 Basic 2D Plotting

One of the other basic uses of mathematics software is easy plotting. Here, we include a brief introduction to the sorts of plotting which will prepare us to use Sage in calculus. (There will be a separate tutorial for more advanced plotting techniques.)

Recall that we can generate a plot using fairly simple syntax. Here, we define a function of  $x$  and plot it between  $-1$  and  $1$ .

```
sage: f(x)=x^3+1
sage: plot(f, (x, -1, 1))
Graphics object consisting of 1 graphics primitive
```

We can give the plot a name, so that if we want to do something with the plot later, we don't have to type out the entire plot command. Remember, this is called *assigning* the plot to the name/variable.

In the next cell, we give the plot the name  $P$ .

```
sage: P=plot(f, (x, -1, 1))
```

One plot is nice, but one might want to superimpose plots as well. For instance, the tangent line to  $f$  at  $x = 0$  is just the line  $y = 1$ , and we might want to show this together with the plot.

So let's plot this line in a different color, and with a different style for the line, but over the same interval.

```
sage: Q=plot(1, (x, -1, 1), color="red", linestyle="--")
sage: Q
Graphics object consisting of 1 graphics primitive
```

Because we put  $Q$  in a line by itself at the end, it shows. We were able to use just one cell to define  $Q$  and show it, by putting each command in a separate line in the same input cell.

Now to show the plots superimposed on each other, we simply add them.

```
sage: P+Q
Graphics object consisting of 2 graphics primitives
```

Suppose we wanted to view a detail of this.

- We could create another plot with different endpoints.
- Another way is to keep the currently created plots, but to set the viewing window using the `show` command, as below.

```
sage: (P+Q).show(xmin=-.1, xmax=.1, ymin=.99, ymax=1.01)
```

Since the axes no longer cross in the frame of reference, Sage shows a short gap between the horizontal and vertical axes.

There are many options one can pass in for various purposes.

- Some require quotes around the values.
- Such as the `color` option when we made a "red" line.
- Some do not.
- Such as the `xmin` in the previous plot, where the minimum  $x$  value was just  $-1$ .

Usually (though not always) quotes are required for option values which are words or strings of characters, and not required for numerical values.

Two of the most useful of these options help in labeling graphs.

- The `axes_labels` option labels the axes.
- As with the word processor, we can use dollar signs (like in LaTeX) to make the labels typeset nicely.
- Here we need both quotes and brackets for proper syntax, since there are two axes to label and the labels are not actually numbers.

```
sage: plot(f, (x, -1, 1), axes_labels=['$x$', '$y$'], legend_label='$f(x)$', show_
↳ legend=True)
Graphics object consisting of 1 graphics primitive
```

- The `legend_label` option is especially useful with multiple plots.
- LaTeX notation works here too.
- In the graphic above, we needed to explicitly ask to show the label. With multiple graphs this should not be necessary.

```
sage: P1 = plot(f, (x, -1, 1), axes_labels=['$x$', '$y$'], legend_label='$f(x)$')
sage: P2 = plot(sin, (x, -1, 1), axes_labels=['$x$', '$y$'], legend_label='$\sin(x)$', color=
↳ 'red')
sage: P1+P2
Graphics object consisting of 2 graphics primitives
```

One additional useful note is that plots of functions with vertical asymptotes may need their vertical viewing range set manually; otherwise the asymptote may really go to infinity!

```
sage: plot(1/x^2, (x, -10, 10), ymax=10)
Graphics object consisting of 1 graphics primitive
```

Remember, you can use the command `plot?` to find out about most of the options demonstrated above.

Below, you can experiment with several of the plotting options.

- Just evaluate the cell and play with the sliders, buttons, color picker, etc., to change the plot options.
- You can access low-level options like the initial number of plotted points, or high-level ones like whether axes are shown or not.
- This uses a feature of Sage called “interacts”, which is a very powerful way to engage students in exploring a problem.

```
sage: x = var('x')
sage: @interact
sage: def plot_example(f=sin(x^2), r=range_slider(-5, 5, step_size=1/4, default=(-3, 3)),
....:                  color=color_selector(widget='colorpicker'),
....:                  thickness=(3, (1..10)),
....:                  adaptive_recursion=(5, (0..10)), adaptive_tolerance=(0.01, (0.
↳ 001, 1)),
....:                  plot_points=(20, (1..100)),
....:                  linestyle=['-', '--', '-.', ':'],
....:                  gridlines=False, fill=False,
....:                  frame=False, axes=True
....:                  ):
....:     show(plot(f, (x, r[0], r[1]), color=color, thickness=thickness,
....:                  adaptive_recursion=adaptive_recursion,
....:                  adaptive_tolerance=adaptive_tolerance, plot_points=plot_points,
....:                  linestyle=linestyle, fill=fill if fill else None),
....:           gridlines=gridlines, frame=frame, axes=axes)
```

### 3.3 Basic 3D Plotting

There are several mechanisms for viewing three-dimensional plots in Sage, but we will stick to the default option in the notebook interface, which is via Java applets from the program [Jmol](#).

Plotting a 3D plot is similar to plotting a 2D plot, but we need to specify ranges for two variables instead of one.

```
sage: g(x,y)=sin(x^2+y^2)
sage: plot3d(g, (x,-5,5), (y,-5,5))
Graphics3d Object
```

There is a lot you can do with the 3D plots.

- Try rotating the plot above by clicking and dragging the mouse inside of the plot.
- Also, right-click (Control-click if you have only one mouse button) just to the right of the plot to see other options in a menu.
- If you have a wheel on your mouse or a multi-touch trackpad, you can scroll to zoom.
- You can also right-click to see other options, such as
- spinning the plot,
- changing various colors,
- and even making the plot suitable for viewing through 3D glasses (under the “style”, then “stereographic” submenus),

When using the `plot3d` command, the first variable range specified is plotted along the usual “x” axis, while the second range specified is plotted along the usual “y” axis.

The plot above is somewhat crude because the function is not sampled enough times - this is fairly rapidly changing function, after all. We can make the plot smoother by telling Sage to sample the function using a grid of 300 by 300 points. Sage then samples the function at 90,000 points!

```
sage: plot3d(g, (x,-5,5), (y,-5,5), plot_points=300)
Graphics3d Object
```

As with 2D plots, we can superimpose 3D plots by adding them together.

Note that in this one, we do not define the functions, but only use expressions (see the first set of topics in this tutorial), so it is wisest to define the variables ahead of time.

```
sage: var('x,y')
(x, y)
sage: b = 2.2
sage: P=plot3d(sin(x^2-y^2), (x,-b,b), (y,-b,b), opacity=.7)
sage: Q=plot3d(0, (x,-b,b), (y,-b,b), color='red')
sage: P+Q
Graphics3d Object
```

As usual, only the last command shows up in the notebook, though clearly all are evaluated. This also demonstrates that many of the same options work for 3D plots as for 2D plots.

We close this tutorial with a cool plot that we define *implicitly* as a 3D contour plot.

```
sage: var('x,y,z')
(x, y, z)
sage: T = golden_ratio
sage: p = 2 - (cos(x + T*y) + cos(x - T*y) + cos(y + T*z) + cos(y - T*z) + cos(z -
↪T*x) + cos(z + T*x))
```



```
sage: r = 4.78
sage: implicit_plot3d(p, (x, -r, r), (y, -r, r), (z, -r, r), plot_points=50, color=
↪ 'yellow')
Graphics3d Object
```

The next tutorial will use all that you have learned about Sage basics, symbolics, and plotting in a specific mathematical venue - the calculus sequence!



## TUTORIAL FOR CALCULUS

This Sage document is one of the tutorials developed for the MAA PREP Workshop “Sage: Using Open-Source Mathematics Software with Undergraduates” (funding provided by NSF DUE 0817071). It is licensed under the Creative Commons Attribution-ShareAlike 3.0 license (CC BY-SA).

This tutorial has the following sections. The first three are based on the topics encountered in a typical three-semester calculus sequence in the United States; the final section is a checkpoint of sorts.

- *Calculus 1*
- *Calculus 2*
- *Calculus 3*
- *‘Exam’*

The tutorial assumes that one is familiar with the basics of Sage, such as outlined in the previous tutorials.

For a refresher, make sure the syntax below for defining a function and getting a value makes sense; then evaluate the cell by clicking the “evaluate” link, or by pressing Shift-Enter (hold down Shift while pressing the Enter key).

```
sage: f(x)=x^3+1
sage: f(2)
9
```

We’ll use this function several times in this tutorial, so it’s important that it is evaluated!

### 4.1 Calculus 1

The calculus is amazing not so much because it solves problems of tangency or area - individual solutions to these problems have been known before. It is amazing because it gives a remarkably comprehensive set of rules for symbolic manipulation for solving such problems in great generality!

Thus, the most typical use of a computer system like Sage in this context is simply to help check (or make less tedious) basic symbolic manipulation of things like derivatives. Let’s first show what our function  $f$  is again, then do things with it.

```
sage: show(f)
```

$$x \mapsto x^3 + 1$$

Something most (but not all) curricula for first-semester calculus do is spend a fair amount of time with limits.

What is the limit of this function as  $x$  approaches 1?

```
sage: lim(f, x=1)
x |--> 2
```

Sage gives the answer we expect. The syntax for limits is pretty straightforward, though it may differ slightly from that of other systems.

Since a limit may exist even if the function is not defined, Sage uses the syntax `x=1` to show the input value approached in all situations.

```
sage: lim((x^2-1)/(x-1), x=1)
2
```

Next, we'll return to the original function, and check the two directional limits.

The syntax uses the extra *keyword* `dir`.

- Notice that we use `dir='right'`, not `dir=right`.
- It's also okay to use `dir='+'`, like we use `dir='-'` below.
- This is basically because otherwise we might have done something like `let right=55` earlier, so instead Sage - and Python - requires us to put `'right'` and `'-'` in quotes to make it clear they aren't variables.

```
sage: lim(f, x=1, dir='-'); lim(f, x=1, dir='right'); f(1)
x |--> 2
x |--> 2
2
```

By comparing with  $f(1)$ , we see that  $f(x)$  is continuous at  $x = 1$ .

This cell also reminds us of something else:

- By separating several commands with semicolons, we can tell Sage to evaluate all of them while still seeing all their outputs.

What we spend the most time on in Calculus 1 is derivatives, and Sage is fully-featured here.

For example, here are three ways to get the basic, single-variable derivative of  $f(x) = x^3 + 1$ .

```
sage: diff(f, x); derivative(f, x); f.derivative(x)
x |--> 3*x^2
x |--> 3*x^2
x |--> 3*x^2
```

Naturally, Sage knows all of the derivatives you want.

```
sage: derivative(sinh(x^2+sqrt(x-1)), x)
1/2*(4*x + 1/sqrt(x - 1))*cosh(x^2 + sqrt(x - 1))
```

And maybe even knows those you don't want. In this case, we put the computation inside `show()` since the output is so long.

```
sage: show(derivative(sinh(x^2+sqrt(x-1)), x, 3))
```

$$\frac{1}{8} \left( 4x + \frac{1}{\sqrt{x-1}} \right)^3 \cosh(\sqrt{x-1} + x^2) - \frac{3}{8} \left( \frac{1}{(x-1)^{\frac{3}{2}}} - 8 \right) \left( 4x + \frac{1}{\sqrt{x-1}} \right) \sinh(\sqrt{x-1} + x^2) + \frac{3 \cosh(\sqrt{x-1} + x^2)}{8(x-1)^{\frac{5}{2}}}$$

A common question is why Sage might not check automatically if there is some “simpler” version. But simplifying an expression, or indeed, even defining what a ‘simple’ expression is, turns out to be a very hard technical and mathematical problem in general. Computers won't solve every problem!

As a brief interlude, let's consider an application of our ability to do some basic differential calculus.

First, as a reminder of plotting from the previous tutorial, try to plot the function  $f(x)$ , together with its tangent line at  $x = 1$ , in the empty cells below. (If you can, do it without looking at the previous tutorial for a reminder.)

Did you get it?

Of course, in general we might want to see tangent lines at lots of different points - for instance, for a class demonstration.

So in the following cell, there are several auxiliary elements:

- We define the plot  $P$  of the original function.
- There is a parameter  $c=1/3$ , which is the  $x$ -value where we want a tangent line.
- We let `fprime` be the derivative function simply by declaring it equal to the derivative.
- We let  $L$  be the tangent line defined by the point-slope formula at  $x = c$ .
- We make  $Q$  be the plot of this line.

Finally, we plot everything together in the last line by adding  $P + Q$ .

```
sage: P=plot(f, (x,-1,1))
sage: c=1/3
sage: fprime=derivative(f,x)
sage: L(x)=fprime(c)*(x-c)+f(c)
sage: Q=plot(L, (x,-1,1), color="red", linestyle="--")
sage: P+Q
Graphics object consisting of 2 graphics primitives
```

You may want to experiment by

- Changing  $c$  to some other value, or
- Changing the function  $f$ , or
- Changing the colors, or
- Changing something else (like the `linestyle` used for the tangent line).

Ideally, it would be *extremely* easy to change that parameter  $c$ . In the cell below, we show our second example of a Sage “interact” (or “Sagelet”).

In this one, dragging a slider will show the tangent line moving.

- Future tutorials will explain this process in more detail; here it's just as an example.
- However, the reader will note how very similar the code for the two cells is.

```
sage: %auto
sage: f(x)=x^3+1
sage: @interact
sage: def _(c=(1/3, (-1,1))):
....:     P=plot(f, (x,-1,1))
....:     fprime=derivative(f,x)
....:     L(x)=fprime(c)*(x-c)+f(c)
....:     Q=plot(L, (x,-1,1), color="red", linestyle="--")
....:     show(P+Q+point((c,f(c)), pointsize=40, color='red'), ymin=0, ymax=2)
```

A very sharp-eyed reader will also have noticed that the previous cell had `%auto` at the very top, and that it was not necessary to evaluate the cell to use it.

- The command `%auto` allows us to have a cell, especially an interactive one, all loaded up as soon as we start - particularly convenient for a classroom situation.
- Such instructions are called *percent directives*. Most are documented in the notebook help one can access at the top of any worksheet.

A final topic in Calculus 1 usually is basic integration. The syntax for indefinite integration is similar to that for differentiation.

```
sage: integral(cos(x), x)
sin(x)
```

We don't get the whole indefinite integral, just a convenient antiderivative.

- (If you were to get a different answer 'by hand', remember that being an antiderivative means the answer is correct *up to a constant* - and deciding whether this is the case with two expressions has the same problems as the issue of "simplification" above.)

Definite integration has similar syntax to plotting.

```
sage: integral(cos(x), (x, 0, pi/2))
1
```

## 4.2 Calculus 2

Second-semester calculus is typically more challenging.

- One reason for that is that the computational problems are not so straightforward as computing derivatives and basic integrals.
- Another reason is that the second semester is usually where the harder versions of problems from the first semester show up.

Nonetheless, Sage can handle this as well.

Sage includes a large number of indefinite integrals (via Maxima), though not all the ones you will find in a comprehensive table.

```
sage: h(x)=sec(x)
sage: h.integrate(x)
x |--> log(sec(x) + tan(x))
```

Since I defined `h` as a function, the answer I get is also a function. If I just want an expression as the answer, I can do the following.

```
sage: integrate(sec(x), x)
log(sec(x) + tan(x))
```

Here is another (longer) example. Do you remember what command would help it look nicer in the browser?

```
sage: integrate(1/(1+x^5), x)
1/5*sqrt(5)*(sqrt(5) + 1)*arctan((4*x + sqrt(5) - 1)/sqrt(2*sqrt(5) + 10))/
↪ sqrt(2*sqrt(5) + 10) + 1/5*sqrt(5)*(sqrt(5) - 1)*arctan((4*x - sqrt(5) - 1)/sqrt(-
↪ 2*sqrt(5) + 10))/sqrt(-2*sqrt(5) + 10) - 1/2*(sqrt(5) + 3)*log(2*x^2 - x*(sqrt(5) +
↪ 1) + 2)/(5*sqrt(5) + 5) - 1/2*(sqrt(5) - 3)*log(2*x^2 + x*(sqrt(5) - 1) + 2)/
↪ (5*sqrt(5) - 5) + 1/5*log(x + 1)
```

Some integrals are a little tricky, of course. If Sage doesn't know the whole antiderivative, it returns as much of it as it (more properly, as Maxima) could do.

```
sage: integral(1/(1+x^10), x)
1/5*arctan(x) - 1/5*integrate((x^6 - 2*x^4 + 3*x^2 - 4)/(x^8 - x^6 + x^4 - x^2 + 1), x)
↪x)
```

```
sage: integral(sinh(x^2+sqrt(x-1)), x) # long time (15s on sage.math, 2012)
integrate(sinh(x^2 + sqrt(x - 1)), x)
```

This last one stumps other systems too.

However, if there is a special function which helps compute the integral, Sage will look for it. In the following case there is no elementary antiderivative, but the `erf` function helps us out.

```
sage: integral(e^(-x^2), x)
1/2*sqrt(pi)*erf(x)
```

Don't forget, if this function is unfamiliar to you (as it might be to students trying this integral), Sage's contextual help system comes to the rescue.

```
sage: erf?
```

There are several ways to do definite integrals in Sage.

The most obvious one is simply turning

$$\int f(x)dx$$

into

$$\int_a^b f(x)dx,$$

as indicated in the Calculus I section.

```
sage: integral(cos(x), (x, 0, pi/2))
1
```

The preferred syntax puts the variable and endpoints together in parentheses.

Just like with derivatives, we can visualize this integral using some of the plotting options from the plotting tutorial.

```
sage: plot(cos(x), (x, 0, pi/2), fill=True, ticks=[[0, pi/4, pi/2], None], tick_formatter=pi)
Graphics object consisting of 2 graphics primitives
```

It is possible to be completely symbolic in doing integration. If you do this, you'll have to make sure you define anything that's a symbolic variable - which includes constants, naturally.

```
sage: var('a, b')
(a, b)
sage: integral(cos(x), (x, a, b))
-sin(a) + sin(b)
```

On the numerical side, sometimes the answer one gets from the Fundamental Theorem of Calculus is not entirely helpful. Recall that  $h$  is the secant function.

```
sage: integral(h, (x, 0, pi/7))
1/2*log(sin(1/7*pi) + 1) - 1/2*log(-sin(1/7*pi) + 1)
```

Here, just a number might be more helpful. Sage has several ways of numerical evaluating integrals.

- Doing a definite integral symbolically, then approximating it numerically
- The `numerical_integral` function
- The `.nintegrate` method

The first one, using the `n` or `N` function for numerical approximation, was also mentioned in the introductory tutorial.

```
sage: N(integral(h, (x, 0, pi/8)))
0.403199719161511
```

The second function, `numerical_integral`, uses a powerful numerical program (the GNU Scientific Library).

- Unfortunately, the syntax for this function is not yet consistent with the rest of Sage.
- Helpfully, the output has two elements - the answer you desire, and its error tolerance.

```
sage: numerical_integral(h, 0, pi/8)
(0.4031997191615114, 4.476416117355069e-15)
```

To access just the number, one asks for the ‘zeroth’ element of this sequence of items. This is done with the following bracket notation.

```
sage: numerical_integral(h, 0, pi/8)[0]
0.4031997191615114
```

Notice that we began counting at zero. This is fairly typical in computer programs (though certainly not universal).

To aid readability (more important than one might think), we often assign the numerical integral to a variable, and then take the zeroth element of that.

```
sage: ni = numerical_integral(h, 0, pi/8)
sage: ni[0]
0.4031997191615114
```

Finally, the `.nintegrate()` method from Maxima gives even more extra information.

- Notice again the period/dot needed to use this.
- It is only possible to use `h(x)`; doing `h.nintegrate()` raises an error.

```
sage: h(x).nintegrate(x, 0, pi/8)
(0.4031997191615114, 4.47641611735507e-15, 21, 0)
```

Second-semester calculus usually also covers various topics in summation. Sage can sum many abstract series; the notation is similar to plotting and integration.

```
sage: var('n') # Don't forget to declare your variables
n
sage: sum((1/3)^n, n, 0, oo)
3/2
```

This is the geometric series, of course.



The next one is the famous result that a row of Pascal's triangle is a power of 2 -

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n,$$

which has many pleasing combinatorial interpretations.

```
sage: k = var('k') # We already declared n, so now we just need k
sage: sum(binomial(n,k), k, 0, n)
2^n
```

Do you remember what to do to see how we typed the nice sum in the text above? That's right, we can double-click the text area/cell to see this.

Sage also can compute Taylor polynomials.

Taylor expansions depend on a lot of things. Whenever there are several inputs, keeping syntax straight is important. Here we have as inputs:

- the function,
- the variable,
- the point around which we are expanding the function, and
- the degree.

In the next cell, we call  $g(x)$  the Taylor polynomial in question.

```
sage: g(x)=taylor(log(x),x,1,6); g(x)
-1/6*(x - 1)^6 + 1/5*(x - 1)^5 - 1/4*(x - 1)^4 + 1/3*(x - 1)^3 - 1/2*(x - 1)^2 + x - 1
```

Notice how close the approximation is to the function on this interval!

```
sage: plot(g, (x,0,2))+plot(log(x), (x,0,2), color='red')
Graphics object consisting of 2 graphics primitives
```

## 4.3 Calculus 3

We have already seen three-dimensional plotting, so it is not surprising that Sage has support for a variety of multi-variable calculus problems.

**Warning:** We will often need to define all variables other than  $x$ .

```
sage: var('y')
y
sage: f(x,y)=3*sin(x)-2*cos(y)-x*y
```

Above, we have defined a typical function of two variables.

Below, we use the separating semicolons to demonstrate several things one might do with such a function, including:

- The gradient vector of all  $\frac{\partial f}{\partial x_i}$
- The Hessian of all possible second derivatives
- A double partial derivative of  $f$  with respect to  $x$ , then  $y$  (that is,  $\frac{\partial f}{\partial y \partial x}$ )

```
sage: f.gradient(); f.hessian(); f.diff(x,y)
(x, y) |--> (-y + 3*cos(x), -x + 2*sin(y))
[(x, y) |--> -3*sin(x)      (x, y) |--> -1]
[      (x, y) |--> -1      (x, y) |--> 2*cos(y)]
(x, y) |--> -1
```

In an effort to make the syntax simpler, the gradient and Hessian are also available by asking for a total derivative. We also ask for nicer output again.

```
sage: show(f.diff()); show(f.diff(2))
```

$$(x, y) \mapsto (-y + 3 \cos(x), -x + 2 \sin(y))$$

$$\left( \begin{array}{cc} (x, y) \mapsto -3 \sin(x) & (x, y) \mapsto -1 \\ (x, y) \mapsto -1 & (x, y) \mapsto 2 \cos(y) \end{array} \right)$$

If we take the determinant of the Hessian, we get something useful for evaluating (the two-dimensional) critical points of  $f$ .

```
sage: show(f.diff(2).det())
```

$$(x, y) \mapsto -6 \sin(x) \cos(y) - 1$$

These ideas are particularly helpful if one wants to plot a vector field.

The following example is of the gradient. The vector plotted in the cell below is the unit vector in the direction  $(1, 2)$ .

```
sage: P=plot_vector_field(f.diff(), (x,-3,3), (y,-3,3))
sage: u=vector([1,2])
sage: Q=plot(u/u.norm())
sage: P+Q
Graphics object consisting of 2 graphics primitives
```

Rather than actually figure out the unit vector in that direction, it's easier to let Sage compute it by dividing the vector by its norm.

The directional derivative itself (in that direction, at the origin) can also be computed in this way.

```
sage: (f.diff()*u/u.norm())(0,0)
3/5*sqrt(5)
```

Another useful type of plot in these situations is a contour plot.

Notice that the one below uses several options. Try to correlate the options with features of the graphic.

```
sage: y = var('y')
sage: contour_plot(y^2 + 1 - x^3 - x, (x,-pi,pi), (y,-pi,pi),\
....:     contours=[-8,-4,0,4,8], colorbar=True, labels=True, label_colors='red')
Graphics object consisting of 1 graphics primitive
```

In this one, we have used options to:

- Explicitly list the contours we want to show,
- Label these contours,
- Place a color bar on the side to show the different levels.

(Incidentally, the `True` and `False` valued options are some of the few non-numerical ones that do *not* need quotes.)

This is another good time to remind us we must explicitly ask for  $y$  to be a variable here, as will be the case a few more times.

As you gain experience in Sage, we will slowly explain less and less of the syntax of commands in these tutorials. You can think of places where not everything is explained as a mini-quiz.

For example, the next example shows how one currently does a multiple integral. What have we done here?

```
sage: integrate(integrate(f, (x, 0, pi)), (y, 0, pi))
6*pi - 1/4*pi^4
```

Answer: notice that `integrate(f, (x, 0, pi))` has been itself placed as the function inside `integrate(..., (y, 0, pi))`.

We could use a 3D plot to help visualize this; these were already mentioned in the symbolics and plotting tutorial.

```
sage: plot3d(f, (x, 0, pi), (y, 0, pi), color='red')+plot3d(0, (x, 0, pi), (y, 0, pi))
Graphics3d Object
```

In addition to multivariate calculus, Calculus 3 often covers parametric calculus of a single variable. Sage can do arbitrary parametric plots, with fairly natural syntax.

This plot shows the tangent line to the most basic Lissajous curve at  $t = 1$ . The commands should be strongly reminiscent of the ones at the beginning of this tutorial.

```
sage: t = var('t')
sage: my_curve(t)=(sin(t), sin(2*t))
sage: PP=parametric_plot(my_curve, (t, 0, 2*pi), color="purple")
sage: my_prime=my_curve.diff(t)
sage: L=my_prime(1)*t+my_curve(1) # tangent line at t=1
sage: parametric_plot(L, (t, -2, 2))+PP
Graphics object consisting of 2 graphics primitives
```

#### Tip:

- After a while, you'll find that giving things names other than  $f$  and  $g$  becomes quite helpful in distinguishing things from each other. Use descriptive names! We have tried to do so here.
- If you are adventurous, try turning this into an interactive cell along the lines of the single variable example earlier!

## 4.4 'Exam'

Before moving out of the calculus world, it is good to have a sort of miniature exam.

In the cell below, we have plotted and given:

- A slope field for a differential equation,
- A solution to an initial value problem,
- And a symbolic formula for that solution.

We assume you have never seen several of the commands before. Can you nonetheless figure out which commands are doing each piece, and what their syntax is? How would you look for help to find out more?

```
sage: y = var('y')
sage: Plot1=plot_slope_field(2-y, (x,0,3), (y,0,20))
sage: y = function('y',x) # declare y to be a function of x
sage: h = desolve(diff(y,x) + y - 2, y, ics=[0,7])
sage: Plot2=plot(h,0,3)
sage: show(expand(h)); show(Plot1+Plot2)
```

$$5e^{(-x)} + 2$$

Ready to see the answers? Don't peek until you've really tried it.

In this cell we do the following:

- Make sure that `y` is indeed a variable for the first plot.
- Create a slope field for the DE for appropriate inputs of  $x$  and  $y$ , and give the plot the name `Plot1`.
- Use the formalism of the function command to get ready for the DE.
- Notice we have here once again used `#` to indicate a comment.
- In this case, in order to use common terminology, we now have told Sage `y` is no longer a variable, but instead a function (abstract) of the variable  $x$ .
- Use the differential equation solving command, with **I\*\*nitial \*\*C\*\*ondition \*\*S** of 2 and 2.
- Plot the solution and give it the name `Plot2`.
- Show a simplification of the symbolic version of the solution (which we didn't know ahead of time!) as well as the sum of the two graphs - the solution against the slope field.

As you gain experience, you will see how to glean what *you* are looking for from examples in the documentation like this - which is one of the real goals of these tutorials.

Congratulations! You are now armed with the basics of deploying Sage in the calculus sequence.

## SAGE INTRODUCTORY PROGRAMMING TUTORIAL

This Sage document is one of the tutorials developed for the MAA PREP Workshop “Sage: Using Open-Source Mathematics Software with Undergraduates” (funding provided by NSF DUE 0817071). It is licensed under the Creative Commons Attribution-ShareAlike 3.0 license (CC BY-SA).

This tutorial will cover the following topics (and various others throughout):

- *Methods and Dot Notation*
- *Lists, Loops, and Set Builders*
- *Defining Functions*
- *Gotchas from names and copies*
- *Appendix: Advanced Introductory Topics*

We will motivate our examples using basic matrices and situations one might want to handle in everyday use of matrices in the classroom.

### 5.1 Methods and Dot Notation

Making a new matrix is not too hard in Sage.

```
sage: A = matrix([[1,2],[3,4]])
```

As we can see, this gives the matrix with rows given in the two bracketed sets.

```
sage: A
[1 2]
[3 4]
```

Some commands are available right off the bat, like derivatives are. This is the determinant.

```
sage: det(A)
-2
```

But some things are not available this way - for instance a row-reduced echelon form. We can ‘tab’ after this to make sure.

```
sage: r
```

So, as we’ve already seen in previous tutorials, *many* of the commands in Sage are “methods” of objects.

That is, we access them by typing:

- the name of the mathematical object,
- a dot/period,
- the name of the method, and
- parentheses (possibly with an argument).

This is a huge advantage, once you get familiar with it, because it allows you to do *only* the things that are possible, and *all* such things.

First, let's do the determinant again.

```
sage: A.det()  
-2
```

Then we do the row-reduced echelon form.

```
sage: A.rref()  
[1 0]  
[0 1]
```

It is very important to keep in the parentheses.

---

**Note:** Things that would be legal without them would be called ‘attributes’, but Sage prefers stylistically to hide them, since math is made of functions and not elements of sets. Or so a category-theorist would say.

```
sage: A.det # Won't work  
<built-in method det of sage.matrix.matrix_integer_dense.Matrix_integer_dense object_  
↪at ...>
```

---

This is so useful because we can use the ‘tab’ key, remember!

```
sage: A.
```

Sometimes you will have surprises. Subtle changes in an object can affect what commands are available, or what their outcomes are.

```
sage: A.echelon_form()  
[1 0]  
[0 2]
```

This is because our original matrix had only integer coefficients, and you can't make the last entry one via elementary operations unless you multiply by a rational number!

```
sage: B = A.change_ring(QQ); B.echelon_form()  
[1 0]  
[0 1]
```

Another question is whether one needs an argument. Remember, it's easy to just read the documentation!

Below, let's see whether we need an argument to get a column.

```
sage: A.column?
```

It looks like we do. Let's input `1` .

```
sage: A.column(1)
(2, 4)
```

Notice that this gives the SECOND column!

```
sage: A
[1 2]
[3 4]
```

What is that about?

## 5.2 Lists, Loops, and Set Builders

### (Especially List Comprehensions!)

In the previous example, we saw that the 1 choice for the column of a matrix gives the *second* column.

```
sage: matrix([[1,2],[3,4]]).column(1)
(2, 4)
```

You might have thought that the would give the first column, but Sage (along with the Python programming language) begins numbering of anything that is like a sequence at zero. We've mentioned this once before, but it's very important to remember.

To reinforce this, let's formally introduce a fundamental object we've seen once or twice before, called a *list*.

You should think of a list as an ordered set, where the elements of the set can be pretty much anything - including other lists.

```
sage: my_list=[2, 'Grover', [3,2,1]]; my_list
[2, 'Grover', [3, 2, 1]]
```

You can access any elements of such a list quite easily using square brackets. Just remember that the counting starts at zero.

```
sage: my_list[0]; my_list[2]
2
[3, 2, 1]
```

There are lots of advanced things one can do with lists.

```
sage: my_list[0:2]
[2, 'Grover']
```

However, our main reason for introducing this is more practical, as we'll now see.

- One of the best uses of the computer in the classroom is to quickly show tedious things.
- One of the most tedious things to do by hand in linear algebra is taking powers of matrices.
- Here we make the first four powers of our matrix 'by hand'.

```
sage: A = matrix([[1,2],[3,4]])
sage: A^0; A^1; A^2; A^3; A^4
[1 0]
[0 1]
```

```
[1 2]
[3 4]
[ 7 10]
[15 22]
[ 37 54]
[ 81 118]
[199 290]
[435 634]
```

This is not terrible, but it's not exactly nice either, particularly if you might want to do something *with* these new matrices.

Instead, we can do what is known as a *loop* construction. See the notation below; it's at least vaguely mathematical.

```
sage: for i in [0,1,2,3,4]:
.....:     A^i
[1 0]
[0 1]
[1 2]
[3 4]
[ 7 10]
[15 22]
[ 37 54]
[ 81 118]
[199 290]
[435 634]
```

What did we do?

- For each  $i$  in the set  $\{0, 1, 2, 3, 4\}$ , return  $A^i$ .

Yeah, that makes sense. The square brackets created a list, and the powers of the original matrix come in the same order as the list.

(The colon in the first line and the indentation in the second line are **extremely** important; they are the basic syntactical structure of Python.)

For the curious: this is better, but still not perfect. It would be best to find a quicker way to write the possible values for  $i$ . There are two ways to do this in Sage.

```
sage: for i in [0..4]:
.....:     det(A^i)
1
-2
4
-8
16
```

```
sage: for i in range(5):
.....:     det(A^i)
1
-2
4
-8
16
```

These ways of constructing lists are very useful - and demonstrate that, like many Sage/Python things, that counting begins at zero and ends at one less than the “end” in things like `range`.



Below, we show that one can get step sizes other than one as well.

```
sage: range(3, 23, 2); [3, 5..21]
[3, 5, 7, 9, 11, 13, 15, 17, 19, 21]
[3, 5, 7, 9, 11, 13, 15, 17, 19, 21]
```

**Note:** It is also important to emphasize that the `range` command does *not* include its last value! For a quick quiz, confirm this in the examples above.

This all works well. However, after a short time this will seem tedious as well (you may have to trust us on this). It turns out that there is a very powerful way to create such lists in a way that very strongly resembles the so-called set builder notation, called a *list comprehension*.

We start with a relatively easy example:

$$\{n^2 \mid n \in \mathbf{Z}, 3 \leq n \leq 12\}$$

Who hasn't written something like this at some point in a course?

This is a natural for the list comprehension, and can be very powerful when used in Sage.

```
sage: [n^2 for n in [3..12]]
[9, 16, 25, 36, 49, 64, 81, 100, 121, 144]
```

That's it. This sort of turns the loop around.

- The notation is easiest if you think of it mathematically; “The set of  $n^2$ , for (all)  $n$  in the range between 3 and 13.”

This is phenomenally useful. Here is a nice plotting example:

```
sage: plot([x^n for n in [2..6]], (x, 0, 1))
Graphics object consisting of 5 graphics primitives
```

Now we apply it to the example we were doing in the first place. Notice we now have a nice concise description of all determinants of these matrices, without the syntax of colon and indentation:

```
sage: [det(A^i) for i in [0..4]]
[1, -2, 4, -8, 16]
```

### 5.2.1 Tables

Finally, getting away from strictly programming, here is a useful tip.

Some of you may be familiar with a way to take such data and put it in tabular form from other programs. The `table` command does this for us:

```
sage: table([ (i, det(A^i)) for i in [0..4] ])
0    1
1    -2
2     4
3    -8
4    16
```

Notice that each element of *this* list is two items in parentheses (a so-called *tuple*).

Even better, we can put a header line on it to make it really clear what we are doing, by adding lists. We've seen keywords like `header=True` when doing some of our plotting and limits. What do you think will happen if you put dollar signs around the labels in the header?

```
sage: table( [['i', 'det(A^i)']] + [ (i, det(A^i)) for i in [0..4] ], header_row=True)
i      det(A^i)
+---+-----+
0      1
1      -2
2       4
3      -8
4      16
```

## 5.3 Defining Functions

### Or, Extending Sage

It is often the case that Sage can do something, but doesn't have a simple command for it. For instance, you might want to take a matrix and output the square of that matrix minus the original matrix.

```
sage: A = matrix([[1,2],[3,4]])
sage: A^2-A
[ 6  8]
[12 18]
```

How might one do this for other matrices? Of course, you could just always do  $A^2 - A$  again and again. But this would be tedious and hard to follow, as with so many things that motivate a little programming. Here is how we solve this problem.

```
sage: def square_and_subtract(mymatrix):
.....:     return mymatrix^2-mymatrix
```

The `def` command has created a new function called `square_and_subtract`. It should even be available using tab-completion.

Here are things to note about its construction:

- The input is inside the parentheses.
- The indentation and colon are crucial, as above.
- There will usually be a return value, given by `return`. This is what Sage will give below the input cell.

```
sage: square_and_subtract(A)
[ 6  8]
[12 18]
```

```
sage: square_and_subtract(matrix([[1.5,0],[0,2]]))
[0.7500000000000000 0.0000000000000000]
[0.0000000000000000 2.0000000000000000]
```

We can get a documentation string available by putting it in triple quotes `"""`.

```
sage: def square_and_subtract(mymatrix):
.....:     """
.....:     Return `A^2-A`
```

```
.....: """
.....:     return mymatrix^2-mymatrix
```

```
sage: square_and_subtract?
```

Pretty cool! And potentially quite helpful to students - and you - especially if the function is complicated. The *A* typesets properly because we put it in backticks (see above).

A very careful reader *may* have noticed that there is nothing that requires the input `mymatrix` to be a matrix. Sage will just try to square whatever you give it and subtract the original thing.

```
sage: square_and_subtract(sqrt(5))
-sqrt(5) + 5
```

This is a typical thing to watch out for; just because you define something doesn't mean it's useful (though in this case it was).

Try to define a function which inputs a matrix and returns the determinant of the cube of the matrix. (There are a few ways to do this, of course!)

## 5.4 Gotchas from names and copies

### Or, What's in a Name

Before we finish the tutorial, we want to point out a few programming-related things that often trip people up.

The first 'gotcha' is that it's possible to clobber constants!

```
sage: i
4
```

Can you figure out why `i=4` ? Look carefully above to see when this happened.

- This gives a valuable lesson; *any time* you use a name there is potential for renaming.

This may seem quite bad, but could be quite logical to do - for instance, if you are only dealing with real matrices. It is definitely something a Sage user needs to know, though.

Luckily, it's possible to restore symbolic constants.

```
sage: reset('i')
sage: i; i^2
I
-1
```

```
sage: type(e)
<type 'sage.symbolic.constants_c.E'>
```

```
sage: type(pi)
<type 'sage.symbolic.expression.Expression'>
```

Variables are another thing to keep in mind. As mentioned briefly in earlier tutorials, in order to maintain maximum flexibility while not allowing things to happen which shouldn't, only `x` is predefined, nothing else.

```
sage: type(x)
<type 'sage.symbolic.expression.Expression'>
```

```
sage: type(y)
Traceback (most recent call last):
...
NameError: name 'y' is not defined
```

**Warning:** There *is* a way to get around this, but it unleashes a horde of potential misuse. See the cells below if you are interested in this.

```
sage: automatic_names(True) # not tested
sage: trig_expand((2*x + 4*y + sin(2*theta))^2) # not tested
4*(sin(theta)*cos(theta) + x + 2*y)^2
```

This only works in the notebook. Now we'll turn it off.

```
sage: automatic_names(False) # not tested
```

Another related issue is that a few names are “reserved” by Python/Sage, and which aren’t allowed as variable names. It’s not surprising that ‘for’ is not allowed, but neither is ‘lambda’ ( $\lambda$ )! People often request a workaround for that.

```
sage: var('lambda')
Traceback (most recent call last):
...
ValueError: The name "lambda" is not a valid Python identifier.
```

There are lots of ways to get around this. One popular, though annoying, way is this.

```
sage: var('lambda_')
lambda_
```

```
sage: lambda_^2-1
lambda_^2 - 1
```

Still, in this one case, showing the expression still shows the Greek letter.

```
sage: show(lambda_^2-1)
```

$$\lambda^2 - 1$$

Finally, there is another thing that can happen if you rename things too loosely.

```
sage: A = matrix(QQ, [[1,2],[3,4]])
sage: B = A
sage: C = copy(A)
```

This actually has just made B and A refer to the same matrix. B isn’t like A, it *is* A. The `copy` command gets around this (though not always).

```
sage: A[0,0]=987
```

```
sage: show([A,B,C])
```

$$\left[ \begin{pmatrix} 987 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 987 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right]$$

This is very subtle if you’ve never programmed before. Suffice it to say that it is safest to let each `=` sign stand for one thing, and to avoid redundant equals (unlike students at times).

## 5.5 Appendix: Advanced Introductory Topics

There are several things which are useful to know about, but which are not always introduced immediately in programming. We give a few examples here, but they are mainly here to make sure you have seen them so that they are not completely surprising when they come up again.

We saw the “block” structure of Python earlier, with the indentation. This gives the opportunity to introduce conditional statements and comparisons. Here, we just give an example for those who have seen conditionals (“if” clauses) before.

```
sage: B = matrix([[0,1,0,0],[0,0,1,0],[0,0,0,1],[0,0,0,0]])
sage: for i in range(5): # all integers from 0 to 4, remember
....:     if B^i==0: # We ask if the power is the zero matrix
....:         print(i)
4
```

We use the double equals sign to test for equality, because `=` assigns something to a variable name. Notice again that colons and indentation are the primary way that Sage/Python indicate syntax, just as commas and spaces do in English.

Another useful concept is that of a *dictionary*. This can be thought of as a mathematical mapping from “keys” to “values”. The order is *not* important and *not* guaranteed. A dictionary is delimited by curly brackets and correspondence is indicated by colons.

Again, we will just give a small example to illustrate the idea.

What if one wants to specify a matrix using just the nonzero entries? A dictionary is a great way to do this.

This one puts 3 as an entry in the (2,3) spot, for example (remember, this is the *third* row and *fourth* column, since we start with zero).

```
sage: D = {(2,3):3, (4,5):6, (6,0):-3}
sage: C = matrix(D)
sage: C
[ 0  0  0  0  0  0]
[ 0  0  0  0  0  0]
[ 0  0  0  3  0  0]
[ 0  0  0  0  0  0]
[ 0  0  0  0  0  6]
[ 0  0  0  0  0  0]
[-3  0  0  0  0  0]
```

That was a lot easier than inputting the whole matrix!

Finally, although Sage tries to anticipate what you want, sometimes it does matter how you define a given element in Sage.

- We saw this above with matrices over the rationals versus integers, for instance.

Here’s an example with straight-up numbers.

```
sage: a = 2
sage: b = 2/1
sage: c = 2.0
sage: d = 2 + 0*I
sage: e = 2.0 + 0.0*I
```

We will not go in great depth about this, either, but it is worth knowing about. Notice that each of these types of numbers has or does not have  $I = \sqrt{-1}$ , decimal points, or division.

```
sage: parent(a)
Integer Ring
sage: parent(b)
Rational Field
sage: parent(c)
Real Field with 53 bits of precision
sage: parent(d)
Symbolic Ring
sage: parent(e)
Symbolic Ring
```

## TUTORIAL FOR ADVANCED 2D PLOTTING

This Sage document is one of the tutorials developed for the MAA PREP Workshop “Sage: Using Open-Source Mathematics Software with Undergraduates” (funding provided by NSF DUE 0817071). It is licensed under the Creative Commons Attribution-ShareAlike 3.0 license (CC BY-SA).

Thanks to Sage’s integration of projects like `matplotlib`, Sage has comprehensive two-dimensional plotting capabilities. This worksheet consists of the following sections:

- *Cartesian Plots*
- *Parametric Plots*
- *Polar Plots*
- *Plotting Data Points*
- *Contour Plots*
- *Vector fields*
- *Complex Plots*
- *Region plots*
- *Builtin Graphics Objects*
- *Saving Plots*

This tutorial assumes that one is familiar with the basics of Sage, such as evaluating a cell by clicking the “evaluate” link, or by pressing Shift-Enter (hold down Shift while pressing the Enter key).

## 6.1 Graphing Functions and Plotting Curves

### 6.1.1 Cartesian Plots

A simple quadratic is easy.

```
sage: plot(x^2, (x,-2,2))  
Graphics object consisting of 1 graphics primitive
```

You can combine “plot objects” by adding them.

```
sage: regular = plot(x^2, (x,-2,2), color= 'purple')  
sage: skinny = plot(4*x^2, (x,-2,2), color = 'green')  
sage: regular + skinny  
Graphics object consisting of 2 graphics primitives
```

**Problem :** Plot a green  $y = \sin(x)$  together with a red  $y = 2 \cos(x)$ . (Hint: you can use `pi` as part of your range.)

Boundaries of a plot can be specified, in addition to the overall size.

```
sage: plot(1+e^(-x^2), xmin=-2, xmax=2, ymin=0, ymax=2.5, figsize=10)
Graphics object consisting of 1 graphics primitive
```

**Problem :** Plot  $y = 5 + 3 \sin(4x)$  with suitable boundaries.

You can add lots of extra information.

```
sage: exponential = plot(1+e^(-x^2), xmin=-2, xmax=2, ymin=0, ymax=2.5)
sage: max_line = plot(2, xmin=-2, xmax=2, linestyle='-.', color = 'red')
sage: min_line = plot(1, xmin=-2, xmax=2, linestyle=':', color = 'red')
sage: exponential + max_line + min_line
Graphics object consisting of 3 graphics primitives
```

You can fill regions with transparent color, and thicken the curve. This example uses several options to fine-tune our graphic.

```
sage: exponential = plot(1+e^(-x^2), xmin=-2, xmax=2, ymin=0, ymax=2.5, fill=0.5,
↳fillcolor='grey', fillalpha=0.3)
sage: min_line = plot(1, xmin=-2, xmax=2, linestyle='-', thickness= 6, color = 'red')
sage: exponential + min_line
Graphics object consisting of 3 graphics primitives
```

```
sage: sum([plot(x^n, (x,0,1),color=rainbow(5)[n]) for n in [0..4]])
Graphics object consisting of 5 graphics primitives
```

**Problem :** Create a plot showing the cross-section area for the following solid of revolution problem: Consider the area bounded by  $y = x^2 - 3x + 6$  and the line  $y = 4$ . Find the volume created by rotating this area around the line  $y = 1$ .

## 6.1.2 Parametric Plots

A parametric plot needs a list of two functions of the parameter; in Sage, we use *square* brackets to delimit the list. Notice also that we must declare `t` as a variable first. Because the graphic is slightly wider than it is tall, we use the `aspect_ratio` option (such options are called *keywords*) to ensure the axes are correct for how we want to view this object.

```
sage: t = var('t')
sage: parametric_plot([cos(t) + 3 * cos(t/9), sin(t) - 3 * sin(t/9)], (t, 0, 18*pi),
↳fill = True, aspect_ratio=1)
Graphics object consisting of 2 graphics primitives
```

**Problem :** These parametric equations will create a hypocycloid.

$$x(t) = 17 \cos(t) + 3 \cos(17t/3)$$

$$y(t) = 17 \sin(t) - 3 \sin(17t/3)$$

Create this as a parametric plot.

Sage automatically plots a 2d or 3d plot, and a curve or a surface, depending on how many variables and coordinates you specify.



```
sage: t = var('t')
sage: parametric_plot((t^2, sin(t)), (t, 0, pi))
Graphics object consisting of 1 graphics primitive
```

```
sage: parametric_plot((t^2, sin(t), cos(t)), (t, 0, pi))
Graphics3d Object
```

```
sage: r = var('r')
sage: parametric_plot((t^2, sin(r*t), cos(r*t)), (t, 0, pi), (r, -1, 1))
Graphics3d Object
```

### 6.1.3 Polar Plots

Sage can also do polar plots.

```
sage: polar_plot(2 + 2*cos(x), (x, 0, 2*pi), color=hue(0.5), thickness=4)
Graphics object consisting of 1 graphics primitive
```

Although they aren't essential, many of these examples try to demonstrate things like coloring, fills, and shading to give you a sense of the possibilities.

More than one polar curve can be specified in a list (square brackets). Notice the automatic graded shading of the fill color.

```
sage: t = var('t')
sage: polar_plot([cos(4*t) + 1.5, 0.5 * cos(4*t) + 2.5], (t, 0, 2*pi),
....:             color='black', thickness=2, fill=True, fillcolor='orange')
Graphics object consisting of 4 graphics primitives
```

Problem: Create a plot for the following problem. Find the area that is inside the circle  $r = 2$ , but outside the cardioid  $2 + 2\cos(\theta)$ .

### 6.1.4 Interactive Demonstration

It may be of interest to see all these things put together in a very nice pedagogical graphic. Even though this is fairly advanced, and so you may want to skip the code, it is not as difficult as you might think to put together.

```
sage: html('<h2>Sine and unit circle (by Jurgis Pralgauskis)</h2> inspired by <a href=
->"http://www.youtube.com/watch?v=Ohp6Okk_tww&feature=related">this video</a>' )
sage: # http://doc.sagemath.org/html/en/reference/sage/plot/plot.html
sage: radius = 100 # scale for radius of "unit" circle
sage: graph_params = dict(xmin = -2*radius,      xmax = 360,
....:                     ymin = -(radius+30), ymax = radius+30,
....:                     aspect_ratio=1,
....:                     axes = False
....:                     )
sage: def sine_and_unit_circle( angle=30, instant_show = True, show_pi=True ):
....:     ccenter_x, ccenter_y = -radius, 0 # center of circle on real coords
....:     sine_x = angle # the big magic to sync both graphs :)
....:     current_y = circle_y = sine_y = radius * sin(angle*pi/180)
....:     circle_x = ccenter_x + radius * cos(angle*pi/180)
....:     graph = Graphics()
....:     # we'll put unit circle and sine function on the same graph
....:     # so there will be some coordinate mangling ;)
```

```

.....: # CIRCLE
.....: unit_circle = circle((ccenter_x, ccenter_y), radius, color="#ccc")
.....: # SINE
.....: x = var('x')
.....: sine = plot( radius * sin(x*pi/180) , (x, 0, 360), color="#ccc" )
.....: graph += unit_circle + sine
.....: # CIRCLE axis
.....: # x axis
.....: graph += arrow( [-2*radius, 0], [0, 0], color = "#666" )
.....: graph += text("$ (1, 0)$", [-16, 16], color = "#666")
.....: # circle y axis
.....: graph += arrow( [ccenter_x, -radius], [ccenter_x, radius], color = "#666" )
.....: graph += text("$ (0, 1)$", [ccenter_x, radius+15], color = "#666")
.....: # circle center
.....: graph += text("$ (0, 0)$", [ccenter_x, 0], color = "#666")
.....: # SINE x axis
.....: graph += arrow( [0, 0], [360, 0], color = "#000" )
.....: # let's set tics
.....: # or http://aghitza.org/posts/tweak_labels_and_ticks_in_2d_plots_using_
↪matplotlib/
.....: # or wayt for http://trac.sagemath.org/sage_trac/ticket/1431
.....: # ['$-\pi/3$', '$2\pi/3$', '$5\pi/3$']
.....: for x in range(0, 361, 30):
.....:     graph += point( [x, 0] )
.....:     angle_label = ". %3d^{\circ} $ " % x
.....:     if show_pi: angle_label += " $(\pi)$ $" % x/180
.....:     graph += text(angle_label, [x, 0], rotation=-90,
.....:         vertical_alignment='top', fontsize=8, color="#000" )
.....: # CURRENT VALUES
.....: # SINE -- y
.....: graph += arrow( [sine_x, 0], [sine_x, sine_y], width=1, arrowsize=3)
.....: graph += arrow( [circle_x, 0], [circle_x, circle_y], width=1, arrowsize=3)
.....: graph += line([circle_x, current_y], [sine_x, current_y]), rgbcolor="#0F0
↪", linestyle = "--", alpha=0.5)
.....: # LABEL on sine
.....: graph += text("$ (%d^{\circ}, %3.2f)$" % (sine_x, float(current_y)/radius), ↪
↪[sine_x+30, current_y], color = "#0A0")
.....: # ANGLE -- x
.....: # on sine
.....: graph += arrow( [0, 0], [sine_x, 0], width=1, arrowsize=1, color='red')
.....: # on circle
.....: graph += disk( (ccenter_x, ccenter_y), float(radius)/4, (0, angle*pi/180), ↪
↪color='red', fill=False, thickness=1)
.....: graph += arrow( [ccenter_x, ccenter_y], [circle_x, circle_y],
.....:     rgbcolor="#cccccc", width=1, arrowsize=1)
.....: if instant_show:
.....:     show (graph, **graph_params)
.....: return graph
sage: #####
sage: # make Interaction
sage: #####
sage: @interact
sage: def _( angle = slider([0..360], default=30, step_size=5,
.....:     label="Pasirinkite kampą: ", display_value=True) ):
.....:     sine_and_unit_circle(angle, show_pi = False)

```

## 6.2 Plotting Data

### 6.2.1 Plotting Data Points

Sometimes one wishes to simply plot data. Here, we demonstrate several ways of plotting points and data via the simple approximation to the Fibonacci numbers given by

$$F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n,$$

which is quite good after about  $n = 5$ .

First, we notice that the Fibonacci numbers are built in.

```
sage: fibonacci_sequence(6)
<generator object fibonacci_sequence at ...>
```

```
sage: list(fibonacci_sequence(6))
[0, 1, 1, 2, 3, 5]
```

The `enumerate` command is useful for taking a list and coordinating it with the counting numbers.

```
sage: list(enumerate(fibonacci_sequence(6)))
[(0, 0), (1, 1), (2, 1), (3, 2), (4, 3), (5, 5)]
```

So we just define the numbers and coordinate pairs we are about to plot.

```
sage: fibonacci = list(enumerate(fibonacci_sequence(6)))
sage: f(n)=(1/sqrt(5))*((1+sqrt(5))/2)^n
sage: asymptotic = [(i, f(i)) for i in range(6)]
sage: fibonacci
[(0, 0), (1, 1), (2, 1), (3, 2), (4, 3), (5, 5)]
sage: asymptotic
[(0, 1/5*sqrt(5)), (1, 1/10*sqrt(5)*(sqrt(5) + 1)), (2, 1/20*sqrt(5)*(sqrt(5) + 1)^
↪2), (3, 1/40*sqrt(5)*(sqrt(5) + 1)^3), (4, 1/80*sqrt(5)*(sqrt(5) + 1)^4), (5, 1/
↪160*sqrt(5)*(sqrt(5) + 1)^5)]
```

Now we can plot not just the two sets of points, but also use several of the documented options for plotting points. Those coming from other systems may prefer `list_plot`.

```
sage: fib_plot=list_plot(fibonacci, color='red', pointsize=30)
sage: asy_plot = list_plot(asymptotic, marker='D',color='black'
↪',thickness=2,plotjoined=True)
sage: show(fib_plot+asy_plot, aspect_ratio=1)
```

Other options include `line`, `points`, and `scatter_plot`. Having the choice of markers for different data is particularly helpful for generating publishable graphics.

```
sage: fib_plot=scatter_plot(fibonacci, facecolor='red', marker='o',markersize=40)
sage: asy_plot = line(asymptotic, marker='D',color='black',thickness=2)
sage: show(fib_plot+asy_plot, aspect_ratio=1)
```

## 6.3 Contour-type Plots

### 6.3.1 Contour Plots

Contour plotting can be very useful when trying to get a handle on multivariable functions, as well as modeling. The basic syntax is essentially the same as for 3D plotting - simply an extension of the 2D plotting syntax.

```
sage: f(x,y)=y^2+1-x^3-x
sage: contour_plot(f, (x,-pi,pi), (y,-pi,pi))
Graphics object consisting of 1 graphics primitive
```

We can change colors, specify contours, label curves, and many other things. When there are many levels, the `colorbar` keyword becomes quite useful for keeping track of them. Notice that, as opposed to many other options, it can only be `True` or `False` (corresponding to whether it appears or does not appear).

```
sage: contour_plot(f, (x,-pi,pi), (y,-pi,pi), colorbar=True, labels=True)
Graphics object consisting of 1 graphics primitive
```

This example is fairly self-explanatory, but demonstrates the power of formatting, labeling, and the wide variety of built-in color gradations (colormaps or `cmap`). The strange-looking construction corresponding to `label_fmt` is a Sage/Python data type called a *dictionary*, and turns out to be useful for more advanced Sage use; it consists of pairs connected by a colon, all inside curly braces.

```
sage: contour_plot(f, (x,-pi,pi), (y,-pi,pi), contours=[-4,0,4], fill=False,
....:      cmap='cool', labels=True, label_inline=True,
....:      label_fmt={-4:"low", 0:"medium", 4: "hi"}, label_colors='black')
Graphics object consisting of 1 graphics primitive
```

Implicit plots are a special type of contour plot (they just plot the zero contour).

```
sage: f(x,y)
-x^3 + y^2 - x + 1
```

```
sage: implicit_plot(f(x,y)==0, (x,-pi,pi), (y,-pi,pi))
Graphics object consisting of 1 graphics primitive
```

A density plot is like a contour plot, but without discrete levels.

```
sage: density_plot(f, (x, -2, 2), (y, -2, 2))
Graphics object consisting of 1 graphics primitive
```

Sometimes contour plots can be a little misleading (which makes for a *great* classroom discussion about the problems of ignorantly relying on technology). Here we combine a density plot and contour plot to show even better what is happening with the function.

```
sage: density_plot(f, (x,-2,2), (y,-2,2))+contour_plot(f, (x,-2,2), (y,-
↪2,2), fill=False, labels=True, label_inline=True, cmap='jet')
Graphics object consisting of 2 graphics primitives
```

It can be worth getting familiar with the various options for different plots, especially if you will be doing a lot of them in a given worksheet or pedagogical situation.

Here are the options for contour plots.

- They are given as an “attribute” - no parentheses - of the `contour_plot` object.
- They are given as a dictionary (see [the programming tutorial](#)).

```
sage: contour_plot.options
{'aspect_ratio': 1,
 'axes': False,
 'colorbar': False,
 'contours': None,
 'fill': True,
 'frame': True,
 'labels': False,
 'legend_label': None,
 'linestyles': None,
 'linewidths': None,
 'plot_points': 100,
 'region': None}
```

Let's change it so that all future contour plots don't have the fill. That's how some of us might use them in a class. We'll also check that the change happened.

```
sage: contour_plot.options["fill"]=False
sage: contour_plot.options
{'aspect_ratio': 1,
 'axes': False,
 'colorbar': False,
 'contours': None,
 'fill': False,
 'frame': True,
 'labels': False,
 'legend_label': None,
 'linestyles': None,
 'linewidths': None,
 'plot_points': 100,
 'region': None}
```

And it works!

```
sage: contour_plot(f, (x,-2,2), (y,-2,2))
Graphics object consisting of 1 graphics primitive
```

We can always access the default options, of course, to remind us.

```
sage: contour_plot.defaults()
{'aspect_ratio': 1,
 'axes': False,
 'colorbar': False,
 'contours': None,
 'fill': True,
 'frame': True,
 'labels': False,
 'legend_label': None,
 'linestyles': None,
 'linewidths': None,
 'plot_points': 100,
 'region': None}
```

### 6.3.2 Vector fields

The syntax for vector fields is very similar to other multivariate constructions. Notice that the arrows are scaled appropriately, and colored by length in the 3D case.

```
sage: var('x,y')
(x, y)
sage: plot_vector_field((-y+x,y*x),(x,-3,3),(y,-3,3))
Graphics object consisting of 1 graphics primitive
```

```
sage: var('x,y,z')
(x, y, z)
sage: plot_vector_field3d((-y,-z,x),(x,-3,3),(y,-3,3),(z,-3,3))
Graphics3d Object
```

3d vector field plots are ideally viewed with 3d glasses (right-click on the plot and select “Style” and “Stereographic”)

### 6.3.3 Complex Plots

We can plot functions of complex variables, where the magnitude is indicated by the brightness (black is zero magnitude) and the argument is indicated by the hue (red is a positive real number).

```
sage: f(z) = exp(z) #z^5 + z - 1 + 1/z
sage: complex_plot(f, (-5,5), (-5,5))
Graphics object consisting of 1 graphics primitive
```

### 6.3.4 Region plots

These plot where an expression is true, and are useful for plotting inequalities.

```
sage: region_plot(cos(x^2+y^2) <= 0, (x, -3, 3), (y, -3, 3), aspect_ratio=1)
Graphics object consisting of 1 graphics primitive
```

We can get fancier options as well.

```
sage: region_plot(sin(x)*sin(y) >= 1/4, (x,-10,10), (y,-10,10), incol='yellow',
↳bordercol='black', borderstyle='dashed', plot_points=250, aspect_ratio=1)
Graphics object consisting of 2 graphics primitives
```

Remember, what command would give full information about the syntax, options, and examples?

## 6.4 Miscellaneous Plot Information

### 6.4.1 Builtin Graphics Objects

Sage includes a variety of built-in graphics objects. These are particularly useful for adding to one’s plot certain objects which are difficult to describe with equations, but which are basic geometric objects nonetheless. In this section we will try to demonstrate the syntax of some of the most useful of them; for most of the the contextual (remember, append ? ) help will give more details.

## Points

To make one point, a coordinate pair suffices.

```
sage: point((3,5))
Graphics object consisting of 1 graphics primitive
```

It doesn't matter how multiple point are generated; they must go in as input via a list (square brackets). Here, we demonstrate the hard (but naive) and easy (but a little more sophisticated) way to do this.

```
sage: f(x)=x^2
sage: points([(0,f(0)), (1,f(1)), (2,f(2)), (3,f(3)), (4,f(4))])
Graphics object consisting of 1 graphics primitive
```

```
sage: points([(x,f(x)) for x in range(5)])
Graphics object consisting of 1 graphics primitive
```

Sage tries to tell how many dimensions you are working in automatically.

```
sage: f(x,y)=x^2-y^2
sage: points([(x,y,f(x,y)) for x in range(5) for y in range(5)])
Graphics3d Object
```

## Lines

The syntax for lines is the same as that for points, but you get... well, you get connecting lines too!

```
sage: f(x)=x^2
sage: line([(x,f(x)) for x in range(5)])
Graphics object consisting of 1 graphics primitive
```

## Balls

Sage has disks and spheres of various types available. Generally the center and radius are all that is needed, but other options are possible.

```
sage: circle((0,1),1,aspect_ratio=1)
Graphics object consisting of 1 graphics primitive
```

```
sage: disk((0,0), 1, (pi, 3*pi/2), color='yellow',aspect_ratio=1)
Graphics object consisting of 1 graphics primitive
```

There are also ellipses and various arcs; see the [full plot documentation](#).

## Arrows

```
sage: arrow((0,0), (1,1))
Graphics object consisting of 1 graphics primitive
```

## Polygons

Polygons will try to complete themselves and fill in the interior; otherwise the syntax is fairly self-evident.

```
sage: polygon([[0,0],[1,1],[1,2]])  
Graphics object consisting of 1 graphics primitive
```

## Text

In 2d, one can typeset mathematics using the `text` command. This can be used to fine-tune certain types of labels. Unfortunately, in 3D the text is just text.

```
sage: text('$\int_0^2 x^2\, dx$', (0.5,2))+plot(x^2, (x,0,2), fill=True)  
Graphics object consisting of 3 graphics primitives
```

### 6.4.2 Saving Plots

We can save 2d plots to many different formats. Sage can determine the format based on the filename for the image.

```
sage: p=plot(x^2, (x,-1,1))  
sage: p  
Graphics object consisting of 1 graphics primitive
```

For testing purposes, we use the Sage standard temporary filename; however, you could use any string for a name that you wanted, like `"my_plot.png"`.

```
sage: name = tmp_filename() # this is a string  
sage: png_savename = name+'.png'  
sage: p.save(png_savename)
```

In the notebook, these are usually ready for downloading in little links by the cells.

```
sage: pdf_savename = name+'.pdf'  
sage: p.save(pdf_savename)
```

Notably, we can export in formats ready for inclusion in web pages.

```
sage: svg_savename = name+'.svg'  
sage: p.save(svg_savename)
```



## PREP QUICKSTART TUTORIALS

Sometimes all one needs to start using Sage in a field is a little help. These “Quickstart” tutorials are designed to get you up and running in one of these fields with a minimum of fuss and muss.

The assumed background in most of these is the first several tutorials in this set of documents. Similarly, keep in mind that these documents are *concise*; the next step from each of them is really the Sage reference manual.

Finally, the last tutorial is a special one introducing the reader to creation of interactive material in Sage. It’s highly recommended that you study the [Programming tutorial](#) before tackling this one.

Contents:

### 7.1 Sage Quickstart for Abstract Algebra

This [Sage](#) quickstart tutorial was developed for the MAA PREP Workshop “Sage: Using Open-Source Mathematics Software with Undergraduates” (funding provided by NSF DUE 0817071). It is licensed under the Creative Commons Attribution-ShareAlike 3.0 license ([CC BY-SA](#)).

As computers are discrete and finite, anything with a discrete, finite set of generators is natural to implement and explore.

#### 7.1.1 Group Theory

Many common groups are pre-defined, usually as permutation groups: that is, explicitly described as subgroups of symmetric groups.

- Every group of order 15 or less is available as a permutation group.
- Sometimes they are available under special names, though.

```
sage: G = QuaternionGroup()
sage: G
Quaternion group of order 8 as a permutation group
```

```
sage: H = AlternatingGroup(5)
sage: H
Alternating group of order 5!/2 as a permutation group
```

```
sage: H.is_simple()
True
```

```
sage: D = DihedralGroup(8)
sage: D
Dihedral group of order 16 as a permutation group
```

We can access a lot of information about groups, such as:

- A list of subgroups up to conjugacy,
- or a stabilizer,
- or other things demonstrated below.

```
sage: for K in D.conjugacy_classes_subgroups():
.....:     print(K)
Subgroup of (Dihedral group of order 16 as a permutation group) generated by [()]
Subgroup of (Dihedral group of order 16 as a permutation group) generated by
→ [(1, 5) (2, 6) (3, 7) (4, 8)]
Subgroup of (Dihedral group of order 16 as a permutation group) generated by
→ [(2, 8) (3, 7) (4, 6)]
Subgroup of (Dihedral group of order 16 as a permutation group) generated by
→ [(1, 2) (3, 8) (4, 7) (5, 6)]
Subgroup of (Dihedral group of order 16 as a permutation group) generated by
→ [(1, 3, 5, 7) (2, 4, 6, 8), (1, 5) (2, 6) (3, 7) (4, 8)]
Subgroup of (Dihedral group of order 16 as a permutation group) generated by
→ [(2, 8) (3, 7) (4, 6), (1, 5) (2, 6) (3, 7) (4, 8)]
Subgroup of (Dihedral group of order 16 as a permutation group) generated by
→ [(1, 2) (3, 8) (4, 7) (5, 6), (1, 5) (2, 6) (3, 7) (4, 8)]
Subgroup of (Dihedral group of order 16 as a permutation group) generated by
→ [(2, 8) (3, 7) (4, 6), (1, 3, 5, 7) (2, 4, 6, 8), (1, 5) (2, 6) (3, 7) (4, 8)]
Subgroup of (Dihedral group of order 16 as a permutation group) generated by
→ [(1, 2, 3, 4, 5, 6, 7, 8), (1, 3, 5, 7) (2, 4, 6, 8), (1, 5) (2, 6) (3, 7) (4, 8)]
Subgroup of (Dihedral group of order 16 as a permutation group) generated by
→ [(1, 2) (3, 8) (4, 7) (5, 6), (1, 3, 5, 7) (2, 4, 6, 8), (1, 5) (2, 6) (3, 7) (4, 8)]
Subgroup of (Dihedral group of order 16 as a permutation group) generated by
→ [(2, 8) (3, 7) (4, 6), (1, 2, 3, 4, 5, 6, 7, 8), (1, 3, 5, 7) (2, 4, 6, 8), (1, 5) (2, 6) (3, 7) (4, 8)]
```

In the previous cell we once again did a for loop over a set of objects rather than just a list of numbers. This can be very powerful.

```
sage: D.stabilizer(3)
Subgroup of (Dihedral group of order 16 as a permutation group) generated by
→ [(1, 5) (2, 4) (6, 8)]
```

```
sage: for K in D.normal_subgroups():
.....:     print(K)
Subgroup of (Dihedral group of order 16 as a permutation group) generated by [()]
Subgroup of (Dihedral group of order 16 as a permutation group) generated by
→ [(1, 5) (2, 6) (3, 7) (4, 8)]
Subgroup of (Dihedral group of order 16 as a permutation group) generated by
→ [(1, 3, 5, 7) (2, 4, 6, 8), (1, 5) (2, 6) (3, 7) (4, 8)]
Subgroup of (Dihedral group of order 16 as a permutation group) generated by
→ [(1, 2) (3, 8) (4, 7) (5, 6), (1, 3, 5, 7) (2, 4, 6, 8), (1, 5) (2, 6) (3, 7) (4, 8)]
Subgroup of (Dihedral group of order 16 as a permutation group) generated by
→ [(2, 8) (3, 7) (4, 6), (1, 3, 5, 7) (2, 4, 6, 8), (1, 5) (2, 6) (3, 7) (4, 8)]
Subgroup of (Dihedral group of order 16 as a permutation group) generated by
→ [(1, 2, 3, 4, 5, 6, 7, 8), (1, 3, 5, 7) (2, 4, 6, 8), (1, 5) (2, 6) (3, 7) (4, 8)]
Subgroup of (Dihedral group of order 16 as a permutation group) generated by
→ [(1, 2, 3, 4, 5, 6, 7, 8), (1, 8) (2, 7) (3, 6) (4, 5)]
```

We can access specific subgroups if we know the generators as a permutation group.

```
sage: L = D.subgroup(["(1,3,5,7) (2,4,6,8)"])
```

```
sage: L.is_normal(D)
True
```

```
sage: Q=D.quotient(L)
sage: Q
Permutation Group with generators [(1,2) (3,4), (1,3) (2,4)]
```

```
sage: Q.is_isomorphic(KleinFourGroup())
True
```

There are some matrix groups as well, both finite and infinite.

```
sage: S = SL(2, GF(3))
sage: S
Special Linear Group of degree 2 over Finite Field of size 3
```

We can print out *all* of the elements of this group.

```
sage: for a in S:
.....:     print(a)
[1 0]
[0 1]
...
[2 2]
[2 1]
```

```
sage: SS = SL(2, ZZ)
```

Of course, you have to be careful what you try to do!

```
sage: SS.list()
Traceback (most recent call last):
...
NotImplementedError: group must be finite
```

```
sage: for a in SS.gens():
.....:     print(a)
[ 0  1]
[-1  0]
...
```

## 7.1.2 Rings

Sage has many pre-defined rings to experiment with. Here is how one would access  $\mathbb{Z}/12\mathbb{Z}$ , for instance.

```
sage: twelve = Integers(12)
sage: twelve
Ring of integers modulo 12
```

```
sage: twelve.is_field()
False
```

```
sage: twelve.is_integral_domain()
False
```

## Quaternions, and generalizations

We can define generalized quaternion algebras, where  $i^2 = a$ ,  $j^2 = b$ , and  $k = i \cdot j$ , all over  $\mathbb{Q}$ :

```
sage: quat = QuaternionAlgebra(-1, -1)
sage: quat
Quaternion Algebra (-1, -1) with base ring Rational Field
```

```
sage: quat.is_field()
False
```

```
sage: quat.is_commutative()
False
```

```
sage: quat.is_division_algebra()
True
```

```
sage: quat2 = QuaternionAlgebra(5, -7)
```

```
sage: quat2.is_division_algebra()
True
```

```
sage: quat2.is_field()
False
```

## Polynomial Rings

Polynomial arithmetic in Sage is a very important tool.

The first cell brings us back to the symbolic world. This is **not the same thing** as polynomials!

```
sage: reset('x') # This returns x to being a variable
sage: (x^4 + 2*x).parent()
Symbolic Ring
```

Now we will turn  $x$  into the generator of a polynomial ring. The syntax is a little unusual, but you will see it often.

```
sage: R.<x> = QQ[]
sage: R
Univariate Polynomial Ring in x over Rational Field
```

```
sage: R.random_element() # random
-5/2*x^2 - 1/4*x - 1
```

```
sage: R.is_integral_domain()
True
```

```
sage: (x^4 + 2*x).parent()
Univariate Polynomial Ring in x over Rational Field
```

```
sage: (x^2+x+1).is_irreducible()
True
```

```
sage: F = GF(5)
sage: P.<y> = F[]
```

```
sage: P.random_element() # random
2*y
```

```
sage: I = P.ideal(y^3+2*y)
sage: I
Principal ideal (y^3 + 2*y) of Univariate Polynomial Ring in y over Finite Field of
↳size 5
```

```
sage: Q = P.quotient(I)
```

```
sage: Q
Univariate Quotient Polynomial Ring in ybar over Finite Field of size 5 with modulus
↳y^3 + 2*y
```

### 7.1.3 Fields

Sage has superb support for finite fields and extensions of the rationals.

#### Finite Fields

```
sage: F.<a> = GF(3^4)
sage: F
Finite Field in a of size 3^4
```

The generator satisfies a Conway polynomial, by default, or the polynomial can be specified.

```
sage: F.polynomial()
a^4 + 2*a^3 + 2
```

```
sage: F.list()
[0, a, a^2, a^3, a^3 + 1, a^3 + a + 1, a^3 + a^2 + a + 1, 2*a^3 + a^2 + a + 1, a^2 +
↳a + 2, a^3 + a^2 + 2*a, 2*a^3 + 2*a^2 + 1, a^3 + a + 2, a^3 + a^2 + 2*a + 1, 2*a^3
↳+ 2*a^2 + a + 1, a^3 + a^2 + a + 2, 2*a^3 + a^2 + 2*a + 1, 2*a^2 + a + 2, 2*a^3 + a^
↳2 + 2*a, 2*a^2 + 2, 2*a^3 + 2*a, 2*a^3 + 2*a^2 + 2, a^3 + 2*a + 2, a^3 + 2*a^2 +
↳2*a + 1, 2*a^2 + a + 1, 2*a^3 + a^2 + a, a^2 + 2, a^3 + 2*a, a^3 + 2*a^2 + 1, a +
↳1, a^2 + a, a^3 + a^2, 2*a^3 + 1, 2*a^3 + a + 2, 2*a^3 + a^2 + 2*a + 2, 2*a^2 + 2*a
↳+ 2, 2*a^3 + 2*a^2 + 2*a, a^3 + 2*a^2 + 2, 2*a + 1, 2*a^2 + a, 2*a^3 + a^2, 2, 2*a,
↳2*a^2, 2*a^3, 2*a^3 + 2, 2*a^3 + 2*a + 2, 2*a^3 + 2*a^2 + 2*a + 2, a^3 + 2*a^2 +
↳2*a + 2, 2*a^2 + 2*a + 1, 2*a^3 + 2*a^2 + a, a^3 + a^2 + 2, 2*a^3 + 2*a + 1, 2*a^3
↳+ 2*a^2 + a + 2, a^3 + a^2 + 2*a + 2, 2*a^3 + 2*a^2 + 2*a + 1, a^3 + 2*a^2 + a + 2,
↳a^2 + 2*a + 1, a^3 + 2*a^2 + a, a^2 + 1, a^3 + a, a^3 + a^2 + 1, 2*a^3 + a + 1, 2*a
↳3 + a + 2 + a + 2, a^2 + 2*a + 2, a^3 + 2*a^2 + 2*a, 2*a^2 + 1, 2*a^3 + a, 2*a^3 + a^
↳2 + 2, 2*a + 2, 2*a^2 + 2*a, 2*a^2 + 2*a, 2*a^3 + 2*a^2, a^3 + 2, a^3 + 2*a + 1, a^3 + 2*a^2 + a
↳+ 1, a^2 + a + 1, a^3 + a^2 + a, 2*a^3 + a^2 + 1, a + 2, a^2 + 2*a, a^3 + 2*a^2, 1]
```

```
sage: (a^3 + 2*a^2 + 2)*(2*a^3 + 2*a + 1)
2*a^3 + a^2 + a + 1
```

$F$  should be the splitting field of the polynomial  $x^{81} - x$ , so it is very good that we get no output from the following cell, which combines a loop and a conditional statement.

```
sage: for a in F:
.....:     if not (a^81 - a == 0):
.....:         print("Oops!")
```

## Field Extensions, Number Fields

Most things you will need in an undergraduate algebra classroom are already in Sage.

```
sage: N = QQ[sqrt(2)]
sage: N
Number Field in sqrt2 with defining polynomial x^2 - 2
```

```
sage: var('z')
z
sage: M.<a>=NumberField(z^2-2)
sage: M
Number Field in a with defining polynomial z^2 - 2
```

```
sage: M.degree()
2
```

```
sage: M.is_galois()
True
```

```
sage: M.is_isomorphic(N)
True
```

## 7.2 Sage Quickstart for Differential Equations

This Sage quickstart tutorial was developed for the MAA PREP Workshop “Sage: Using Open-Source Mathematics Software with Undergraduates” (funding provided by NSF DUE 0817071). It is licensed under the Creative Commons Attribution-ShareAlike 3.0 license ([CC BY-SA](#)).

Solving differential equations is a combination of exact and numerical methods, and hence a great place to explore with the computer. We have already seen one example of this in the calculus tutorial, which is worth reviewing.

### 7.2.1 Basic Symbolic Techniques

```
sage: y = function('y')(x)
sage: de = diff(y,x) + y - 2
sage: h = desolve(de, y)
```

Forgetting about plotting for the moment, notice that there are three things one needs to solve a differential equation symbolically:

- an abstract function `y` ;
- a differential equation, which here we put in a separate line;
- the actual **differential equation solve** command (bold for the acronym `desolve` ).

```
sage: show(expand(h))
```

$$ce^{(-x)} + 2$$

Since we did not specify any initial conditions, Sage (from Maxima) puts in a parameter. If we want to put in an initial condition, we use `ics` (for **initial conditions**). For example, we set `ics=[0,3]` to specify that when  $x = 0$ ,  $y = 3$ .

```
sage: h = desolve(de, y, ics=[0,3]); h
(2*e^x + 1)*e^(-x)
```

And of course we have already noted that we can plot all this with a slope field.

```
sage: y = var('y') # Needed so we can plot
sage: Plot1=plot_slope_field(2-y, (x,0,3), (y,0,5))
sage: Plot2=plot(h,x,0,3)
sage: Plot1+Plot2
Graphics object consisting of 2 graphics primitives
```

**Note:** Regarding symbolic functions versus symbolic variables:

- If you wanted to make `y` an abstract function again instead of a variable, you'd have to do that separately. A differential equation requires a `function` but plotting requires a `var` .
- Another option is to let `z` be the name of the vertical axis variable.
- Either way something will have to give, since in common speaking about these things we treat `y` as both a variable and a function, which is much trickier to accomplish with a computer.

There are many other differential equation facilities in Sage. We can't cover all the variants of this in a quickstart, but the documentation is good for symbolic solvers.

```
sage: desolvers?
```

For instance, Maxima can do systems, as well as use Laplace transforms, and we include versions of these wrapped for ease of use.

In all differential equation solving routines, it is important to pay attention to the syntax! In the following example, we have placed the differential equation in the body of the command, and had to specify that `f` was the **dependent variable** (`dvar`), as well as give initial conditions  $f(0) = 1$  and  $f'(0) = 2$ , which gives the last list in the example.

```
sage: f=function('f')(x)
sage: desolve_laplace(diff(f,x,2) == 2*diff(f,x)-f, dvar = f, ics = [0,1,2])
x*e^x + e^x
```

```
sage: g(x)=x*e^x+e^x
sage: derivative(g,x,2)-2*derivative(g,x)+g
x |--> 0
```

## 7.2.2 Numerical and Power Series Methods

There are also numerical methods.

For instance, one of the options above was `desolve_rk4`. This is a fourth-order Runge-Kutta method, and returns appropriate (numerical) output. Here, we *must* give the dependent variable *and* initial conditions.

```
sage: y = function('y')(x)
sage: de = diff(y,x) + y -2
sage: h = desolve_rk4(de, y, step=.05, ics=[0,3])
```

It can be fun to compare this with the original, symbolic solution. We use the `points` command from the advanced plotting tutorial.

```
sage: h1 = desolve(de, y, ics=[0,3])
sage: plot(h1, (x,0,5), color='red')+points(h)
Graphics object consisting of 2 graphics primitives
```

The primary use of numerical routines from here is pedagogical in nature.

For more advanced numerical routines, we primarily use the GNU scientific library. Using this is a little more sophisticated, but gives a wealth of options.

```
sage: ode_solver?
```

We can even do power series solutions. In order to do this, we must first define a special *power series ring*, including the precision.

```
sage: R.<t> = PowerSeriesRing(QQ, default_prec=10)
sage: a = -1 + 0*t
sage: b = 2 + 0*t
sage: h=a.solve_linear_de(b=b,f0=3,prec=10)
```

This power series solution is pretty good for a while!

```
sage: h = h.polynomial()
sage: plot(h,-2,5)+plot(2+e^-x, (x,-2,5), color='red', linestyle=':', thickness=3)
Graphics object consisting of 2 graphics primitives
```

This was just an introduction; there are a lot of resources for differential equations using Sage elsewhere, including a book by David Joyner, who wrote much of the original code wrapping Maxima for Sage to do just this.

## 7.3 Sage Quickstart for Graph Theory and Discrete Mathematics

This [Sage](#) quickstart tutorial was developed for the MAA PREP Workshop “Sage: Using Open-Source Mathematics Software with Undergraduates” (funding provided by NSF DUE 0817071). It is licensed under the Creative Commons Attribution-ShareAlike 3.0 license ([CC BY-SA](#)).

As computers are discrete and finite, topics from discrete mathematics are natural to implement and use. We’ll start with Graph Theory.

### 7.3.1 Graph Theory

The pre-defined `graphs` object provides an abundance of examples. Just tab to see!



```
sage: graphs.[tab]
```

Its companion `digraphs` has many built-in examples as well.

Visualizing a graph is similar to plotting functions.

```
sage: G = graphs.HeawoodGraph()
sage: plot(G)
Graphics object consisting of 36 graphics primitives
```

Defining your own graph is easy. One way is the following.

- Put a vertex next to a list (recall this concept from the programming tutorial) with a colon, to show its adjacent vertices. For example, to put vertex 4 next to vertices 0 and 2, use `4: [0, 2]`.
- Now combine all these in curly braces (in the advanced appendix to the programming tutorial, this is called a *dictionary*).

```
sage: H=Graph({0:[1,2,3], 4:[0,2], 6:[1,2,3,4,5]})
sage: plot(H)
Graphics object consisting of 18 graphics primitives
```

Adjacency matrices, other graphs, and similar inputs are also recognized.

Graphs have “position” information for location of vertices. There are several different ways to compute a layout, or you can compute your own. Pre-defined graphs often come with “nice” layouts.

```
sage: H.set_pos(H.layout_circular())
sage: plot(H)
Graphics object consisting of 18 graphics primitives
```

Vertices can be lots of things, for example the codewords of an error-correcting code.

---

**Note:** Technical caveat: they need to be “immutable”, like Python’s tuples.

---

Here we have a matrix over the integers and a matrix of variables as vertices.

```
sage: a=matrix([[1,2],[3,4]])
sage: var('x y z w')
(x, y, z, w)
sage: b=matrix([[x,y],[z,w]])
sage: a.set_immutable()
sage: b.set_immutable()
sage: K=DiGraph({a:[b]})
sage: show(K, vertex_size=800)
```

Edges can be labeled.

```
sage: L=graphs.CycleGraph(5)
sage: for edge in L.edges():
....:     u = edge[0]
....:     v = edge[1]
....:     L.set_edge_label(u, v, u*v)
sage: plot(L, edge_labels=True)
Graphics object consisting of 16 graphics primitives
```

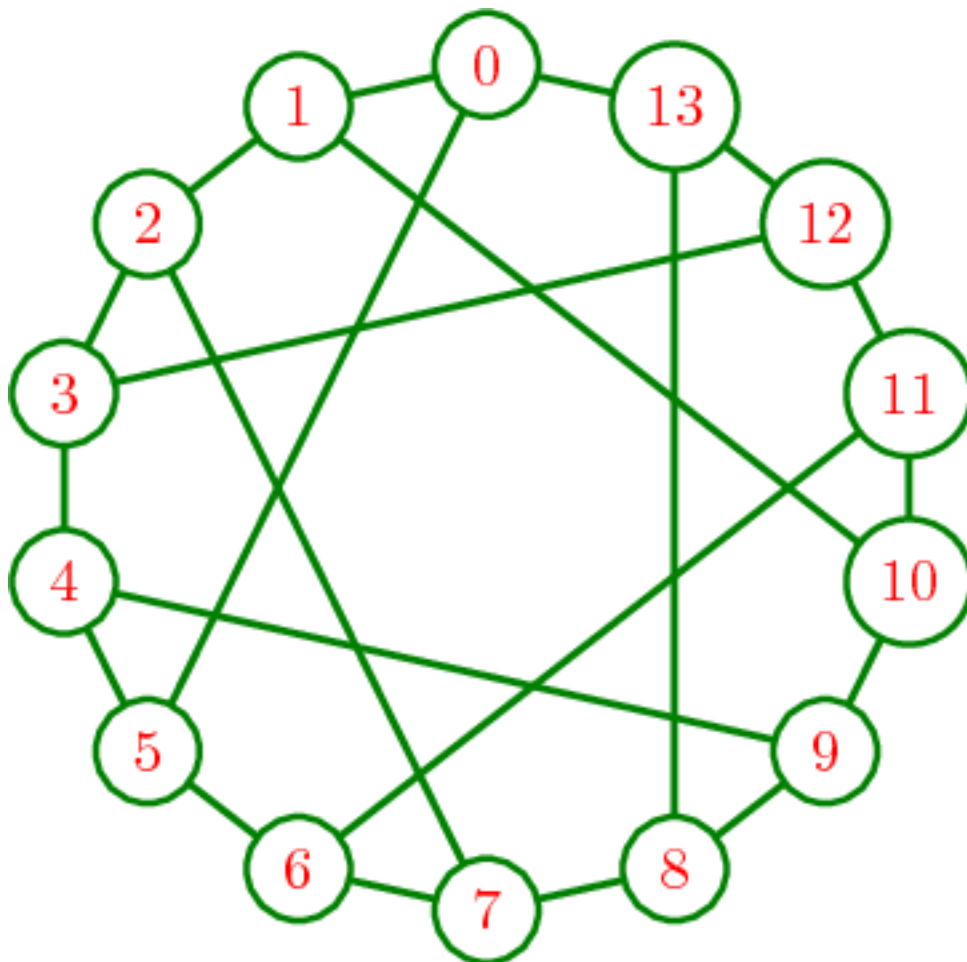
There are natural connections to other areas of mathematics. Here we compute the automorphism group and eigenvalues of the skeleton of a cube.

```
sage: C = graphs.CubeGraph(3)
sage: plot(C)
Graphics object consisting of 21 graphics primitives
```

```
sage: Aut=C.automorphism_group()
sage: print("Order of automorphism group: {}".format(Aut.order()))
Order of automorphism group: 48
sage: print("Group: \n{}".format(Aut)) # random
Group:
Permutation Group with generators [('010','100'),('011','101'), ('001','010'),('101',
↪ '110'), ('000','001'),('010','011'),('100','101'),('110','111')]
```

```
sage: C.spectrum()
[3, 1, 1, 1, -1, -1, -1, -3]
```

There is a huge amount of LaTeX support for graphs. The following graphic shows an example of what can be done; this is the Heawood graph.



Press 'tab' at the next command to see all the available options.

```
sage: sage.graphs.graph_latex.GraphLatex.set_option?
```

## 7.3.2 More Discrete Mathematics

Discrete mathematics is a broad area, and Sage has excellent support for much of it. This is largely due to the “sage-combinat” group. These developers previously developed for MuPad (as “mupad-combinat”) but switched over to Sage shortly before MuPad was sold.

### Simple Combinatorics

Sage can work with basic combinatorial structures like combinations and permutations.

```
sage: pets = ['dog', 'cat', 'snake', 'spider']
sage: C=Combinations(pets)
sage: C.list()
[[], ['dog'], ['cat'], ['snake'], ['spider'], ['dog', 'cat'], ['dog', 'snake'], ['dog',
→ 'spider'], ['cat', 'snake'], ['cat', 'spider'], ['snake', 'spider'], ['dog', 'cat',
→ 'snake'], ['dog', 'cat', 'spider'], ['dog', 'snake', 'spider'], ['cat', 'snake',
→ 'spider'], ['dog', 'cat', 'snake', 'spider']]
```

```
sage: for a, b in Combinations(pets, 2):
....:     print("The {} chases the {}".format(a, b))
The dog chases the cat.
The dog chases the snake.
The dog chases the spider.
The cat chases the snake.
The cat chases the spider.
The snake chases the spider.
```

```
sage: for pair in Permutations(pets, 2):
....:     print(pair)
['dog', 'cat']
['dog', 'snake']
['dog', 'spider']
['cat', 'dog']
['cat', 'snake']
['cat', 'spider']
['snake', 'dog']
['snake', 'cat']
['snake', 'spider']
['spider', 'dog']
['spider', 'cat']
['spider', 'snake']
```

Of course, we often want these for numbers, and these are present as well. Some are familiar:

```
sage: Permutations(5).cardinality()
120
```

Others somewhat less so:

```
sage: D = Derangements([1,1,2,2,3,4,5])
sage: D.list()[:5]
[[2, 2, 1, 1, 4, 5, 3], [2, 2, 1, 1, 5, 3, 4], [2, 2, 1, 3, 1, 5, 4], [2, 2, 1, 3, 4, 1,
→ 5, 1], [2, 2, 1, 3, 5, 1, 4]]
```

And some somewhat more advanced – in this case, symmetric polynomials.

```
sage: s = SymmetricFunctions(QQ).schur()
sage: a = s([2,1])
sage: a.expand(3)
x0^2*x1 + x0*x1^2 + x0^2*x2 + 2*x0*x1*x2 + x1^2*x2 + x0*x2^2 + x1*x2^2
```

Various functions related to this are available as well.

```
sage: binomial(25,3)
2300
```

```
sage: multinomial(24,3,5)
589024800
```

```
sage: falling_factorial(10,4)
5040
```

Do you recognize this famous identity?

```
sage: var('k,n')
(k, n)
sage: sum(binomial(n,k),k,0,n)
2^n
```

### Cryptography (for education)

This is also briefly mentioned in the *Number theory quickstart*. Sage has a number of good pedagogical resources for cryptography.

```
sage: # Two objects to make/use encryption scheme
sage: #
sage: from sage.crypto.block_cipher.sdes import SimplifiedDES
sage: sdes = SimplifiedDES()
sage: bin = BinaryStrings()
sage: #
sage: # Convert English to binary
sage: #
sage: P = bin.encoding("Encrypt this using S-DES!")
sage: print("Binary plaintext:  {}\n".format(P))
sage: #
sage: # Choose a random key
sage: #
sage: K = sdes.list_to_string(sdes.random_key())
sage: print("Random key:  {}\n".format(K))
sage: #
sage: # Encrypt with Simplified DES
sage: #
sage: C = sdes(P, K, algorithm="encrypt")
sage: print("Encrypted:  {}\n".format(C))
sage: #
sage: # Decrypt for the round-trip
sage: #
sage: plaintext = sdes(C, K, algorithm="decrypt")
sage: print("Decrypted:  {}\n".format(plaintext))
sage: #
sage: # Verify easily
```



## 7.4.1 Matrices and Vectors

We can make a matrix easily by passing a list of the rows. Don't forget to use tab-completion to see routines that are possible.

```
sage: A = matrix([[1,2,3],[4,5,6]]); A
[1 2 3]
[4 5 6]
```

But there are lots of other ways to make matrices. Each of these shows what is assumed with different input; can you figure out how Sage interprets them before you read the documentation which the command `matrix?` provides?

It's a good idea to get in the habit of telling Sage what ring to make the matrix over. Otherwise, Sage guesses based on the elements, so you may not have a matrix over a field! Here, we tell Sage to make the ring over the rationals.

```
sage: B = matrix(QQ, 3, 2, [1,2,3,4,5,6]); B
[1 2]
[3 4]
[5 6]
```

```
sage: C = matrix(QQ, 3, [1,2,3,4,5,6]); C
[1 2]
[3 4]
[5 6]
```

```
sage: D = matrix(CC, 20, range(400)); D
20 x 20 dense matrix over Complex Field with 53 bits of precision (use the '.str()'
↪method to see the entries)
```

Don't forget that when viewing this in the notebook, you can click to the left of the matrix in order to cycle between “wrapped”, “unwrapped” and “hidden” modes of output.

```
sage: print(D.str())
[0.0000000000000000  1.0000000000000000  2.0000000000000000  3.0000000000000000  4.
↪0000000000000000  5.0000000000000000  6.0000000000000000  7.0000000000000000  8.
↪0000000000000000  9.0000000000000000  10.0000000000000000  11.0000000000000000  12.
↪0000000000000000  13.0000000000000000  14.0000000000000000  15.0000000000000000  16.
↪0000000000000000  17.0000000000000000  18.0000000000000000  19.0000000000000000]
[ 20.0000000000000000  21.0000000000000000  22.0000000000000000  23.0000000000000000  24.
↪0000000000000000  25.0000000000000000  26.0000000000000000  27.0000000000000000  28.
↪0000000000000000  29.0000000000000000  30.0000000000000000  31.0000000000000000  32.
↪0000000000000000  33.0000000000000000  34.0000000000000000  35.0000000000000000  36.
↪0000000000000000  37.0000000000000000  38.0000000000000000  39.0000000000000000]
[ 40.0000000000000000  41.0000000000000000  42.0000000000000000  43.0000000000000000  44.
↪0000000000000000  45.0000000000000000  46.0000000000000000  47.0000000000000000  48.
↪0000000000000000  49.0000000000000000  50.0000000000000000  51.0000000000000000  52.
↪0000000000000000  53.0000000000000000  54.0000000000000000  55.0000000000000000  56.
↪0000000000000000  57.0000000000000000  58.0000000000000000  59.0000000000000000]
[ 60.0000000000000000  61.0000000000000000  62.0000000000000000  63.0000000000000000  64.
↪0000000000000000  65.0000000000000000  66.0000000000000000  67.0000000000000000  68.
↪0000000000000000  69.0000000000000000  70.0000000000000000  71.0000000000000000  72.
↪0000000000000000  73.0000000000000000  74.0000000000000000  75.0000000000000000  76.
↪0000000000000000  77.0000000000000000  78.0000000000000000  79.0000000000000000]
[ 80.0000000000000000  81.0000000000000000  82.0000000000000000  83.0000000000000000  84.
↪0000000000000000  85.0000000000000000  86.0000000000000000  87.0000000000000000  88.
↪0000000000000000  89.0000000000000000  90.0000000000000000  91.0000000000000000  92.
↪0000000000000000  93.0000000000000000  94.0000000000000000  95.0000000000000000  96.
↪0000000000000000  97.0000000000000000  98.0000000000000000  99.0000000000000000]
```

```

[ 100.000000000000 101.000000000000 102.000000000000 103.000000000000 104.
↪000000000000 105.000000000000 106.000000000000 107.000000000000 108.
↪000000000000 109.000000000000 110.000000000000 111.000000000000 112.
↪000000000000 113.000000000000 114.000000000000 115.000000000000 116.
↪000000000000 117.000000000000 118.000000000000 119.000000000000]
[ 120.000000000000 121.000000000000 122.000000000000 123.000000000000 124.
↪000000000000 125.000000000000 126.000000000000 127.000000000000 128.
↪000000000000 129.000000000000 130.000000000000 131.000000000000 132.
↪000000000000 133.000000000000 134.000000000000 135.000000000000 136.
↪000000000000 137.000000000000 138.000000000000 139.000000000000]
[ 140.000000000000 141.000000000000 142.000000000000 143.000000000000 144.
↪000000000000 145.000000000000 146.000000000000 147.000000000000 148.
↪000000000000 149.000000000000 150.000000000000 151.000000000000 152.
↪000000000000 153.000000000000 154.000000000000 155.000000000000 156.
↪000000000000 157.000000000000 158.000000000000 159.000000000000]
[ 160.000000000000 161.000000000000 162.000000000000 163.000000000000 164.
↪000000000000 165.000000000000 166.000000000000 167.000000000000 168.
↪000000000000 169.000000000000 170.000000000000 171.000000000000 172.
↪000000000000 173.000000000000 174.000000000000 175.000000000000 176.
↪000000000000 177.000000000000 178.000000000000 179.000000000000]
[ 180.000000000000 181.000000000000 182.000000000000 183.000000000000 184.
↪000000000000 185.000000000000 186.000000000000 187.000000000000 188.
↪000000000000 189.000000000000 190.000000000000 191.000000000000 192.
↪000000000000 193.000000000000 194.000000000000 195.000000000000 196.
↪000000000000 197.000000000000 198.000000000000 199.000000000000]
[ 200.000000000000 201.000000000000 202.000000000000 203.000000000000 204.
↪000000000000 205.000000000000 206.000000000000 207.000000000000 208.
↪000000000000 209.000000000000 210.000000000000 211.000000000000 212.
↪000000000000 213.000000000000 214.000000000000 215.000000000000 216.
↪000000000000 217.000000000000 218.000000000000 219.000000000000]
[ 220.000000000000 221.000000000000 222.000000000000 223.000000000000 224.
↪000000000000 225.000000000000 226.000000000000 227.000000000000 228.
↪000000000000 229.000000000000 230.000000000000 231.000000000000 232.
↪000000000000 233.000000000000 234.000000000000 235.000000000000 236.
↪000000000000 237.000000000000 238.000000000000 239.000000000000]
[ 240.000000000000 241.000000000000 242.000000000000 243.000000000000 244.
↪000000000000 245.000000000000 246.000000000000 247.000000000000 248.
↪000000000000 249.000000000000 250.000000000000 251.000000000000 252.
↪000000000000 253.000000000000 254.000000000000 255.000000000000 256.
↪000000000000 257.000000000000 258.000000000000 259.000000000000]
[ 260.000000000000 261.000000000000 262.000000000000 263.000000000000 264.
↪000000000000 265.000000000000 266.000000000000 267.000000000000 268.
↪000000000000 269.000000000000 270.000000000000 271.000000000000 272.
↪000000000000 273.000000000000 274.000000000000 275.000000000000 276.
↪000000000000 277.000000000000 278.000000000000 279.000000000000]
[ 280.000000000000 281.000000000000 282.000000000000 283.000000000000 284.
↪000000000000 285.000000000000 286.000000000000 287.000000000000 288.
↪000000000000 289.000000000000 290.000000000000 291.000000000000 292.
↪000000000000 293.000000000000 294.000000000000 295.000000000000 296.
↪000000000000 297.000000000000 298.000000000000 299.000000000000]
[ 300.000000000000 301.000000000000 302.000000000000 303.000000000000 304.
↪000000000000 305.000000000000 306.000000000000 307.000000000000 308.
↪000000000000 309.000000000000 310.000000000000 311.000000000000 312.
↪000000000000 313.000000000000 314.000000000000 315.000000000000 316.
↪000000000000 317.000000000000 318.000000000000 319.000000000000]
[ 320.000000000000 321.000000000000 322.000000000000 323.000000000000 324.
↪000000000000 325.000000000000 326.000000000000 327.000000000000 328.
↪000000000000 329.000000000000 330.000000000000 331.000000000000 332.
↪000000000000 333.000000000000 334.000000000000 335.000000000000 336.
↪000000000000 337.000000000000 338.000000000000 339.000000000000]

```

```
[ 340.0000000000000  341.0000000000000  342.0000000000000  343.0000000000000  344.
↪0000000000000  345.0000000000000  346.0000000000000  347.0000000000000  348.
↪0000000000000  349.0000000000000  350.0000000000000  351.0000000000000  352.
↪0000000000000  353.0000000000000  354.0000000000000  355.0000000000000  356.
↪0000000000000  357.0000000000000  358.0000000000000  359.0000000000000]
[ 360.0000000000000  361.0000000000000  362.0000000000000  363.0000000000000  364.
↪0000000000000  365.0000000000000  366.0000000000000  367.0000000000000  368.
↪0000000000000  369.0000000000000  370.0000000000000  371.0000000000000  372.
↪0000000000000  373.0000000000000  374.0000000000000  375.0000000000000  376.
↪0000000000000  377.0000000000000  378.0000000000000  379.0000000000000]
[ 380.0000000000000  381.0000000000000  382.0000000000000  383.0000000000000  384.
↪0000000000000  385.0000000000000  386.0000000000000  387.0000000000000  388.
↪0000000000000  389.0000000000000  390.0000000000000  391.0000000000000  392.
↪0000000000000  393.0000000000000  394.0000000000000  395.0000000000000  396.
↪0000000000000  397.0000000000000  398.0000000000000  399.0000000000000]
```

```
sage: E = diagonal_matrix( [0..40,step=4] ); E
```

```
[ 0  0  0  0  0  0  0  0  0  0  0  0]
[ 0  4  0  0  0  0  0  0  0  0  0  0]
[ 0  0  8  0  0  0  0  0  0  0  0  0]
[ 0  0  0 12  0  0  0  0  0  0  0  0]
[ 0  0  0  0 16  0  0  0  0  0  0  0]
[ 0  0  0  0  0 20  0  0  0  0  0  0]
[ 0  0  0  0  0  0 24  0  0  0  0  0]
[ 0  0  0  0  0  0  0 28  0  0  0  0]
[ 0  0  0  0  0  0  0  0 32  0  0  0]
[ 0  0  0  0  0  0  0  0  0 36  0  0]
[ 0  0  0  0  0  0  0  0  0  0 40]
```

```
sage: column_matrix(QQ, [[1,2,3],[4,5,6],[7,8,9]])
```

```
[1 4 7]
[2 5 8]
[3 6 9]
```

You can also combine matrices in different ways.

```
sage: F1=matrix(QQ,2,2,[0,1,1,0])
sage: F2=matrix(QQ,2,2,[1,2,3,4])
sage: F3=matrix(QQ,1,2,[3,1])
sage: block_matrix(2,2,[F1,F2,0,F3])
```

```
[0 1|1 2]
[1 0|3 4]
[---+---]
[0 0|3 1]
```

```
sage: F1.augment(F2)
```

```
[0 1 1 2]
[1 0 3 4]
```

```
sage: F1.stack(F2)
```

```
[0 1]
[1 0]
[1 2]
[3 4]
```



```
sage: block_diagonal_matrix([F1,F2])
[0 1|0 0]
[1 0|0 0]
[---+---]
[0 0|1 2]
[0 0|3 4]
```

Vectors are rows or columns, whatever you please, and Sage interprets them as appropriate in multiplication contexts.

```
sage: row = vector( (3, -1, 4) )
sage: col = vector( QQ, [4, 5] )
sage: row; col
(3, -1, 4)
(4, 5)
```

```
sage: F = matrix(QQ, 3, 2, range(6)); F
[0 1]
[2 3]
[4 5]
```

```
sage: F*col
(5, 23, 41)
```

```
sage: row*F
(14, 20)
```

Although our “vectors” (especially over rings other than fields) might be considered as elements of an appropriate free module, they basically behave as vectors for our purposes.

```
sage: ring_vec = vector(SR, [2, 12, -4, 9])
sage: field_vec = vector( QQ, (2, 3, 14) )
sage: ring_vec; field_vec
(2, 12, -4, 9)
(2, 3, 14)
```

```
sage: type( ring_vec )
<class 'sage.modules.vector_symbolic_dense.FreeModule_ambient_field_with_category.
↪element_class'>
sage: type( field_vec )
<type 'sage.modules.vector_rational_dense.Vector_rational_dense'>
```

## 7.4.2 Left-Handed or Right-handed?

Sage “prefers” rows to columns. For example, the `kernel` method for a matrix  $A$  computes the left kernel – the vector space of all vectors  $v$  for which  $v \cdot A = 0$  – and prints out the vectors as the rows of a matrix.

```
sage: G = matrix(QQ, 2, 3, [[1,2,3],[2,4,6]])
sage: G.kernel()
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[ 1 -1/2]
```

```
sage: G.left_kernel()
Vector space of degree 2 and dimension 1 over Rational Field
```

```
Basis matrix:  
[  1 -1/2]
```

The `right_kernel` method computes the space of vectors  $w$  so that  $A \cdot w = 0$ , of course.

### 7.4.3 Vector Spaces

Since Sage knows the kernel is a vector space, you can compute things that make sense for a vector space.

```
sage: V=G.right_kernel()  
sage: V  
Vector space of degree 3 and dimension 2 over Rational Field  
Basis matrix:  
[  1  0 -1/3]  
[  0  1 -2/3]
```

```
sage: V.dimension()  
2
```

Here we compute the coordinate vector of  $(1, 4, -3)$  relative to  $V$ :

```
sage: V.coordinate_vector([1,4,-3])  
(1, 4)
```

Here we get the basis matrix (note that the basis vectors are the *rows* of the matrix):

```
sage: V.basis_matrix()  
[  1  0 -1/3]  
[  0  1 -2/3]
```

Or we can get the basis vectors explicitly as a list of vectors:

```
sage: V.basis()  
[  
(1, 0, -1/3),  
(0, 1, -2/3)  
]
```

---

**Note:** Kernels are **vector spaces** and bases are “**echelonized**” (canonicalized).

This is why the `ring` for the matrix is important. Compare the kernels above with the kernel using a matrix which is only defined over the integers.

```
sage: G = matrix(ZZ, 2, 3, [[1,2,3],[2,4,6]])  
sage: G.kernel()  
Free module of degree 2 and rank 1 over Integer Ring  
Echelon basis matrix:  
[ 2 -1]
```

### 7.4.4 Computations

Here are some more computations with matrices and vectors.

As you might expect, random matrices are random.

```
sage: H = random_matrix(QQ, 5, 5, num_bound = 10, den_bound = 4)
sage: H.det() # random
15416
sage: H.eigenvalues() # random
[-10.08361801792048?, -2.682220984496031?, 4.739405672111427?, -1.320116668180795? -
↪ 10.88676412262347?*I, -1.320116668180795? + 10.88676412262347?*I]
```

According to the *Numerical analysis quickstart*, the question marks indicate that the actual number is inside the interval found by incrementing and decrementing the last digit of the printed number. So 9.1? is a number between 9.0 and 9.2. Sage knows exactly what number this is (since it's a root of a polynomial), but uses interval notation to print an approximation for ease of use.

The `eigenvectors_right` command prints out a list of (eigenvalue, [list of eigenvectors], algebraic multiplicity) tuples for each eigenvalue.

```
sage: H.eigenvectors_right() # random
[(-10.08361801792048?, [(1, -0.3820692683963385?, -0.4659857618614747?, -0.
↪ 1264082922197715?, -0.3548156445133095?)], 1), (-2.682220984496031?, [(1, -1.
↪ 855347152382563?, -0.4203899923232704?, 0.004411201577480876?, -0.5050698736445243?
↪)], 1), (4.739405672111427?, [(1, 0.3284800982819703?, 2.059182569319718?, -1.
↪ 428547399599918?, 0.5455069936349178?)], 1), (-1.320116668180795? - 10.
↪ 88676412262347?*I, [(1, 0.710831790589076? + 0.2646474741698805?*I, 0.
↪ 4504038344112447? + 3.145667601780920?*I, 2.763061217778457? + 0.9994136057023008?
↪ *I, 3.092272491890536? - 2.105461094305392?*I)], 1), (-1.320116668180795? + 10.
↪ 88676412262347?*I, [(1, 0.710831790589076? - 0.2646474741698805?*I, 0.
↪ 4504038344112447? - 3.145667601780920?*I, 2.763061217778457? - 0.9994136057023008?
↪ *I, 3.092272491890536? + 2.105461094305392?*I)], 1)]
```

It may be more convenient to use the `eigenmatrix_right` command, which gives a diagonal matrix of eigenvalues and a column matrix of eigenvectors.

```
sage: D,P=H.eigenmatrix_right()
sage: P*D-H*P
[0 0 0 0 0]
[0 0 0 0 0]
[0 0 0 0 0]
[0 0 0 0 0]
[0 0 0 0 0]
```

## 7.4.5 Matrix Solving

We can easily solve linear equations using the backslash, like in Matlab.

```
sage: A=random_matrix(QQ,3) # random
sage: v=vector([2,3,1])
sage: A,v # random
(
[ 0 -1  1]
[-1 -1 -1]
[ 0  2  2], (2, 3, 1)
)
sage: x=A\v; x # random
(-7/2, -3/4, 5/4)
sage: A*x # random
(2, 3, 1)
```

For *lots* more (concise) information, see the Sage [Linear Algebra Quick Reference](#).

## 7.5 Sage Quickstart for Multivariable Calculus

This Sage quickstart tutorial was developed for the MAA PREP Workshop “Sage: Using Open-Source Mathematics Software with Undergraduates” (funding provided by NSF DUE 0817071). It is licensed under the Creative Commons Attribution-ShareAlike 3.0 license ([CC BY-SA](#)).

Because much related material was covered in the calculus tutorial, this quickstart is designed to:

- Give concrete examples of some things not covered there, without huge amounts of commentary, and
- Remind the reader of a few of the things in that tutorial.

### 7.5.1 Vector Calculus

In Sage, vectors are primarily linear algebra objects, but they are slowly becoming simultaneously analytic continuous functions.

#### Dot Product, Cross Product

```
sage: v = vector(RR, [1.2, 3.5, 4.6])
sage: w = vector(RR, [1.7, -2.3, 5.2])
sage: v*w
17.910000000000000
```

```
sage: v.cross_product(w)
(28.780000000000000, 1.5800000000000000, -8.710000000000000)
```

#### Lines and Planes

Intersect  $x - 2y - 7z = 6$  with  $\frac{x-3}{2} = \frac{y+4}{-3} = \frac{z-1}{1}$ . One way to plot the equation of a line in three dimensions is with `parametric_plot3d`.

```
sage: # designed with intersection at t = 2, i.e. (7, -10, 3)
sage: var('t, x, y')
(t, x, y)
sage: line = parametric_plot3d([2*t+3, -3*t-4, t+1], (t, 0, 4), color='red')
sage: plane = plot3d((1/5)*(-12+x-2*y), (x, 4, 10), (y, -13, -7), opacity=0.5)
sage: intersect=point3d([7, -10, 3], color='black', size=30)
sage: line+plane+intersect
Graphics3d Object
```

#### Vector-Valued Functions

We can make vector-valued functions and do the usual analysis with them.

```
sage: var('t')
t
sage: r=vector((2*t-4, t^2, (1/4)*t^3))
```

```
sage: r
(2*t - 4, t^2, 1/4*t^3)
```

```
sage: r(t=5)
(6, 25, 125/4)
```

The following makes the derivative also a vector-valued expression.

```
sage: velocity = r.diff(t) # velocity(t) = list(r.diff(t)) also would work
sage: velocity
(2, 2*t, 3/4*t^2)
```

Currently, this expression does *not* function as a function, so we need to substitute explicitly.

```
sage: velocity(t=1)
(2, 2, 3/4)
```

```
sage: T=velocity/velocity.norm()
```

```
sage: T(t=1).n()
(0.683486126173409, 0.683486126173409, 0.256307297315028)
```

Here we compute the arclength between  $t = 0$  and  $t = 1$  by integrating the normalized derivative. As pointed out in the [calculus tutorial](#), the syntax for `numerical_integral` is slightly nonstandard – we just put in the endpoints, not the variable.

```
sage: arc_length = numerical_integral(velocity.norm(), 0, 1)
sage: arc_length
(2.3169847271197814, 2.572369791753217e-14)
```

We can also plot vector fields, even in three dimensions.

```
sage: x,y,z=var('x y z')
sage: plot_vector_field3d((x*cos(z), -y*cos(z), sin(z)), (x,0,pi), (y,0,pi),
↳ (z,0,pi), colors=['red', 'green', 'blue'])
Graphics3d Object
```

If we know a little vector calculus, we can also do line integrals. Here, based on an example by Ben Woodruff of BYU-Idaho, we compute a number of quantities in a physical three-dimensional setting.

Try to read through the entire code. We make an auxiliary function that will calculate  $\int (\text{integrand}) dt$ . We'll use our formulas relating various differentials to  $dt$  to easily specify the integrands.

```
sage: density(x,y,z)=x^2+z
sage: r=vector([3*cos(t), 3*sin(t), 4*t])
sage: tstart=0
sage: tend=2*pi
sage: t = var('t')
sage: t_range=(t,tstart,tend)
sage: def line_integral(integrand):
....:     return RR(numerical_integral((integrand).subs(x=r[0], y=r[1], z=r[2]),
↳ tstart, tend)[0])
sage: _ = var('x,y,z,t')
sage: r_prime = diff(r,t)
sage: ds=diff(r,t).norm()
sage: s=line_integral(ds)
sage: centroid_x=line_integral(x*ds)/s
```

```

sage: centroid_y=line_integral(y*ds)/s
sage: centroid_z=line_integral(z*ds)/s
sage: dm=density(x,y,z)*ds
sage: m=line_integral(dm)
sage: avg_density = m/s
sage: moment_about_yz_plane=line_integral(x*dm)
sage: moment_about_xz_plane=line_integral(y*dm)
sage: moment_about_xy_plane=line_integral(z*dm)
sage: center_mass_x = moment_about_yz_plane/m
sage: center_mass_y = moment_about_xz_plane/m
sage: center_mass_z = moment_about_xy_plane/m
sage: Ix=line_integral((y^2+z^2)*dm)
sage: Iy=line_integral((x^2+z^2)*dm)
sage: Iz=line_integral((x^2+y^2)*dm)
sage: Rx = sqrt(Ix/m)
sage: Ry = sqrt(Iy/m)
sage: Rz = sqrt(Iz/m)

```

Finally, we can display everything in a nice *table*. Recall that we use the `r"stuff"` syntax to indicate “raw” strings so that backslashes from LaTeX won’t cause trouble.

```

sage: html.table([
....:     [r"Density  $\delta(x,y)$ ", density],
....:     [r"Curve  $\vec{r}(t)$ ", r],
....:     [r"$t$ range", t_range],
....:     [r"$\vec{r}'(t)$", r_prime],
....:     [r"$ds$, a little bit of arclength", ds],
....:     [r"$s$ - arclength", s],
....:     [r"Centroid (constant density)  $\left(\frac{1}{m}\int x\,ds, \frac{1}{m}\int y\,ds, \frac{1}{m}\int z\,ds\right)$ ", (centroid_x, centroid_y, centroid_z)],
....:     [r"$dm=\delta ds$ - a little bit of mass", dm],
....:     [r"$m=\int \delta ds$ - mass", m],
....:     [r"average density  $\frac{1}{m}\int ds$ ", avg_density.n()],
....:     [r"$M_{yz}=\int x\,dm$ - moment about $yz$ plane", moment_about_yz_plane],
....:     [r"$M_{xz}=\int y\,dm$ - moment about $xz$ plane", moment_about_xz_plane],
....:     [r"$M_{xy}=\int z\,dm$ - moment about $xy$ plane", moment_about_xy_plane],
....:     [r"Center of mass  $\left(\frac{1}{m}\int xdm, \frac{1}{m}\int ydm, \frac{1}{m}\int zdm\right)$ ", (center_mass_x, center_mass_y, center_mass_z)],
....:     [r"$I_x = \int (y^2+z^2) dm$", Ix], [r"$I_y=\int (x^2+z^2) dm$", Iy], [mp(r"$I_z=\int (x^2+y^2) dm$", Iz)],
....:     [r"$R_x=\sqrt{I_x/m}$", Rx], [mp(r"$R_y=\sqrt{I_y/m}$", Ry)], [mp(r"$R_z=\sqrt{I_z/m}$", Rz)]
....: ])

```

## 7.5.2 Functions of Several Variables

This connects directly to other issues of multivariable functions.

How to view these was mostly addressed in the various plotting tutorials. Here is a reminder of what can be done.

```

sage: # import matplotlib.cm; matplotlib.cm.datad.keys()
sage: # 'Spectral', 'summer', 'blues'
sage: g(x,y)=e^-x*sin(y)
sage: contour_plot(g, (x, -2, 2), (y, -4*pi, 4*pi), cmap = 'Blues', contours=10,
....: colorbar=True)
Graphics object consisting of 1 graphics primitive

```

## Partial Differentiation

The following exercise is from Hass, Weir, and Thomas, University Calculus, Exercise 12.7.35. This function has a local minimum at  $(4, -2)$ .

```
sage: f(x, y) = x^2 + x*y + y^2 - 6*x + 2
```

Quiz: Why did we *not* need to declare the variables in this case?

```
sage: fx(x, y) = f.diff(x)
sage: fy(x, y) = f.diff(y)
sage: fx; fy
(x, y) |--> 2*x + y - 6
(x, y) |--> x + 2*y
```

```
sage: f.gradient()
(x, y) |--> (2*x + y - 6, x + 2*y)
```

```
sage: solve([fx==0, fy==0], (x, y))
[[x == 4, y == -2]]
```

```
sage: H = f.hessian()
sage: H(x, y)
[2 1]
[1 2]
```

And of course if the Hessian has positive determinant and  $f_{xx}$  is positive, we have a local minimum.

```
sage: html("$f_{xx}=%s"%H(4, -2)[0, 0])
sage: html("$D=%s"%H(4, -2).det())
```

Notice how we were able to use many things we've done up to now to solve this.

- Matrices
- Symbolic functions
- Solving
- Differential calculus
- Special formatting commands
- And, below, plotting!

```
sage: plot3d(f, (x, -5, 5), (y, -5, 5)) + point((4, -2, f(4, -2)), color='red', size=20)
Graphics3d Object
```

## Multiple Integrals and More

Naturally, there is lots more that one can do.

```
sage: f(x, y) = x^2*y
sage: # integrate in the order dy dx
sage: f(x, y).integrate(y, 0, 4*x).integrate(x, 0, 3)
1944/5
sage: # another way to integrate, and in the opposite order too
```

```
sage: integrate( integrate(f(x,y), (x, y/4, 3)), (y, 0, 12) )
1944/5
```

```
sage: var('u v')
(u, v)
sage: surface = plot3d(f(x,y), (x, 0, 3.2), (y, 0, 12.3), color = 'blue', opacity=0.3)
sage: domain = parametric_plot3d([3*u, 4*(3*u)*v,0], (u, 0, 1), (v, 0,1), color =
↪'green', opacity = 0.75)
sage: image = parametric_plot3d([3*u, 4*(3*u)*v, f(3*u, 12*u*v)], (u, 0, 1), (v,
↪0,1), color = 'green', opacity = 1.00)
sage: surface+domain+image
Graphics3d Object
```

Quiz: why did we need to declare variables this time?

## 7.6 Sage Quickstart for Numerical Analysis

This Sage quickstart tutorial was developed for the MAA PREP Workshop “Sage: Using Open-Source Mathematics Software with Undergraduates” (funding provided by NSF DUE 0817071). It is licensed under the Creative Commons Attribution-ShareAlike 3.0 license (CC BY-SA).

Sage includes many tools for numerical analysis investigations.

The place to begin is the default decimal number types in Sage.

### 7.6.1 Basic Analysis

- The `RealField` class using arbitrary precision (implemented with `MPFR`).
- The default real numbers (`RR`) is `RealField(53)` (i.e., 53 bits of precision).
- But you can make them live at whatever precision you wish.

```
sage: ring=RealField(3)
```

To print the actual number (without rounding off the last few imprecise digits to only display correct digits), call the `.str()` method with the option `truncate=False`.

```
sage: print(ring('1').nextabove())
1.2
```

```
sage: print(ring('1').nextabove().str(truncate=False))
1.2
sage: print(ring('1').nextbelow().str(truncate=False))
0.88
```

Let's change our precision.

```
sage: ring=RealField(20)
sage: print(ring('1').nextabove().str(truncate=False))
1.0000019
sage: print(ring('1').nextbelow().str(truncate=False))
0.99999905
```





The fraction part of the normalized number is called the *mantissa* (or *significand* ).

```
sage: f=(x*2^(-3)).frac() # .frac() gets the fraction part, the part after the decimal
sage: mantissa=f.str(base=2)
sage: mantissa
'0.1010100100000000000000000000000000000000000000000000000'
```

Since we multiplied by  $2^{-3}$  to normalize the number, we need to store this information. We add 1023 in order to not have to worry about storing negative numbers. In the below, `c` will be our exponent.

```
sage: c=(3+1023).str(base=2)
sage: c
'100000000010'
```

Note that  $c$  has 11 bits, which is exactly what we want.

```
sage: len(c)
```

Evaluating `mantissa[2:54]` will give the first 52 binary digits after the decimal point of the mantissa. Note that we don't need to store the leading 1 before the decimal point because it will always be there from the way we normalized things. This lets us get 53-bit precision using only 52 bits of storage.

```
sage: len(mantissa[2:54])
52
```

Since the original number was positive, our sign bit is zero.

```
sage: sign='0'
```

So here is our 64-bit double-precision floating point number.

```
sage: sign+' '+c+' '+mantissa[2:54] # the [2:] just chops off the '0.', since we just_
↪ need to store the digits after the decimal point
'0 10000000010 101010010000000000000000000000000000000000000000000000000000'
```

```
sage: len(sign+c+mantissa[2:54]) # it's 64 bits!
```

Here we convert back to our original number from the floating point representation that we constructed.

```
sage: ((-1)^(int(sign)) * 2^(int(c,base=2)-1023)*(1+RR(mantissa[:54], base=2)))
13.2812500000000
```

```
sage: x
13.2812500000000
```

So they agree!

Sage uses a cutting-edge numerical library, MPFR, to carry out precise floating point arithmetic using any precision a user specifies. MPFR has a slightly different convention for normalization. In MPFR, we normalize by multiplying by an appropriate power of 2 to make the mantissa an integer, instead of a binary fraction. This allows us to use big integer libraries and sophisticated techniques to carry out calculations at an arbitrary precision.

```
sage: x.sign_mantissa_exponent()
(1, 7476679068876800, -49)
```



```
sage: R=RealField(100) # 100 bits
sage: g=fast_callable(f, vars=[x], domain=R)
sage: plot(g, (x, 0, 2e4))
Graphics object consisting of 1 graphics primitive
```

## 7.7 Sage Quickstart for Number Theory

This Sage quickstart tutorial was developed for the MAA PREP Workshop “Sage: Using Open-Source Mathematics Software with Undergraduates” (funding provided by NSF DUE 0817071). It is licensed under the Creative Commons Attribution-ShareAlike 3.0 license ([CC BY-SA](#)).

Since Sage began life as a project in algebraic and analytic number theory (and this continues to be a big emphasis), it is no surprise that functionality in this area is extremely comprehensive. And it’s also just enjoyable to explore elementary number theory by computer.

### 7.7.1 Modular Arithmetic

Conveniently, the ring of integers modulo  $n$  is always available in Sage, so we can do modular arithmetic very easily. For instance, we can create a number in  $\mathbb{Z}/11\mathbb{Z}$ . The `type` command tells us that  $a$  is not a regular integer.

```
sage: a = mod(2,11); a; type(a); a^10; a^1000000
2
<type 'sage.rings.finite_rings.integer_mod.IntegerMod_int'>
1
1
```

Note that we verified Fermat’s “Little Theorem” in the previous cell, for the prime  $p = 11$  and input  $a = 2$ .

Recalling the basic programming construct called a *loop*, we can verify this for all  $a$  in the integers modulo  $p$ . Here, instead of looping over an explicit list like `[0, 1, 2, 3, . . . 8, 9, 10]`, we loop over a Sage object.

```
sage: for a in Integers(11):
....:     print("{} {}".format(a, a^10))
0 0
1 1
2 1
3 1
4 1
5 1
6 1
7 1
8 1
9 1
10 1
```

Notice that `Integers(11)` gave us an algebraic object which is the ring of integers modulo the prime ideal generated by the element 11.

This works for much bigger numbers too, of course.

```
sage: p=random_prime(10^200,proof=True)
sage: Zp=Integers(p) # Here we give ourselves shorthand for the modular integers
sage: a=Zp(2) # Here we ask for 2 as an element of that ring
sage: p; a; a^(p-1); a^(10^400)
```

```
8312788012695811618307874983564775723984228960862912671896833078489764688168961070553362809593882411
2
1
2142806294038015609753838672291339055944542026496540965623271466415613614492017381893641542784495375
```

Whenever you encounter a new object, you should try tab-completion to see what you can do to it. Try it here, and pick one (hopefully one that won't be too long!).

```
sage: Zp.
```

Here's one that sounds interesting.

```
sage: Zp.zeta?
```

And we use it to find the fifth roots of unity in this field. We use the *list comprehension* (set-builder) notation from the programming tutorial this time.

```
sage: root_list = Zp.zeta(5, all=True); root_list
[801997703885633241003345486263452400812942732891098664369966525253282686529225088929460685386415383
↪ 69839783895572286297568834485025073557885364348071061715465477061873400359794989367423407683971299
↪ 57407444219199061498801298672323590163238592201574572482836619025676869537007609315386800852204337
↪ 41936641877539676652531567723249367917108638987131879521606243741814402095343724540844607212999297
```

Are these really fifth roots of unity?

```
sage: [root^5 for root in root_list]
[1, 1, 1, 1]
```

Luckily, it checked out. (If you didn't get any, then your random prime ended up being one for which there *are* no fifth roots of unity – try doing the sequence over again!)

## 7.7.2 More basic functionality

Similarly to the situation in linear algebra, there is much more we can access at the elementary level, such as

- primitive roots,
- ways to write a number as a sum of squares,
- Legendre symbols,
- modular solving of basic equations,
- etc.

A good way to use Sage in this context is to allow students to experiment with pencil and paper first, then use Sage to see whether patterns they discover hold true before attempting to prove them.

```
sage: p = 13
sage: primitive_root(p); two_squares(p); is_prime(p)
2
(2, 3)
True
```

This makes it easy to construct elementary cryptographic examples as well. Here is a standard example of a Diffie-Hellman key exchange, for instance. If we didn't do the second line, exponentiation would be impractical.

```

sage: p=random_prime(10^20,10^30) # a random prime between these numbers
sage: q=mod(primitive_root(p),p) # makes the primitive root a number modulo p, not an
↪integer
sage: n=randint(1,p) # Alice's random number
sage: m=randint(1,p) # Bob's random number
sage: x=q^n; y=q^m
sage: x; y; x^m; y^n
66786436189350477660
77232558812003408270
45432410008036883324
45432410008036883324

```

The final line of the cell first requests Alice and Bob's (possibly) public information, and then verifies that the private keys they get are the same.

It is hard to resist including just one interact. How many theorems do you see here?

```

sage: @interact
sage: def power_table_plot(p=(7,prime_range(50))):
....:     P=matrix_plot(matrix(p-1,[mod(a,p)^b for a in range(1,p) for b in
↪srange(p)]),cmap='jet')
....:     show(P)

```

This is a graphic giving the various powers of integers modulo  $p$  as colors, not numbers. The columns are the powers, so the first column is the zeroth power (always 1) and the second column gives the colors for the numbers modulo the given prime (first power).

One more very useful object is the prime counting function  $\pi(x)$ . This comes with its own custom plotting.

```

sage: prime_pi(100); plot(prime_pi,1,100)
25
Graphics object consisting of 1 graphics primitive

```

A very nice aspect of Sage is combining several aspects of mathematics together. It can be very eye-opening to students to see analytic aspects of number theory early on. (Note that we have to reassign  $x$  to a variable, since above it was a cryptographic key!)

```

sage: var('x')
x
sage: plot(prime_pi,2,10^6,thickness=2)+plot(Li,2,10^6,color='red')+plot(x/ln(x),2,10^
↪6,color='green')
Graphics object consisting of 3 graphics primitives

```

## 7.7.3 Advanced Number Theory

For those who are interested, more advanced number-theoretic objects are easy to come by; we end with a brief sampler of these.

In the first example,  $K$  is the field extension  $\mathbf{Q}(\sqrt{-14})$ , where the symbol  $a$  plays the role of  $\sqrt{-14}$ ; we discover several basic facts about  $K$  in the next several cells.

```

sage: K.<a> = NumberField(x^2+14); K
Number Field in a with defining polynomial x^2 + 14

```

```

sage: K.discriminant(); K.class_group().order(); K.class_group().is_cyclic()
-56

```

```
4
True
```

Various zeta functions are also available; here is a complex plot of the Riemann zeta.

```
sage: complex_plot(zeta, (-30,30), (-30,30))
Graphics object consisting of 1 graphics primitive
```

## 7.7.4 Cryptography

Sage supports various more advanced cryptographic procedures as well as some basic pedagogical ones natively. This example is adapted from the documentation.

```
sage: from sage.crypto.block_cipher.sdes import SimplifiedDES
sage: sdes = SimplifiedDES(); sdes
Simplified DES block cipher with 10-bit keys
```

```
sage: bin = BinaryStrings()
sage: P = [0,1,0,0,1,1,0,1] # our message
sage: K = sdes.random_key() # generate a random key
sage: C = sdes.encrypt(P, K) # encrypt our message
sage: plaintext = sdes.decrypt(C, K) # decrypt it
sage: plaintext # print it
[0, 1, 0, 0, 1, 1, 0, 1]
```

See also the cryptography example in the *discrete math quickstart*.

## 7.8 Sage Quickstart for Statistics

This Sage quickstart tutorial was developed for the MAA PREP Workshop “Sage: Using Open-Source Mathematics Software with Undergraduates” (funding provided by NSF DUE 0817071). It is licensed under the Creative Commons Attribution-ShareAlike 3.0 license (CC BY-SA).

Although Sage began as a project in algebra and geometry, it has many functions for statistics and finance. Particularly due to the R project being a component of Sage, we have very powerful statistical techniques at our disposal.

### 7.8.1 Basic Descriptive Statistics

Some basic statistical functions are built right in.

```
sage: mean([1,2,3,5])
11/4
```

```
sage: std([1,2,2,4,5,6,8]) # The standard deviation
sqrt(19/3)
```

Once we get beyond such things, there are usually several ways to accomplish things – which can be complicated, but also powerful.

## 7.8.2 Distributions

Let's generate a random sample from a given type of continuous distribution using native Python random generators.

- We use the simplest method of generating random elements from a log normal distribution with (normal) mean 2 and  $\sigma = 3$ .
- Notice that there is really no way around making some kind of loop.

```
sage: my_data=[lognormvariate(2,3) for i in range(10)]
sage: my_data # random
[13.189347821530054, 151.28229284782799, 0.071974845847761343, 202.62181449742425, 1.
↪ 9677158880100207, 71.959830176932542, 21.054742855786007, 3.9235315623286406, 4129.
↪ 9880239483346, 16.41063858663054]
```

We can check whether the mean of the log of the data is close to 2.

```
sage: mean([log(item) for item in my_data]) # random
3.0769518857697618
```

Here is an example using the [Gnu scientific library](#) under the hood.

- Let `dist` be the variable assigned to a continuous Gaussian/normal distribution with standard deviation of 3.
- Then we use the `.get_random_element()` method ten times, adding 2 each time so that the mean is equal to 2.

```
sage: dist=RealDistribution('gaussian',3)
sage: my_data=[dist.get_random_element()+2 for _ in range(10)]
sage: my_data # random
[3.18196848067, -2.70878671264, 0.445500746768, 0.579932075555, -1.32546445128, 0.
↪ 985799587162, 4.96649083229, -1.78785287243, -3.05866866979, 5.90786474822]
```

For now, it's a little annoying to get histograms of such things directly. Here, we get a larger sampling of this distribution and plot a histogram with 10 bins.

```
sage: my_data2 = [dist.get_random_element()+2 for _ in range(1000)]
sage: T = stats.TimeSeries(my_data2)
sage: T.plot_histogram(normalize=False,bins=10)
Graphics object consisting of 10 graphics primitives
```

To access discrete distributions, we access another part of Sage which has statistics built in: [Scipy](#).

- We have to import this module.
- We use `binom_dist` to denote the binomial distribution with 20 trials and 5% expected failure rate.
- The `.pmf(x)` method gives the probability of  $x$  failures, which we then plot in a bar chart for  $x$  from 0 to 20. (Don't forget that `range(21)` means all integers from *zero to twenty*.)

```
sage: import scipy.stats
sage: binom_dist = scipy.stats.binom(20,.05)
sage: bar_chart([binom_dist.pmf(x) for x in range(21)])
Graphics object consisting of 1 graphics primitive
```

The `bar_chart` function performs some of the duties of histograms.

Scipy's statistics can do other things too. Here, we find the median (as the fiftieth percentile) of an earlier data set. (We use a Python `int` to work around a bug in Numpy.)



```
sage: scipy.stats.scoreatpercentile(my_data, int(50)) # random
0.51271641116183286
```

The key thing to remember here is to look at the documentation!

- Particularly for Scipy, not everything in Sage is “wrapped” with an easy command, so you may have to do some experimentation.
- Improving this documentation would be a great way to get students involved.

### 7.8.3 Using R from within Sage

There are several other pieces of Sage that have statistical capabilities, but by far the most important is the [R project](#), which is the industry and academic standard for statistical analysis of *all* kinds.

There are several ways to access R.

- One of the easiest is to just put `r()` around things you want to make into statistical objects, and then ...
- Use R commands via `r.method()` to pass them on to Sage for further processing.

The following example of the Kruskal-Wallis test comes directly from the examples in `r.kruskal_test?` in the notebook.

```
sage: x=r([2.9, 3.0, 2.5, 2.6, 3.2]) # normal subjects
sage: y=r([3.8, 2.7, 4.0, 2.4])      # with obstructive airway disease
sage: z=r([2.8, 3.4, 3.7, 2.2, 2.0]) # with asbestosis
sage: a = r([x,y,z]) # make a long R vector of all the data
sage: b = r.factor(5*[1]+4*[2]+5*[3]) # create something for R to tell which subjects
↪are which
sage: a; b # show them
[1] 2.9 3.0 2.5 2.6 3.2 3.8 2.7 4.0 2.4 2.8 3.4 3.7 2.2 2.0
[1] 1 1 1 1 1 2 2 2 2 3 3 3 3
Levels: 1 2 3
```

```
sage: r.kruskal_test(a,b) # do the KW test!
Kruskal-Wallis rank sum test

data: sage17 and sage33
Kruskal-Wallis chi-squared = 0.7714, df = 2, p-value = 0.68
```

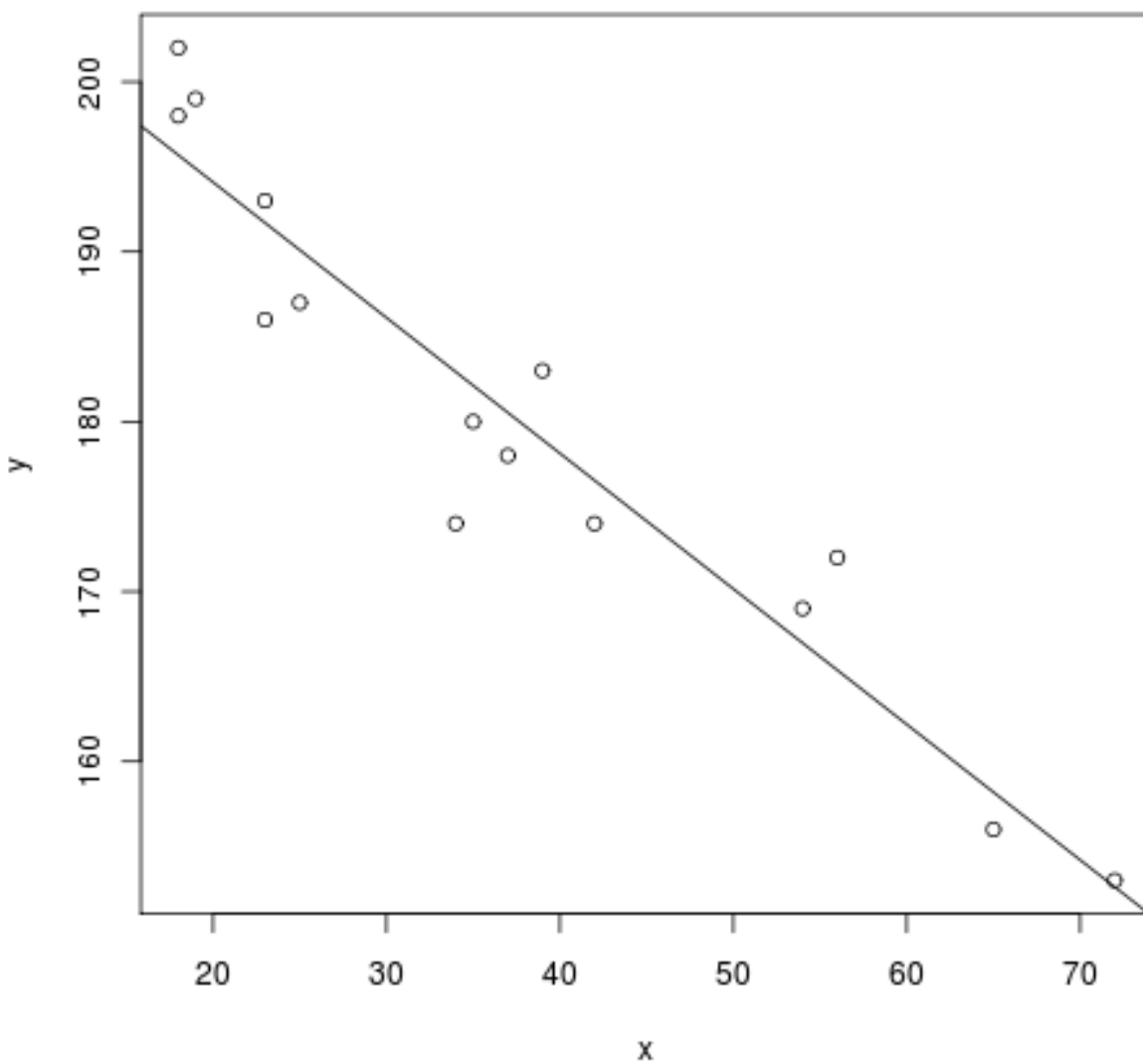
Looks like we can’t reject the null hypothesis here.

The best way to use R seriously is to simply ask each individual cell to evaluate completely in R, using a so-called “percent directive”. Here is a sample linear regression from John Verzani’s [simpleR](#) text. Notice that R also uses the `#` symbol to indicate comments.

```
sage: %r
....: x = c(18,23,25,35,65,54,34,56,72,19,23,42,18,39,37) # ages of individuals
....: y = c(202,186,187,180,156,169,174,172,153,199,193,174,198,183,178) # maximum
↪heart rate of each one
....: png() # turn on plotting
....: plot(x,y) # make a plot
....: lm(y ~ x) # do the linear regression
....: abline(lm(y ~ x)) # plot the regression line
....: dev.off() # turn off the device so it plots
Call:
lm(formula = y ~ x)
```

```
Coefficients:
(Intercept)      x
  210.0485    -0.7977

null device
      1
```



To get a whole worksheet to evaluate in R (and be able to ignore the % ), you could also drop down the `r` option in the menu close to the top which currently has `sage` in it.

(There is also yet another Python interface to R called the [rpy2](#) interface, but we do not currently recommend its use with Sage.)

## 7.9 Sage Interact Quickstart

This Sage quickstart tutorial was developed for the MAA PREP Workshop “Sage: Using Open-Source Mathematics Software with Undergraduates” (funding provided by NSF DUE 0817071).

Invaluable resources are the Sage wiki <http://wiki.sagemath.org/interact> (type “sage interact” into Google), <http://interact.sagemath.org> (a collection of contributed interacts), and the [interact documentation](#).

### 7.9.1 Start with just one command

How would one create an interactive cell? First, let’s focus on a new thing to do! Perhaps we just want a graph plotter that has some options.

So let’s start by getting the commands for what you want the output to look like. Here we just want a simple plot.

```
sage: plot(x^2, (x, -3, 3))
Graphics object consisting of 1 graphics primitive
```

Then abstract out the parts you want to change. We’ll be letting the user change the function, so let’s make that a variable  $f$ .

```
sage: f=x^3
sage: plot(f, (x, -3, 3))
Graphics object consisting of 1 graphics primitive
```

This was important because it allowed you to step back and think about what you would really be doing.

Now for the technical part. We make this a `def` function - see the [programming tutorial](#).

```
sage: def myplot(f=x^2):
....:     show(plot(f, (x, -3, 3)))
```

Let’s test the `def` function `myplot` by just calling it.

```
sage: myplot()
```

If we call it with a different value for  $f$ , we should get a different plot.

```
sage: myplot(x^3)
```

So far, we’ve only defined a new function, so this was review. To make a “control” to allow the user to interactively enter the function, we just preface the function with `@interact`.

```
sage: @interact
sage: def myplot(f=x^2):
....:     show(plot(f, (x, -3, 3)))
```

**Note:** Technically what `@interact` does is wrap the function, so the above is equivalent to:

```
def myplot(..): ...
myplot=interact(myplot)
```

Note that we can still call our function, even when we’ve used `@interact`. This is often useful in debugging it.

```
sage: myplot(x^4)
```

## 7.9.2 Adding Complexity

We can go ahead and replace other parts of the expression with variables. Note that `_` is the function name now. That is a just convention for throw-away names that we don't care about.

```
sage: @interact
sage: def _(f=x^2, a=-3, b=3):
....:     show(plot(f, (x,a,b)))
```

If we pass `('label', default_value)` in for a control, then the control gets the label when printed. Here, we've put in some text for all three of them. Remember that the text must be in quotes! Otherwise Sage will think that you are referring (for example) to some variable called "lower", which it will think you forgot to define.

```
sage: @interact
sage: def _ (f='$f$', x^2), a=('lower', -3), b=('upper', 3)):
....:     show(plot(f, (x,a,b)))
```

We can specify the type of control explicitly, along with options. See [below](#) for more detail on the possibilities.

```
sage: @interact
sage: def _ (f=input_box(x^2, width=20, label="$f$")):
....:     show(plot(f, (x,-3,3)))
```

Here we demonstrate a bunch of options. Notice the new controls:

- `range_slider`, which passes in *two* values, `zoom[0]` and `zoom[1]`
- `True/False` gets converted to checkboxes for the end user

```
sage: @interact
sage: def _ (f=input_box(x^2,width=20),
....: color=color_selector(widget='colorpicker', label=""),
....: axes=True,
....: fill=True,
....: zoom=range_slider(-3,3,default=(-3,3))):
....:     show(plot(f, (x,zoom[0], zoom[1]), color=color, axes=axes, fill=fill))
```

There is also one button type to *disable automatic updates*.

The previous interact was a bit ugly, because all of the controls were stacked on top of each other. We can control the layout of the widget controls in a grid (at the top, bottom, left, or right) using the `layout` parameter.

```
sage: @interact(layout=dict(top=[['f', 'color']],
....: left=[['axes'], ['fill']],
....: bottom=[['zoom']])))
sage: def _ (f=input_box(x^2,width=20),
....: color=color_selector(widget='colorpicker', label=""),
....: axes=True,
....: fill=True,
....: zoom=range_slider(-3,3, default=(-3,3))):
....:     show(plot(f, (x,zoom[0], zoom[1]), color=color, axes=axes, fill=fill))
```

### 7.9.3 Control Types

There are many potential types of widgets one might want to use for interactive control. Sage has all of the following:

- boxes
- sliders
- range sliders
- checkboxes
- selectors (dropdown lists or buttons)
- grid of boxes
- color selectors
- plain text

We illustrate some more of these below. For complete detail, see the official [interact documentation](#).

```
sage: @interact
sage: def _(frame=checkbox(True, label='Use frame')):
....:     show(plot(sin(x), (x,-5,5)), frame=frame)
```

```
sage: var('x,y')
sage: colormaps=sage.plot.colors.colormaps.keys()
sage: @interact
sage: def _(cmap=selector(colormaps)):
....:     contour_plot(x^2-y^2, (x,-2,2), (y,-2,2), cmap=cmap).show()
```

```
sage: var('x,y')
sage: colormaps=sage.plot.colors.colormaps.keys()
sage: @interact
sage: def _(cmap=selector(['RdBu', 'jet', 'gray', 'gray_r'], buttons=True),
sage: type=['density', 'contour']):
....:     if type=='contour':
....:         contour_plot(x^2-y^2, (x,-2,2), (y,-2,2), cmap=cmap, aspect_ratio=1).show()
....:     else:
....:         density_plot(x^2-y^2, (x,-2,2), (y,-2,2), cmap=cmap,
↪ frame=True, axes=False, aspect_ratio=1).show()
```

By default, ranges are sliders that divide the range into 50 steps.

```
sage: @interact
sage: def _(n=(1,20)):
....:     print(factorial(n))
```

You can set the step size to get, for example, just integer values.

```
sage: @interact
sage: def _(n=slider(1,20, step_size=1)):
....:     print(factorial(n))
```

Or you can explicitly specify the slider values.

```
sage: @interact
sage: def _(n=slider([1..20])):
....:     print(factorial(n))
```

And the slider values don't even have to be numbers!

```
sage: @interact
sage: def _(fun=('function', slider([sin,cos,tan,sec,csc,cot]))):
....:     print(fun(4.39293))
```

Matrices are automatically converted to a grid of input boxes.

```
sage: @interact
sage: def _(m=('matrix', identity_matrix(2))):
....:     print(m.eigenvalues())
```

Here's how to get vectors from a grid of boxes.

```
sage: @interact
sage: def _(v=('vector', input_grid(1, 3, default=[[1,2,3]], to_value=lambda x:
↪vector(flatten(x)))):
....:     print(v.norm())
```

## 7.9.4 The option not to update

As a final problem, what happens when the controls get so complicated that it would be counterproductive to see the interact update for each of the changes one wants to make? Think changing the endpoints and order of integration for a triple integral, for instance, or the example below where a whole matrix might be changed.

In this situation, where we don't want any updates until we specifically say so, we can use the `auto_update=False` option. This will create a button to enable the user to update as soon as he or she is ready.

```
sage: @interact
sage: def _(m=('matrix', identity_matrix(2)), auto_update=False):
....:     print(m.eigenvalues())
```