1 General Linear Model

We will study inference problem for GLM into the chapter.

Definition 1 *General Linear Model(GLM)*

Let Y be an $nx1$ observable random vector, X be an nxp matrix $(n > p)$ of known fixed numbers. β be a $px1$ vector of unknown parameter; ϵ be an $nx1$ unobservable random (error) vector with $E(\epsilon) = 0$ and $Cov(\epsilon) = \Sigma$; and let those quantities be related by:

$$Y = x\beta + \epsilon \tag{1.1}$$

These specification define a GLM
We will discuss estimation and hypothesis testing mostly for Case 1 and estimate Case 2 in the following sections of this chapter.

2 Point Estimation of α^2 and Linear Function of β: Case 1

Theorem 2.1 *Let $Y = x\beta + \epsilon$ as specified in Defn 1.1.1*
Assume $\epsilon \sim N_n(0, \alpha^2 1)$ Then the following results follow:

1. $\beta = X'Y$ *is the MLE for* β *where* $X' = (X'X)^{-1}X'$
2. $\alpha^2 = \dfrac{1}{n-p}Y'(1-K)Y$ *is the MLE for* α^2, *where* $K = X(X'X)^{-1}X' = XX'$

3 $\beta \sim N_{10}(\beta, \alpha^2 C)$, *where* $C = \big(X'X\big)^{-1}$

4 $(n-p)\alpha^2/\alpha^2 = U \sim X^2_{n-p}$

5 β *and* α *are independent*

6 β *and* α *are sufficient statistics for* β *and* α^2

7 β *and* α *are complete statistics*

Proof: $\epsilon \sim N_n 0, \alpha^2 1) \Longrightarrow Y \sim N_n(x\beta, \alpha^2 1)$
$\Longrightarrow$ *The likelihood function is:*

$$L(\beta, \alpha^2|Y) = \Big(\frac{1}{\sqrt{2\pi\alpha^2}}\Big)^n e^{-\dfrac{1}{2\alpha^2}(y-x\beta)'(y-x\beta)}$$

$$\Longrightarrow \ln L(\beta,\alpha^2) = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\alpha^2 - \frac{1}{2\alpha^2}(y-x\beta)'(y-x\beta)$$

where the parameter space is:

$$\omega = (\beta,\alpha^2); \alpha^2 > 0, -\infty < \beta_i < \infty, i = 1, ...p$$

$$\frac{\partial \ln L(\beta,\alpha^2)}{\partial \beta} = +\frac{2}{\alpha^2}X'(y-X\beta) = \frac{1}{\alpha^2}(X'y - X'X\beta) = 0$$

$$\frac{\partial \ln L(\beta,\alpha^2)}{\partial \beta} = -\frac{n}{2\alpha^2}X'(y-X\beta) = \frac{1}{2\alpha^4}(y - X'\beta) = 0$$

$$\Longrightarrow \begin{cases} x'y - X'X\beta = 0 \\ (y-x\beta)'(y-x\beta) - n\alpha^2 = 0 \end{cases} \text{"}normalequations\text{"}$$

$$\Longrightarrow \begin{cases} X'X\beta = X'y \\ (\alpha^2 = \frac{1}{n}(y-X\beta)'(y-x\beta) \end{cases} \text{"}aresolutions\text{"}$$

for the above normal equations.

As X has rank P, $X'X$ is of full rank. That is $(X'X)^{-1}$ exists. Then the MLEs are obtained as:

$$\hat{\beta} = (X'X)^{-1}X'\underline{y} = X - \underline{Y}$$

$$\hat{\alpha}^2 = \frac{1}{n}(\underline{y} - X(X'X)^{-1}X'\underline{y})'(\underline{y} - X(X'X)^{-1}X'\underline{y})$$

$$= \frac{1}{n}\underline{y'}\big[I - X(X'X)^{-1}X'\big]\big[I - X(X'X)^{-1}X'\big]\underline{y}$$

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where $X = (X'X)^{-1}X'$

$$\Longrightarrow \hat{\beta} = X'\underline{Y} istheMLEof \underline{\beta} \qquad isproved$$

also

$$\Longrightarrow \hat{\beta} = X'\underline{Y} istheMLEof \underline{\beta} \qquad (1) isproved$$

$\hat{\alpha}^2 = \frac{1}{n}\underline{y'}\big[I - X(X'X)^{-1}X'\big]\underline{y}$ *is a function(a constant multiple) of the MLE of $\hat{\alpha}^2$ (2) is proved*

AS $\hat{\beta} = X'\underline{y} = (X'X)^{-1}X'\underline{y}$ is a linear form of y.

$$\Longrightarrow \underline{\beta} \sim N_p\big[(X'X)^{-1}X\beta, \alpha^2(X'X)^{-1}X'X(X'X)^{-1}\big]$$

$$= N_p(\beta, \alpha^2(X'X)^{-1}) = N_p(\beta, \alpha^2 C), C = (X'X)^{-1}$$

$\implies$ *(3) is proved*

As $U = \dfrac{(n-p)\hat{\alpha}^2}{\alpha^2} = \dfrac{1}{\alpha}\underline{y'}\big[I - X(X'X)^{-1}X'\big]\underline{y}$ *is a quadratic function of* $\underline{y}$ *with* $\underline{y} \sim N_p(\beta, \alpha^2 I)$ *and* $A = \big[I - X(X'X)^{-1}X'\big]\dfrac{1}{\alpha^2}$ *Now,*

$$A\sum = \frac{1}{\alpha^2}\big[I - X(X'X)^{-1}X'\big]^2 I = I - X(X'X)^{-1}X'$$

$$= \big[I - X(X'X)^{-1}X'\big]$$

$$A\sum A\sum = \big[I - X(X'X)^{-1}X'\big]\big[I - X(X'X)^{-1}X'\big]$$

$$= I - X(X'X)^{-1}X' - X(X'X)^{-1}X' + X(X'X)^{-1}X'X$$