Chapter 6. General Linear Model

Sec 6.1. Intro

We will study inference problem for GLM in this chapter.

Definition 6.1.1 General Linear Model (GLM).
Let $\underline{Y}$ be an $n\times 1$ observable random vector, X be an $n\times p$ matrix $(n>p)$ of known fixed numbers; $\underline{\beta}$ be a $p\times 1$ vector of unknown parameters; $\underline{\varepsilon}$ be an $n\times 1$ unobservable random (error) vector with $E(\underline{\varepsilon})=\underline{0}$ and $Cov(\underline{\varepsilon})=\Sigma$; and let these quantities be related ~~as given~~ by:

$$\underline{Y}=X\underline{\beta}+\underline{\varepsilon} \qquad (6.1.1)$$

These specifications define a GLM.

Remark. (1) For this chapter, we assume X is of its full rank p.
(2) Two cases will be considered:
Case 1: $\underline{\varepsilon}\sim N_n(\underline{0},\sigma^2 I)$, σ^2 unknown
Case 2. $\underline{\varepsilon}$ follows an unknown distribution with $E(\underline{\varepsilon})=\underline{0}$ and $Cov(\underline{\varepsilon})=\sigma^2 I$, σ^2 unknown.

We will discuss estimation and hypothesis testing mostly for Case 1 (Sec 6.1–6.9) and estimation for case 2 (Sec 6.10) in the follow sections of this chapter.

Sec 6.2. Point Estimation of σ^2 and Linear Functions of β: Case I.

Y?

Thm 6.2.1 Let $\underline{Y} = X\underline{\beta} + \underline{\varepsilon}$ as specified in Defn 6.1.1. Assume $\underline{\varepsilon} \sim N_n(\underline{0}, \sigma^2 I)$. Then the following results follow:

(1) $\hat{\underline{\beta}} = X^- \underline{Y}$ is the MLE for $\underline{\beta}$, where $X^- = (X'X)^{-1}X'$.

(2) $\hat{\sigma}^2 = \frac{1}{n-p}\underline{Y}'(I-K)\underline{Y}$ is the MLE for σ^2, where $K = X(X'X)^{-1}X' = XX^-$.
based estimator

(3). $\hat{\underline{\beta}} \sim N_p(\underline{\beta}, \sigma^2 C)$, where $C = (X'X)^{-1}$.

(4) $(n-p)\hat{\sigma}^2/\sigma^2 = U \sim \chi^2_{n-p}$.

(5) $\hat{\underline{\beta}}$ and $\hat{\sigma}^2$ are independent.

(6) $\hat{\underline{\beta}}$ and $\hat{\sigma}^2$ are sufficient statistics for $\underline{\beta}$ and σ^2.

(7) $\hat{\underline{\beta}}$ and $\hat{\sigma}^2$ are complete statistics.

Proof: $\underline{\varepsilon} \sim N_n(\underline{0}, \sigma^2 I) \Rightarrow \underline{Y} \sim N_n(X\underline{\beta}, \sigma^2 I)$

$\Rightarrow$ The likelihood function is:

$$L(\underline{\beta}, \sigma^2 | \underline{Y}) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}(\underline{y}-X\underline{\beta})'(\underline{y}-X\underline{\beta})}$$

$$\Rightarrow \ln L(\underline{\beta}, \sigma^2) = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}(\underline{y}-X\underline{\beta})'(\underline{y}-X\underline{\beta})$$

where the parameter space is:

$$\Omega = \{(\underline{\beta}, \sigma^2): \sigma^2 > 0, -\infty < \beta_i < \infty, i=1,\cdots,p\}$$

$$\frac{\partial \ln L(\underline{\beta}, \sigma^2)}{\partial \underline{\beta}} = +\frac{2}{2\sigma^2}X'(\underline{y}-X\underline{\beta}) = \frac{1}{\sigma^2}(X'\underline{y} - X'X\underline{\beta}) \overset{set}{=} 0$$

$$\frac{\partial \ln L(\underline{\beta}, \sigma^2)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}(\underline{y}-X\underline{\beta})'(\underline{y}-X\underline{\beta}) \overset{set}{=} 0$$

$$\Rightarrow \begin{cases} X'\underline{y} - X'X\underline{\beta} = 0 \\ (\underline{y}-X\underline{\beta})'(\underline{y}-X\underline{\beta}) - n\sigma^2 = 0 \end{cases}$$

"normal equations"

$$\Longrightarrow \begin{cases} X'X\hat{\beta} = X'y \\ \hat{\sigma}^2 = \frac{1}{n}(\underline{y} - X\underline{\beta})'(\underline{y} - X\underline{\beta}) \end{cases}$$ are solutions

for the above normal equations.

As X has rank p, $X'X$ is of full rank. That is, $(X'X)^{-1}$ exists. Then the MLEs are obtained as:

$$\tilde{\beta} = (X'X)^{-1}X'\underline{y} = X^{-}Y$$

$$\tilde{\sigma}^2 = \frac{1}{n}(\underline{y} - X(X'X)^{-1}X'\underline{y})'(\underline{y} - X(X'X)^{-1}X'\underline{y})$$

$$= \frac{1}{n}\underline{y}'[I - X(X'X)^{-1}X'][I - X(X'X)^{-1}X']\underline{y}$$

$$= \frac{1}{n}\underline{y}'[I - X(X'X)^{-1}X']\underline{y} \qquad \uparrow \text{idempotent}$$

~~where~~ Let $X^{-} = (X'X)^{-1}X'$

$\Longrightarrow$ $\hat{\beta} = X^{-}Y$ is the MLE of $\underline{\beta}$ (1) is proved.

also $\Longrightarrow$ $\hat{\sigma}^2 = \frac{1}{n-p}\underline{y}'[I - X(X'X)^{-1}X']\underline{y}$ is a funct^n (a constant multiple) of the MLE $\tilde{\sigma}^2$, and hence the MLE of σ^2. (2) is proved.

As $\hat{\underline{\beta}} = X^{-}\underline{y} = (X'X)^{-1}X'\underline{y}$ is a linear form ~~function~~ of $\underline{y}$

$$\Longrightarrow \hat{\underline{\beta}} \sim N_p((X'X)^{-1}X'X\underline{\beta}, \sigma^2(X'X)^{-1}X'X(X'X)^{-1}]$$

$$= N_p(\underline{\beta}, \sigma^2(X'X)^{-1}) = N_p(\underline{\beta}, \sigma^2 C), \quad C = (X'X)^{-1}.$$

$\Longrightarrow$ (3) is proved.

As $U = \frac{(n-p)\hat{\sigma}^2}{\sigma^2} = \frac{1}{\sigma^2}\underline{y}'[I - X(X'X)^{-1}X']\underline{y}$ is a quadratic function of $\underline{y}$ with $\underline{y} \sim N_n(X\underline{\beta}, \underbrace{\sigma^2 I}_{\Sigma})$ and

$A = [I - X(X'X)^{-1}X']\frac{1}{\sigma^2}$.

Now, $A\Sigma = \frac{1}{\alpha^2}[I - X(X'X)^{-1}X']\alpha^2 I = I - X(X'X)^{-1}X'$

$$= [I - X(X'X)^{-1}X']$$

$$A\Sigma A\Sigma = [I - X(X'X)^{-1}X'][I - X(X'X)^{-1}X']$$

$$= I - X(X'X)^{-1}X' - X(X'X)^{-1}X' + X(X'X)^{-1}X'X(X'X)^{-1}X'$$

$$= I - X(X'X)^{-1}X'$$

$$= A\Sigma$$

that is, $A\Sigma$ is idempotent

and $\text{rank}(A\Sigma) = \text{rank}[I - X(X'X)^{-1}X']$

$$\overset{\text{Thm 1.8.5}}{=} \text{trace}[I - X(X'X)^{-1}X']$$

$$= \text{trace}(I) - \text{trace}[X(X'X)^{-1}X']$$

$$= n - \text{trace}[(X'X)^{-1}X'X]$$

$$= n - \text{trace}[I_p] = n - p$$

$$\xRightarrow{\text{Thm 4.4.3}} \quad U \sim \chi'^2_{n-p,\,\lambda}$$

$$\lambda = \frac{1}{2}(X\beta)'\frac{1}{\alpha^2}[I - X(X'X)^{-1}X'](X\beta)$$

$$= \frac{1}{2\alpha^2}[\beta'X'X\beta - \beta'X'X\underbrace{(X'X)^{-1}X'X}\beta]$$

$$= 0.$$

That is, indeed, $U \sim \chi^2_{n-p}$. (4) is proved.

Further, as $\hat{\beta} = \underbrace{(X'X)^{-1}X'}_{B}y$ is a linear form of y while $\hat{\alpha}^2$ is a quadratic form of y, where $y \sim N_n(X\beta, \underset{\underset{\alpha^2 I}{\Vert}}{\alpha^2 I})$, $\hat{\alpha}^2 = y'\underbrace{\frac{1}{n-p}[I - X(X'X)^{-1}X']}_{A}y$

As $B\Sigma A = (X'X)^{-1}X'\Sigma\frac{1}{n-p}[I - X(X'X)^{-1}X']$

$$= \frac{\alpha^2}{n-p}[(X'X)^{-1}X' - \underbrace{(X'X)^{-1}X'X}_{I}(X'X)^{-1}X']$$

$$= 0 \xRightarrow{\text{Thm 4.5.2}} \hat{\beta} \perp\!\!\!\perp \hat{\alpha}^2.$$

(6-5)

In the proof of Thm 6.2.1;

Lastly, we want to prove the sufficiency and completeness of $\hat{\beta}$ and $\hat{\alpha}^2$.

Note again:

$$f_{\underline{Y}}(\underline{y};\beta,\alpha^2)=\frac{1}{(\sqrt{2\pi\alpha^2})^n}\exp\{-\frac{1}{2\alpha^2}(\underline{Y}-X\beta)'(\underline{Y}-X\beta)\}$$

where:

$$\begin{aligned}
&(\underline{Y}-X\beta)'(\underline{Y}-X\beta)\\
&=[(\underline{Y}-X\hat{\beta})'+(X\hat{\beta}-X\beta)'][(\underline{Y}-X\hat{\beta})+(X\hat{\beta}-X\beta)]\\
&=(\underline{Y}-X\hat{\beta})'(\underline{Y}-X\hat{\beta})-(\beta-\hat{\beta})'\underbrace{X'(\underline{Y}-X\hat{\beta})}_{=0}\\
&\quad-\underbrace{(\underline{Y}-X\hat{\beta})'X}_{0}(\beta-\hat{\beta})+(\beta-\hat{\beta})'X'X(\beta-\hat{\beta})\\
&=(\underline{Y}-X\hat{\beta})'(\underline{Y}-X\hat{\beta})+(\beta-\hat{\beta})'X'X(\beta-\hat{\beta})\\
&=n\hat{\alpha}^2+\beta'X'X\beta-\hat{\beta}'X'X\beta-\beta'X'X\hat{\beta}+\hat{\beta}'X'X\hat{\beta}\\
&=n\hat{\alpha}^2+\beta'X'X\beta-2\beta'X'X\hat{\beta}+\underbrace{\underline{Y}'\underline{Y}-n\hat{\alpha}^2}\\
&=\underline{Y}'\underline{Y}+\beta'X'X\beta-2\beta'X'X\hat{\beta}
\end{aligned}$$

pay attention starting here

$n\hat{\alpha}^2=\underline{y}'[I-X(X'X)^{-}X']\underline{y}$

$=\underline{y}'\underline{y}-\underline{y}'X(X'X)^{-}X'\underline{y}$

(1) $\Rightarrow \hat{\beta}'X'X\hat{\beta}=\underline{y}'\underline{y}-n\hat{\alpha}^2$

Then:

$$f_{\underline{Y}}(\underline{Y};\beta,\alpha^2)=\left(\frac{1}{\sqrt{2\pi\alpha^2}}\right)^n\exp\left\{-\frac{1}{2\alpha^2}\underline{Y}'\underline{Y}-\frac{\beta'X'X\beta}{2\alpha^2}+\frac{\beta'X'X\hat{\beta}}{\alpha^2}\right\}$$

$$=\underbrace{\left(\frac{1}{\sqrt{2\pi\alpha^2}}\right)^n e^{-\frac{\beta'X'X\beta}{2\alpha^2}}}_{h(\underline{\theta})}\cdot\underbrace{1}_{g(\underline{Y})}\cdot\exp\left\{\underbrace{-\frac{1}{2\alpha^2}}_{P_2(\underline{\theta})}\underbrace{\underline{Y}'\underline{Y}}_{S_2(\underline{Y})}+\underbrace{\frac{\beta'X'X}{\alpha^2}}_{P_1(\underline{\theta})}\underbrace{\hat{\beta}}_{S_1(\underline{Y})}\right\}$$

$\xRightarrow{\text{Defn 2.6.3}}$ $f_{\underline{Y}}\in$ exponential family

$\xRightarrow{\text{Thm 2.7.8}}$ $S_1(\underline{Y})=\hat{\beta}$ and $S_2(\underline{Y})=\underline{Y}'\underline{Y}$ are complete sufficient statistics.

Now, $\hat{\alpha}^2=\frac{1}{n}\underline{Y}'[I-XX^{-}]\underline{Y}=\frac{1}{n}[\underline{Y}'\underline{Y}-\underline{Y}'XX^{-}\underline{Y}]=\frac{1}{n}[\underline{Y}'\underline{Y}-\hat{\beta}'X'X\hat{\beta}]$

$=\frac{1}{n}[S_2(\underline{Y})-S_1'(\underline{Y})X'XS_1(\underline{Y})]$, $\{\hat{\alpha}^2,\hat{\beta}\}$ is 1-1 of $\{S_2,S_1\}$

$\xRightarrow{\text{Thm 2.7.6}}$ $\hat{\alpha}^2,\hat{\beta}$ are complete sufficient statistics. $\Longrightarrow$ (6) & (7) are proved.

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$E(\hat{\beta}) = E[X^- \underline{y}] = X^- E(\underline{y}) = (X'X)^{-1}X' X\underline{\beta} = \beta$

$E(\hat{\sigma}^2) = \frac{1}{n-p} E[\underline{y}'(I-K)\underline{y}] = \frac{1}{n-p} E[\text{trace}(\underline{y}'(I-K)\underline{y})]$

$= \frac{1}{n-p} E[\text{trace}(I-K)\underline{y}\underline{y}'] = \frac{1}{n-p}\text{trace}[(I-K)E(\underline{y}\underline{y}')]$

$= \frac{1}{n-p}\text{trace}[(I-K)[Cov(\underline{y}) + \mu_n\mu_n']]$

$= \frac{1}{n-p}\text{trace}[(I-K)(\sigma^2 I + X\beta\beta'X')]$

$= \frac{1}{n-p}\text{trace}[\sigma^2 I - \sigma^2 K + X\beta\beta'X' - X(X'X)^{-1}X'X\beta\beta'X']$

$= \frac{1}{n-p}\sigma^2[n - \text{trace}(K)] = \sigma^2$

Combining Thm 2.7.6 & Thm 6.2.1, we can have the following theorem.

Note: $\hat{\underline{\beta}}$ and $\hat{\sigma}^2$ are unbiased for β and σ^2. (HW problem)

<u>Thm 6.2.2</u> Let $\underline{Y} = X\underline{\beta} + \underline{\varepsilon}$ be defined by (6.1.1) and $\underline{\varepsilon} \sim N_n(\underline{0}, \sigma^2 I)$. Let $t(\underline{\beta}, \sigma^2)$ be any function of the parameters $\underline{\beta}$ and σ^2 for which an unbiased estimator exists. Then there exists a function of the sufficient and complete statistics $\hat{\underline{\beta}}$ and $\hat{\sigma}^2$, $q(\hat{\underline{\beta}}, \hat{\sigma}^2)$, that is also an unbiased estimator of $t(\underline{\beta}, \sigma^2)$. In addition, $q(\hat{\underline{\beta}}, \hat{\sigma}^2)$ is the UMVUE for $t(\underline{\beta}, \sigma^2)$.

Guide students to read example 6.2.1. <See 6-6(1)

Large sample properties of $\hat{\beta}$ and $\hat{\sigma}^2$ are given in Thm 6.2.3 and Cor 6.2.3.

skip

<u>Thm 6.2.3</u> Consider the sequence of GLMs:

$$\underline{Y}_n = X_n\underline{\beta} + \underline{\varepsilon}_n, \quad \underline{\varepsilon}_n \sim N_n(\underline{0}_n, \sigma^2 I_n), \quad n = p+1, p+2, \ldots$$

where $\underline{Y}_n$ is an $n\times 1$ vector, X_n is $n\times p$ of rank p (for each n), $\underline{\beta}$ is $p\times 1$, $\underline{\varepsilon}_n$ is $n\times 1$. Let $\hat{\underline{\beta}}_n$ and $\hat{\sigma}^2_n$ be the MLEs (with $\hat{\sigma}^2_n$ adjusted for unbiaseness) of $\underline{\beta}$ and σ^2 in the nth model, where

$$\hat{\underline{\beta}}_n = (X_n'X_n)^{-1}X_n'\underline{Y}_n$$

$$\hat{\sigma}^2_n = \frac{1}{n-p}\underline{Y}_n'[I_n - X_n(X_n'X_n)^{-1}X_n']\underline{Y}_n, \quad n = p+1, p+2, \ldots$$

(1) If $\lim_{n\to\infty}(X_n'X_n)^{-1} = \underline{0}$, then for any constant vector $\underline{\ell}$ of $p\times 1$, the sequence of estimators $\{\underline{\ell}'\hat{\underline{\beta}}_n\}$ is a MSE (and simple) consistent estimator of $\underline{\ell}'\underline{\beta}$.

Example 6.2.1 Simple linear regression model

$Y_i = \beta_0 + \beta x_i + \varepsilon_i$, $\varepsilon_i \sim iid\ N(0, \sigma^2)$, $i = 1, \cdots, n$

Find ~~Are there~~ UMVUEs for the following parameters:

~~1. β 2. β_0 3. σ^2~~ (1) $2\beta - 3\beta_0$ (2) $5\sigma^2 + 8\beta$

(3) ~~[illegible]~~ $\beta_0 + 1.94\sigma$ (4) β_0/σ^2 ~~8. $\log^3|\beta|$~~ 5. $\log^3|\beta|$

SL: Let $\underline{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$ $X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}_{n\times 2 \text{ of rank } 2}$ $\beta = \begin{bmatrix} \beta_0 \\ \beta \end{bmatrix}$ $\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}$

Then $X'X = \begin{bmatrix} n & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix}$ $X'\underline{Y} = \begin{bmatrix} \Sigma Y_i \\ \Sigma x_i Y_i \end{bmatrix}$

$$(X'X)^{-1} = \frac{1}{\underbrace{n\Sigma x_i^2 - (\Sigma x_i)^2}_{= n\Sigma(x_i - \bar{x})^2}} \begin{bmatrix} \Sigma x_i^2 & -\Sigma x_i \\ -\Sigma x_i & n \end{bmatrix}$$

$$\hat{\underline{\beta}} = X^- Y = (X'X)^{-1} X'\underline{Y}$$

$$= \frac{1}{n\Sigma(x_i - \bar{x})^2} \begin{bmatrix} (\Sigma Y_i)(\Sigma x_i^2) - (\Sigma x_i)(\Sigma x_i Y_i) \\ n\Sigma x_i Y_i - (\Sigma x_i)(\Sigma Y_i) \end{bmatrix}$$

$= n\Sigma(x_i - \bar{x})(Y_i - \bar{Y})$

$$\Rightarrow \hat{\underline{\beta}} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} \bar{Y} - \hat{\beta}\bar{x} \\ \dfrac{\Sigma(x_i - \bar{x})(Y_i - \bar{Y})}{\Sigma(x_i - \bar{x})^2} \end{bmatrix}$$

That is ~~with~~

$$\hat{\beta} = \frac{\Sigma(x_i - \bar{x})(Y_i - \bar{Y})}{\Sigma(x_i - \bar{x})^2}$$

$= \Sigma x_i y_i - n\bar{x}\bar{y} = \Sigma x_i y_i - \frac{1}{n}\Sigma x_i \Sigma y_i$

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}\bar{x} \qquad \hat{\beta}_0 = \frac{\Sigma y_i \Sigma x_i^2 - \Sigma x_i \Sigma x_i y_i}{n\Sigma(x_i - \bar{x})^2}$$

$$\hat{\sigma}^2 = \frac{1}{n-2}\left[\underline{Y}'\underline{Y} - \hat{\underline{\beta}}'X'\underline{Y}\right] = \frac{1}{n-2}\left[\Sigma(Y_i - \bar{Y})^2 - \frac{[\Sigma(x_i - \bar{x})(Y_i - \bar{Y})]^2}{\Sigma(x_i - \bar{x})^2}\right]$$

are ~~UMVUEs~~ complete sufficient statistics.

6-6

2

$\hat{\beta}$ is UMVUE for β and $\hat{\alpha}^2$ is UMVUE for α^2. In particular, $\hat{\beta}_0$ and $\hat{\beta}$ are UMVUE for β_0, β. The 4 parameters are functions of the complete suff. statistics, ~~then~~ From Thm 6.2.2, as long as we can find an unbiased estimator for each of the parameters using the complete sufficient statistics, these estimators will be the UMVUEs.

(1.) As $E(\hat{\beta}) = \beta$, $E(\hat{\beta}_0) = \beta_0$, $E(\hat{\beta}) = \beta$, ~~$E(\hat{\alpha}^2) = \alpha^2$~~

$$E(2\hat{\beta} - 3\hat{\beta}_0) = 2E(\hat{\beta}) - 3E(\hat{\beta}_0) = 2\beta - 3\beta_0$$

$$\Rightarrow \quad 2\hat{\beta} - 3\hat{\beta}_0 = 2\,\frac{\Sigma(X_i-\bar{X})(Y_i-\bar{Y})}{\Sigma(X_i-\bar{X})^2} - 3(\bar{Y} - \hat{\beta}\bar{X})$$

$$= 2\,\frac{\Sigma(X_i-\bar{X})(Y_i-\bar{Y})}{\Sigma(X_i-\bar{X})^2} - 3\left(\bar{Y} - \frac{\Sigma(X_i-\bar{X})(Y_i-\bar{Y})}{\Sigma(X_i-\bar{X})^2}\bar{X}\right)$$

is unbiased for $2\beta - 3\beta_0 \xrightarrow{\text{Thm 6.2.2}}$ Also UMVUE.

(2) As $E(\hat{\alpha}^2) = \alpha^2$, and $E(\hat{\beta}) = \beta$

$$E[5\hat{\alpha}^2 + 8\hat{\beta}] = 5\alpha^2 + 8\beta$$

$$\Rightarrow \quad 5\hat{\alpha}^2 + 8\hat{\beta} = \frac{5}{n-2}\left[\Sigma(Y_i-\bar{Y})^2 - \frac{[\Sigma(X_i-\bar{X})(Y_i-\bar{Y})]^2}{\Sigma(X_i-\bar{X})^2}\right] + \frac{8\,\Sigma(X_i-\bar{X})(Y_i-\bar{Y})}{\Sigma(X_i-\bar{X})^2}$$

is unbiased for $5\alpha^2 + 8\beta$, and hence an UMVUE for $5\alpha^2 + 8\beta$

6-6 (3)

3.) For $\beta_0 + 1.94\alpha$: as long as we can find an unbiased estimator for α, we get the work done.

Note that : $U = \dfrac{(n-2)\hat{\alpha}^2}{\alpha^2} \sim \chi^2_{n-2}$

$$E(U^{\frac{1}{2}}) = \int_0^\infty U^{\frac{1}{2}} \cdot \frac{1}{\Gamma(\frac{n-2}{2})2^{\frac{n-2}{2}}} u^{\frac{n-4}{2}} e^{-u/2} du$$

$$= \int_0^\infty \frac{1}{\Gamma(\frac{n-2}{2})2^{\frac{n-2}{2}}} u^{\frac{n-3}{2}} e^{-u/2} du$$

$$= \frac{2^{\frac{1}{2}}\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})} \int_0^\infty \underbrace{\frac{1}{\Gamma(\frac{n-1}{2})2^{\frac{n-1}{2}}} u^{\frac{(n-1)-2}{2}} e^{-\frac{u}{2}}}_{\text{pdf of } \chi^2_{n-1}} du$$

$$= \frac{2^{\frac{1}{2}}\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})}$$

$$\Rightarrow E(U^{\frac{1}{2}}) = E\left[\sqrt{n-2}\ \frac{\hat{\alpha}}{\alpha}\right] = \frac{2^{\frac{1}{2}}\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})}$$

$$\Rightarrow E\left[\frac{\Gamma(\frac{n-2}{2})\sqrt{n-2}}{\sqrt{2}\,\Gamma(\frac{n-1}{2})}\hat{\alpha}\right] = \alpha$$

$\leftarrow T$

$\Rightarrow$ $\hat{\beta}_0 + 1.94T$ is unbiased for $\beta_0 + 1.94\alpha$ and hence UMVUE for $\beta_0 + 1.94\alpha$

4) $\leftarrow$ HW problem

For β_0/α^2 : we need to find an unbiased estimator for α^2 first ~~and since $\hat{\beta}_0$ and W are independent~~

W

$$U = \frac{(n-2)\hat{\alpha}^2}{\alpha^2} \sim \chi^2_{n-2}$$

Note :

$$E\left(\frac{1}{U}\right) = \int_0^\infty \frac{1}{\Gamma(\frac{n-2}{2})2^{\frac{n-2}{2}}} u^{\frac{n-4-2}{2}} e^{-u/2}\, du$$

$$= \frac{\Gamma(\frac{n-4}{2})}{2\cdot\Gamma(\frac{n-2}{2})} \int_0^\infty \underbrace{\frac{1}{\Gamma(\frac{n-4}{2})\, 2^{\frac{n-4}{2}}} u^{\frac{n-4-2}{2}} e^{-\frac{u}{2}}}_{\text{pdf of } \chi^2_{n-4}}\, du$$

$$= \frac{\Gamma(\frac{n-4}{2})}{2\Gamma(\frac{n-2}{2})} = \frac{\Gamma(\frac{n-4}{2})}{2\cdot\frac{n-4}{2}\cdot\Gamma(\frac{n-4}{2})} = \frac{1}{n-4}.$$

$$\Longrightarrow \quad E\left[\frac{\sigma^2}{(n-2)\hat{\sigma}^2}\right] = \frac{1}{n-4}$$

$$\Longrightarrow \quad E\Big[\underbrace{\frac{n-4}{(n-2)\hat{\sigma}^2}}_{W}\Big] = \frac{1}{\sigma^2}$$

W is unbiased for $\frac{1}{\sigma^2}$

Since W is a function of $\hat{\sigma}^2$ only, and $\hat{\sigma}^2$ is independent of $\hat{\beta}_0$.

$$\Longrightarrow \quad E[\hat{\beta}_0 W] = E[\hat{\beta}_0]\cdot E[W]$$

$$= \beta_0 \cdot \frac{1}{\sigma^2} = \frac{\beta_0}{\sigma^2}.$$

$\Longrightarrow$ $\hat{\beta}_0 \frac{(n-4)}{(n-2)\hat{\sigma}^2}$ is unbiased for β_0/σ^2 and hence UMVUE for β_0/σ^2.

5. Currently, no unbiased estimator of $\log^3|\beta|$ exists.

Don't prove (2). too lengthy with an additional Thm needed.

(2). The sequence of estimators $\{\hat{\sigma}_n^2\}$ is a MSE (and simple) consistent estimator of σ^2.

Proof: (1). $\cancel{\lim_{n\to\infty}} E[\underline{l}'\hat{\beta}_n] = \cancel{\lim_{n\to\infty}} E[\underline{l}'(X_n'X_n)^{-1}X_n'Y_n]$

$$= \underline{l}'(X_n'X_n)^{-1}X_n'E[Y_n]$$
$$= \underline{l}'(X_n'X_n)^{-1}X_n'X_n\beta$$
$$= \underline{l}'\beta \Longrightarrow \lim_{n\to\infty} E[\underline{l}'\hat{\beta}_n] = \lim_{n\to\infty} \underline{l}'\beta = \underline{l}'\beta$$

That is, $\underline{l}'\hat{\beta}_n$ is unbiased for $\underline{l}'\beta$ and hence asymptotically unbiased for $\underline{l}'\beta$.

$$Var[\underline{l}'\hat{\beta}_n] = Var[\underline{l}'(X_n'X_n)^{-1}X_n'Y_n]$$
$$= \underline{l}'(X_n'X_n)^{-1}X_n' Var[Y_n][\underline{l}'(X_n'X_n)^{-1}X_n']'$$
$$= \underline{l}'(X_n'X_n)^{-1}X_n'\sigma^2 I X_n(X_n'X_n)^{-1}\underline{l}$$
$$= \underline{l}'(X_n'X_n)^{-1}\underline{l}$$

Since $\lim_{n\to\infty}(X_n'X_n)^{-1} = 0 \Longrightarrow \lim_{n\to\infty} Var[\underline{l}'\hat{\beta}_n] = 0$

$\xRightarrow{\text{Thm2.7.9}}$ $\{\underline{l}'\hat{\beta}_n\}$ is a MSE consistent estimator of $\underline{l}'\beta$.

(2). $E(\hat{\sigma}_n^2) = \sigma^2$ ~~(shown before)~~

$$E(\hat{\sigma}_n^2) = \frac{1}{n-p} E\{tr[Y_n'[I_n - X_n(X_n'X_n)^{-1}X_n']Y_n\}$$
$$= \frac{1}{n-p} E\{tr([I_n - X_n(X_n'X_n)^{-1}X_n']Y_nY_n')\}$$
$$= \frac{1}{n-p} tr\{([I_n - X_n(X_n'X_n)^{-1}X_n']E[Y_nY_n']\}$$
$$= \frac{1}{n-p} tr\{([I_n - X_n(X_n'X_n)^{-1}X_n'][Cov(Y_n) + X_n\beta(X_n\beta)'])\}$$
$$= \frac{1}{n-p} tr\{[I_n - X_n(X_n'X_n)^{-1}X_n'][\sigma^2 I + X_n\beta\beta'X_n']\}$$

$$= \frac{1}{n-p} \operatorname{tr}\Big\{\big([I_n - X_n(X_n'X_n^{-1})]X_n'\sigma^2 + \cancel{X_n\beta\beta'X_n'} - \cancel{X_n\beta\beta'X_n'}\big)\Big\}$$

$$= \frac{1}{n-p}\sigma^2 \operatorname{tr}\{I_n - X_n(X_n'X_n)^{-1}X_n'\}$$

$$= \frac{1}{n-p}\sigma^2\cdot[n-p] = \sigma^2.$$

$$Var(\hat\sigma_n^2) = E(\hat\sigma_n^4) - [E(\hat\sigma_n^2)]^2$$

$$= \frac{1}{(n-p)^2}\Big\{2\operatorname{tr}\big(\underbrace{[I - X_n(X_n'X_n)^{-1}X_n']}_{A}\sigma^2 I\big)^2 + 4(X_n\beta)'A\sigma^2 I_n A X_n\beta\Big\}$$

$A \leftarrow A^2 = A$, $\operatorname{tr}(A) = n-p$

additional Thm

$$= \frac{1}{(n-p)^2}\Big\{2\operatorname{tr}[\sigma^4 A] + 4\sigma^2\beta'X_n'[I - X_n(X_n'X_n)^{-1}X_n']X_n\beta\Big\}$$

$$= \frac{1}{(n-p)^2}\Big\{2\sigma^2\operatorname{tr}(A) + \underbrace{4\sigma^2\beta'X_n'X_n\beta - 4\sigma^2\beta'\underbrace{X_n'X_n(X_n'X_n)^{-1}}_{I}X_n'X_n\beta}_{0}\Big\}$$

$$= \frac{2\sigma^2\operatorname{tr}(A)}{(n-p)^2} = \frac{2\sigma^2(n-p)}{(n-p)^2} = \frac{2\sigma^2}{n-p} \longrightarrow 0 \quad \text{as } n\to\infty$$

$\Longrightarrow$ $\hat\sigma_n^2$ is MSE consistent for σ^2. //

Cor 6.2.3 Under the conditions of ~~$\ell'\beta$ and $a(x)$ for~~ Thm 6.2.3, each element of $\hat\beta_n$ is a MSE (and simple) consistent estimator of the corresponding element in β.

(3)

Sec ~~6.2.1~~ Point Estimation of $\underline{\ell}'\underline{\beta}$ and $\mu(\underline{x})$ for case 1.

For a given vector $\underline{x}$ in domain D, it follows that

$$\mu(\underline{x}) = \underline{\beta}'\underline{x} = \sum_{i=1}^{p} \beta_i x_i$$

and this is a given linear combination of the β_i.

Thm 6.2.2 says that any function $g(\hat{\underline{\beta}}, \hat{\sigma}^2)$ that is unbiased for $t(\underline{\beta}, \sigma^2)$ must be the UMVUE of $t(\underline{\beta}, \sigma^2)$.

Hence, we can choose $g(\hat{\underline{\beta}}, \hat{\sigma}^2) = \hat{\underline{\beta}}'\underline{x} = \hat{\mu}(\underline{x})$, then $\mu(\underline{x}) = \underline{\beta}'\underline{x}$ is indeed an UMVUE of $\mu(\underline{x}) = \underline{\beta}'\underline{x}$.

Similary, if we chose $g(\hat{\underline{\beta}}, \hat{\sigma}^2) = \underline{\ell}'\hat{\underline{\beta}}$, where $\underline{\ell}$ is $p \times 1$ constant vector, then, the UMVUE of $\underline{\ell}'\underline{\beta}$ is $\underline{\ell}'\hat{\underline{\beta}}$.

eg 6.2.2. student read. — easy.

eg 6.2.3 Alternative way of finding UMVUE for $\underline{\ell}'\underline{\beta}$ if not using Thm 6.2.2. Instead, use Thm 2.7.4 work a long-way to get the UMVUE of $\underline{\ell}'\underline{\beta}$ being $\underline{\ell}'\hat{\underline{\beta}}$.

6-11

Sec 6.3 Test of the Hypothesis $H\beta = \underline{h}$: case 1.
(& 6.11)

In the general linear models such as the multiple linear regression model, we may be interested in testing some hypothesis regarding some of the regression coefficients in some kind of combinations, which usually leads to a testing of the so called "linear general hypothesis"

eg. In the following multiple linear regression model:

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \varepsilon_i, \ i=1, \cdots, n$$

The usual model utility test is to test

$H_0: \underline{\beta}_1 = \underline{0}$ vs $H_1: \underline{\beta}_1 \neq \underline{0}$ with $\underline{\beta}_1 = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_4 \end{pmatrix}$

Note: Let $\underline{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_4 \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \underline{\beta}_1 \end{bmatrix}$. $H = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ Then above is to test: $H_0: H\underline{\beta} = \underline{0}$ vs $H_1: H\underline{\beta} \neq \underline{0}$

If we want to test: $H_0: \beta_1 = \beta_2$ and $\beta_3 = \beta_4$ ①

vs H_1: not H_0 ②

We can let a matrix $H = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

and then testing of ① vs ② is equivalent to testing $H_0: H\underline{\beta} = \underline{0}$ ← $\underline{h} = \underline{0}$

vs $H_1: H\underline{\beta} \neq \underline{0}$

If want to test $\beta_1 = \beta_3 = 6$, $\beta_2 = \beta_4 = -6$

For this reason, we derive the following thm for testing generalize linear hypothesis in GLM

$H = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad \underline{h} = \begin{bmatrix} 6 \\ -6 \\ 6 \\ -6 \end{bmatrix}$ $\quad H\underline{\beta} = \underline{h}$