Thomas Fiorilla (trf40)

Srikanth Kundeti (sk1799)

Anthony Tiongson (ast119)

Ethan Wang (ew360)

Professor McMahon

CS440 Intro to AI

Assignment 3

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Question 3 [25 points]:

a) Prove that

$$P(X \mid MB(X)) = \alpha P(X \mid U_1, \ldots, U_m) \prod_{Y_i}^{n} P(Y_i \mid Z_{i1}, \ldots)$$

where $MB(X)$ is the Markov Blanket of variable X.

The probability of the Markov Blanket of variable X can be written as:

$$P(U_1, \ldots, U_m, X, Y_1, \ldots, Y_n, Z_{11}, \ldots, Z_{nj})$$

$$= P(U_1 \cap \ldots \cap U_m \cap X \cap Y_1 \cap \ldots \cap Y_n \cap Z_{11} \cap \ldots \cap Z_{nj})$$

$$= [\prod_{i=1}^{m} P(U_i)]P(X \mid U_1, \ldots, U_m)[\prod_{i=1}^{n} P(Y_i \mid X, Z_{i1}, \ldots, Z_{ij}) \prod_{k=1}^{j} P(Z_{ik})]$$

The Markov Blanket of variable X is the set:

$$MB(X) = \{U_1, \ldots, U_i, Y_1, \ldots, Y_n, Z_{11}, \ldots, Z_{nj}\}$$

The conditional probability of X given $MB(X)$ can be written as:

$$P(x \mid MB(x)) = \frac{P(x \cap MB(x))}{P(MB(x))}$$

The numerator $P(x \cap MB(x))$ can be equivalently expressed as the probability of the Markov Blanket of variable x:

$$[\prod_{i=1}^{m} P(u_i)]P(x \mid u_1, \ldots, u_m)[\prod_{i=1}^{n} P(y_i \mid x, z_{i1}, \ldots, z_{ij})\prod_{k=1}^{j} P(z_{ik})]$$

The denominator $P(MB(x))$ can be written as:

$$P(u_1, \ldots, u_m, y_1, \ldots, y_n, z_{11}, \ldots, z_{nj})$$

$$= P(u_1 \cap \ldots \cap u_m \cap y_1 \cap \ldots \cap y_n \cap z_{11} \cap \ldots \cap z_{nj})$$

$$= [\prod_{i=1}^{m} P(u_i)][\sum_x \prod_{i=1}^{n} P(y_i \mid x, z_{i1}, \ldots, z_{ij})\prod_{k=1}^{j} P(z_{ik})]$$

Consequently, we have:

$$P(x \mid MB(x)) = \frac{[\prod_{i=1}^{m} P(u_i)]P(x \mid u_1, \ldots, u_m)[\prod_{i=1}^{n} P(y_i \mid x, z_{i1}, \ldots, z_{ij})\prod_{k=1}^{j} P(z_{ik})]}{[\prod_{i=1}^{m} P(u_i)][\sum_x \prod_{i=1}^{n} P(y_i \mid x, z_{i1}, \ldots, z_{ij})\prod_{k=1}^{j} P(z_{ik})]}$$

$$= \frac{[\prod_{i=1}^{m} P(u_i)]P(x \mid u_1, \ldots, u_m)[\prod_{i=1}^{n} P(y_i \mid x, z_{i1}, \ldots, z_{ij})][\prod_{i=1}^{n}\prod_{k=1}^{j} P(z_{ik})]}{[\prod_{i=1}^{m} P(u_i)][\sum_x \prod_{i=1}^{n} P(y_i \mid x, z_{i1}, \ldots, z_{ij})][\prod_{i=1}^{n}\prod_{k=1}^{j} P(z_{ik})]}$$

$$= \frac{P(x \mid u_1, \ldots, u_m)\prod_{i=1}^{n} P(y_i \mid x, z_{i1}, \ldots, z_{ij})}{\sum_x \prod_{i=1}^{n} P(y_i \mid x, z_{i1}, \ldots, z_{ij})}$$

The ratio $\dfrac{1}{\sum_x \prod_{i=1}^{n} P(y_i \mid x, z_{i1}, \ldots, z_{ij})}$ represents the normalization constant α.

Thus, we have proven:

$$P(x \mid MB(x)) = \alpha P(x \mid u_1, \ldots, u_m)[\prod_{i=1}^{n} P(y_i \mid x, z_{i1}, \ldots, z_{ij})]$$

$Q.E.D.$

b) Consider the query

$P(Rain \mid Sprinkler = true, WetGrass = true)$

in the Rain/Sprinkler network and how MCMC would answer it. How many possible states are there for the approach to consider given the network and the available evidence variables?

Considering the number of evidence variables is 2, $Sprinkler = true$ and $WetGrass = true$, and the nonevidence variables for the query are $Rain$ and $Cloudy$ since the Markov Blanket for node $Rain$ is the set $\{Sprinkler, WetGrass, Cloudy\}$, the initial state for the network can be represented as $[true, true, R_i, C_i]$ where R_i and C_i can either be randomly initialized as $true$ or $false$. This gives us 4 possible states for the initial approach to consider: $[true, true, false, false]$, $[true, true, false, true]$, $[true, true, true, false]$, and $[true, true, true, true]$.

c) Using the query in 3b, calculate the transition matrix Q that stores the probabilities $P(y \rightarrow y')$ for all the states y, y'. If the Markov Chain has n states, then the transition matrix has size $n \times n$ and you should compute n^2 probabilities.

For the query $P(Rain \mid Sprinkler = true, WetGrass = true)$, we know the Markov Chain has 4 states so the resulting transition matrix Q will be a 4×4 matrix with 16 probabilities for $y \rightarrow y'$ where y represents a "current" state for the nonevidence variables $Rain$ and $Cloudy$ and y' is a possible next state that it can transition to. Of the 16 probabilities, we know that 4 will equal 0 for when the following state has both nonevidence variables change, like from $Rain, Cloudy \rightarrow \neg Rain, \neg Cloudy$, since in Markov Chain Monte Carlo only a single variable is allowed to change at each transition. The matrix Q can then be initially visualized as:

$$
\begin{array}{r c}
 & \begin{array}{cccc} Rain, Cloudy & Rain, \neg Cloudy & \neg Rain, Cloudy & \neg Rain, \neg Cloudy \end{array} \\
\begin{array}{r} Rain, Cloudy \\ Rain, \neg Cloudy \\ \neg Rain, Cloudy \\ \neg Rain, \neg Cloudy \end{array} & \begin{bmatrix} X & X & X & 0 \\ X & X & 0 & X \\ X & 0 & X & X \\ 0 & X & X & X \end{bmatrix}
\end{array}
$$

To calculate the transition probabilities, we first calculate the the Gibbs sampling for $Rain$ and $Cloudy$ conditioned on the evidence variables $Sprinkler$ and $WetGrass$ in the respective Markov Blankets of $Rain$ and $Cloudy$:

$$P(Rain \mid Cloudy, Sprinkler, WetGrass) = \alpha P(Rain \mid Cloudy) P(WetGrass \mid Rain, Sprinkler)$$

$$= \alpha(0.8)(0.99) = \frac{0.792}{0.972} = \frac{22}{27}$$

$$P(\neg Rain \mid Cloudy, Sprinkler, WetGrass) = \alpha P(\neg Rain \mid Cloudy) P(WetGrass \mid \neg Rain, Sprinkler)$$

$$= \alpha(0.2)(0.9) = \frac{0.18}{0.972} = \frac{5}{27}$$

$$P(Rain \mid \neg Cloudy, Sprinkler, WetGrass) = \alpha P(Rain \mid \neg Cloudy) P(WetGrass \mid Rain, Sprinkler)$$

$$= \alpha(0.21)(0.99) = \frac{0.198}{0.918} = \frac{11}{51}$$

$$P(\neg Rain \mid \neg Cloudy, Sprinkler, WetGrass) = \alpha P(\neg Rain \mid \neg Cloudy) P(WetGrass \mid \neg Rain, Sprinkler)$$

$$= \alpha(0.8)(0.9) = \frac{0.72}{0.918} = \frac{40}{51}$$

$$P(Cloudy \mid Rain, Sprinkler) = \alpha P(Cloudy) P(Rain \mid Cloudy) P(Sprinkler \mid Cloudy)$$

$$= \alpha(0.5)(0.8)(0.1) = \frac{0.04}{0.09} = \frac{4}{9}$$

$$P(\neg Cloudy \mid Rain, Sprinkler) = \alpha P(\neg Cloudy) P(Rain \mid \neg Cloudy) P(Sprinkler \mid \neg Cloudy)$$

$$= \alpha(0.5)(0.2)(0.5) = \frac{0.05}{0.09} = \frac{5}{9}$$

$$P(Cloudy \mid \neg Rain, Sprinkler) = \alpha P(Cloudy)P(\neg Rain \mid Cloudy)P(Sprinkler \mid Cloudy)$$

$$= \alpha(0.5)(0.2)(0.1) = \frac{0.01}{0.21} = \frac{1}{21}$$

$$P(\neg Cloudy \mid \neg Rain, Sprinkler) = \alpha P(\neg Cloudy)P(\neg Rain \mid \neg Cloudy)P(Sprinkler \mid \neg Cloudy)$$

$$= \alpha(0.5)(0.8)(0.5) = \frac{0.20}{0.21} = \frac{20}{21}$$

We can use these values to compute each transition. For transitions where the transition state equals the initial state, both samplings are taken into account in the summation $0.5(GibbsSampling_1) + 0.5(GibbsSampling_2)$. For transitions where only one nonevidence variable changes, only the Gibbs sampling of the nonevidence variable that changed is considered. The resulting matrix Q:

$$\begin{array}{cc} & \begin{array}{cccc} Rain, Cloudy & Rain, \neg Cloudy & \neg Rain, Cloudy & \neg Rain, \neg Cloudy \end{array} \\ \begin{array}{c} Rain, Cloudy \\ Rain, \neg Cloudy \\ \neg Rain, Cloudy \\ \neg Rain, \neg Cloudy \end{array} & \begin{bmatrix} \frac{17}{27} & \frac{5}{18} & \frac{5}{54} & 0 \\ \frac{2}{9} & \frac{59}{153} & 0 & \frac{20}{51} \\ \frac{11}{27} & 0 & \frac{22}{189} & \frac{10}{21} \\ 0 & \frac{11}{102} & \frac{1}{42} & \frac{310}{357} \end{bmatrix} \end{array}$$