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CS440 Intro to AI

Assignment 3

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Question 3 [25 points]:

a) Prove that

$$P(X \mid MB(X)) = \alpha P(X \mid U_1, \ldots, U_m) \prod_{Y_i}^{n} P(Y_i \mid Z_{i1}, \ldots)$$

where $MB(X)$ is the Markov Blanket of variable X.

The probability of the Markov Blanket of variable X can be written as:

$$P(U_1, \ldots, U_m, X, Y_1, \ldots, Y_n, Z_{11}, \ldots, Z_{nj})$$

$$= P(U_1 \cap \ldots \cap U_m \cap X \cap Y_1 \cap \ldots \cap Y_n \cap Z_{11} \cap \ldots \cap Z_{nj})$$

$$= [\prod_{i=1}^{m} P(U_i)]P(X \mid U_1, \ldots, U_m)[\prod_{i=1}^{n} P(Y_i \mid X, Z_{i1}, \ldots, Z_{ij}) \prod_{k=1}^{j} P(Z_{ik})]$$

The Markov Blanket of variable X is the set:

$$MB(X) = \{U_1, \ldots, U_i, Y_1, \ldots, Y_n, Z_{11}, \ldots, Z_{nj}\}$$

The conditional probability of X given $MB(X)$ can be written as:

$$P(x \mid MB(x)) = \frac{P(x \cap MB(x))}{P(MB(x))}$$

The numerator $P(x \cap MB(x))$ can be equivalently expressed as the probability of the Markov Blanket of variable x:

$$[\prod_{i=1}^{m} P(u_i)]P(x \mid u_1, \ldots, u_m)[\prod_{i=1}^{n} P(y_i \mid x, z_{i1}, \ldots, z_{ij})\prod_{k=1}^{j} P(z_{ik})]$$

The denominator $P(MB(x))$ can be written as:

$$P(u_1, \ldots, u_m, y_1, \ldots, y_n, z_{11}, \ldots, z_{nj})$$

$$= P(u_1 \cap \ldots \cap u_m \cap y_1 \cap \ldots \cap y_n \cap z_{11} \cap \ldots \cap z_{nj})$$

$$= [\prod_{i=1}^{m} P(u_i)][\sum_{x} \prod_{i=1}^{n} P(y_i \mid x, z_{i1}, \ldots, z_{ij})\prod_{k=1}^{j} P(z_{ik})]$$

Consequently, we have:

$$P(x \mid MB(x)) = \frac{[\prod_{i=1}^{m} P(u_i)]P(x \mid u_1, \ldots, u_m)[\prod_{i=1}^{n} P(y_i \mid x, z_{i1}, \ldots, z_{ij})\prod_{k=1}^{j} P(z_{ik})]}{[\prod_{i=1}^{m} P(u_i)][\sum_{x} \prod_{i=1}^{n} P(y_i \mid x, z_{i1}, \ldots, z_{ij})\prod_{k=1}^{j} P(z_{ik})]}$$

$$= \frac{[\prod_{i=1}^{m} P(u_i)]P(x \mid u_1, \ldots, u_m)[\prod_{i=1}^{n} P(y_i \mid x, z_{i1}, \ldots, z_{ij})][\prod_{i=1}^{n}\prod_{k=1}^{j} P(z_{ik})]}{[\prod_{i=1}^{m} P(u_i)][\sum_{x} \prod_{i=1}^{n} P(y_i \mid x, z_{i1}, \ldots, z_{ij})][\prod_{i=1}^{n}\prod_{k=1}^{j} P(z_{ik})]}$$

$$= \frac{P(x \mid u_1, \ldots, u_m)\prod_{i=1}^{n} P(y_i \mid x, z_{i1}, \ldots, z_{ij})}{\sum_{x} \prod_{i=1}^{n} P(y_i \mid x, z_{i1}, \ldots, z_{ij})}$$

The ratio $\frac{1}{\sum_x \prod_{i=1}^{n} P(y_i \mid x, z_{i1}, \ldots, z_{ij})}$ represents the normalization constant α.

Thus, we have proven:

$$P(x \mid MB(x)) = \alpha P(x \mid u_1, \ldots, u_m)[\prod_{i=1}^{n} P(y_i \mid x, z_{i1}, \ldots, z_{ij})]$$

$Q.E.D.$

b) Consider the query

$P(Rain \mid Sprinkler = true, WetGrass = true)$

in the Rain/Sprinkler network and how MCMC would answer it. How many possible states are there for the approach to consider given the network and the available evidence variables?

Considering the number of evidence variables is 2, $Sprinkler = true$ and $WetGrass = true$, and the non-evidence variables for the query are $Rain$ and $Cloudy$ since the Markov Blanket for node $Rain$ is the set $\{Sprinkler, WetGrass, Cloudy\}$, the initial state for the network can be represented as $[true, true, R_i, C_i]$ where R_i and C_i can either be randomly initialized as $true$ or $false$. This gives us 4 possible states for the initial approach to consider.

c) Using the query in 3b, calculate the transition matrix Q that stores the probabilities $P(y \to y')$ for all the states y, y'. If the Markov Chain has n states, then the transition matrix has size $n \times n$ and you should compute n^2 probabilities.