FINAN 6510
FINANCIAL APPLICATIONS OF STATISTICAL MODELS
"TAKE-HOME" FINAL EXAM

Submitted By:
Tony Trotter

Exercise 1. A certain company's board of directors has 13 members. How many ways are there to choose a president, vice president, secretary, and treasurer of the board, where no member of the board can hold more than one office?

Order matters and without replacement

$n = 13; r = 4$

$$\frac{n!}{(n-r)!} = \frac{13!}{(13-4)!} = \frac{13!}{9!} = \frac{13 * 12 * 11 * 10 * \cancel{9!}}{\cancel{9!}} = 17{,}160$$

Exercise 2. The G20 (or Group of Twenty) is an international economic forum for the governments of 19 countries and the European Union. The European Union is itself a block of 28 nations that are located primarily in Europe. How many ways are there to select 11 countries in the G20 to serve on an economic council if 6 are selected from among the 19 member nations not belonging to the European Union and the others are selected from the remaining 28 member nations (those belonging to the European Union)?

Order does not matter and without replacement

$$\binom{19}{6} * \binom{28}{5} = \frac{19!}{6!\,(19-6)!} * \frac{28!}{5!\,(28-5)!} = \frac{19!}{6! * 13!} * \frac{28!}{5! * 23!} = \frac{19 * 18 * 17 * 16 * 15 * 14 * \cancel{13!}}{6! * \cancel{13!}} *$$

$$\frac{28 * 27 * 26 * 25 * 24 * \cancel{23!}}{5! * \cancel{23!}} = 2{,}666{,}532{,}960$$

Exercise 3. How many different anagrams (including nonsensical words) can be made from the letters in the word STATISTICS, using all the letters?

$Statistics = 10\ letters; S\ is\ repeated\ 3\ times; T\ is\ repeated\ 3\ times\ and\ I\ is\ repeated\ 2\ times$

$$\frac{10!}{3! * 3! * 2!} = 50{,}400$$

Exercise 4. Prior knowledge about a stock indicates that the probability θ that the price will rise on any given day is either 0.25 or 0.75. Past data from similar stocks suggest that θ is equally likely to be 0.25 or 0.75 (i.e., $P(\theta = 0.25) = P(\theta = 0.75) = 0.5$). Let A be the event that the price of the stock rises on each of 5 consecutive days. Assuming that the price changes are independent across days (so that the probability that the price rises on each of 5 consecutive days is θ^5), find the probability $P(\theta = 0.25|A)$ that $\theta = 0.25$ given 5 consecutive price increases.

$$P(\theta = .25|A) = \frac{P(A|\theta = .25)P(\theta = .25)}{P(A)} = \frac{P(A|\theta = .25)P(\theta = .25)}{P(A|\theta = .25)P(\theta = .25) + P(A|\theta = .75)P(\theta = .75)}$$

$$= \frac{(.25)^5(.5)}{(.25)^5(.5) + (.75)^5(.5)} \approx .004099$$

Exercise 5. A chartered financial analyst can choose any one of Routes A, B, or C to get to work. The probabilities that he/she will arrive at work on time using Routes A, B, and C are 0.41, 0.57, and 0.64, respectively. Suppose that the probability that he/she chooses Route C is 0.3. Assuming that the analyst is twice as likely to choose Route A as he/she is to choose Route B, and supposing that he/she has just arrived at work on time, what is the probability that he/she chose Route B?

$$P(A) + P(B) + P(C) = 1; = P(A) + \frac{1}{2}P(A) + .3 = 1; = 1.5P(A) = .7; = P(A) = \frac{.7}{1.5} = .467$$

$$P(B) = .7 - .467 = .2333$$

$$P(C) = .3$$

$$P(A) = .4667$$

$$P(B|T) = \frac{P(T|B)P(B)}{P(T|B)P(B) + P(T|A)P(A) + P(T|C)P(C)} = \frac{(.57)(.2333)}{(.57)(.2333) + (.41)(.4667) + (.64)(.3)}$$

$$\approx .25755$$

Exercise 6. A modified random walk model for a stock price assumes that at each time step the price can either increase by a fixed amount $\Delta > 0$, decrease by this same fixed amount $\Delta > 0$, or remain unchanged. Suppose that P_1, $0 < P_1 < 1$ is the probability of an increase, that P_2, $0 < P_2 < 1$ is the probability of a decrease, and that $P_3 = 1 - P_1 - P_2$ is the probability of no change. Let X be the discrete random variable representing the change in a single step.

(a) Find the expected value of the random variable X.

(b) Find the variance of the random variable X.

$$a: E(X) = P_1\Delta + P_2(-\Delta) + 0;$$

$$= P_1\Delta - P_2\Delta$$

$$b: V(X) = E(X^2) - \left(E(X)\right)^2$$

$$= P_1\Delta^2 + P_2(-\Delta)^2 - (P_1\Delta - P_2\Delta)^2$$

$$= \Delta^2 P_1 + \Delta^2 P_2 - (\Delta P_1 - \Delta P_2)^2$$

$$(\Delta P_1 - \Delta P_2)^2 \colon \ \Delta^2 P_1^2 - 2\Delta^2 P_1 P_2 + \Delta^2 P_2^2$$

$= \Delta^2 P_1 + \Delta^2 P_2 - (\Delta^2 P_1^2 - 2\Delta^2 P_1 P_2 + \Delta^2 P_2^2)$

$-(\Delta^2 P_1^2 - 2\Delta^2 P_1 P_2 + \Delta^2 P_2^2) : -\Delta^2 P_1^2 + 2\Delta^2 P_1 P_2 - \Delta^2 P_2^2$

$= P_1 \Delta^2 + P_2 \Delta^2 - \Delta^2 P_1^2 + 2\Delta^2 P_1 P_2 - \Delta^2 P_2^2$

Exercise 7. The number of defaults in one year within a certain portfolio of bonds is found to be a Poisson random variable with parameter $\lambda = 5$ (i.e., the portfolio has an expectation of 5 defaults per year).

(a) Find the probability that the bond portfolio will have fewer than 2 defaults during the upcoming year.

(b) Describe the distribution of the continuous random variable representing the time between successive defaults.

(c) Find the expected time between successive defaults.

(d) Calculate the probability of there being less than 8 months between two successive defaults.

$a\text{:}\ P(X < 2) = \frac{5^0 e^{-5}}{0!} + \frac{5^1 e^{-5}}{1!} \approx .04042$

$b\text{:}\ \ T \sim Exp(5)$

$c\text{:}\ \ E(T) = \frac{1}{5}\ of\ a\ year$

$d\text{:}\ P\left(T < \frac{2}{3}\right) = \int_0^{\frac{2}{3}} 5e^{-5t}\, dt\ \approx .96432$

Exercise 8. A biased coin has a 0.6 chance of coming up heads when flipped. Find the probability of flipping 3 or fewer heads in 10 flips. What is the expected number of heads in 10 flips?

$X \sim B(10, .6)$

The probability mass function of X is:

$P(X = x) = \binom{10}{x} *.6^x * (1 - .6)^{(10-x)}$

$= P(X \leq 3) = \sum_{x=0}^{3} \frac{10!}{x!\,(10-x)!} *.6^x * (1 - .6)^{(10-x)} = .054762$

$E(X) = n * p = 10 * .6 = 6$

Exercise 9. Let X be a random variable representing a quantitative daily market dynamic (such as new information about the economy). Suppose that today's stock price S_0 for a certain company is \$150 and that tomorrow's price S_1 can be modeled by the equation $S_1 = S_0 \cdot e^X$. Assume that X is normally distributed with a mean of 0 and a variance of 0.5.

(a) Find the probability that X is less than or equal to 0.1.

(b) Suppose the daily dynamics X_i, $i = 1, 2, 3, 4, 5$ of each of the next five consecutive days are independently and identically distributed as X so that $Y \triangleq X_1 + X_2 + X_3 + X_4 + X_5$ represents the stock's five-day logarithmic return.

(i) Describe the distribution of the continuous random variable Y.

(ii) Calculate the probability that Y is less than or equal to 0.1.

(c) Describe the distribution of the random variable $\frac{S_1}{S_0}$ representing the ratio of tomorrow's price to today's price.

(d) What is the expected value of the random variable $\frac{S_1}{S_0}$?

(e) What is the variance of the random variable $\frac{S_1}{S_0}$?

a: $P(\leq .1) \approx .5562$

b:

i. Sum of normal is normal distribution: $Y \sim N(0,1)$

ii. $Y \sim N(0+0+0+0+0, .5+.5+.5+.5+.5) N(0,2.5), \textit{the standard normal distribution.}$

So, $p(X \leq 0.1) = \int_{-\infty}^{.1} f(x)dx = .5252$

c: $S_1 = S_0 e^X; \frac{S_1}{S_0} = e^X = ln\left(\frac{S_1}{S_0}\right) = ln(e^x) = lognormal$ $\qquad \frac{S_1}{S_0} \sim Ln(0, .5)$

d: $E\left(\frac{S_1}{S_0}\right) = e^{0+\frac{.5}{2}} \approx 1.28403$

e: $V\left(\frac{S_1}{S_0}\right) = \left(e^{(2)(0)+.5}\right)\left(e^{.5} - 1\right) \approx 1.0696$

Exercise 10. Let the dynamics X_i, $i = 1, 2, \ldots, d$ be independently and identically distributed as $Z \sim N(0, 1)$. One approach for modeling the short-term interest rate r_t at any time t is given by defining $r_t \triangleq X_1^2 + X_2^2 + \ldots + X_d^2$.

(a) Describe the distribution of the continuous random variable r_t.

(b) Find the probability that $r_t \in (0, 0.07]$ if $d = 3$.

(c) Find the probability that $r_t \in (0, 0.07]$ if $d = 5$.

a: $r_t \sim \chi^2(d)$

b: $P(r_t \leq .07) \approx .004824$

c: $P(r_t \leq .07) \approx .000067$

Exercise 11. (Essay Question) Write one page summarizing what you learned in this class. Explain how the fields of Probability and Statistics can be applied to the world of Finance, and describe how you might use these fields in your own chosen career path.

This has been a very valuable class to me for many reasons. First, it has given me a more rigorous introduction to Probability and Statistics than I got in two semesters of 'Business Statistics' that were required as part of my undergraduate degree in Finance. For example, I never knew how to calculate or understand *odds* before this class. Similarly, Bayes' Law and its potential applications in Finance and Business was never covered in my previous statistics classes; I look forward to doing additional study of Bayes' Law and aspire to be able to use it effectively in my future career.

I especially appreciated your lecture on the differences between Inductive Empiricism and Deductive Rationalism. I had never heard as clear of a breakdown of induction and deduction as you gave in your lecture. It fundamentally changed the way I think. Also, I'm glad to have been introduced to the concept of using statistical distributions to model, explain and predict events.

Probability and statistics play a role in every field of human activity. Increasingly, they are the quantitative tools used by professionals in economics and finance to solve previously unsolved problems. Knowledge of modern probability and statistics is essential for the development of financial theories and for the testing of their validity through robust analysis of real-world data. For example, probability and statistics are currently being used to shape effective monetary and fiscal policies and to develop pricing models for financial assets such as equities, bonds, currencies, and derivative securities.[1]

As I've researched Bayes Law, I've found several ways that financial professionals can use it to help them make better financial decisions. For example, Bayes Law can be used to predict changes in interest rates; it can also be used to help companies better predict their net income. Finally, Bayes Law is currently being used to help financial companies make better decisions about how much and to whom to extend credit.[2]

My current goal is to get a PhD in Finance. One of the unsolved problems I've thought of researching as part of a PhD is how to achieve mean-variance optimization of portfolios that contain both publicly traded assets as well as private investments that aren't traded on public exchanges. As I mentioned earlier, I'm also excited to think about ways business strategists could use Bayes' Law to integrate probabilistic thinking into their decision-making processes. This class has introduced me to an important set of tools that will assist me in pursuing these goals as I go forward.

---

[1] Brown, S. (2014, Dec 21). Probability and Statistics with Applications in Finance and Economics. Retrieved July 3, 2022, From Probability and Statistics with Applications in Finance and Economics (hindawi.com)

[2] O'Connell, B. (2018, Dec. 3). What is Bayes Theorem and Why is it Important in Business and Finance? Retrieved July 3, 2022 From What Is Bayes Theorem and Why Is it Important for Business and Finance? - TheStreet