

FEDERAL BUDGET
HIGHER EDUCATION
NORTH – OSSETIAN STATE UNIVERSITY

Faculty of Mathematics and Information Technology

Bachelor's direction

01/03/02 Applied Mathematics and Informatics

SCIENTIFIC RESEARCH:

“Numerical Integration and
Numerical Methods for Solving
Nonlinear Equations”

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Introduction

Numerical methods (computational methods, methods of computational) - a section of computational mathematics and, studying approximate ways to solve types of mathematical problems, which are either not solved, or difficult to solve by precise analytical methods (computational mathematics in a sense). mathematics relates to the range of issues related to computer use and programming. Dividing the methods of computation into analytical and numerical several conditional. It is necessary to emphasize the computer-oriented nature of numerical methods - in the end, their implementation is related to the use of computer technology and programming. Analytical solution of a problem in the form of separate formulaic ratios is an exception rather than a rule because of the complex and approximate nature of the models under study.

Numerical integration

The task of numerical integration is to replace the original function $f(x)$ for which it is difficult or impossible to write down the original in analytics, some approximation function $q(x)$ (polynomial). То есть:

$$\phi(x) = \sum_{i=1}^n c_i \varphi_i(x) \quad I = \int_a^b f(x) dx = \int_a^b \varphi(x) dx + R$$

where there is $R = \int_a^b r(x) dx$ *a prior error of the method* at the integration interval, and $r(x)$ is a prior error of the method on a separate step of integration.

Rectangle method

There are methods of left, right and middle rectangles. The essence of the method is clear from Figure 1. At each step of integration, the function is approximated by a zero-degree polynomial, a segment parallel to the axis of the abscissus.

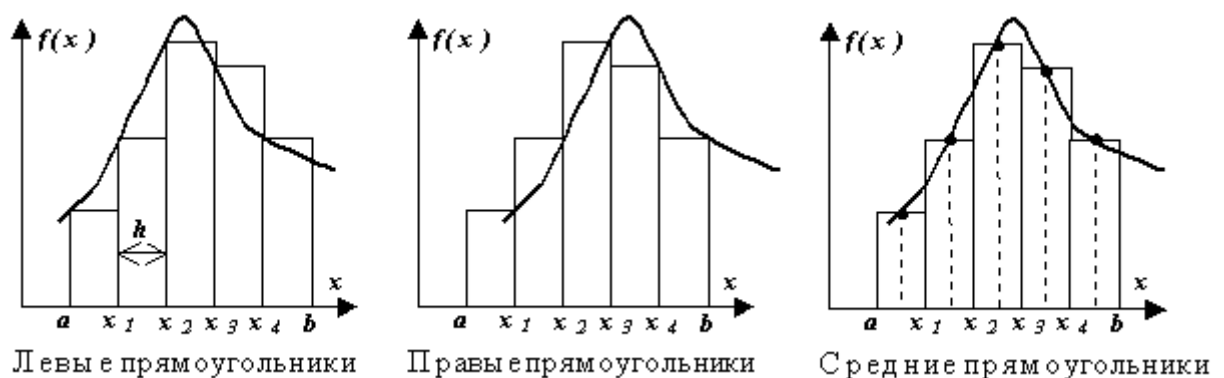


Figure 1

Let's remove the rectangle method formula from *the* $f(x)$ decomposition analysis to Taylor's row near a certain point $x \times x_i$.

$$f(x)|_{x=x_i} = f(x_i) + (x - x_i)f'(x_i) + \frac{(x - x_i)^2}{2!} f''(x_i) + \dots$$

Let's take a look at the integration range from x_i to $x_i + h$, where h is the integration step..

Let's calculate ...
$$\int_{x_i}^{x_i+h} f(x) dx = f(x_i)h + \frac{h^2}{2} f'(x_i) + \frac{h^3}{3 \cdot 2!} f''(x_i) + \dots$$

= . We got a formula $f(x_i)h + \frac{h^2}{2} f'(x_i) + O(h^3)$ $f(x_i)h + r_i$ of right (or left) rectangles and a priori estimate of the error of r on a separate step of integration.

In the case of an equal h step across the entire integration range, the common formula has a look

$$\int_a^b f(x) dx = h \sum_{i=0}^{n-1} f(x_i) + R$$

Here n - the number of integration interval splits, .

$$R = \sum_{i=0}^{n-1} r_i = \frac{h}{2} \sum_{i=0}^{n-1} f'(x_i) = \frac{h}{2} \int_a^b f'(x) dx$$

Here, at each interval, the function value is considered at the point, that is.

$$\bar{x} = x_i + \frac{h}{2} \int_{x_i}^{x_i+h} f(x) dx = hf(\bar{x}) + r_i$$

$$r = \frac{h^3}{24} f'''(\bar{x}), R = \frac{h^2}{24} \int_a^b f'''(x) dx$$

The trapezia method

The approximation in this method is done by first-degree polyina. метода ясна из рисунка . At a single interval

$$\int_{x_i}^{x_i+h} f(x) dx = \frac{h}{2} (f(x_i) + f(x_i+h)) + r_i$$

In the case of a uniform mesh ($h = \text{const}$)



Figure 1

$$\int_a^b f(x) dx = h \left(\frac{1}{2} f(x_0) + \sum_{i=1}^{n-1} f(x_i) + \frac{1}{2} f(x_n) \right) + R$$

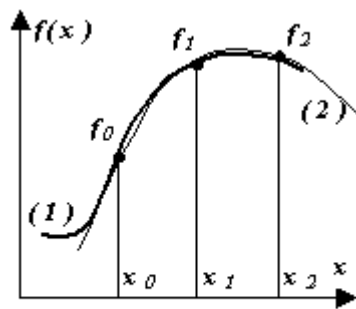
At the same time, as well. The error of the trapezal method is twice as high as that of the medium rectanglemethod method! In practice, however, you can only find the average value at the elementary interval in the functions that are set analytically (rather than tabs), so it is not always possible to use the method of medium rectangles. Due to the different errors in the formulas of trapezes and medium rectangles, the true value of the integral usually lies between the two

estimates.

$$r_i = -\frac{h^3}{12} f'''(x_i) \quad R = -\frac{h^3}{12} \int_a^b f'''(x) dx$$

The Simpson Method

The integral function $f(x)$ is replaced by second-degree $P(x)$ interpolation polyina, a parabola that passes through three nodes, for example, as shown in the picture ((1) - function, (2) - pauline).



Метод Симпсона

Figure 2

Consider the two steps of integration ($h = x_1 - x_0$), i.e. three nodes x_0, x_1, x_2 , through which we will conduct parabola, using Newton's equation:

$$P(x) = f_0 + \frac{x - x_0}{h} (f_1 - f_0) + \frac{(x - x_0)(x - x_1)}{2h^2} (f_0 - 2f_1 + f_2)$$

Let $z = x - x_0$, then

$$\begin{aligned} P(z) &= f_0 + \frac{z}{h} (f_1 - f_0) + \frac{z(z-h)}{2h^2} (f_0 - 2f_1 + f_2) = \\ &= f_0 + \frac{z}{2h} (-3f_0 + 4f_1 - f_2) + \frac{z^2}{2h^2} (f_0 - 2f_1 + f_2) \end{aligned}$$

Now, taking advantage of the resulting ratio, we calculate the integral at this interval:

$$\int_{x_0}^{x_2} P(x) dx = \int_0^{2h} P(z) dz = 2hf_0 + \frac{(2h)^2}{4h}(-3f_0 + 4f_1 - f_2) + \frac{(2h)^3}{6h^2}(f_0 - 2f_1 + f_2) =$$

$$= 2hf_0 + h(-3f_0 + 4f_1 - f_2) + \frac{4h}{3}(f_0 - 2f_1 + f_2) = \frac{h}{3}(6f_0 - 9f_0 + 12f_1 - 3f_2 + 4f_0 - 8f_1 + 4f_2)$$

As a result,

$$\int_{x_0}^{x_2} f(x) dx = \frac{h}{3}(f_0 + 4f_1 + f_2) + r$$

For a uniform grid and an even number of steps n , Simpson's formula takes the form of:

$$\int_a^b f(x) dx = \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \dots + 2f_{n-2} + 4f_{n-1} + f_n) + R$$

Here, and $r = -\frac{h^5}{90} f^{IV}(x_i)$ $R = -\frac{h^4}{180} \int_a^b f^{IV}(x) dx$ in the assumption of continuity of the fourth derivative of the integral function.

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$$\int_{0,6}^{1,4} \frac{\sqrt{x^2 + 0,5} dx}{2x + \sqrt{x^2 + 2,5}}$$

$$\int_{0,2}^{0,8} \frac{\cos(x^2 + 1) dx}{2 + \sin(2x + 0,5)}$$

The text of the program

```
using System;
using System.Collections.Generic;
using System.Linq;
using System.Text;
using System.Threading.Tasks;

namespace ConsoleApplication6
{
    class Program
    {
        static double Y(double x)
        {
            return x * x;

            // 1. Math.Sqrt(x * x + 0.5) / (2 * x + Math.Sqrt(x * x + 2.5))
            // 2.Math.Cos(x * x + 1) / (2 + Math.Sin(2 * x + 0.5))
        }

        static void LT(double a, double b, double h)
        { //левые
            double sum = 0;

            for (double i = a; i < b - h / 2; i += h)
            {
                sum += Y(i) * h;
            }
            Console.WriteLine("MJIT " + sum);
        }

        static void RT(double a, double b, double h)
        { //правые
            double sum = 0;
            for (double i = a; i < b - h / 2; i += h)
            {
                sum += Y(i + h) * h;
            }
            Console.WriteLine("MIIT " + sum);
        }

        static void CT(double a, double b, double h)
        { //центральные
            double sum = 0;

            for (double i = a; i < b - h / 2; i += h)
            {
                sum += Y(i + h / 2) * h;
            }
            Console.WriteLine("MCT " + sum);
        }

        static void MS(double a, double b, double h)
        {
            double x, s;
            s = 0; x = a + h;
            while (x < b)
            {
                s = s + 4 * Y(x);
                x = x + h;
                s = s + 2 * Y(x);
                x = x + h;
            }
            s = h / 3 * (s + Y(a) - Y(b));
            Console.WriteLine("MC {0}", s);
        }
    }
}
```



```
static void Main(string[] args)
{
    double a = 0;
    double b = 1;
    int n = 10;

    double h = (b - a) / n;
    MS(a, b, h);
    MT(a, b, h);
    LT(a, b, h);
    RT(a, b, h);
    CT(a, b, h);
}
```

Numerical methods for solving non-linear equations

Half-division method(the bisection method)

The half-division method is also known as the **dichotomy method**.. В данном методе интервал делится ровно пополам.

This approach ensures a guaranteed convergence of the method regardless of the complexity of the function - and this is a very important feature. The disadvantage of the method is the same - the method will never come together faster, i.e. the convergence of the method is always equal to convergence in the worst case.

Half-dividing method:

One of the easiest ways to find the roots of a single argument function..

2. Used to find the values of a valid-valued function defined by a criterion (this may be a comparison of a minimum, maximum or a specific number).

Half-division method as function root search method

Statement of the method

Before using the method to find the roots of the function, you need to separate the roots in one of the known ways, for example, by graphic method. Separating the roots is necessary if it is not known at what point you need to look for the root.

Let's assume that the root of the function is separated on the segment. The challenge is to find and refine this root by the method of half-division. In other words, you want to find a near-value of the root with a set accuracy. $f(x) = 0$ $[a, b]$

Let the function be continuous on the cut, and $f[a,b]f(a) \cdot f(b) < 0$, $\epsilon = 0,01$
 $t \in [a,b]$ - the only root of the equation. $f(x) = 0$, $a \leq t \leq b$

(We don't consider a case where the roots are several, $t \in [a,b]$ or there is more than one. ϵ можно взять и другое достаточно малое положительное число, например, $0,001$)

Divide the cut in half. Let's get a point and two segments. $[a,b] c = \frac{a+b}{2}$, $a < c < b$
 $[a,c]$, $[c,b]$

If, the root is found. $f(c) = 0$ $t = c$

If not, then from the two segments received and it is necessary to choose one such that, that is, $[a,c][c,b][a_1;b_1] f(a_1) \cdot f(b_1) < 0$

- $[a_1;b_1] = [a,c]$ If $f(a) \cdot f(c) < 0$
- $[a_1;b_1] = [c,b]$ If $f(c) \cdot f(b) < 0$

In order to find the approximate value of the root with precision, it is necessary to stop the process of half-division at such a step, on which you can

calculate. $[a_1;b_1] c_1 = \frac{a_1+b_1}{2}$ $\epsilon > 0$ $n |b_n - c_n| < \epsilon$ $x = \frac{a_n+b_n}{2}$ $t \approx x$

Algorithm

Let the root of the equation

$$f(x) = 0$$

separated on a segment of $[a, b]$, i.e. $f(a)f(b) < 0$ (and $f'(x)$ retains the sign (Figure 2.6). (

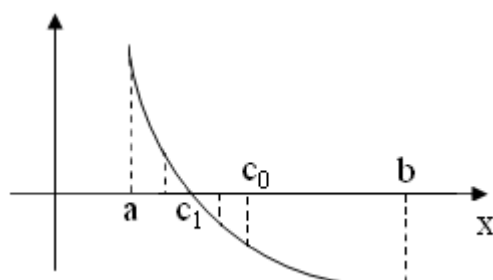


Fig. 2.6

As an initial approximation of the root, let's take point $from_0$ - the middle of the segment.

$$c_0 = \frac{a+b}{2}.$$

If $f(from_0)$ is 0, then c_0 is the desired root of the equation, if

$$f(c_0) \neq 0,$$

[that's from two segments of] $the a, c_0]$ and $c_0, b]$, choose the one at the ends of which the function takes the value of the different characters.

The new segment is divided in half again and then we follow the above. The length of each new segment is half the length of the previous segment, i.e. for n steps will be reduced by 2^n times.

We stop the calculations if the length of the segment is c_k to, with $a, q_1,]$ will be less than the specified error, i.e.:

$$|c_k - c_{k+1}| < \varepsilon$$

Block-scheme algorithm to solve equations by Newton's method

Given on rice. 2.7

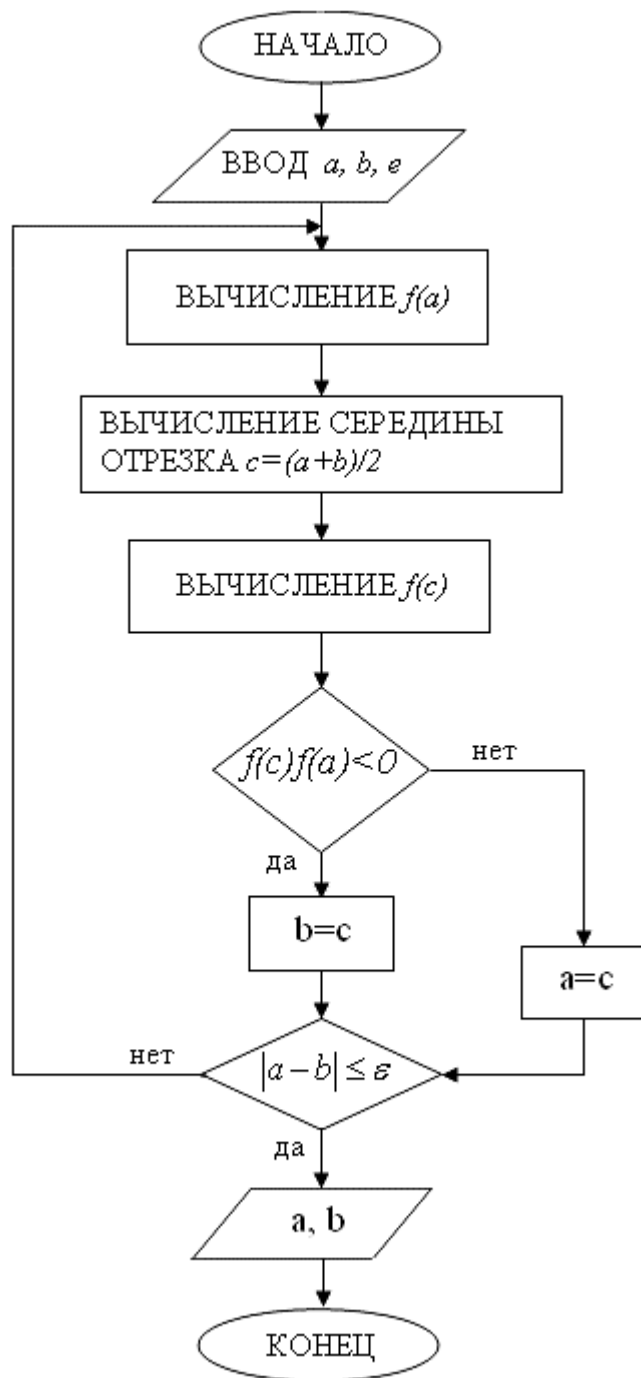


Figure 2.7 Half-Division Unit Block Scheme

Individual version number 22

$$2x + \cos x = 0,5(\text{Figure1})$$

$$F(x) = 2x + \cos x - 0,5$$

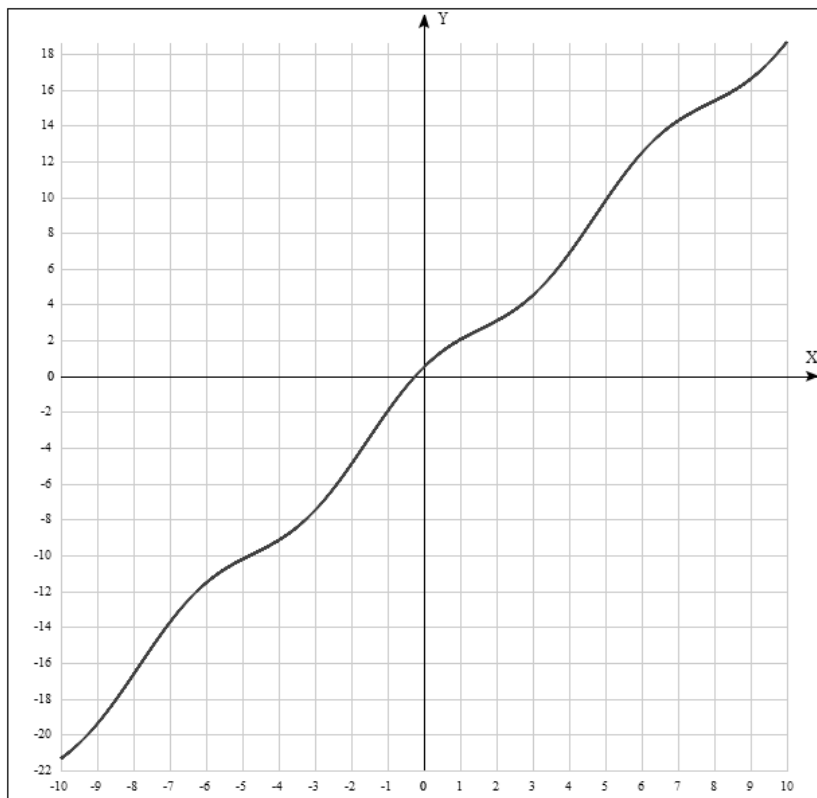


Figure 1

$$x^3 - 0,2x^2 + 0,3x - 1,2 = 0 \text{ (Figure 2)}$$

$$F(x) = x^3 - 0.2x^2 + 0.3x - 1.2$$

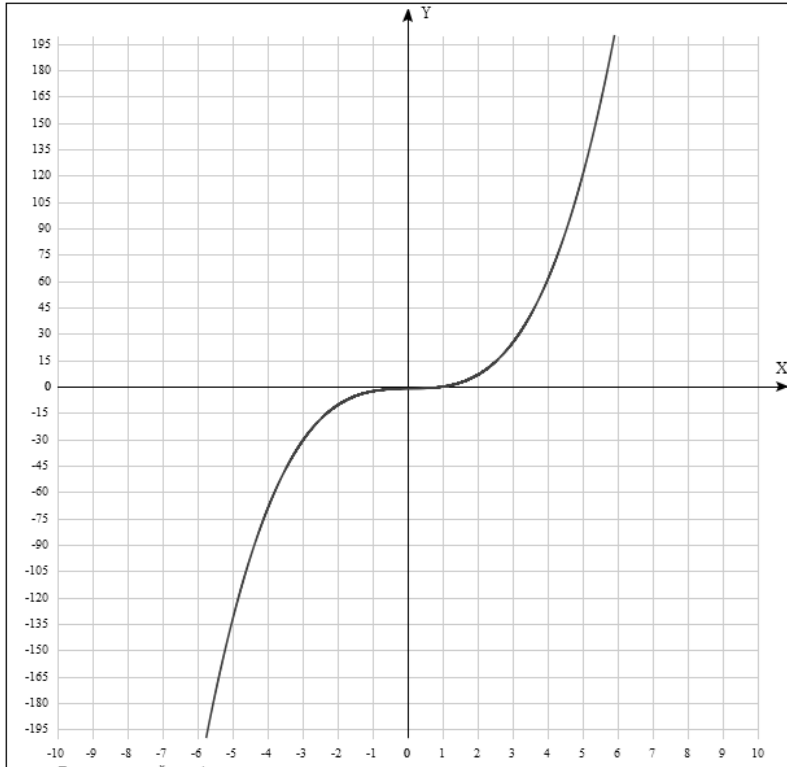


Figure 2

The text of the program

```
using System;
using System.Collections.Generic;
using System.Linq;
using System.Text;
using System.Threading.Tasks;

namespace ConsoleApplication9
{
    class Program
    {
        static double f(double x)
        {
            return 2*x + Math.Cos(x) - 0.5;
        }

        static void Main(string[] args)
        {

            double a, b, c, epsilon = 0.001;
            a = -1;
            b = 0;
            int count = 0;
            while (b - a > epsilon)
            {
                c = (a + b) / 2;
                if (f(b) * f(c) < 0)
                    a = c;
                else
                    b = c;

                count++;
            }
            Console.WriteLine("x = " + (a + b) / 2);
            Console.WriteLine("n = " + count);

        }
    }
}
```

Program results

$2x + \cos x = 0,5$	x -0.23583984375 n q 10
---------------------	----------------------------

$x^3 - 0,2x^2 + 0,3x - 1,2 = 0$	x q 1,03369140625 n q 12
---------------------------------	-----------------------------

The method of successive approximation (iteration method)

The method of iteration is a numerical method of solving mathematical problems, the approaching method of the solution of the limit of the sequence of the system of linear algebraic equations. convergence and the very fact of convergence of the method depends on the choice of the initial approximation of the root x_0 .

The simple iteration method is used to solve private-view equations where the equation can be written as a

$$F(x) = x - f(x) \geq 0.$$

Figure 3. 1 is given by the geo-ethical interpretation of the method.

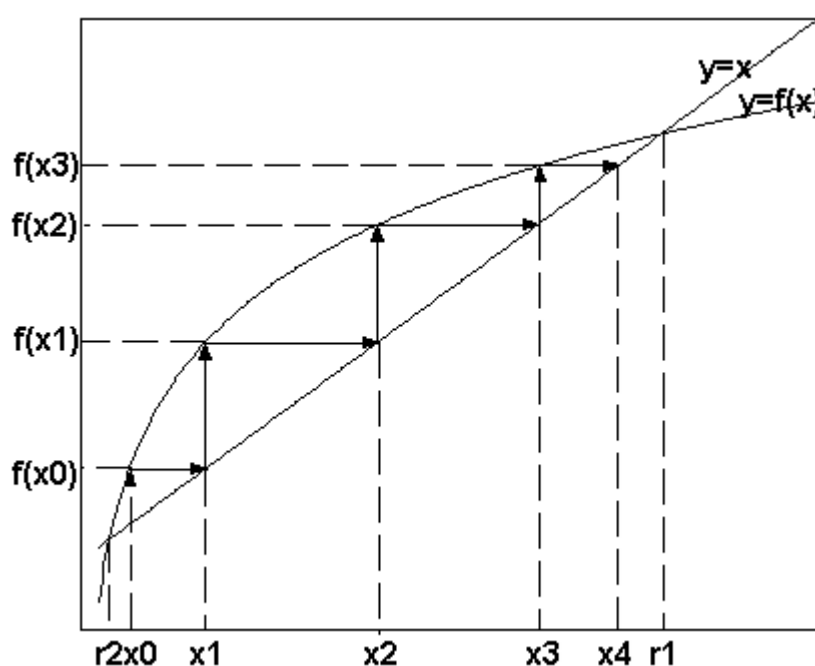


Figure 3.1 Geometric interpretation of simple iterations

The picture shows two graphs for the functions of $U(x)$ and $y'(x)$. The intersection points of these curves determine the roots of the $x-f(x)$ equation.

These roots are marked r_1 and r_2 . The following procedure is proposed to find the root by simple iterations. Let's select the free x_0 point on the x axis.

Continuing this process, you can see that each subsequent approach to the root is determined through the previous formula

$$x_{n+1} = f(x_n) \quad (3.1)$$

The following statement (without proof) is true. If the $f'(x_0)$ is at the intersection of the charts of $y'x$ and $y'f(x)$, then the iterative process (3.1) converges to this point.

In our example, this point is the point of x'_1 . Such roots are called attractive for the method of simple iterations

The root of r_2 in our example is repulsive and cannot be found by simple iterations (3.1). могут быть с его помощью найдены.

Algorithm block scheme to solve equations by simple iterations are given to rice. 3.2.

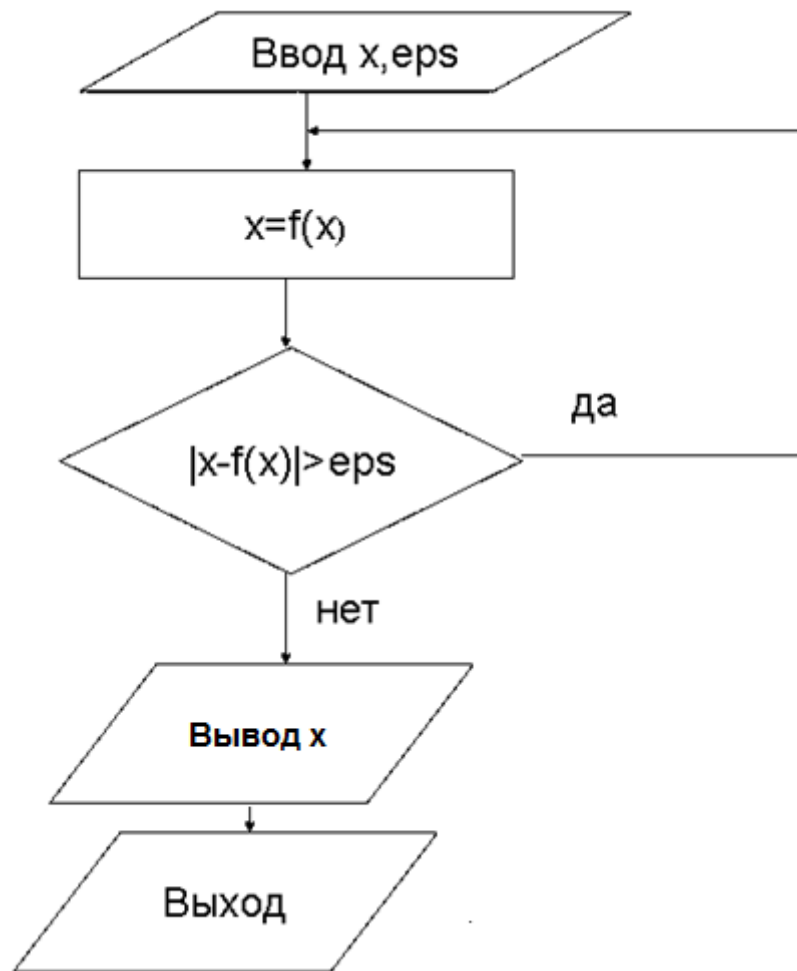


Figure 3.2

Example

$$2x + \lg(2x + 3) = 1$$

$$x = \frac{1}{2} - \frac{1}{2} \lg(2x + 3)$$

$$\varphi(x) = \frac{1}{2} - \frac{1}{2} \lg(2x + 3) = \frac{1}{2} - \frac{1}{2} \frac{\ln(2x + 3)}{\ln 10}$$

$$\varphi'(x) = -\frac{1}{2 \ln 10} \frac{1}{2x + 3} = \frac{1}{\ln 10 (2x + 3)}$$

$$C_0 = \frac{0.5 + 0}{2} = 0.25$$

$$|\varphi'(0.25)| \approx 0.29$$

0.29 < 1 The convergence condition is met

$$C_{n+1} = \frac{1}{2} - \frac{1}{2} \lg(2C_n + 3)$$

The answer is: $x \approx 0.23$ – результат 2 итерации

Individual version number 22

$$2x + \cos x = 0,5 \text{ Graph (Figure 1)}$$

$$2x = 0.5 - \cos x$$

$$x = \frac{0.5 - \cos x}{2}$$

$$\varphi(x) = \frac{0.5 - \cos x}{2}$$

$$C_0 = \frac{-1 + 0}{2} = -0.5$$

$$\varphi'(x) = \frac{\sin x}{2}$$

$$|\varphi'(-0.5)| \approx 0.239$$

$0.239 < 1$ The convergence condition is met

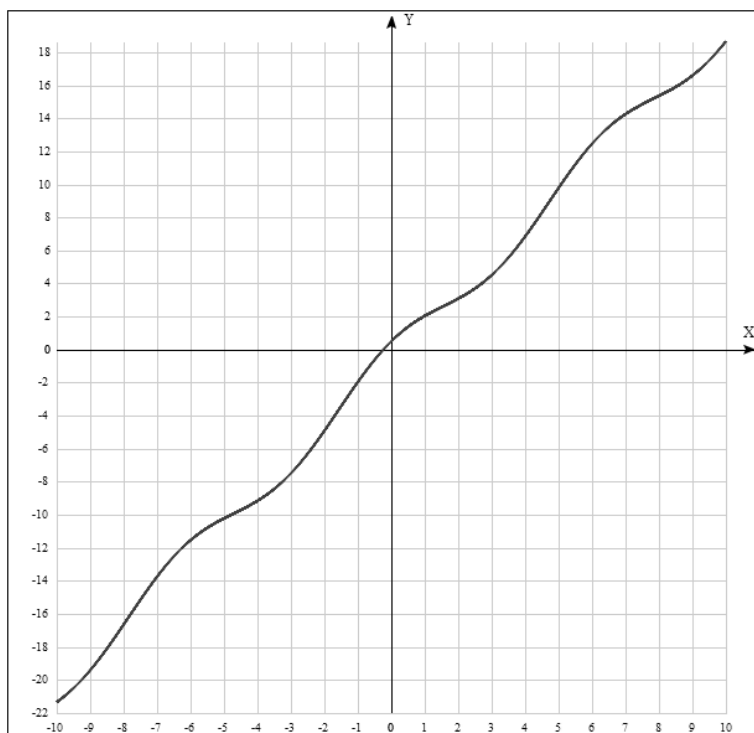


Figure 1

$$x^3 - 0,2x^2 + 0,3x - 1,2 = 0 \text{ (Figure 2)}$$

$$x^3 = 0.2x^2 - 0.3x + 1.2$$

$$x = \sqrt[3]{(0.2x^2 - 0.3x + 1.2)}$$

$$\varphi(x) = \sqrt[3]{(0.2x^2 - 0.3x + 1.2)}$$

$$C_0 = \frac{-2 + 2}{2} = 0$$

$$\varphi'(x) = \frac{1}{3} (0.2x^2 - 0.3x + 1.2)^{-\frac{2}{3}} (0.4x - 0.3)$$

$|\varphi'(0)| < 1$ – условие сходимости **ВЫПОЛНЯЕТСЯ**

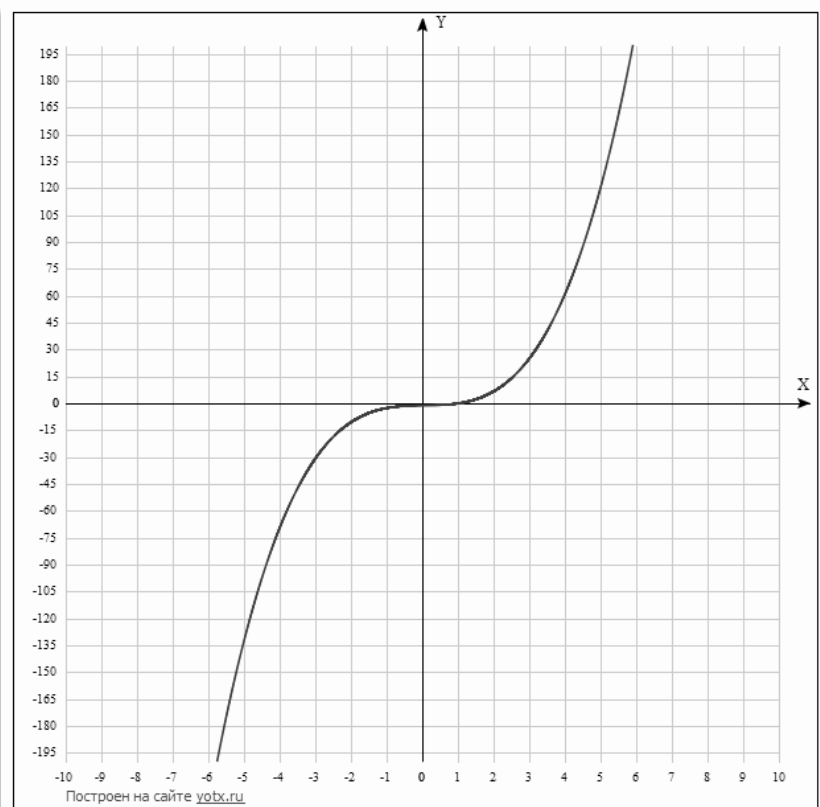


Figure 2

The text of the program

```
using System;
using System.Collections.Generic;
using System.Linq;
using System.Text;
using System.Threading.Tasks;

namespace math
{
    class Program
    {
        static double f(double x)
        {
            return (0.5 - Math.Cos(x))/2;
        }

        static void Main(string[] args)
        {
            double c = -0.5;
            double eps = 0.001;
            int count = 0;
            double x;
            while (true)
            {
                count++;
                x = f(c);
                if (Math.Abs(x - c) < eps)
                    break;
                else
                    c = x;
            }
            Console.WriteLine("x = " + x);
            Console.WriteLine("n = " + count);

        }
    }
}
```

Program results

$2x + \cos x = 0,5$	x -0.236194691993718 n q 4
$x^3 - 0,2x^2 + 0,3x - 1,2 = 0$	x q 1,0313170464277 n q 9

Newton Method (touch-tap method)

астрономом Исааком Ньютоном 1643—1727 Newton's Method, Newton's algorithm (also known as the математиком tangent method), is an iterative numerical method of finding a root (zero) of a given function. the method сходимостью is based on successive approximations and задач оптимизации производной градиента is based on the principles of a simple iteration. multidimensional space.

Justification

To numerically solve the equation $f(x) = 0$ by simply reiteration, it is necessary to lead to the following form: $x = \varphi(x)$ where φ - compressing display. For the best convergence of the method at the point of ищут x^* $\varphi'(x^*) = 0$ тогда: the next approximation must fulfill the $\varphi(x) = x + \alpha(x)f(x)$ condition.

$$\varphi'(x^*) = 1 + \alpha'(x^*)f(x^*) + \alpha(x^*)f'(x^*) = 0$$

Assuming that the approximation point is "close enough" to the root, \tilde{x} and that the given function is continuous, ($f(x^*) \approx f(\tilde{x}) = 0$) the final formula for this $\alpha(x)$ is:

$$\alpha(x) = -\frac{1}{f'(x)}$$

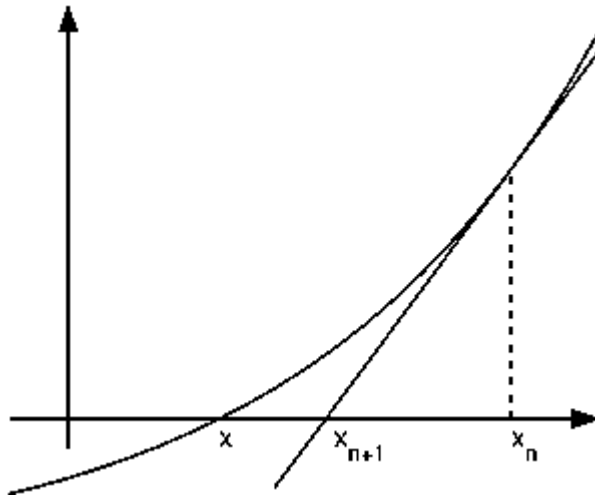
With this in mind, the function $\varphi(x)$ is defined by the expression:

$$\varphi(x) = x - \frac{f(x)}{f'(x)}$$

This function in the vicinity отображение of the root is compressive, and the algorithm for finding a numerical solution to the equation $f(x) = 0$ boils down to an iterative calculation procedure:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Geometric interpretation



The illustration of Newton's method depicts a $f(x)$ function that needs to be found, tangential at the point of the next approach. Here we can see, what the subsequent approximation is better than the previous one. x_n .

The basic idea of the method is this: the initial approximation near the intended root is set, after which the tangent to the study function is built at the point of approximation, for which there is an intersection with the axis of the abscissus. This point is taken as the next approach. And so on, until the necessary accuracy is achieved.

Let it be defined on $f(x) : [a, b] \rightarrow \mathbb{R}$ the segment a, b , and the function that is validated on it.

$$f'(x_n) = \operatorname{tg} \alpha = \frac{\Delta y}{\Delta x} = \frac{f(x_n) - 0}{x_n - x_{n+1}} = \frac{0 - f(x_n)}{x_{n+1} - x_n},$$

where the angle is the tangent at point x_n .

Hence the expression the desired expression for has the appearance of: x_{n+1}

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

The iteration process starts with some initial approximation of x_0 (the closer to zero, the better, but if there are no assumptions about finding a solution, you can

narrow down the scope of possible values by applying an intermediate theorem).
значениях).

Algorithm

Set by the initial approximation of x_0 .

So far, the stop condition, which can be met in the son-in-law or $|x_{n+1} - x_n| < \varepsilon$
 $|f(x_{n+1})| < \varepsilon$

(i.e. a margin of error in the pre-Elys), calculate a new approximation: пред

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Block-scheme algorithm to solve equations by Newton's method

Given on Figure 4

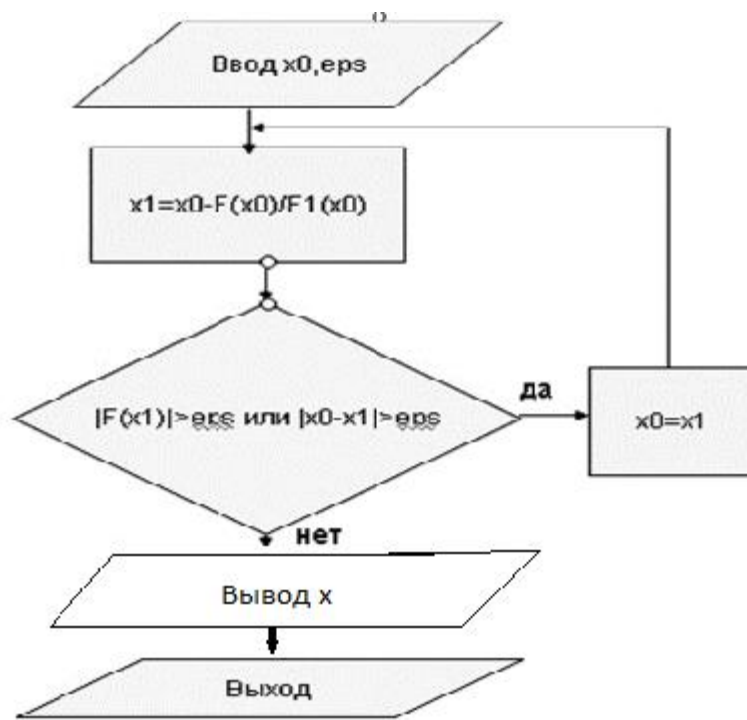


Figure 4

Example

1) $2x + \lg(2x + 3) - 1 = 0$

$F(x) = 2x + \lg(2x + 3) - 1$

$$F' = 2 + \frac{2}{\ln 10(2x + 3)}$$

$$C_{n+1} = C_n - \frac{2C_n + \lg(2C_n + 3) - 1}{2} = 2 + \frac{2}{\ln 10(2C_n + 3)}$$

$$C_0 = 0.25$$

$$x \approx 0.2$$

$$2) x^3 - 2x^2 + 7x + 3 = 0$$

$$F(x) = x^3 - 2x^2 + 7x + 3$$

$$F' = 3x^2 - 4x + 7$$

$$C_{n+1} = C_n - \frac{C_n^3 - 2C_n^2 + 7C_n + 3}{3C_n^2 - 4C_n + 7}$$

$$C_0 = -0.5$$

$$x \approx -0.38$$

Individual version number 22

$$1) 2x + \cos x = 0,5$$

$$F(x) = 2x + \cos x - 0,5$$

$$F'(x) = 2 - \sin x$$

$$C_0 = \frac{-1 + 0}{2} = -0.5$$

$$2) x^3 - 0.2x^2 + 0.3x - 1.2 = 0$$

$$F(x) = x^3 - 0.2x^2 + 0.3x - 1.2$$

$$F'(x) = 3x^2 - 0.4x + 0.3$$

$$C_0 = \frac{1 + 0}{2} = 0.5$$

The text of the program

```
using System;
using System.Collections.Generic;
using System.Linq;
using System.Text;
using System.Threading.Tasks;

namespace math
{
    class Program
    {
        static double f(double x)
        {
            return 2 * x + Math.Cos(x) - 0.5;
        }
        static double p(double x)
        {
            return 2 - Math.Sin(x);
        }
        static void Main(string[] args)
        {
            double eps = 0.001;
            double c = -0.5;
            double x;
            int count = 0;
            while (true)
            {
                count++;
                x = c - f(c) / p(c);

                if (Math.Abs(x - c) < eps)
                    break;
                else
                    c = x;
            }
            Console.WriteLine("x = " + x);
            Console.WriteLine("n = " + count);
        }
    }
}
```

Program results

$2x + \cos x = 0,5$	x -0.24896707789014 n q 1
$x^3 - 0,2x^2 + 0,3x - 1,2 = 0$	x q 1,03339326044463 n q 3

Conclusion

In calculating integrals by numerical integration methods, Simpson's method proved to be the most accurate method of calculating integral.

In the course of solving the equation numerical methods, Newton's method proved to be the most effective.

	$2x + \cos x = 0,5$	$x^3 - 0,2x^2 + 0,3x - 1,2 = 0$
Dichotomy Method	10	12
Iteration Method	4	9
Newton Method	1	3

It is clear from the table that Newton's method of response was calculated for a smaller amount of iteration than by iteration method and by dichotomy method.

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