INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Revisited Progressive Second Price Auction for Charging Telecommunication Networks

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N°4176

Mai 2001

THÈME 1

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--4176--FR+ENG

Revisited Progressive Second Price Auction for Charging Telecommunication Networks

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Thème 1 — Réseaux et systèmes
Projet Armor

Rapport de recherche n° 4176 — Mai 2001 — 13 pages

Abstract: Flat-rate charging of telecommunication networks does not seem relevant anymore as it creates congestion (like in the Internet). Among usage-based charging schemes, auctionning for bandwidth looks very promising. We review here the tremendous work called progressive second price auction and give a modification in the allocation rule in order to prove and improve the results published in the literature.

Key-words: Auctionning, Fairness, Internet economics, Optimization, Pricing.

(Résumé : tsvp)

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Une reformulation des enchères progressives de second prix pour la tarification des réseaux de télécommunication

Résumé : Le mode actuel de tarification des réseaux de télécommunication, basé sur une utilisation illimitée, ne semble plus pertinent car il engendre de la congestion (comme dans l'Internet). Parmi les méthodes de tarification basées sur l'utilisation, les enchères pour la bande passante semble très prometteuses. Nous rappelons ici le travail imposant appelé *progressive second price auction* et nous donnons une modification dans la règle d'allocation afin de corriger et améliorer les résultats de la littérature.

Mots-clé : Economie de l'Internet, Enchères, Equité, Optimisation, Tarification

1 Introduction

Pricing is one of the big challenges of the new ATM or IPv6 networks. Indeed, the current charging mode of the Internet, mainly based on a subscription fee and an unlimited connection time, is not viable if we wish to give different quality of services for different kinds of applications, like video, telephony, html... Charging becomes tricky due to this large range of applications, whereas in the previous networks, we had only one or very few different applications. Moreover, as some of these applications require a big amount of bandwidth with respect to the others, it seems natural to charge them more, especially in case of congestion.

The new trend is then to introduce a usage-based charging scheme. In the literature, telecommunication network pricing is the subject of a tremendous work. Pricing with bandwidth reservation has been studied for instance in [3, 12]. Pricing without bandwidth reservation has been studied in several directions. A proposal called Paris Metro Pricing (PMP) [11] suggests to decompose the network in several separate networks with different connection fees and such that each network works like the current one. It is expected that the most expensive networks will be less congested. In another proposition [1, 2, 4, 10], the user chooses a priority class (with a given price), and paquets are served according to this priority. Also, transfer rates can be adjusted according to the willingness to pay of the user and according to the network congestion [5, 6, 8]. Finally, auctioning for priority in a model called "smart market" has also proved to be an interesting solution [9].

In [7, 13, 14, 15], the costly auctions for individual packets are replaced by auctions for bandwidth during intervals of time. A good analysis of this scheme based on game theory is provided, including fairness properties. The behavior of the system is essentially real-time, and not model-based. The work made in [7, 13, 14, 15] is tremendous, ranging from mathematical theory for a single resource and, more impressive, a whole network, to the implementation, resulting in two patents.

Nevertheless, we argue that a technical mistake is included in the theory. In the present paper, we point out this problem, we modify the allocation rule in order to remove it, and show that the results are not altered by the changes.

2 Auctions for a single resource

2.1 Allocation rule

To explain how the auction works, consider a single ressource of capacity Q and I players competing for it. Player i's bid is $s_i = (q_i, p_i)$ where q_i is the capacity player i is asking and p_i is the unit price he is proposing. A bid profile is $s = (s_1, \cdots, s_I)$. Let $s_{-i} = (s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_I)$ be the profile where player i's bid is excluded from the game. We will sometimes write s as $s = (s_i; s_{-i})$ in order to emphasize player i's bid. For $y \geq 0$ define

$$\underline{Q}_i(y; s_{-i}) = \left[Q - \sum_{k \neq i \ : p_k \geq y} q_k\right]^+ .$$

The progressive second price allocation rule [7, 13] gives to player i a bandwidth

$$a_i(s) = \min(q_i, \underline{Q}_i(p_i; s_{-i})) \tag{1}$$

and set the total cost to

$$c_i(s) = \sum_{j \neq i} p_j [a_j(0; s_{-i}) - a_j(s_i; s_{-i})]. \tag{2}$$

Thus the highest bids are allocated the desired quantity and the cost is given by the declared willingness to pay (bids) of the users who are excluded by i's presence.

We are going to modify the allocation rule (1) based on two purposes: efficiency and, more important, mathematical accuracy.

First then, a drawback occurs with this allocation procedure (1) in case of users betting the same amount per bandwidth unit and such that the desired resources are not available for all of them. Consider the following example from [13] as an illustration. If $Q = 100$, $I = 2$, $s_1 = (60, 4)$, $s_2 = (70, 4)$, the allocation will be $a_1(s) = 30$ and $a_2(s) = 40$, leading to an under-use of the bandwidth with respect to what was asked. It is stressed in [13] that not allocating the remaining bandwidth is just a technicallity since the users will prefer to change their bids so that it will not happen at equilibrium. Nevertheless, we believe that this allocation rule is not optimal because, in practical situations, players are steadily entering and leaving the game, leading to frequent "transient" situations. A complete bandwidth allocation is then desirable.

Second, an important problem occurs when using this allocation rule. Defining

$$Q_i(y; s_{-i}) = \left[Q - \sum_{k \neq i \ : p_k > y} q_k \right]^+$$

and

$$P_i(z, s_{-i}) = \inf\{y \geq 0 \ : \ Q_i(y, s_{-i}) \geq z\},$$

it is said in [13] that it is readily apparent that

$$c_i(s) = \int_0^{a_i(s)} P_i(z, s_{-i}) dz. \tag{3}$$

This equality is then used in most of the proofs about efficiency, convergence and stability of the allocation rule. However, we can point out that Equation (3) is not true in the case of players bidding the same amount per bandwidth unit. Indeed, considering the case $Q = 100$, $I = 4$, $s_1 = (60, 4)$, $s_2 = (70, 4)$, $s_3 = (40, 2)$ and $s_4 = (10, 1)$, we have $(a_1(s), c_1(s)) = (30, 180)$, $(a_2(s), c_2(s)) = (40, 200)$ and $(a_3(s), c_3(s)) = (a_4(s), c_4(s)) = (0, 0)$. As

$$Q_1(y, s_{-1}) = \begin{cases} 0 & \text{if } 0 \leq y < 2 \\ 30 & \text{if } 2 \leq y < 4 \\ 100 & \text{if } 4 \leq y, \end{cases}$$

we have

$$P_1(z, s_{-1}) = \begin{cases} 0 & \text{if } z = 0 \\ 2 & \text{if } 0 < z \leq 30 \\ 4 & \text{if } 30 < z \leq 100. \end{cases}$$

Then

$$\int_0^{a_1(s)} P_1(z, s_{-1}) dz = \int_0^{30} P_1(z, s_{-1}) dz = 60 \neq c_1(s).$$

In order to circumvent this problem (as well as the "transient" one), we define a new allocation rule:

$$a_i(s) = \min\left(q_i, \frac{q_i}{\sum_{k \ : p_k = p_i} q_k} Q_i(p_i; s_{-i}) \right) \tag{4}$$

and we keep the total cost to

$$c_i(s) = \sum_{j \neq i} p_j [a_j(0; s_{-i}) - a_j(s_i; s_{-i})].$$

By doing this, the remaining bandwidth is shared between users betting the same unit price proportionally to their request. Moreover, we can remark that in the case of players betting different amounts per bandwidth unit, this allocation rule is equivalent to the one of Equation (1). We then have the following result:

Lemma 1 *Using Equation (4), we have*

$$c_i(s) = \int_0^{a_i(s)} P_i(z, s_{-i}) dz.$$

Proof: the problem using allocation rule (1) occurs when several players bet the same amount per bandwidth unit and the desired resources are not available for all of them (otherwise, allocation rules (1) and (4) are identical). The problem is that a player pays not only for the players betting *less than* him that he is excluding (which is the meaning of the integral), but also for the bandwidth that is not allocated anymore to *equal* bids.

Assume without loss of generality that players 1 to K (with $K \leq I$) are betting the same unit price p and that these players are giving the highest bid (removing higher bidders and diminishing Q to Q_1 would set this last assumption). For player k ($1 \leq k \leq K$), let $a_k^0(s)$ be the amount that is allocated to players betting less than p when player k is excluded from the game. Thus $c_k(s) = \int_0^{a_k^0(s)} P_k(z, s_{-k})dz + p_k(a_k(s) - a_k^0(s)) = \int_0^{a_k(s)} P_k(z, s_{-k})dz$ because $P_k(z, s_{-k}) = p_k$ when z lies between $a_k^0(s)$ and $a_k(s)$. ■

Remark: the lemma remains true whatever the allocation rule is among the players betting the same amount (not necessarily proportionally as done, but also equally...) provided that the remaining bandwidth is allocated to them.

It remains to show that the algorithm using this new allocation keeps the convergence, stability and efficiency properties of [7, 13, 14, 15].

2.2 Algorithm and properties

Assume that player i attemps to maximize his utility $u_i(s) = \theta_i(a_i(s)) - c_i(s)$ where θ_i is the valuation function that player i gives to his allocation.

We use the following smoothness assumptions for function θ_i:

- $\theta_i(0) = 0$,
- θ_i is differentiable,
- $\theta_i' \geq 0$, is non-increasing and continuous,
- $\exists \gamma_i > 0$, $\forall z \geq 0$, $\theta_i'(z) > 0 \Rightarrow \forall \eta < z, \theta_i'(z) < \theta_i'(\eta) - \gamma_i(z - \eta)$.

It is assumed that there is a bid fee ε each time a player submit a bid and that player i has a budget constraint b_i such that $c_i(s_i, s_{-i}) \leq b_i$. Let $\mathcal{S}_i(s_{-i})$ be the set of player i's bids verifying this constraint. The stability issue which is studied is ε-Nash equilibrium, a more general notion than the usual Nash equilibrium. Basically, define player i's ε-best replies as the set

$$\mathcal{S}_i^\varepsilon(s_{-i}) = \{s_i \in \mathcal{S}_i(s_{-i}) : u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) - \varepsilon, \forall s_i' \in \mathcal{S}_i(s_{-i})\}.$$

An ε-Nash equlibrium is a fixed point of $\mathcal{S}^\varepsilon = \prod_{i=1}^{I} \mathcal{S}_i^\varepsilon(s_{-i})$.

Define a truthful bid as a bid $s_i = (q_i, p_i)$ such that $p_i = \theta_i'(q_i)$. The key Proposition 1 of [7, 13] that we recall now remains true using our new allocation rule.

Proposition 1 *(Incentive compatibility) Let*

$$G_i(s_{-i}) = \sup\left\{z : z \leq Q_i(\theta_i'(z), s_{-i}) \text{ and } \int_0^z P_i(\eta, s_{-i}) d\eta \leq b_i\right\}.$$

Under the above assumptions on θ_i and $\forall 1 \leq i \leq I, \forall s_{-i}$ such that $Q_i(0, s_{-i}) = 0$, $\forall \varepsilon > 0$, there exists a truthful ε-best reply

$$t_i(s_{-i}) = (v_i = [G_i(s_{-i}) - \varepsilon/\theta_i'(0)]^+, \omega_i = \theta_i'(v_i)).$$

As already remarked, the old and new allocation rules differ only when two players bet the same amount per bandwidth unit *and the required bandwidth is not available for both of them.* Thus the proof of Proposition 1 is the one given in [7, 13] if we are able to prove that the above situation does not occur. Our key result is the following theorem.

Theorem 1 *If each player places his bid as described in Proposition 1, then it is not possible that several players bid the same unit price and that the total required bandwidth at this unit price is not available.*

Proof: Assume that players 1 to $I-1$ have already placed their bids (q_i, ω_i) and that they are sorted such that $\omega_1 > \cdots > \omega_{I-1}$ (if an equal price is given for several players, we just aggregate the bids and consider that we have less players).

Let $J = \min\{i : Q \leq \sum_{j=1}^{i} q_j\}$ be the number of players who are given some bandwidth before player I places his bid. We then have

$$Q_I(\theta_I'(z), s_{-I}) = \begin{cases} Q & \text{if } \theta_I'(z) \geq \omega_1 \\ Q - \sum_{j=1}^{i} q_j & \text{if } \omega_i > \theta_I'(z) \geq \omega_{i+1} \ (1 \leq i < J) \\ 0 & \text{if } \theta_I'(z) < \omega_J. \end{cases}$$

From the assumptions given at the beginning of the section, θ_I' is invertible on the set $[0, z_I^*]$, z_I^* being the point where θ_I begins to flatten (if any). We assume that $\omega_1 \leq \theta_I'(0)$ in order that $\theta_I^{'-1}(\omega_1)$ is defined. If it is not the case, just remove the players i such that $\omega_i > \theta_I'(0)$ and replace each time Q by $Q - q_i$, because these players are "untouchable" for player I. Hence, $\forall z \in [0, z_I^*]$,

$$Q_I(\theta_I'(z), s_{-I}) = \begin{cases} Q & \text{if } z \leq \theta_I^{'-1}(\omega_1) \\ Q - \sum_{j=1}^{i} q_j & \text{if } \theta_I^{'-1}(\omega_i) < z \leq \theta_I^{'-1}(\omega_{i+1}) \ (1 \leq i < J) \\ 0 & \text{if } z > \theta_I^{'-1}(\omega_J). \end{cases}$$

Let $G_I'(s_{-I}) = \sup\{z \ : z \leq Q_I(\theta_I'(z), s_{-I})\}$ be defined as $G_I(s_{-I})$ but without the budget constraint. Define also $G_I''(s_{-I})$ as the maximum bandwidth that player I can get with his budget constraint b_I. We the have

$$G_I(s_{-I}) = \min(G_I'(s_{-I}), G_I''(s_{-I})).$$

Try to compute $G_I'(s_{-I})$ first. We have two cases shown in Figure 2.2, depending on whether $Q_I(\theta_I'(z), s_{-I})$ meets the line $y = z$ or not. Define k_I as the integer such that $k_I = 0$ if $\theta_I^{'-1}(\omega_1) > Q - q_1$ and

$$k_I = \max\left\{i \ : \theta_I^{'-1}(\omega_i) \leq Q - \sum_{j=1}^{i} q_j\right\}$$

otherwise. The two cases of Figure 2.2 can be translated to

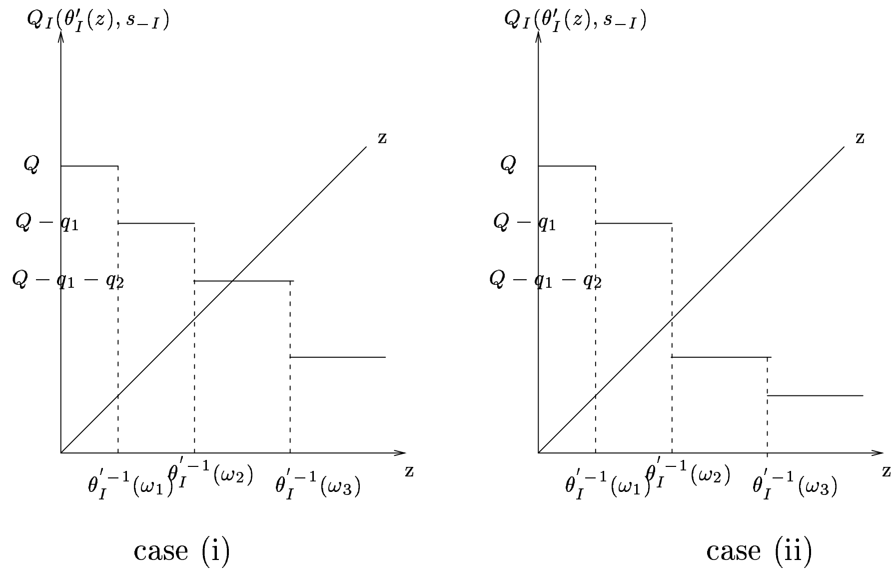

Figure 1: Two situations when computing $G'_I(s_{-I})$

- case (i) : $\theta_I'^{-1}(\omega_{k_I+1}) \geq Q - \sum_{j=1}^{k_I} q_j$;
- case (ii) : $\theta_I'^{-1}(\omega_{k_I+1}) < Q - \sum_{j=1}^{k_I} q_j$.

Then

$$G'_I(s_{-I}) = \begin{cases} Q - \sum_{j=1}^{k_I} q_j & \text{in case (i)} \\ \theta_I'^{-1}(\omega_{k_I+1}) & \text{in case (ii).} \end{cases}$$

Assume first that $G''_I(s_{-I}) \geq G'_I(s_{-I})$ so that $G_I(s_{-I}) = G'_I(s_{-I})$. In case (i), $G_I(s_{-I}) = Q - \sum_{j=1}^{k_I} q_j$, so

$$v_I = Q - \sum_{j=1}^{k_I} q_j - \varepsilon/\theta'_I(0) \text{ and } \omega_I = \theta'_I\left(Q - \sum_{j=1}^{k_I} q_j - \varepsilon/\theta'_I(0)\right)$$

(the case $v_I = 0$ does not need to be considered for our problem). If there exists j, $1 \leq j < I$ such that $\omega_I = \omega_j$, then necessarily $j \leq k_I$, but then the total required bandwidth at price $\omega_I = \omega_j$ or higher is $v_I + \sum_{i=1}^{j} q_i < Q$, so that the situation of concern (not enough bandwidth for all users betting the same unit price) does not occur. Consider now case (ii), i.e., $G_I(s_{-I}) = \theta_I^{'-1}(\omega_{k_I+1})$. Then

$$v_I = \theta_I^{'-1}(\omega_{k_I+1}) - \varepsilon/\theta_I'(0) \text{ and } \omega_I = \theta_I'(\theta_I^{'-1}(\omega_{k_I+1}) - \varepsilon/\theta_I'(0)) > \omega_{k_I+1}.$$

In this case also, if there exists j, $1 \leq j < I$, such that $\omega_I = \omega_j$, then necessarily $j \leq k_I$ and the total required bandwidth at price ω_I or higher verifies $v_I + \sum_{i=1}^{j} q_i < Q$.

If the budget constraint is used, i.e., $G_I''(s_{-I}) < G_I'(s_{-I})$, then

$$v_I = G_I''(s_{-I}) - \varepsilon/\theta_I'(0) < G_I'(s_{-I}) - \varepsilon/\theta_I'(0).$$

The bandwidth allocated to player I is then less than the one allocated in the previous case, hence the total required bandwidth at price ω_I or higher is less than Q, which completes the proof. It is impossible that two players bid the same unit price and the aggregated required bandwidth exceeds the remaining capacity. ∎

Remark: According to Theorem 1, Proposition 1 is also true with the previous allocation rule. Nevertheless, the proof was not correct because of the misuse of Equation (3).

Introduce an additional player, labelled 0, which represents the minimal price that the seller is willing to buy the bandwidth from himself. His bid is $s_0 = (Q, p_0)$, meaning that he is not going to sell the bandwidth for less than the reserve price p_0. Then, by the same arguments than the one in [13], we have the following result:

Proposition 2 *(ε-Nash equilibrium) Under the above assumptions there exists an ε-Nash equilibrium.*

Moreover, the procedure is efficient in the following way (written differently from [7, 13] for the definition of $\mathcal{A}$ below)

Proposition 3 *(Efficiency)Let s^* be an ε-Nash equilibrium and $a^* = a(s^*)$. Let $\mathcal{A} = \{a'(s^*) = (a_1'(q_1, p_1), \cdots, a_I'(q_I, p_I)) : \forall i, a_i'(q_i, p_i) \leq q_i$ and $a_i'(q_i) = q_i$*

if $\sum_i q_i \leq Q$ *and* $\sum_i a'_i(q_i, p_i) = Q$ *if* $\sum_i q_i \geq Q\}$. *If* $b_i = \infty$, $\theta'_i < \infty$ *and* $\exists \kappa$ *such that* $\forall, z, z', z > z' \geq 0$, $\theta'_i(z) - \theta'_i(z') > -\kappa(z - z')$ *, we have*

$$\max_{\mathcal{A}} \left(\sum_i \theta_i(a'_i(q_i, p_i)) - \sum_i \theta_i(a^*_i) \right) = O(\sqrt{\varepsilon \kappa}).$$

Proof: See [7, 13]. From our new formulation, it can be easily checked that $\sum_{k\ :a'_k > a^*_k} (a'_k - a^*_k) = \sum_{k\ :a'_k < a^*_k} (a^*_k - a'_k)$. ■

The game is played dynamically. Each player makes his bid asynchronously each 1 every second for instance and submits a new bid if and only if he increases his utility by more than ε. The following result from [7, 13] is preserved using our allocation rule.

Proposition 4 *The dynamic game is converging to a truthful ε-Nash equilibrium.*

Note as a last remark that when a player is allocated a bandwidth for the next one second time interval, he is not sure to keep it during that interval. Indeed, as the game is is played asynchronously, another player can bet during that interval and take all or a part of his bandwidth. The way the bandwidth is taken from the previous players has not been specified in [7, 13, 14, 15] in the dynamic game but we can expect that it is done by removing the smallest bids.

3 Networked auctions

The game is extended in [13, 15] to networked auctions and the same properties are obtained. In this networked game, players can be raw bandwidth sellers, end-users or service providers buying and selling bandwidth to each others. Each player is acting like in the single node case, trying to optimize his utility $u_i = \theta_i \circ e_i(a) - \sum_j c^j_i$ where e_i is a function called the *expected bottleneck* depending on the type of player and c^j_i is the total cost charged to player i by seller j.

By the same kind of argument than the one used in Theorem 1, the results remain true when switching to our allocation rule, the main differences being at the beginning of the proof of Proposition 4 of [15] (which is also Proposition 7 of [13]).

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Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
http://www.inria.fr
ISSN 0249-6399