Automatica 46 (2010) 1276–1283

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

An efficient Nash-implementation mechanism for network resource allocation[☆]

Rahul Jain [a,b,*], Jean Walrand [b,c]

[a] Electrical Engineering Department, University of Southern California, Los Angeles, CA 90089, United States

[b] ISE Department, University of Southern California, Los Angeles, CA 90089, United States

[c] EECS Department, University of California, Berkeley, CA 94720, United States

ARTICLE INFO

Article history:
Received 22 January 2009
Received in revised form
15 January 2010
Accepted 21 April 2010
Available online 9 June 2010

Keywords:
Game theory
Nash equilibrium
Mechanism design
Network auctions
Bandwidth allocation

ABSTRACT

We propose a mechanism for auctioning bundles of multiple divisible goods in a network where buyers want the same amount of bandwidth on each link in their route. Buyers can specify multiple routes (corresponding to a source–destination pair). The total flow can then be split among these multiple routes. We first propose a one-sided VCG-type mechanism. Players do not report a full valuation function but only a two-dimensional bid signal: the maximum quantity that they want and the per-unit price they are willing to pay. The proposed mechanism is a weak Nash implementation, i.e., it has a non-unique Nash equilibrium that implements the social-welfare maximizing allocation. We show the existence of an efficient Nash equilibrium in the corresponding auction game, though there may exist other Nash equilibria that are not efficient. We then generalize this to arbitrary bundles of various goods. Each buyer submits a bid separately for each good but their utility function is a general function of allocations of bundles of various divisible goods. We then present a double-sided auction mechanism for multiple divisible goods. We show that there exists a Nash equilibrium of this auction game which yields the efficient allocation with strong budget balance.

1. Introduction

Many network resource allocation problems involve multiple divisible resources (i.e., those that can be divided infinitely, e.g., bandwidth when it is available in any real fraction of Mbps) which are to be shared among many entities. The allocation of resources is to be done to achieve a global objective (such as maximization of the sum of individual objective functions). There are information asymmetries: each entity knows only its own objective function (henceforth, called a utility function) and the system administrator knows the class to which the utility functions belong but does not actually know the individual realized utility functions. The system administrator, given this limited information, is to design a system that determines an allocation to the various entities that achieves a global objective. Any such design is possible only if some information indicative of the entities' utility functions is elicited from them, and used to determine the allocation. However, each of the entities is an independent, self-interested and strategic player, and thus may attempt to manipulate the system to its advantage by misreporting information about its utility function. A basic question then is how can we design "rules of interaction" or a game that, despite strategic behavior on the part of players, and without a priori knowledge of the realized utility function by the system administrator, still achieves an allocation that maximizes the global objective function.

We can see this as an "inverse game theory" problem, i.e., how to design games that achieve certain objectives. A theory that studies the design of "strategy-proof" resource allocation mechanisms has been under development since the 1960s (Fudenberg & Tirole, 1991(chap 8) and Mas-Colell, Whinston, & Green, 1995(chap 23) are good references). In this paper, we are interested in solving network resource allocation and exchange problems in a particular environment. Our formulation is motivated by the problem of resource allocation in communication networks where service providers want bandwidth on a whole route, and hence the same bandwidth on all links in the route.

The first problem is allocating multiple divisible network resources among strategic agents. Let there be L divisible goods available in quantities $C_1, \ldots, C_L$. Let $r \subseteq [1 : L]$ denote a bundle of goods, such as those links that form a *route*. Let there be n agents and let R_i for agent i denote a set of bundles, such as the set of routes between a source–destination pair. For each agent, his allocation might be split between $r \in R_i$ (such as multiple routes),

[☆] The material in this paper was partially presented at the 45th IEEE Conference on Decision and Control, San Diego, CA, USA, December 13–15, 2006, and the 2nd IEEE International Workshop on Bandwidth on Demand, Salvador da Bahia, Brazil, April 11, 2008. This paper was recommended for publication in revised form by Associate Editor Michèle Breton under the direction of Editor Berç Rüstem.

* Corresponding author at: Electrical Engineering Department, University of Southern California, Los Angeles, CA 90089, United States. Tel.: +1 213 740 2246; fax: +1 213 740 4447.

E-mail addresses: rahul.jain@usc.edu (R. Jain), wlr@eecs.berkeley.edu (J. Walrand).

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doi:10.1016/j.automatica.2010.05.013

but within each r, the share of allocation on good l for route r has to be the same for all $l \in r$ (such as requiring the same capacity on all links on a route). All the goods belong to the system administrator who must determine how the goods should be allocated among the agents. Each agent derives different satisfaction from owning a certain quantity of the various goods, i.e., they have different utility functions. The system administrator would like to allocate the various goods among the agents to maximize the sum of utility derived by all the agents. However, user utilities are unknown to the system administrator. Thus, he must elicit some information from the agents to determine the optimal allocation. This can be done through an auction mechanism wherein each agent is asked to reveal a bid signal representative of its utility function. However, each agent is selfish, acts strategically and may have an incentive to misrepresent its bid signal. Thus, we must design an auction mechanism that is robust to such strategic manipulation by the agents. We then generalize this to the case where the allocated bundle to a buyer can be arbitrary.

The second problem addresses a more general environment for multilateral trading among many buyers and many sellers. We will assume that each buyer wants capacity between a source–destination pair (over multiple routes). Each seller sells goods individually (i.e., without forming bundles), and for simplicity we will assume that each seller sells only one type of good, though there may be multiple sellers selling the same good. We will require each buyer and each seller to reveal a bid signal representative of his utility or cost function. And our goal is to determine an allocation of resources that maximizes the social welfare (sum of utility derived by all buyers minus sum of cost incurred by all the sellers). Each of the agents has his own utility and cost function, and acts strategically. Thus, it might be difficult to obtain an optimal allocation. Our goal is to design an exchange mechanism which despite strategic behavior by the participants yields an allocation that maximizes the social welfare.

Literature review. Mechanism design for resource allocation is a classical problem studied by Economists and Operation Researchers. Unfortunately, most of the fundamental results are negative (such as the various impossibility theorems that specify economic environments for which it is impossible to design mechanisms with certain specified properties). The Vickrey–Clarke–Groves (VCG) mechanism (Vickrey, 1961) is the most prominent positive result.

Attention was drawn to similar resource allocation problem in networks by the work of Kelly (1997), Low and Varaiya (1993) and Mackie-Mason and Varian (1995). This work was largely motivated by the need to design and analyze distributed pricing signal-based network congestion control algorithms. In particular, Kelly (1997) and Kelly, Maullo, and Tan (1998) showed that when agents in a network do not act strategically, the resource allocation problem can be solved efficiently in a distributed manner. In fact, it was suggested that the internet transport control protocol (TCP) can be understood to be doing exactly such a distributed optimization. This work inspired a mechanism design (the Kelly mechanism) for allocation of divisible goods (such as bandwidth in a network) (Maheswaran & Basar, 2004). This mechanism was analyzed for the case when users are strategic in Johari and Tsitsiklis (2004). It was discovered that, with a single divisible good, the Kelly mechanism can result in an efficiency loss of up to 25%, i.e., the value of the social welfare function at the equilibrium outcome allocation is 25% less than the one determined by a centralized mechanism with complete knowledge of all the players' utility functions. It was however shown in Hajek and Yang (unpublished) that, in the network version of Kelly mechanism (a player submits a single bid for bandwidth on all links that constitute his route), the efficiency loss can be arbitrarily bad.

Following up on this work, a generalized class of proportional allocation (ESPA) mechanisms was introduced and analyzed in Maheswaran and Basar (2004). It was shown that these are efficient for allocation of a single divisible good. Such ESPA mechanisms require one-dimensional bid signals and have a unique Nash equilibrium at which the allocation is efficient. However, the mechanisms trade off dominant-strategy implementation, a very desirable property, for ease in implementation as compared to the VCG class of mechanisms. In Hajek and Yang (unpublished), a similar generalization of the proportional allocation mechanism was proposed for a single divisible good.

In Johari and Tsitsiklis (2009), a general convex VCG-type mechanism was introduced that required one-dimensional bid signals. It was established that there exists one Nash equilibrium at which the corresponding allocation is efficient. Conditions were provided under which the Nash equilibrium is unique and the outcome is guaranteed to be efficient. A proposal, very similar in spirit, and really a sub-case of the above, was presented in Yang and Hajek (2007). Both these mechanism require that the pseudo-utility functions that the players report be twice continuously differentiable. A mechanism for a single good, in the same spirit but with non-differentiable pseudo-utility functions, was first reported in Semret (1999). Note that all the mechanisms mentioned above are single-sided, i.e., they only involve the auctioneer and multiple buyers. Double-sided mechanisms with both buyers and sellers are of interest for actual bandwidth exchanges.

Our contribution. This paper is directly related to the work of Lazar and Semret (Lazar & Semret, 1997; Semret, 1999). They proposed a VCG-style auction mechanism for a single divisible good (Semret, 1999). Attempts have been made to generalize this mechanism to multiple divisible goods so that it can be useful for network resource allocation problems (Bitsaki, Stamoulis, & Courcoubetis, 2005; Lazar & Semret, 1997; Maille & Tuffin, 2004). The setting of Lazar and Semret (1997) addresses the case where agents want bundles of links (goods), and a different auction is held for every link. However, each agent's utility only depends on the minimum allocation it obtains on any link in its route. A slightly different setting is provided in Semret (1999, chap 3), wherein sellers place ask bids to sell bandwidth on individual links. Moreover, a buyer has to effectively bid separately for bandwidth in each link in its route. Thus, there is a separate double auction for each link. Such auctions when agents have complementarities across goods can lead to outcomes where an agent does not get all goods in its bundle, and thus might end up with zero valuation for his allocation.

In fact, Bitsaki et al. (2005) considers the PSP mechanism of Lazar and Semret (1997) and Semret (1999) as it would be implemented in a network context wherein agents make separate bids for each link in a route. This however makes the bidding strategies of agents very complex. It is more desirable to have a mechanism wherein agents can make a single bid on a whole path (or an end-to-end route). In Maille and Tuffin (2004), a variation of the basic PSP mechanism is provided for a single good which uses a higher-dimensional bid-signal space. This allows for a one-shot mechanism that achieves efficiency at an equilibrium without the time-consuming bidding convergence phase of the PSP mechanism. However, the proposed mechanism worked only for a single good and thus had limitations in terms of topology.

The proposals in this paper are inspired by Semret (1999). We propose a VCG-style mechanism, but instead of reporting their types (or complete utility functions), agents only report a two-dimensional bid: a per-unit price β and the maximum quantity d that the agent is willing to buy at that price. Note that this corresponds to a valuation function $\hat{v}(x) = \beta \cdot \min\{x, d\}$ which is continuous, concave, non-decreasing but non-differentiable. The mechanism determines an allocation which maximizes the social welfare corresponding to the reported utility functions. The payment of each agent is exactly the externality it imposes on the others through its participation, just as in the VCG mechanism. What is remarkable here is that for divisible goods, when the utility functions are strictly increasing, strictly concave and differentiable, it suffices for agents to report only a quantity and their marginal valuation at that quantity (instead of the full valuation function) for the mechanism to yield the efficient outcome at a Nash equilibrium. What is lost is the dominant-strategy implementation of VCG mechanisms, i.e., truthful reporting of utility functions is not a dominant strategy equilibrium: each agent may not have knowledge of the utility functions of others, nor of the actions being taken by them. The main reason for this is that dominant-strategy implementation requires the bid-message space to be rich enough so that each agent can report his exact utility function, which for divisible goods he must be able to specify his utility u at each real value x.

2. Problem statement

Consider L divisible goods, $\mathcal{L} = \{1, \ldots, L\}$, with C_l units of good l being available. Let Γ be the power set of $\mathcal{L}$. Let there be n buyers. Buyer i wants a bundle of goods $r \in R_i \subseteq \Gamma$ and wants the same quantity x_i of all goods in this bundle. We call such bundles routes. For example, a buyer might desire any route between a source–destination pair. R_i would then denote all the routes r between this source–destination pair. Moreover, we would allow that the buyer's total flow is split between various routes in R_i, e.g., buyer i might receive $z_{i1} = x_i/2$ on route $r_1 \in R_i$ and $z_{i2} = x_i/2$ on route $r_2 \in R_i$ for a total of x_i.

We will assume that each buyer has a quasi-linear utility function $u_i(x_i, \omega_i) = \upsilon_i(x_i) - \omega_i$, where ω_i is the payment made by buyer i and $\upsilon_i(x_i)$ is a strictly increasing, strictly concave and twice differentiable valuation function. Denote $x = (x_1, \ldots, x_n)'$ and $C = (C_1, \ldots C_L)'$. We will denote by z_{ir} the flow of i carried on route $r \in R_i$. We will use the notation $H_{ir} = \mathbb{1}\,(r \in R_i)$ and $A_{lr} = \mathbb{1}\,(l \in r)$, where $\mathbb{1}$ is the indicator function. When there are multiple sellers as well, seller j selling capacity on link l_j, we denote $B_{jl} = \mathbb{1}(l_j = l)$.

We will call $S(x) = \sum_{i=1}^{n} \upsilon_i(x_i)$ the *social welfare function*, which is a strictly increasing concave function. We will require capacity constraints

$$\sum_{i,r} A_{lr} z_{ir} \le C_l, \quad \forall l \in \mathcal{L}, \tag{1}$$

$$x_i = \sum_{r \in R_i} H_{ir} z_{ir}, \quad \forall i, \tag{2}$$

$$x_i, z_{ir} \ge 0, \quad \forall i, r. \tag{3}$$

The first constraint simply says that it is not possible to allocate more than the available quantity of any good, the second constraint says that total flow allocation to a buyer equals the sum of flow allocations to him on various routes $r \in R_i$, and the third constraint says that only non-negative allocations are allowed. The three constraints together determine a convex domain. Let λ_l and ν_i be the Lagrange multipliers corresponding to constraints (1) and (2).

System objective: To determine an allocation x^* that satisfies

$$\begin{aligned} \max\quad & S(x) \\ & Az \le C, \\ & Hz = x, \\ & x, z \ge 0. \end{aligned} \tag{4}$$

We will call such an allocation *efficient*.

Observe that this is a convex optimization problem. Thus, a solution exists and moreover it is unique. It is characterized by the following set of conditions:

$$\begin{aligned} & (\upsilon_i'(x_i^*) - \nu_i^*)x_i^* = 0, \quad \forall i \\ & \upsilon_i'(x_i^*) - \nu_i^* \le 0, \quad \forall i \\ & \left(C_l - \sum_{i,r} A_{lr} z_{ir}^*\right)\lambda_l^* = 0, \quad \forall l \\ & C_l - \sum_{i,r} A_{lr} z_{ir}^* \ge 0, \quad \forall l \\ & \nu_i^* - \sum_{l \in r} \lambda_l^* \le 0, \quad \forall r \in R_i, \forall i \\ & \left(\nu_i^* - \sum_{l \in r} \lambda_l^*\right) z_{ir}^* = 0, \quad \forall r \in R_i, \forall i \\ & \lambda_l^*, \nu_i^*, x_i^*, z_{ir}^* \ge 0, \quad \forall l, \forall r \in R_i, \forall i. \end{aligned} \tag{5}$$

The above conditions are derived from the KKT necessary and sufficient conditions for optimality in convex programs.

Note that it is possible for a system administrator to achieve this objective only if he knows the valuation functions of all the agents exactly. This however may not be true in distributed systems with selfish agents who may not reveal their actual valuation functions. In that case, we need an incentivized mechanism $((\tilde{x}_1, P_1)), \ldots, (\tilde{x}_n, P_n))$ which asks agent i to report a signal b_i indicative of its valuation function υ_i, and determines an allocation $\tilde{x}_i$ and a payment P_i to be made by it.

Agent's objective: To pick a b_i to maximize its net utility $u_i(b_i; b_{-i}) = \upsilon_i(x_i(b_i, b_{-i})) - P_i(b_i, b_{-i})$, where b_{-i} are the bid signals of all the other agents.

This gives rise to a strategic game between the agents. The allocation and payment rule is to be designed in such a way that each agent reports a signal that enables the system administrator to determine the allocation x^* even without knowing the actual valuation functions.

3. The network second-price (NSP) mechanism with multiple routes

We now propose a mechanism to be used by the system administrator (also called the auctioneer) to allocate multiple divisible goods available in certain quantities among many buyers.

The buyers specify $R_1, \ldots, R_n \subseteq \Gamma$ and corresponding bids $b_1, \ldots, b_n$. The bid $b_i = (\beta_i, d_i)$ specifies the maximum per-unit price β_i that i is willing to pay and demands up to d_i units of R_i. Denote $d = (d_1, \ldots, d_n)'$.

Note that any buyer i derives zero utility if he gets flow on a route $r \notin R_i$. Thus, he has no incentive to not truthfully report R_i.

The auctioneer then determines an allocation $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ as a solution of the following optimization problem:

$$\begin{aligned} \max\quad & \sum_i \beta_i x_i \\ \text{s.t.}\quad & Az \le C, \\ & Hz = x, \\ & x \le d, \\ & x, z \ge 0. \end{aligned} \tag{6}$$

Let $\tilde{x}^{-i}$ denote the solution of the above with $d_i = 0$. Then, the payment to be made by buyer i is

$$P_i(b_i, b_{-i}) = \sum_{j \ne i} \beta_j(\tilde{x}_j^{-i} - \tilde{x}_j). \tag{7}$$

The above defines the network second-price (NSP) mechanism. This is a VCG-style mechanism (Vickrey, 1961) where the players, instead of reporting their type or a full valuation function, only report the parameters (β_i, d_i) of the revealed valuation function $\hat{\upsilon}_i(x) = \beta_i \min(x, d_i)$. The payment of i is the "externality" or the decrease in social welfare that the buyer imposes on all the other players by his participation based on this revealed valuation function.

Note that the solution to the auction optimization (6) need not be unique. This problem occurs in most auctions that determine the allocation by solving an optimization problem that may not have a unique solution including the VCG mechanism. One way to get around this problem is to use any deterministic "tie-breaking" rule to pick one among the many optimal solutions. Since the players compute a strategy knowing full well the properties the allocation will have (namely, that it will solve the auction optimization problem), and the payment that they will have to make based on the bids and the allocation, this does not affect the analysis.

The payoff of buyer i is

$$u_i(b_i, b_{-i}) = v_i(\tilde{x}_i(b)) - P_i(b).$$

Recall that an allocation x^{**} is *efficient* if it is a solution of the optimization (3). Note that such an allocation cannot be changed to improve any player's payoff without decreasing some other player's payoff and hence is *Pareto-efficient*.[1]

The strategy space of buyer i is $\mathcal{B}_i = [0, \infty) \times [0, C^i]$, where $C^i = \sum_{r \in R_i} \min_{l \in r} C_l$.

A *Nash equilibrium* is a bid profile $b^* = (b_1^*, \dots, b_n^*)$ such that

$$u_i(b_i^*, b_{-i}^*) \geq u_i(b_i, b_{-i}^*), \quad \forall b_i \in \mathcal{B}_i.$$

Nash equilibria which yield efficient allocation will be said to be *efficient*. For any Nash equilibrium allocation x^*, we will say that its *relative efficiency* is

$$\eta := \sum_i v_i(x_i^*) / \sum_i v_i(x^{**}).$$

Note that this will lie in [0,1], where $\eta = 1$ will mean that full efficiency is achieved. The worst (least) relative efficiency over all Nash equilibria of game is its called the *price of anarchy*. The best (highest) relative efficiency over all Nash equilibria of a game is called its *price of stability* (Anshelevich et al., 2004).

3.1. Properties of the NSP mechanism

We first note the KKT conditions for the auction optimization problem. Let λ_l be the Lagrange multiplier corresponding to the first (capacity) constraint for good l, ν_i be the Lagrange multiplier corresponding to the second (flow balance) constraint and μ_i be the Lagrange multiplier corresponding to the third (demand) constraint in the auction optimization (6).

$$
\begin{aligned}
&(\beta_i - \nu_i^* - \mu_i^*)x_i^* = 0, \quad \forall i \qquad (8)\\
&\beta_i - \nu_i^* - \mu_i^* \leq 0, \quad \forall i\\
&\left(C_l - \sum_{i,r} A_{lr} z_{ir}^*\right) \lambda_l^* = 0, \quad \forall l\\
&C_l - \sum_{i,r} A_{lr} z_{ir}^* \geq 0, \quad \forall l\\
&d_i - x_i^* \geq 0, \quad \forall i\\
&(d_i - x_i^*)\mu_i^* = 0, \quad \forall i\\
&\nu_i^* - \sum_{l \in r} \lambda_l^* \leq 0, \quad \forall r \in R_i, \forall i\\
&\left(\nu_i^* - \sum_{l \in r} \lambda_l^*\right) z_{ir}^* = 0, \quad \forall r \in R_i, \forall i\\
&\lambda_l^*, \nu_i, \mu_i, x_i^*, z_{ir}^* \geq 0, \quad \forall l, \forall r \in R_i, \forall i.
\end{aligned}
$$

3.1.1. Existence of an efficient Nash equilibrium

We first show the existence of a Nash equilibrium in the corresponding resource allocation game by construction.

Theorem 1. *There exists a Nash equilibrium b^* of the NSP mechanism whose corresponding allocation x^* is efficient (i.e., $\eta(x^*) = 1$).*

The proof is by construction. We relegate it to the Appendix. Note that the above result implies the existence of an ε-efficient ε-Nash equilibrium, a result obtained in Semret (1999) for the special case of a single good.

Remarks. (1) It is worth noting here that a unique Nash equilibrium such as achieved in dominant-strategy implementation mechanisms (e.g., the VCG mechanism) is not possible here since such mechanisms require reporting of the whole utility function. This is practically impossible with divisible goods with general utility functions (unless they are finitely parameterizable).

(2) There is a multiplicity of equilibria but, as we discuss next, some of the Nash equilibria can be eliminated by introducing reserve prices. If the mechanism is repeated, then there is the possibility of eliminating all but the efficient Nash equilibrium. This would require a learning scheme which is outside the scope of this paper, and will be addressed in future work.

3.1.2. Inefficient Nash equilibria and reserve prices

However, not all Nash equilibria of the NSP mechanism are efficient. We show the existence of an inefficient one through an example.

Example 1. Consider two players with linear valuation functions, $v_i(x) = \theta_i x$ for one good with $C = 1$, and with $\theta_1 > \theta_2$. Thus, the efficient allocation is $(1, 0)$. Let player 2 bid $\beta_2 = (\theta_1, 1-\epsilon)$ and player 1 bid $\beta_1 = (\theta_2, \epsilon)$. The allocation is $(\epsilon, 1-\epsilon)$ and the payments are $(0, 0)$. It is easy to check that it is a Nash equilibrium. Further, the relative efficiency is $(\theta_2(1-\epsilon) + \theta_1\epsilon)/\theta_1$. For ϵ and θ_2 arbitrarily small, this can be made arbitrarily close to zero.

Note that, in the example above, we assumed that the valuation functions are linear. Theorem 1 assumes that the utility functions are strictly concave. However, one can imagine strictly concave valuation functions arbitrarily close to being linear. Thus, for any $0 < \epsilon < 1$, there exist valuation functions and Nash equilibria in the two-player auction game above which have relative efficiency smaller than ϵ.

But note that this arbitrarily large efficiency loss can be mitigated by introducing reserve prices and eliminating some of the inefficient Nash equilibria.

Example 2. Let p be a reserve price, the price that any participant has to pay. Then, in the example above, the players bid $\beta_1 = (\theta_1, d_1)$ and $\beta_2 = (\theta_2, d_2)$ with $\theta_1 > \theta_2$ if there is a d_2 such that

$$v_2(d_2) - \theta_2(d_2 - (1 - d_1)) - p \geq v_2(1 - d_1) - p \geq 0.$$

The inequality follows because, with such bids, player 2 prefers to be the "winner" and get d_2 and pay $p + \theta_2(d_2 + d_1 - 1)$. Similarly, player 1 bids $\beta_1 < \beta_2$ and a d_1 such that

$$v_1(1 - d_2) - p \geq v_1(d_1) - \theta_1(d_1 + d_2 - 1) - p \geq 0.$$

And again this inequality follows because player 1 prefers to "lose" and get $1 - d_2$ and pay only the reserve price. The two above yield that $d_1 \leq 1 - p/\theta_2$ and $d_2 \leq 1 - p/\theta_1$. Thus, d_2 cannot be arbitrarily close to 1, and clearly, the worst relative efficiency of any Nash equilibria has now improved.

This idea extends to general networks. However, unless the auctioneer has some *a priori* information about user valuation functions (such as a distribution on user types), it cannot be guaranteed that reserve pricing will not eliminate the efficient Nash equilibrium as well.

4. The NSP mechanism for arbitrary bundles

We now consider a slightly different setting. There are still L divisible goods, $\mathcal{L} = \{1, \dots, L\}$, with C_l units of good l being available. And there are n buyers. But now each buyer wants an arbitrary bundle, i.e., buyer i wants $x_i = (x_{i1}, \dots, x_{iL})$. Its valuation function now is $v_i(x_{i1}, \dots, x_{iL})$, which depends on amounts of various goods obtained. We still assume that these functions are

[1] Note that we are after "allocative" efficiency (social welfare maximization) here, which also happens to be a Pareto-efficient allocation. However, there would be other Pareto-efficient allocations as well.

nice, in the sense that they are strictly increasing, strictly concave and twice differentiable in each argument.

We will call $S(x_1, \ldots, x_n) = \sum_{i=1}^{n} v_i(x_i)$ the *social welfare function*, which is a strictly increasing concave function.

Our *system objective* then is to determine an allocation x^* that satisfies

$$\begin{aligned} \max \quad & S(x_1, \ldots, x_n) \\ & \sum_i x_{il} \le C_l, \quad \forall l, \\ & x_{il} \ge 0, \quad \forall i, l. \end{aligned} \tag{9}$$

We will call such an allocation *efficient*.

As before, this is a convex optimization problem and thus, a solution exists and is unique. Let λ_l be the Lagrange multipliers corresponding to the capacity constraint. Then, the optimal solution is characterized by the following set of conditions:

$$\left(\frac{\partial v_i}{\partial x_{il}} - \lambda_l^*\right) x_{il}^* = 0, \quad \forall i, l \tag{10}$$

$$\frac{\partial v_i}{\partial x_{il}} - \lambda_l^* \le 0, \quad \forall i, l$$

$$\left(C_l - \sum_i x_{il}^*\right) \lambda_l^* = 0, \quad \forall l$$

$$C_l - \sum_i x_{il}^* \ge 0, \quad \forall l$$

$$\lambda_l^*, x_{il}^* \ge 0, \quad \forall i, l.$$

The buyers specify bids $b_1, \ldots, b_n$, where $b_i = (\beta_i, d_i)$, where $\beta_i = (\beta_{i1}, \ldots, \beta_{iL})$, $d_i = (d_{i1}, \ldots, d_{iL})$, which specifies the maximum per-unit price β_{il} that i is willing to pay for good l and demands up to d_{il} units of it.

The auctioneer then determines an allocation $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ as a solution of the following optimization problem:

$$\begin{aligned} \max \quad & \sum_{i,l} \beta_{il} x_{il} \\ \text{s.t.} \quad & \sum_{il} x_{il} \le C_l, \quad \forall i, \\ & 0 \le x_{il} \le d_{il}, \quad \forall i, l. \end{aligned} \tag{11}$$

Let $\tilde{x}^{-i}$ denote the solution of the above with $d_i = 0$. Then, the payment to be made by buyer i is

$$P_i(b_i, b_{-i}) = \sum_{j \ne i} \beta_j (\tilde{x}_j^{-i} - \tilde{x}_j). \tag{12}$$

This defines the NSP mechanism for arbitrary bundles.

As before, the payoff of buyer i is

$$u_i(b_i, b_{-i}) = v_i(\tilde{x}_i(b)) - P_i(b).$$

The strategy space of buyer i is $\mathcal{B}_i = [0, \infty)^L \times \times_l [0, C_l]$. The Nash equilibrium is then defined as before.

We can now show existence of a Nash equilibrium in the corresponding resource allocation game by construction.

Theorem 2. *There exists a Nash equilibrium b^* of the NSP mechanism whose corresponding allocation x^* is efficient (i.e., $\eta(x^*) = 1$).*

The proof is given in the Appendix.

Remarks. As for the mechanism in the previous section, the NSP mechanism for arbitrary bundles can also have multiple Nash equilibria. From the same examples as in the previous section, one can conclude that some of the inefficient Nash equilibria can be eliminated by introducing reserve prices.

5. The NSP double-sided mechanism

Consider L divisible goods, $\mathcal{L} = \{1, \ldots, L\}$, with C_l units of good l being available. Let Γ be the power set of $\mathcal{L}$. Let there be n buyers. Buyer i wants a bundle of goods $r \in R_i \subseteq \Gamma$ and wants the same quantity (or flow) x_i of all goods in this bundle r. Let there be $m \ge L$ sellers; seller j sells only one good l_j and there can be more than one seller selling the same good. We will assume that each buyer has valuation function $v_i(x)$, which is strictly increasing, strictly concave and differentiable. And each seller has cost $c_j(y)$, which is strictly increasing, convex and differentiable. Note that this also includes the case where the costs are linear.

The buyers specify $R_1, \ldots, R_n \subseteq \Gamma$ and corresponding bids $b_1, \ldots, b_n$. The bid $b_i = (\beta_i, d_i)$ specifies the maximum per-unit price β_i that i is willing to pay and demands up to d_i units of the bundle R_i. Denote $d = (d_1, \ldots, d_n)'$. Seller j specifies the good l_j and an ask-bid $a_j = (\alpha_j, s_j)$, where α_j is the minimum per-unit price that j is willing to accept and can supply up to s_j units of the good l_j. Denote $s = (s_1, \ldots, s_m)'$.

The auctioneer then determines an allocation $(\tilde{x}, \tilde{y})$ as a solution of the following optimization problem:

$$\begin{aligned} \max \quad & \sum_i \beta_i x_i - \sum_j \alpha_j y_j \\ & Az \le By, \\ & Hz = x, \\ & x \le d, \\ & y \le s, \\ & x, y, z \ge 0. \end{aligned} \tag{13}$$

Let $(\tilde{x}^{-i}, \tilde{y}^{-i})$ denote the solution to the above with $d_i = 0$ and $(\bar{x}^{-j}, \bar{y}^{-j})$ denote the solution to the above with $s_j = 0$.

Then, the money transfer (the payment) to be made by buyer i is

$$\tilde{T}_i(b_i, b_{-i}, a) = \sum_{k \ne i} \beta_k (\tilde{x}_k^{-i} - \tilde{x}_k) - \sum_j \alpha_j (\tilde{y}_j^{-i} - \tilde{y}_j). \tag{14}$$

and the money transfer to be made by seller j (negative would mean transfer to the seller)

$$\bar{T}_j(b, a_j, a_{-j}) = \sum_i \beta_i (\bar{x}_i^{-j} - \tilde{x}_i) - \sum_{k \ne j} \alpha_k (\bar{y}_k^{-j} - \tilde{y}_k). \tag{15}$$

Recall that these transfer are the "externality" that the agents impose on the others through their participation.

The payoff of buyer i is

$$\tilde{u}_i(b_i, b_{-i}, a) = v_i(\tilde{x}_i(b, a)) - \tilde{T}_i(b, a),$$

and the payoff of seller j is

$$\bar{u}_j(b, a_j, a_{-j}) = -\bar{T}_j(b, a) - c_j(\tilde{y}_j(b, a)).$$

We will say that an allocation (x^{**}, y^{**}) is *efficient* if it is a solution of the following optimization problem:

$$\max \quad \sum_i v_i(x_i) - \sum_j c_j(y_j) \tag{16}$$

$$Az \le By, \tag{17}$$

$$Hz = x, \tag{18}$$

$$x, y, z \ge 0. \tag{19}$$

Such an allocation is necessarily *Pareto-efficient* since no player can unilaterally improve his payoff without making another player worse off. The strategy space of the buyer i is $\mathcal{B}_i = [0, \infty) \times [0, \infty)$. The strategy space of seller j is $\mathcal{A}_j = [0, \infty) \times [0, \infty)$.

A Nash equilibrium for this game is defined as before, and we say that it is *efficient* if the corresponding allocation is efficient.

We can show the existence of a Nash equilibrium in the double-sided mechanism by construction. Moreover, an important property a double-sided mechanism must have is *strong budget balance*, i.e., $\sum_i \tilde{T}_i = \sum_j \bar{T}_j$. In words, the total payments made by the buyer equal the total payments made to the sellers.

Theorem 3. *There exists an efficient Nash equilibrium (x^*, y^*) with strong budget balance in the NSP double-sided mechanism.*

The proof can be found in the Appendix.

Remarks. (1) While the above theorem established the existence of an efficient, strongly budget-balanced Nash equilibrium, there might exist other Nash equilibria which are either not efficient, or not budget balanced, or neither.

(2) We also note that the double-sided NSP mechanism can be extended in a natural way (as in Section 4) to the case where buyers have arbitrary bundles while sellers sell on individual links. We leave it to the reader to confirm that the analogue of Theorem 3 will still hold.

6. Conclusions and further work

We have proposed a mechanism for the allocation of multiple divisible goods such as bandwidth in a communication network. The mechanism is VCG-like and the players are only asked to report two numbers: a price per unit, and the maximum quantity demanded, as opposed to the VCG mechanism, which requires the full valuation function. Our mechanism is a generalization of that presented in Semret (1999) to the network case. We show the existence of a Nash equilibrium where the allocation is efficient. This immediately implies the existence of an ε-Nash equilibrium (where each player, given strategies of all other players, chooses a response which is within ε of the best response) which is ε-efficient (i.e., an allocation which is within $\kappa\varepsilon$ of the social welfare maximizing allocation, where κ is a constant). However, not all Nash equilibria are efficient, as we show through an example. A distributed, computationally efficient algorithm that yields an ε-efficient ε-Nash equilibrium for the single good case was presented in Semret (1999) while we presented its generalization to a multiple goods setting with two players in Dimakis, Jain, and Walrand (2006). The generalization to multiple players is not available yet, and it is not clear if using such an algorithm is a Nash equilibrium at all. We also present a double-sided mechanism which has a Nash equilibrium with efficient allocation and strong budget balance. We note that since, for each of the mechanisms presented, we demonstrated the existence of an efficient Nash equilibria, the price of stability (PoS) (see Anshelevich et al. (2004)) of the mechanisms is 1, and the price of anarchy is 0 as shown by Example 1.

Our work is also related to Johari and Tsitsiklis (2009). They present a limited communication VCG-like mechanism that yields an efficient Nash equilibrium and gives conditions under which all equilibria are efficient, some of which are restrictive. Further, while they require the revealed utility functions to be differentiable for every parameter, our revealed utility functions are not differentiable and hence this is not a particular case of their mechanism. Further, experimental work in electricity markets has shown that mechanisms which express both quantity and per-unit price, such as in our mechanism, work better than one-dimensional bid mechanisms (see Elmaghraby and Oren (1999) for a discussion).

Appendix. Proofs of theorems

Proof of Theorem 1. Let x^{**} be an efficient allocation. Then, there exist $\nu_1, \ldots, \nu_n \geq 0, \lambda_1, \ldots, \lambda_L \geq 0$ such that $v_i'(x_i^{**}) \leq \nu_i = \sum_{l\in r:r\in R_i} \lambda_l, \forall r \in R_i, \forall i$ with equality if $x_i^{**} > 0$ (a strict inequality is possible with $x_i^{**} = 0$). Consider the strategy profile $d_i = x_i^{**}$ and $\beta_i = v_i'(d_i)$. Note that x^{**} is an auction outcome with λ's and ν's as above, and $\mu_i = 0, \forall i$, i.e., these solve (8). This implies that $\forall i$ such that $x_i^{**} > 0$

$$\beta_i = \sum_{l\in r:r\in R_i} \lambda_l, \quad \forall r \in R_i. \tag{20}$$

Consider a buyer i with $x_i^{**} > 0$. Given the bids b_{-i} of the others as fixed, if buyer i changes his bid b_i to decrease his allocation x_i^{**} by a $\delta > 0$ (a buyer i with $x_i^{**} = 0$ cannot decrease his allocation), then the allocation of all the other players does not change since all of them already receive the maximum quantity they ask for. From Eq. (7), we get that the payment of player i does not change. However, since v_i is strictly increasing and concave, his utility reduces by $v_i(x_i^{**}) - v_i(x_i^{**} - \delta)$. Thus, his net payoff actually reduces.

Now, consider a buyer i (with $x_i^{**} \geq 0$) changes his bid to b_i' such that he increases his allocation x_i^{**} by a $\delta > 0$. Denote the change on route r by δ_r so that $\delta = \sum_{r\in R_i} \delta_r$. Let the resulting allocation be x'^*.

Denote $L_i = \{l \in r : r \in R_i\}$, the set of links on which buyer i's traffic can flow for some route $r \in R_i$. Let $L_{-i} = \{l \in s : s \in R_j, j \neq i\}$, the set of links on which any other buyer j's traffic can flow for some route $s \in R_j$. We first note that, for $l \in L_i \cap L_{-i}$,

$$\lambda_l \cdot \sum_{(j,s):j\neq i, s\in R_j, l\in s} (z_{js}^{**} - z_{js}'^*) \geq \lambda_l \cdot \sum_{r\in R_i:l\in r} (z_{ir}'^* - z_{ir}^{**}). \tag{21}$$

This is because, if $\lambda_l > 0$, then the capacity constraint on l is tight, and any increase in i's share of bandwidth on l is at the expense of other buyers i, who also want l. (Note that we have an inequality because buyer i's flow need not completely obtain the capacity vacated by the other buyers.) If the capacity constraint on l is not tight, then $\lambda_l = 0$, and the above still holds.

We further note that, for $l \in L_i \setminus L_{-i}$, i.e., the set of links on which no other buyer's traffic can flow, if the capacity constraint is tight, then an increase in i's flow cannot come from increases for $r \in R_i$ such that $l \in r$ (though there *may* be a decrease, hence the inequality). If the capacity constraint is not tight, then $\lambda_l = 0$. In other words,

$$\lambda_l \cdot \sum_{r\in R_i:l\in r} (z_{ir}'^* - z_{ir}^{**}) \leq 0, \quad \forall l \in L_i \setminus L_{-i}. \tag{22}$$

The change in buyer i's payment (later denoted ΔP_i) as he changes his bid to b_i' to increase his allocation by δ is given by

$$
\begin{aligned}
P_i(b_i', b_{-i}) - P_i(b_i, b_{-i}) &= \sum_{j\neq i} (x_j^{**} - x_j'^*)\beta_j \\
&= \sum_{j\neq i} \sum_{s\in R_j} (z_{js}^{**} - z_{js}'^*) \sum_{l\in s} \lambda_l \\
&= \sum_{(j,s):j\neq i, s\in R_j} \sum_{l\in s} (z_{js}^{**} - z_{js}'^*)\lambda_l \\
&\geq \sum_{(j,s):j\neq i, s\in R_j} \sum_{l\in s\cap L_i} (z_{js}^{**} - z_{js}'^*)\lambda_l && (23) \\
&= \sum_{l\in L_i\cap L_{-i}} \lambda_l \cdot \sum_{(j,s):j\neq i, s\in R_j, l\in s} (z_{js}^{**} - z_{js}'^*) && (24) \\
&\geq \sum_{l\in L_i\cap L_{-i}} \lambda_l \cdot \sum_{r\in R_i:l\in r} (z_{ir}'^* - z_{ir}^{**}) && (25) \\
&\geq \sum_{l\in L_i} \lambda_l \cdot \sum_{r\in R_i:l\in r} (z_{ir}'^* - z_{ir}^{**}) && (26) \\
&= \sum_{r\in R_i} \sum_{l\in r} (z_{ir}'^* - z_{ir}^{**}) \cdot \lambda_l \\
&= \sum_{r\in R_i} (z_{ir}'^* - z_{ir}^{**}) \cdot \left(\sum_{l\in r} \lambda_l\right) \\
&= \sum_{r\in R_i} (z_{ir}'^* - z_{ir}^{**}) \cdot \nu_i \\
&= \delta \cdot \nu_i.
\end{aligned}
$$

The first equality follows by definition. The second equality follows by definition, by Eq. (20) and by noting that since buyer i increases his bid, for any buyer $j \neq i$, $x_j^{**} \geq x_j^{'*}$ and $z_{js}^{**} \geq z_{js}^{'*}$. The third follows merely by compactifying notation and writing a double sum over j and s as a single sum over (j, s). Inequality (23) follows because the sum over l has fewer terms than before. Equality (24) is arrived at by rearrangement of terms. Inequality (25) follows from inequality (21). Inequality (26) follows from inequality (22). The remaining are obvious.

Now, since v_i is strictly concave,

$$v_i(x_i^{**} + \delta) - v_i(x_i^{**}) < \nu_i \cdot \delta \leq \Delta P_i.$$

This holds even if $x_i^{**} = 0$.

Thus, given the bids b_{-i} of all the other players, the best response of player i is to bid b_i so that he obtains x_i^{**}. This implies that $b = (b_1, \ldots, b_n)$ is a Nash equilibrium and the corresponding allocation is efficient. □

Proof of Theorem 2. Let x^{**} be an efficient allocation. Then, there exist $\lambda_1, \ldots, \lambda_L \geq 0$ such that $\frac{\partial v_i(x_i^{**})}{\partial x_{il}} \leq \lambda_l$, $\forall i, l$ (with equality if $x_{il}^{**} > 0$). Consider the strategy profile $d_{il} = x_{il}^{**}$ and $\beta_{il} = \frac{\partial v_i(d_{il})}{\partial x_{il}}$. Note that this implies that

$$\beta_{il} = \lambda_l, \quad \forall i, l : x_{il}^{**} > 0.$$

Given the bids b_{-i} of the others as fixed, suppose that buyer i changes his bid b_i to change his allocation x_i^{**} to some other $x_i^{'*}$ by some $\Delta_l = x_{il}^{'*} - x_{il}^{**}$. Without loss of generality, assume that there is some $\bar{l}$ such that $\Delta_l < 0$ for $l \leq \bar{l}$ and $\Delta_l \geq 0$ for $l > \bar{l}$. Define

$$\begin{aligned} \tilde{x}_{il} &= x_{il}^{**} - |\Delta_l|, \quad \text{for } l \leq \bar{l}, \\ &= x_{il}^{**}, \quad \text{for } l > \bar{l}. \end{aligned} \tag{27}$$

We consider two cases: (i) buyer i changes his bid to $\tilde{b}_i$ to change his allocation from x_i^{**} to $\tilde{x}_i$, and (ii) buyer i changes his bid to b_i' to change his allocation from $\tilde{x}_i$ to $x_i^{'*}$.

So, first consider that the buyer changes his bid to change his allocation from x_i^{**} to $\tilde{x}_i$. Then, buyer i now gets $|\Delta_l|$ less for goods $l \leq \bar{l}$ and the same as before for the other goods. The allocation of all the other players does not change since all of them already receive the maximum quantity they ask for. From Eq. (12), we get then that the payment of player i does not change. However, since v_i is strictly increasing and concave in each argument, his valuation strictly reduces by $v_i(x_i^{**}) - v_i(\tilde{x}_i)$. Thus,

$$v_i(\tilde{x}_i) - v_i(x_i^{**}) < P_i(\tilde{b}_i, b_{-i}) - P_i(b_i, b_{-i}) = 0. \tag{28}$$

Now, suppose that buyer i changes his bid from $\tilde{b}_i$ to b_i' such that he changes his allocation from $\tilde{x}_i$ to $x_i^{'*}$. Note that now his allocation changes by $\Delta_l \geq 0$ for $l > \bar{l}$. Then, the change in his payment is given by

$$\begin{aligned} P_i(b_i', b_{-i}) - P_i(\tilde{b}_i, b_{-i}) &= \sum_{j \neq i} \sum_{l > \bar{l} : \tilde{x}_{jl} > 0} \beta_{jl}(\tilde{x}_{jl} - x_{jl}^{'*}) \\ &= \sum_{l > \bar{l}} \lambda_l \sum_{j \neq i : \tilde{x}_{jl} > 0} (\tilde{x}_{jl} - x_{jl}^{'*}) \\ &= \sum_{l > \bar{l}} \lambda_l \cdot \Delta_l, \end{aligned}$$

where the last equality follows since the total change in allocation of all other players on an item $l > \bar{l}$ is Δ_l, which is how much more buyer i gets of l. Now, v_i is strictly concave in each argument. Thus,

$$v_i(x_i^{'*}) - v_i(\tilde{x}_i) < \sum_{l > \bar{l}} \lambda_l \cdot \Delta_l = P_i(b_i', b_{-i}) - P_i(\tilde{b}_i, b_{-i}). \tag{29}$$

From (28) and (29), we get that

$$v_i(x_i^{'*}) - v_i(x_i^{**}) < P_i(b_i', b_{-i}) - P_i(b_i, b_{-i}),$$

which implies that, given the bids b_{-i} of all the other players, the best response of player i is to bid b_i so that he obtains x_i^{**}. Thus, $b = (b_1, \ldots, b_n)$ is a Nash equilibrium and the corresponding allocation is efficient. □

Proof of Theorem 3. Let (x^{**}, y^{**}) be an efficient allocation. Then, there exist $\nu_1, \ldots, \nu_n \geq 0$ and $\lambda_1, \ldots, \lambda_L \geq 0$ such that $v_i'(x_i^{**}) \leq \nu_i = \sum_{l \in r : r \in R_i} \lambda_l$, $\forall i$ and $c_j'(y_j^{**}) \geq \lambda_{l_j}$, $\forall j$. Consider the strategy profile $d_i = x_i^{**}$, $\beta_i = v_i'(d_i)$, $s_j = y_j^{**}$ and $\alpha_j = c_j'(s_j)$. Note that this implies that

$$\beta_i = \sum_{l \in r : r \in R_i} \lambda_l, \quad \forall i : x_i^{**} > 0 \quad \text{and} \quad \alpha_j = \lambda_{l_j}, \quad \forall j : y_j^{**} > 0. \tag{30}$$

Consider a buyer i with $x_i^{**} > 0$. Given the bids (b_{-i}, a) of the others as fixed, if buyer i changes his bid b_i to decrease his allocation x_i^{**} by a $\Delta > 0$, then note that the allocation of all the other buyers does not change but some sellers on links $L_i = \{l \in r, r \in R_i\}$ sell less.

From Eq. (14), we get the change in payment of buyer i (later denoted $\Delta\tilde{T}_i$) is

$$\begin{aligned} \tilde{T}_i(b_i', b_{-i}, a) - \tilde{T}_i(b, a) &= \sum_{l \in L_i} \sum_{j : l_j = l, y_j^{**} > 0} \alpha_j(y_j^{'*} - y_j^{**}) \\ &= \sum_{l \in L_i} \sum_{j : l_j = l, y_j^{**} > 0} \lambda_{l_j}(y_j^{'*} - y_j^{**}) \\ &= \sum_{r \in R_i : x_{ir}^{**} > 0} \sum_{l \in r} \lambda_l (x_{ir}^{'*} - x_{ir}^{**}) \\ &= -\sum_{r \in R_i : x_{ir}^{**} > 0} \sum_{l \in r} \lambda_l \, \Delta_r \\ &= -\sum_{r \in R_i : x_{ir}^{**} > 0} \Delta_r \cdot \nu_i = -\Delta \cdot \nu_i. \end{aligned}$$

The first equality is obtained just by taking differences of the two payments, the second equality by (30), and the third follows by an argument similar to (21). The last two are obvious. Since v_i is strictly increasing and concave, we get that

$$v_i(x_i^{**} - \Delta) - v_i(x_i^{**}) < -\Delta \cdot \nu_i = \Delta\tilde{T}_i, \tag{31}$$

i.e., his net payoff decreases.

Now, suppose the buyer i, with $x_i^{**} \geq 0$, changes his bid to b_i' such that it increases his allocation x_i^{**} by a $\Delta > 0$, then note that, while the allocation of all the sellers remains unchanged, that of some buyers decreases. Let the resulting allocation of buyers be $x^{'*}$. Then, as in the proof of Theorem 1,

$$\begin{aligned} &\tilde{T}_i(b_i', b_{-i}, a) - \tilde{T}_i(b, a) \\ &= \sum_{(j,s) : j \neq i, s \in R_j, z_{js}^{**} > 0} \sum_{l \in s} (z_{js}^{**} - z_{js}^{'*}) \lambda_l \\ &\geq \sum_{(j,s) : j \neq i, s \in R_j, z_{js}^{**} > 0} \sum_{l \in s \cap L_i} (z_{js}^{**} - z_{js}^{'*}) \lambda_l \\ &= \sum_{l \in L_i \cap L_{-i}} \lambda_l \cdot \sum_{r \in R_i : l \in r} (z_{ir}^{'*} - z_{ir}^{**}) \\ &= \sum_{l \in L_i} \lambda_l \cdot \sum_{r \in R_i : l \in r} (z_{ir}^{'*} - z_{ir}^{**}) \\ &= \sum_{r \in R_i} \sum_{l \in r} (z_{ir}^{'*} - z_{ir}^{**}) \cdot \lambda_l = \Delta \cdot \nu_i. \end{aligned} \tag{32}$$

The reasoning is same as before. Further, since v_i is strictly increasing and concave, we have

$$v_i(x_i^{**}+\Delta)-v_i(x_i^{**})<\nu_i\cdot\Delta\leq\Delta\tilde{T}_i. \tag{33}$$

From (31) and (33), we get that, given the bids (b_{-i}, a) of all the other players, the best response of a buyer i is to bid b_i so that he obtains x_i^{**}.

Now consider a seller j with $y_j^{**}\geq 0$. Suppose a seller j changes his bid to increase y_j^{**} by a $\Delta>0$. This will not affect the allocation of the buyers but some sellers selling good l might get affected. Clearly, the net change in payment to the seller is $-\Delta\bar{T}_j=\Delta\cdot\lambda_l$ (follows easily from (30)) and, since c_j is strictly increasing and convex, we get that

$$c_j(y_j^{**}+\Delta)-c_j(y_j^{**})>\lambda_l\cdot\Delta=-\Delta\bar{T}_j. \tag{34}$$

And if any seller j (selling l), with $y_j^{**}>0$, were to change his bid to decrease his allocation by $\Delta>0$, then the allocation to other sellers does not change but some buyers get Δ less. Thus, the net change in seller j's transfer is given by

$$\begin{aligned}\Delta\bar{T}_j&=\sum_{i:l\in L_i}\sum_{r\in R_i:l\in r,z_{ir}^{**}>0}\beta_i(z_{ir}^{**}-z_{ir}^{'*}) \qquad (35)\\ &\geq\sum_{i:l\in L_i}\sum_{r\in R_i:l\in r,z_{ir}^{**}>0}\lambda_l(z_{ir}^{**}-z_{ir}^{'*})\\ &=\lambda_l\cdot\Delta.\end{aligned}$$

And again, by strict convexity of c_j,

$$c_j(y_j^{**}-\Delta)-c_j(y_j^{**})>-\lambda_l\Delta\geq-\Delta\bar{T}_j. \tag{36}$$

From (34) and (36), we get that a_j is a best response of seller j to bids of other players (b, a_{-j}).

Thus, (b, a) is a Nash equilibrium. Moreover, the corresponding allocation is efficient.

To prove strong budget balance at this Nash equilibrium, we note that

$$\begin{aligned}&\sum_{k:l\in R_k}\tilde{x}_k=\sum_{j:l=L_j}\tilde{y}_j,\quad\forall l,\\ &\sum_{k\neq i:l\in R_k}\tilde{x}_k^{-i}=\sum_{j:l=L_j}\tilde{y}_j^{-i},\quad\forall i:l\in R_i\\ &\sum_{k\neq j:l=L_k}\tilde{y}_k^{-j}=\sum_{i:l\in R_i}\tilde{x}_i^{-i},\quad\forall j:l=L_j.\end{aligned}$$

We can now write the payments for all i and j as

$$\tilde{T}_i=\sum_l\lambda_l\left(\sum_{k\neq i:l\in R_k}(\tilde{x}_k^{-i}-\tilde{x}_k)-\sum_{j:l=L_j}(\tilde{y}_j^{-i}-\tilde{y}_j)\right),$$

and

$$\bar{T}_j=\sum_l\lambda_l\left(\sum_{i:l\in R_i}(\bar{x}_i^{-j}-\tilde{x}_i)-\sum_{k\neq j:l=L_k}(\bar{y}_k^{-j}-\tilde{y}_k)\right),$$

which using the facts noted above yield

$$\tilde{T}_i=\sum_{l\in R_i}\lambda_l\tilde{x}_i,\quad\text{and}\quad\bar{T}_j=\lambda_{L_j}\tilde{y}_j,$$

from which we easily get $\sum_i\tilde{T}_i=\sum_j\bar{T}_j$, i.e., strong budget balance at the Nash equilibrium (b, a). □

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Rahul Jain is an Assistant Professor of Electrical Engineering at the University of Southern California, Los Angeles, CA. Prior to joining USC, he was a postdoctoral member of the research staff at the Mathematical Sciences Division of the IBM T.J. Watson Research Center, Yorktown Heights, NY. He received his Ph.D. in EECS in 2004, and an M.A. in Statistics in 2002, both from the University of California, Berkeley. He also received an M.S. in ECE from Rice University in 1999 and completed his undergraduate work with a B.Tech in EE in 1997 from the Indian Institute of Technology, Kanpur. He won a Best Paper Award at The ValueTools Conference 2009 and was a recipient of the CAREER Award from the National Science Foundation in 2010. He has diverse research interests, with current focus on Game Theory and Economics for Networks, and Stochastic Control and Learning.

Jean Walrand received his Ph.D. in EECS from UC Berkeley and has been on the faculty of that department since 1982. He is the author of *An Introduction to Queueing Networks* (Prentice Hall, 1988) and of *Communication Networks: A First Course* (2nd ed, McGraw-Hill, 1998) and co-author of High-Performance *Communication Networks* (2nd ed, Morgan Kaufman, 2000) and of *Communication Networks: A Concise Introduction* (Morgan & Claypool, 2010). His research interests include stochastic processes, queuing theory, communication networks, game theory and the economics of the Internet. Prof. Walrand is a Fellow of the Belgian American Education Foundation and of the IEEE and a recipient of the Lanchester Prize and of the Stephen O. Rice Prize.