CS325 Winter 2013: HW 6

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Problem: Below is a definition of the graph isomorphism problem.

> **Input:** two graphs, $G_1 = (V_1, E_1)$, and $G_2 = (V_2, E_2)$.
>
> **Question:** Can the nodes of G_1 be renamed s.t. G_1 becomes G_2 ?
> In other words, is there a one-to-one function $f : V_1 \to V_2$ such that for any edge $(x, y) \in E_1$ if only if $(f(x), f(y)) \in E_2$. Show that the graph Isomorphism problem is in NP.

Solution To show that Graph Isomorphism (GI) is in NP we will show that 3-SAT reduces to GI. Each literal in 3-SAT is a node in a graph, and each clause is a group of nodes. Moving a clause from the last position to the first will create a new graph isomorphic to the original, and it will take NP to compare each node.

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8.4 Consider the CLIQUE problem restricted to graphs in which every vertex has degree at most 3. Call this problem CLIQUE-3.

1. Prove that CLIQUE -3 is in NP.

2. What is wrong with the following proof of NP-completeness for CLIQUE -3? We know that the CLIQUE problem in general graphs is NP-complete, so it is enough to present a reduction from CLIQUE -3 to CLIQUE. Given a graph G with vertices of degree ≤ 3, and a parameter g, the reduction leaves the graph and the parameter unchanged: clearly the output of the reduction is a possible input for the CLIQUE problem. Furthermore, the answer to both problems is identical. This proves the correctness of the reduction and, therefore, the NP-completeness of CLIQUE -3.

3. It is true that the VERTEX COVER problem remains NP-complete even when restricted to graphs in which every vertex has degree at most 3. Call this problem VC -3. What is wrong with the following proof of NP-completeness for CLIQUE -3? We present a reduction from VC -3 to CLIQUE -3. Given a graph G = (V, E) with node degrees bounded by 3, and a parameter b, we create an instance of CLIQUE -3 by leaving the graph unchanged and switching the parameter to $|V| - b$. Now, a subset $C \subset V$ is a vertex cover in G if and only if the complementary set $V - C$ is a clique in G. Therefore G has a vertex cover of size $\leq b$ if and only if it has a clique of size $\geq |V| - b$. This proves the correctness of the reduction and, consequently, the NP-completeness of CLIQUE-3.

4. Describe an $O(|V|4)$ algorithm for CLIQUE-3.

Solution:

1. A) Given a clique in a graph, all that is required to prove the certificate is that there exists and edge between every pair of verticies, which is easily done in polynomial time.

2. B) The reduction is done in the wrong order. To prove that clique-3 is at least as hard as clique, clique must be reduced to clique-3.

3. C) The statement "Now, a subset C $\subseteq$ V is a vertex cover in G if and only if the complementary set V - C is a clique in G" is incorrect. This would be correct if independent set was used instead of clique.

4. D) With the restriction that every vertex in our clique problem is limited to a degree of at most 3, we know that there cannot exist a clique that is larger than 4 verticies. To assume that more than 4 verticies

can exist in a clique-3 graph breaks the definition of 3-clique and its requirement of being fully connected to every other vertex in the graph. Therefore, we know that no solution exists for magnitudes of greater than 4. With that knowledge we can restrict our search of solutions to $O(|v|^4)$ time.

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Problem: In the HITTING SET problem, we are given a family of sets $S_1, S_2, \ldots, S_n$ and a budget b, and we wish to find a set H of size $\leq b$ which intersects every S_i , if such an H exists. In other words, we want $H \cap S_i = \oslash$ for all i. Show that HITTING SET is NP-complete.

Solution: This can be thought of as a vertex-cover. Given a graph $G = (V, E)$ we can describe each edge as a two element set that consists of the two vertices it connects. In other words, $e_i = (u, w)$ where e is an edge and (u, w) are verticies. This builds a family of sets from our given graph G, which allows us to intuitively reduce this to an NP graph problem. With this method of thinking finding a vertex cover uses the same method as the hitting set problem.

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6.26 from book: Proving NP-completeness by generalization. For each of the problems below, prove that it is NP- complete by showing that it is a generalization of some NP-complete problem we have seen in this chapter.

1. SUBGRAPH ISOMORPHISM: Given as input two undirected graphs G and H, determine whether G is a subgraph of H (that is, whether by deleting certain vertices and edges of H we obtain a graph that is, up to renaming of vertices, identical to G), and if so, return the corresponding mapping of $V(G)$ into $V(H)$.

2. LONGEST PATH : Given a graph G and an integer g, find in G a simple path of length g.

3. DENSE SUBGRAPH: Given a graph and two integers a and b, find a set of a vertices of G such that there are at least b edges between them.

4. SPARSE SUBGRAPH: Given a graph and two integers a and b, find a set of a vertices of G such that there are at most b edges between them.

Solution:

1. A) The subgraph isomorphism can be reduced to the clique problem.
2. B) The longest path problem can be reduced to the rudrata path problem.
3. D) The dense subgraph problem can be reduced to the clique problem.
4. E) The sparse subgraph problem can be reduced to the independent set problem.