CS325 Winter 2013: HW 3

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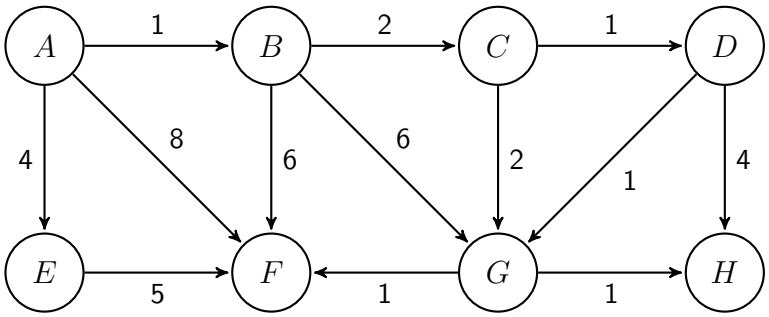
January 31, 2013

4.1

Problem: Suppose Dijkstra's algorithm is run on the following graph, starting at node A.



(a) Draw a table showing the intermediate distance values of all the nodes at each iteration of the algorithm.

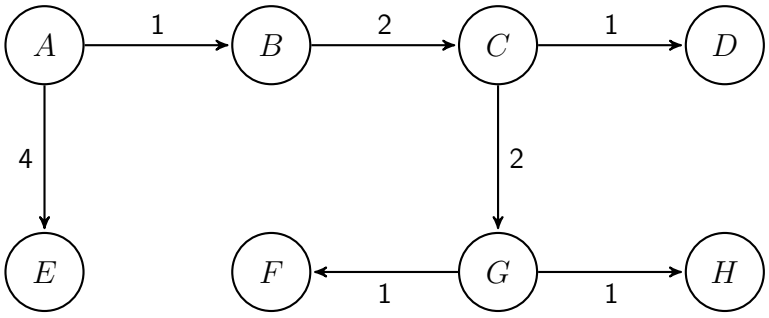
(b) Show the final shortest-path tree.

Solution:

(a)

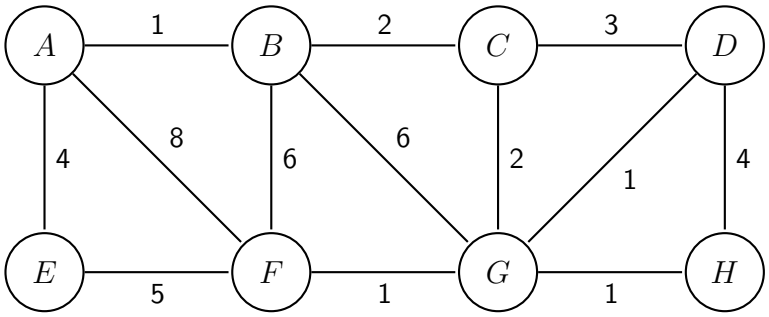
	Iteration						
Node	0	1	2	3	4	5	6
A	0	0	0	0	0	0	0
B	1	1	1	1	1	1	1
C	∞	3	3	3	3	3	3
D	∞	∞	4	4	4	4	4
E	4	4	4	4	4	4	4
F	8	7	7	7	7	6	6
G	∞	7	5	5	5	5	5
H	∞	∞	∞	8	8	6	6

(b)



5.2

Problem: Suppose we want to find the minimum spanning tree of the following graph.



(a) Run Prim's algorithm; whenever there is a choice of nodes, always use alphabetic ordering (e.g., start from node A). Draw a table showing the intermediate values of the cost array.

(b) Run Kruskal's algorithm on the same graph. Show how the disjoint-sets data structure looks at every intermediate stage (including the structure of the directed trees), assuming path compression is used.

Solution:

(a)

Set S	A	B	C	D	E	F	G	H
$\{\}$	$0/\text{nil}$	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil
A		$1/A$	∞/nil	∞/nil	$4/A$	$6/B$	$6/B$	∞/nil
A, B			$2/B$	∞/nil	$4/A$	$6/B$	$2/C$	∞/nil
A, B, C				$3/C$	$4/A$	$6/B$	$2/C$	∞/nil
A, B, C, D					$4/A$	$6/B$	$1/D$	$4/D$
A, B, C, D, E						$5/E$	$1/D$	$4/D$
A, B, C, D, E, F							$1/D$	$4/D$
A, B, C, D, E, F, G						$1/G$		$1/G$
A, B, C, D, E, F, G, H								$1/G$

(b)

Figure 1: After makeset(A), makeset(B),...,makeset(H):

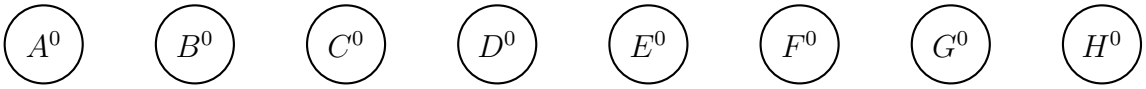


Figure 2: After union(A,B), union(G,F), union(G,D), union(G,H):

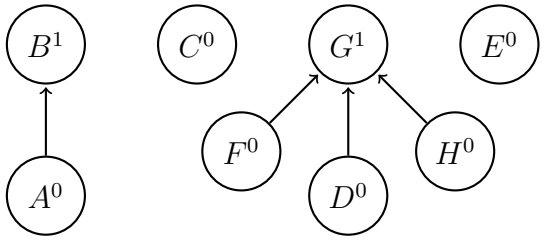


Figure 3: After union(C,B):

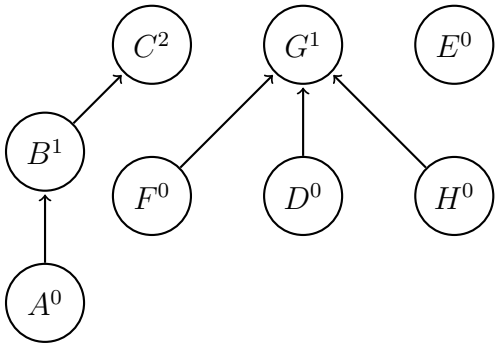


Figure 4: After union(G,C):

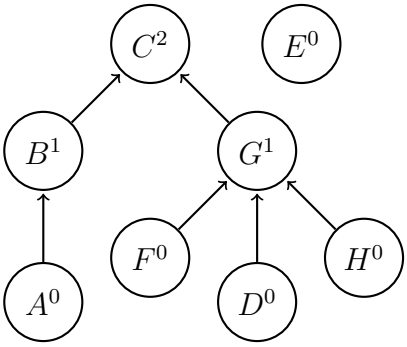
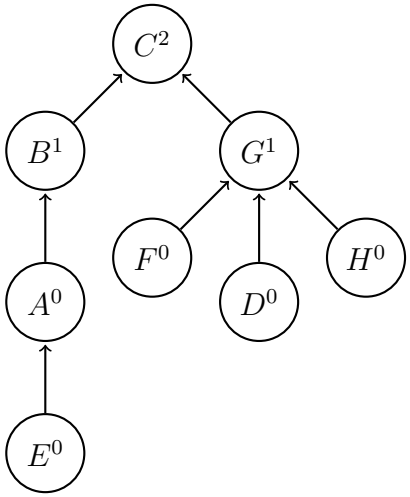


Figure 5: After union(E,A):



5.5

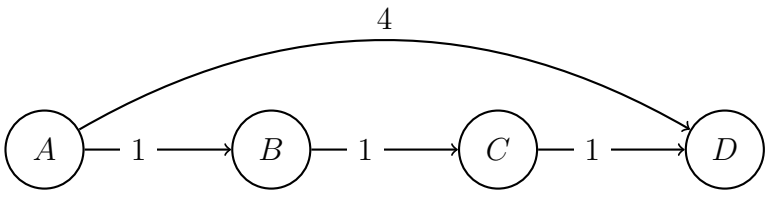
Problem: Consider an undirected graph $G = (V, E)$ with nonnegative edge weights $w_e \geq 0$. Suppose that you have computed a minimum spanning tree of G, and that you have also computed shortest paths to all nodes from a particular node $s \in V$. Now suppose each edge weight is increased by 1: the new weights are $w_e = w_e + 1$.

(a) Does the minimum spanning tree change? Give an example where it changes or prove it cannot change.

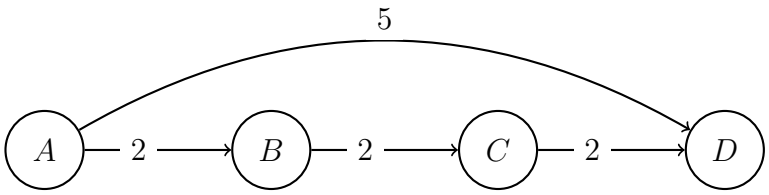
(b) Do the shortest paths change? Give an example where they change or prove they cannot change.

Solution:

Proof. Part B: Proof by counter example. Consider the graph:



In this graph the shortest path from A to D is from A to B to C to D with a weight of 3. Now consider the same graph, but with all of the edges having a weight increased by 1.



The previously shortest path now has a weight of 6, where the direct route from A to D only has a weight of 5. Thus the shortest path has changed and the counter example is proved. □

Proof. Part A: Creating a shortest path tree can be thought of as distances from one node to another. The distance formula can be derived with the use of triangles. In uniformly increasing the weight of the edges of triangles, the ratio of the edges change. Hence it is possible, as we have just demonstrated, to have the shortest path tree change when uniformly increasing the weight of all edges. The minimal spanning tree however, can be thought of as the balance in a graph. Since increasing the graph uniformly does not alter the balance, we know that the minimal spanning tree will not change.

□

5.7

Problem: Show how to find the *maximum* spanning tree of a graph, that is, the spanning tree of largest total weight.

Solution:

Proof. This can be accomplished in a similar way that a minimal spanning tree of a graph is created. Of course there must be a difference to achieve this, and we can choose one of two options. We can select the maximal instead of minimal edge at every iteration, or we can make all of the edge weights negative and use an existing algorithm. □

5

Problem: Consider the Change Problem in Austria. The input to this problem is an integer L. The output should be the minimum cardinality collection of coins required to make L shillings of change (that is, you want to use as few coins as possible). In Austria the coins are worth 1, 5, 10, 20, 25, 50 Shillings. Assume that you have an unlimited number of coins of each type. Formally prove or disprove that the greedy algorithm (that takes as many coins as possible from the highest denominations) correctly solves the Change Problem. So for example, to make change for 234 Shillings the greedy algorithms would take four 50 shilling coins, one 25 shilling coin, one 5 shilling coin, and four 1 shilling coins.

Solution:

Proof. Proof by counter example. When the Shilling total is equal to 40, the optimal solution would use two 20 shilling coins, so $n = 2$ where n is the number of coins. According to the greedy method of taking as many coins as possible from the highest denomination available, a shilling total of 40 would have change made with one 25 shilling coin, one 15 shilling coin, and one 5 shilling coin making $n = 3$ which is not optimal. □

6

Problem: Consider a long quiet country road with houses scatter very sparsely along it (We can picture the road as a long line segment). You want to place cell phone base stations at certain points along the road, so that every house is within four miles of one of the base stations. Give an efficient algorithm that achieves this goal, using as few stations as possible. Show that the algorithm achieve the optimal solution using the "stay ahead" argument.

Solution:

Proof. Approaching from the Left or from the Right, While: At the first house that you come across that is uncovered, add four miles and place a tower

Let $T_g(i)$ denote the total number of houses covered by the first i base stations when approached from left to right in the greedy algorithm. Let $T_o(i)$ denote the total number of houses covered by the first i base stations when approached from left to right in the optimal algorithm. "Greedy stays ahead" Theorem: $T_g(i) \geq T_o(i)$ If the theorem is true, then T_g is optimal.

Base case: $i = 1$
$T_g(1) \geq T_o(1)$ is true because the greedy algorithm is creating maximal coverage with the first tower that is placed at mile j. Any other tower placement would have less coverage, and therefore would be open to a counter example.

Because the only other placements of the base stations would either be to the west of j (there may be a house in between j and $j + 4$ that didn't get covered), or east of j (the first house is not covered $\implies$ optimal is infeasible)

Inductive step:
Assume: $T_g(k) \geq T_o(k), \forall k \leq i$
Prove: $T_g(k + 1) \geq T_o(k + 1)$

We know $T_g(k) \geq T_o(k)$. The k base station of T_g is placed at x, and T_o is placed at y. We know that $x \geq y$ from the greedy stays ahead theorem. This means that the starting point of x is at least as far as y, and will either continually tie y or exceed y. By the inductive assumption we have that $x \geq y$. Now let $x*$ be the $k + 1$th base station of greedy. Let $y*$ be the $k + 1$th base station of T_o. Because $x \geq y$ the only possibility is for x* to be at least as far as y*. This means that $T_g(k + 1) \geq T_o(k + 1)$. By induction, T_g is optimal.

□