H1 # Lecture 4: Elementary Sorts

H2 Rules of the Game

H3 Sorting Problem

To create an algorithm that implements a *static* method `sort()`, which sorts an array of ***any*** type of data

H3 Callbacks

Question:

How can `sort()` *know how to compare data of different type (both primitive and object) without any information about the type of an item's key?*

Solution:

- Client passes array of objects to `sort()` function
- The `sort()` function *calls back* object's `compareTo()` method as needed

> ***Callback***:
>
> A reference to executable code

H4 Implementing Callbacks in Different Languages

- *Java* : interfaces
- *C* : function pointers
- *C++* : class-type funtors
- *C#* : delegates
- *Python, Perl, ML, JavaScript* : first-class functions

H4 Roadmap

Client:

```java
import java.io.File;
public class FileSorter {
    public static void main(String[] args) {
        File directory = new File(args[0]);
        File[] files = directory.listFiles();
        Insertion.sort(files);
        for (int i = 0; i < files.length; i++)
    }
}
```

Object Implemtation:

```java
public class File implements Comparable<File> {
    ...
    public int compareTo(File b) {
        ...
        return -1;
        ...
        return +1;
        ...
        return 0;
    }
}
```

`Comparable` ***Interface*** *(built in to Java)*:

```java
public interface Comparable<Item> {
    public int compareTo(Item that);
}
```

> ***Note that:***
>
> In *Java*, there's an implicit mechanism that says that any such array of object is going to have the `compareTo()` method, then the `sort()` function calls back the `compareTo()` method associated with the objects in the array whenever it needs to compare two items.

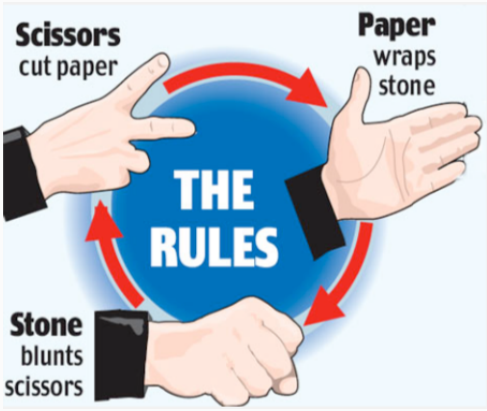
`sort()` ***Implementation***:

```java
public static void sort(Comparable[] a) {
    int N = a.length;
    for (int i = 0; i < N; i++) {
        for (int j = i; j > 0; j--) {
            if (a[j].compareTo(a[j-1]) < 0) {
                // KEY POINT: no dependence on File data type
                exch(a, j, j-1);
            } else {
                break;
            }
        }
    }
}
```

H3 Total Order

A ***total order*** is a binary relation $\leq$ that satisfies:

- Antisymmetry : if $v \leq w$ and $w \leq v$, then $v = w$
- Transitivity : if $v \leq w$ and $w \leq x$, then $v \leq x$
- Totality : either $v \leq w$ or $w \leq v$ or both

> *Suprising Fact* 😲:
>
> The `<=` operator for *double* is not a total order because `(Double.NaN <= Double.NaN)` is false, which violates ***totality***
>
> *Rock, Paper and Scissors* game is also not a total order because it has an intrasitive relation: $w > v$ and $v > x$, but w isn't necessarily greater than x
>
> 
>

H4 Sample Question

Question:

Consider the data type `Temperature` defined below. Which of the following required properties of the `Comparable` interface does the `compareTo()` method violate?*

```java
public class Temperature implements Comparable<Temperature>
{
    private final double degrees;
```

```
3
4       public Temperature(double degrees) {
5           if (Double.isNaN(degrees))
6               throw new IllegalArgumentException();
7           this.degrees = degrees;
8       }
9
10      public int compareTo(Temperature that) {
11          double EPSILON = 0.1;
12          if (this.degrees < that.degrees - EPSILON) return
    -1;
13          if (this.degrees > that.degrees + EPSILON) return
    +1;
14          return 0;
15      }
16      ...
17  }
```

Answer: **Transitivity**

Suppose a, b and c refer to objects corresponding to tmeperatures of 10.6°, 10.08° and 10.00°

respectively. Then, `a.compareTo(b) <= 0` and `b.compareTo(c) <= 0`, but `a.compareTo(c) > 0`. For this reason, you must not introduce a fudge factor when comparing two floatin-point numbers if you want to imlement the `Comparable` interface.

H3 `Comparable` API

Implement `compareTo()` so that `v.compareTo(w)`

- is a *total order*
- returns a negative integer, zero, or positive integer if v is less than, equal to, or greater than w, respectively
- Throws an exception if compatible types (or either is `null`)

> *Note that:*
>
> Built-in `Comparable` types: `Integer`, `Double`, `String`, `Date`, `File`, ...
>
> User-defined `Comparable` types: inplement the `Comparable` interface

H4 Implementing the Comparable Interface

`Data` ***data type*** *(simplified version of* `java.util.Date`*)*

```
1   public class Date implements Comparable<Date> { // only
    compare Date to another Date
2       private final int month, day, year;
3
4       public Date(int m, int d, int y) {
5           month = m;
```

```
        day = d;
        year = y;
    }

    public int compareTo(Date that) {
        if (this.year  < that.year ) return -1;
        if (this.year  > that.year ) return +1;
        if (this.month < that.month) return -1;
        if (this.month > that.month) return +1;
        if (this.day   < that.day  ) return -1;
        if (this.day   > that.day  ) return +1;
        return 0;
    }
}
```

Helper Functions

`less()`: is item v less than w?

```
private static boolean less(Comparable v, Comparable w) {
    return v.compareTo(w) <0;
}
```

`exch()`: swap item in an array `a[]` at index i with the one at index j

```
private static void exch(Comparable[] a, int i, int j) {
    Comparable swap = a[i];
    a[i] = a[j];
    a[j] = swap;
}
```

Testing

Test if an array is sorted

```
private static boolean isSorted(Comparable[] a) {
    for (int i = 1; i < a.length; i++) {
        if (less(a[i], a[i-1])) return false;
    }
    return true;
}
```

Question:

If the sorting algorithm passes the test, did it correctly sort the array?

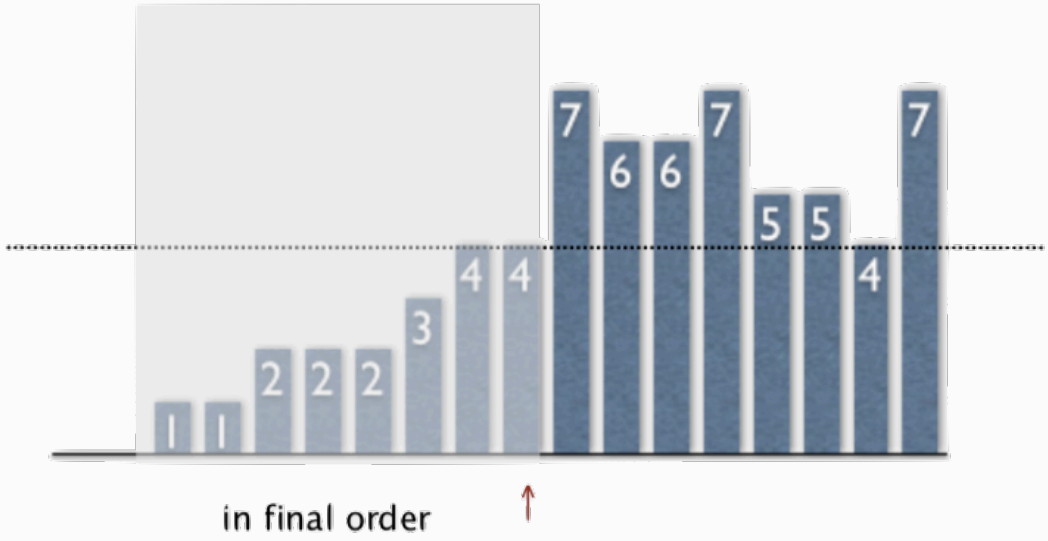
Answer:

Not always. If the values of all items in an `int[]` are set to 0, the test would be passed, suggesting that it is important to use helper functions `less(` and `exch()` to ensure this testing method works.

H2 Selection Sort

- In iteration i, find index `min` of smallest remaining entry
- Swap `a[i]` and `a[min]`

Animations



The pointer scans from left to right.

Invariants:

- Entries the left of pointer fixed and in *ascending order*
- No entry to right of pointer is smaller than any entry to the left of pointer

H3 Inner Loop

To maintain algorithm invariants:

- Move the pointer to the right:

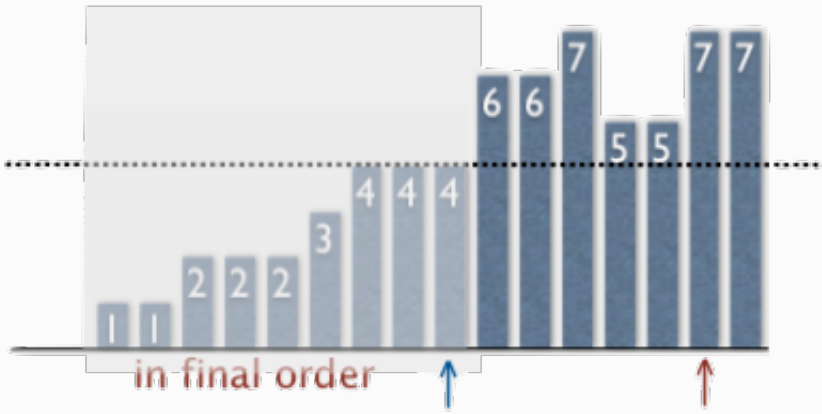
```
1  i++
```

- Identify index of minimum entry on right

```
int min = i;
for (int j = i+1; j < N; j++) {
    if (less(a[j],a[min])) {
        min = j;
    }
}
```

- Exchange into position

```
exch(a,i,min);
```



H3 Java Implementation

```
public class Selection {
    public static void sort(Comparable[] a) {
        int N = a.length;
        for (int i = 0l i < N; i++) {
            int min = i;
            for (int j = i+1; j < N; j++) {
                if (less(a[j],a[min])) {
                    min = j;
                }
            }
            exch(a,i,min);
```

```
12            }
13        }
14
15        private static boolean less(Comparable v, Comparable w)
    {
16            /* as before */
17        }
18
19        private static void exch(Comparable[] a, int i, int j)
    {
20            /* as before */
21        }
22    }
```

H3 Mathematical Analysis

Proposition: Selection sort uses $(N-1)+(N-2)+\cdots+1+0 \sim \frac{N^2}{2}$ compares and N exchanges

Trace of selection sort (array contents just after each exchange)

Running Time:

- quadratic time
- running time is insensitive to input
- even if input is partially- or fully-sorted

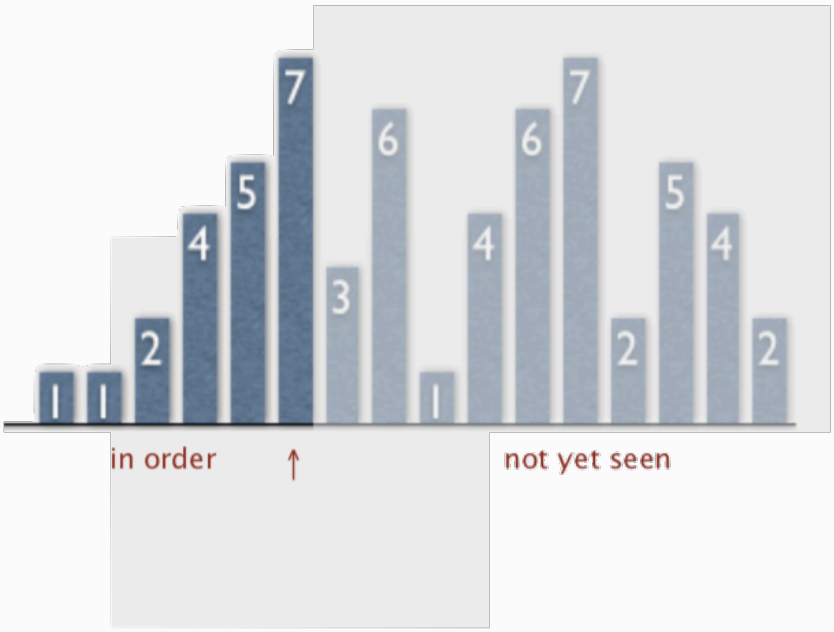
Data Movement:

- linear number of exchanges
- data movement is minimal

Insertion Sort

In iteration i, swap `a[i]` with each larger entry to its left

Animations



Pointer scans from left to right.

Invariants:

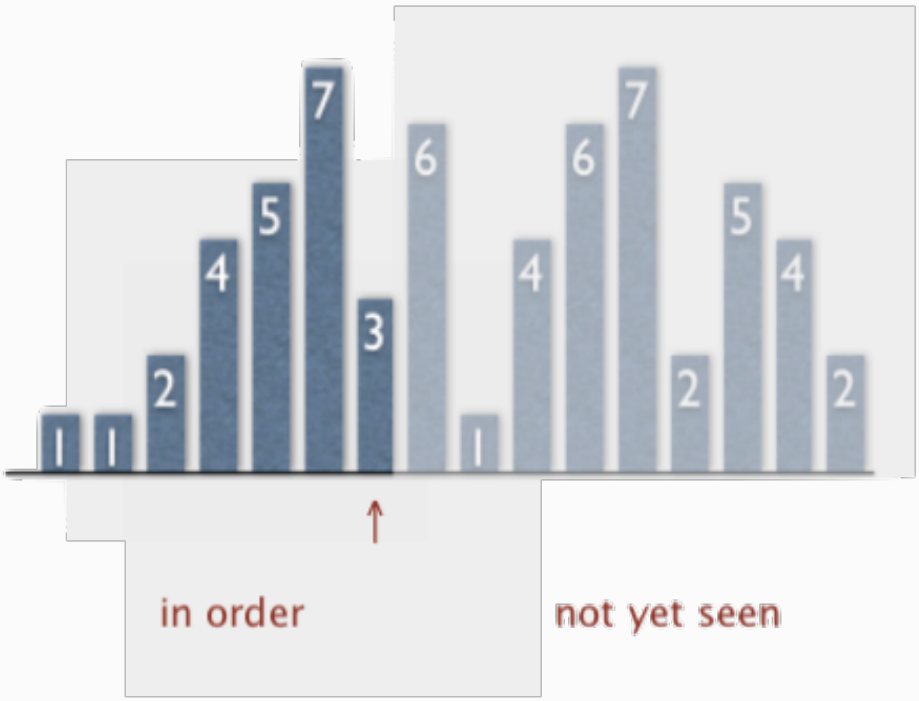
- entries to the left of pointer (including pointer) are in *ascending order*
- entries to the right of pointer have not yet been seen

Inner Loop

To maintain algorithm invariants:

- Move the pointer to the right

```
1  i++;
```



- Moving from right to left, exchange `a[i]` with each larger entry to its left

```
1  for (int j = i; j > 0; j--) {
2      if (less(a[j],a[j-1])) {
3          exch(a,j,j-1);
4      } else break;
5  }
```

H3 Java Implementation

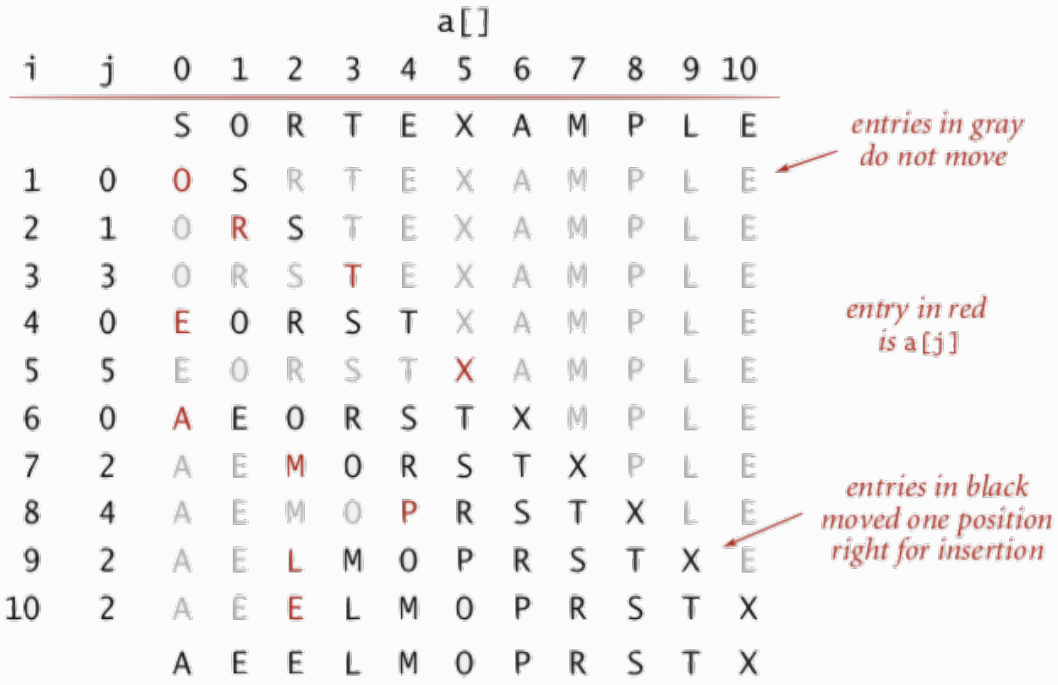
```
1  public class Insertion {
2      public static void sort(Comparable[] a) {
3          int N  = a.length;
```

```
4           for (int i = 0; i < N; i++) {
5               for (int j = i; j > 0; j--) {
6                   if (less(a[j],a[j-1])) {
7                       exch(a,j,j-1);
8                   } else break;
9               }
10          }
11      }
12
13      private static boolean less(Comparable v, Comparable w)
    {
14          /* as before */
15      }
16
17      private static void exch(Comparable[] a, int i, int j)
    {
18          /* as before */
19      }
20  }
```

H3 Mathematical Analysis

Proposition: to sort a randomly-ordered array with distinct keys, insertion sort uses $\sim \frac{N^2}{4}$ compares and $\sim \frac{N^2}{4}$ exchanegs on average

Proof: expect each entry to move halfway back on average:

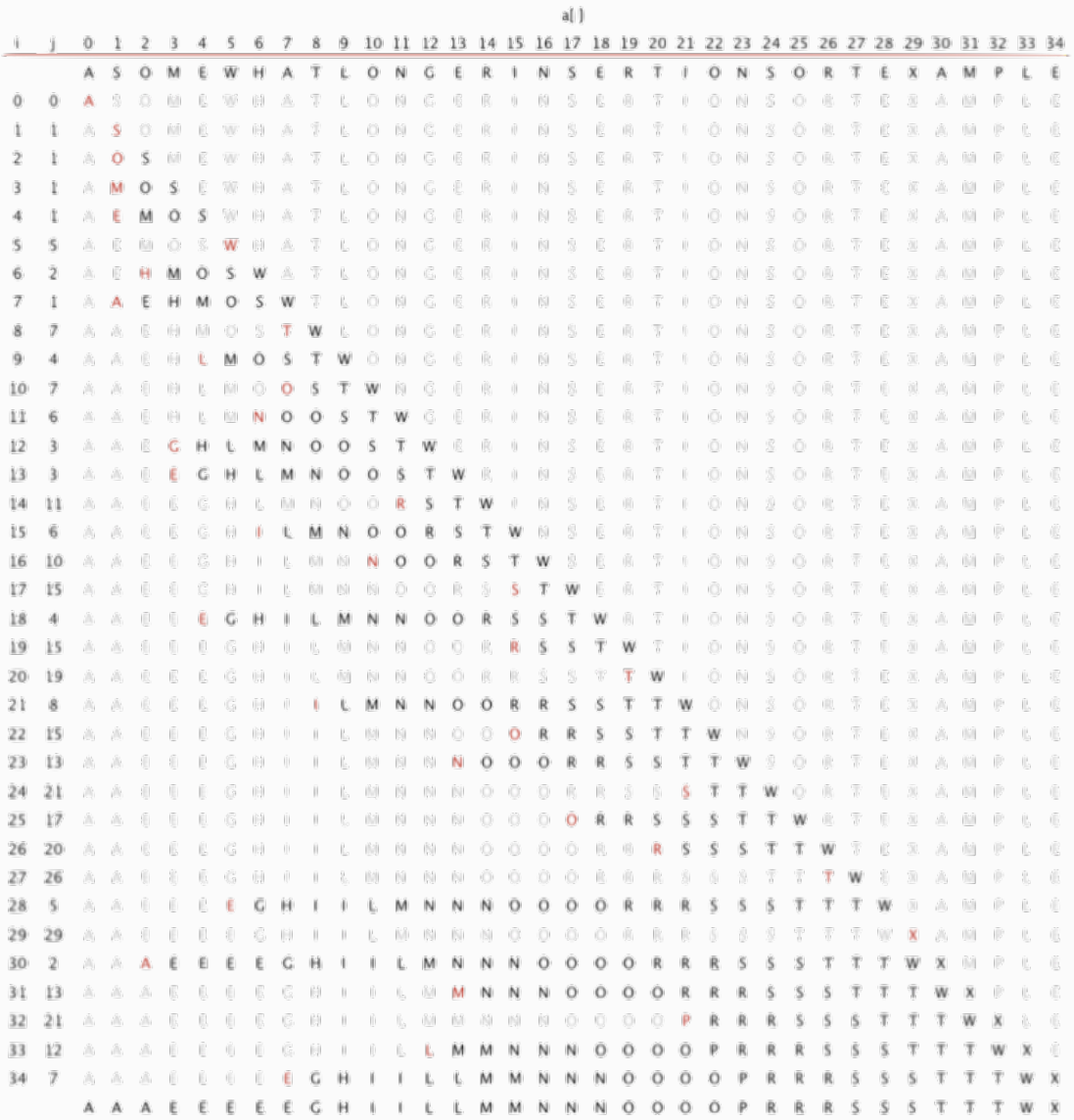


Trace of insertion sort (array contents just after each insertion)

> *Note that*:

> Each entry moves halfway back on average, which means half of the entries are expected to be black below the diagonal. There are $\frac{N^2}{2}$ entries below the diagonal and the half of that is $\frac{N^2}{4}$

H4 Trace



H4 Best Case

If the array is in ***ascending order***, insertion sort makes $N-1$ comapres and 0 exchanges

> *Note that*:
>
> In this case the compares are just a validation on the ascending order

H4 Worst Case

If the array is in ***descending order*** (and no duplicates), insertion sort makes $\sim \frac{N^2}{2}$ compares and $\sim \frac{N^2}{2}$ exchanges

H4 Partially-Sorted Array

Define: an array is partially sorted if the number of inversions is $\leq cN$

> *Inversion*:

> An ***inversion*** is a pair of keys that are out of order
>
> *For instance:*
>
> $$A\ E\ E\ L\ M\ O\ T\ R\ X\ P\ S$$
>
> There are 6 inversions:
>
> $$\begin{array}{l} T-R \\ T-P \\ T-S \\ R-P \\ X-P \\ X-S \end{array}$$

Examples:

- A subarray of size 10 appended to a sorted subarray of size N
- An array of size N with only 10 entries out of place

Proposition: for partially-sorted arrays, insertion sort runs in ***linear time***

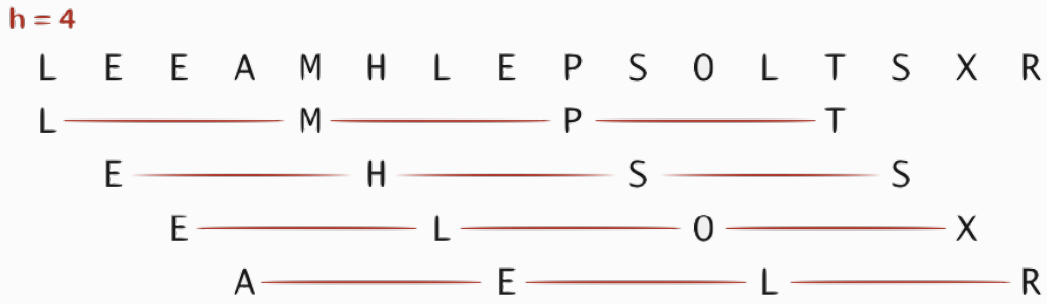
Proof:

- Number of exchanges equals the number of inversions
- Number of compares equals the number of exchanges plus $(N-1)$

H2 Shellsort

Idea:

Move entries more than one position at a time by h***-sorting*** the array.



> *h-Sorted Array*:
>
> An h-sorted array is h interleaved sorted subsequences

Shellsort:

h-sort array for ***decreasing sequence*** of values of h

Animation

input	S	H	E	L	L	S	O	R	T	E	X	A	M	P	L	E
13-sort	P	H	E	L	L	S	O	R	T	E	X	A	M	S	L	E
4-sort	L	E	E	A	M	H	L	E	P	S	O	L	T	S	X	R
1-sort	A	E	E	E	H	L	L	L	M	O	P	R	S	S	T	X

H3 h-Sorting

Questions:

How to h-sort an array?

Answer:

Insertion sort, with ***stride length*** h

> *Why Insertion Sort?*
>
> - For big increments, the subarrays will be small, which means any sorting algorithms would work well
> - For small increments after those big ones, the array will be partially sorted, which means the insertion sort will perform well

H3 Example: Increments 7, 3, 1

input

S O R T E X A M P L E

7-sort

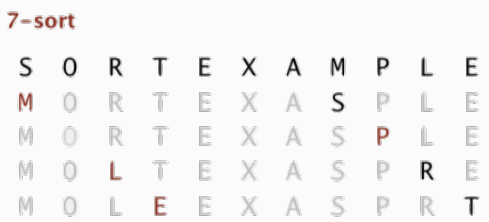
S O R T E X A M P L E
M O R T E X A S P L E
M O R T E X A S P L E
M O L T E X A S P R E
M O L E E X A S P R T

3-sort

M O L E E X A S P R T
E O L M E X A S P R T
E E L M O X A S P R T
E E L M O X A S P R T
A E L E O X M S P R T
A E L E O X M S P R T
A E L E O P M S X R T
A E L E O P M S X R T
A E L E O P M S X R T

Intuition

Proposition: a g-sorted array ***remains*** g-sorted after h-sorting it



> *Challenge*:
>
> It looks trivial to prove but it's actually more subtle

H3 Increment Sequence

> *Power of Two:*
>
> $$1, 2, 4, 8, 16, 32, \ldots$$
>
> It will wind up not comparing the elements in even positions with those in odd positions until the 1-sort. So the performance can be ***bad***.

> *Power of Two Minus One* [Shell]:
>
> $$1, 3, 7, 15, 31, 63, \ldots$$
>
> It performes ***okay***.

> $3x + 1$ [Knuth]:
>
> $$1, 4, 13, 40, 121, 364, \ldots$$
>
> It works ***well*** and it is ***easy to compute***.

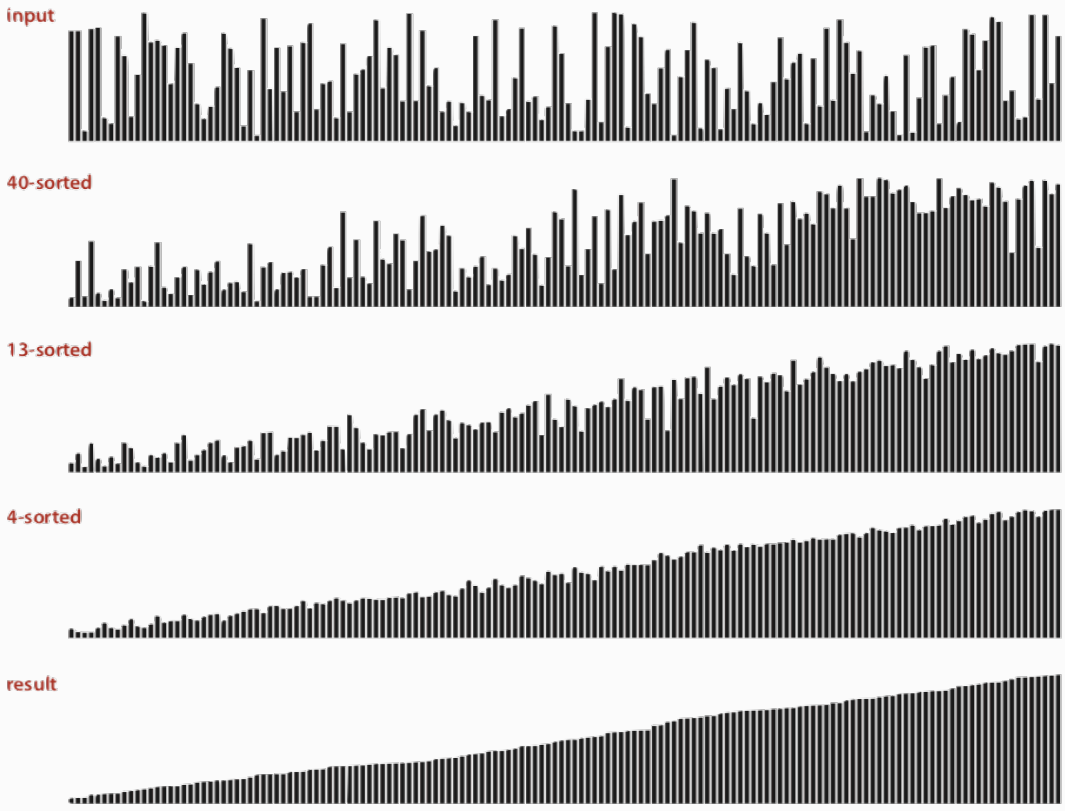
> *Sedgewick's Sequence* [Sedgewick]:
>
> $$1, 5, 19, 41, 109, 209, 505, 929, 2161, 3905, \ldots$$
>
> It's ***good*** and ***tough to beat*** in empirical studies.

H3 Java Implementation

```
1  public class Shell {
2      public static void sort(Comparable[] a) {
3          int N  = a.length;
4
5          // 3x+1 increment  sequence
6          int h = 1;
7          while (h < N/3) {
8              h = 3*h + 1; // 1, 4, 13, 40, 121, 364, ...
9          }
10
11         while (h >= 1) {
12             // h-sort the array
13             // insertion sort
14             for (int i = h; i < N; i++) {
15                 for (int j = i; j >= h && less(a[j],a[j-
   h]); j -= h)
16                     exch(a,j,j-h);
17             }
18
```

```
19                h = h/3; // move to next increment
20            }
21        }
22
23        private static boolean less(Comparable v, Comparable w)
    {
24            /* as before */
25        }
26
27        private static void exch(Comparable[] a, int i, int j)
    {
28            /* as before */
29        }
30    }
```

Visual Trace



Analysis

Proposition: the worst-case number of compares used by shellsort with the $3x + 1$ increments is $O(N^{\frac{3}{2}})$

Property:

Number of compares used by shellsort with the $3x + 1$ increments is at most by a small multiple of N times the number of increments used.

N	compares	$N^{1.289}$	$2.5N \lg N$

5,000	93	58	106
10,000	209	143	230
20,000	467	349	495
40,000	1022	855	1059
80,000	2266	2089	2257

> measured in thousands

> *Remark*:
>
> An accurate model has ***not yet*** been discovered

H4 Sample Question

Question:

*How many compares does shellsort (using $3x + 1$ increment sequence) make on an input array that is **already sorted**?*

Answer:

Linearithmic. Since successive increment values of h differ by at least a factor of 3, there are $\sim \log_3 N$ increment values. For each increment value h, the array is already h-sorted so it will make $\sim N$ compares (insertion sort).

H3 Why Are We Interested In Shellsort?

It is an exmaple of ***simple idea leading to substantial performance gains***.

It's useful in practice:

- Fast unless array size is huge (used for small subarrays: *bzip2* , */linux/kernel/groups.c*)
- Tiny, fixed footprint for code (used in some embedded systems: *uClibc*)
- Hardware sort prototype

It's a ***simple*** algorithm with yet ***non-trivial*** performance. That raises interesting questions:

- Asymptotic growth rate?
- Best sequence of increments?
- Average-case performace?

> *Lesson*:
>
> Some good algorithms are still waiting discovery

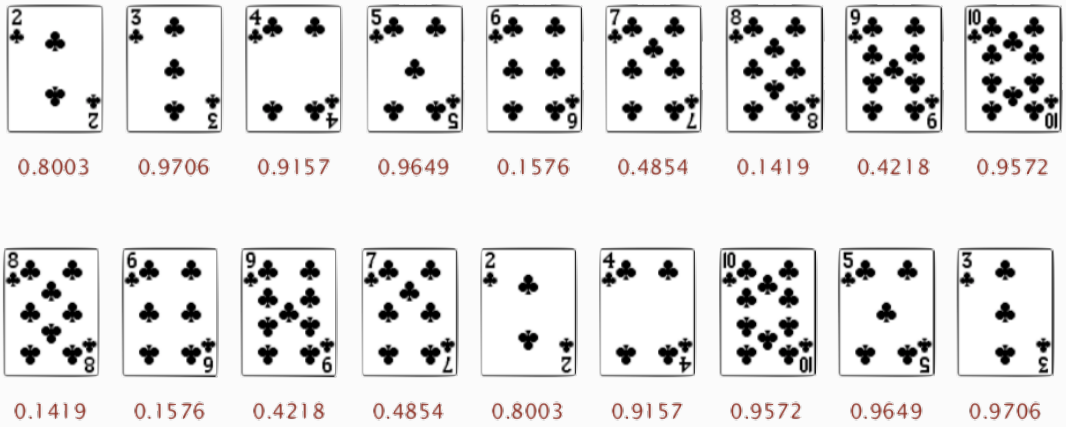
H2 Shuffling

Goal: rearrange array so that result is a ***uniformly random permutation***.

H3 Shuffle Sort

- generate a random real number for each array entry
- sort the array

> *Fun Fact*: 😲
>
> Useful for shuffling columns in a spreadsheet



Proposition: shuffle sort produces a uniformly random permutation (assuming real number uniformly at random) of the input array, provided no duplicate values

H3 War Stroy: Microsoft

Microsoft antitrust probe by EU. Microsoft then agreed to provide a ***randomised ballot*** screen for users to select browser in Windows 7:

Solution:

Implement shuffle sort by making comparator ***always return a random answer***.

```
1  public int compareTo(Browser that) {
2      double r = Math.random();
3      if (r < 0.5) return -1;
4      if (r > 0.5) return +1;
5      return 0
6  }
```

But *IE* appeared last ***50%*** of the time.

H3 Knuth Shuffle

- In iteration i, pick integer r between 0 and i uniformly at random.

- Swap `a[i]` and `a[r]`

H4 Java Implementation

```java
public class StdRandom {
    ...
    public static void shuffle(Object[] a) {
        int N = a.length;
        for (int i = 0; i < N; i++) {
            int r = StdRandom.uniform(i + 1);
            exch(a,i,r);
        }
    }
}
```

> *Note that*:
>
> - Common bug: pick between 0 and $N - 1$
> - Correct variant: pick between i and $N - 1$

Proposition: Knuth shuffling algorithm produces a uniformly random (assuming integers uniformly at random) permutation of the input array in linear time.

H3 War Story: Online Poker

How We Learned to Cheat at Online Poker: A Study in Software Security

H4 Algorithm with Bugs

```
FOR i := 1 TO 52 DO BEGIN
    r := RANDOM(51) + 1 // between 1 and 51
    swap := card[r]
    card[r] := card[i]
    card[i] := swap;
END
```

> *Bug #1*:
>
> Random number r never 52, so 52^{nd} card can't end up in 52^{nd} place

> *Bug #2*:
>
> Shuffle not uniform, it should be between 1 and i

> *Bug #3*:
>
> `RANDOM()` uses 32-bit seed, giving only 2^{32} possible shuffles

> *Bug #4*:
>
> The seed is the milliseconds since midnight, which gives only 86.4 million shuffles

Exploit:

After **seeing 5 cards** and **synchronising with server clock,** you can get all the future cards in real time in a programme.

> *"The generation of random number is too important to be left to chance."*
>
> [Rober R. Coveyou]

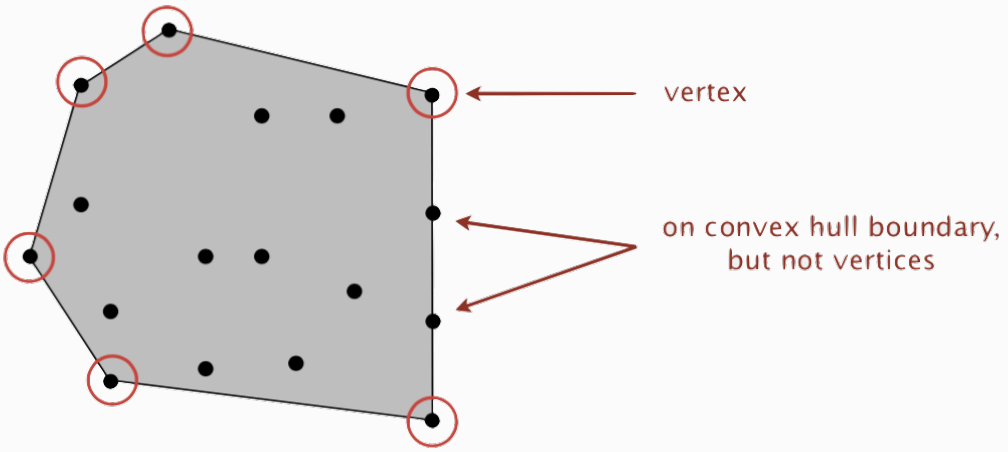
H3 Best Practice for Shuffling

- Use a *hardware* random-number generator that has passed both the FIPS 140-2 and the NIST statistical test suites
- Continuously monitor statistic properties: hardware random-number generators are fragile and fail silently
- Use an unbiased shufling algorithm

> *Lesson of the Day:* 💡
>
> Don't think that it's easy to shuffle a deck of cards

H2 Convex Hull

The ***convex hull*** of a set of N points is the smallest primeter fence enclosing the points



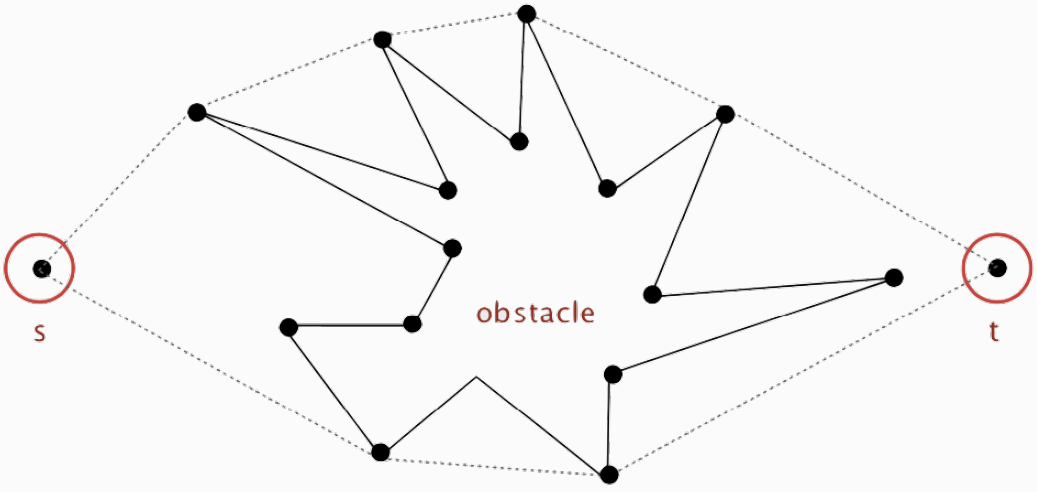
> *Equivalent Definitions*:
>
> - Smallest convex set containing all the points
> - Smallest area convex polygon enclosing the points
> - Convex polygon enclosing the points, whose vertices are points in set

Convex Hull Output: sequence of vertices in **counterclockwise** order

H3 Application

H4 Robot Motion Planning

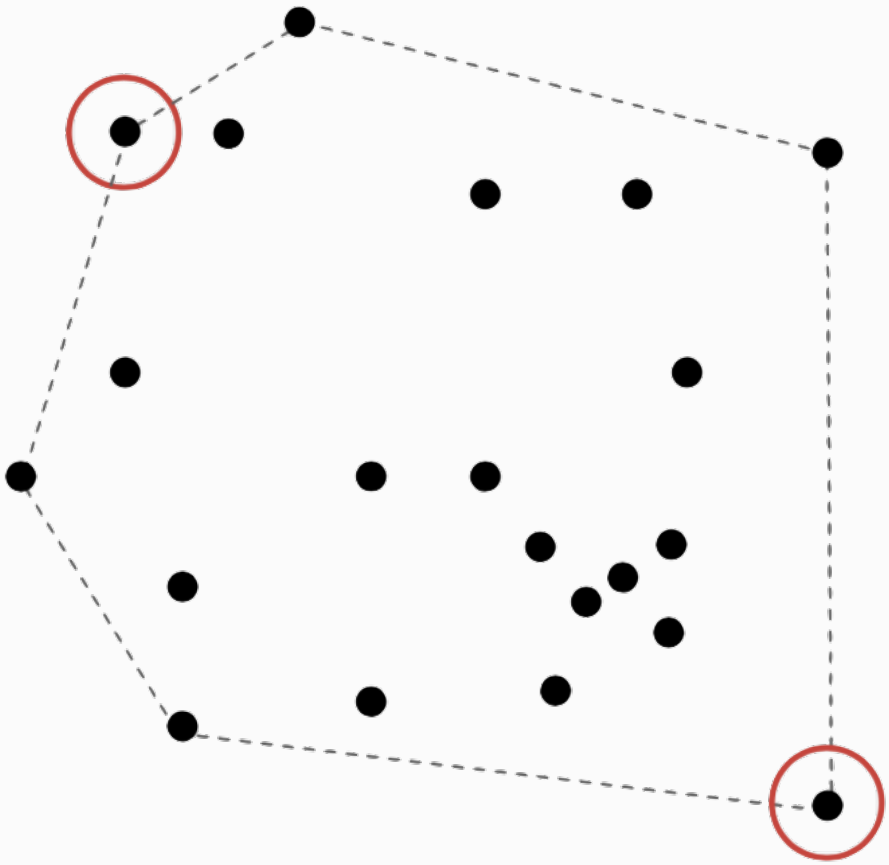
Find shortest path in the plane from s to t that avoids a polygonal obstacle



> *Fact*:
>
> Shortest path is either straight line from s to t or it is one of two polygonal chains of convex hull

H4 Furthest Pair

Given N pointers in the plane, find a pair of points with the largest Euclidean distance between them



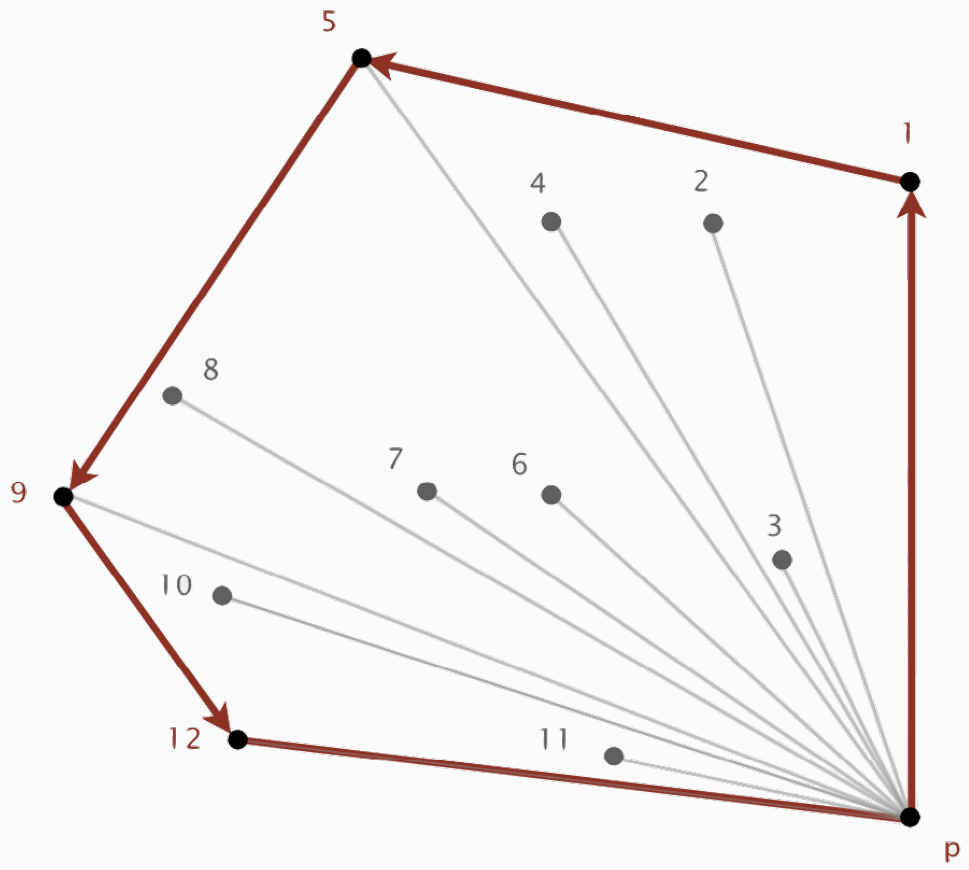
> *Fact*:

> Furthest pair of points are **extreme points** on convex hull

H3 Geometric Properties

> *Fact*:
>
> Can traverse the convex hull by making ***only*** counterclockwise turns

> *Fact*:
>
> The vertices of convex hull appear in increasing order of polar angle with respect to point p with lowest y-coordinate



H3 Gramham Scan

- choose point p with smallest y-coordinate
- *sort* points by polar angle with p
- consider points in order; discard unless it create a counterclockwise turn

H4 Implementation Challenges

Question: *How to find point p with smllest y-coordinate?*

Answer : Define a total order, comparing by y-coordinate.

Question : *How to sort points by polar angle with respect to p?*

Answer: Define a total order for each point p

Question : *How to determine whether $p_1 \rightarrow p_2 \rightarrow p_3$ is a counterclockwise turn?*

Answer : Computational Geometry

Question: *How to sort efficiently?*

Answer: Mergesort sorts in $N \log N$ times

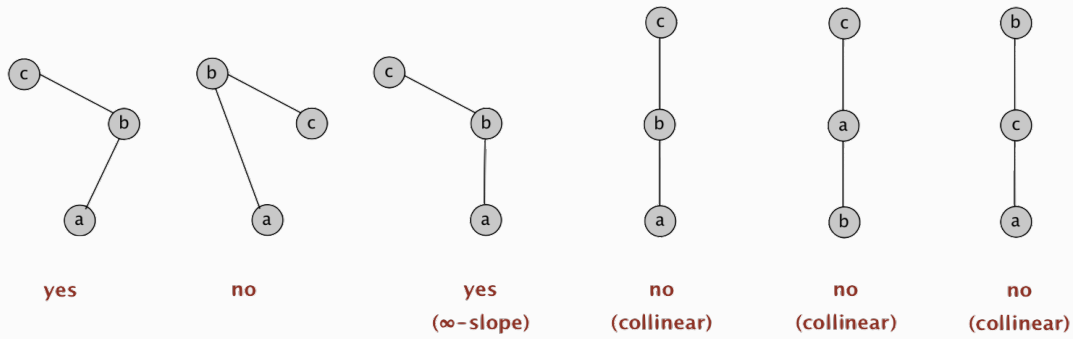
Question: *How to handle degeneracies (three or more points on a line)?*

Answer: Requires some care, but not hard

H4 Implementing *Counterclockwise*

Decoded Question:

In sequence $a \to b \to c$, is c to the ***left*** of the ray $a \to b$?



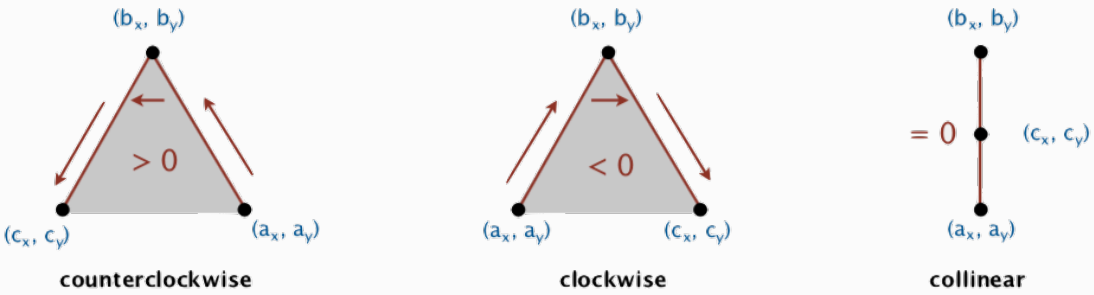
> Lesson:
>
> Geometric primitives are tricky to implement
>
> - Dealing with degenerate cases
> - Coping with floating-point precision

Linear Algebra Hint:

Determinant (or ***cross product***) gives $2 \times$ signed area of planar triangle

$$2 \times Area(a, b, c) = \begin{vmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ c_x & c_y & 1 \end{vmatrix} = (b_x - a_x)(c_y - a_y) - (b_y - a_y)(c_x - a_x)$$

- If signed area > 0, then $a \to b \to c$ is counterclockwise
- If signed area < 0, then $a \to b \to c$ is clockwise
- If signed area $= 0$, then $a \to b \to c$ are collinear



H3 *Immutable Point* Data Type

```
1  public class Point2D {
2      private final double x;
3      private final double y;
```

```
4
5       public Point2D(double x, double y) {
6           this.x = x;
7           this.y = y;
8       }
9
10      ...
11
12      public static int ccw(Point2D a, Point2D b, Point2D c)
    {
13          double area2 = (b.x - a.x)*(c.y - a.y) - (b.y -
    a.y)*(c.x - a.x);
14          if (area2 < 0) return -1; // clockwise
15          else if (area > 0) return 1; // counterclockwise
16          else return 0; // collinear
17      }
18  }
```