Lecture 6: Quicksort

Quicksort

- *Java* `sort` for primitive types
- *C* qsort, *Unix* , *Visual C++* , *Python* , *Matlab* , *Chrome JavaScript*

Animation

Basic Plan

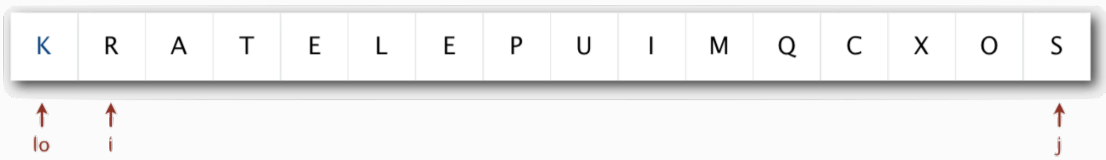
- ***shuffle*** the array

- *partition* so that, for some `j`:
    - entry `a[j]` is in place
    - no larger entry to the left of `j`
    - no smaller entry to the right of `j`
- *sort* each piece recursively

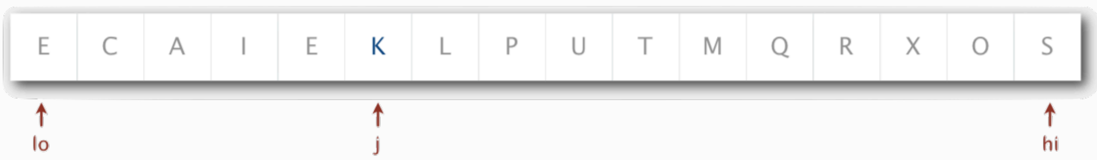
Partitioning

- maintain two pointers `i` and `j`
- repeat until `i` and `j` cross
    - scan `i` from left to right so long as `a[i] < a[lo]`
    - scan `j` from right to left so long as `a[j] > a[lo]`
    - exchange `a[i]` with `a[j]`
- when pointers cross, exchange `a[lo]` with `a[j]`

Before Partitioning



After Partitioning




```
private static int partition(Comparable[] a, int lo, int
hi) {
    int i = lo, j = hi+1;
    while (true) {
        /* find item on left to swap*/
        while (less(a[++i], a[lo]))
            if (i == hi) break;

```

```
            /* find item on right to swap*/
            while (less(a[lo], a[--j]))
                if (j == lo) break;

            if (i >= j) break; // check if pointers cross
            exch(a, i, j); // swap
        }

        exch(a, lo, j); // swap with partitioning item
        return j; // return index of item now known to be in place
    }
```

Java Implementation

```
public class Quick {
    private static int partition(Comparable[] a, int lo, int high) {
        /* see above */
    }

    public static void sort(Comparable[] a) {
        StdRandom.shuffle(a);
        sort(a, 0, a.length-1);
    }

    private static void sort(Comparable[] a, int lo, int hi) {
        if (hi <= lo) return;
        int j = partition(a, lo, hi);
        sort(a, lo, j-1);
        sort(a, j+1, hi);
    }
}
```

Implementation Details

> *Partitioning In-Place:*
>
> Using an extra array makes partitioning easier (and stable), but is not worth the cost

> *Terminating The Loop:*
>
> Testing whether the pointers cross is a bit trickier than it might seem

> ***Staying In Bounds:***
>
> The `(j == lo)` test is redundant, but the `(i == hi)` test is not

> ***Preserving Randomness:***
>
> Shuffling is needed for performance guarantee

> ***Equal Keys:***
>
> When duplicates are present, it is (*counter-intuitively*) better to stop on keys equal to the partitioning item's key

Trace

	lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
initial values				Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
random shuffle				K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
	0	5	15	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
	0	3	4	E	C	A	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	2	2	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	0	1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	1		1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	4		4	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	6	6	15	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	7	9	15	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	7	7	8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	8		8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	10	13	15	A	C	E	E	I	K	L	M	O	P	S	Q	R	T	U	X
	10	12	12	A	C	E	E	I	K	L	M	O	P	R	Q	S	T	U	X
	10	11	11	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	10		10	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	14	14	15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	15		15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result				A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

no partition for subarrays of size 1

Quicksort trace (array contents after each partition)

Running Time Analysis

Best Case

Number of compares is $\sim N \lg N$

lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
											a[ ]						
initial values			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
random shuffle			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
0	7	14	D	A	C	B	F	E	G	H	L	I	K	J	N	M	O
0	3	6	B	A	C	D	F	E	G	H	L	I	K	J	N	M	O
0	1	2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
0		0	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
2		2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
4	5	6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
4		4	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
6		6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
8	11	14	A	B	C	D	E	F	G	H	J	I	K	L	N	M	O
8	9	10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
8		8	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
10		10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
12	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12		12	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Worst Case

Number of compares is $\sim \frac{1}{2}N^2$

			a[ ]														
lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
random shuffle			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	0	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	1	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
2	2	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
3	3	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
4	4	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	5	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
6	6	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
7	7	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
8	8	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
9	9	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
10	10	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
11	11	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12	12	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
13	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Average Case

Proposition: The avarage number of comapres C_N to quicksort an array of N distinct keys is $\sim 2N \ln N$ (and the number of exchanges is $\sim \frac{1}{3} N \ln N$)

Proof:

C_N statisfies the recurrence $C_0 = C_1 = 0$ and for $N \geq 2$:

$$C_N = (N+1) + \frac{C_0 + C_{N-1}}{N} + \frac{C_1 + C_{N-2}}{N} + \cdots + \frac{C_{N-1} + C_0}{N}$$

> Note that:
>
> - $(N+1)$ is the partitioning
> - $C_0 \ldots C_{N-1}$ are the left subarrays
> - $C_{N-1} \ldots C_0$ are the right subarrays
> - determinator N is the partioning probability

Multiply both sides by N:

$$NC_N = N(N+1) + 2(C_0 + C_1 + \cdots + C_{N-1})$$

Subtract the same equation for $N - 1$

$$
\begin{aligned}
(N-1)C_{N-1} &= N(N-2) + 2(C_0 + C_1 + \cdots + C_{N-2}) \\
NC_N - (N-1)C_{N-1} &= N(N+1) - N(N-2) + 2C_{N_1} \\
&= 2N + 2C_{N-1}
\end{aligned}
$$

Rearrange terms and divide by $N(N+1)$

$$
\frac{C_N}{N+1} = \frac{C_{N-1}}{N} + \frac{2}{N+1}
$$

Repratedly apply above equation

$$
\begin{aligned}
\frac{C_N}{N+1} &= \frac{C_{N-1}}{N} + \frac{2}{N+1} \\
&= \frac{C_{N-2}}{N-1} + \frac{2}{N} + \frac{2}{N+1} \\
&= \frac{C_{N-3}}{N-2} + \frac{2}{N-1} + \frac{2}{N} + \frac{2}{N+1} \\
&= \frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \cdots + \frac{2}{N+1}
\end{aligned}
$$

Approximate sum by an integral

$$
\begin{aligned}
C_N &= 2(N+1)(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{N+1}) \\
&\sim 2(N+1)\int_3^{N+1} \frac{1}{x}dx \\
&= 2(N+1)\ln N \approx 1.39N\lg N
\end{aligned}
$$

Performance Characteristics

Worst Case: Number of comapres is quadratic

- $N + (N-1) + (N-2) + \cdots + 1 \sim \frac{1}{2}N^2$
- less likely

Average Case: Number of compares is $\sim 1.39N\lg N$

- 39% more compares than mergesort
- but faster than mergesort in practice because of less data movement

Random Shuffle

- Probabilistic guarantee against worst case
- basis for math model that can be validated with experiments

Caveat Emptor: Many textbook implementations go quadratic if array

- is sorted or reverse sorted
- has many duplicates (even if randomised)

Properties

In Place

Propostition: Quicksort is an in-place sorting algorithm

Proof:

- Partitioning: constant extra space
- Depth of recursion: logarithmic extra space (with high probability), can guarantee logarithmic depth by recurring on smaller subarray before larger subarray

Instability

Proposition: Quicksort is not stable

Proof:

i	j	0	1	2	3
		B_1	C_1	C_2	A_1
1	3	B_1	C_1	C_2	A_1
1	3	B_1	A_1	C_2	C_1
0	1	A_1	B_1	C_2	C_1

Practical Improvement

Insertion Sort Small Subarrays

- even quicksort has too much overhead for tiny subarrays
- cutoff to insertion sort for ≈ 10 items
- note: could delay insertion sort until one pass at end

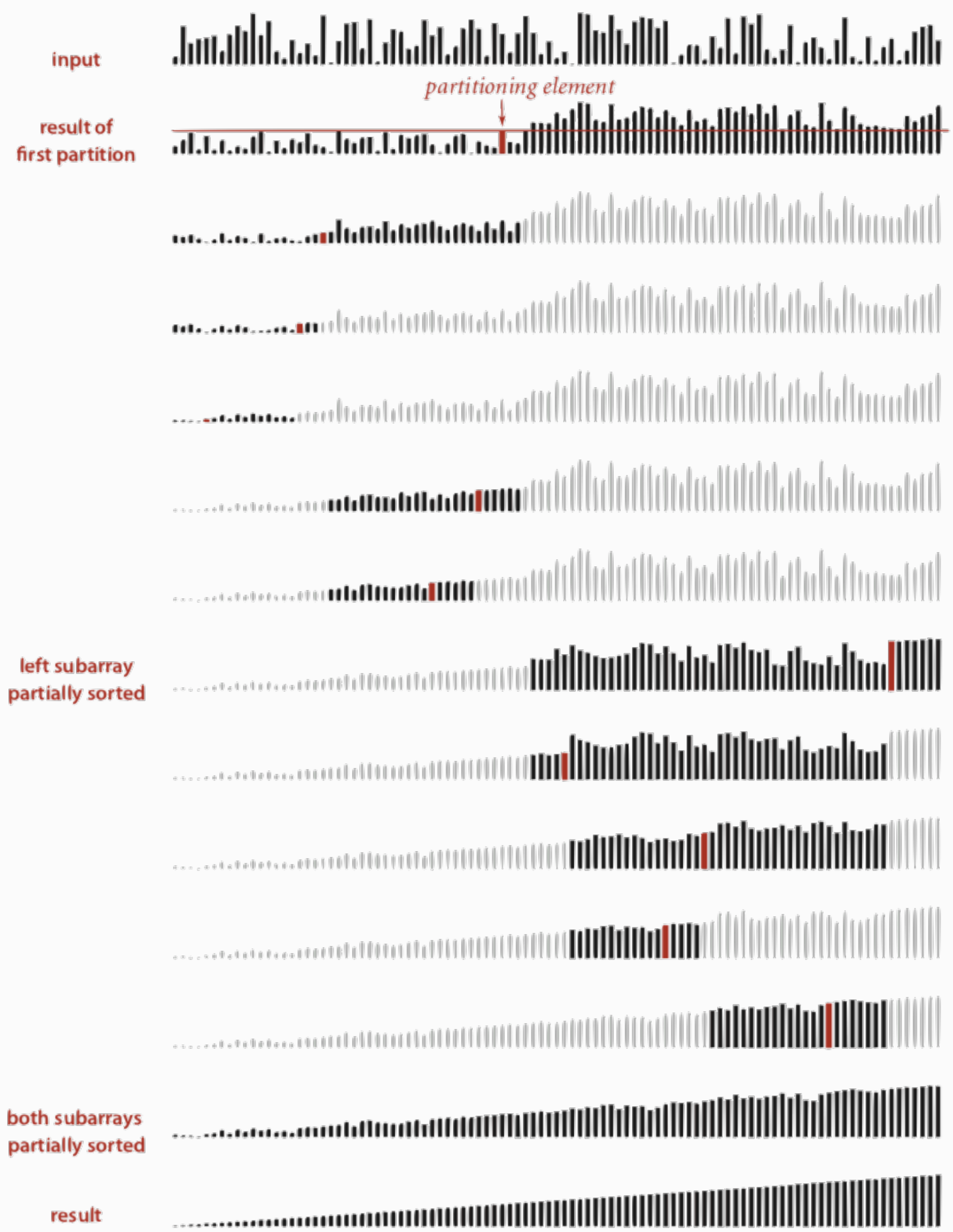
```java
private static void sort(Comparable[] a, int lo, int hi) {
    if (hi <= lo + CUTOFF - 1) {
        Insertion.sort(a, lo, hi);
        return;
    }
    int j = partition(a, lo, hi) {
        sort(a, lo, j-1);
        sort(a, j+1, hi);
    }
}
```

Median of Sample

- best choice of pivot item is median
- estimate true median by taking median of sample
- Median-of-3 (random) items

```java
private static void sort(Comparable[] a, int lo, int hi) {
    if (hi <= lo) return;

    int m = medianOf3(a, lo, lo + (hi - lo)/2, hi);
    swap(a, lo, m);

    int j = partition(a, lo, hi);
    sort(a, lo, j-1);
    sort(a, j+1, hi);
}
```

Visualisation



Selection

Goal: Given an array of N items, find a k^{th} smallest item

Theory:

- Easy $N \lg N$ uppoer bound, by sorting the array in the first place
- Easy kN upper bound for $k = 1, 2, 3, \dots,$ k pass entry to the array
- Easy N lower bound since all items have to be visited so that no item will be missed

Question: *is there a lieanr-time algorithm for each k?*

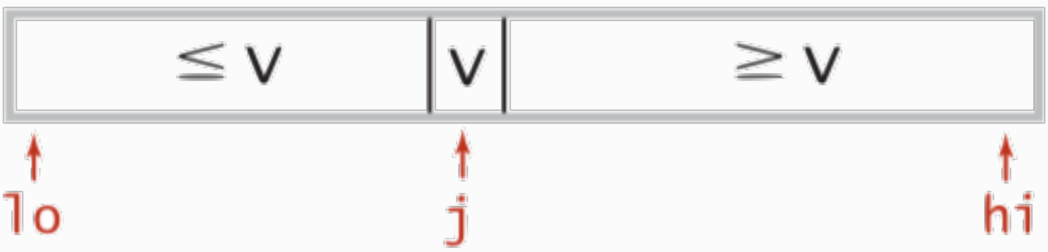
Quick-Select

Partition array so that:

- entry `a[j]` is in place
- no larger entry to the left of `j`
- no smaller entry to the right of `j`

Repeat in **one** subarray, depending on `j`, finished when `j` equals `k`

```
public static Comparable select(Comparable[] a, int k) {
    StdRandom.shuffle(a);
    int lo = 0, hi = a.length - 1;
    while (hi > lo) {
        int j = partition(a, lo, hi);
        if (j < k) lo = j + 1;
        else if (j > k) hi = j - 1;
        else return a[k];
    }
    return a[k];
}
```



Running Time Analysis

Proposition: Quick-select takes linear time on average

Proof:

- Intuitively, each partitioning step splits array approximately in half: $N + \frac{N}{2} + \frac{N}{4} + \cdots + 1 \sim 2N$ compares
- Formal analysis similar to quicksort analysis yields:

$$C_N = 2N + 2k\ln(\frac{N}{k}) + 2(N-k)\ln\frac{N}{N-k}$$

> $(2+2\ln 2)N$ to find the median

Remark:

Quick-select uses $\sim \frac{1}{2}N^2$ compares in the ***worst case***, but (as with quicksort) the random shuffle provides a probabilistic guarantee

> ***Compared-Based Selection***:
> A compared-based selection algorithm has worst-case running which is linear
> *[Blum, Floyd, Pratt, Rivest, Tarjan, 1973]*
>
> **Remark**: but constants are too high, so the algorithm is not used in practice

> ***Lessons***:
>
> - Still worthwhile to seek practical linear-time (worst-case) algorithm.
> - Until one is discovered, use quick-select if you don't need a full sort

Duplicate Keys

Mergesort with Duplicate Keys:

Between $\frac{1}{2}N\lg N$ and $N\lg N$ compares

Quicksort wth Duplicate Keys:

- algoirthm goes quadratic unless partitioning stops on equal keys
- 1990s C user found this defect in `qsort()`

Problem

> ***Mistake***: put all items equal to the partitioining item on one side
>
> ***Consequence***: $\sim \frac{1}{2}N^2$ compares when all keys equal

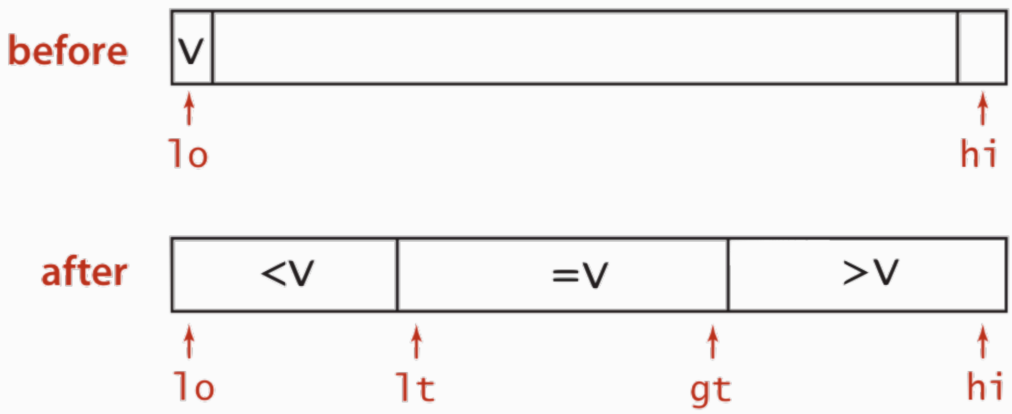
> ***Recommanded***: stop scans on items equal to the partitioning item
>
> ***Consequences***: $\sim N\lg N$ compares when all keys equal

> ***Desirable***: put all items equal to the partitioning item in place

3-Way Partitioning

Goal: partition array into 3 parts so that:

- entries between `lt` and `gt` equal to partition item `v`
- no larger entries to left of `lt`
- no smaller entries to right of `gt`

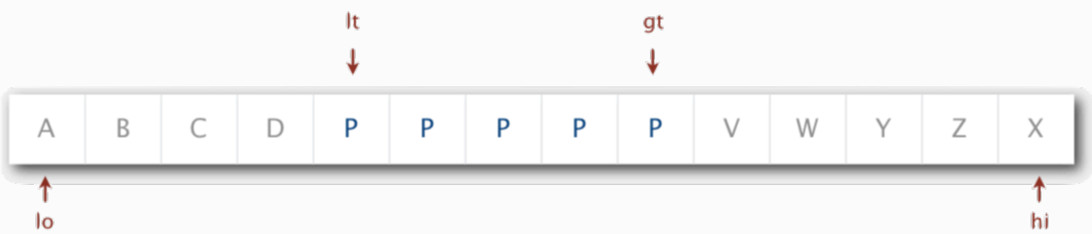


Dijkstra 3-Way Partitioning

- let `v` be partitioning item `a[lo]`
- scan `i` from left to right
  - `(a[i] < v)` : exchange `a[lt]` with `a[i]` ; increment both `lt` and `i`
  - `(a[i] > v)` : exchange `a[gt]` with `a[i]` ; decrement `gt`
  - `(a[i] == v)` : increment `i`

Before:

After:



Java Implementation

```java
private static void sort(Comparable[] a, int lo, int hi) {
    if (hi <= lo) return;
    int lt = lo, gt  = hi;
    Comparable v = a[lo];
    int i = lo;
    while (i <= gt) {
        int cmp = a[i].compareTo(v);
        if (cmp < 0) exch(a, lt++, i++);
        else if (cmp > 0) exch(a, i, gt--);
        else i++;
```

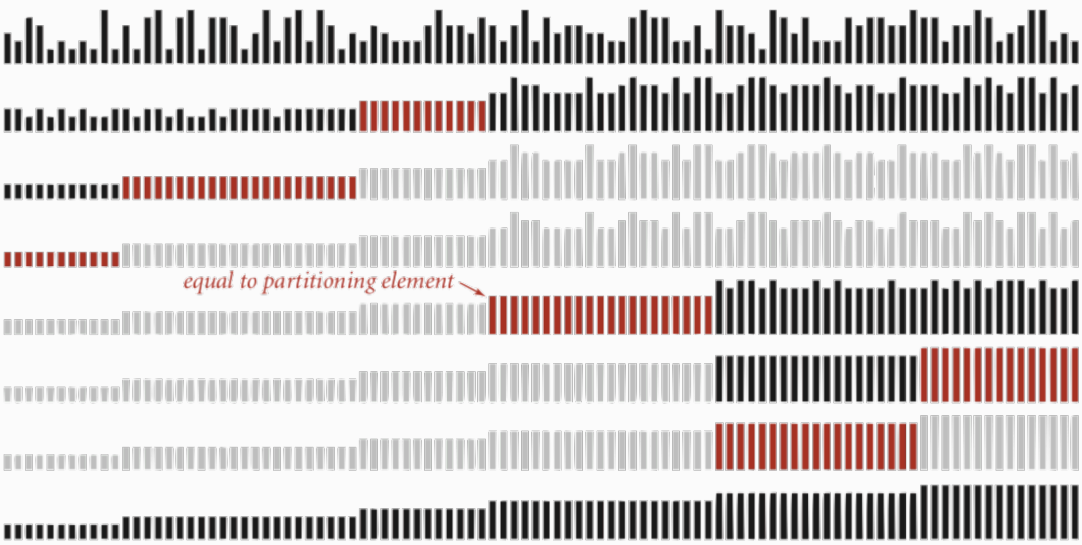
```
    }

    sort(a, lo, lt-1);
    sort(a, gt+1, hi);
}
```

Trace

lt	i	gt	0	1	2	3	4	5	6	7	8	9	10	11
0	0	11	R	B	W	W	R	W	B	R	R	W	B	R
0	1	11	R	B	W	W	R	W	B	R	R	W	B	R
1	2	11	B	R	W	W	R	W	B	R	R	W	B	R
1	2	10	B	R	R	W	R	W	B	R	R	W	B	W
1	3	10	B	R	R	W	R	W	B	R	R	W	B	W
1	3	9	B	R	R	B	R	W	B	R	R	W	W	W
2	4	9	B	B	R	R	R	W	B	R	R	W	W	W
2	5	9	B	B	R	R	R	W	B	R	R	W	W	W
2	5	8	B	B	R	R	R	W	B	R	R	W	W	W
2	5	7	B	B	R	R	R	R	B	R	W	W	W	W
2	6	7	B	B	R	R	R	R	B	R	W	W	W	W
3	7	7	B	B	B	R	R	R	R	R	W	W	W	W
3	8	7	B	B	B	R	R	R	R	R	W	W	W	W
3	8	7	B	B	B	R	R	R	R	R	W	W	W	W

3-way partitioning trace (array contents after each loop iteration)

Visualisation



Lower Bound

If there are n distinct keys and the i^{th} one occurs x_i times, any compare-based sorting algorithm must use at least

$$\lg(\frac{N!}{x_1!x_2!\ldots x_n!}) \sim -\sum_{i=1}^{n} x_i \lg \frac{x_i}{N}$$

compares in the worst case

> - $N \lg N$ when all distinct ($x_i = 1$)
> - linear when only a constant number of distinct keys

Proposition: Quicksort with 3-way partitioning is ***entropy-optimal***

> *Entropy-Optimal*:
>
> Propotional to lower bound

Buttom Line:

Randomised quicksort with 3-way partitioning reduces running time from ***linearithmic*** to ***linear*** in broad class of applications

Sorting Summary

algorithms	in-place?	stable>	worst	average	best	remark
selection	✔		$\frac{N^2}{2}$	$\frac{N^2}{2}$	$\frac{N^2}{2}$	N exchanges
insertion	✔	✔	$\frac{N^2}{2}$	$\frac{N^2}{4}$	N	use for small N or partially order
shell	✔		?	?	N	tight code, subquadratic
merge		✔	$N \lg N$	$N \lg N$	$N \lg N$	$N \log N$ guarantee, stable
quick	✔		$\frac{N^2}{2}$	$2N \ln N$	$N \lg N$	$N \log N$ probabilistic guarantee, fastest in practice
3-way quick	✔		$\frac{N^2}{2}$	$2N \ln N$	N	improves quicksort in presence of duplicate keys
???	✔	✔	$N \lg N$	$N \lg N$	N	holy sorting grail 🏆