Lecture 5: Mergesort

Top-Down Mergesort

- *Java* sort for *objects*
- *Perl* , *C++* stable sort, *Python* stable sort, *Firefox JavaScript* , ...

Basic Plan

- Divide array into two halves
- *Recursively* sort each half
- Merge two halves

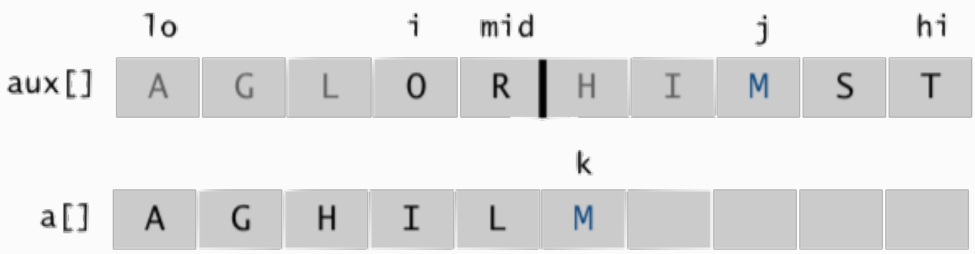
Animations

input	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sort left half	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
sort right half	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge results	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Mergesort overview

Abstract In-Place Merge

Goal: given two sorted subarrays `a[lo]` to `a[mid]` and `a[mid+1]` to `a[hi]`, replace with sorted subarray `a[lo]` to `a[hi]`.



- make a copy of `a[]` to an ***auxiliary array*** `aux[]`
- maintain 3 indices:
  - `i` as the entry for the left subarray of `aux[]`
  - `j` as the entry for the right subarray of `aux[]`
  - `k` as the entry for `a[]`
- compare `aux[i]` with `aux[j]`, put whichever that is smaller to `a[k]`
- if `aux[i]` equals `aux[j]`, put `aux[i]` to `a[k]`
- if any subarray eliminated, put the remainder of the other subarray to the remainder of `a[]`
- increment whichever index that points at the moved element, and `k`

```java
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi) {
    assert isSorted(a,lo,mid); // precondition: a[lo..mid] sorted
    assert isSorted(a,mid+1,hi); // precondition: a[mid+1..hi] sorted

    // copy
    for (int k = lo; k <= hi; k++) {
        aux[k] = a[k];
    }

    // merge
    int i = lo, j = mid+1;
    for (int k = lo; k <= hi; k++) {
        if (i > mid) a[k] = aux[j++];
        else if (j > hi) a[k] = aux[i++];
        else if (less(aux[j],aux[i])) a[k] = aux[j++];
        else a[k] = aux[i++];
    }

    assert isSorted(a,lo,hi); // postcondition: a[lo..hi] sorted

}
```

> *Assertion*:
>
> Statement to test assumptions about your program.
>
> - helps detect logic bugs
> - documents code
>
> ***Java `assert` Statement:***
>
> Throws *Exception* unless boolean condition is `true`
>
> ```java
> assert isSorted(a,lo,hi);
> ```
>
> ***Runtime:***
>
> ```
> java -ea MyProgram # enable assertions
> java -da MyProgram # disable assertions (default)
> ```
>
> So no cost in production code.
>
> ***Best Practices***:
>
> - use assertions to check internal invariants
> - assume assertions will be disabled in production code - do not use for external argument checking

Java Implementation

```
public class Merge {
    private static void merge(...) {
        /* as before */
    }

    private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
        if (hi <= lo) return; // base case
        int mid = lo + (hi - lo) / 2;
        sort(a,aux,lo,mid); // recursive case: sort the first half
        sort(a,aux,mid+1,hi); // recursive case: sort the other half
        merge(a,aux,lo,mid,hi);
    }

    // interface
    public static void sort(Comparable[] a) {
        aux = new Comparable[a.length];
        sort(a,aux,0,a.length-1);
    }
}
```

Trace

Running Time Analysis

Proposition: mergesort uses at most $N \lg N$ compares and $6N \lg N$ array accesses to sort any array of size N

Proof Sketch:

The number of compares $C(N)$ and array accesses $A(N)$ to mergesort an array of size N satisfy the recurrences:

$$C(N) \leq C([\frac{N}{2}]) + C([\frac{N}{2}]) + N$$

for $N > 1$, wirh $C(1) = 0$

> ***Note that***:
>
> The N item in the polynomial is the maximal number of compares to merge, happens when neither of the two halves exhaust.

$$A(N) \leq A([\frac{N}{2}]) + A([\frac{N}{2}]) + 6N$$

for $N > 1$, wirh $C(1) = 0$

> ***Note that***:
>
> The $6N$ item in the polynomial is the maximal number of array accesses to merge.

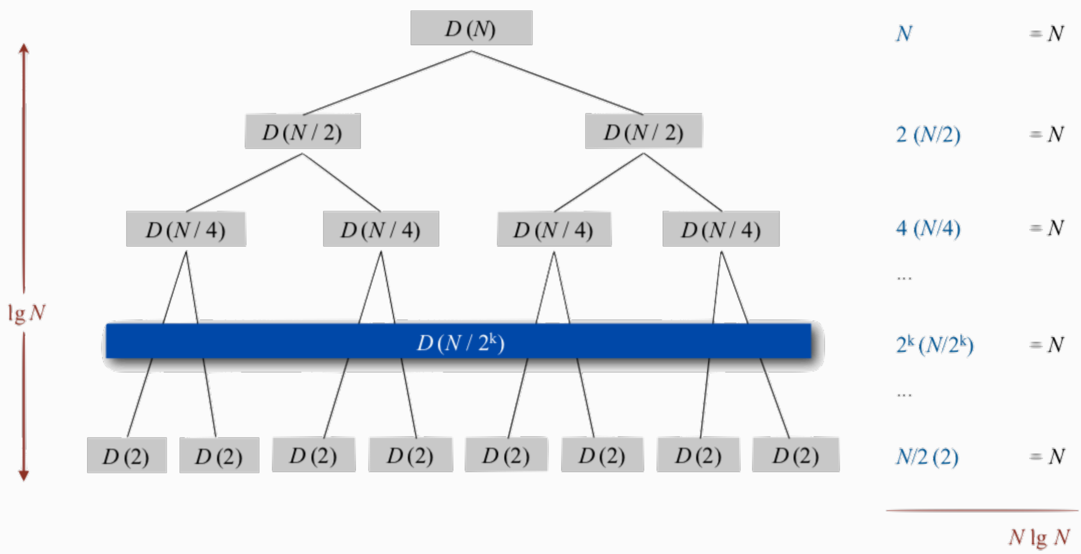
Solve the recurrence when N is a power of 2. The result holds for all N

$$D(N) = 2D(\frac{N}{2}) + N$$

for $N > 1$, with $D(1) = 0$

Proposition: if $D(N)$ satisfies $D(N) = 2D(\frac{N}{2}) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

Proof By Picture



The ***depth*** is the *number of times you divide N to take it down to 2., which is $\lg N$.

> Assuming N is a power of 2

Proof By Expansion

$$
\begin{aligned}
D(N) &= 2D(\frac{N}{2}) + N \\
\frac{D(N)}{N} &= \frac{2D(\frac{N}{2})}{N} + 1 \\
&= \frac{D(\frac{N}{2})}{\frac{N}{2}} + 1 \\
&= \frac{D(\frac{N}{4})}{\frac{N}{4}} + 1 + 1 \\
&= \frac{D(\frac{N}{8})}{\frac{N}{8}} + 1 + 1 + 1 \\
&\dots \\
&= \frac{D(\frac{N}{N})}{\frac{N}{N}} + 1 + 1 + \dots + 1 \\
&= \lg N
\end{aligned}
$$

Proof By Induction

- *Base case* : $N = 1$
- *Inductive hypothesis* : $D(N) = N \lg N$
- *Goal* : show that $D(2N) = (2N) \lg(2N)$

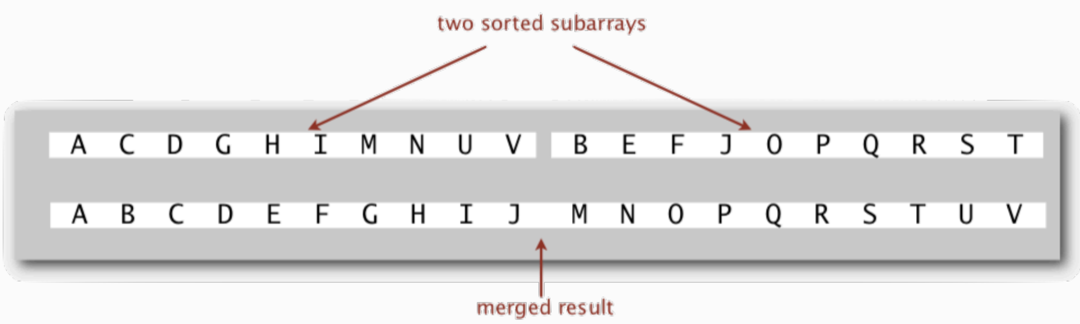
$$
\begin{aligned}
D(2N) &= 2D(N) + 2N \\
&= 2N \lg N + 2N \\
&= 2N(\lg(2N) - 1) + 2N \\
&= (2N) \lg(2N)
\end{aligned}
$$

Memory Analysis

Proposition: Mergesort uses extra space proportional to N

Proof:

The array `aux[]` needs to be of size N for the last merge



> ***In-Place***:
>
> A sorting algorithm is ***in-place*** if it uses $\le c \log N$ extra memory.
>
> **Examples**: insertion sort, selection sort, shellsort

> ***Fun Fact***:
>
> There are methods for in-place merge but they are relatively too complex to be used in practice. But it's possible that there is a perfect solution out there waiting for dicovery.

Practical Improvements

Small Subarrays

Mergesort has too much overhead for tiny subarrays. The recursive nature means that there are going to be lots of subarrays to be sorted.

Solution: ***Cutoff*** to insertion sort for subarrays with ≈ 7 items

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    // test for cutoff
    if (hi <= lo + CUTOFF - 1) {
        Insertion.sort(a,lo,hi);
        return
    }

    // original mergesort
    int mid = lo + (hi - lo) / 2;
    sort(a,aux,lo,mid);
    sort(a,aux,mid+1,hi);
    merge(a,aux,lo,mid,hi);
}
```

Sorted Array

Stop if the array is already sorted.

Test for if the array is sorted or not:

- Is the biggest item in the first half $\leq$ smallest item in the second half?
- helps for partially-ordered arrays


```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo) return;
    int mid = lo + (hi - lo) / 2;
    sort(a,aux,lo,mid);
    sort(a,aux,mid+1,hi);
    if (!less(a[mid+1],a[mid])) return; // check
    merge(a,aux,lo,mid,hi);
}
```

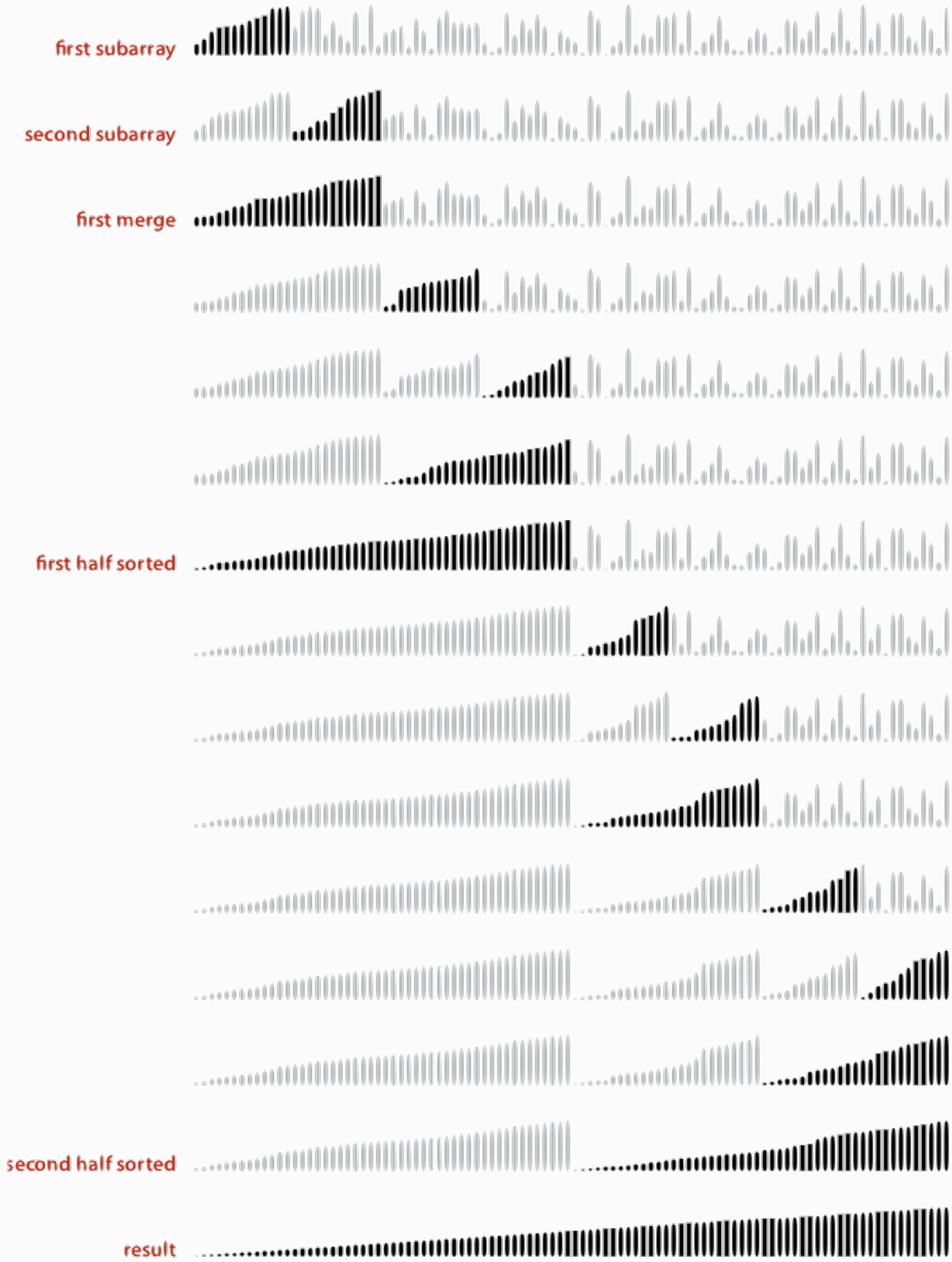
Auxiliary Array

Eliminate the ***copy*** to the auxiliary array, which saves time but not space, by switching the role of the input and auxiliary array in each recursive call. *(Sort an array and put the result in the other)*

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi) {
    int i = lo, j = mid + 1;
    for (int k = lo; k <= hi; k++) {
        // merge from a[] to aux[]
        if (j > mid) aux[k] = a[j++];
        else if (j > hi) aux[k] = a[i++];
        else if (less(a[j],a[i])) aux[k] = a[j++];
        else aux[k] = a[i++];
    }
}

private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo) return;
    int mid = lo + (hi - lo) / 2;
    // switch roles of aux[] and a[]
    sort(aux,a,lo,mid);
    sort(aux,a,mid+1,hi);
    merge(a,aux,lo,mid,hi);
    /* Note that:
     * sort(a) initialises aux[] ans sets aux[i] = a[i] for each i
     */
}
```

Visualisation



Buttom-Up Mergesort

Basic Plan

- Pass through array, merging subarrays of size 1
- Repeat for subarrays of size $2, 4, 8, 16, \dots$

	a[i]															
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sz = 1																
merge(a, aux, 0, 0, 1)	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 2, 2, 3)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 4, 4, 5)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 6, 6, 7)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 8, 8, 9)	E	M	G	R	E	S	O	R	E	T	X	A	M	P	L	E
merge(a, aux, 10, 10, 11)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
merge(a, aux, 12, 12, 13)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
merge(a, aux, 14, 14, 15)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	E	L
sz = 2																
merge(a, aux, 0, 1, 3)	E	G	M	R	E	S	O	R	E	T	A	X	M	P	E	L
merge(a, aux, 4, 5, 7)	E	G	M	R	E	O	R	S	E	T	A	X	M	P	E	L
merge(a, aux, 8, 9, 11)	E	G	M	R	E	O	R	S	A	E	T	X	M	P	E	L
merge(a, aux, 12, 13, 15)	E	G	M	R	E	O	R	S	A	E	T	X	E	L	M	P
sz = 4																
merge(a, aux, 0, 3, 7)	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
merge(a, aux, 8, 11, 15)	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
sz = 8																
merge(a, aux, 0, 7, 15)	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

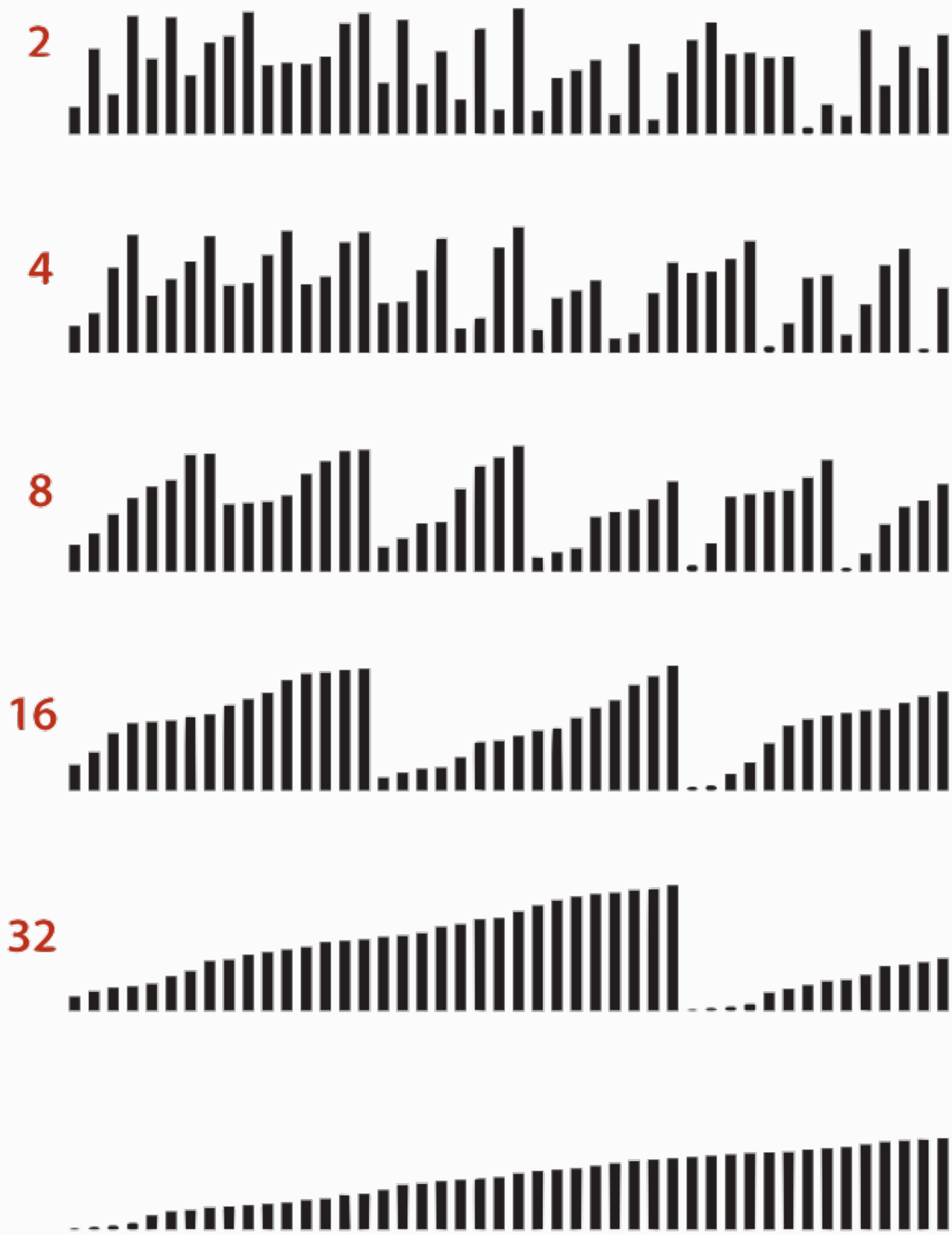
Java Implementation

```java
public class MergeBU {
    private static void merge(...) {
        /* as before */
    }

    public static void sort(Comparable[] a) {
        int N = a.length;
        Comparable[] aux = new Comparable[N];
        for (int sz = 1; sz < N; sz = sz+sz)
            for (int lo = 0; lo < N-sz; lo += sz+sz)
                merge(a,aux,lo,lo+sz-1,Math.min(lo+sz+sz-1,N-1));
                /* Math.min() is for the case when
                 * a remainder of a[] can't be covered by sz
                 */
    }
}
```

> *Note that*:
>
> It's a simple and non-recursive version of mergesort. But about 10% slower than recursive, top-down mergesort on typical systems

Trace



Sorting Complexity

For a sorting problem:

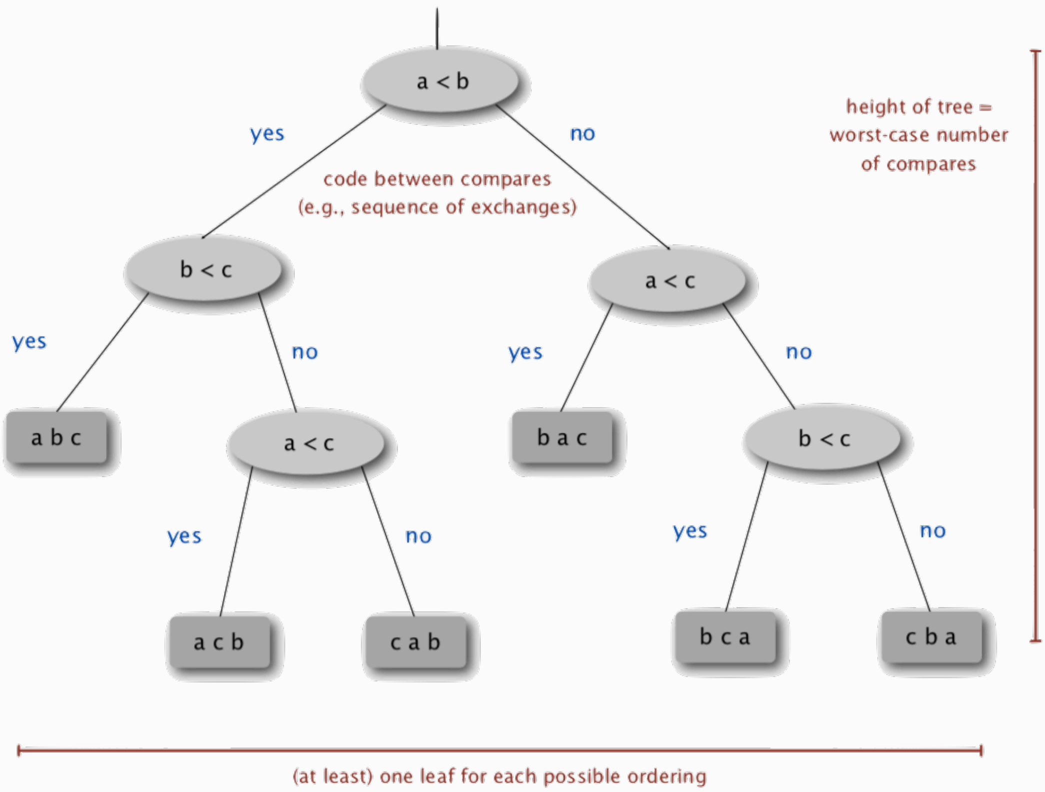
Parameters	Argument
Model of Computation	Decision tree - *can access information only through compares (e.g. Java* `Comparable` *framework)*
Cost Model	Number of compares
Upper Bound	$\sim N \lg N$ from **mergesort**

Lower Bound	?
Optimal Algorithm	?

> ***Computational Complexity***:
>
> ***Framework*** to study ***efficiency*** of algorithms for solving a particular problem X.
>
> | Parameters | Meaning |
> |---|---|
> | Model of Computation | Allowable **operations** |
> | Cost Model | Operation count(s) |
> | Upper Bound | Cost guarantee provided by **some** algorithms of X |
> | Lower Bound | Proven limit on cost guarantee of **all** algorithms of X |
> | Optimal Algorithm | Algorithm with best (lower bound ~ upper bound) possible cost guarantee for X |

Decision Tree

For 3 distinct item a, b, c



Compared-Based Lower Bound

Proposition: Any compare-based sorting algorithms must use at least $\lg(N!) \sim N \lg N$ compares in the worst-case.

Proof:

- assume array consists of N distinct values a_1 through a_N .
- worst case dictated by height h of decision tree
- binary tree of height h has at most 2^h leaves
- $N!$ different ordering so at least $N!$ leaves

By using ***Stirling's*** formula

$$2^h \geq \#leaves \geq N! \rightarrow h \geq \lg(N!) \sim N \lg N$$

So the sorting complexity:

Parameters	Argument
Model of Computation	Decision tree - *can access information only through compares (e.g. Java* `Comparable` *framework)*
Cost Model	Number of compares
Upper Bound	$\sim N \lg N$ from **mergesort**
Lower Bound	$\sim N \lg N$
Optimal Algorithm	***mergesort***

Complexity Result In Real Context

Mergesort is ***optimal*** with respect to numer of compares but it is ***not optimal*** with respect to space usage.

> *Lesson*:
>
> Use theory as a guide
>
> **Example**: don't try to design sorting algorithm that guarantees $\frac{1}{2}N \lg N$ compares, since the lower bound says no.
>
> **Example**: it might be possible to design a sorting algorithm with $\sim N \lg N$ compares and optimal space usage.

Limitation on Lower Bound Analysis

Lower bound may not hold if the algorithm has information about:

- the initial order of the input
- the distribution of key values
- the representation of the keys

Partially-Ordered Arrays

Depending on the ***initial order*** of the input, we might not need $N \lg N$ compares as insertion sort only requires $N - 1$ compares if input array is sorted.

Duplicate Keys

Depending on the input distribution of dupliactes, we may not need $N \lg N$ compares

(Stay tuned for *3-Way Quicksort*)

Digital Properties of Keys

We can use digit/charater compares instead of key compares for numbers and strings. (Stay tuned for *Radix Sorts*)

Comparators

A compelling reason why use `Comparator` *interface* rather than `Comparable` *interface* is that `Comparator` supports multiple ordering of a given data type.

`Comparator` *Interface*

Sort using an alternative order.

```
public interface Comparator<Item> {
    public int compare(Key v, Key w)
}
```

Required property: must be a total order

Ex. Sort strings by:

- Natural order. `Now is the time`
- Case insensitive. `is Now the time`
- Spanish. `café cafetero cuarto churro nube ñoño` (pre-1994 order for digraphs ch and ll and rr)
- British phone book. `McKinley Mackintosh`
- . . .

System Sort

To use with *Java* system `sort()`:

- create `Comparator` *object*
- pass as second argument to `Arrays.sort()`

```
String[] a;
...
Arrays.sort(a); // uses natural order
...
// uses alternate order defined by Comparator<String> object
Arrays.sort(a, String.CASE_INSENSITIVE_ORDER);
...
Arrays.sort(a, Collator.getInstance(new Locale("es")));
...
Arrays.sort(a, new BritishPhoneBookOrder())
```

> *Note that*:
>
> `Comparator` decouples the definition of the data type from the definition of what it means to compare two objects of that type.

Customised Sorting

To support `Comparator` in our sort implementation

- use `Object` instead of `Comparable`
- pass `Comparator` to `sort()` and `less()` and use it in `less()`

Insertion Sort using `Comparator`: Java Implementation

```java
public static void sort(Object[] a, Comparator comparator) {
    int N = a.length;
    for (int i = 0; i < N; i++) {
        for (int j = i; j > 0 && less(comparator, a[j], a[j-1]); j--)
            exch(a,j,j-1);
    }
}

private static boolean less(Comparator c, Object v, Object w) {
    return c.compare(v,w) < 0;
}

private static void exch(Object[] a, int i, int j) {
    Object swap = a[i];
    a[i] = a[j];
    a[j] = swap;
}
```

Implementing `Comparator` *Interface*

To implement a `Comparator`

- define a (nested) class that implements the `Comparator` *interface*
- implement the `compare()` method

```java
public class Student {
    public static final Comparator<Student> BY_NAME = new ByName();
    public static final Comparator<Student> BY_SECTION = new BySection();

    ...
    private static class ByName implements Comparator<Student> {
        public int compare(Student v, Student w) {
            return v.name.compareTo(w.name);
        }
    }
```

```
    private static class BySection implements Comparatoe<Student> {
        public int compare(Student v, Student w) {
            return v.section - w.section;
            // this technique works here since no danger of overflow
        }
    }
}
```

`Arrays.sort(a, Student.BY_NAME);`

Andrews	3	A	664-480-0023	097 Little
Battle	4	C	874-088-1212	121 Whitman
Chen	3	A	991-878-4944	308 Blair
Fox	3	A	884-232-5341	11 Dickinson
Furia	1	A	766-093-9873	101 Brown
Gazsi	4	B	766-093-9873	101 Brown
Kanaga	3	B	898-122-9643	22 Brown
Rohde	2	A	232-343-5555	343 Forbes

`Arrays.sort(a, Student.BY_SECTION);`

Furia	1	A	766-093-9873	101 Brown
Rohde	2	A	232-343-5555	343 Forbes
Andrews	3	A	664-480-0023	097 Little
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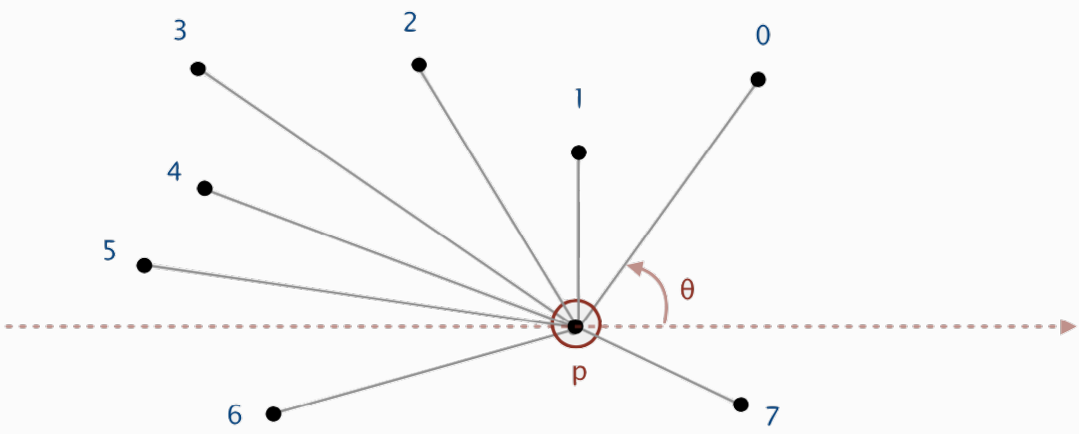
Kanaga	3	B	898-122-9643	22 Brown
Battle	4	C	874-088-1212	121 Whitman
Gazsi	4	B	766-093-9873	101 Brown

Polar Order

Given a point p, order points by polar angle they make with p



`Arrays.sort(points, p.POLAR_ORDER);`

Application: ***Graham Scan*** algorithm for convex hull

Trigonometric Solution

Compute polar angle θ with respect to p using `atan2()`

Drawback: evaluating a trigonometric function is expensive

Counterclockwise-Based Solution

- if q_1 is above p and q_2 is below p, then q_1 makes smaller polar angle
- if q_1 is below p and q_2 is above p, then q_1 makes larger polar angle

- otherwise, $ccw(p, q_1, q_2)$ identifies which of q_1 or q_2 makes larger angle

```java
public class Point2D {
    public final Comparator<Point2D> POLAR_ORDER = new PolarOrder();
    private final double x,y;
    ...
    private static int ccw(Point2D a, Point2D b, Point2D c) {
        /* as in previous lecture */
    }

    private class PolarOrder implements Comparator<Pointer2D> {
        public int compare(Point2D q1, Point2D q2) {
            double dy1 = q1.y - y;
            double dy2 = q2.y - y;

            if (dy1 == 0 && dy2 == 0) { ... } // p,q1,q2 horizontal
            else if (dy1 >= 0 && dy2 < 0) return -1; // q1 above p; q2 below p
            else if (dy2 >= 0 && dy 1 <0) return +1; // q1 below p; q2 above p
            else return -ccw(Point2D.this, q1, q2);
            // Point2D.this accesses invoking point from within inner class
        }
    }
}
```

Stability

Sorting the students by *name* and then by *section*

`Selection.sort(a, Student.BY_NAME);`

Andrews	3	A	664-480-0023	097 Little
Battle	4	C	874-088-1212	121 Whitman
Chen	3	A	991-878-4944	308 Blair
Fox	3	A	884-232-5341	11 Dickinson
Furia	1	A	766-093-9873	101 Brown
Gazsi	4	B	766-093-9873	101 Brown
Kanaga	3	B	898-122-9643	22 Brown
Rohde	2	A	232-343-5555	343 Forbes

`Selection.sort(a, Student.BY_SECTION);`

Furia	1	A	766-093-9873	101 Brown
Rohde	2	A	232-343-5555	343 Forbes
Chen	3	A	991-878-4944	308 Blair
Fox	3	A	884-232-5341	11 Dickinson
Andrews	3	A	664-480-0023	097 Little
Kanaga	3	B	898-122-9643	22 Brown
Gazsi	4	B	766-093-9873	101 Brown
Battle	4	C	874-088-1212	121 Whitman

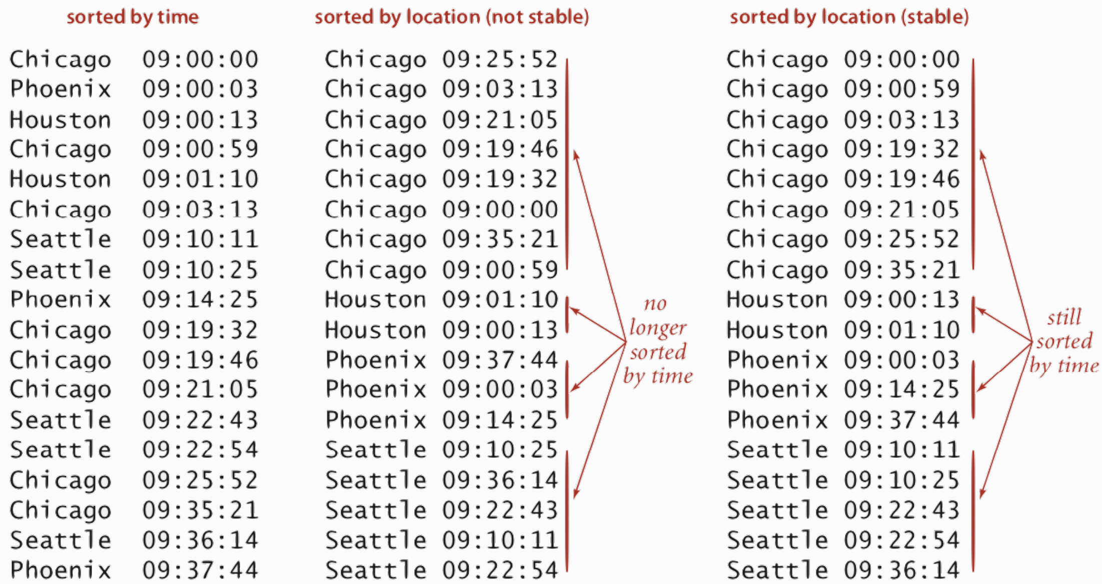
Problem: students in *section* 3 no longer sorted by *name*

> *Stability*:

> A ***stable*** sort preserves the relative order of items with equal keys

Question: which sorts are stable?

Answer: insertion sort and mergesort



> ***Note that***:
>
> Need to carefully check code ("less than" vs "less than or equal to")

Stability of Insertion Sort

Proposition: Insertion sort is **stable**

```
public class Insetion {
    public static void sort(Comparable[] a) {
        int N = a.length;
        for (int i = 0; i < N; i++)
            for (int j = i; j > 0 && less(a[j], a[j-1]); j--)
                exch(a,j,j-1)
    }
}
```

Proof:

Equal items never move past each other

i	j	0	1	2	3	4
0	0	B_1	A_1	A_2	A_3	B_2
1	0	A_1	B_1	A_2	A_3	B_2
2	1	A_1	A_2	B_1	A_3	B_2
3	2	A_1	A_2	A_3	B_1	B_2
4	4	A_1	A_2	A_3	B_1	B_2
		A_1	A_2	A_3	B_1	B_2

Stability of Selection Sort

Proposition: Selection sort is **not stable**

```java
public class Selection {
    public static void sort(Comparable[] a) {
        int N = a.length;
        for (int i = 0; i < N; i++) {
            int min = i;
            for (int j = i+1; j < N; j++)
                if (less(a[j], a[min]))
                    min = j;
            exch(a, i, min);
        }
    }
}
```

Disproof by Counter Example

Long-distance exchange might move an item past some equal item

i	min	0	1	2
0	2	B_1	B_2	A
1	1	A	B_2	B_1
2	2	A	B_2	B_1
		A	B_2	B_1

Stability of Mergesort

Proposition: Mergesort is **stable**

```
public class Merge {
    private static Comparable[] aux;
    private static void merge(Comparable[] a, int lo, int mid, int hi)
    {  /* as before */  }

    private static void sort(Comparable[] a, int lo, int hi) {
        if (hi <= lo) return;
        int mid = lo + (hi - lo) / 2;
        sort(a, lo, mid);
        sort(a, mid+1, hi);
        merge(a, lo, mid, hi);
    }

    public static void sort(Comparable[] a)
    {  /* as before */  }
}
```

Proof:

`merge()` operation is stable, it takes from left subarray if equal keys

```java
private static void merge(...) {
    for (int k = lo; k <= hi; k++)
        aux[k] = a[k];

    int i = lo, j = mid+1;
    for (int k = lo; k <= hi; k++)
    {
        if      (i > mid)              a[k] = aux[j++];
        else if (j > hi)               a[k] = aux[i++];
        else if (less(aux[j], aux[i])) a[k] = aux[j++];
        else                           a[k] = aux[i++];
    }
}
```

0	1	2	3	4	5	6	7	8	9	10
A_1	A_2	A_3	B	D	A_4	A_5	C	E	F	G

Sample Question

Question:

Given an array of points, which approach would be least useful for removing duplicate points? Assume the point data type has the following three orders:

- A natural order that compares by x -coordinate and breaks ties by y -coordinate.
- One comparator that compares by x -coordinate; another by y -coordiante.

Note: `quicksort` is an efficient, but *unstable*, sorting algorithm.

Answer:

`Mergesort` by x-coordinate; `quicksort` by y-coordinate

Since `quicksort` is not stable, if you `mergesort` by x*-coordinate and then quicksort by y-coordinate, there is no guarantee that equal points will be adjacent in the sorted order.