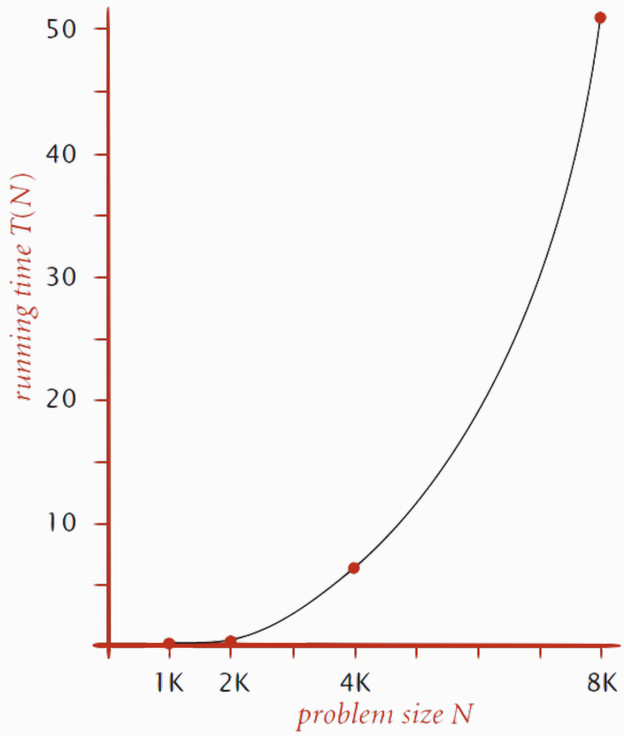
Lecture 2: Analysis of Algorithms

Data Analysis

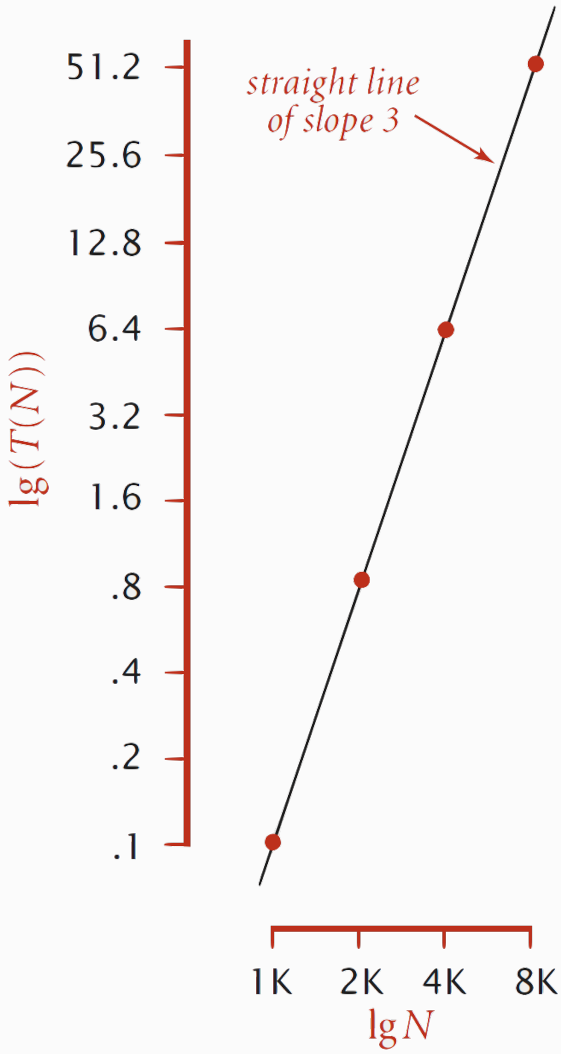
Standard Plot

Plot running time $T(N)$ vs. input size N



Log-log Plot

Plot running time $T(N)$ vs. input size N using log-log scale



Conduct ***regression analysis*** to fit straight line through data points: aN^b ("***power law***")

$$\lg(T(N)) = b(\lg(N)) + c$$
$$b = 2.999$$
$$c = -33.2103$$
$$T(N) = 2^{-33.2103} N^2.999$$

> ***Hypothesis***: *the running time is about* $1.006 \times 10^{-10} \times N^{2.999}$ *seconds*

Prediction and Validation

Predictions

- 51.0 seconds for $N = 8000$
- 408.1 seconds for $N = 16000$

Observations

N	time (seconds)
8000	51.1
8000	51.0
8000	51.1

16000	410.8

Vlalidates hypothesis

Doubling Hypothesis

Doubling hypothesis is a quick way to estimate b in a power-law relationship

Run programme, ***doubling*** the size of input:

N	time (seconds)	ratio	$\lg$(ratio)
250	0.0	-	-
500	0.0	4.8	2.3
1000	0.1	6.9	2.8
2000	0.8	7.7	2.9
4000	6.4	8.0	3.0
8000	51.1	8.0	3.0

> $\lg(ratio)$ converges to constant $b \approx 3$

Running time is about aN^b with $b = \lg(ratio)$, solve for a with a sufficently large value of N.

> ***Hypothesis***:
>
> Running time is about $0.998 \times 10^{-10} \times N^3$ seconds

Experimental Algorithms

Determinants of b:

System independent effects:

- Algorithm
- Input data

Determinants of a

System independent effects:

- Algorithm
- Input data

System dependent effects:

- Hardware: CPU, memory, cache, ...
- Software: complier, interpreter, garbage collector, ...
- System: operating system, network, other apps, ...

Mathematical Models for Running Time

Cost Model

Use some basic operation as a proxy for running time.

Example: 2-Sum

```
int count = 0;
for (int i = 0; i < N; i++) {
     for (int j = i+1; j < N; j++){
          if (a[i] + a[j] == 0) count ++;
     }
}
```

Operation	Frequency
Variable Declaration	$N+2$
Assignment Statement	$N+2$
Less than Comparison	$\frac{1}{2}(N+1)(N+2)$
Equal to Comparison	$0+1+2+\cdots+(N-1)=\frac{1}{2}N(N-1)=\binom{N}{2}$
Array access - *Cost Model*	$N(N-1)$
Increment	$\frac{1}{2}N(N-1)$ to $N(N-1)$

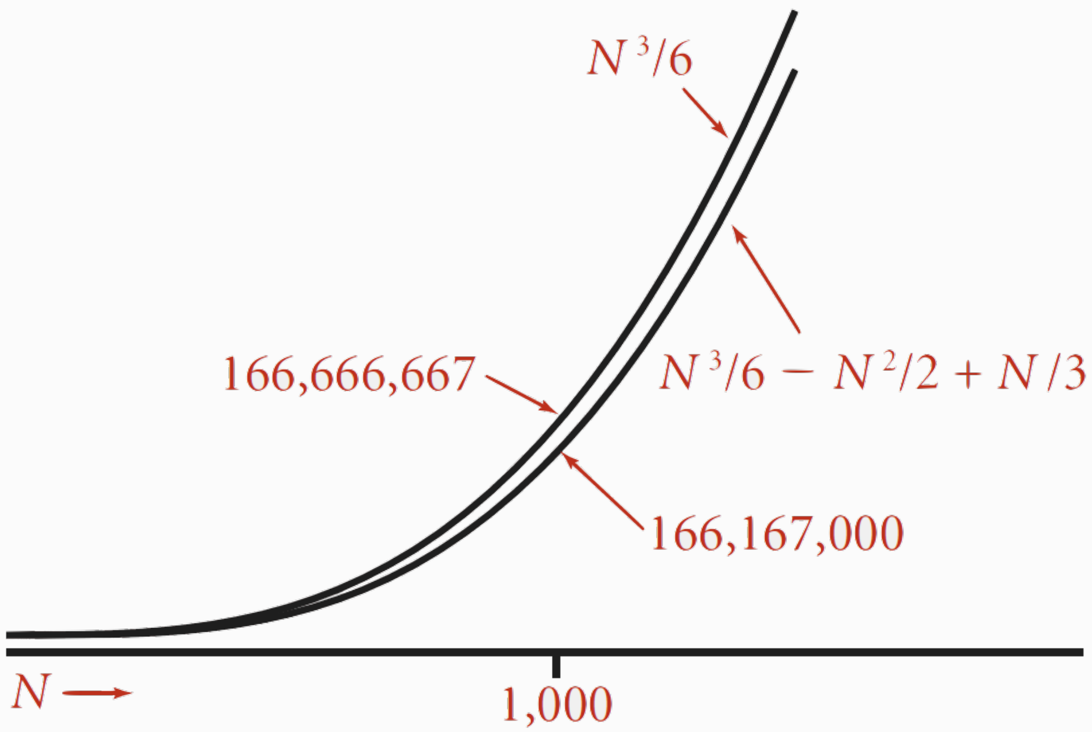
> Assuming compiler/JVM does not optimise array accesses away

Tilde Notation

- Estimating running time (or memory) as a function of input size N
- Ignore lower order terms
  - when N is large, terms are negligible
  - when N is small, omit

Examples

$$\frac{1}{6}N^3+20N+16\sim\frac{1}{6}N^3$$
$$\frac{1}{6}N^3+100N^{\frac{4}{3}}+56\sim\frac{1}{6}N^3$$
$$\frac{1}{6}N^3-\frac{1}{2}N^2+\frac{1}{3}N\sim\frac{1}{6}N^3$$



Leading-term approximation

Discard lower-order terms

> ***Technical Definition***
>
> $f(N) \sim g(N)$ means:
>
> $$\lim_{N\to\infty} \frac{f(N)}{g(N)} = 1$$

2-Sum Example with Tilde Notation

Operation	Frequency	Tilde Notation
Variable Declaration	$N+2$	$\sim N$
Assignment Statement	$N+2$	$\sim N$
Less than Comparison	$\frac{1}{2}(N+1)(N+2)$	$\sim \frac{1}{2}N^2$
Equal to Comparison	$0+1+2+\cdots+(N-1) = \frac{1}{2}N(N-1) = \binom{N}{2}$	$\sim \frac{1}{2}N^2$
Array access - *Proxy*	$N(N-1)$	$\sim N^2$
Increment	$\frac{1}{2}N(N-1)$ to $	$\sim \frac{1}{2}N^2$ to $\sim N^2$

Example: 3-Sum

```
int count = 0;
for (int i = 0; i < N; i++){
    for (int j = i+1; j < N; j++ ){
        if (a[i] + a[j] + a[k] == 0) count++;
    }
}
```

Operation	Frequency	Tilde Notation
Equal to Comparison	$\binom{N}{3} = \frac{N(N-1)(N-2)}{3!}$	$\sim \frac{1}{6}N^3$
Array access - *Proxy*	$3 \times \binom{N}{3}$	$\sim \frac{1}{2}N^3$

> ***Note that:***
>
> Each *Equal to* Comparison has 3 array accesses

> ***Estimating a Discrete Sum***
>
> Replace the sum with an integral
>
> $$\sum_{i=1}^{N} i \sim \int_{x=1}^{N} x\,dx \sim \frac{1}{2}N^2$$
> $$\sum_{i=1}^{N} i^k \sim \int_{x=1}^{N} x^k\,dx \sim \frac{1}{k+1}N^{K+1}$$
> $$\sum_{i=1}^{N} \frac{1}{i} \sim \int_{x=1}^{N} \frac{1}{x}\,dx = \ln N$$
> $$\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=j}^{N} 1 \sim \int_{x=1}^{N}\int_{y=x}^{N}\int_{z=1y}^{N} dz\,dy\,dx \sim \frac{1}{6}N^3$$

Sample Question:

How many array accesses does the following code fragment make as a function of n?

(Assume the compiler does not optimize away any array accesses in the innermost loop.)

```
int sum = 0;
for (int i = 0; i < n; i++)
    for (int j = i+1; j < n; j++)
        for (int k = 1; k < n; k = k*2)
            if (a[i] + a[j] >= a[k]) sum++;
```

Running Time of k-loop:

Geometric Progression with $a = 1, r = 2$:

$$2^T = N$$
$$T = \lg N$$

3 array accesses for `a[i], a[j], a[k]`:

Running Time of k-loop:

$$3 \lg N$$

Number of time k-loop is Executed:

The equivalent of *2-Sum* problem

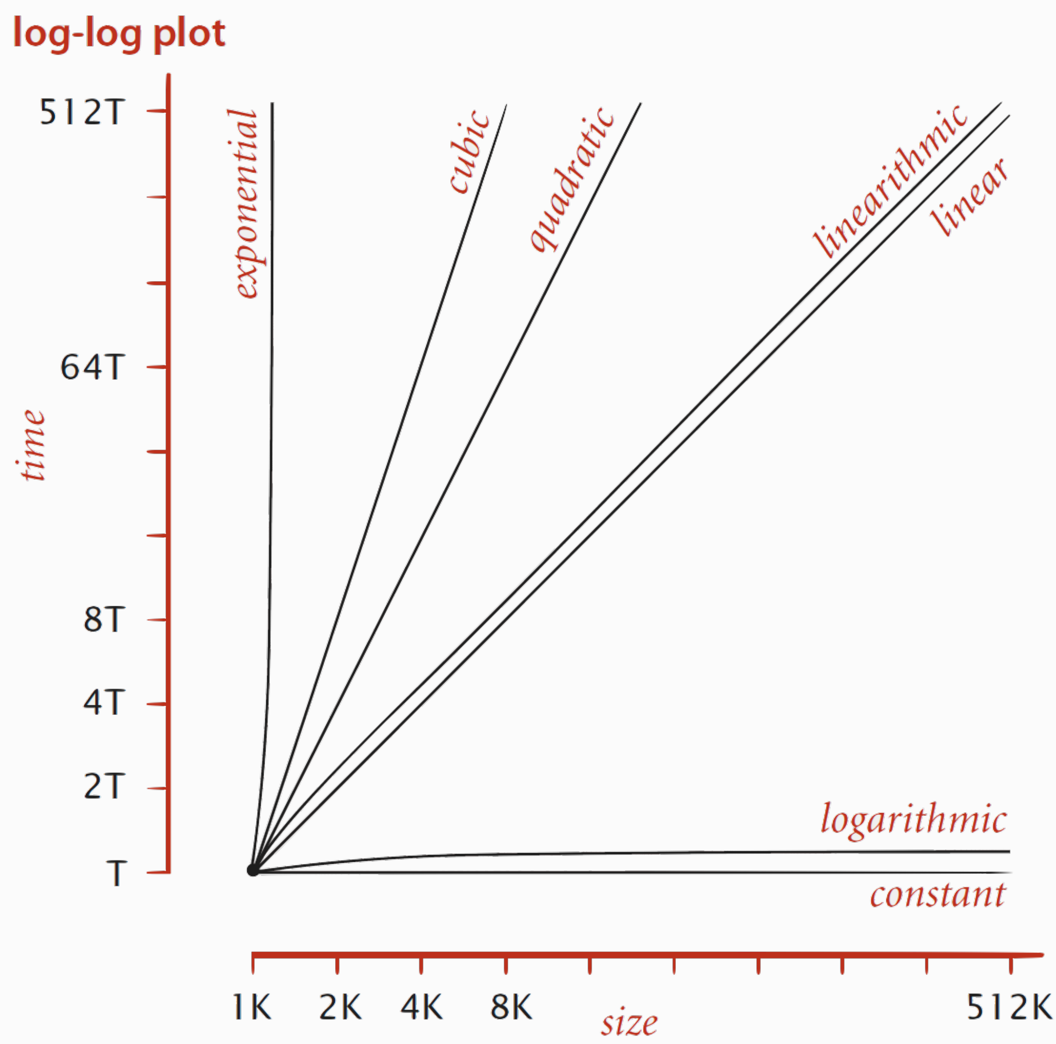
$$\binom{N}{2} = \frac{1}{2} N^2$$

Array Accesses of the Function:

$$\frac{1}{2} N^2 \times 3 \lg N = \frac{3}{2} N^2 \lg N$$

Order-of-Growth Classification

Order-of-Growth	Name	Typical Code Framework	Description	Example	$\frac{T(2N)}{T(N)}$
1	Constant	`a = b + c`	Statement	Addition of two numbers	1
$\log N$	Logarithmic	`while (N > 1) { N = N/2; ...}`	Divide in half	Binary Search	~ 1
N	Linear	`for (int i = 0; i < N; i++)` `{...}`	Loop	Find the max	2
$N \log N$	Linearithmic	`mergeSort()`	Divide and conquer	Merge Sort	~ 2
N^2	Quadratic	`for (int i = 0; i < N; i++)` `for (int j = 0; j < N; j++)` `{...}`	Double loop	Check all pairs	4
N^3	Cubic	`for (int i = 0; i < N; i++)` `for (int j = 0; j < N; j++)` `for (int k = 0; k < N; k++)` `{...}`	Triple loop	check all triples	8
2^N	Exponential	`combinatorial()`	Exhaustive search	Checks all subsets	$T(N)$



Typical orders of growth

Binary Search

Java Implementation

```java
public static int binarySearch(int[] a, int key) {
     int lo = 0, hi = a.length-1;
     while (lo <= hi) {
          int mid = lo + (hi - lo) / 2;
          if (key < a[mid]) hi = mid - 1;
          else if (key > a[mid]) lo = mid + 1;
          else return mid;
     }
     return -1;
}
```

> *Function Facts* 👀
>
> - Firt binary search published in 1946
> - First bug-free one published in 1962
> - Bug in Java's `Arrays.binarySearch()` discovered in 2006

Propostion

Binary search uses at most $1 + \lg N$ key compares to search in a sorted array of size N

Recurrence

Define $T(N)$ as the number of key compares to binary search a sorted subarray of size $\leq N$

$$T(N) \leq T(\frac{N}{2}) + 1$$

for $N > 1$, with $T(1) = 1$

> Omitting odd N case

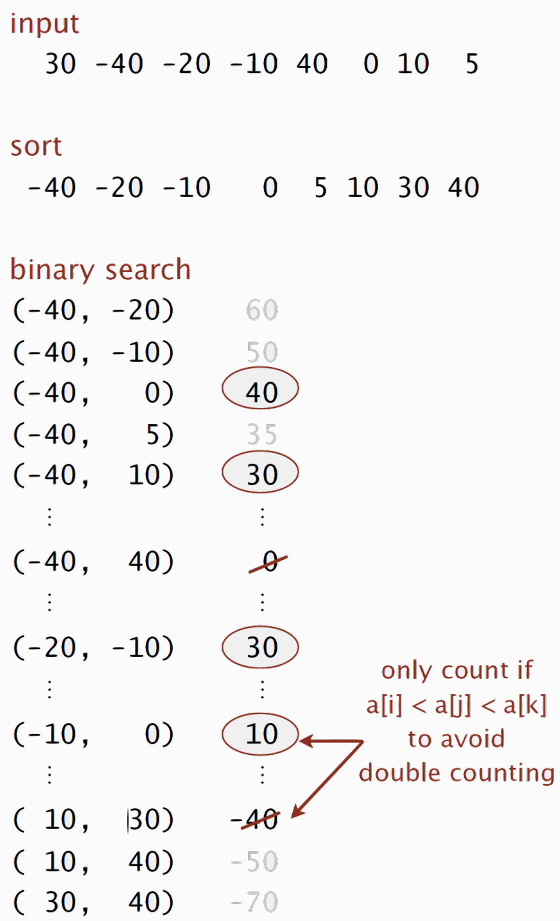
Proof Sketch

$$\begin{aligned} T(N) &\leq T(\frac{N}{2}) + 1 \\ &\leq T(\frac{N}{4}) + 1 + 1 \\ &\leq T(\frac{N}{8}) + 1 + 1 + 1 \\ &\ldots \\ &\leq T(\frac{N}{N}) + 1 + 1 + \cdots + 1 \\ &= 1 + \lg N \end{aligned}$$

An $N^2 \log N$ Algorithm for 3-Sum

Sorting based algorithm

1. ***Sort*** the N (distinct) numbers
2. For each pair of numbers `a[i]` and `a[j]`, ***binary search*** for `-(a[i] + a[j])`



Analysis

- N^2 with ***insertion sort***
- $N^2 \lg N$ with ***binary search***

Theory of Algorithms

Cases

Best Case: Lower bound of cost

- Determined by "easiest" input
- Provides a goal for all inputs

Worst Case: Upper bound on cost

- Determined by "most difficult" input
- Provides a guarantee for all inputs

Average Case: Expected cost for random input

- Need a ***model*** for "random" input
- Provides a way to predict performance

Example: Array Accesses for Brute-Force 3-Sum

- **Best** : $\sim \frac{1}{2}N^3$
- **Average** : $\sim \frac{1}{2}N^3$
- **Worst** : $\sim \frac{1}{2}N^3$

Example: Compares for Binary Search

- **Best** : ~ 1
- **Average** : $\sim \lg N$
- **Worst** : $\sim \lg N$

> ***Actual Data Might Not Match Input Model?***
>
> Need to understand input to effectively process it
>
> - Approach 1: design for the worst case
> - Approach 2: randomise, depend on probabilistic guarantee

Goals

- Eastablish "difficulty" of a problem
- Develop "optimal" algorithms

Approaches

- Suppress details in analysis: analyse "to within a constant factor"
- Eliminate variability in input model by focusing on the worst case

Optimal Algorithm

- ***Performance guarantee*** (to within a constant factor) for ***any input***
- ***No algorithm*** can provide a better performance guarantee

Commonly-Used Notations

Notation	Provides	Example	Shorthand for	Used to
Tilde	Leading term	$\sim 10N^2$	$10N^2, 10N^2 + 22N \log N, 10N^2 + 2N + 37$	Provide approximate model
Big Theta	Asympototic growth rate	$\Theta(N^2)$	$\frac{1}{2}N^2, 10N^2, 5N^2 + 22N \log N + 3N$	Classify algorithms
Big Oh	$\Theta(N^2)$ and smaller	$O(N^2)$	$10N^2, 100N, 22N \log N + 3N$	Develop upper bounds
Big Omega	$\Theta(N^2)$ and larger	$\Omega(N^2)$	$\frac{1}{2}N^2, N^5, N^3 + 22N \log N + 3N$	Develop lower bounds

> ***Note that***:
>
> It is a common mistake to interpret big-Oh as an approximate model. It develops the upper bounds instead

Example 1

Goals

- Establish "difficulty" of a problem and develop "optimal" algorithms
- **Example** : 1-Sum: *"Is there a 0 in the array?"*

Upper Bound: a specific alogrithm

- **Example** : Brute-force 1-Sum
- Running time of the optimal algorithm for 1-Sum is $O(N)$

Lower Bound: proof that no algorithm can do better

- For any algorithm, it has to examine all N entries
- Running time of the optimal algorithm for 1-Sum is $\Omega(N)$

Derived Optimal Algorithm

- Lower bound equals upper bound (to within a constant factor)
- **Conclusion** : Brute-force 1-Sum is optimal: its running time is $\Theta(N)$

Example 2

Goals

- Establish "difficulty" of a problem and develop "optimal" algorithms
- **Example** : 3-Sum

Upper Bound: a specific alogrithm

- **Example** : Brute-force 3-Sum
- **However** , Sorted-based 3-Sum performance better
- Running time of the optimal algorithm for 1-Sum is $O(N^2 \log N)$

Lower Bound: proof that no algorithm can do better

- For any algorithm, it has to examine all N entries
- Running time of the optimal algorithm for 1-Sum is $\Omega(N)$

Open Problems

- Maybe better algorithm
- Subquadratic algorithm?
- Quadratic lower bound?

Algorithm Design Approach

Start

- ***Develop*** an algorithm
- ***Prove*** a lower bound

Gap?

- ***Lower*** the upper bound - discover a new algorithm
- ***Raise*** the lower bound - more difficult

Cavests

- Overly pessimistic to focus on worst case?
- Need better than "to within a constant factor" to predict performance

Memory

Basics

Unit	Coversion
Bit	$\in \{0,1\}$
Byte	8 bits
Megabyte (MB)	2^{20} bytes

Gigabyte (GB)	2^{30} bytes

> ***64-bit Machine***: we assume 64-bit machine with ***8 bytes pointers***
> - can address more memory
> - pointers use more space - *but some JVMs "compress" ordinary object pointers to 4 bytes to avoid this cost*

Typical Memory Usage for Primitive Types and Arrays

Primitive Types

type	bytes
`boolean`	1
`byte`	1
`char`	2
`int`	4
`float`	4
`long`	8
`double`	8

One-dimensional Arrays

type	bytes
`char[]`	$2N + 24$
`int[]`	$4N + 24$
`double[]`	$8N + 24$

Two-dimensional Arrays

type	bytes
`char[][]`	$\sim 2MN$
`int[][]`	$\sim 4MN$
`double[][]`	$\sim 8MN$

Typical Memory Usage for Objects in Java

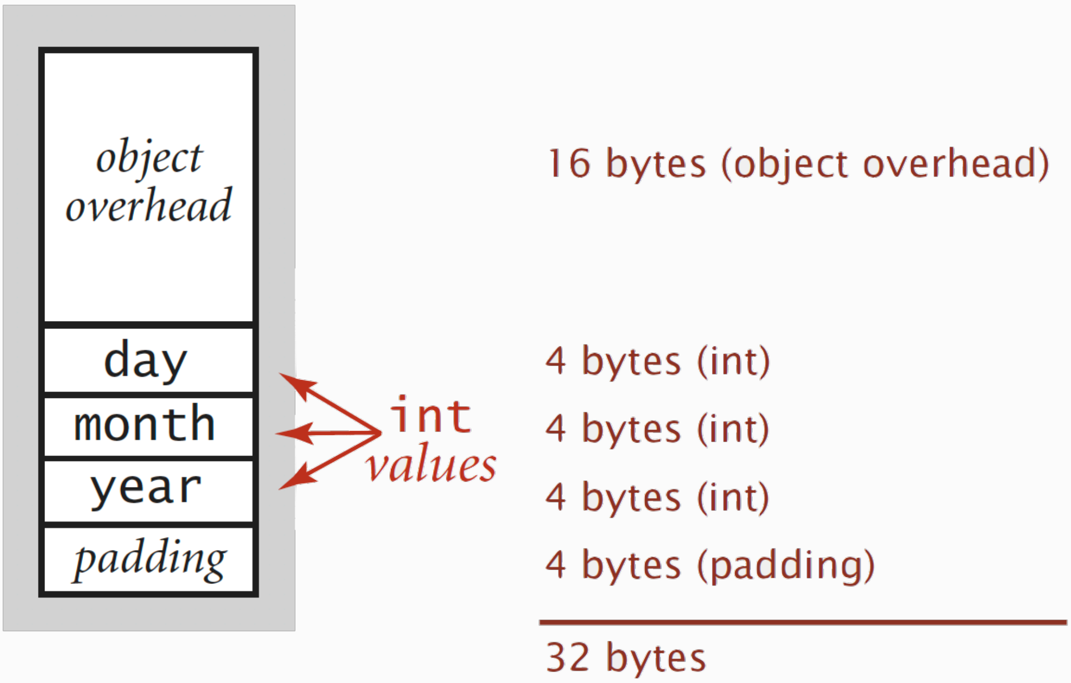
object	bytes
Object overhead	16

Reference	8
Padding	*each object uses a multiple of 8 bytes*

Example 1

A *Date* object uses 32 bytes of memory

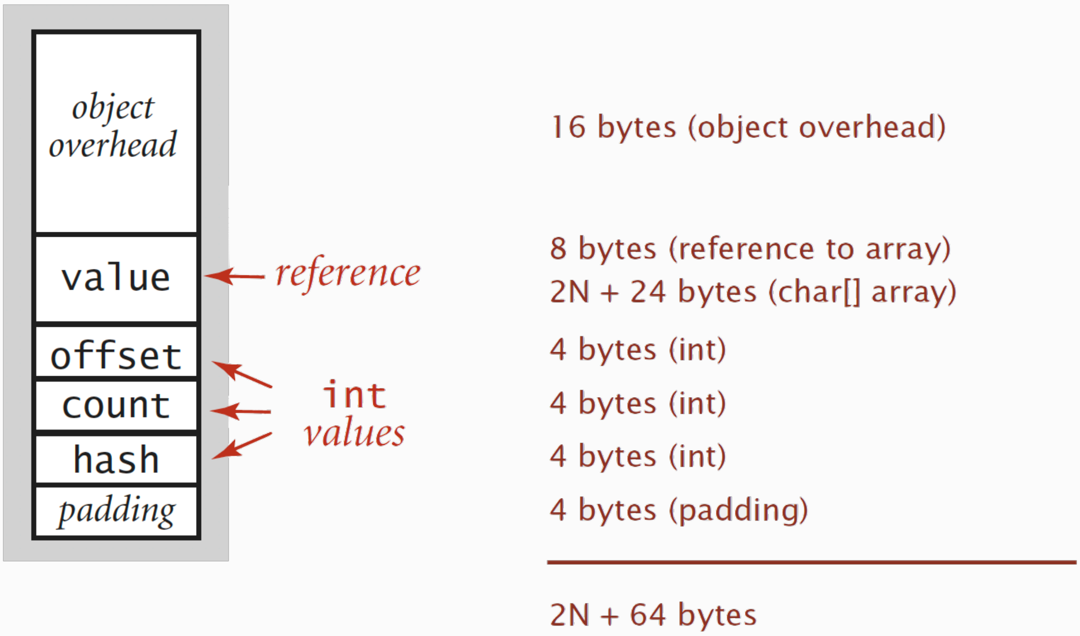
```java
public class Date {
      private int day;
      private int month;
      private int year;
      ...
}
```



Example 2

A virgin *String* of length N uses $\sim 2N$ bytes of memory

```java
public class String {
      private char[] value;
      private int offset;
      private int count;
      private int hash;
      ...
}
```



Shallow Memory Usage: don't count referenced objects

Deep Memory Usage: if array entry or instance variable is a reference, add memory (recursively) for referenced object