Lecture 1: Union-Find

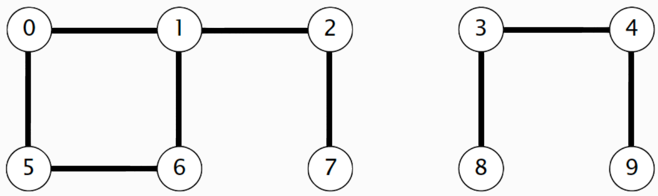
Dynamic Connectivity

Problem

Given a set of N objects, design efficient ***data structure*** for union-find:

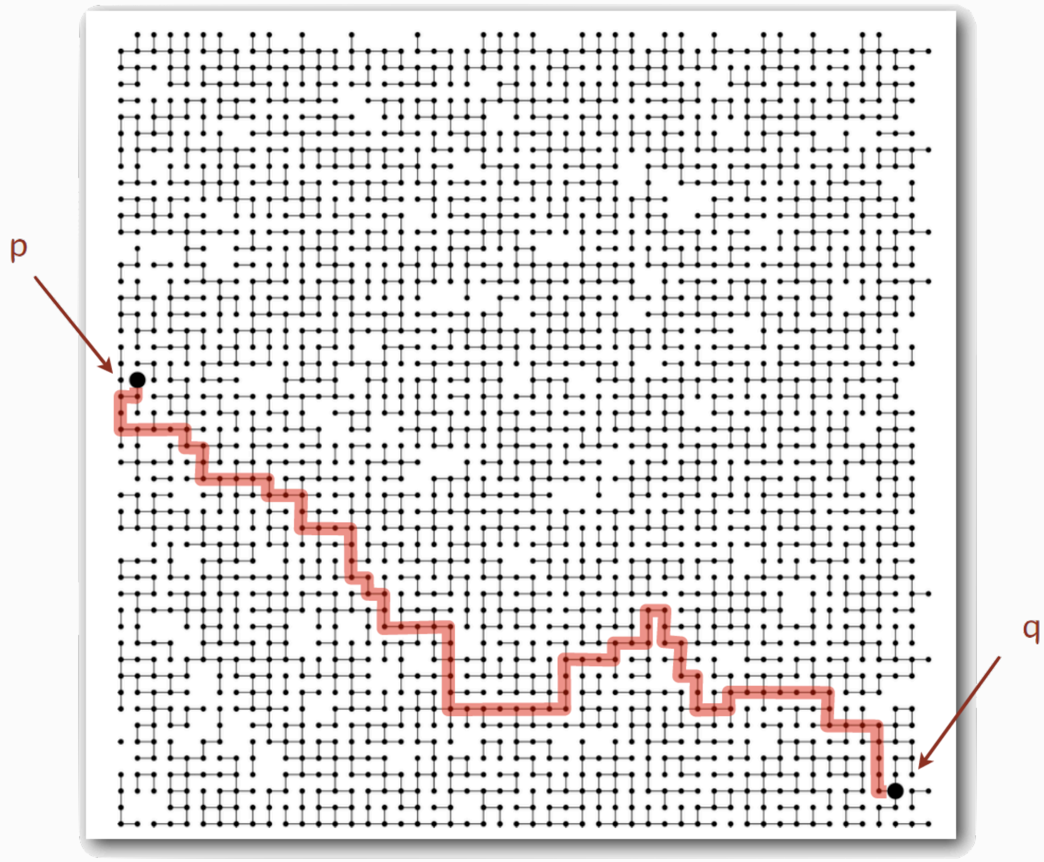
- `union()` command: ***connects*** two objects
- `find()` (`connected()`) query: is there a path connecting the two objects.

```
union(4, 3)
union(3, 8)
union(6, 5)
union(9, 4)
union(2, 1)
connected(0, 7)   ✗
connected(8, 9)   ✔
union(5, 0)
union(7, 2)
union(6, 1)
union(1, 0)
connected(0, 7)   ✔
```



> ***Note that***:
>
> - Number of ***objects*** N can be ***huge***
> - Number of ***operations*** M can be ***huge***
> - Find queris and union commands may be ***intermixed***

Example: *Is there a path from p to q?*



Modeling

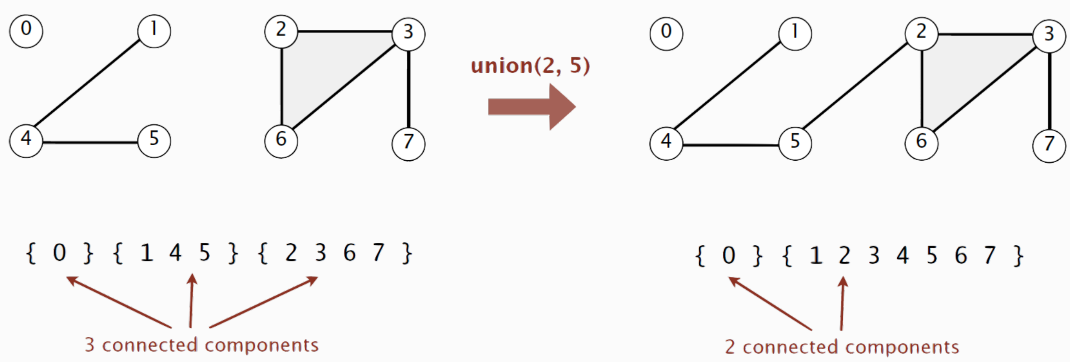
Connections:

Assume "*is connected to*" is an equivalence relation:

- ***Reflexive*** : p is connected to p
- ***Symmetric*** : if p is connected ed to q , then q is connected to p
- ***Transitivie*** : if p is connected to q and q is connected r , then p is connected to r

Connected Components:

Connected components are the maximal ***sets*** of objects that are ***mutually connected***.



`find()` ***query***: *checks if the wo objects are in the same component*

`union()` ***command***: *replace components containing two objects with their union*

Union-Find Data Type (API)

```java
public class UF{
      public UF(int N){
            /*
            initialise union-find data structure with N objects
            */
      }

      public void union(int p, int q){
            /*
            add connection between p and q
            */
      }

      public boolean connected(int p, int q){
            /*
            checks if p and q are in the same component
            */
      }

      public int find (int p){
            /*
            component identifier for p
            */
      }

      public int count(){
            /*
            returns the number of components
            */
      }
}
```

Dynamic-Connectivity Client

- Read in number of objects N from standard input
- Repeat:
    - read in pair of integers from standard input
    - if they are not yet connected, connect them and print out pair

```java
public static void main(String[] args) {
      int N = StdIn.readInt();
      UF uf = new UF(N);

      while (!StdIn.isEmpty()) {
            int p = StdIn.readInt();
```

```java
        int q = StdIn.readInt();

        if (!uf.connected(p, q)) {
            uf.union(p, q);
            StdOut.println(p + " " + q);
        }
    }
}
```

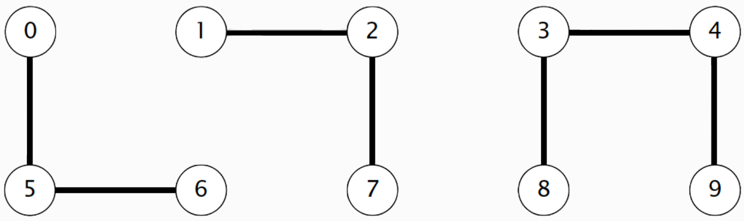
Quick Find (*Eager Approach*)

Data Strcuture

- `int[] id` of size N
- ***Interpretations*** : p and q are connected ***if and only if*** (iff) they have the same *id*

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	1	8	8

0, 5 and 6 are connected
1, 2, and 7 are connected
3, 4, 8, and 9 are connected



Commands

`find()`: checks if p and q have the same *id*

`union()`: to merge components containing p and q, changes all entries whose *id* equials `id[p]` to `id[q]`

Java Implementation

```java
public class QuickFindUF{
    private int[] id;

    public QuickFindUF(int N) {
        id = new int[N];
        for (int i = 0; i < N;i++ ){
            id[i] = i;
        }
    }

    public boolean connected(int p, int q){
        return id[p] == id[q]l
```

```java
    }

    public void union(int p, int q){
        int pid = id[p];
        int qid = id[q];
        for (int i = 0; i < id.length; i++){
            if (id[i] == pid){
                id[i] = qid;
            }
        }

    }
}
```

Cost Model

Method	Time Complexity
`initialise`	$O(N)$
`union()`	$O(N)$
`connected()`	$O(1)$

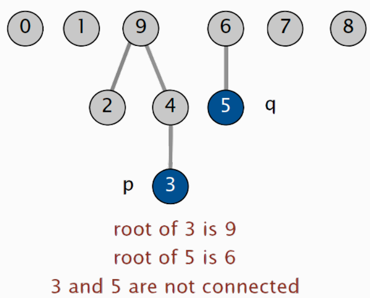
> Too ***expensive***: takes N^2 array accesses to process sequence of N union commands on N objects

Quick Union (*Lazy Approach*)

Data Structure

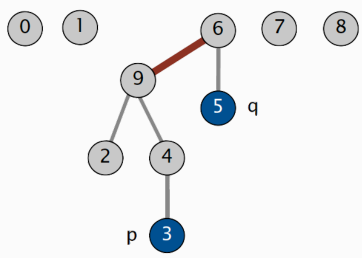
- `int[] id` of sieze N
- ***Interpretation*** : `id[i]` is parent of `i`
- ***Root*** of i is `id[id[id[...id[i]...]]]`

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	6

only one value changes



Commands

`find()`: checks if p and q have the same root

`union()`: to merge components containing p and q, sets the *id* of p's root to the id of q's root

Java Implementation

```java
public class QuickUnionUF{
    private int[] id;

    public QuickUnionUF(int N){
        id = new int[N];
        for (int i = 0; i < N; i++){
            id[i] = i;
        }
    }

    private int root(int i){
        while(i != id[i]){
            i = id[i];
        }
        return i
    }

    public boolean connected(int p, int q){
        return root(p) == root(q)
    }

    public void union(int p, int q){
        int i = root(p);
        int j = root(q);
        id[i] = j;
    }
}
```

Cost Model

Method	Time Complexity
`initialise`	$O(N)$
`union()`	$O(N)$ (includes cost of finding roots)
`connected()`	$O(N)$ (worst case)

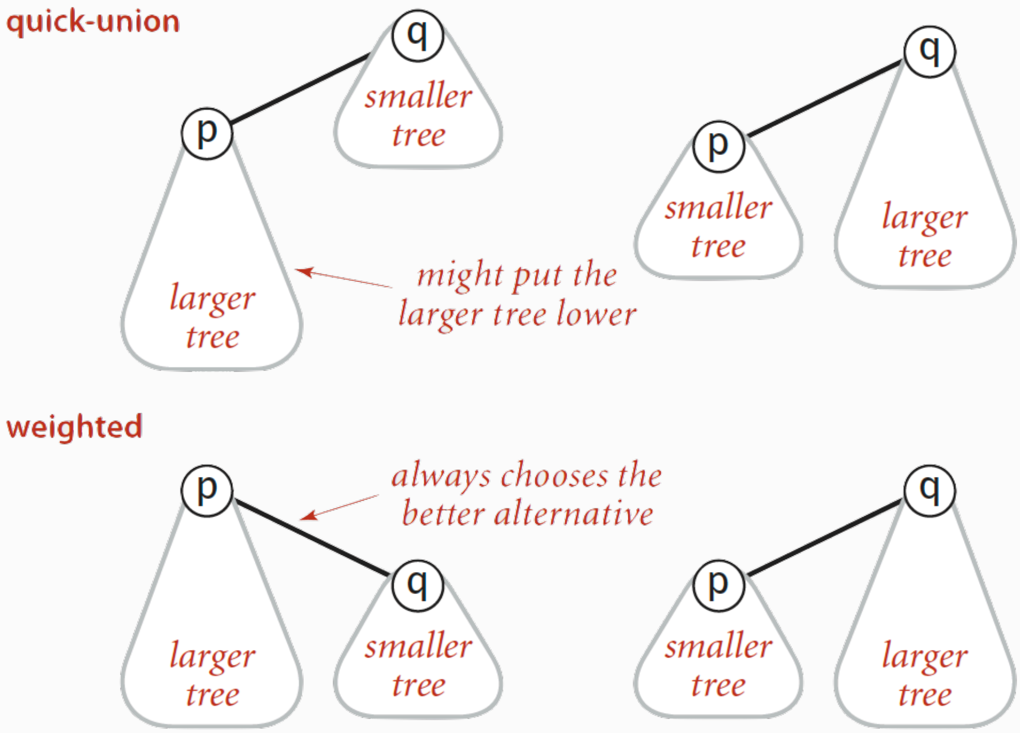
> ***Quick-find*** defects:
>
> - Union too expensive (N array accesses)
> - Trees are flat, but too expensive tio keep them flat

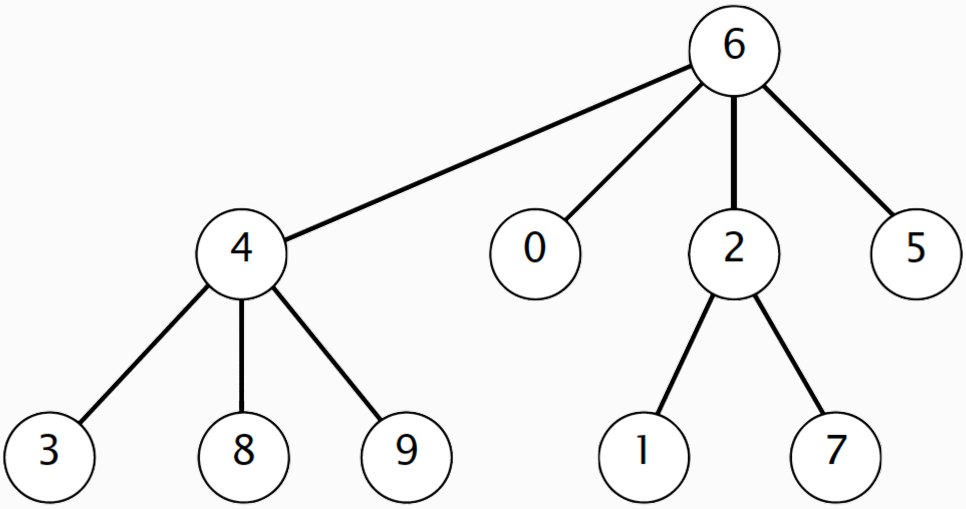
> *Quick-union* defects:
>
> - Tress can get tall
> - Find too expensive (could be N array access)

Quick Union Improvement

Improvement 1: Weighted Quick Union

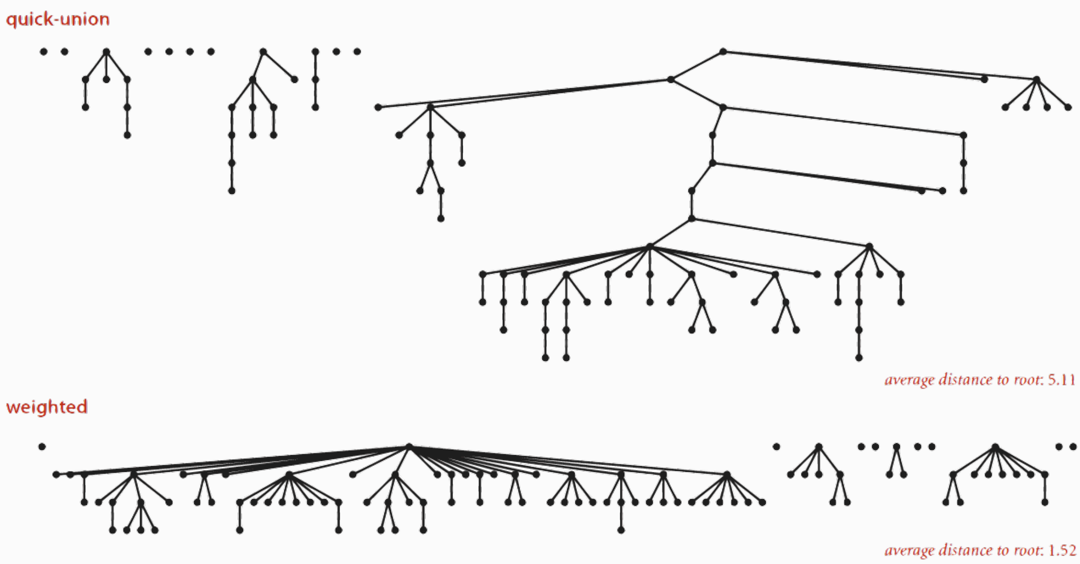
- Modify quick-union to ***avoid tall trees***
- Keep track of ***size*** of each tree (number of objects)
- ***Balance*** by *linking root of smaller tree to root of larger tree*





	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	6	6	6	2	4	4

Comparison



Quick-union and weighted quick-union (100 sites, 88 `union()` operations)

Data Structure

Same as quick-union, but maintain extrac array `sz[i]` to count number of objects in the tree rooted at `i`

Commands

`connected()`: itendtical to quick-union

`union()`: modify quick-union to:

- Link root of smaller tree to root of larger tree
- Update the `sz[]` arrya

Java Implementation

```java
public class QuickUnionUF{
	private int[] id;
	private int[] sz;

	public QuickUnionUF(int N){
		id = new int[N];
		sz = new int[N]
		for (int i = 0; i < N; i++){
			id[i] = i;
		}
		for (int i = 0; i < N; i++){
			sz[i] = 1;
		}
	}

	private int root(int i){
		while(i != id[i]){
			i = id[i];
		}
		return i
	}

	public boolean connected(int p, int q){
		return root(p) == root(q)
	}

	public void union(int p, int q){
		int i = root(p);
		int j = root(q);
		if (i == j) {
			return;
		}
		if (sz[i] < sz[j]) {
			id[i] = j;
			sz[j] += sz[i];
		} else {
			id[j] = i;
			sz[i] += sz[j]
		}
	}
}
```

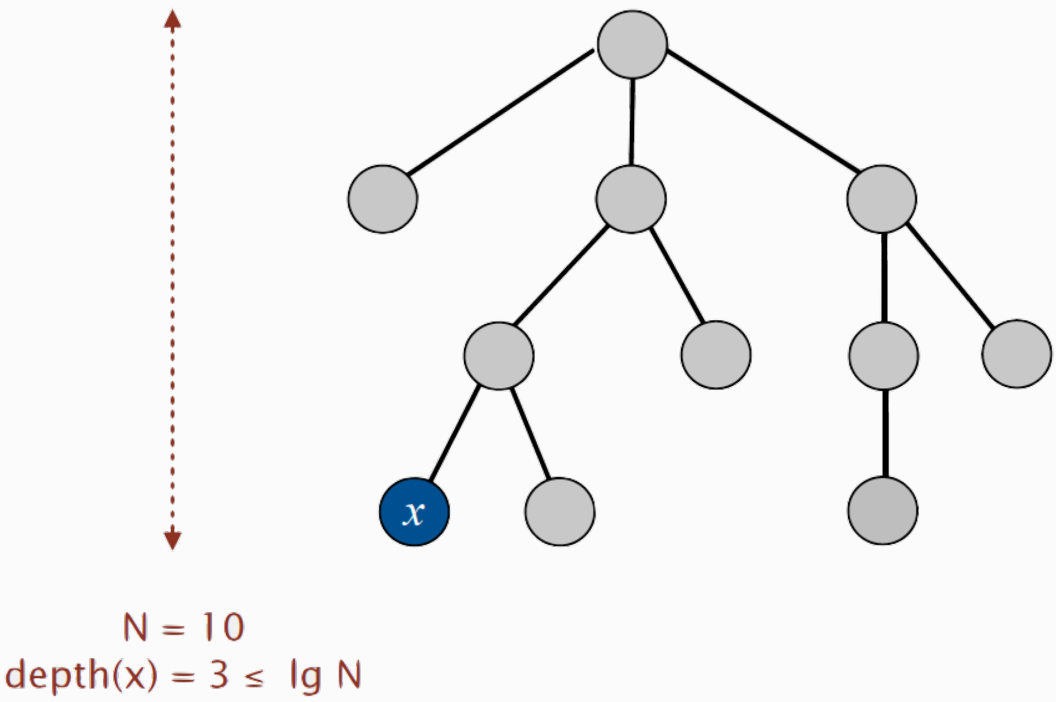
Running Time

`connected()`: takes time proportional to depth of p and q

`union()`: takes constant time, given roots

Proposition

Depth of any node x is ***at most*** $\log_2 N$ (denote $\lg N$)

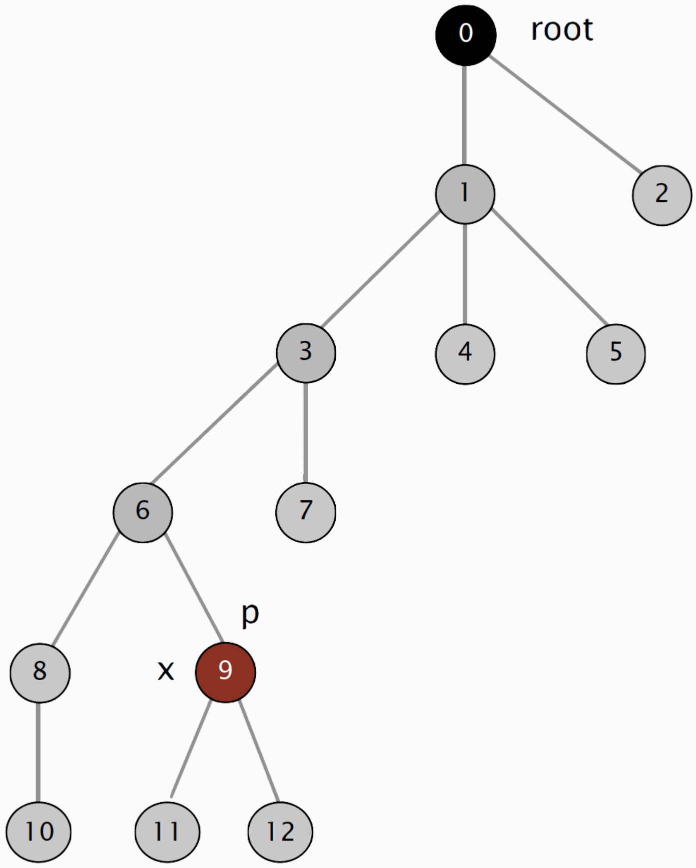


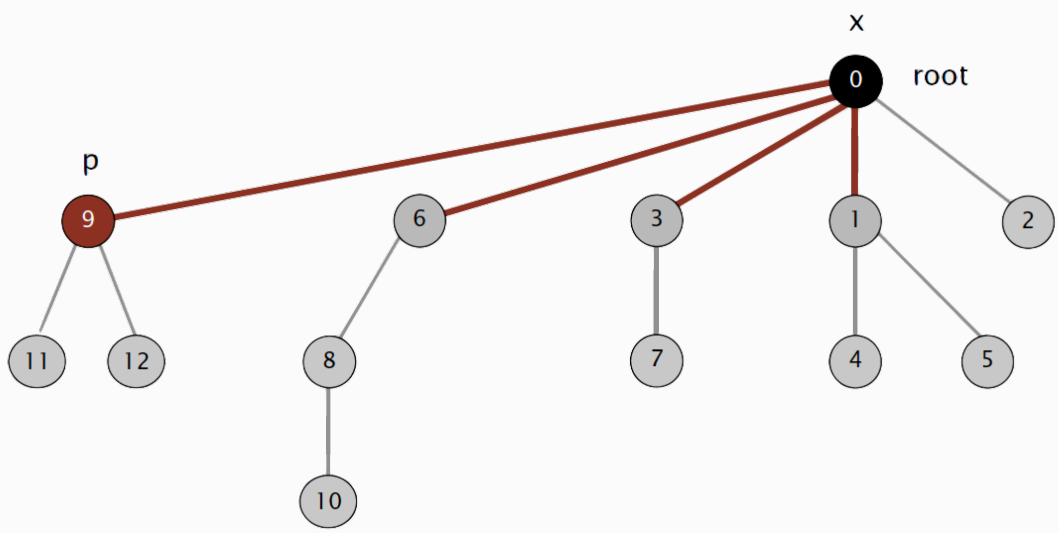
> *Proof*
>
> *When does depth of x increase?* It increase by 1 when tree T_1 containing x is merged into another tree T_2
>
> - The size of the tree containing x at least doubles since $|T_2| \geq |T_1|$
> - Size of tree containing x can double at most $\lg N$ times because if you start with 1:
>
> $$\begin{aligned} 1 \times 2^{\lg N} &= x \\ \lg x &= \lg N \\ x &= N \end{aligned}$$

Method	Time Complexity
`initialise`	$O(N)$
`union()`	$O(\lg N)$
`connected()`	$O(\lg N)$

Improvement 2: Quick Union with Path Compression

Just after computing the root of p, set the id of each examined node to point to that root.





Java Implementation

- **Two-Pass Implementation**: add second loop to `root()` to set the `id[]` of each examined node to the root
- Simpler One-Pass Variant: Make every other node in path ***point to its granparent*** (thereby halving path length)

```java
private int root(int i) {
      while (i != id[i]){
            id[i] = id[id[i]];
            i = id[i]
      }
      return i;
}
```

Weighted Quick-Union with Path Compression: Amortised Analysis

Proposition

Starting from an empty data structure, any sequence of M union-find operations on N objects makes $\leq c(N + M \lg^* N)$ array accesses.

- Analysis can be imprvoed to $N + M\alpha(M, N)$.
- Simple algorithm with fascinating mathematics

> $\lg^* N$ is the number of times you have to take the lg of N to get 1.
>
> | N | $\lg^* N$ |
> | --- | --- |
> | 1 | 0 |
> | 2 | 1 |
> | 4 | 2 |
> | 16 | 3 |
> | 65536 | 4 |
> | 2^{65536} | 5 |

Summary

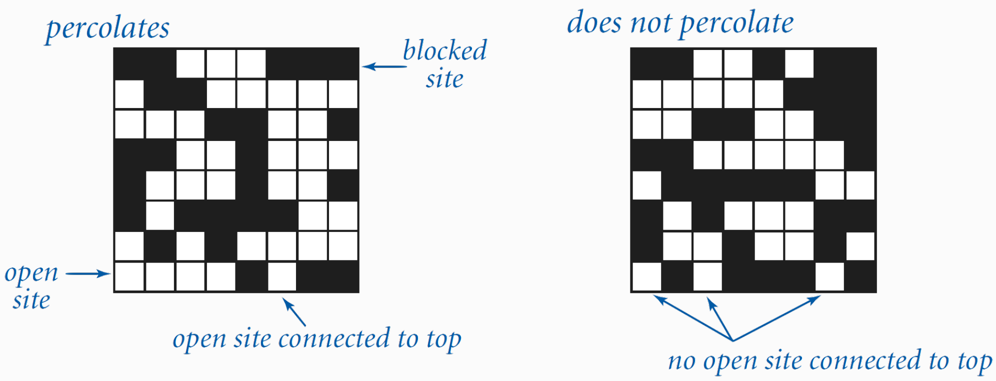
Algorithm	**Worst-case Time**
Quick-Find	MN
Quick-Union	MN
Wighted QU	$N + M \log N$
QU + Path Compression	$N + M \log N$
Weighted QU + Path Compression	$N + M \lg^* N$

Application: Percolation

Modelling

- N -by- N ***grid*** of sites
- Each ***site*** is *open* with probability p (or *blocked* with probability $1 - p$)

- System ***percolates*** iff *top and bottom are connected by open sites* .



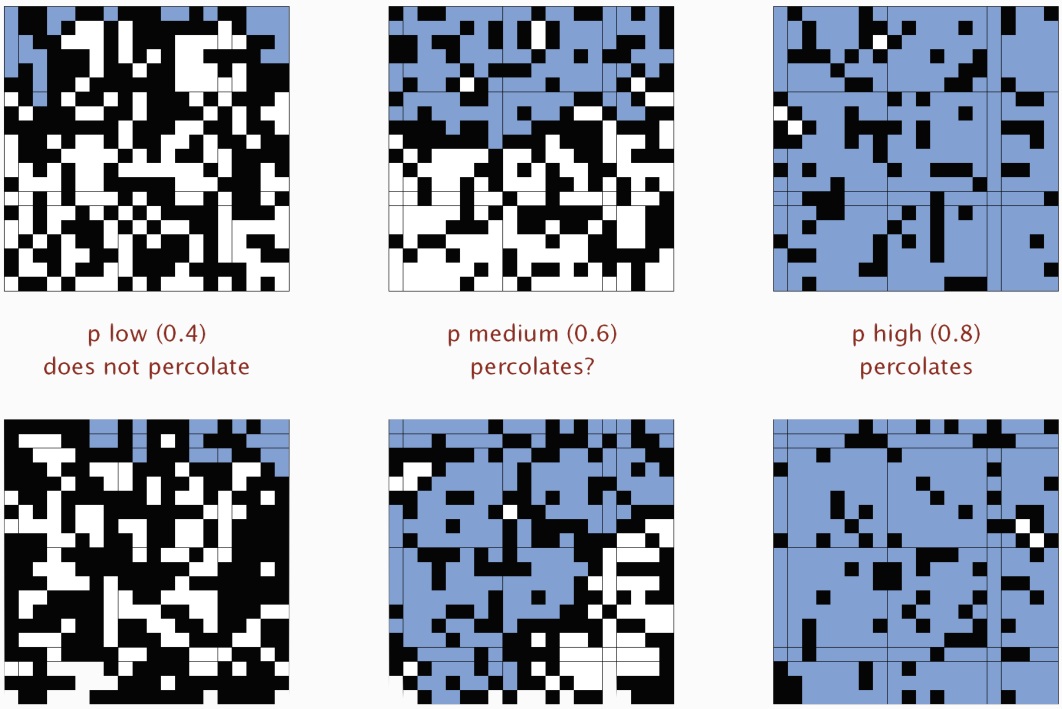
N = 8

Example for Physical Systems

Model	System	Vacant site	Occupied site	Percolates
Electricity	Material	Conductor	Insulated	Conducts
Fluid Flow	Material	Empty	Blocked	Porous
Social Interaction	Population	Person	Empty	Communicates

Likelihood of Percolation

Depends on site vacancy probability p



p low (0.4)
does not percolate

p medium (0.6)
percolates?

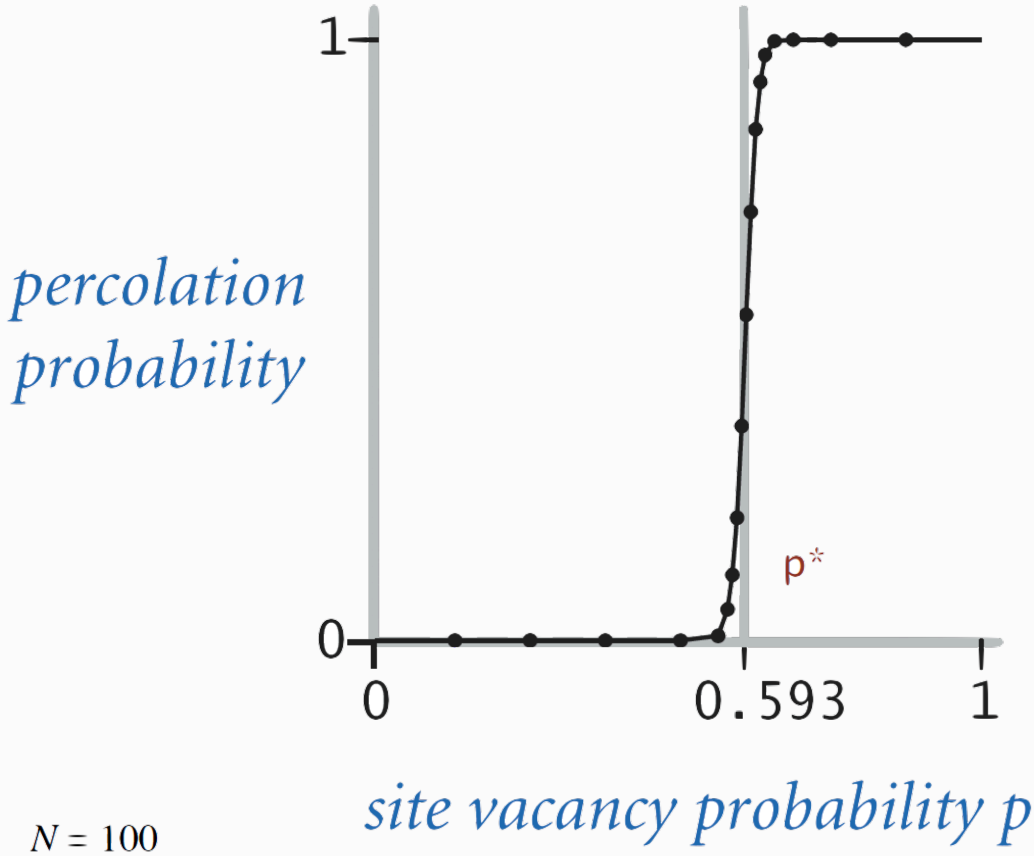
p high (0.8)
percolates

Percolation Phase Transition

When N is large, theory guarantees a sharp threshold p^*

- $p > p^*$: almost certainly percolates
- $p < p^*$: almost certainly does not percolates

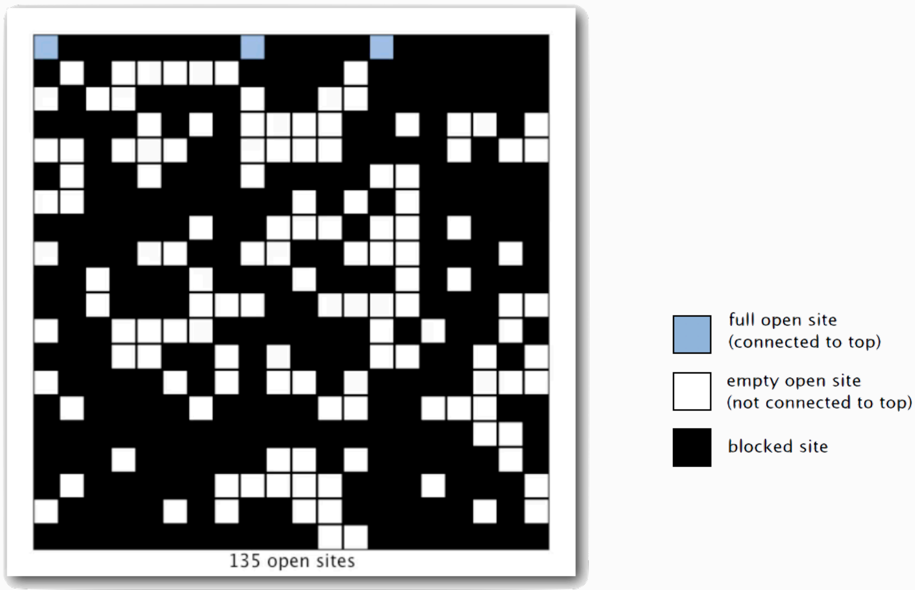
Question: What is the value of p^*



$N = 100$

Monte Carlo Simulation

- Initialise N -by- N whole grid to be *blocked*
- Declare random sites *open* until top conneceted to bottom
- Vacancy percentage estimates p^*



$N = 20$

Dynamic Connectivity Solution to Estimate Percolation Threshold

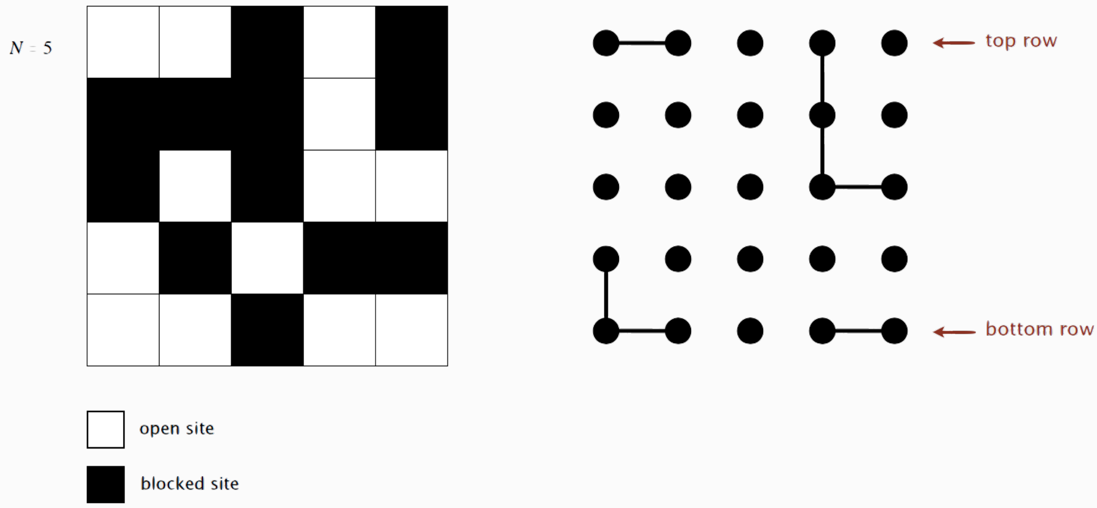
Question: How to check whether an N-by-N system percolates?

- Create an object for each site and index from 0 to $N^2 - 1$
- Sites are in same component if connected by open sites

- ***Percolates*** iff any site on <u>bottom</u> row is connected to site on <u>top</u> row

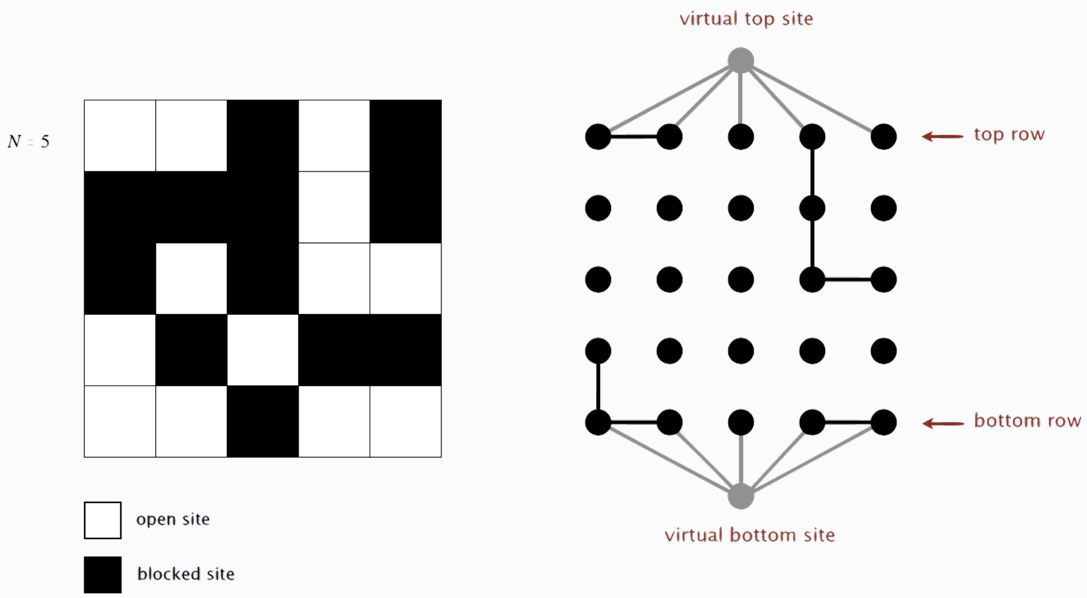
Brute-Force Algorithm

N^2 calls to `connected()`



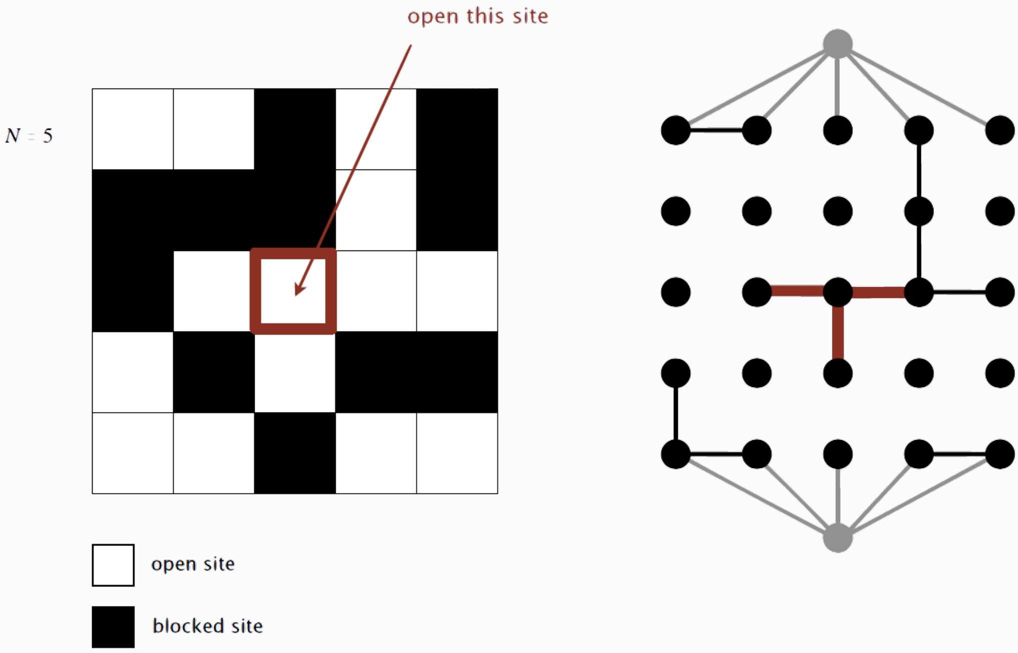
Efficient Algorithm

Only 1 call to `connected()`



Question: How to model *opening a new site*?

Mark new site as *open*, connect it to all of its adjacent *open* sites - up tp 4 calls to `union()`



open this site
$N = 5$
open site
blocked site