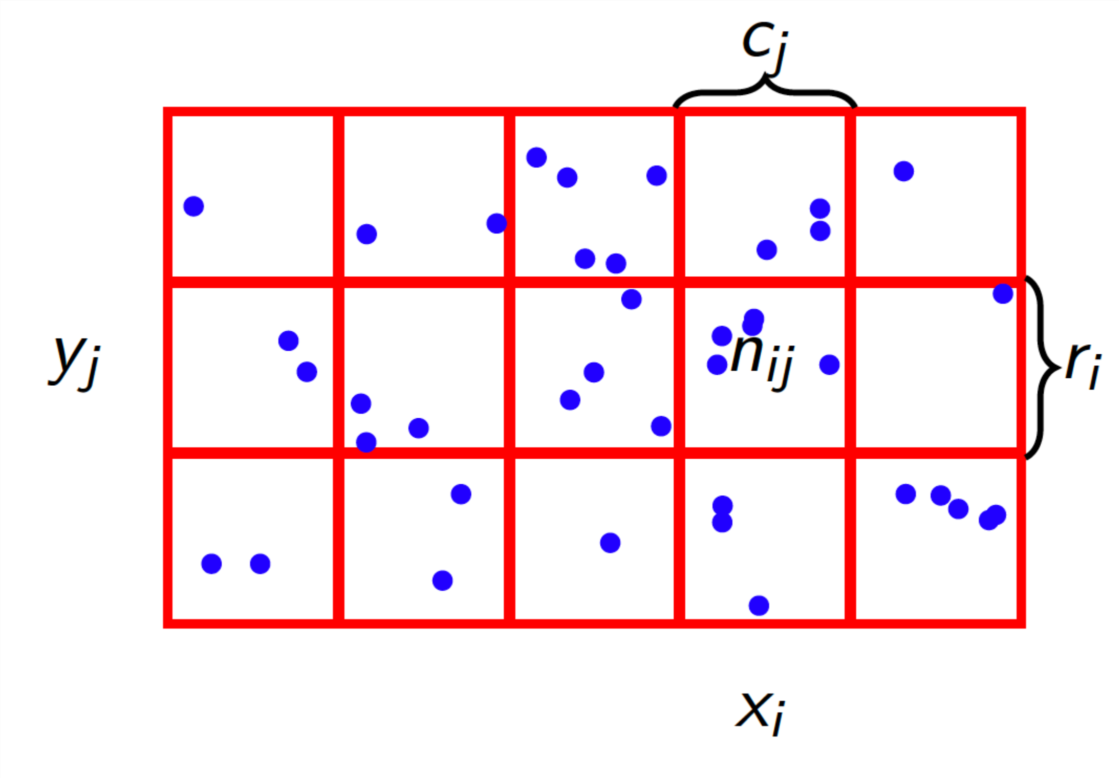
Supplement A: Probability Theory

Frequentist Probability



> p_{XY} is ***probability mass function*** over values that random variables X and Y can take.

$$Pr(X = x_i, Y = y_j) = p_{XY}(x_i, y_j)$$

X is a random variable that can take any value x_i, so does Y with y_i

If we sample (X, Y) a large number of times N:

- n_{ij} is the number of times $X = x_i, Y = y_j$
- c_i is the number of times $X = x_i$
- r_j is the number of times $Y = y_j$

> ***Probability mass functions*** capture the relative frequency of outcomes

Probabilities

Marginal Probability:

$$Pr(X = x_i) = p_X(x_i) = \frac{c_i}{N}$$

Joint Probability:

$$Pr(X = x_i, Y = y_j) = p_{XY}(x_i, y_j) = \frac{n_{ij}}{N}$$

Conditional Probability:

$$Pr(Y = y_j | X = x_i) = p_{Y|X}(y_j|x_i) = \frac{n_{ij}}{c_i}$$

Rule

Sum Rule:

$$p_X(x) = \frac{c_i}{N} = \frac{1}{N}\sum_j n_{ij} = \sum_j p_{xy}(x_i, y_j)$$

> ***Omit subscripts for brevity:***
>
> $$p(x) = \sum_y p(x, y)$$

Product Rule:

$$p_{XY}(x_i, y_j) = \frac{n_{ij}}{N} = \frac{n_{ij}}{c_i} \times \frac{c_i}{N} = p_{Y|X}(y_j|x_i)p_X(x_i)$$

> ***Omit subscripts for brevity:***
>
> $$p(x, y) = p(y|x)p(x)$$

Application in 3 Random Variables Case:

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ p(x, y, z) &= p(x, y|z)p(z) \\ &= p(y, z|x)p(x) \end{aligned}$$

If $p_{XY}(x, y) = p_X(x)p_Y(y)$, we say X and Y are ***independent***

Example: Blood Test for Disease

Randome Variables:

- ***A*** disease status (***ill*** or ***healthy***)
- ***B*** blood test (***+ve*** or ***-ve***)

$$p_{AB}(a, b) = p_{A|B}(a|b)p_B(b) = p_{B|A}(b|a)p_A(a)$$

$$\begin{aligned} p_A(ill) &= Pr(person\ has\ disease) = 1\% \\ p_B(+ve) &= Pr(person\ has\ +ve\ blood\ test) = 10\% \\ p_{B|A}(+ve|ill) &= Pr(blood\ test\ is\ +ve\ given\ person\ is\ ill) = 70\% \\ p_{A|B}(ill|+ve) &= Pr(person\ is\ ill\ given\ blood\ test\ is\ +ve) = 7\% \end{aligned}$$

Bayes Rule

$$p(y|x) = \frac{p(x|y)p(y)}{p(x)}$$

$p(x) = \sum_y p(x|y)p(y)$ ***normalises*** the equation

Practically:

$$p(y|x) \propto p(x|y)p(y)$$
$$posterior \propto likelihood \times prior$$

> ***Note that:***
>
> The ***proportional symbol***, $\propto$, means that *when varying the parameter of interest* (y) *in the expression on the left the only terms that vary are those on the right*. In this case, the numerator from ***Bayes Rule***, $p(x)$, does not depend on y and so can be ignored. Another way to think about this is that the ***shape*** of the function $p(y|x)$ is the same as $p(x|y)p(y)$ (for fixed x and varying y)

Frequentist v. Bayesian

Frequentist

In the ***frequentist*** perspective, probability distributions represent ***expected outcomes given a large number of trails***, e.g.

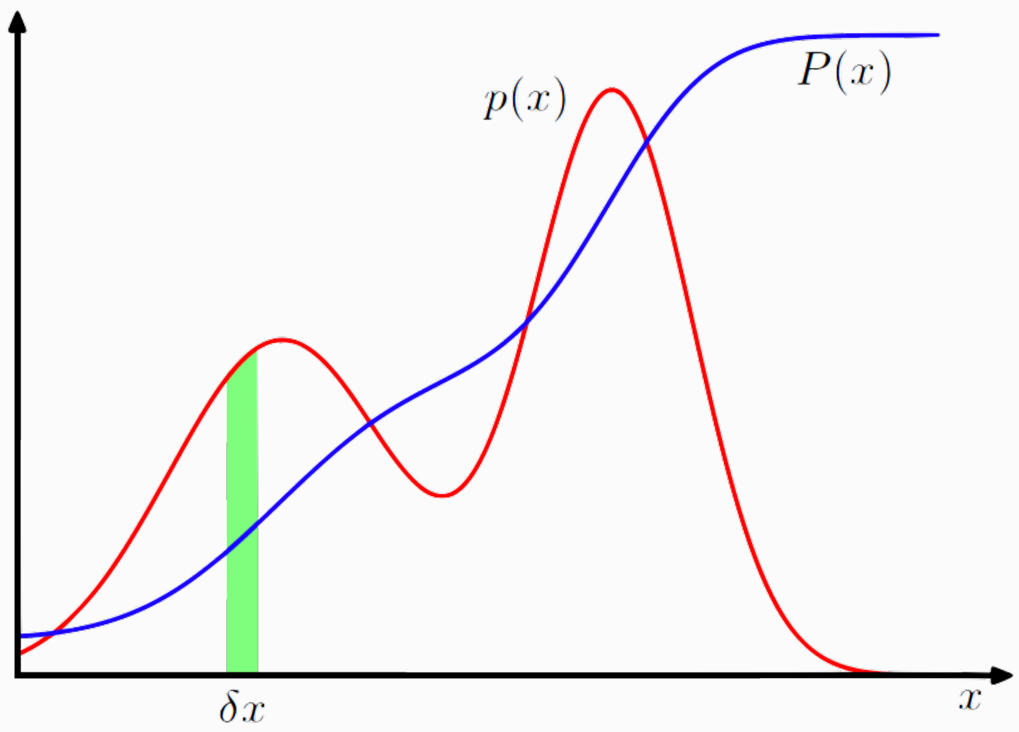
$$\mathbb{E}[X] = \lim_{N\to\infty} \frac{1}{N} \sum_n x_n$$

Bayesian

Bayesian Inference involves shifting the perspective in order to reason about vents that may ***happen only once***, in which ***probability is a measure of belief***

> This is a key concept and a powerful one, We are no longer restricted to thinking about probability as measuring the relative frequencies of particular outcomes in repeatable experiments. Instead, probability is a measure of belief in, or plausibility of, an outcome

Probability Densities



Probability Density:

$$Pr(X \in (a,b)) = \int_a^b p_X(x)dx$$

Cumulative Distribution:

$$P_X(z) = Pr(X < z) = \int_{-\infty}^{z} p_X(x)dx$$

> ***Constraints:*** $p_X(x) \geq 0$ and $\int_{-\infty}^{\infty} p_X(x)dx = 1$

For vectors of continuous variables $\mathbf{X} = (X_1, \ldots, X_k)$ then

$$p_{\mathbf{X}} = p_{X_1,...,X_k}$$

Probability Density and Bayes

The ***Sum***, ***Product*** and ***Bayes Rules*** still apply:

Product Rule:

$$p(x,y) = p(y|x)p(x)$$

Sum Rule:

$$p(x) = \int p(x,y)dy$$

Bayes Rule:

$$p(x|y) = \frac{p(y|x)p(x)}{p(y)}$$

> ***Note that:***
>
> Apparently, $p_X(x), p(X = x), p(x)$ and $p(X)$ are used ***interchangeably***. Remember X is the ***random variable***, and x is the ***value it takes***.

Expectations

The average value of random variable X under probability distribution $p_X(x)$ is called the expectation of X.

For discrete X:

$$\mathbb{E}[x] = \sum_x x p_X(x)$$

For continuous X:

$$\mathbb{E}[x] = \int x p_X(x)dx$$

We can approximate $\mathbb{E}[x]$ with samples $\{x_i\}_{i=1}^N$ using:

$$\mathbb{E}[x] \approx \frac{1}{N}\sum_i x_i$$

> ***Note that:***
>
> When performing approximation, the probability weighting is unknown and so can no longer explicitly weigh the terms. Instead, the obersed frequency of particular values (discrete) or within particular ranges (continuous) is used as a subtitle.

Expectations of Functions

The average value of any function of X, say $f(X)$, under $p_X(x)$ is called the expectation of f.

For discrete X:

$$\mathbb{E}[f] = \sum_x f(X)p_X(x)$$

For continuous X:

$$\mathbb{E}[f] = \int f(x)p_X(x)dx$$

We can approximate $\mathbb{E}[f(x)]$ with samples $\{x_i\}_{i=1}^N$ using:

$$\mathbb{E}[f] \approx \frac{1}{N}\sum_i f(x_i)$$

Variance

The ***variance*** of $f(X)$ is a measure of its ***variability/spread***:

$$\begin{aligned}\text{var}[f] &= \mathbb{E}[(f(x) - \mathbb{E}[f(x)])^2] \\ &= \mathbb{E}[f(x)^2] - \mathbb{E}[f(x)]^2\end{aligned}$$

In particular, the ***variance*** of X itself (denoted σ_X^2 or σ^2) is:

$$\text{var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \sigma_X^2$$

Standard deviation is *the square root of the variance*, typically denoted σ_x or σ

Covariance

Covariance measures the extent to which two variables (X and Y) vary together

$$\text{cov}[X, Y] = \mathbb{E}_{XY}[(x - \mathbb{E}[x])(y - \mathbb{E}[y])]$$

Here $\mathbb{E}_{XY}$ indicates the *expectation over* ***joint distribution*** p_{XY}, e.g. for continuous random variables:

$$\mathbb{E}_{XY}[f(x, y)] = \int\int f(x, y)p_{XY}(x, y)dxdy$$

Correlation is a related measure between -1 and 1:

$$\text{corr}[X, Y] = \frac{\text{cov}[X, Y]}{\sigma_X \sigma_Y}$$

More on Expectations

When it is unclear which probability distribution is being considered for a random variable, this can be specified with a ***subscript***. E.g. ***Cross-entropy*** for discrete random variable X between true distribution p_X and artificial distribution q_X is given by:

$$\mathbb{E}_{x \sim p_X}[-\ln q_X(x)] = -\sum_x \ln q_X(x) p_X(x)$$

Conditional Expectations are expectations under some conditional probability distribution, $p_{X|Y}$ where $Y = y$, e.g. for continuous random variables:

$$\mathbb{E}[f(x)|Y = y] = \mathbb{E}_{X \sim p_{X|Y}(.|y)}[f(X)] = \int f(x) p_{X|Y}(x|y) dx$$

> ***Note that:***
>
> Care should be taken with conditional probabilities and condition expectations. In the value to the right of the conditioning line is impossible then the probability distribution, and as a consequence the expectation, is ***undefined***.