Original Heuristic that Dominates the Manhattan Heuristic

Settings. Denote the state G for a Hua Rong Dao sliding puzzle as a set by:

$$G = \{E, C, H, V, S\}, \text{ with } |\cup_{X \in G} X| = 20$$

with the conditions: $|G| = 5$, $\forall X \in G \subset B = \{(x, y) : (x, y) \in \mathbb{Z} \times \mathbb{Z}, 0 \leq x \leq 4, 0 \leq y \leq 3\}$, $|E| = 2, |C| = 4, |H| + |V| = 10, |S| = 4$, and for C, there exists $(x, y) \in B$ such that

$$C = \{(x, y), (x, y + 1), (x + 1, y), (x + 1, y + 1)\},$$

and for H, V, there exists $P = \{(x, y) : (x, y) \in B\}$ with $|P| = 5$, $P_H \sqcup P_V = P$ such that

$$H = \{(x, y) : (x, y) \in P_H\} \sqcup \{(x, y + 1) : (x, y) \in P_H\},$$

$$V = \{(x, y) : (x, y) \in P_V\} \sqcup \{(x + 1, y) : (x, y) \in P_V\} \text{ and } H \sqcup V.$$

For $X \in G$, denote $X_{TL} \subset X$ such that $E_{TL} = E$, $C_{TL} = \{(x, y) \in C : \arg_{(x,y) \in C} \min |x+y|\}$, $S_{TL} = S, H_{TL} = P_H, V_{TL} = P_V$. In descriptive terms, $X \in G$ is the set of the indices corresponding to the locations occupied by the pieces of X in the game board B. E, C, H, V, S are the empty pieces, Caocao, horizontal pieces, vertical pieces, single pieces. $\cdot_{TL}$ refers to the most top-left indices, which is how I inplemented in my `hrd.py` in the order of *row* then *column*. $\sum_{X \in G} |X_{TL}| = 8$ since there are in total 8 pieces. The goal of Hua Rong Dao is to slide the pieces such that $C_{TL} = \{(3, 1)\}$. It is too tedious to describe sliding in math.

Proof. Let $h_M(G)$ denote the manhattan heuristic function for a state G. We may claim without proof that it is both admissible and consistent. Let $n_{obs}(G)$ denote the number of pieces intersecting with the target region $T = \{(3, 1), (3, 2), (4, 1), (4, 2)\}$, where $\cdot_{obs}$ is the abbreviation for "obstacles". We construct the original heuristic function:

$$h(G) = h_M(G) + n_{obs}(G).$$

We argue that h is both admissible and consistent. For one move from G_1 to G_2 in HRD, only one piece is moved, and pieces other than the empty slots do not move, and the piece moved must be: either Caocao or *not* Caocao, thus, 1) there is a change of the least number of moves to reach the goal state with $n_{obs}(G_1) = n_{obs}(G_2)$, the difference is then reduced to $h_M(G_1)$ and $h_M(G_2)$, but we have that $h_M(\cdot)$ is both admissible and consistent; 2) the second case Caocao is not moved so $h_M(G_1) = h_M(G_2)$, notice that the piece moved either does not affect $n_{obs}(G_1)$ or $n_{obs}(G_2)$ or moves to intersect with T so that $n_{obs}(G_2) - n_{obs}(G_1) = 1$ and this still gives $h_M(G_2) + n_{obs}(G_2) \leq h^*(G_2)$ which implies h in this case still admissible since we still have at least $n_{obs}(G_2)$ moves to clear out the "obstacles" and then at least $h_M(G_2)$ moves for Caocao to completely occupy T. $0 \leq h(G_1) - h(G_2) \leq 1 = g(G_1, G_2)$ immediately implies the consistency of h since G_1, G_2 are neighboring states. For all G, $h(G) \geq h_M(G)$ by construction, and trivially there exists G such that $h(G) > h_M(G)$ when there is at least an obstacle intersecting region T in state G. Therefore, h dominates h_M as desired. $\square$