Bits, Bytes and Integers

Introduction to Computer Systems

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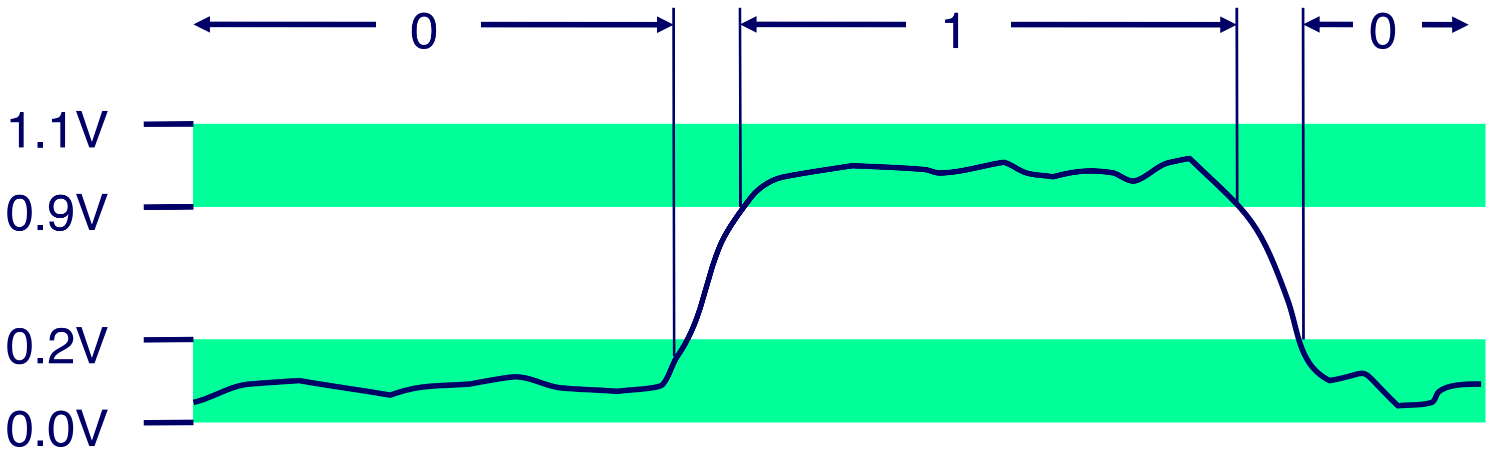
Today: Bits, Bytes, and Integers

- **Representing information as bits**
- **Bit-level manipulations**
- **Integers**
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
  - Summary
- **Representations in memory, pointers, strings**

Everything is bits

- **Each bit is 0 or 1**
- **By encoding/interpreting sets of bits in various ways**
  - Computers determine what to do (instructions)
  - … and represent and manipulate numbers, sets, strings, etc…
- **Why bits? Electronic Implementation**
  - Easy to store with bi-stable elements
  - Reliably transmitted on noisy and inaccurate wires



For example, can count in binary

- **Base 2 Number Representation**
  - Represent 15213_{10} as 11101101101101_2
  - Represent 1.20_{10} as $1.0011001100110011[0011]\ldots_2$
  - Represent 1.5213×10^4 as $1.1101101101101_2 \times 2^{13}$

Encoding Byte Values

- **Byte = 8 bits**
  - Binary 00000000_2 to 11111111_2
  - Decimal: 0_{10} to 255_{10}
  - Hexadecimal 00_{16} to FF_{16}
    - Base 16 number representation
    - Use characters '0' to '9' and 'A' to 'F'
    - Write $FA1D37B_{16}$ in C as
      - 0xFA1D37B
      - 0xfa1d37b

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Example Data Representations

C Data Type	Typical 32-bit	Typical 64-bit	x86-64
`char`	1	1	1
`short`	2	2	2
`int`	4	4	4
`long`	4	8	8
`float`	4	4	4
`double`	8	8	8
`long double`	–	–	10/16
pointer	4	8	8

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Boolean Algebra

- **Developed by George Boole in 19th Century**
  - Algebraic representation of logic
    - Encode "True" as 1 and "False" as 0

And

- A&B = 1 when both A=1 and B=1

&	0	1
0	0	0
1	0	1

Or

- A|B = 1 when either A=1 or B=1

|	0	1
0	0	1
1	1	1

Not

- ~A = 1 when A=0

~	
0	1
1	0

Exclusive-Or (Xor)

- A^B = 1 when either A=1 or B=1, but not both

^	0	1
0	0	1
1	1	0

General Boolean Algebras

- **Operate on Bit Vectors**
  - Operations applied bitwise

```
  01101001     01101001     01101001
& 01010101   | 01010101   ^ 01010101   ~ 01010101
  --------     --------     --------     --------
  01000001     01111101     00111100     10101010
```

- **All of the Properties of Boolean Algebra Apply**

Example: Representing & Manipulating Sets

- **Representation**
  - Width w bit vector represents subsets of {0, …, w–1}
  - $a_j = 1$ if $j \in A$

    - 01101001 { 0, 3, 5, 6 }
    - *76543210*

    - 01010101 { 0, 2, 4, 6 }
    - *76543210*

- **Operations**

&	Intersection	01000001	{ 0, 6 }
|	Union	01111101	{ 0, 2, 3, 4, 5, 6 }
^	Symmetric difference	00111100	{ 2, 3, 4, 5 }
~	Complement	10101010	{ 1, 3, 5, 7 }

Bit-Level Operations in C

- **Operations &, |, ~, ^ Available in C**
  - Apply to any “integral” data type
    - `long, int, short, char, unsigned`
  - View arguments as bit vectors
  - Arguments applied bit-wise
- **Examples (Char data type)**
  - `~0x41` → `0xD6`
    - `~01000001`$_2$ → `10111110`$_2$
  - `~0x00` → `0xFF`
    - `~00000000`$_2$ → `11111111`$_2$
  - `0x69 & 0x55` → `0x41`
    - `01101001`$_2$ `&` `01010101`$_2$ → `01000001`$_2$
  - `0x69 | 0x55` → `0x7D`
    - `01101001`$_2$ `|` `01010101`$_2$ → `01111101`$_2$

Contrast: Logic Operations in C

- **Contrast to Logical Operators**
  - `&&, ||, !`
    - View 0 as “False”
    - Anything nonzero as “Tr
    - Always return 0 or 1
    - Early termination

Watch out for && vs. & (and || vs. |)… one of the more common oopsies in C programming

- **Examples (char data type)**
  - `!0x41` → `0x00`
  - `!0x00` → `0x01`
  - `!!0x41` → `0x01`

  - `0x69 && 0x55` → `0x01`
  - `0x69 || 0x55` → `0x01`
  - `p && *p` (avoids null pointer access)

Shift Operations

- **Left Shift: `x << y`**
  - Shift bit-vector **`x`** left **`y`** positions
    - – Throw away extra bits on left
    - Fill with 0's on right
- **Right Shift: `x >> y`**
  - Shift bit-vector **`x`** right **`y`** positions
    - Throw away extra bits on right
  - Logical shift
    - Fill with 0's on left
  - Arithmetic shift
    - Replicate most significant bit on left
- **Undefined Behavior**
  - Shift amount < 0 or ≥ word size

Argument `x`	01100010
<< 3	00010*000*
Log. >> 2	*00*011000
Arith. >> 2	*00*011000

Argument `x`	10100010
<< 3	00010*000*
Log. >> 2	*00*101000
Arith. >> 2	*11*101000

An Example

Listing 2 Tagging stack pointers upon allocation.

```
// assuming: 'ptr' is target allocation
region_base = ptr & (~((1 << 24)-1));
distance = ptr - region_base;
jumps = distance >> size_class_power;
tag = jumps & 15;
ptr = ptr | (tag << 56);
```

F. Gorter, T. Kroes, H. Bos and C. Giuffrida, "Sticky Tags: Efficient and Deterministic Spatial Memory Error Mitigation using Persistent Memory Tags," 2024 IEEE Symposium on Security and Privacy (SP), San Francisco, CA, USA, 2024, pp. 4239-4257

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Encoding Integers

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

Sign Bit

```
short int x =  15213;
short int y = -15213;
```

- **C `short` 2 bytes long**

	Decimal	Hex	Binary
`x`	15213	`3B 6D`	`00111011 01101101`
`y`	-15213	`C4 93`	`11000100 10010011`

- **Sign Bit**
  - For 2's complement, most significant bit indicates sign
    - 0 for nonnegative
    - 1 for negative

Two-complement Encoding Example (Cont.)

```
x =      15213: 00111011 01101101
y =     -15213: 11000100 10010011
```

Weight	15213		-15213	
1	1	1	1	1
2	0	0	1	2
4	1	4	0	0
8	1	8	0	0
16	0	0	1	16
32	1	32	0	0
64	1	64	0	0
128	0	0	1	128
256	1	256	0	0
512	1	512	0	0
1024	0	0	1	1024
2048	1	2048	0	0
4096	1	4096	0	0
8192	1	8192	0	0
16384	0	0	1	16384
-32768	0	0	1	-32768
Sum		**15213**		**-15213**

Numeric Ranges

- **Unsigned Values**
  - *UMin* = 0
    000…0
  - *UMax* = $2^w - 1$
    111…1

- **Two's Complement Values**
  - *TMin* = -2^{w-1}
    100…0
  - *TMax* = $2^{w-1} - 1$
    011…1
- **Other Values**
  - Minus 1
    111…1

Values for *W* = 16

	Decimal	Hex	Binary
`UMax`	65535	`FF FF`	`11111111 11111111`
`TMax`	32767	`7F FF`	`01111111 11111111`
`TMin`	-32768	`80 00`	`10000000 00000000`
`-1`	-1	`FF FF`	`11111111 11111111`
`0`	0	`00 00`	`00000000 00000000`

Values for Different Word Sizes

	W			
	8	**16**	**32**	**64**
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

- **Observations**
  - $|TMin| = TMax + 1$
    - Asymmetric range
  - $UMax = 2 * TMax + 1$

- **C Programming**
  - #include <limits.h>
  - Declares constants, e.g.,
    - ULONG_MAX
    - LONG_MAX
    - LONG_MIN
  - Values platform specific

Unsigned & Signed Numeric Values

X	B2U(*X*)	B2T(*X*)
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	–8
1001	9	–7
1010	10	–6
1011	11	–5
1100	12	–4
1101	13	–3
1110	14	–2
1111	15	–1

- **Equivalence**
  - Same encodings for nonnegative values
- **Uniqueness**
  - Every bit pattern represents unique integer value
  - Each representable integer has unique bit encoding
- **$\Rightarrow$ Can Invert Mappings**
  - $U2B(x) = B2U^{-1}(x)$
    - Bit pattern for unsigned integer
  - $T2B(x) = B2T^{-1}(x)$
    - Bit pattern for two's comp integer

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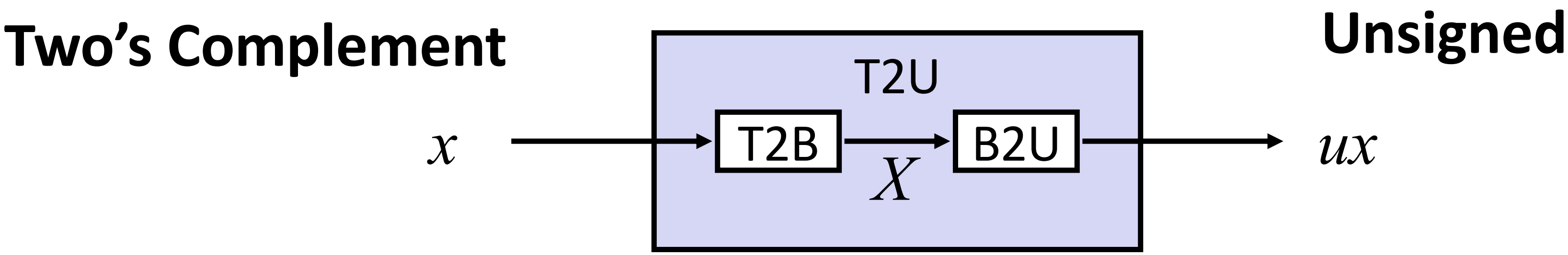
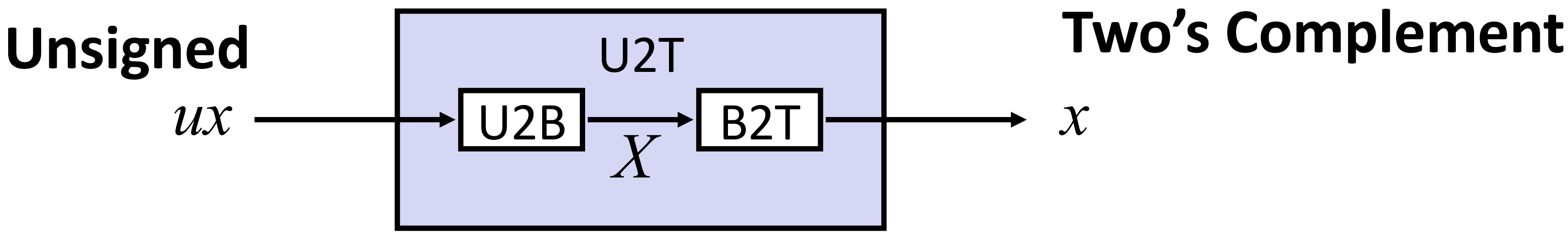
Mapping Between Signed & Unsigned



Maintain Same Bit Pattern



Maintain Same Bit Pattern

- **Mappings between unsigned and two's complement numbers:**
  Keep bit representations and reinterpret

Mapping Signed ↔ Unsigned

Bits	Signed	Unsigned
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	-8	8
1001	-7	9
1010	-6	10
1011	-5	11
1100	-4	12
1101	-3	13
1110	-2	14
1111	-1	15

→ T2U →

← U2T ←

Mapping Signed ↔ Unsigned

Bits	Signed	Unsigned
0000	0	0
0001	1	1
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1001	-7	9
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1100	-4	12
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1110	-2	14
1111	-1	15

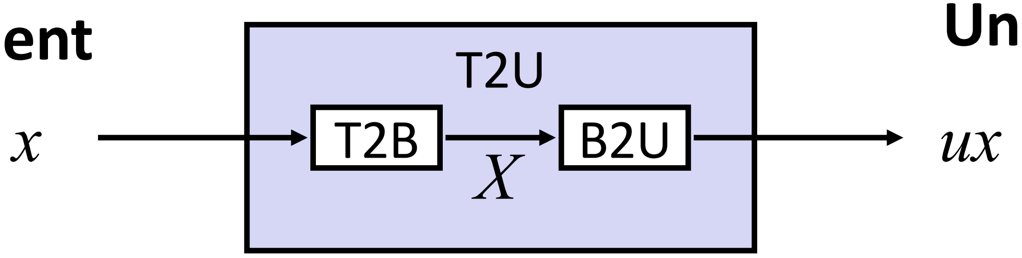
=

+/- 16

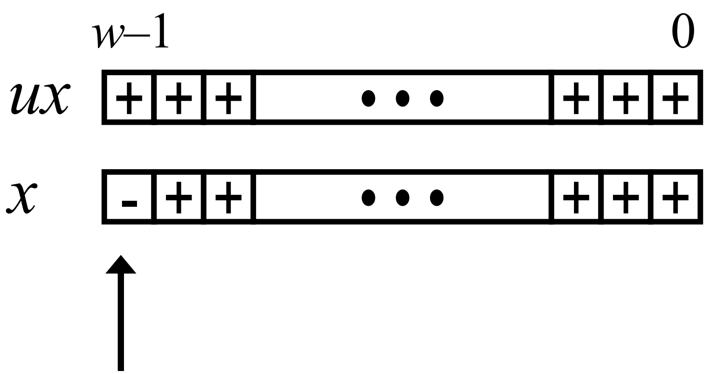
Relation between Signed & Unsigned

Two's Complement

Unsigned



Maintain Same Bit Pattern



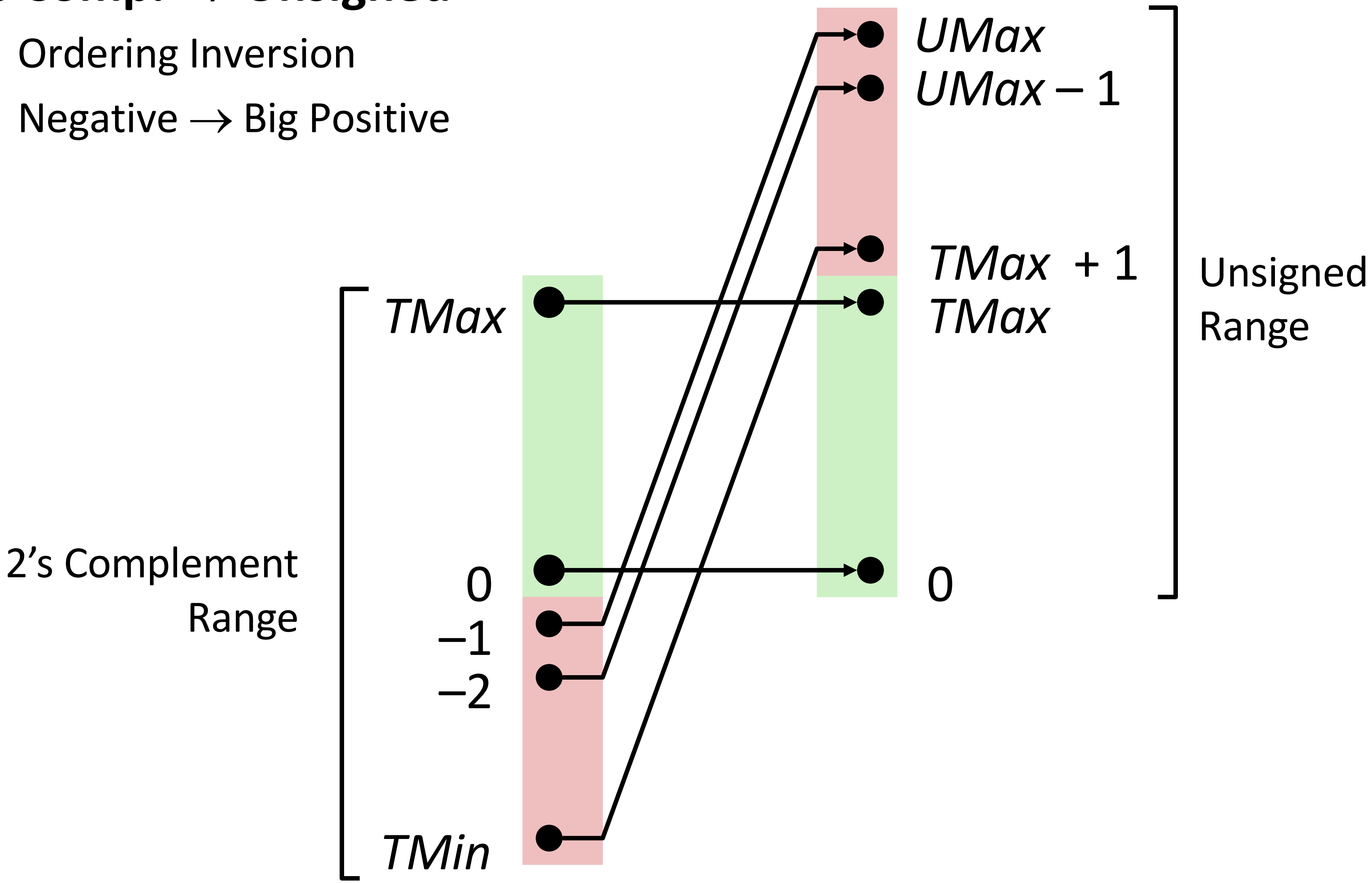
Large negative weight

becomes

Large positive weight

Conversion Visualized

- **2's Comp. → Unsigned**
  - Ordering Inversion
  - Negative → Big Positive



Signed vs. Unsigned in C

- **Constants**
  - By default are considered to be signed integers
  - Unsigned if have "U" as suffix

    ```
    0U, 4294967259U
    ```

- **Casting**
  - Explicit casting between signed & unsigned same as U2T and T2U

    ```
    int tx, ty;
    unsigned ux, uy;
    tx = (int) ux;
    uy = (unsigned) ty;
    ```

  - Implicit casting also occurs via assignments and procedure calls

    ```
    tx = ux;
    uy = ty;
    ```

Casting Surprises

- **Expression Evaluation**
  - If there is a mix of unsigned and signed in single expression, ***signed values implicitly cast to unsigned***
  - Including comparison operations **<, >, ==, <=, >=**
  - Examples for $W = 32$: **TMIN = -2,147,483,648 , TMAX = 2,147,483,647**

Constant$_1$	**Constant$_2$**	**Relation**	**Evaluation**
0	0U	**==**	**unsigned**
-1	0	**<**	**signed**
-1	0U	**>**	**unsigned**
2147483647	-2147483647-1	**>**	**signed**
2147483647U	-2147483647-1	**<**	**unsigned**
-1	-2	**>**	**signed**
(unsigned)-1	-2	**>**	**unsigned**
2147483647	2147483648U	**<**	**unsigned**
2147483647	(int) 2147483648U	**>**	**signed**

Summary
Casting Signed ↔ Unsigned: Basic Rules

- **Bit pattern is maintained**
- **But reinterpreted**
- **Can have unexpected effects: adding or subtracting 2^w**

- **Expression containing signed and unsigned int**
  - `int` is cast to `unsigned`!!

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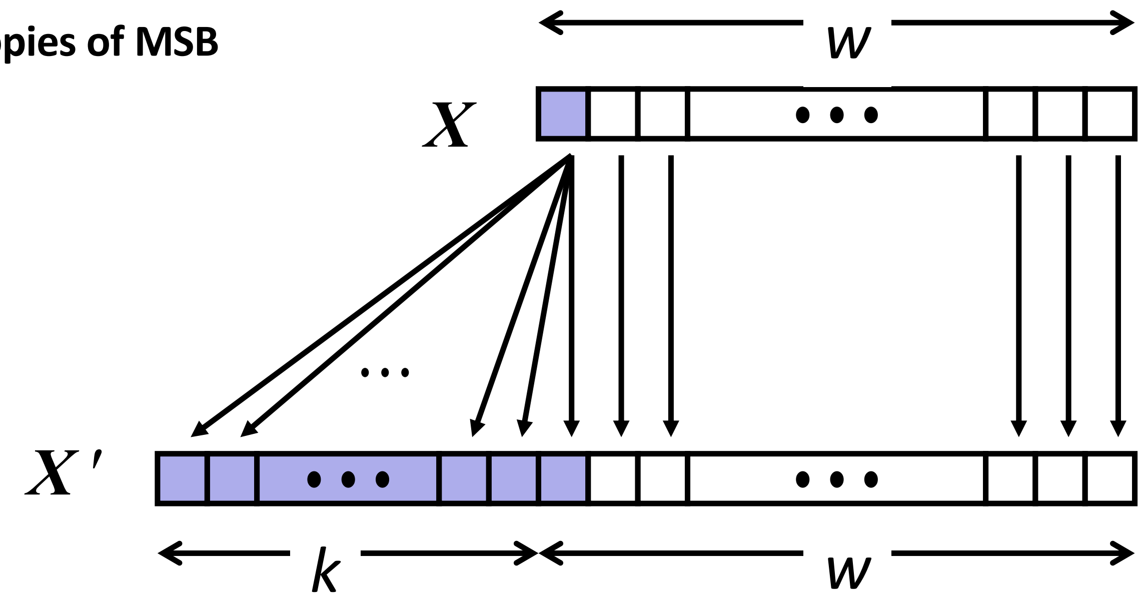
Sign Extension

- **Task:**
  - Given *w*-bit signed integer *x*
  - Convert it to *w*+*k*-bit integer with same value
- **Rule:**
  - Make *k* copies of sign bit:
  - $X' = \underbrace{x_{w-1}, \ldots, x_{w-1}}_{k \text{ copies of MSB}}, x_{w-1}, x_{w-2}, \ldots, x_0$



Sign Extension Example

```
short int x =  15213;
int       ix = (int) x;
short int y = -15213;
int       iy = (int) y;
```

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- Converting from smaller to larger integer data type
- C automatically performs sign extension

Summary:
Expanding, Truncating: Basic Rules

- **Expanding (e.g., short int to int)**
  - Unsigned: zeros added
  - Signed: sign extension
  - Both yield expected result

- **Truncating (e.g., unsigned to unsigned short)**
  - Unsigned/signed: bits are truncated
  - Result reinterpreted
  - Unsigned: mod operation
  - Signed: similar to mod
  - For small numbers yields expected behavior

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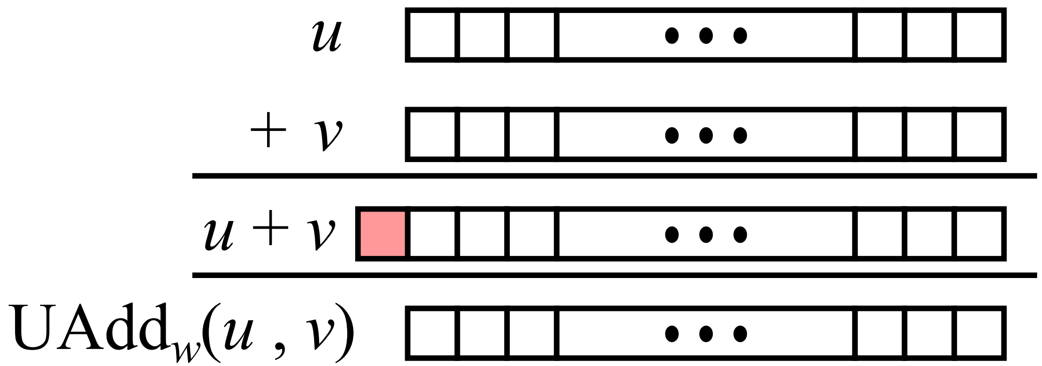
Unsigned Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



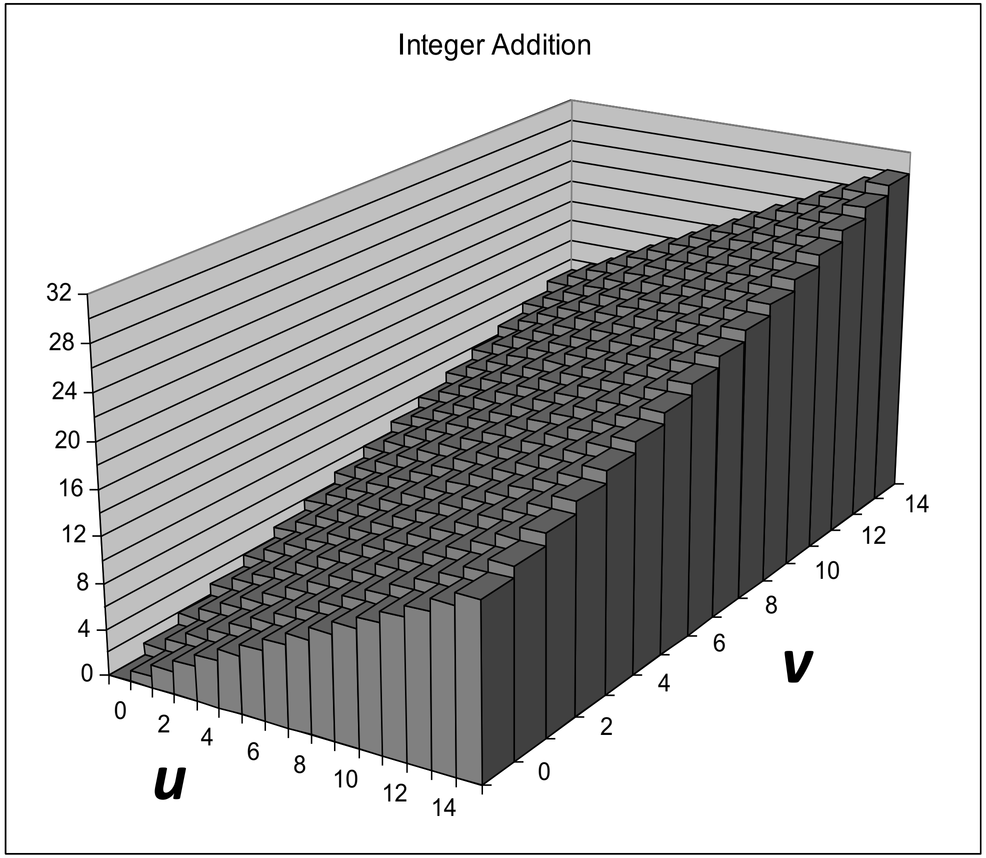
- **Standard Addition Function**
  - Ignores carry output
- **Implements Modular Arithmetic**

$$s \quad = \quad \mathrm{UAdd}_w(u\ ,\ v) \quad = \quad (u + v) \bmod 2^w$$

Visualizing (Mathematical) Integer Addition

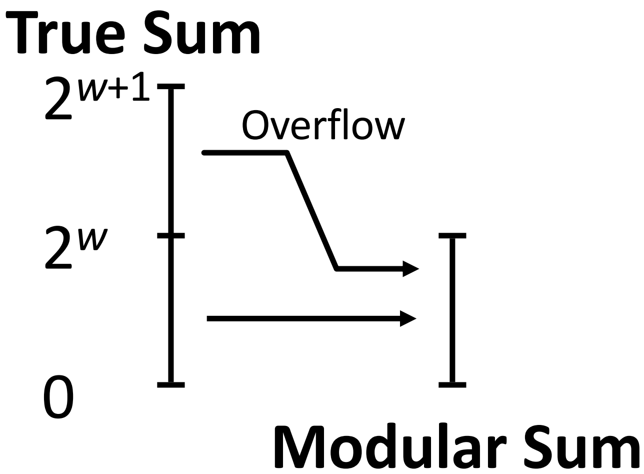
- **Integer Addition**
  - 4-bit integers *u*, *v*
  - Compute true sum $\mathrm{Add}_4(u, v)$
  - Values increase linearly with *u* and *v*
  - Forms planar surface

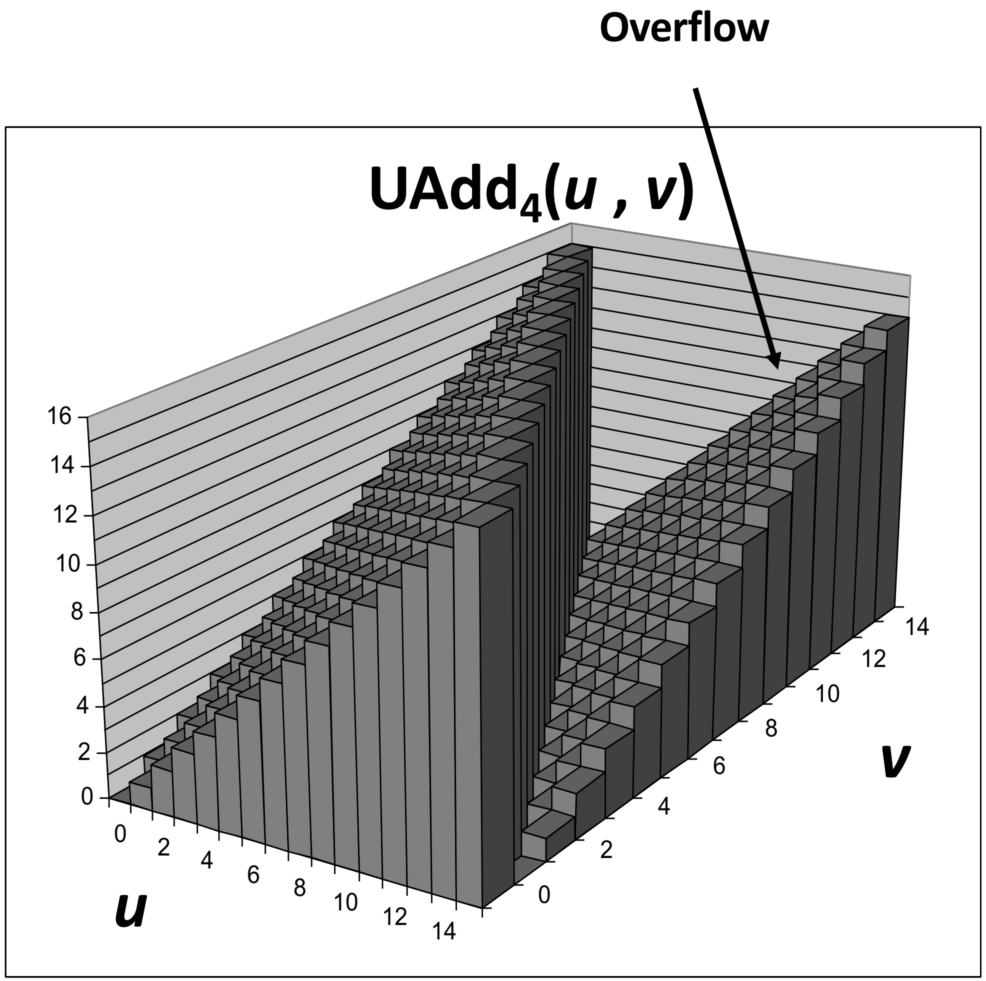
$\mathrm{Add}_4(u, v)$



Visualizing Unsigned Addition

- **Wraps Around**
  - If true sum ≥ 2^w
  - At most once



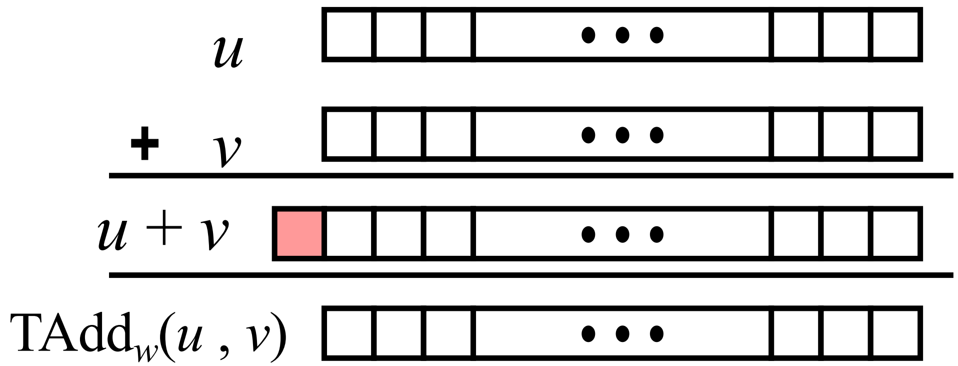


Two's Complement Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



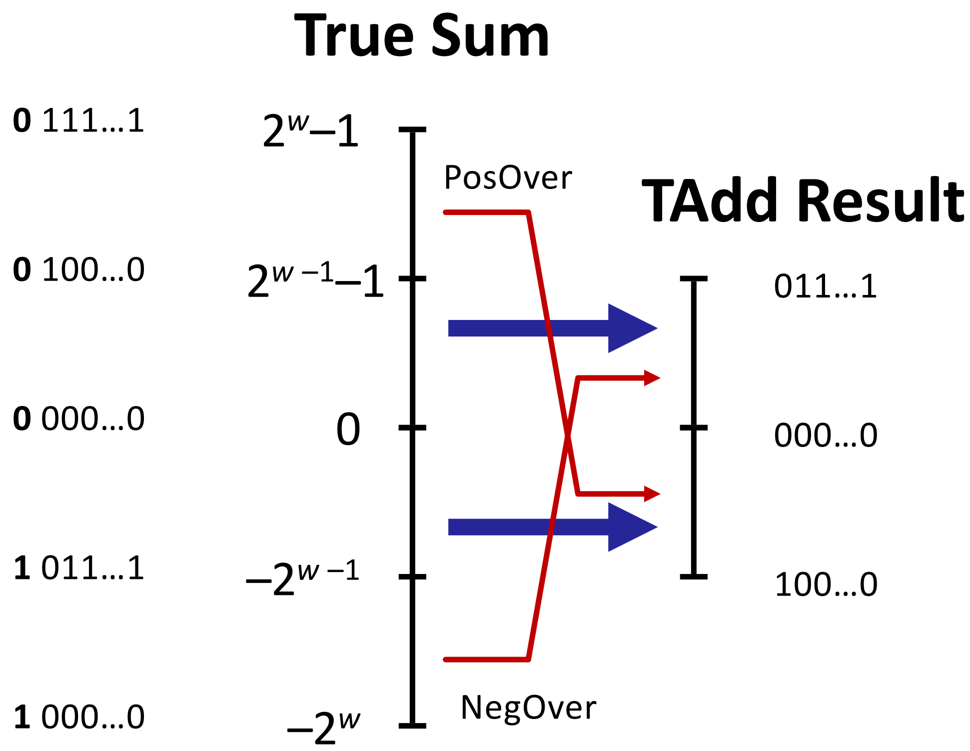
- **TAdd and UAdd have Identical Bit-Level Behavior**
  - Signed vs. unsigned addition in C:

```
int s, t, u, v;
s = (int) ((unsigned) u + (unsigned) v);
t = u + v
```

  - Will give `s == t`

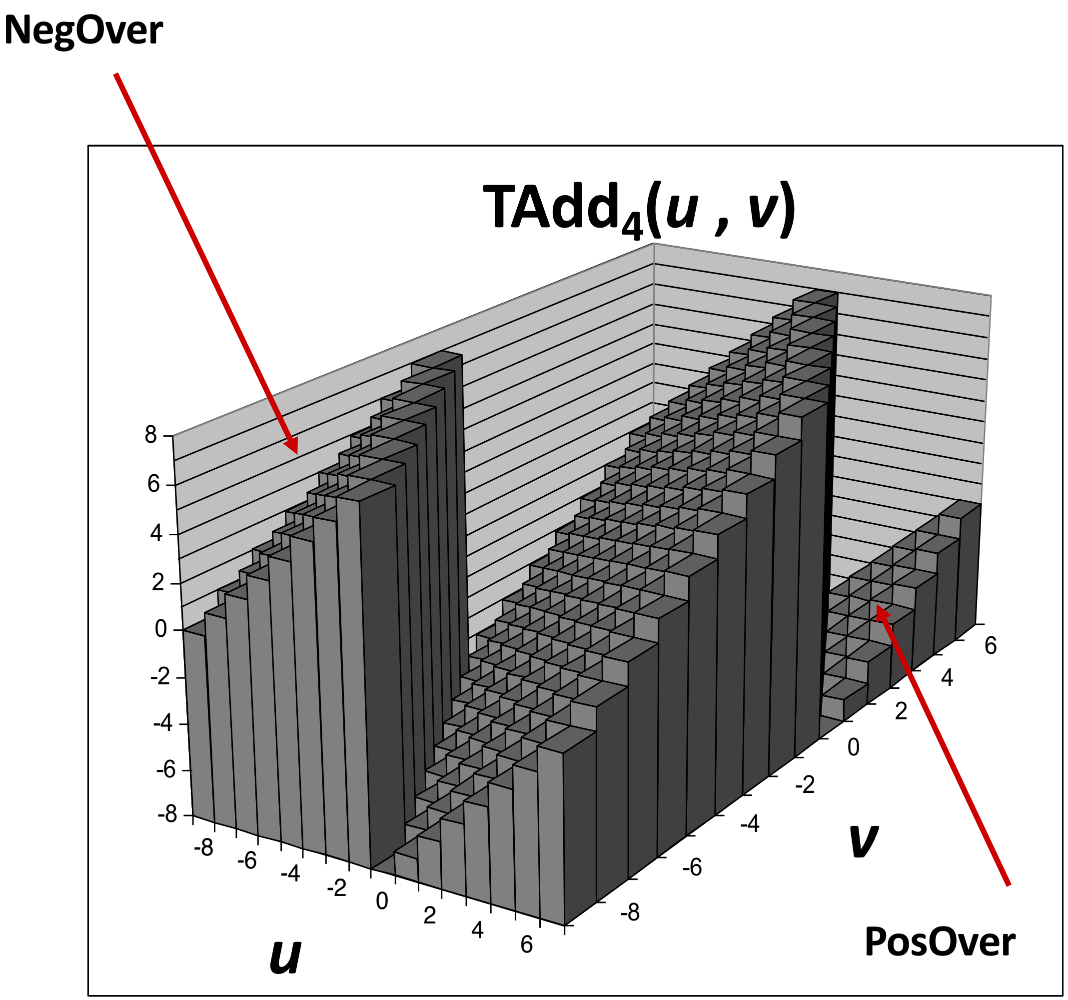
TAdd Overflow

- **Functionality**
  - True sum requires w+1 bits
  - Drop off MSB
  - Treat remaining bits as 2's comp. integer



Visualizing 2's Complement Addition

- **Values**
  - 4-bit two's comp.
  - Range from -8 to +7
- **Wraps Around**
  - If sum $\geq 2^{w-1}$
    - Becomes negative
    - At most once
  - If sum $< -2^{w-1}$
    - Becomes positive
    - At most once



Multiplication

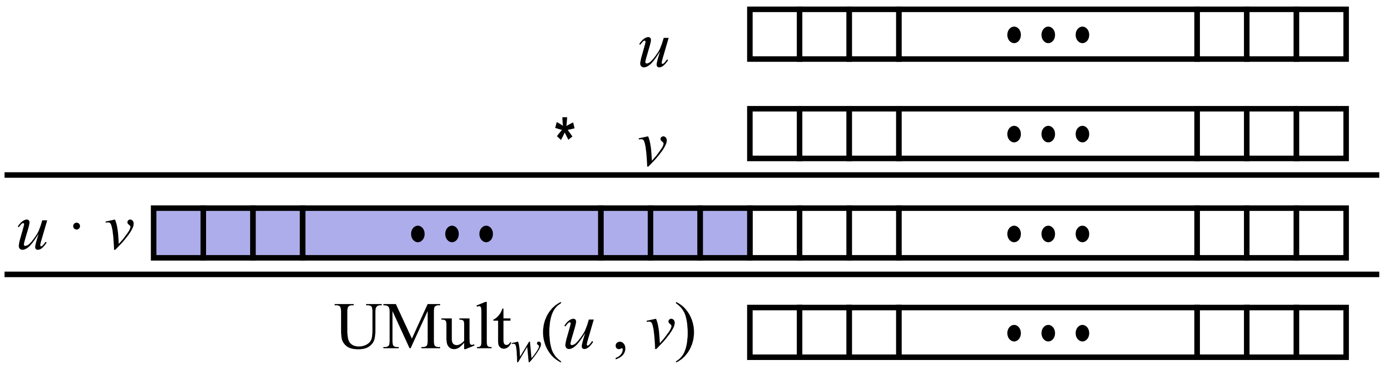
- **Goal: Computing Product of *w*-bit numbers *x*, *y***
  - Either signed or unsigned
- **But, exact results can be bigger than *w* bits**
  - Unsigned: up to 2*w* bits
    - Result range: $0 \le x * y \le (2^w - 1)^2 = 2^{2w} - 2^{w+1} + 1$
  - Two's complement min (negative): Up to 2*w*-1 bits
    - Result range: $x * y \ge (-2^{w-1})*(2^{w-1}-1) = -2^{2w-2} + 2^{w-1}$
  - Two's complement max (positive): Up to 2*w* bits, but only for $(TMin_w)^2$
    - Result range: $x * y \le (-2^{w-1})^2 = 2^{2w-2}$
- **So, maintaining exact results…**
  - would need to keep expanding word size with each product computed
  - is done in software, if needed
    - e.g., by "arbitrary precision" arithmetic packages

Unsigned Multiplication in C

Operands: *w* bits

True Product: 2**w* bits

Discard *w* bits: *w* bits



- **Standard Multiplication Function**
  - Ignores high order *w* bits
- **Implements Modular Arithmetic**

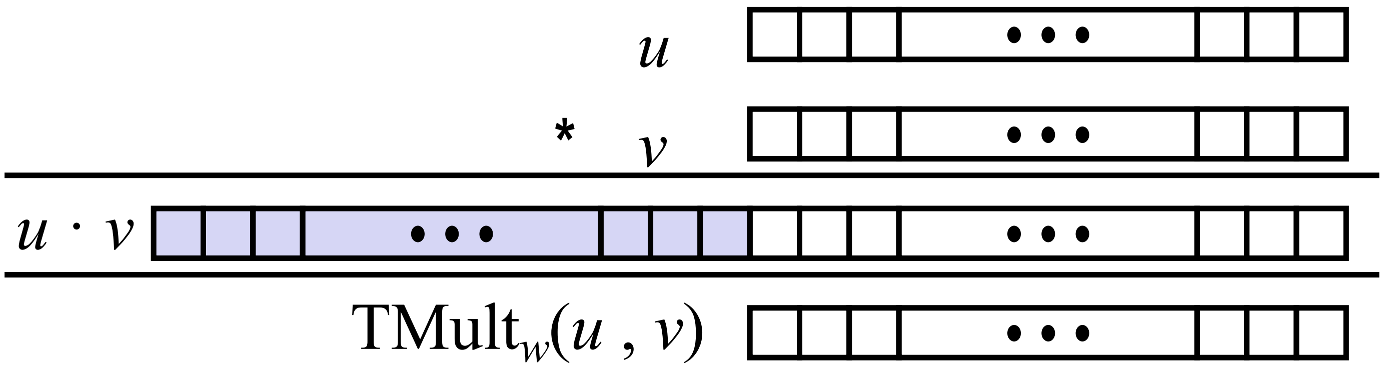
  $\text{UMult}_w(u, v) = \quad (u \cdot v) \mod 2^w$

Signed Multiplication in C

Operands: *w* bits

True Product: 2**w* bits

Discard *w* bits: *w* bits



- **Standard Multiplication Function**
  - Ignores high order *w* bits
  - Some of which are different for signed vs. unsigned multiplication
  - Lower bits are the same

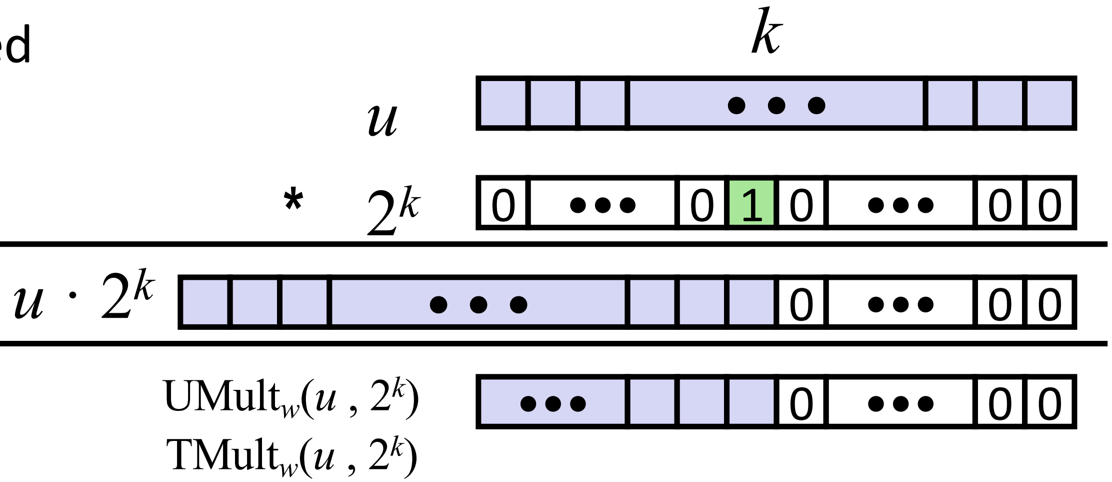
Power-of-2 Multiply with Shift

- **Operation**
  - `u << k` gives `u * 2`^{k}
  - Both signed and unsigned

Operands: *w* bits

True Product: *w*+*k* bits

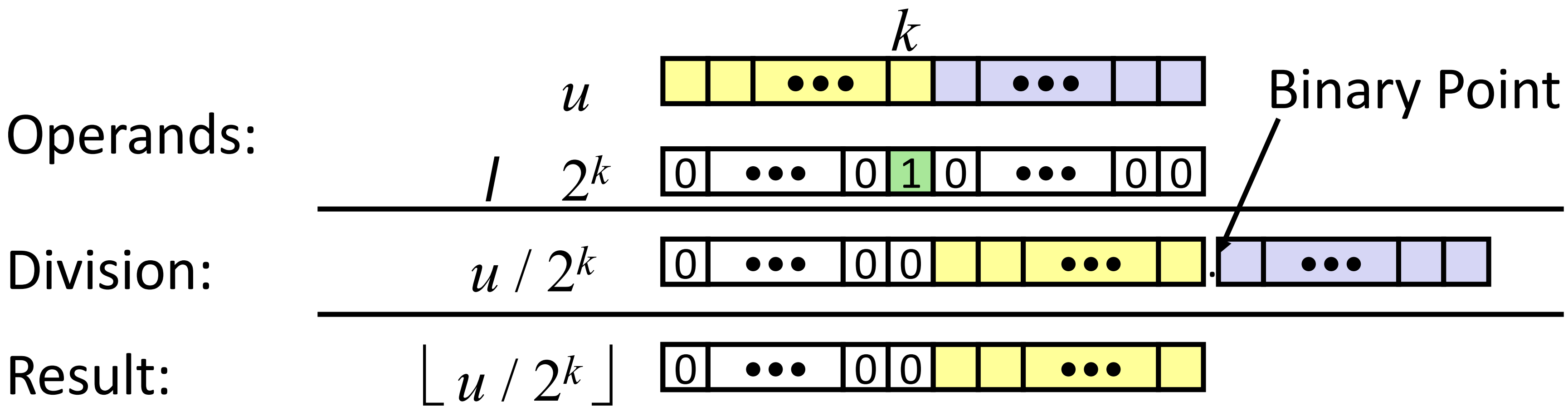
Discard *k* bits: *w* bits



- **Examples**
  - `u << 3 == u * 8`
  - `(u << 5) - (u << 3) == u * 24`
  - Most machines shift and add faster than multiply
    - Compiler generates this code automatically

Unsigned Power-of-2 Divide with Shift

- **Quotient of Unsigned by Power of 2**
  - `u >> k` gives $\lfloor \texttt{u} / 2^k \rfloor$
  - Uses logical shift

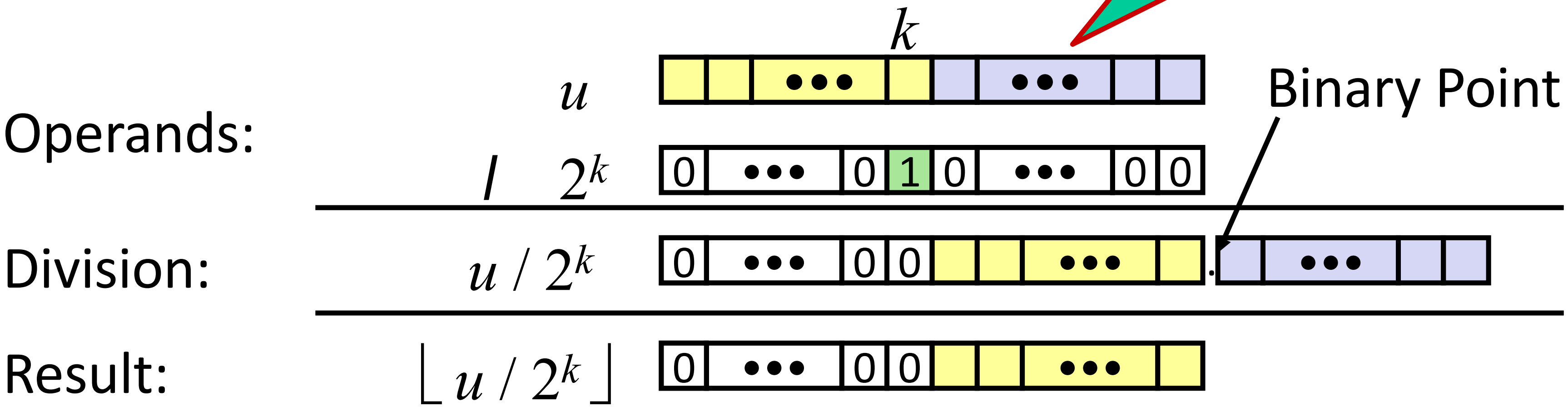


	Division	Computed	Hex	Binary
`x`	15213	15213	`3B 6D`	`00111011 01101101`
`x >> 1`	7606.5	7606	`1D B6`	`00011101 10110110`
`x >> 4`	950.8125	950	`03 B6`	`00000011 10110110`
`x >> 8`	59.4257813	59	`00 3B`	`00000000 00111011`

Signed Power-of-2 Divide with Shi

Add a bias:
(1<<k)-1

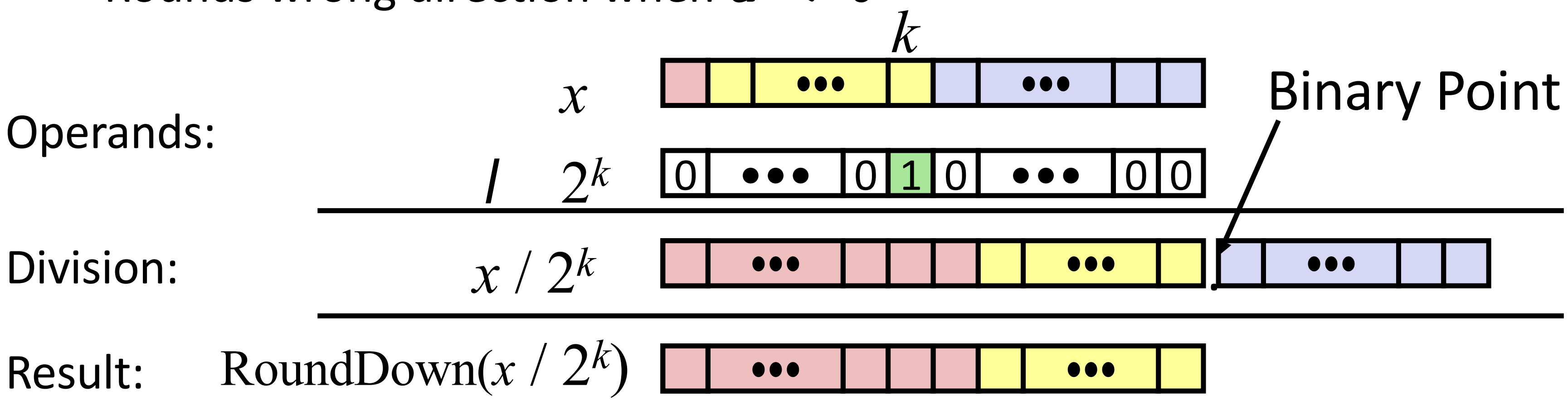
- **Quotient of Signed by Power of 2**
  - `u >> k` gives $\lfloor$ `u / 2`^{k} $\rfloor$
  - Uses arithmetic shift



	Division	Computed	Hex	Binary
`x`	-12340	-12340	`AF AA`	`11001111 11001100`
`x >> 1`	-6170.0	-6170	`E7 E6`	`11100111 11100110`
`x >> 4`	-771.25	-772	`FA FA`	`11111100 11111100`
`x >> 8`	-48.203125	-49	`FF AF`	`11111111 11001111`

Signed Power-of-2 Divide with Shift

- **Quotient of Signed by Power of 2**
  - `x >> k` gives $\lfloor$ `x / ` $2^k \rfloor$
  - Uses arithmetic shift
  - Rounds wrong direction when `u < 0`

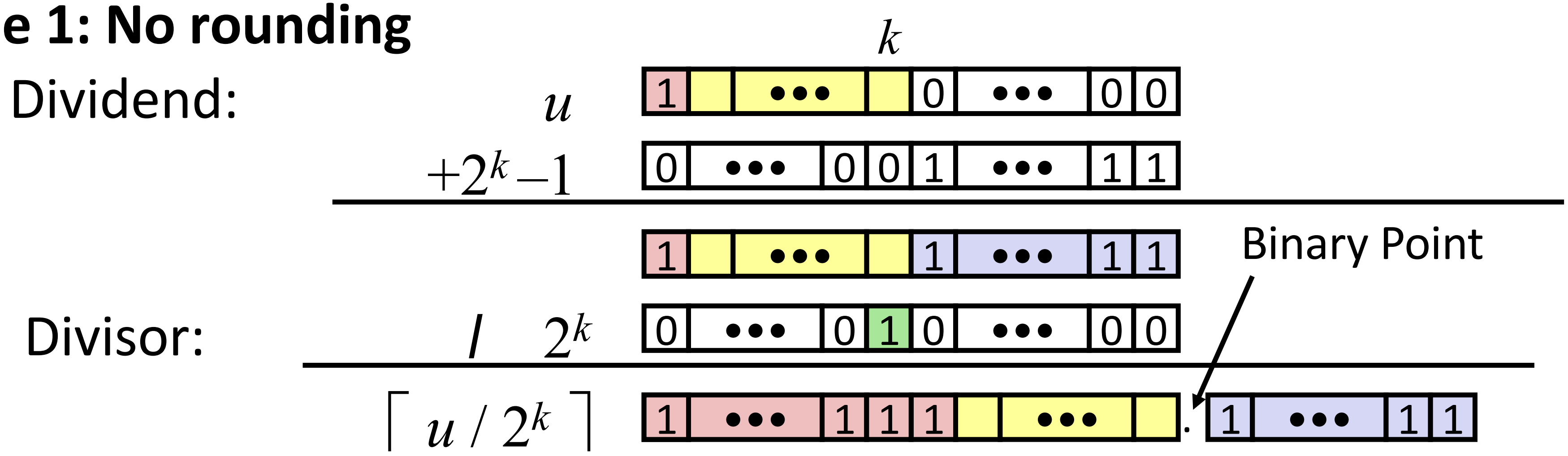


	Division	Computed	Hex	Binary
`y`	-15213	-15213	`C4 93`	`11000100 10010011`
`y >> 1`	-7606.5	-7607	`E2 49`	`11100010 01001001`
`y >> 4`	-950.8125	-951	`FC 49`	`11111100 01001001`
`y >> 8`	-59.4257813	-60	`FF C4`	`11111111 11000100`

Correct Power-of-2 Divide

- **Quotient of Negative Number by Power of 2**
  - Want $\lceil$ **`x / 2`**^{k} $\rceil$ (Round Toward 0)
  - Compute as $\lfloor$ **`(x+2`**^{k}**`-1) / 2`**^{k} $\rfloor$
    - In C: **`(x + (1<<k)-1) >> k`**
    - Biases dividend toward 0

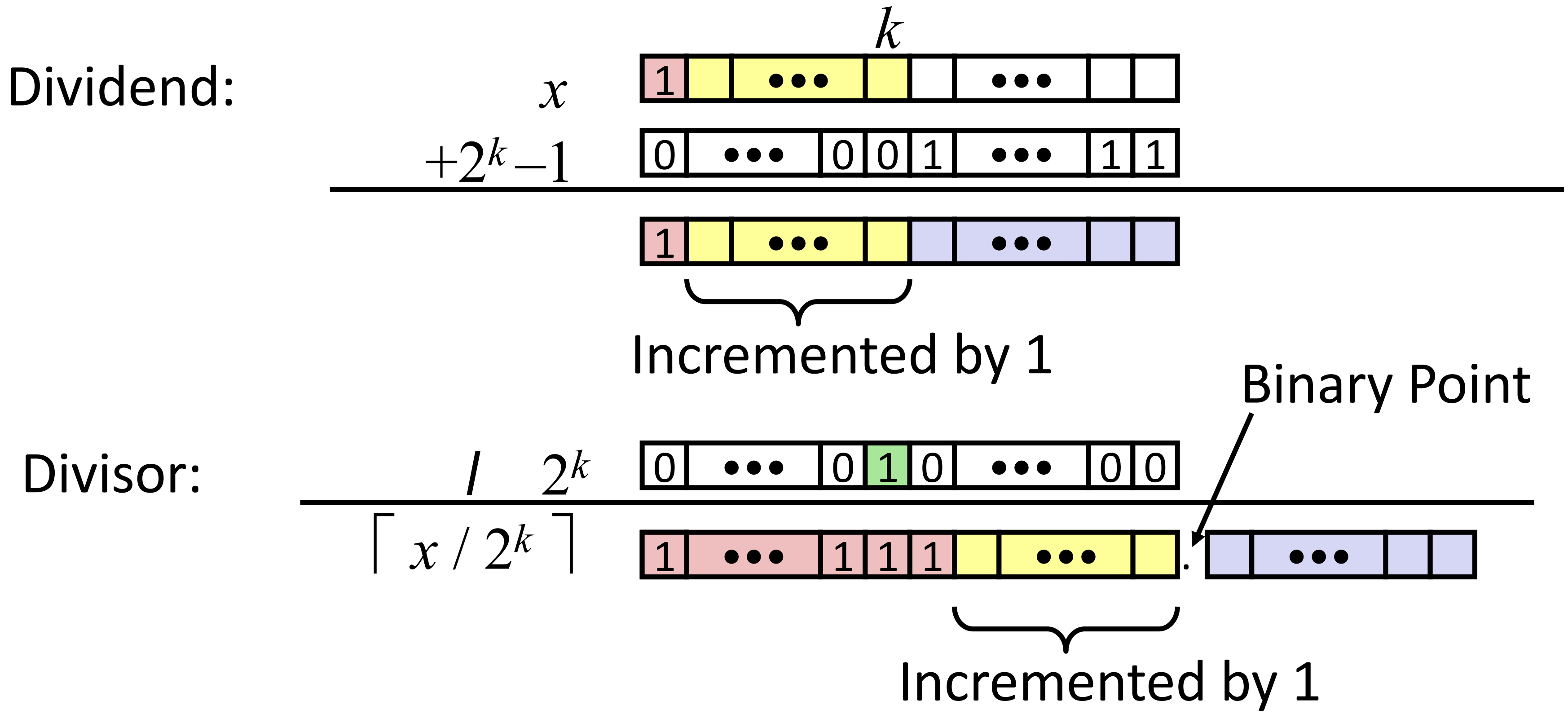
Case 1: No rounding



Biasing has no effect

Correct Power-of-2 Divide (Cont.)

Case 2: Rounding



Biasing adds 1 to final result

Today: Bits, Bytes, and Integers

- Representing information as bits
- Bit-level manipulations
- **Integers**
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
  - **Summary**
- Representations in memory, pointers, strings

Arithmetic: Basic Rules

- **Addition:**
  - Unsigned/signed: Normal addition followed by truncate, same operation on bit level
  - Unsigned: addition mod 2^w
    - Mathematical addition + possible subtraction of 2^w
  - Signed: modified addition mod 2^w (result in proper range)
    - Mathematical addition + possible addition or subtraction of 2^w

- **Multiplication:**
  - Unsigned/signed: Normal multiplication followed by truncate, same operation on bit level
  - Unsigned: multiplication mod 2^w
  - Signed: modified multiplication mod 2^w (result in proper range)

Why Should I Use Unsigned?

- ***Don't*** **use without understanding implications**
  - Easy to make mistakes

```
unsigned i;
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

  - Can be very subtle

```
#define DELTA sizeof(int)
int i;
for (i = CNT; i-DELTA >= 0; i-= DELTA)
  . . .
```

Counting Down with Unsigned

- **Proper way to use unsigned as loop index**

```
unsigned i;
for (i = cnt-2; i < cnt; i--)
  a[i] += a[i+1];
```

- **See Robert Seacord, *Secure Coding in C and C++***
  - C Standard guarantees that unsigned addition will behave like modular arithmetic
    - 0 – 1 → *UMax*
- **Even better**

```
size_t i;
for (i = cnt-2; i < cnt; i--)
  a[i] += a[i+1];
```

  - Data type `size_t` defined as unsigned value with length = word size
  - Code will work even if `cnt` = *UMax*
  - What if `cnt` is signed and < 0?

Why Should I Use Unsigned? (cont.)

- ***Do* Use When Performing Modular Arithmetic**
  - Multiprecision arithmetic
- ***Do* Use When Using Bits to Represent Sets**
  - Logical right shift, no sign extension

5.17 ◆◆

The library function `memset` has the following prototype:

```
void *memset(void *s, int c, size_t n);
```

This function fills `n` bytes of the memory area starting at `s` with copies of the low-order byte of `c`. For example, it can be used to zero out a region of memory by giving argument 0 for `c`, but other values are possible.

The following is a straightforward implementation of `memset`:

```
/* Basic implementation of memset */
void *basic_memset(void *s, int c, size_t n)
{
    size_t cnt = 0;
    unsigned char *schar = s;
    while (cnt < n) {
        *schar++ = (unsigned char) c;
        cnt++;
    }
    return s;
}
```

Implement a more efficient version of the function by using a word of data type `unsigned long` to pack eight copies of `c`, and then step through the region using word-level writes. You might find it helpful to do additional loop unrolling as well. On our reference machine, we were able to reduce the CPE from 1.00 for the straightforward implementation to 0.127. That is, the program is able to write 8 bytes every clock cycle.

Here are some additional guidelines. To ensure portability, let K denote the value of `sizeof(unsigned long)` for the machine on which you run your program.

- You may not call any library functions.
- Your code should work for arbitrary values of `n`, including when it is not a multiple of K. You can do this in a manner similar to the way we finish the last few iterations with loop unrolling.
- You should write your code so that it will compile and run correctly on any machine regardless of the value of K. Make use of the operation `sizeof` to do this.
- On some machines, unaligned writes can be much slower than aligned ones. (On some non-x86 machines, they can even cause segmentation faults.) Write your code so that it starts with byte-level writes until the destination address is a multiple of K, then do word-level writes, and then (if necessary) finish with byte-level writes.
- Beware of the case where `cnt` is small enough that the upper bounds on some of the loops become negative. With expressions involving the `sizeof` operator, the testing may be performed with unsigned arithmetic. (See Section 2.2.8 and Problem 2.72.)

```
#include <stdio.h>

/* 将字符打包进一个long类型变量中，每次循环做处理两个long的空间 */
void *align_pack_2_memset(void *s, int c, size_t n) {
    unsigned long word = 0;
    unsigned char *schar;
    unsigned long *slong;
    size_t cnt = 0;
    long i;

    /* n过小时，用题目中给出的基本方法处理即可 */
    if (n < 3 * sizeof(word))
        return basic_memset(s, c, n);

    /* 将多个c打包装进word中 */
    for (i = 0; i < sizeof(word); i++)
        word = (word << 8) | (c & 0xFF);

    /* 找到地址对齐的起点 */
    schar = (unsigned char *) s;
    while ((unsigned long) schar % sizeof(word) != 0) {
        *schar++ = (unsigned char) c;
        cnt++;
    }

    /* 一次循环处理2*sizeof(word)个字符 */
    slong = (unsigned long *) schar;
    while (cnt < n-2*sizeof(word)+1) {
        *slong = word;
        *(slong+1) = word;
        cnt += 2*sizeof(word);
        slong += 2;
    }

    /* 处理剩余的一些字节 */
    schar = (unsigned char *) slong;
    while (cnt < n) {
        *schar++ = (unsigned char) c;
        cnt++;
    }
    return s;
}
```

Integer C Puzzles

```
1.  x < 0              →  ((x*2) < 0)
2.  ux >= 0
3.  x & 7 == 7         →  (x<<30) < 0
4.  ux > -1
5.  x > y              →   -x < -y
6.  x * x >= 0
7.  x > 0 && y > 0     →   x + y > 0
8.  x >= 0             →   -x <= 0
9.  x <= 0             →   -x >= 0
10. (x|-x)>>31 == -1
11. ux >> 3 == ux/8
12. x >> 3 == x/8
13. x & (x-1) != 0
```

Initialization

```
int x = foo();

int y = bar();

unsigned ux = x;

unsigned uy = y;
```

Today: Bits, Bytes, and Integers

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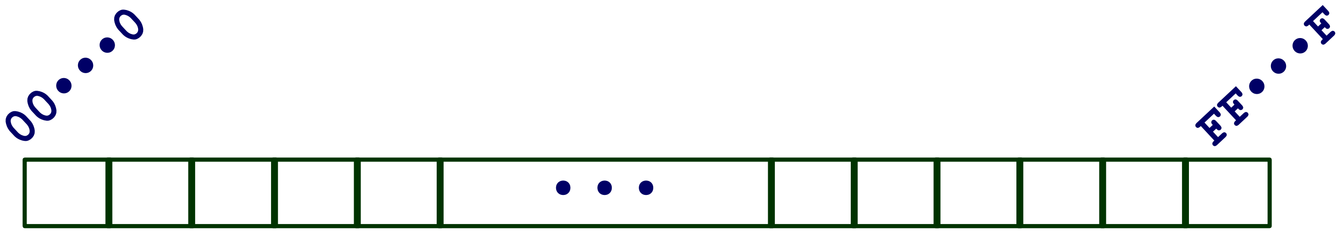
  - Summary
- **Representations in memory, pointers, strings**

Byte-Oriented Memory Organization



- **Programs refer to data by address**
  - Conceptually, envision it as a very large array of bytes
    - In reality, it's not, but can think of it that way
  - An address is like an index into that array
    - and, a pointer variable stores an address

- **Note: system provides private address spaces to each "process"**
  - Think of a process as a program being executed
  - So, a program can clobber its own data, but not that of others

Machine Words

- **Any given computer has a "Word Size"**
  - Nominal size of integer-valued data
    - and of addresses

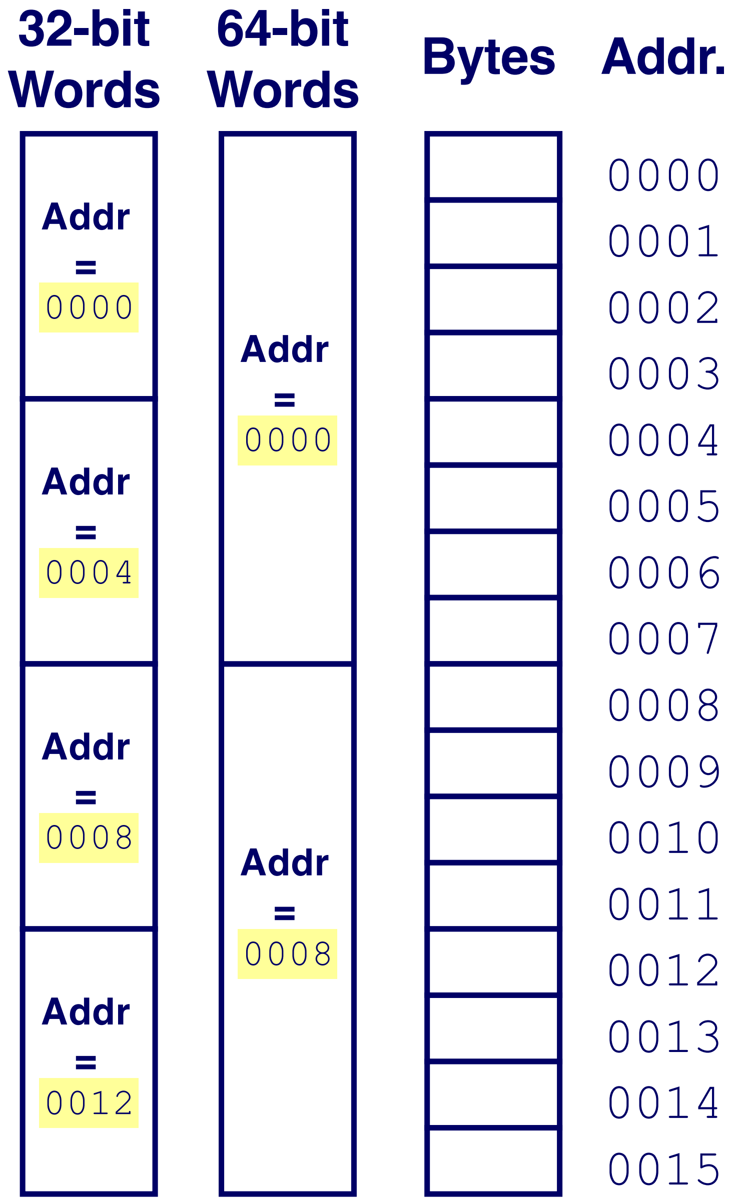
  - Until recently, most machines used 32 bits (4 bytes) as word size
    - Limits addresses to 4GB (2^{32} bytes)

  - Increasingly, machines have 64-bit word size
    - Potentially, could have 16 EB (exabytes) of addressable memory
    - That's 18.4 X 10^{18}

  - Machines still support multiple data formats
    - Fractions or multiples of word size
    - Always integral number of bytes

Word-Oriented Memory Organization

- **Addresses Specify Byte Locations**
  - Address of first byte in word
  - Addresses of successive words differ by 4 (32-bit) or 8 (64-bit)



Example Data Representations

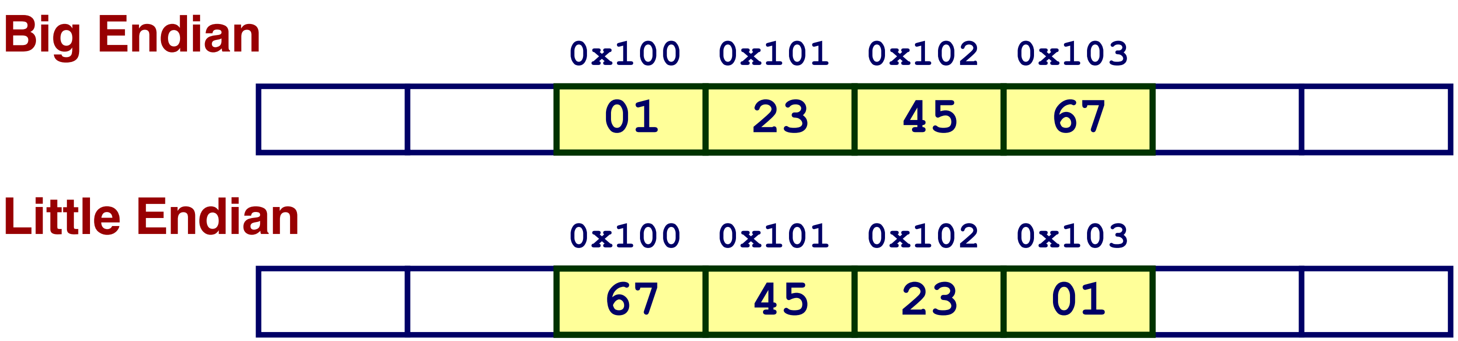
C Data Type	Typical 32-bit	Typical 64-bit	x86-64
`char`	1	1	1
`short`	2	2	2
`int`	4	4	4
`long`	4	8	8
`float`	4	4	4
`double`	8	8	8
`long double`	−	−	10/16
`pointer`	4	8	8

Byte Ordering

- **So, how are the bytes within a multi-byte word ordered in memory?**
- **Conventions**
  - Big Endian: Sun, PPC Mac, Internet
    - Least significant byte has highest address
  - Little Endian: x86, ARM processors running Android, iOS, and Windows
    - Least significant byte has lowest address

Byte Ordering Example

- **Example**
  - Variable x has 4-byte value of 0x01234567
  - Address given by &x is 0x100



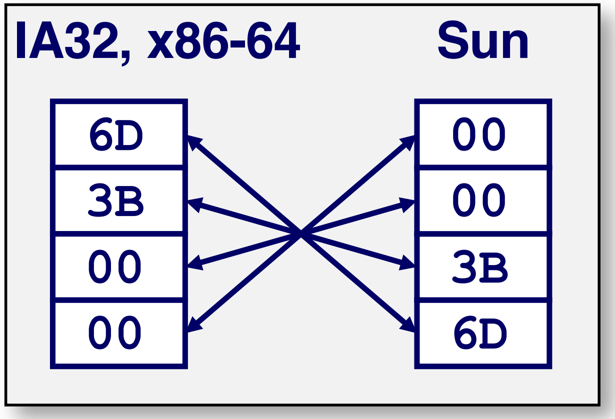
Representing Integers

Decimal: `15213`

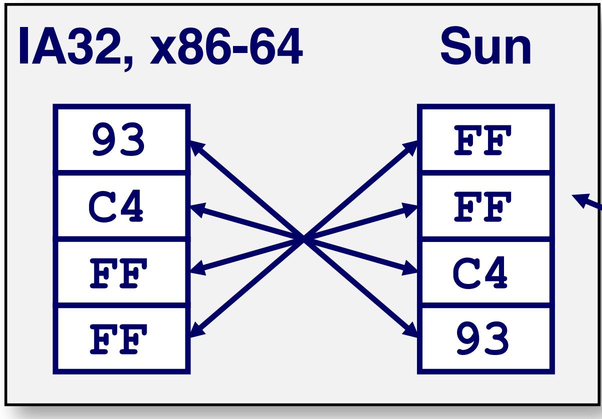
Binary: `0011 1011 0110 1101`

Hex: `3 B 6 D`

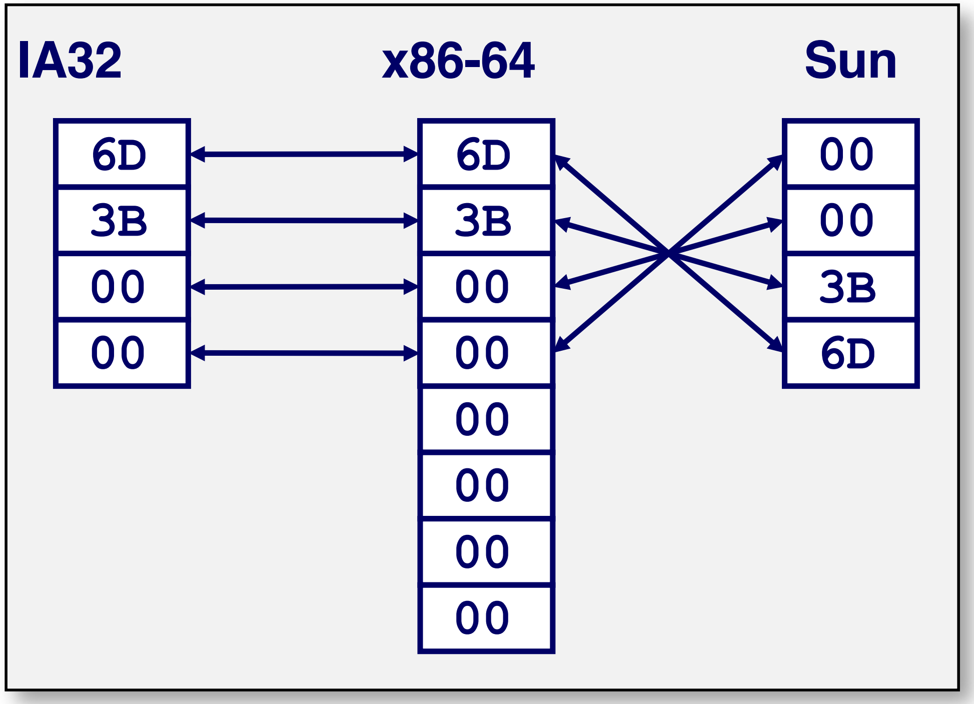
```
int A = 15213;
```




```
int B = -15213;
```




```
long int C = 15213;
```



Two's complement representation

Examining Data Representations

- **Code to Print Byte Representation of Data**
  - Casting pointer to unsigned char * allows treatment as a byte array

```
typedef unsigned char *pointer;

void show_bytes(pointer start, size_t len){
  size_t i;
  for (i = 0; i < len; i++)
    printf(”%p\t0x%.2x\n",start+i, start[i]);
  printf("\n");
}
```

Printf directives:
%p: Print pointer
%x: Print Hexadecimal

show_bytes Execution Example

```
int a = 15213;
printf("int a = 15213;\n");
show_bytes((pointer) &a, sizeof(int));
```

Result (Linux x86-64):

```
int a = 15213;
0x7fffb7f71dbc    6d
0x7fffb7f71dbd    3b
0x7fffb7f71dbe    00
0x7fffb7f71dbf    00
```

Reading Byte-Reversed Listings

- **Disassembly**
  - Text representation of binary machine code
  - Generated by program that reads the machine code
- **Example Fragment**

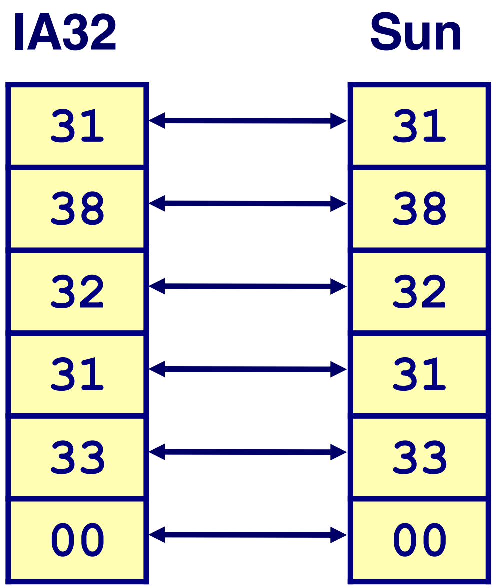
Address	Instruction Code	Assembly Rendition	
8048365:	5b	pop	%ebx
8048366:	81 c3 ab 12 00 00	add	$0x12ab,%ebx
804836c:	83 bb 28 00 00 00 00	cmpl	$0x0,0x28(%ebx)

- **Deciphering Numbers**
  - Value: 0x12ab
  - Pad to 32 bits: 0x000012ab
  - Split into bytes: 00 00 12 ab
  - Reverse: ab 12 00 00

Representing Strings

```
char S[6] = "18213";
```

- **Strings in C**
  - Represented by array of characters
  - Each character encoded in ASCII format
    - Standard 7-bit encoding of character set
    - Character “0” has code 0x30
      - Digit *i* has code 0x30+*i*
  - String should be null-terminated
    - Final character = 0
- **Compatibility**
  - Byte ordering not an issue



Today: Bits, Bytes, and Integers

- **Representing information as bits**
- **Bit-level manipulations**
- **Integers**
  - Representation: unsigned and signed
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  - Summary
- **Representations in memory, pointers, strings**

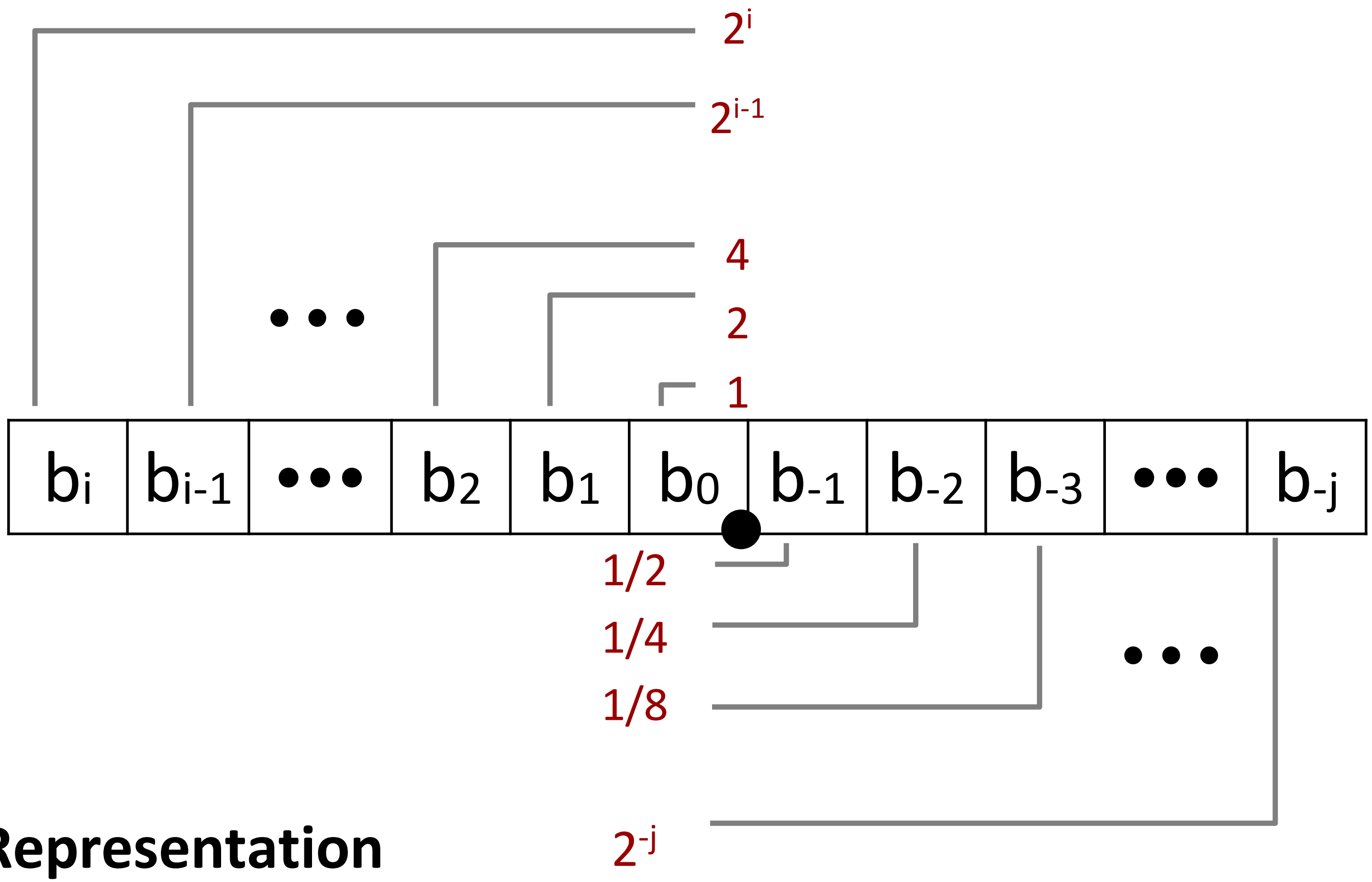
Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

Fractional binary numbers

- **What is 1011.101_2?**

Fractional Binary Numbers



- **Representation**
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number: $$\sum_{k=-j}^{i} b_k \times 2^k$$

Fractional Binary Numbers: Examples

- **Value** **Representation**

Value	Representation
5 3/4	101.11_2
2 7/8	10.111_2
1 7/16	1.0111_2

- **Observations**
  - Divide by 2 by shifting right (unsigned)
  - Multiply by 2 by shifting left
  - Numbers of form $0.111111\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Representable Numbers

- **Limitation #1**
  - Can only exactly represent numbers of the form $x/2^k$
    - Other rational numbers have repeating bit representations
  - Value Representation
    - 1/3 `0.0101010101[01]`…$_2$
    - 1/5 `0.001100110011[0011]`…$_2$
    - 1/10 `0.0001100110011[0011]`…$_2$

- **Limitation #2**
  - Just one setting of binary point within the *w* bits
    - Limited range of numbers (very small values? very large?)

Today: Floating Point

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IEEE Floating Point

- **IEEE Standard 754**
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs

- **Driven by numerical concerns**
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation

- **Numerical Form:**

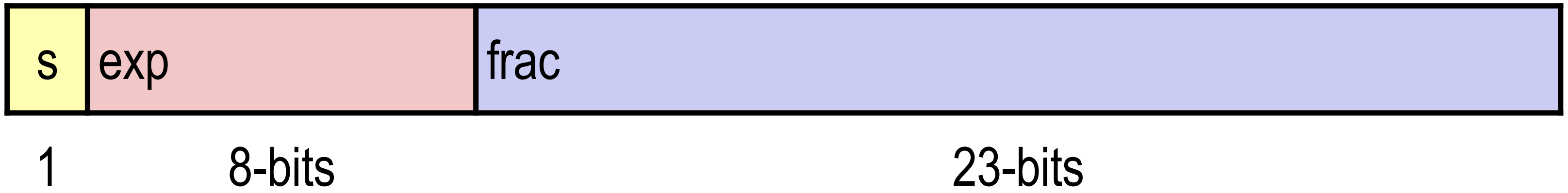
$$(-1)^s \, M \, 2^E$$

  - Sign bit s determines whether number is negative or positive
  - Significand M normally a fractional value in range [1.0,2.0).
  - Exponent E weights value by power of two

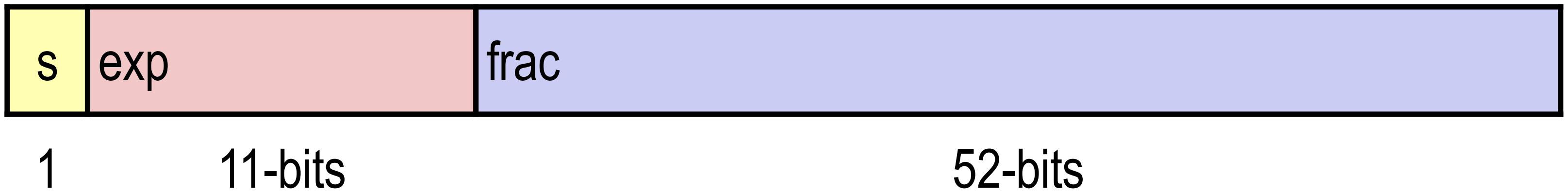
- **Encoding**
  - MSB s is sign bit s
  - exp field encodes E (but is not equal to E)
  - frac field encodes M (but is not equal to M)

Precision options

- **Single precision: 32 bits**



- **Double precision: 64 bits**



- **Extended precision: 80 bits (Intel only)**

"Normalized" Values

$v = (-1)^s M \, 2^E$

- **When: exp ≠ 000…0 and exp ≠ 111…1**

- **Exponent coded as a biased value: E = Exp – Bias**
  - Exp: unsigned value of exp field
  - Bias = $2^{k-1} - 1$, where k is number of exponent bits
    - Single precision: 127 (Exp: 1…254, E: -126…127)
    - Double precision: 1023 (Exp: 1…2046, E: -1022…1023)

- **Significand coded with implied leading 1: M = $1.\text{xxx…x}_2$**
  - xxx…x: bits of frac field
  - Minimum when frac=000…0 (M = 1.0)
  - Maximum when frac=111…1 (M = 2.0 – ε)
  - Get extra leading bit for "free"

Normalized Encoding Example

$v = (-1)^s M 2^E$
$E = Exp - Bias$

- Value: `float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- Significand
  $M = 1.\underline{1101101101101}_2$
  `frac` $= \underline{1101101101101}0000000000_2$

- Exponent
  $E = 13$
  $Bias = 127$
  $Exp = 140 = 10001100_2$

- Result:

s	exp	frac
0	10001100	11011011011010000000000

Denormalized Values

$v = (-1)^s M 2^E$
$E = 1 - \text{Bias}$

- **Condition: exp = 000…0**

- **Exponent value: E = 1 – Bias (instead of E = 0 – Bias)**
- **Significand coded with implied leading 0: $M = 0.xxx\ldots x_2$**
  - `xxx…x`: bits of `frac`
- **Cases**
  - `exp` = 000…0, `frac` = 000…0
    - Represents zero value
    - Note distinct values: +0 and –0 (why?)
  - `exp` = 000…0, `frac` ≠ 000…0
    - Numbers closest to 0.0
    - Equispaced

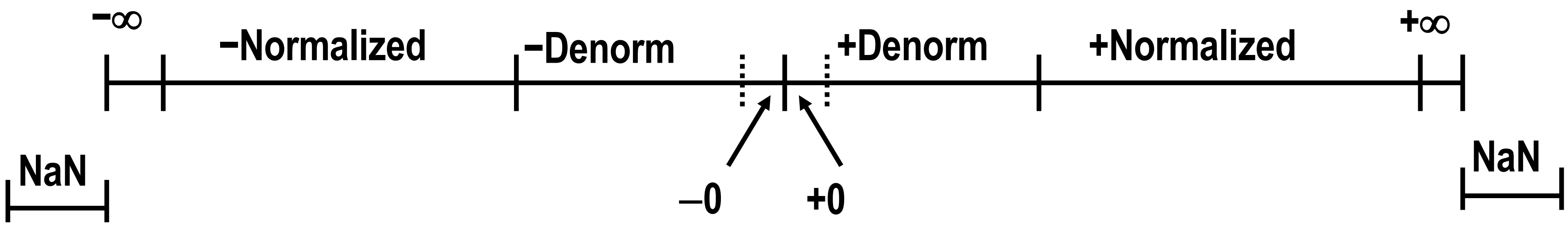
Special Values

- **Condition: `exp` = `111`…`1`**

- **Case: `exp` = `111`…`1`, `frac` = `000`…`0`**
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - E.g., $1.0/0.0 = -1.0/-0.0 = +\infty$, $1.0/-0.0 = -\infty$

- **Case: `exp` = `111`…`1`, `frac` ≠ `000`…`0`**
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), $\infty - \infty$, $\infty \times 0$

Visualization: Floating Point Encodings



Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- **Example and properties**
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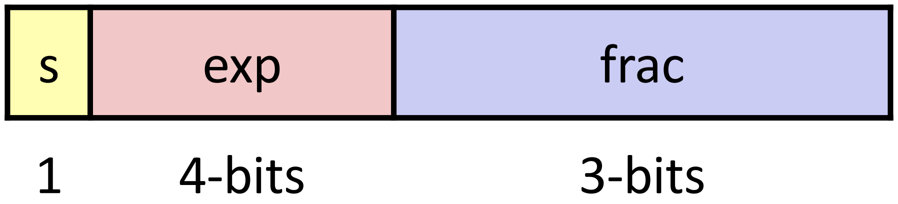
Tiny Floating Point Example



- **8-bit Floating Point Representation**
  - the sign bit is in the most significant bit
  - the next four bits are the exponent, with a bias of 7
  - the last three bits are the `frac`

- **Same general form as IEEE Format**
  - normalized, denormalized
  - representation of 0, NaN, infinity

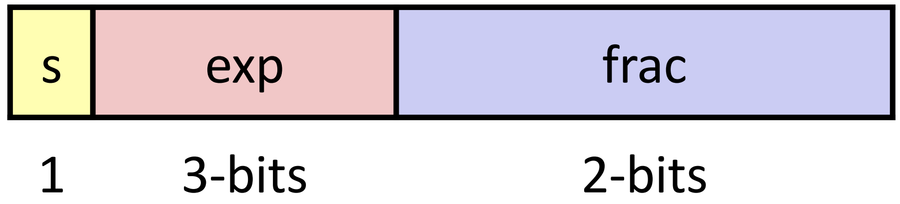
Dynamic Range (Positive Only)

$v = (-1)^s M 2^E$
n: E = Exp – Bias
d: E = 1 – Bias

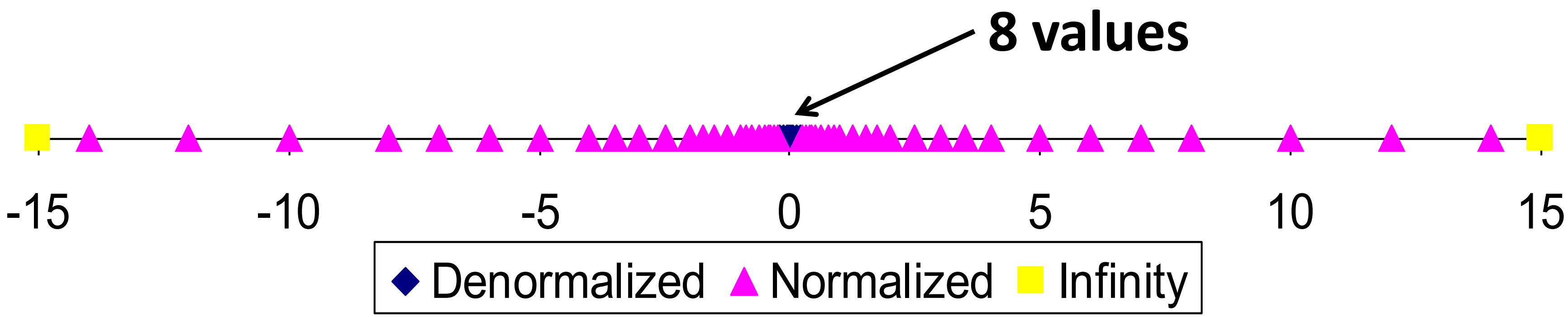
	s	exp	frac	E	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Distribution of Values

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is $2^{3-1}-1 = 3$

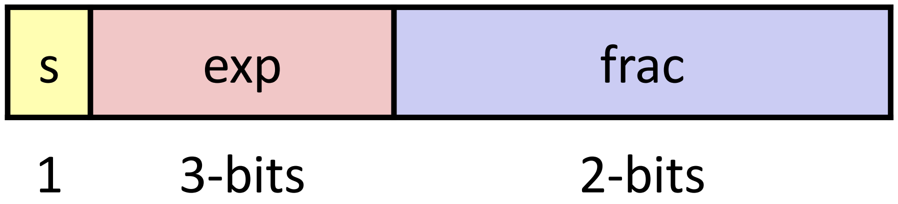


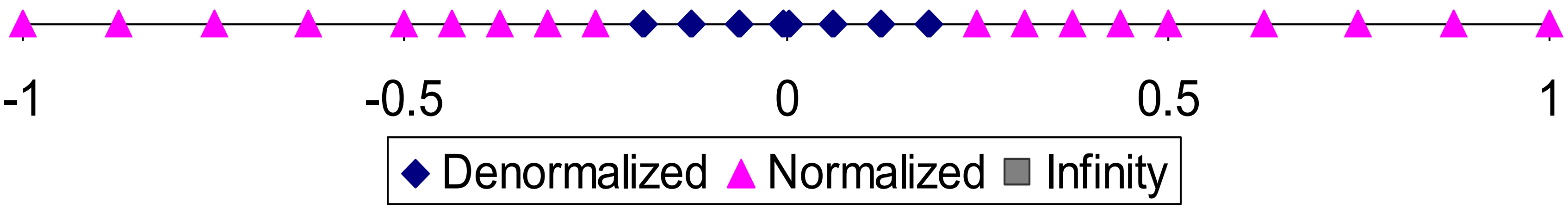
- **Notice how the distribution gets denser toward zero.**



Distribution of Values (close-up view)

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3





Special Properties of the IEEE Encoding

- **FP Zero Same as Integer Zero**
  - All bits = 0

- **Can (Almost) Use Unsigned Integer Comparison**
  - Must first compare sign bits
  - Must consider −0 = 0
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- **Rounding, addition, multiplication**
- Floating point in C
- Summary

Floating Point Operations: Basic Idea

- $\mathtt{x} +_{\mathtt{f}} \mathtt{y} = \mathtt{Round}(\mathtt{x} + \mathtt{y})$

- $\mathtt{x} \times_{\mathtt{f}} \mathtt{y} = \mathtt{Round}(\mathtt{x} \times \mathtt{y})$

- **Basic idea**
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into **`frac`**

Rounding

- **Rounding Modes (illustrate with $ rounding)**

	$1.40	**$1.60**	**$1.50**	**$2.50**	**–$1.50**
Towards zero	$1	$1	$1	$2	–$1
Round down (−∞)	$1	$1	$1	$2	–$2
Round up (+∞)	$2	$2	$2	$3	–$1
Nearest Even (default)	$1	$2	$2	$2	–$2

Closer Look at Round-To-Even

- **Default Rounding Mode**
  - Hard to get any other kind without dropping into assembly
  - All others are statistically biased
    - Sum of set of positive numbers will consistently be over- or under-estimated

- **Applying to Other Decimal Places / Bit Positions**
  - When exactly halfway between two possible values
    - Round so that least significant digit is even
  - E.g., round to nearest hundredth

7.8949999	7.89	(Less than half way)
7.8950001	7.90	(Greater than half way)
7.8950000	7.90	(Half way—round up)
7.8850000	7.88	(Half way—round down)

Rounding Binary Numbers

- **Binary Fractional Numbers**
  - "Even" when least significant bit is **0**
  - "Half way" when bits to right of rounding position = **100**…$_2$

- **Examples**
  - Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

FP Multiplication

- $(-1)^{s1}$ M1 2^{E1} x $(-1)^{s2}$ M2 2^{E2}
- **Exact Result:** $(-1)^{s}$ M 2^{E}
  - Sign s: s1 ^ s2
  - Significand M: M1 x M2
  - Exponent E: E1 + E2

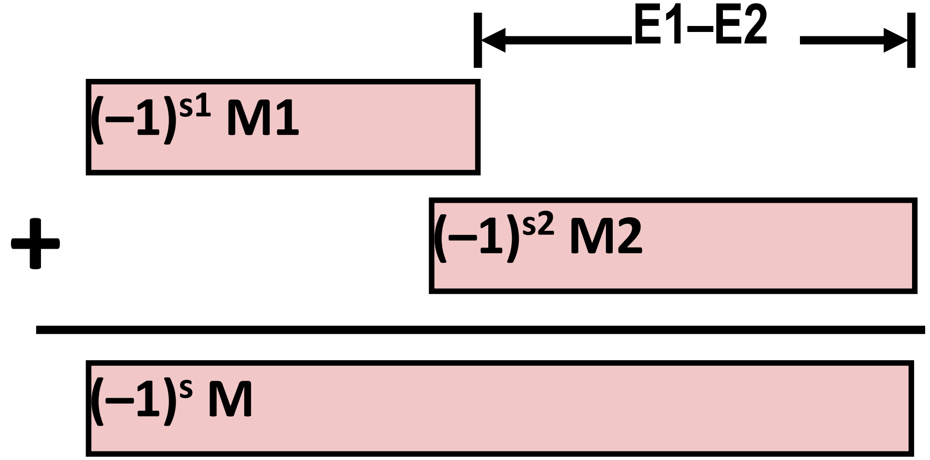
- **Fixing**
  - If M ≥ 2, shift M right, increment E
  - If E out of range, overflow
  - Round M to fit `frac` precision

- **Implementation**
  - Biggest chore is multiplying significands

Floating Point Addition

- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ +\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
  - Assume E1 > E2

Get binary points lined up



- **Exact Result:** $(-1)^{s}\ M\ \ 2^{E}$
  - Sign s, significand M:
    - Result of signed align & add
  - Exponent E: E1

- **Fixing**
  - If M ≥ 2, shift M right, increment E
  - if M < 1, shift M left k positions, decrement E by k
  - Overflow if E out of range
  - Round M to fit `frac` precision

Mathematical Properties of FP Add

- **Compare to those of Abelian Group**
  - Closed under addition? **Yes**
    - But may generate infinity or NaN
  - Commutative? **Yes**
  - Associative? **No**
    - Overflow and inexactness of rounding
    - `(3.14+1e10)-1e10 = 0, 3.14+(1e10-1e10) = 3.14`
  - 0 is additive identity? **Yes**
  - Every element has additive inverse? **Almost**
    - Yes, except for infinities & NaNs
- **Monotonicity**
  - $a \geq b \Rightarrow a+c \geq b+c$? **Almost**
    - Except for infinities & NaNs

Mathematical Properties of FP Mult

- **Compare to Commutative Ring**
  - Closed under multiplication? **Yes**
    - But may generate infinity or NaN
  - Multiplication Commutative? **Yes**
  - Multiplication is Associative? **No**
    - Possibility of overflow, inexactness of rounding
    - Ex: `(1e20*1e20)*1e-20= inf, 1e20*(1e20*1e-20)= 1e20`
  - 1 is multiplicative identity? **Yes**
  - Multiplication distributes over addition? **No**
    - Possibility of overflow, inexactness of rounding
    - `1e20*(1e20-1e20)= 0.0, 1e20*1e20 – 1e20*1e20 = NaN`
- **Monotonicity**
  - $a \geq b$ & $c \geq 0 \Rightarrow a * c \geq b * c$? **Almost**
    - Except for infinities & NaNs

Today: Floating Point

- Background: Fractional binary numbers
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- Summary

Floating Point in C

- **C Guarantees Two Levels**
  - **`float`** single precision
  - **`double`** double precision

- **Conversions/Casting**
  - Casting between **`int`**, **`float`**, and **`double`** changes bit representation
  - **`double`/`float` → `int`**
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - **`int` → `double`**
    - Exact conversion, as long as **`int`** has ≤ 53 bit word size
  - **`int` → `float`**
    - Will round according to rounding mode

Floating Point Puzzles

- **For each of the following C expressions, either:**
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;
float f = …;
double d = …;
```

Assume neither d nor f is NaN

```
1. x == (int)(float) x
2. x == (int)(double) x
3. f == (float)(double) f
4. d == (double)(float) d
5. f == -(-f);
6. 2/3 == 2/3.0
7. d < 0.0       ⇒  ((d*2) < 0.0)
8. d > f         ⇒  -f > -d
9. d * d >= 0.0
10. (d+f)-d == f
```

浮点数除0的问题

```
#include <conio.h>
#include <stdio.h>
int  main()
{
        int  a=1,b=0;
        printf( "Division  by  zero:%d\n ",a/b);
        getchar();
        return  0;
}
int  main()
{
        double  x=1.0,y=-1.0,z=0.0;
        printf( "division  by  zero:%f  %f\n ",x/z,y/z);
        getchar();
        return  0;
}
```

问题一:为什么整除int型会产生错误? 是什么错误?

二:用double型的时候结果为1.#INF00和-1.#INF00,作何解释???

这是网上的一个帖子

为什么整数除0会发生异常?

为什么浮点数除0不会出现异常?

浮点运算中，一个有限数除以0，结果为正无穷大（负无穷大）

举例：Ariana火箭爆炸

- 1996年6月4日，Ariana 5火箭初次航行，在发射仅仅37秒钟后，偏离了飞行路线，然后解体爆炸，火箭上载有价值5亿美元的通信卫星。
- 原因是在将一个64位浮点数转换为16位带符号整数时，产生了溢出异常。溢出的值是火箭的水平速率，这比原来的Ariana 4火箭所能达到的速率高出了5倍。在设计Ariana 4火箭软件时，设计者确认水平速率决不会超出一个16位的整数，但在设计Ariana 5时，他们没有重新检查这部分，而是直接使用了原来的设计。
- 在不同数据类型之间转换时，往往隐藏着一些不容易被察觉的错误，这种错误有时会带来重大损失，因此，编程时要非常小心。

举例：爱国者导弹定位错误

- 1991年2月25日，海湾战争中，美国在沙特阿拉伯达摩地区设置的爱国者导弹拦截伊拉克的飞毛腿导弹失败，致使飞毛腿导弹击中了一个美军军营，杀死了美军28名士兵。其原因是由于爱国者导弹系统时钟内的一个软件错误造成的，引起这个软件错误的原因是浮点数的精度问题。
- 爱国者导弹系统中有一内置时钟，用计数器实现，每隔0.1秒计数一次。程序用0.1的一个24位定点二进制小数x来乘以计数值作为以秒为单位的时间
- 这个x的机器数是多少呢？
- 0.1的二进制表示是一个无限循环序列：

  0.00011[0011]…，x=0.000 1100 1100 1100 1100 1100B。

显然，x是0.1的近似表示，0.1-x

= 0.000 1100 1100 1100 1100 1100 [1100]… - 0.000 1100 1100 1100 1100 1100B

即为：

= 0.000 0000 0000 0000 0000 0000 1100 [1100]…B

= $2^{-20}\times 0.1 \approx 9.54\times 10^{-8}$ 这就是机器值与真值之间的误差！

举例：爱国者导弹定位错误

已知在爱国者导弹准备拦截飞毛腿导弹之前，已经连续工作了**100**小时，飞毛腿的速度大约为**2000**米/秒，则由于时钟计算误差而导致的距离误差是多少？

100小时相当于计数了$100\times60\times60\times10=36\times10^{5}$次，因而导弹的时钟已经偏差了$9.54\times10^{-8}\times36\times10^{5}\approx 0.343$秒

因此，距离误差是2000×0.343秒 ≈ 687米

举例：爱国者导弹定位错误

- 若x用float型表示，则x的机器数是什么？0.1与x的偏差是多少？系统运行100小时后的时钟偏差是多少？在飞毛腿速度为2000米/秒的情况下，预测的距离偏差为多少？
  - 0.1= 0.0 0011[0011]B=+1.1 0011 0011 0011 0011 0011 00B×2^{-4}，故x的机器数为0 011 1101 1 100 1100 1100 1100 1100 1100
  - Float型仅24位有效位数，后面的有效位全被截断，故x与0.1之间的误差为：|x-0.1|=0.000 0000 0000 0000 0000 0000 0000 1100 [1100]…B。这个值等于2^{-24}×0.1 ≈ 5.96×10^{-9}。100小时后时钟偏差5.96×10^{-9}×36×10^{5} ≈ 0.0215秒。距离偏差0.0215×2000≈43米。比爱国者导弹系统精确约16倍。

举例：爱国者导弹定位错误

- 若用32位二进制定点小数x=0.000 1100 1100 1100 1100 1100 1100 1101 B表示0.1，则误差比用float表示误差更大还是更小？
  - 当x=0.000 1100 1100 1100 1100 1100 1100 1101 B时，与0.1之间的误差约为：|x-0.1|=0.000 0000 0000 0000 0000 0000 0000 0000 00 1100 [1100]…B。这个值等于$2^{-30}\times 0.1 \approx 9.31\times 10^{-11}$。100小时后时钟偏差$9.31\times 10^{-11}\times 36\times 10^{5} \approx 0.000335$秒。预测的距离偏差仅为$0.000335\times 2000 \approx 0.67$米。

举例：浮点数运算的精度问题

- 从上述结果可以看出：
  - 用32位定点小数表示0.1 ，比采用float精度高64倍
  - 用float表示在计算速度上更慢，必须先把计数值转换为IEEE 754格式浮点数，然后再对两个IEEE 754格式的数相乘，故采用float比直接将两个二进制数相乘要慢
- **Ariana 5**火箭和爱国者导弹的例子带来的启示

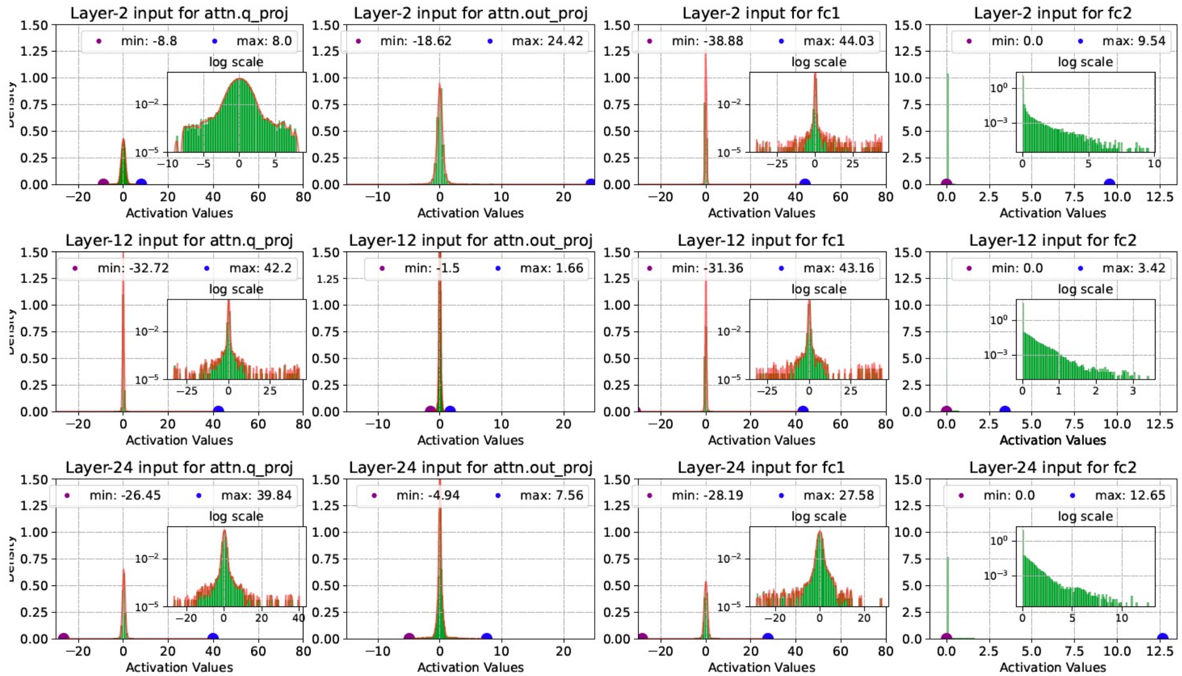
✓ 程序员应对底层机器级数据的表示和运算有深刻理解

✓ 计算机世界里，经常是“差之毫厘，失之千里”，需要细心再细心，精确再精确

✓ 不能遇到小数就用浮点数表示，有些情况下（如需要将一个整数变量乘以一个确定的小数常量），可先用一个确定的定点整数与整数变量相乘，然后再通过移位运算来确定小数点

拓展：大语言模型中的量化（激活值的量化）

- **Motivation**
  - 激活值的均匀量化（比如INT4， INT8）显著劣化模型质量
    - 由于outlier的存在



opt1.3B的激活值的分布

拓展：大语言模型中的量化（激活值的量化）

- **Microsoft 23年的工作**
  - 激活*x*——使用FP8格式进行量化
    - 与值的分布一致
    - H100 NVIDIA硬件支持
  - 参数*w*——使用FP4格式进行量化
    - 计算*x***w*之前，FP4需要转成FP8
    - 优化转换过程的速度

- **NVIDIA定义的FP8格式有两种主要类型：**
  - **E4M3：提供了较高的动态范围，适用于需要较大数值范围的应用场景**
    - **1位符号（S）**
    - **4位指数（E）**
    - **3位尾数（M）**
  - **E5M2：提供更大的动态范围但相对较小的精度，适合那些对精度要求不高但需要更大数值范围的情况**
    - **1位符号（S）**
    - **5位指数（E）**
    - **2位尾数（M）**

From Microsoft:
Xiaoxia Wu, Zhewei Yao, Yuxiong He: ZeroQuant-FP: A Leap Forward in LLMs Post-Training W4A8 Quantization Using Floating-Point Formats. CoRR abs/2307.09782 (2023)

讨论：为什么这样设计浮点数?

- 保证浮点数操作的代数完备性
- **trade-off** 的思想
  - 32 位浮点数 vs 32 位定点数
- 巧妙的设计使得数值分布平滑过渡

Summary

- **IEEE Floating Point has clear mathematical properties**
- **Represents numbers of form M x 2^E**
- **One can reason about operations independent of implementation**
  - As if computed with perfect precision and then rounded
- **Not the same as real arithmetic**
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers