On the Statistics of Character Table of S_n

Tony Zhang, Atharva Kulkarni, Arnav Shah

University of Michigan

Last update February 18, 2024

Motivations

Definition

The character of group element $g \in G$ is, $\chi(g) = Tr(\rho(g))$ where $\rho : G \to GL_n(\mathbf{C})$ is the group representation [1]

- Studying character tables is incredibly useful as the trace of similar matrices is the same, thus the character of an element is invariant under a change of basis
- We aim to improve upon existing algorithms to compute higher order character tables of S_n and analyze various statistics of them (eg. if the dimensions of the irreducible representations converges)

Creating Character Tables using Partitions

University of Michigan LoG(M)

Tony Zhang, Atharva Kulkarni, Arnav Shah

Definition ([2, Defintion 1])

A *partition* $\lambda = (\lambda_1, \ldots, \lambda_k)$ of a natural number n is a decreasing sequence $\lambda_1 \geq \cdots \geq \lambda_k$ of natural numbers that sums to n.

- Every conjugacy class $\sigma \in S_n$ is determined by its cycle type, and the lengths of the cycles in its cycle decomposition give a partition of n. Thus, a bijective correspondence exists between partitions of n and conjugacy classes of S_n.
- For example, take $n = 3$. The partitions of n will be $(1, 1, 1), (2, 1)$, and (3). We can map each partition to $\{id\}$, $\{(12), (23), (31)\}$, $\{(123), (132)\}$, respectively.
- Similarly, there is a bijective correspondence between partitions of n and irreducible representations of S_n. For more details, please see [2].

Creating Character Tables using Partitions (cont.)

- Thus, for every natural number n, one can organize the data of all values of irreducible characters on conjugacy classes of S_n in a square table, called the character table, with rows and columns indexed by the partitions of n.
- The character table of S_3 can be seen below in table 4:

	(1,1,1)	(2,1)	(3)
(3)	1	1	1
(2,1)	2	0	-1
(1,1,1)	1	-1	1

Table: Character Value Table of S3

Heatmap of character table for $n = 6$

University of Michigan LoG(M)

Tony Zhang, Atharva Kulkarni, Arnav Shah

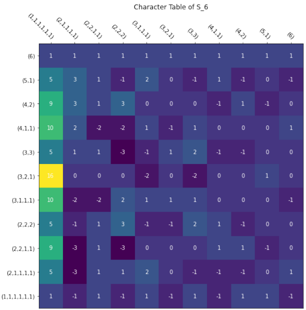


Figure: Heatmap of Character Table for $n = 6$ [1]

- Rows are labeled with partitions corresponding to irreducible representations, columns are labeled with partitions corresponding to conjugacy classes.

[1] See the program

Frobenius Formula

University of Michigan LoG(M)

Tony Zhang, Atharva Kulkarni, Arnav Shah

Theorem (Frobenius Formula)

- *Given an integer partition* $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ *of* n*, let* χ^λ *be the corresponding irreducible character of* S_n.
- *Let* χ^λ_μ *be short for the value of* χ^λ *at any* g *with cycle type* μ*, denote* $l_j = \lambda_j + k - j$*, and* i_j *the number of times* j *appears in* μ*, so* $\sum_j i_j j = n$
- *We have the following Frobenius Formula:*
  $\chi^\lambda_\mu =$ *coeff. of* $x_1^{l_1} x_2^{l_2} \cdots x_k^{l_k}$ *in* $\Delta(x) P_\mu(x)$
  where $\Delta(x) = \prod_{1 \le i < j \le k} (x_i - x_j)$,
  $P_\mu(x) = \prod_j P_j(x_1, \cdots, x_k)^{i_j}$*, where*
  $P_j(x_1, \cdots, x_k) = x_1^j + \cdots + x_k^j$ *is the* j*-th sum.*

Heatmap of character table for $n = 20$

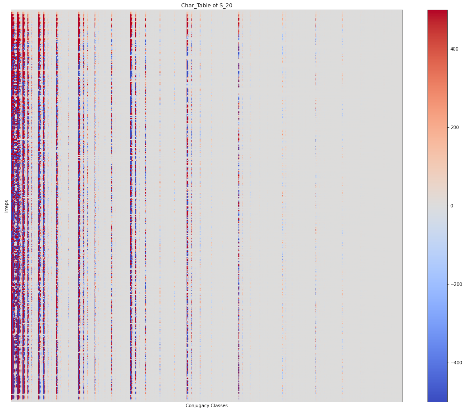
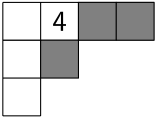


Figure: Heatmap of Character Table for $n = 20$ (values truncated within ± 500)[2]

[2] See the program

Hooks

- The hook h associated to a box b in the Young diagram of λ consists of the box b together with all the boxes directly to its right and directly below it.
- The hook length of h, denoted by $l(h)$, is the number of boxes contained in the hook.
- The height of the hook h, denoted by $\text{ht}(h)$, is one less than the number of rows in the Young diagram of λ that contain a box of h.
- Associated to each hook is a border strip, denoted $\text{bs}(h)$, which is the connected region of boundary boxes of the Young diagram running from the rightmost to the bottom-most box of h.



Murnaghan-Nakayama Rule

University of Michigan LoG(M)

Tony Zhang, Atharva Kulkarni, Arnav Shah

Theorem (The Murnaghan–Nakayama rule)

Let n and t be positive integers, with $t \leq n$. Let $\sigma \in S_n$ be of the form $\sigma = \tau \cdot \rho$, where ρ is a t-cycle, and τ is a permutation of S_n with support disjoint from ρ. Let λ be a partition of n. Then

$$\chi^\lambda(\sigma) = \sum_{h \in \lambda,\, \ell(h)=t} (-1)^{ht(h)} \chi^{\lambda \setminus bs(h)}(\tau).$$

- $\chi^\lambda(\sigma)$ denotes the value of the character of the irreducible representation of S_n corresponding to the partition λ, evaluated on the conjugacy class of σ
- $\lambda \setminus \mathrm{bs}(h)$ denotes the partition of $n - t$ obtained by removing the border strip $\mathrm{bs}(h)$ from the Young diagram of λ

Notion of Abacus

University of Michigan LoG(M)

Tony Zhang, Atharva Kulkarni, Arnav Shah

- An abacus is a bi-infinite sequence of 0's and 1's beginning with an infinite sequence of 1's and ending with an infinite sequence of 0's.
- E.g.:

$$\ldots, 1, \ldots, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 0, \ldots, 0, \ldots$$

- Now, an abacus has a one-to-one correspondence with a partition.
- For a given partition of an integer n, we can draw its corresponding Young diagram and trace its border starting from the bottom-left corner to the top-right corner.
- When we move horizontally and vertically, we denote it as a 0 or 1, respectively. This process can be easily reversed as well.

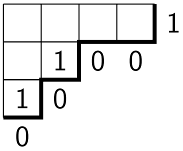
Example

University of Michigan LoG(M)

Tony Zhang, Atharva Kulkarni, Arnav Shah

- As an illustration, consider the partition (4,2,1) of 7
- Following Figure 11, tracing its border as previously mentioned, we move right once, up once, right once, up once, right twice, and lastly up once.
- Our string obtained will be 0,1,0,1,0,0,1 and the corresponding abaci will be:

$$\ldots, 1, \ldots, 1, 1, \mathbf{0,\ 1,\ 0,\ 1,\ 0,\ 0,\ 1}, 0, 0, \ldots, 0, \ldots$$



Bibliography

Persi Diaconis.
Group representations in probability and statistics.
Lecture Notes-Monograph Series, 11:i–192, 1988.

Yufei Zhao.
Young tableaux and the representations of the symmetric group.
01 2008.