On the Statistics of Character Table of S_n

Tony Zhang, Atharva Kulkarni, Arnav Shah
Mentors: Sarah Peluse, Karthik Ganapathy

University of Michigan

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Motivations

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Tony Zhang, Atharva Kulkarni, Arnav Shah
Mentors: Sarah Peluse, Karthik Ganapathy

Definition

The character of group element $g \in G$ is, $\chi(g) = Tr(\rho(g))$ where $\rho : G \to GL_n(\mathbf{C})$ is the group representation.

- Character values capture the different behaviors of different conjugacy classes, just like a "periodic table" for symmetric groups.
- Vector spaces with symmetries are fundamental objects which show up in math, physics, etc.
- We aim to improve upon existing algorithms to compute higher order character tables of S_n and analyze various statistics of them.

	(1,1,1)	(2,1)	(3)
(3)	1	1	1
(2,1)	2	0	-1
(1,1,1)	1	-1	1

Character Table of S6

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1	1	1	1	1	1	1	1	1	1	1
5	1	1	-1	2	0	-1	1	-1	0	-1
9	3	1	3	0	0	0	-1	1	-1	0
10	2	-2	-2	1	-1	1	0	0	0	1
5	1	1	-3	-1	1	2	-1	-1	0	0
16	0	0	0	-2	0	-2	0	0	1	0
10	-2	-2	2	1	1	1	0	0	0	-1
5	-1	1	3	-1	-1	2	1	-1	0	0
9	-3	1	-3	0	0	0	1	1	-1	0
5	-3	1	1	2	0	-1	-1	-1	0	1
1	-1	1	-1	1	-1	1	-1	1	1	-1

Table: Character Value Table of S6

Creating Character Tables using Partitions

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Tony Zhang, Atharva Kulkarni, Arnav Shah
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Definition

A *partition* $\lambda = (\lambda_1, \ldots, \lambda_k)$ of a natural number n is a decreasing sequence $\lambda_1 \geq \cdots \geq \lambda_k$ of natural numbers that sums to n.

- A natural bijective correspondence exists between partitions of n and conjugacy classes of S_n.
- Similarly, there is a bijective correspondence between partitions of n and irreducible representations of S_n.
- Thus, for every natural number n, we can create character tables with rows and columns indexed by the partitions of n.

Frobenius Formula

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Tony Zhang, Atharva Kulkarni, Arnav Shah
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Theorem (Frobenius Formula (adapted from Zhao))

- *Given an integer partition* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ *of* n, *let* χ^λ *be the corresponding irreducible character of* S_n.
- *Let* χ^λ_μ *be short for the value of* χ^λ *at any* g *with cycle type* μ, *denote* $l_j = \lambda_j + k - j$, *and* i_j *the number of times* j *appears in* μ, *so* $\sum_j i_j j = n$
- *We have the following Frobenius Formula:*
  $\chi^\lambda_\mu =$ *coeff. of* $x_1^{l_1} x_2^{l_2} \cdots x_k^{l_k}$ *in* $\Delta(x) P_\mu(x)$
  where $\Delta(x) = \prod_{1 \le i < j \le k} (x_i - x_j)$,
  $P_\mu(x) = \prod_j P_j(x_1, \cdots, x_k)^{i_j}$, *and*
  $P_j(x_1, \cdots, x_k) = x_1^j + \cdots + x_k^j$ *is the* j*-th power sum.*

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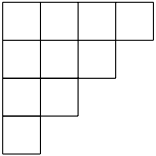
Young Diagram

Definition

A Young diagram corresponding to a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ is a diagram of "boxes" which has λ_1 boxes in the first row, λ_2 boxes in the second row, $\cdots$, λ_k boxes in the k-th row.

Example: the Young diagram corresponding to $\lambda = (4, 3, 2, 1)$



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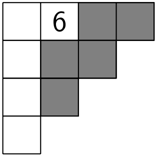
Hooks

- The hook h in the Young diagram of λ consists of the box b together with all the boxes directly to its right and directly below it.
- The hook length, $l(h)$, is the number of boxes contained in the hook.
- The height of the hook, $\mathrm{ht}(h)$, is one less than the number of rows that contain a box of h.
- Border strip, $\mathrm{bs}(h)$, is the connected region of boundary boxes running from the rightmost to the bottom-most box of h.



Murnaghan-Nakayama Rule

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Mentors: Sarah Peluse, Karthik Ganapathy

Theorem (The Murnaghan–Nakayama rule (adapted from Peluse & Soundarajan))

Let n and t be positive integers, with $t \leq n$. Let $\sigma \in S_n$ be of the form $\sigma = \tau \cdot \rho$, where ρ is a t-cycle, and τ is a permutation of S_n with support disjoint from ρ. Let λ be a partition of n. Then

$$\chi_\sigma^\lambda = \sum_{h \in \lambda,\, \ell(h) = t} (-1)^{ht(h)} \chi_\tau^{\lambda \setminus bs(h)}.$$

Notion of Abacus [Peluse and Soundarajan]

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- An abacus is a bi-infinite sequence of 0's and 1's beginning with an infinite sequence of 1's and ending with an infinite sequence of 0's.
- E.g.:

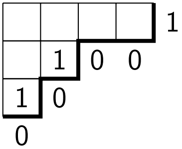
$$\ldots, 1, \ldots, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 0, \ldots, 0, \ldots$$

- Now, an abacus has a one-to-one correspondence with a partition.
- For a given partition of an integer n, we can draw its corresponding Young diagram and trace its border starting from the bottom-left corner to the top-right corner.
- When we move horizontally and vertically, we denote it as a 0 or 1, respectively. This process can be easily reversed as well.

Example

- As an illustration, consider the partition (4,2,1) of 7
- Following Figure 10, tracing its border as previously mentioned, we move right once, up once, right once, up once, right twice, and lastly up once.
- Our string obtained will be 0,1,0,1,0,0,1 and the corresponding abaci will be:

$$\ldots, 1, \ldots, 1, 1, \mathbf{0,\ 1,\ 0,\ 1,\ 0,\ 0,\ 1}, 0, 0, \ldots, 0, \ldots$$



Heatmap of character table for $n = 6$

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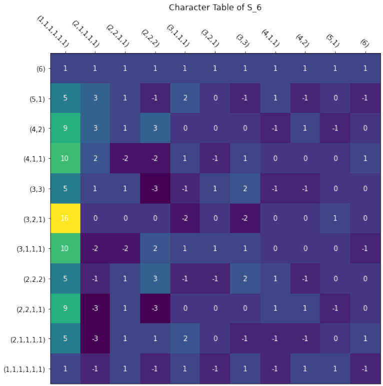

Figure: Heatmap of Character Table for $n = 6$ [1]

[1] See the program

More heatmaps!

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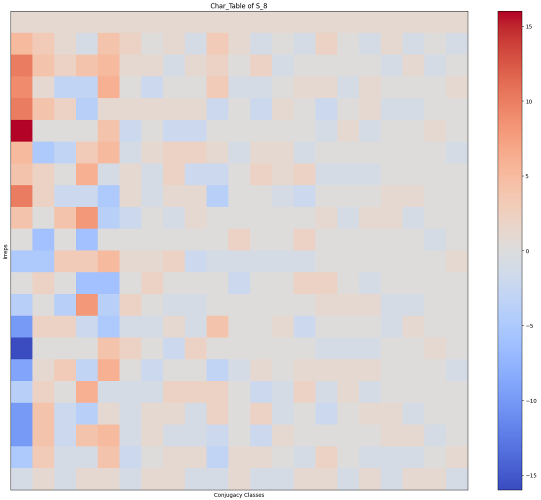

Figure: Heatmap of Character Table for $n = 8$ [2]

More heatmaps!

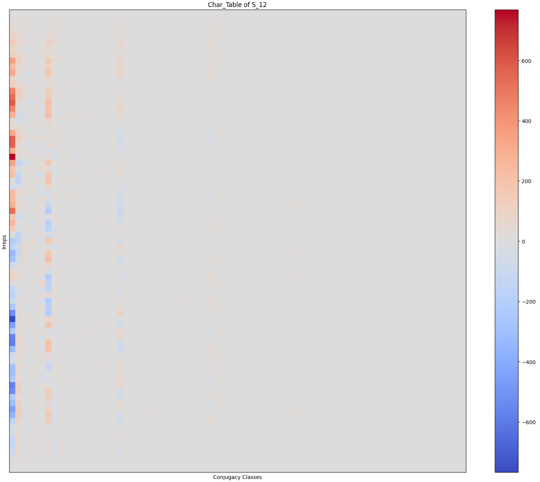

Figure: Heatmap of Character Table for $n = 12$ [3]

[3]

More heatmaps!

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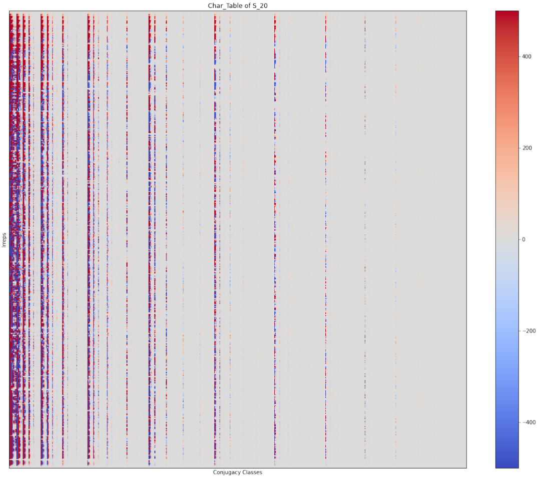

Figure: Heatmap of Character Table for $n = 20$ (truncated ± 500) [4]

[4] See the program

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Number of Zeroes

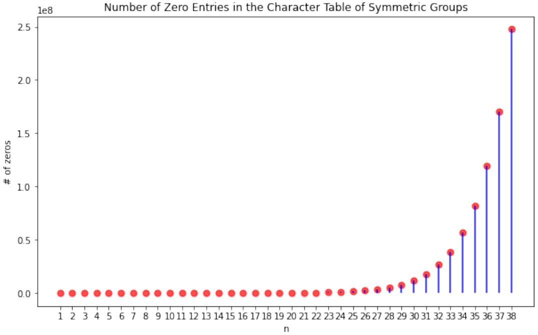

Figure: Number of Zero Entries in the Character Table of Symmetric Groups

Density of Zeroes Increasing?

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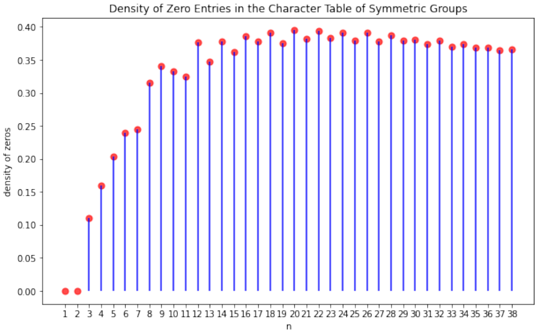

Figure: Density of Zero Entries in the Character Table of Symmetric Groups

Distribution of Values In a Column

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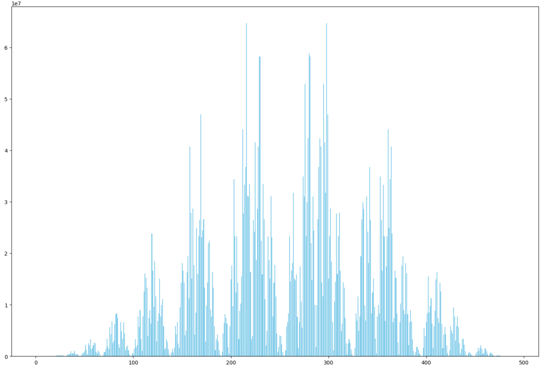

Figure: Distribution of Size of Character Values of First Column of S19

Next Steps

- Analyze the column associated with the staircase partition, as it has no repeated parts and its entries are generally smaller
- Look at the moments of the entries in the character table to see if they resemble some well known distribution

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