第十章　重积分

一. 将二重积分 $I=\iint\limits_D f(x,y)d\sigma$ 化为累次积分(两种形式), 其中 D 给定如下:

1. D: 由 $y^2=8x$ 与 $x^2=8y$ 所围之区域.

2. D: 由 x = 3, x = 5, x－2y + 1 = 0 及 x－2y + 7 = 0 所围之区域.

3. D: 由 $x^2+y^2\le 1$, y ≥ x 及 x > 0 所围之区域.

4. D: 由|x| + |y| ≤ 1 所围之区域.

解. 1. $I=\iint\limits_D f(x,y)d\sigma=\int_0^2 dx\int_{x^2}^{\sqrt{8x}} f(x,y)dy=\int_0^4 dy\int_{\frac{y^2}{8}}^{\sqrt{y}} f(x,y)dx$

2. $I=\iint\limits_D f(x,y)d\sigma=\int_3^5 dx\int_{\frac{x+1}{2}}^{\frac{x+7}{2}} f(x,y)dy$

$$=\int_2^3 dy\int_3^{2y-1} f(x,y)dx+\int_3^5 dy\int_3^5 f(x,y)dx+\int_5^6 dy\int_{2y-7}^5 f(x,y)dx$$

3. $I=\iint\limits_D f(x,y)d\sigma=\int_0^{\sqrt{2}/2} dx\int_x^{\sqrt{1-x^2}} f(x,y)dy$

$$=\int_0^{\sqrt{2}/2} dy\int_0^y f(x,y)dx+\int_{\sqrt{2}/2}^1 dy\int_0^{\sqrt{1-y^2}} f(x,y)dx$$

4. $I=\iint\limits_D f(x,y)d\sigma=\int_{-1}^0 dx\int_{-x-1}^{x+1} f(x,y)dy+\int_0^1 dx\int_{x-1}^{1-x} f(x,y)dy$

$$=\int_{-1}^0 dy\int_{-y-1}^{y+1} f(x,y)dx+\int_0^1 dy\int_{y-1}^{1-y} f(x,y)dx$$

二. 改变下列积分次序:

1. $\int_0^a dx\int_{\frac{a^2-x^2}{2a}}^{\sqrt{a^2-x^2}} f(x,y)dy$　　　　2. $\int_0^1 dx\int_0^{x^2} f(x,y)dy+\int_1^3 dx\int_0^{\frac{3-x}{2}} f(x,y)dy$

3. $\int_{-1}^0 dx\int_{-x}^{2-x^2} f(x,y)dy+\int_0^1 dx\int_x^{2-x^2} f(x,y)dy$

解: 1. $\int_0^a dx\int_{\frac{a^2-x^2}{2a}}^{\sqrt{a^2-x^2}} f(x,y)dy=\int_0^{\frac{a}{2}} dy\int_{\sqrt{a^2-2ay}}^{\sqrt{a^2-y^2}} f(x,y)dx+\int_{\frac{a}{2}}^a dy\int_0^{\sqrt{a^2-y^2}} f(x,y)dx$

2. $\int_0^1 dx\int_0^{x^2} f(x,y)dy+\int_1^3 dx\int_0^{\frac{3-x}{2}} f(x,y)dy=\int_0^1 dy\int_{\sqrt{y}}^{3-2y} f(x,y)dx$

3. $\int_{-1}^0 dx\int_{-x}^{2-x^2} f(x,y)dy+\int_0^1 dx\int_x^{2-x^2} f(x,y)dy$

$$=\int_0^1 dy\int_{-y}^{y} f(x,y)dx+\int_0^2 dy\int_{-\sqrt{2-y}}^{\sqrt{2-y}} f(x,y)dx$$

三. 将二重积分 $I=\iint\limits_D f(x,y)d\sigma$ 化为极坐标形式的累次积分，其中:

1. D: $a^2 \le x^2+y^2 \le b^2,\ y \ge 0,\ (b > a > 0)$
2. D: $x^2+y^2 \le y,\ x \ge 0$
3. D: $0 \le x+y \le 1,\ 0 \le x \le 1$

解. 1. $I=\iint\limits_D f(x,y)d\sigma=\int_0^{\pi} d\theta\int_a^b f(\rho\cos\theta,\rho\sin\theta)\rho d\rho$

2. $I=\iint\limits_D f(x,y)d\sigma=\int_0^{\frac{\pi}{2}} d\theta\int_0^{\sin\theta} f(\rho\cos\theta,\rho\sin\theta)\rho d\rho$

3. $I=\iint\limits_D f(x,y)d\sigma=\int_{-\frac{\pi}{4}}^{0} d\theta\int_0^{\frac{1}{\cos\theta}} f(\rho\cos\theta,\rho\sin\theta)\rho d\rho$

$$+\int_0^{\frac{\pi}{2}} d\theta\int_0^{\frac{1}{\cos\theta+\sin\theta}} f(\rho\cos\theta,\rho\sin\theta)\rho d\rho$$

四. 求解下列二重积分:

1. $\int_1^2 dx\int_{\sqrt{x}}^{x}\sin\frac{\pi x}{2y}dy+\int_2^4 dx\int_{\sqrt{x}}^{2}\sin\frac{\pi x}{2y}dy$

2. $\int_0^1 dx\int_0^{\sqrt{x}} e^{-\frac{y^2}{2}}dy$

3. $\iint\limits_D \frac{y}{x^6}dxdy$, D: 由 $y=x^4-x^3$ 的上凸弧段部分与 x 轴所形成的曲边梯形

4. $\iint\limits_D \frac{xy}{x^2+y^2}dxdy$, D: $y \ge x$ 及 $1 \le x^2+y^2 \le 2$

解.

1. $\int_1^2 dx\int_{\sqrt{x}}^{x}\sin\frac{\pi x}{2y}dy+\int_2^4 dx\int_{\sqrt{x}}^{2}\sin\frac{\pi x}{2y}dy=\int_1^2 dy\int_y^{y^2}\sin\frac{\pi x}{2y}dx=-\frac{2}{\pi}\int_1^2 y\cos\frac{\pi x}{2y}\Bigg|_y^{y^2}dy$

$$=-\frac{2}{\pi}\int_1^2 y\cos\frac{\pi y}{2}dy=-\frac{4}{\pi^2}\int_1^2 yd\sin\frac{\pi y}{2}$$

$$=-\frac{4}{\pi^2}y\sin\frac{\pi y}{2}\Bigg|_1^2+\frac{4}{\pi^2}\int_1^2\sin\frac{\pi y}{2}dy$$

$$=\frac{4}{\pi^2}-\frac{8}{\pi^3}\cos\frac{\pi y}{2}\Bigg|_1^2=\frac{4}{\pi^3}(\pi+2)$$

2. $\int_0^1 dx\int_0^{\sqrt{x}} e^{-\frac{y^2}{2}}dy=\int_0^1 e^{-\frac{y^2}{2}}dy\int_{y^2}^1 dx=\int_0^1 e^{-\frac{y^2}{2}}dy-\int_0^1 y^2 e^{-\frac{y^2}{2}}dy$

$$=\int_0^1 e^{-\frac{y^2}{2}}dy+\int_0^1 yde^{-\frac{y^2}{2}}=\int_0^1 e^{-\frac{y^2}{2}}dy+ye^{-\frac{y^2}{2}}\Bigg|_0^1-\int_0^1 e^{-\frac{y^2}{2}}dy=e^{-\frac{1}{2}}$$

3. $\iint\limits_D \frac{y}{x^6}dxdy$, D: 由 $y=x^4-x^3$ 的上凸弧段部分与 x 轴所形成的曲边梯形.

解. $y'=4x^3-3x^2$, $y''=12x^2-6x=6x(2x-1)<0$. 解得 $0<x<\frac{1}{2}$. 此时图形在 x 轴下方. 所以

$$\iint\limits_D \frac{y}{x^6}dxdy=\int_0^{1/2}\int_{x^4-x^3}^0 \frac{y}{x^6}dy=\frac{1}{2}\int_0^{1/2}\frac{y^2}{x^6}\Bigg|_{x^4-x^3}^0 dx=-\frac{1}{2}\int_0^{1/2}\frac{(x^4-x^3)^2}{x^6}dx=-\frac{7}{48}$$

4. $\iint\limits_D \frac{xy}{x^2+y^2}dxdy$, D: $y\ge x$ 及 $1\le x^2+y^2\le 2$.

解. 使用极坐标变换

$$\iint\limits_D \frac{xy}{x^2+y^2}dxdy=\int_{\pi/4}^{5\pi/4}d\theta\int_1^{\sqrt{2}}\frac{\rho\cos\theta\rho\sin\theta}{\rho^2}\rho d\rho$$

$$=\frac{1}{2}\int_{\pi/4}^{5\pi/4}\sin 2\theta d\theta\int_1^{\sqrt{2}}\rho d\rho=0$$

五. 计算下列二重积分:

1. $\iint\limits_D \sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}dxdy$, D: $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$.

解. 令 $x=a\rho\cos\theta$, $y=b\rho\sin\theta$. 雅可比行列式为

$$\frac{\partial(x,y)}{\partial(\rho,\theta)}=\begin{vmatrix} x'_\rho & x'_\theta \\ y'_\rho & y'_\theta \end{vmatrix}=\begin{vmatrix} a\cos\theta & -a\rho\sin\theta \\ b\sin\theta & b\rho\cos\theta \end{vmatrix}=ab\rho$$

$$\iint\limits_D \sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}dxdy=\int_0^{2\pi}d\theta\int_0^1\sqrt{1-\rho^2}\,ab\rho d\rho=-2\pi ab\frac{1}{3}(1-\rho^2)^{\frac{3}{2}}\Bigg|_0^1=\frac{2}{3}\pi ab$$

2. $\iint\limits_D \ln(x^2+y^2)dxdy$, D: $\varepsilon^2 \le x^2+y^2 \le 1$, 并求上述二重积分当 $\varepsilon \to 0^+$ 时的极限.

解. $\iint\limits_D \ln(x^2+y^2)dxdy = \int_0^{2\pi} d\theta \int_\varepsilon^1 \ln\rho^2 \rho d\rho = \pi\int_\varepsilon^1 \ln\rho^2 d\rho^2$

$$=\pi(\rho^2\ln\rho^2-\rho^2)\Big|_\varepsilon^1 = \pi(-\varepsilon^2\ln\varepsilon^2+\varepsilon^2-1)$$

所以 $\lim\limits_{\varepsilon\to 0^+}\iint\limits_D \ln(x^2+y^2)dxdy = -\pi$.

3. $\int_0^a dx\int_0^x \dfrac{f'(y)}{\sqrt{(a-x)(x-y)}}dy$

解. $\int_0^a dx\int_0^x \dfrac{f'(y)}{\sqrt{(a-x)(x-y)}}dy = \int_0^a f'(y)dy\int_y^a \dfrac{dx}{\sqrt{(a-x)(x-y)}}$

$$=\int_0^a f'(y)dy\int_y^a \frac{dx}{\sqrt{-ay+2\cdot\frac{a+y}{2}x-x^2}} = \int_0^a f'(y)dy\int_y^a \frac{d(x-\frac{a+y}{2})}{\sqrt{\frac{(a-y)^2}{4}-(x-\frac{a+y}{2})}}$$

$$=\int_0^a f'(y)\arcsin\left.\left(\frac{x-\frac{a+y}{2}}{\frac{a-y}{2}}\right)\right|_y^a dy = \int_0^a f'(y)\pi dy = \pi(f(a)-f(0))$$

4. $\iint\limits_D \sqrt{\dfrac{1-x^2-y^2}{1+x^2+y^2}}dxdy$, D: $x^2+y^2\le 1$, $x\ge 0$, $y\ge 0$.

解. $\iint\limits_D \sqrt{\dfrac{1-x^2-y^2}{1+x^2+y^2}}dxdy = \iint\limits_{D_{\rho,\theta}} \sqrt{\dfrac{1-\rho^2}{1+\rho^2}}\rho d\rho xd\theta = \dfrac{\pi}{4}\int_0^1\sqrt{\dfrac{1-t}{1+t}}dt$

$$\underline{\underline{令\sqrt{\frac{1-t}{1+t}}=u}}\ \pi\int_0^1\frac{u^2}{(1+u^2)^2}du\ \underline{\underline{令u=\tan\theta}}\ \pi\int_0^{\pi/4}\frac{\tan^2\theta\sec^2\theta}{\sec^4\theta}d\theta$$

$$=\pi\int_0^{\pi/4}\sin^2\theta d\theta = \frac{\pi}{8}(\pi-2).$$

六. 求证: $\iint\limits_D f(xy)dxdy = \ln 2\int_1^2 f(u)du$, 其中 D 是由 xy = 1, xy = 2, y = x 及 y = 4x(x > 0, y > 0)所围成之区域.

证明: 令 u = xy, y = vx. 即 $x=\sqrt{\dfrac{u}{v}}$, $y=\sqrt{uv}$. $\dfrac{\partial(x,y)}{\partial(u,v)}=\dfrac{1}{2v}$. 所以

$$\iint_D f(xy)dxdy=\iint_{D_{u,v}} f(u)\frac{1}{2v}dudv=\int_1^2 f(u)du\int_1^4\frac{1}{2v}dv=\ln 2\int_1^2 f(u)du$$

七. 求证: $\iint\limits_{x^2+y^2\le 1} f(x+y)dxdy=\int_{-\sqrt{2}}^{\sqrt{2}}\sqrt{2-u^2}\,f(u)du$

证明: 令 $u=x+y$, $v=x-y$. $\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y\end{vmatrix}}=-\frac{1}{2}$. 所以

$$\iint\limits_{x^2+y^2\le 1} f(x+y)dxdy=\iint\limits_{u^2+v^2\le 2} f(u)\frac{1}{2}dudv=\int_{-\sqrt{2}}^{\sqrt{2}} f(u)\left[\int_0^{\sqrt{2-u^2}}dv\right]du$$

$$=\int_{-\sqrt{2}}^{\sqrt{2}}\sqrt{2-u^2}\,f(u)du$$

八. 设 $f(t)$是半径为 t 的圆周长, 试证:

$$\frac{1}{2\pi}\iint\limits_{x^2+y^2\le a^2} e^{-\frac{x^2+y^2}{2}}dxdy=\frac{1}{2\pi}\int_0^a f(t)e^{-\frac{t^2}{2}}dt$$

证明: 左 $=\frac{1}{2\pi}\iint\limits_{x^2+y^2\le a^2} e^{-\frac{x^2+y^2}{2}}dxdy=\frac{1}{2\pi}\int_0^{2\pi}d\theta\int_0^a e^{-\frac{\rho^2}{2}}\rho d\rho$

$$=\frac{1}{2\pi}\int_0^a 2\pi\rho e^{-\frac{\rho^2}{2}}d\rho=\frac{1}{2\pi}\int_0^a f(\rho)e^{-\frac{\rho^2}{2}}d\rho$$ =右

九. 设 m, n 均为正整数, 其中至少有一个是奇数, 证明:

$$\iint\limits_{x^2+y^2\le a^2} x^m y^n dxdy=0$$

证明: 区域 D 既对 x 轴对称, 又对 y 轴对称.

当 m 为奇数时 $x^m y^n$ 为对于 x 的奇函数, 所以二重积分为 0;

当 n 为奇数时 $x^m y^n$ 为对于 y 的奇函数, 所以二重积分为 0.

十. 设函数 $f(x)$在$[0, t]$上连续, 令$F(t)=\int_0^t dz\int_0^z dy\int_0^y (y-z)^2 f(x)dx$, 证明:

$$\frac{dF}{dt}=\frac{1}{3}\int_0^t (t-x)^3 f(x)dx$$

解: 先计算

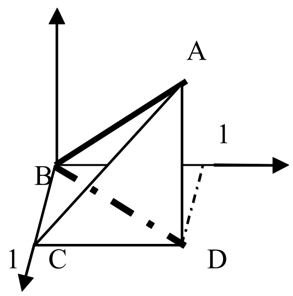
$$\int_0^z dy\int_0^y (y-z)^2 f(x)dx = \int_0^z f(x)\left[\int_x^z (y-z)^2 dy\right]dx = \frac{1}{3}\int_0^z f(x)(y-z)^3\Big|_x^z dx$$

$$= \frac{1}{3}\int_0^z (z-x)^3 f(x)dx$$

所以 $\dfrac{dF}{dt} = \dfrac{d}{dt}\left\{\int_0^t [\frac{1}{3}\int_0^z (z-x)^3 f(x)dx]dz\right\} = \dfrac{1}{3}\int_0^t (t-x)^3 f(x)dx$

十一. 计算: $\int_0^1 dx\int_0^x dy\int_0^y \dfrac{\sin z}{1-z}dz$

解. 因为 $\int \dfrac{\sin z}{1-z}dz$ 不能积成有限形式, 所以必须更换积分次序. 四面体 $A-BCD$ 为所求的积分区域.



由图知 $\int_0^1 dx\int_0^x dy\int_0^y \dfrac{\sin z}{1-z}dz = \iint\limits_{D_{y,z}} \dfrac{\sin z}{1-z}\int_y^1 dx = \iint\limits_{D_{y,z}} \dfrac{\sin z}{1-z}(1-y)dydz$

$$= \int_0^1 \frac{\sin z}{1-z}dz\int_z^1 (1-y)dy = \frac{1}{2}\int_0^1 (1-z)\sin z dz = \frac{1}{2}(1-\sin z)$$

十二. $\iiint\limits_{\Omega} (2y+\sqrt{x^2+z^2})dxdydz$, Ω: 由 $x^2+y^2+z^2=a^2$, $x^2+y^2+z^2=4a^2$, 及

$x^2-y^2+z^2=0$ $(y \ge 0, a > 0)$所围成.

解. 令 $y = r\cos\varphi, z = r\cos\theta\sin\varphi, x = r\sin\theta\sin\varphi$. 则

$dxdydz = r^2\sin\varphi d\theta dr d\varphi$. 于是

$$\iiint\limits_{\Omega} (2y+\sqrt{x^2+z^2})dxdydz = \int_0^{\pi/4} d\varphi\int_0^{2\pi} d\theta\int_a^{2a} (2r\cos\varphi + r\sin\varphi)r^2\sin\varphi dr$$

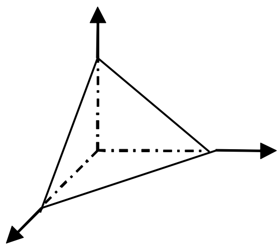
$$= 2\pi\frac{r^4}{4}\Big|_a^{2a}\cdot\int_0^{\pi/4} (2\cos\varphi+\sin\varphi)d\varphi$$

$$= \frac{15\pi}{2}a^4\left[-\cos^2\varphi\Big|_0^{\pi/4} + \int_0^{\pi/4}\frac{1-\cos 2\varphi}{2}d\varphi\right] = \frac{15a^4\pi}{16}(2+\pi)$$

十三. 计算下列三重积分:

1. $\iiint\limits_{\Omega}(1+x+y+z)^{-3}dv$, Ω: 由 $x+y+z=1, x=0, y=0$ 及 $z=0$ 所围成.

解.



$$\iiint\limits_{\Omega}(1+x+y+z)^{-3}dv=\int_0^1dx\int_0^{1-x}dy\int(1+x+y+z)^{-1}dz$$

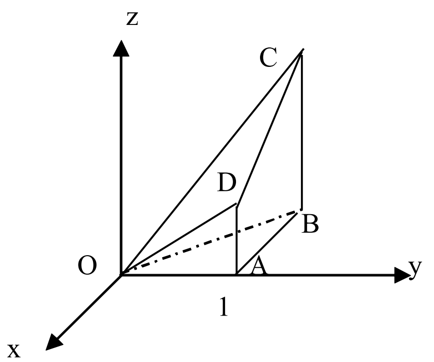
$$=-\frac{1}{2}\int_0^1dx\int_0^{1-x}(1+x+y+z)^{-2}\Big|_0^{1-x-y}dy=\frac{1}{2}\int_0^1dx\int_0^{1-x}[(1+x+y)^{-2}-2^{-2}]dy$$

$$=\frac{1}{2}\int_0^1\left[-(1+x+y)\Big|_0^{1-x}-\frac{1}{4}(1-x)\right]dx=\frac{1}{2}\int_0^1[(1+x)^{-1}-\frac{1}{2}-\frac{1}{4}(1-x)]dx$$

$$=\frac{1}{2}\left\{\ln(1+x)\Big|_0^1-\frac{1}{2}+\frac{1}{8}(1-x)^2\Big|_0^1\right\}=\frac{1}{2}\left\{\ln 2-\frac{1}{2}-\frac{1}{8}\right\}=\frac{1}{2}\left[\ln 2-\frac{5}{8}\right]$$

2. $\iiint\limits_{\Omega}e^{x+y+z}dv$, Ω: $y=1, y=-x, x=0, z=0$ 及 $z=-x$ 所围形体.

解.



四面体 $O-ABCD$ 为积分区域.

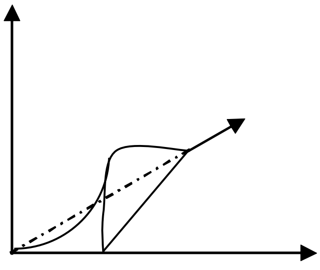
$$\iiint\limits_{\Omega}e^{x+y+z}dv=\iint\limits_{D_{xy}}e^{x+y}\int_0^{-x}e^z dz=\iint\limits_{D_{xy}}e^{x+y}(e^{-x}-1)dxdy=\iint\limits_{D_{xy}}(e^y-e^{x+y})dxdy$$

$$=\int_0^1\left[\int_{-y}^0(e^y-e^{x+y})dx\right]dy=\int_0^1(ye^y-e^{x+y})\Big|_{-y}^0dy=\int_0^1(ye^y-e^y+1)dy$$

$$=ye^y\Big|_0^1-2\int_0^1e^ydy+1=e-2e+2+1=3-e$$

3. $\iiint\limits_{\Omega}xydv$, Ω: $z=xy, x+y=1$ 及 $z=0$ 所围形体.

解.

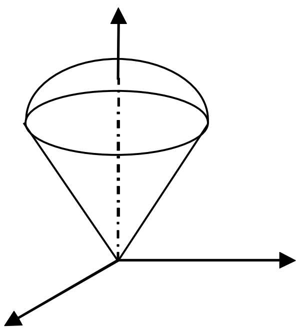


$$\iiint\limits_{\Omega} xydv=\iint\limits_{D_{xy}} xydxdy\int_0^{xy}dz=\iint\limits_{D_{xy}} x^2y^2dxdy=\int_0^1 x^2\left[\int_0^{1-x}y^2dy\right]dx=\int_0^1 x^2\frac{1}{3}(1-x)^3dx$$

$$=\frac{1}{3}\int_0^1 x^2(1-3x+3x^2-x^3)dx=\frac{1}{3}\left(\frac{1}{3}-\frac{3}{4}+\frac{3}{5}-\frac{1}{6}\right)=\frac{1}{3}\cdot\frac{20-45+36-10}{60}=\frac{1}{180}$$

4. $\iiint\limits_{\Omega} z\sqrt{x^2+y^2+z^2}dv,\Omega:$ 由$x^2+y^2+z^2=1$与$z=\sqrt{3(x^2+y^2)}$围成的空间区域.

解.



解$\begin{cases}x^2+y^2+z^2=1\\ z=\sqrt{3(x^2+y^2)}\end{cases}$ 得 $z=\dfrac{\sqrt{3}}{2}$.

方法一: $\iiint\limits_{\Omega} z\sqrt{x^2+y^2+z^2}dv$

$$=\int_0^{\frac{\sqrt{3}}{2}} z\left[\iint\limits_{D_{xy}}\sqrt{x^2+y^2+z^2}dxdy\right]dz+\int_{\frac{\sqrt{3}}{2}}^1 z\left[\iint\limits_{D_{xy}}\sqrt{x^2+y^2+z^2}dxdy\right]dz$$

$$=\int_0^{\frac{\sqrt{3}}{2}} z\left[\int_0^{\frac{z}{\sqrt{3}}}\sqrt{r^2+z^2}rdr\right]dz+2\pi\int_{\frac{\sqrt{3}}{2}}^1 z\left[\int_0^{\sqrt{1-z^2}}\sqrt{r^2+z^2}rdr\right]dz$$

$$=\frac{2}{3}\pi\int_0^{\frac{\sqrt{3}}{2}} z\sqrt{\frac{z^2}{3}+z^2}^{\,3}dz-\frac{2}{3}\pi\int_0^{\frac{\sqrt{3}}{2}} z\cdot z^3dz+\frac{2}{3}\pi\int_{\frac{\sqrt{3}}{2}}^1 z\sqrt{r^2+z^2}\Big|_0^{\sqrt{1-z^2}}dz$$

$$=\frac{2}{3}\pi\int_0^{\frac{\sqrt{3}}{2}}\left(\frac{4}{3}\right)^{\frac{3}{2}}z^4dz-\frac{2}{3}\pi\int_0^{\frac{\sqrt{3}}{2}}z^4dz+\frac{2}{3}\pi\int_{\frac{\sqrt{3}}{2}}^1 zdz-\frac{2}{3}\pi\int_{\frac{\sqrt{3}}{2}}^1 z^4dz$$

$$=\frac{2}{3}\pi\left(\frac{4}{3}\right)^{\frac{3}{2}}\frac{z^5}{5}\Bigg|_0^{\frac{\sqrt{3}}{2}}+\frac{2}{3}\pi\frac{z^2}{2}\Bigg|_{\frac{\sqrt{3}}{2}}^{1}-\frac{2}{3}\pi\frac{1}{5}$$

$$=\frac{2}{15}\pi\left(\frac{4}{3}\right)^{\frac{3}{2}}\left(\frac{\sqrt{3}}{2}\right)^5+\frac{\pi}{3}-\frac{\pi}{3}\cdot\frac{3}{4}-\frac{2}{15}\pi=\frac{3}{30}\pi-\frac{\pi}{20}=\frac{\pi}{20}$$

方法二：用球坐标变换

$$\iiint_\Omega z\sqrt{x^2+y^2+z^2}\,dv=\iiint_\Omega r\cos\varphi\cdot r\cdot r^2\sin\varphi d\theta d\varphi dr$$

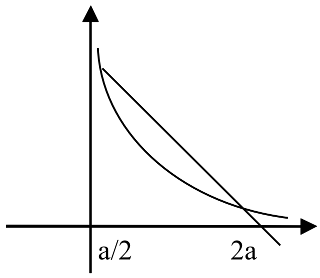
$$=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{6}}\sin\varphi\cos\varphi\int_0^1 r^4dr=-2\pi\cdot\left(-\frac{1}{4}\cos 2\varphi\right)_0^{\frac{\pi}{6}}\cdot\frac{1}{5}r^4\Big|_0^1$$

$$=2\pi\left(\frac{1}{4}-\frac{1}{4}\cdot\frac{1}{2}\right)\cdot\frac{1}{5}=\frac{\pi}{20}$$

十四. 求由下列曲线所围图形的面积.

1. $xy=a^2, x+y=\frac{5}{2}a \quad (a>0)$

解:



求解联立方程$\begin{cases}xy=a^2\\ x+y=\frac{5}{2}a\end{cases}$, 得 $x=\frac{a}{2}, x=2a$. 所以面积 S 为

$$S=\iint_D dxdy=\int_{\frac{a}{2}}^{2a}\left[\int_{\frac{a^2}{x}}^{\frac{5}{2}a-x}dy\right]dx=\int_{\frac{a}{2}}^{2a}\left(\frac{5}{2}a-x-\frac{a^2}{x}\right)dx$$

$$=\left(\frac{5}{2}ax-\frac{1}{2}x^2-a^2\ln x\right)_{\frac{a}{2}}^{2a}=\frac{15}{8}a^2-2a^2\ln 2.$$

2. $(x^2+y^2)=a(x^3-3xy^2), (a>0)$

解. 由表达式可知图形关于 y 轴对称, 所以总面积为上半平面部分的面积的二倍. 化成极坐标, 得

$$r=a\cos\theta(4\cos^2\theta-3)$$

因为 $r>0$, 所以

$$\cos\theta(4\cos^2\theta-3)\geq 0$$

求解 $\begin{cases}\cos\theta \ge 0 \\ 4\cos^2\theta - 3 \ge 0\end{cases}$ 或 $\begin{cases}\cos\theta \le 0 \\ 4\cos^2\theta - 3 \le 0\end{cases}$，且 $0 \le \theta \le \pi$

解得 $\quad 0 \le \theta \le \dfrac{\pi}{6}$ 或 $\dfrac{\pi}{2} \le \theta \le \dfrac{5\pi}{6}$. 于是面积 S 为

$$S = 2\iint\limits_{D_{xy}(y\ge 0)} dxdy = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{6}} r^2(\theta)d\theta + \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} r^2(\theta)d\theta\right)$$

$$= \int_0^{\frac{\pi}{6}} a^2\cos^2\theta(4\cos^2\theta - 3)^2 d\theta + \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} a^2\cos^2\theta(4\cos^2\theta - 3)^2 d\theta$$

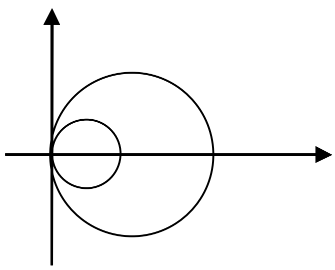
$$\xlongequal{\text{第二式中令}\theta = \pi - \varphi} \int_0^{\frac{\pi}{6}} a^2\cos^2\theta(4\cos^2\theta - 3)^2 d\theta$$

$$+ \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} a^2\cos^2\theta(4\cos^2\theta - 3)^2 d\theta$$

$$= \int_0^{\frac{\pi}{2}} a^2\cos^2\theta(4\cos^2\theta - 3)^2 d\theta = \frac{\pi a^2}{4}.$$

十五. 求曲面 $z = \sqrt{x^2 + y^2}$ 夹在二曲面 $x^2 + y^2 = y, x^2 + y^2 = 2y$ 之间的部分的面积.

解. 该曲面在 xoy 平面上的投影区域为



所以所求面积为

$$S = \iint\limits_{D_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy = \iint\limits_{D_{xy}} \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}}\, dxdy$$

$$= \iint\limits_{D_{xy}} \sqrt{2}\, dxdy = \sqrt{2}(\pi - \frac{\pi}{4}) = \frac{3\sqrt{2}}{4}\pi$$

十六. 求用平面 $x + y + z = b$ 与曲面 $x^2 + y^2 + z^2 - xy - xz - yz = a^2$ 相截所得的截断面之面积.

解. 作变换

$$\begin{cases} x' = \dfrac{1}{\sqrt{2}}x - \dfrac{1}{\sqrt{2}}z \\ y' = \dfrac{1}{\sqrt{6}}x - \dfrac{2}{\sqrt{6}}y + \dfrac{1}{\sqrt{6}}z \\ z' = \dfrac{1}{\sqrt{3}}x + \dfrac{1}{\sqrt{3}}y + \dfrac{1}{\sqrt{3}}z \end{cases}$$

上式是正交变换, 所以$0x'\,y'\,z'$也是直角坐标系. 在新坐标系下平面方程为

$$z' = \frac{1}{\sqrt{3}}(x + y + z) = \frac{1}{\sqrt{3}}b$$

反解变换式可得

$$\begin{cases} x = \dfrac{1}{\sqrt{2}}x' + \dfrac{1}{\sqrt{6}}y' + \dfrac{1}{\sqrt{3}}z' \\ y = \dfrac{-\sqrt{6}}{3}y' + \dfrac{1}{\sqrt{3}}z' \\ z = -\dfrac{1}{\sqrt{2}}x' + \dfrac{1}{\sqrt{6}}y' + \dfrac{1}{\sqrt{3}}z' \end{cases}$$

代入曲面方程后得到

$$x'^2 + y'^2 = \frac{2}{3}a^2$$

正交变换不改变面积, 所以

$$S = \iint\limits_{x'^2 + y'^2 \le \frac{2}{3}a^2} dx'\,dy' = \frac{2}{3}\pi a^2$$

十七. 求下列曲面所围形体的体积.

1. $z = xy, x + y + z = 1, z = 0.$

解. 曲顶的曲面为 $z = xy$ 及 $x + y + z = 1$. 所以所求体积必须分成二部分. 该二部分在 xoy 平面上的投影区域分别为 D_1, D_2. 于是体积 V 为

$$V = \iint\limits_{D_1} xydxdy + \iint\limits_{D_2} (1 - x - y)dxdy$$

$$= \int_0^1 xdx \int_0^{\frac{1-x}{1+x}} ydy + \int_0^1 dx \int_{\frac{1-x}{1+x}}^{1-x} (1 - x - y)dy$$

$$= \left(-\frac{11}{4}4\ln 2\right) + \left(\frac{25}{6} - 6\ln 2\right) = \frac{17}{12} - 2\ln 2$$

2. $z = x^2 + y^2, x^2 + y^2 = x, x^2 + y^2 = 2x, z = 0$

解. $V=\iint\limits_{D_{xy}}(x^2+y^2)dxdy=\int_0^{\pi}d\theta\int_{\cos\theta}^{2\cos\theta}r^2rdr$

$$=\frac{1}{4}\int_0^{\pi}(16\cos^4\theta-\cos^4\theta)d\theta=\frac{45}{32}\pi\ .$$

3. $z=8-x^2-y^2, z=x^2+y^2$

解. 解联立方程$\begin{cases}z=8-x^2-y^2\\ z=x^2+y^2\end{cases}$, 得 $z=4$.

$$V=\iiint\limits_{\Omega}dv=\int_0^4dz\iint\limits_{D_{xy}}dxdy+\int_4^8dz\iint\limits_{D_{xy}}dxdy=\int_0^4\pi zdz+\int_4^8(8-z)dz=16\pi.$$

十八. 将三重积分$\iiint\limits_{\Omega}f(x,y,z)dv$化为柱面坐标的累次积分, 其中Ω是由$x^2+y^2=z^2$, $z=$ 1 及 $z=4$ 所围成.

解.

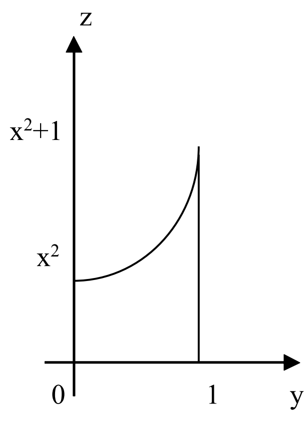
$$I=\int_0^{2\pi}d\theta\int_0^1rdr\int_1^4f(r\cos\theta,r\sin\theta,z)dz+\int_0^{2\pi}d\theta\int_1^4rdr\int_r^4f(r\cos\theta,r\sin\theta,z)dz$$

十九. 改变下列三重积分的积分次序:

1. $\int_0^1dx\int_0^1dy\int_0^{x^2+y^2}f(x,y,z)dz$, 2. $\int_0^1dx\int_0^{1-x}dy\int_0^{x+y}f(x,y,z)dz$

解. 1. 因为$\iiint\limits_{V}f(x,y,z)dv=\iint\limits_{D_{xz}}dxdz\int_{y_1(x,z)}^{y_2(x,z)}f(x,y,z)dy=\int_{y_1}^{y_2}dy\iint\limits_{D_{xz}(y)}f(x,y,z)dxdz$. 应该注意最后这个积分的积分区域和 y 有关, 因此内层的二重积分为 y 的函数.

当 x 取自[0, 1]时, 该积分区域 V 在 yoz 平面上的投影区域如图:



于是

$$\int_0^1dx\int_0^1dy\int_0^{x^2+y^2}f(x,y,z)dz$$

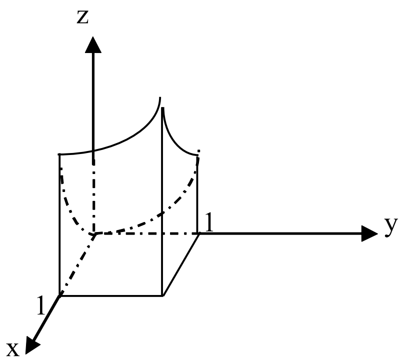
$$=\int_0^1dx\left[\int_0^{x^2}dz\int_0^1f(x,y,z)dy+\int_0^1dz\int_{x^2}^{x^2+1}dz\int_{\sqrt{z-x^2}}^1f(x,y,z)dy\right]$$

由于 x, y 的轮换对称性, 立即可得

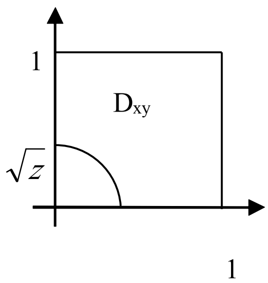
$$\int_0^1dx\int_0^1dy\int_0^{x^2+y^2}f(x,y,z)dz$$

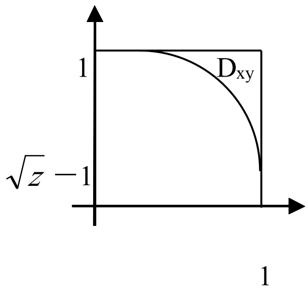
$$=\int_0^1dy\left[\int_0^{y^2}dz\int_0^1f(x,y,z)dx+\int_0^1dy\int_{y^2}^{y^2+1}dz\int_{\sqrt{z-y^2}}^1f(x,y,z)dx\right]$$

该题的积分区域如下图:



对于 $z = x^2 + y^2$，当 $x = 0, y = 1$ 时, $z = 1$; 当 $x = 1, y = 0$ 时, $z = 1$. 当 $x = 1, y = 1$ 时, $z = 2$. 所以当 z 取自[0, 1]时, V 在 xoy 平面上的投影 D_{xy} 如左图; z 取自[1, 2]时, V 在 xoy 平面上的投影 D_{xy} 如右图.





于是

$$\int_0^1 dx \int_0^1 dy \int_0^{x^2+y^2} f(x,y,z)dz$$

$$= \int_0^1 dz \left[\int_0^{\sqrt{z}} dy \int_{\sqrt{z-y^2}}^1 f(x,y,z)dx + \int_{\sqrt{z}}^1 dy \int_0^1 f(x,y,z)dx \right]$$

$$+ \int_1^2 dz \int_{\sqrt{z-1}}^1 dy \int_{\sqrt{z-y^2}}^1 f(x,y,z)dx$$

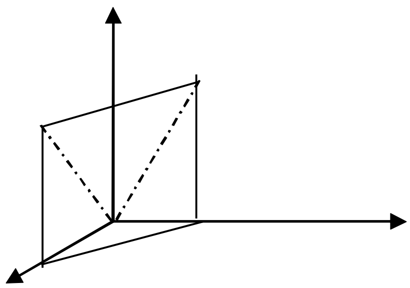
由 x, y 的对称性, 直接可得

$$\int_0^1 dx \int_0^1 dy \int_0^{x^2+y^2} f(x,y,z)dz$$

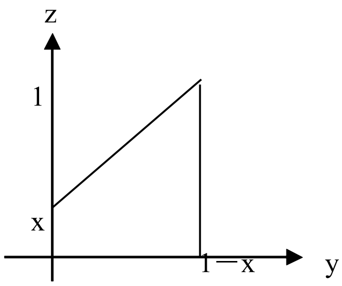
$$= \int_0^1 dz \left[\int_0^{\sqrt{z}} dx \int_{\sqrt{z-x^2}}^1 f(x,y,z)dy + \int_{\sqrt{z}}^1 dx \int_0^1 f(x,y,z)dy \right]$$

$$+ \int_1^2 dz \int_{\sqrt{z-1}}^1 dx \int_{\sqrt{z-x^2}}^1 f(x,y,z)dy$$

2. 积分区域如下图:



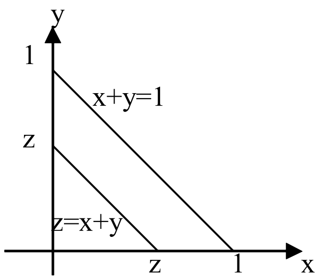
当 x 取自[0, 1]时积分区域 V 在 yoz 平面的投影如图:



于是

$$\int_0^1 dx\int_0^{1-x} dy\int_0^{x+y} f(x,y,z)dz$$
$$=\int_0^1 dx\left\{\int_0^x dz\int_0^{1-x} f(x,y,z)dy+\int_x^1 dz\int_{z-x}^{1-x} f(x,y,z)dy\right\}$$

当 z 取自[0, 1]时, V 在 xoy 平面的投影区域如图:



$$\int_0^1 dx\int_0^{1-x} dy\int_0^{x+y} f(x,y,z)dz$$
$$=\int_0^1 dz\left\{\int_0^z dy\int_{z-y}^{1-y} f(x,y,z)dx+\int_z^1 dy\int_0^{1-y} f(x,y,z)dx\right\}$$

二十. 已知质量为 M, 半径为 a 的球上任一点的密度与该点到球心的距离成正比, 求球关于切线的转动惯量.

解. 设直线 l 和 z 轴平行，l 和 xoy 平面的交点坐标为 x_1 和 y_1，则物体绕 l 的转动惯量为：

$$I=\iiint\limits_{\Omega}[(x-x_1)^2+(y-y_1)^2]\rho(x,y,z)dxdydz \qquad (1)$$

将球心放在原点，则密度 ρ（x,y,z）$=kr$，r 为点（x,y,z）到球心的距离。因为球的质量为 M, 所以

$$M=\iiint_{\Omega} krdxdydz = k\int_0^{2\pi} d\theta\int_0^{\pi}\sin\phi d\phi\int_0^R rr^2 dr = 2k\pi\times 2\times\left.\frac{r^4}{4}\right|_0^R = k\pi R^4$$

所以 $k=\dfrac{M}{\pi R^4}$， $\rho=\dfrac{Mr}{\pi R^4}$

取球的切线为平行于 z 轴，与 xoy 平面的交点坐标为（0，R），该切线为 l，球体绕 l 转动的转动惯量为

$$I_l=\iiint_{\Omega}[(x^2+(y-R)^2]krdxdydz=\iiint_{\Omega}(x^2+y^2-2yR+R^2)krdxdydz$$

$$=R^2\iiint_{\Omega}krdxdydz+\iiint_{\Omega}kr(x^2+y^2)dxdydz-\iiint_{\Omega}2krRydxdydz$$

$$=R^2M+k\int_0^{2\pi}d\theta\int_0^{\pi}\sin^3\phi\int_0^R r^5dr-0$$

$$=R^2M+k\times2\pi\times\frac{4}{3}\times\frac{1}{6}R^6=R^2M+\frac{4}{9}k\pi R^6=R^2M+\frac{4}{9}\frac{M}{\pi R^4}\pi R^6$$

$$=\frac{13}{9}R^2M$$