

Test Lecture Notes

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Executive Summary

This lecture provides a comprehensive review of fundamental **linear algebra concepts**, emphasizing their practical importance in engineering and computing. The discussion progresses from **vectors** to **matrices** and finally to **subspaces**. Two key examples are presented to illustrate these concepts: one involving an **invertible matrix** A and another with a **non-invertible matrix** C . The lecture demonstrates how to solve systems of linear equations in the form $Ax = b$ and explores the geometric interpretation of these solutions. The concept of **linear combinations** is introduced, showing how they relate to matrix multiplication: $Ax = x_1u + x_2v + x_3w$. The lecturer emphasizes the importance of recognizing **invertible** versus **non-invertible matrices** and their implications for solution spaces. The idea of a **basis** for a vector space is introduced, along with the concept of **subspaces**. The lecture concludes by previewing future course content, highlighting the significance of the **matrix transpose** and the product $A^T A$ in various applications.

Detailed Notes

1. Introduction to Linear Algebra Review

The lecture begins with an emphasis on the importance of linear algebra in modern computation and engineering. The lecturer argues that linear algebra is more relevant than calculus in many real-world applications, as it deals with discrete systems and matrices, which are fundamental in digital computations.

2. Fundamental Concepts of Linear Algebra

2.1 Vectors and Linear Combinations

The lecture starts with the most basic concept in linear algebra: vectors and their linear combinations.

Linear Combination: A linear combination of vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} is expressed as:

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$$

where a , b , and c are scalars.

2.2 Transition from Vectors to Matrices

The lecturer introduces a specific example using three vectors in \mathbb{R}^3 :

$$\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

These vectors are then combined into a matrix A :

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

(Concept Explained: This matrix A is called a "difference matrix" because it computes differences between adjacent elements when multiplied by a vector.)

2.3 Matrix-Vector Multiplication

The lecturer explains that multiplying matrix A by a vector \mathbf{x} is equivalent to taking a linear combination of the columns of A :

$$A\mathbf{x} = x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w}$$

The result of this multiplication is:

$$A\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{pmatrix}$$

3. Solving Systems of Linear Equations

3.1 Forward Problem: $A\mathbf{x} = \mathbf{b}$

The lecturer demonstrates the forward problem by multiplying A with a specific vector:

$$A \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

3.2 Inverse Problem: Solving for \mathbf{x}

The lecturer then introduces the inverse problem: given \mathbf{b} , find \mathbf{x} such that $A\mathbf{x} = \mathbf{b}$. For the matrix A , the solution is:

$$\mathbf{x} = \begin{pmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{pmatrix}$$

3.3 Inverse Matrix

The solution can be written as a matrix multiplication:

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{b}$$

(Concept Explained: A^{-1} is the inverse matrix of A . It "undoes" the action of A , transforming from \mathbf{b} back to \mathbf{x} .)

4. Invertible vs. Non-invertible Matrices

4.1 Example of a Non-invertible Matrix

The lecturer introduces a second matrix C :

$$C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

This matrix is not invertible, which leads to several important observations:

- There exists a non-zero vector \mathbf{x} such that $C\mathbf{x} = \mathbf{0}$
- The equation $C\mathbf{x} = \mathbf{b}$ is not always solvable
- When a solution exists, it is not unique

4.2 Conditions for Solvability

For the equation $C\mathbf{x} = \mathbf{b}$ to be solvable, \mathbf{b} must satisfy:

$$b_1 + b_2 + b_3 = 0$$

5. Geometric Interpretation

5.1 Subspaces

The lecturer introduces the concept of subspaces:

- For matrix A , the combinations of its columns span all of \mathbb{R}^3
- For matrix C , the combinations of its columns form a plane in \mathbb{R}^3

(Concept Explained: A subspace is a subset of a vector space that is closed under addition and scalar multiplication. It can be visualized as a line, plane, or higher-dimensional analogue passing through the origin.)

5.2 Basis and Independence

The lecturer introduces the concepts of basis and independence:

- The columns of A form a basis for \mathbb{R}^3 - they are linearly independent and span the entire space
- The columns of C are linearly dependent - the third column is a combination of the first two

6. Future Course Topics

The lecturer hints at future topics in the course:

- Rectangular matrices and their properties
- The importance of $A^T A$ in many applications
- Applications in networks, involving voltage and current laws

The lecture concludes with an emphasis on the course's approach to learning, encouraging students to engage with the material actively and ask questions freely.

Main

Key Points

- The lecturer emphasizes the **importance of linear algebra** in modern computing and engineering.
- Linear algebra progresses from **vectors** to **matrices** to **subspaces**.
- **Linear combinations** are the key operations in linear algebra: multiplying vectors by scalars and adding/subtracting them.
- Matrices are formed by putting vectors into columns, allowing for efficient representation of linear combinations.
- The equation $Ax = b$ represents a system of linear equations, where A is the matrix, x is the input vector, and b is the output vector.
- **Invertible matrices** allow for unique solutions to $Ax = b$. The inverse matrix A^{-1} satisfies $x = A^{-1}b$.
- **Non-invertible matrices** lead to systems that may have no solution, infinite solutions, or solutions only under certain conditions.
- A **subspace** is a set of vectors closed under linear combinations. Examples in \mathbb{R}^3 include planes, lines, the origin, and the whole space.
- **Independent vectors** form a basis for a space. For n -dimensional space, n independent vectors are needed.
- The **rank** of a matrix is related to the number of independent columns or rows.
- In applications, the matrix $A^T A$ often appears, which is always square and symmetric.

Action Items

- Review linear algebra if you haven't had an official course.
- Check the MIT OpenCourseWare website (ocw.mit.edu) for additional linear algebra resources.
- Look at the 1806 website for more linear algebra review materials, including old exams.
- Complete the homework: three problems from section 1.1 (specific problems not mentioned).

AI Lecture Study Guide: Introduction to Linear Algebra

1. Vectors and Linear Combinations

Linear algebra fundamentally deals with vectors and their combinations. A vector is an ordered list of numbers, typically represented as a column.

1.1 Linear Combinations

A linear combination of vectors is formed by multiplying each vector by a scalar and adding the results. For vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} , a linear combination is expressed as:

$$\mathbf{b} = x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w}$$

where x_1 , x_2 , and x_3 are scalars.

1.2 Geometric Interpretation

In 3D space, linear combinations of two independent vectors form a plane, while combinations of three independent vectors can span the entire 3D space.

2. Matrices and Matrix Operations

2.1 Matrix Definition

A matrix is a rectangular array of numbers. In linear algebra, vectors are often arranged as columns of a matrix.

2.2 Matrix-Vector Multiplication

When a matrix A multiplies a vector \mathbf{x} , the result is a linear combination of the columns of A :

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3$$

where \mathbf{a}_1 , \mathbf{a}_2 , and \mathbf{a}_3 are the columns of A .

2.3 Example: Difference Matrix

A first difference matrix A and its operation on a vector \mathbf{x} :

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad A\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}$$

3. Solving Systems of Linear Equations

3.1 Matrix Equation Form

A system of linear equations can be written as a matrix equation $A\mathbf{x} = \mathbf{b}$.

3.2 Inverse Matrix

For an invertible matrix A , the solution is given by $\mathbf{x} = A^{-1}\mathbf{b}$, where A^{-1} is the inverse of A .

3.3 Example: Sum Matrix (Inverse of Difference Matrix)

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

4. Invertible vs. Non-invertible Matrices

4.1 Invertible Matrices

An invertible matrix has the following properties:

- It has a unique solution for every \mathbf{b} in $A\mathbf{x} = \mathbf{b}$
- Its columns form a basis for the space
- The matrix equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$

4.2 Non-invertible Matrices

A non-invertible (singular) matrix has the following properties:

- The equation $A\mathbf{x} = \mathbf{b}$ may have no solution or infinitely many solutions
- Its columns are linearly dependent
- There exist non-zero solutions to $A\mathbf{x} = \mathbf{0}$

5. Subspaces

5.1 Definition

A subspace is a subset of a vector space that is closed under linear combinations. In \mathbb{R}^3 , subspaces can be:

- The entire space \mathbb{R}^3
- A plane through the origin
- A line through the origin
- The zero vector $\{\mathbf{0}\}$

5.2 Column Space

The column space of a matrix A is the subspace spanned by its columns, denoted as $C(A)$.

6. Basis and Independence

6.1 Linear Independence

Vectors are linearly independent if none can be expressed as a linear combination of the others.

6.2 Basis

A basis is a set of linearly independent vectors that span a space. For \mathbb{R}^n , a basis consists of n linearly independent vectors.

Practice Problems

1. Given the vectors $\mathbf{u} = [1, -1, 0]^T$, $\mathbf{v} = [0, 1, -1]^T$, and $\mathbf{w} = [0, 0, 1]^T$, determine if they form a basis for \mathbb{R}^3 .

2. For the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1} and verify that $AA^{-1} = I$.

3. Given the matrix $C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, determine if it's invertible. If not, find a non-zero vector \mathbf{x} such that $C\mathbf{x} = \mathbf{0}$.

Practice Problems Solutions

1. To determine if \mathbf{u} , \mathbf{v} , and \mathbf{w} form a basis for \mathbb{R}^3 , we need to check if they are linearly independent and span \mathbb{R}^3 . We can do this by forming a matrix with these vectors as columns and calculating its determinant:

$$\det \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = 1 \neq 0$$

Since the determinant is non-zero, the vectors are linearly independent and form a basis for \mathbb{R}^3 .

2. We found earlier that:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

To verify $AA^{-1} = I$, we multiply:

$$AA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

3. To determine if C is invertible, we can check if there's a non-zero vector \mathbf{x} such that $C\mathbf{x} = \mathbf{0}$. Let $\mathbf{x} = [x_1, x_2, x_3]^T$. Then:

$$C\mathbf{x} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives us the system of equations:

$$\begin{cases} x_1 - x_3 = 0 \\ -x_1 + x_2 = 0 \\ -x_2 + x_3 = 0 \end{cases}$$

Adding these equations, we get $0 = 0$, which is always true. This means the equations are dependent, and there are non-zero solutions. One such solution is $\mathbf{x} = [1, 1, 1]^T$. Therefore, C is not invertible.

Further Study

To deepen your understanding of linear algebra, consider exploring these related topics:

- Eigenvalues and eigenvectors
- Orthogonality and projections
- Singular Value Decomposition (SVD)
- Applications of linear algebra in computer graphics and machine learning

Additional resources:

- Gilbert Strang's book "Introduction to Linear Algebra"
- MIT OpenCourseWare: Linear Algebra course materials
- 3Blue1Brown's YouTube series on the essence of linear algebra

Appendix

Transcript

The following content is provided under a Creative Commons license. Your support will help MIT OpenCourseWare continue to offer high quality educational resources for free. To make a donation or to view additional materials from hundreds of MIT courses, visit MIT OpenCourseWare at ocw.mit.edu. OK. So this is the one and only review, you could say, of linear algebra. My website, I just think linear algebra is very important. You may have got that idea. And my website even has a little essay called Too Much Calculus. Because I think it's crazy for all the US universities to do this pretty much. You get semester after semester, differential calculus, integral calculus, ultimately differential equations. You run out of steam before the good stuff or you run out of time. And anybody who computes, who's living in the real world, is using linear algebra. You're taking a differential equation, you're taking your model, making it discrete, and computing with matrices. I mean, that's the world's digital now, not analog. So I hope it's OK to start the course with linear algebra. And so, but many engineering curricula don't fully recognize that. And so if you haven't had a official course, linear algebra, stay with 18085. This is a good way to learn it. You sort of learning what's important. So my review would be, and then this is, so future Wednesdays will be in our regular room for homework, review, questions of all kinds. And today, questions too. Shall I just fire away for the first half of the time to give you a sense of how I see the subject or least within that limited time? And then questions are totally welcome. Always welcome, actually. So I'll just start right up. So essentially, linear algebra progresses, starting with vectors to matrices. And then finally to subspaces. So that's like the abstraction, you could say abstraction, but it's not difficult, that you want to see. Until you see the idea of a subspace, you haven't really got linear algebra. OK, so I'll start at the beginning. What do you do with vectors? Answer, you take their linear combinations. That's what you can do with a vector. You can multiply it by a number, and you can add or subtract. So that's the key operation. Suppose I have vectors u , v , and w . Let me take three of them. And so I can take their combinations. So some combination will be, say, some number times u plus some number times v plus some number times w . So these numbers are called scalars. So these would be called scalars. And the whole thing is a linear combination. Let me abbreviate those words, linear combination. And you get some answer, say b . OK. But let's pin it down, make this whole discussion specific. Yeah, I started it a little early, I think. So let me, I'm going to take three vectors, u , v , and w . And take their combinations. So my u is going to be, they're carefully chosen. My u is going to be 1 minus 1, 0. And I'll take vectors in three dimensions. So that means their combinations will be in three dimensions. R^3 , three-dimensional space. So that will be u , and then v . So let's take 0, I think, 1 and minus 1. OK. Suppose I stopped there and took their linear combinations. It's very helpful to see a picture in three-dimensional space. I mean, the great thing about linear algebra, it moves into n -dimensional space, 10-dimensional, 100-dimensional, where we can't visualize. But yet, our instinct is right if we just follow. So what's your instinct? If I took those two vectors and notice they're not on the same line. One isn't a multiple of the other. They go in different directions. If I took their combinations, x_1 times u plus x_2 times v . Oh, now this is, oh, let me push. This is a serious question. If I took all their combinations. So let me try to draw a little bit. OK. I'm in three-dimensional space. And u goes somewhere, maybe there. And v goes somewhere, maybe here. Now suppose I take all the combinations. So I could multiply that first guy by any number. That would fill the line. I could multiply that second guy v . So this was u , and this was v . I could multiply that by any number x_2 . That would fill its line. Each of those lines I would later call a one-dimensional subspace, just a line. But now what happens if I take all combinations of the two? What

do you think? You get a plane. Get a plane. If I take anything on this line and anything on this line and add them up, you can see that I'm not going to fill 3D. But I'm going to fill a plane. And that maybe takes a little thinking. It just, then it becomes sort of, yeah, you see that that's what it has to be. OK. Now I'm going to have a third vector. OK. My third vector will be 001. OK. So that 001 is 0 in the x, 0 in the y, and 1 in the z direction. So there's w. OK. Now I want to take their combinations. So let me do that very specifically. How do I take combinations? This is important. Seems it's very simple, but important. I like to think of taking the combinations of some vectors. I'm always putting vectors into the columns of a matrix. So now I'm going to move to step two, matrix. I'm going to move to step two, and maybe I'll put it, well, not better, put it here. OK. Step two is the matrix has those vectors in its columns. So in this case, it's 3 by 3. OK. That's my matrix that I'm going to call it A. And now I'm going to, how do I take combinations of vectors? I, well, I should have maybe done it in detail here, but let me, I'll just do it with the matrix here. Watch this now. I'm going to, if I multiply A by the vector of x's, what that does? So this is now A times x. So very important of matrix times the vector. What does it do? The output is just what I want. This is the output. It takes x_1 times the first column plus x_2 times the second plus x_3 times the third. That's the way matrix multiplication works by columns. And you don't always see that. Because what do you see? You know how to probably know how to multiply that matrix by that vector. Let me ask you to do it. OK. What do you get? So everybody does it a component at a time. So what's the first component of the answer? x_1 . Yeah. How do you get that? It's a row times the vector. And when I say times, I really mean that dot product. This time plus this plus this is x_1 . And what about the second row? Minus x_1 plus x_2 , or I'll just say x_2 minus x_1 . And the third guy, the third component would be x_3 minus x_2 . OK. So right away, I'm going to call this matrix a difference matrix. It always helps to give names to things. So this is a difference matrix. Because it takes differences of the x's. And I would even say a first difference matrix. Because it's just straightforward difference. And we'll see second differences in class Friday. OK. So that's what A does. But remember my first point was that when a matrix multiplies a vector, the result is a combination of the columns. And that's not always because, see, I'm looking at the picture not just by numbers. With numbers, I'm just doing this stuff. But now I'm stepping back a little bit and saying, I'm combining, it's this vector times x_1 . That vector times x_1 plus this vector times x_2 plus that one times x_3 added together gives me this. Same, nothing complicated here. It's just look at it by vectors also. OK. Now suppose, so it's a little interesting. Already. Here we multiply these vectors by numbers. x_1 , x_2 , x_3 . That was our thinking here. Now our thinking here is a little we've switched slightly. Now I'm multiplying the matrix times the numbers in x. Just a slight switch. Multiply the matrix times the number. And I get some answer B, which is this. This is B. OK. And of course, I can do a specific example, like suppose I take the squares to be in x. So suppose I take A times the first three squares, 1, 4, 9. What answer would I get? Just to keep it clear that we're very specific here. So what would be the output B? I think of this as the input, the 1, 4, 9, the x's. Now the machine is a multiply by A. And there, here's the output. And what would be the output? What numbers am I going to get there? Yeah. 1, 3, something? 1, 3, 5. Which is actually a little neat. That you find the differences of the squares are the odd numbers. That appealed to me in school somehow. And that was already a bad sign, right? This dumb kid notices it. That you take the differences of square numbers, right? Whatever. OK. So now is a big step. This was the forward direction, right? The input, and there is the output. But now the real reality, that's easy and important. But the more deep problem is, what if I give you B and ask for x? So again, we're switching the direction here. OK. We're solving an equation now, or three equations, and three unknowns, $Ax = B$. So if I give you this B, can you get x? OK. How do you solve three equations? So this is just we're looking backwards. Now, that won't be too hard for this matrix, for this particular matrix that I chose, because it's triangular, we'll be able to go backwards. So let me do that. Let me take B to B. It's a vector. It's got three components. And now I'm going to go backwards to find x, or we will. OK. So do you see the three equations? Here they are. x_1 is B_1 . This is B_2 . That difference is B_3 . Those are my three equations. Three unknown, x is three known right hand sides. Or I think of it as A times x as a matrix times x, giving a vector B. What's the answer? OK. As I said, we're going to be able to do this. We're going to be able to solve this system easily, because it's already triangular. And it's actually lower triangular. So that means we'll start from the top. So the answer is the solution will be what? Let's make room for it. x_1 , x_2 , and x_3 . I want to find. And what's the answer? What can we just go from the top to bottom now? What's x_1 ? B_1 . Great. What's x_2 ? So x_2 minus x_1 . These are my equations. So what's x_2 minus x_1 ? Well, it's B_2 . So what is x_2 ? B_1 plus B_2 , right? And what's x_3 ? What do we need there for x_3 ? So I'm looking at the

third equation. That'll determine x_3 . When I see it this way, I see those ones. And I see it multiplying x_3 . And what do I get? Yeah. So x_3 minus this guy is B_3 . So I have to add in another B_3 , right? I'm doing sort of substitution down as I go. Once I learned that x_1 was B_1 , I used it there to find x_2 . And now I'll use x_2 to find x_3 . And what do I get again? x_3 is I put the x_2 over there. I think you've got it B_1 plus B_2 plus B_3 . OK. So that's a solution. Not difficult because the matrix was triangular. But let's think about that solution. That solution is a matrix times B . When you look at that, so this is like a good early step in linear algebra. When I look at that, I see a matrix multiplying B . I mean, you take that step up to seeing a matrix. And you can just read it off. So let me say, what's the matrix there? That's multiplying B to give that answer. From remember, the columns of this matrix, well, I don't know how you want to read it off. But one way is to think the columns of that matrix are multiplying B_1 , B_2 , and B_3 to give this. So what's the first column of the matrix? It's whatever I'm reading off. The coefficients really of B_1 here, 1, 1, 1. And what's the second column of the matrix? 0, 1, 1. Good. 0, B_2 is 1, 1. And the third is 0, 0, 1. Good. OK. Now, so lots of things to comment here. This is, let me write up again here. This is x . That was the answer. It's a matrix times B . And what's the name of that matrix? It's the inverse matrix. If Ax gives B , the inverse matrix does it the other way round. x is a inverse B . Let me just put that over here. If Ax is B , then x should be a inverse B . You could, so we had inverse, I wrote down inverse this morning, but without saying the point. But so you see how that comes. I mean, if I want to go formally, I multiply both sides by A inverse. If there is an A inverse, that's a critical thing, as we saw. Is the matrix invertible? The answer here is yes. There is an inverse. And what does that really mean? The inverse is the thing that takes us from B back to x . Think of A as multiplying by A as kind of a mapping, mathematicians use the word, or transform. Transform would be good. Transform from x to B . And this is the inverse transform. So it doesn't happen to be the discrete Fourier transform or a wavelet transform. Well, actually, we could give it a name. This is kind of a difference transform, right? That's what A did. Took differences. So what does A inverse do? It takes sums. It takes sums. That's why you see 1, 1, 1, 1, 1 along the rows, because it's just adding, and you see it here, and fully displayed. It's a sum matrix. I might as well call it S for sum. So that matrix, that sum matrix, is the inverse of the difference matrix. Yeah. OK. And maybe I just sent a hit on calculus earlier. You could say the calculus is all about one thing, and it's inverse. The derivative is A , and what's S ? In calculus. The integral. The whole subject is about one operation. Now, admittedly, it's not a matrix. It operates on functions instead of just little vectors. But that's the main point. The fundamental theorem of calculus is telling us that integrations, the inverse of difference, H . OK. So this is good, and I could put in, if I put in B equal 1, 3, 5, for example, just to put in some numbers. If I put in B equal 1, 3, 5, what would the x that comes out B ? 1, 4, 9. Because it takes us back. Here previously, we took differences of 1, 4, 9, got 1, 3, 5. Now, if we take sums of 1, 3, 5, we get 1, 4, 9. OK. Now, we have a system of linear equation. Now, I want to step back and see what was good about this matrix, somehow it has an inverse. A , x equal B has a solution, in other words. And it has only one solution, right? Because we worked it out. We had no choice. That was it. So there's just one solution. There's always one, and only one solution. It's like a perfect transform from the x 's to the b 's and back again. So that's what an invertible matrix is. It's a perfect map from one set of x 's to the b 's, and you can get back again. OK. Right. Yeah, questions always. OK. Now, I think I'm ready for another example. There are only two examples. And actually, this example, these two examples are on the 1806 web page. If you want to, some people asked after class, OK, how to get sort of a review of linear algebra? Well, the 1806 website would be definitely a possibility. So that's, well, I'll put down the open courseware website, m-m-i-t-e-d-u, and then you would look at the linear algebra course, or the math one. What is it? Web.math.edu, is that it? No, maybe that's an MIT. Not, so is it math? I can't live withoutedu at the end, right? Yeah. Is it justedu? Whatever. And then that, yeah. OK. So that website has, well, all the old exams you could ever want, if you wanted any. And it has this example, and you click on starting with two matrices. And this is one of them. OK, ready for the other. So here comes the second matrix, second example that you can contrast. All right, second example is going to have the same u . Let me put our matrix, I'm going to call it, what am I going to call it? Maybe c . OK, so it will have the same u and the same v . But I'm going to change w . And that's going to make all the difference. My w , I'm going to make that into w . OK. So now I have three vectors. I can take their combinations. I can look at the equation cx equal b . I can try to solve it. All the normal stuff. With those combinations of those three vectors. OK, so, and we'll see a difference. OK, so now what happens if I do, can I even like do just a little a race to deal with c now? How does c differ? If I change this multiplication from a to c to this new matrix, then what we've done is to put in a minus 1. That's the only change we made. And what's the

change in cx ? This, I've changed the first rows. I'm going to change the first row of the answer to what? x_1 minus x_3 . You could say again, as I said this morning, you've sort of changed the boundary condition maybe. You've made this difference equation somehow circular. That's why you're using that letter c . Yeah? Is it different? Oh, yes. I didn't get it right here. Thank you. Thank you very much. Absolutely. I mean, that would have been another matrix that we could think about, but it wouldn't have made the point I wanted. So, I'm, thanks, that's absolutely great. So, let me, yeah, so now it's correct here. And this is correct. And I can look at equations. But can I solve them? Can I solve them? And you're guessing already, no, we can't do it. Because, so now, let me maybe go to a board, work below, because I hate to erase that was so great. That being able to solve it in a nice clear solution and some matrix coming in. But now, how about this one? OK. So, one comment I should have made here. Suppose the bees were zero. Suppose I was looking at originally at a times x equal all zeros. What's x ? If all the bees were zero, and this was the one that dealt with the matrix a , if all the bees are zero, then the x 's are zero. The only way to get zero right-hand sides, bees, was to have zero x 's. It didn't, if you wanted to get zero out, you had to put zero in. Well, you can always put zero and get zero out. But here, you can put other vectors in and get zero out. So, I want to say there's a solution with zeros out coming out of C . But some non-zeroes are going in. And of course, we know from this morning that that's a signal that it's a different sort of matrix. There won't be an inverse. We've got questions. Now, what is, what are the, tell me all the solutions? All the solutions. So, actually, not just one, well, you could tell me one. Tell me one for it. One, one, one. Now, tell me all. C , C , C , yeah. That whole line through one, one, one. And that would be normal. So, this is a line of solutions. A line of solutions, I think, of one, one, one is in some solution space. And then all multiples, that whole line. Later, I would say, it's a subspace. When I say what that word subspace means, it's just this linear algebra has done its job beyond just one, one, one, okay. So, and again, it's just, it's a fact of, if we only know the differences, yeah, you can see different ways that this has got problems. So, that's C times x . Now, we can, one way to see a problem is to say, we can get the answer of all zeros by putting constants. All that's saying is, in words, the differences of a constant vector are all zeros, right? That's all that happened. Another way to see a problem, if I had, if I had this system of equations, how would you see that there's a problem? And how would you see that there is sometimes an answer and even decide when? I don't know if you can take a quick look. If I had three equations, x_1 minus x_3 is b_1 . This equals b_2 . This equals b_3 . Why, yeah. Do you see something that I can do to the left sides? That's important somehow. Suppose I add those left hand sides. What do I get? So, and I'm allowed to do that, right? If I've got three equations, I'm allowed to add them. And I would get zero, if I add, I get zero equals, b , adding, I have to add the right sides, of course. b_1 plus b_2 plus b_3 . That's, I hesitate to say a fourth equation because it's not independent of those three, but it's a consequence of those three. So, actually, this is telling me when I could get an answer and when I couldn't. If I get zero on the left side, I have to have zero on the right side, or I'm lost. So, I could actually solve this when b_1 plus b_2 plus b_3 is zero. That would be, that would, now, so I've taken a step there. I've said that, okay, we're in trouble often. But in case the right side adds up to zero, then we're not. And if you'll allow me to jump to a mechanical meaning of this, if these were springs or something, masses, and these were forces on them. So, I'm solving for displacements of masses that we'll see very soon, and these are forces. What I'm saying is, what that equation is saying is, because they're sort of cyclical, it's somehow saying that if the forces add up to zero, if the resultant force is zero, then you're okay. The springs and masses don't take off, or start spinning, or whatever. So, there's a physical meaning for that condition that it's okay, provided, if the b 's add up to zero. But of course, if the b 's don't add up to zero, we're lost. Right, yeah, yeah. Okay, so ax equal b could be solved. Cx equal b could be solved sometimes, but not always. And so you see that we're seeing the, the difficulty with C is showing up several ways. It's showing up in a C times a vector x giving zero. That's bad news, because no C inverse can bring you back. I mean, it's like you can't come back from zero. Once you've got to zero, C inverse could never bring you back to x . Right? A took x into b up there, and then A inverse brought back x . But here, there's no way to bring back that x , because I can't multiply zero by anything and get back to x . So that's why I see it's got troubles here. Here I see it's got troubles because if I add the left sides, I get zero. And therefore, the right sides must add to zero. So you've got trouble several ways. Let's see another way. Let's see geometrically why we're in trouble. Okay, so let me draw a picture to go with that picture. So there's three dimensional space. I didn't change u . I didn't change v , but I changed w to minus one. What does that mean? Minus one sort of going this way, maybe. Zero one is the z direction. Somehow I changed it to there. So this is w star, maybe a different w . This is the w that gave me problems. What's the problem for the, how does the

picture show the problem? Oh, I'm not sure. I don't, my, my, what's the problem with those three vectors, those three columns of c ? And why are they, yeah? They're in the same plane. w star gave us nothing new. We had the combinations of u and v made a plane and w star happened to fall in that plane. So this is a plane here, somehow, and, and, and, go through the origin, of course. What is that plane? This is all combinations, all combinations of u , v , and the third guy, w star. So this is the, this is the, this is the, this is the, this is the, u , v , and the third guy, w star. Right. Is a plane and I drew a triangle, but of course, I should draw the plane goes out to infinity. But the point is there are lots of b 's, lots of right hand sides, not on that plane. Okay. Now what, what would the, if I drew all combinations of u , v , w , the original w , what have I got? So let me bring that picture back for a moment. If I took all combinations of those, does w lie in the plane of u and v ? No, right. I would call it independent. These three vectors are independent. These three, u , v , and w star, I would call dependent. Because I did not, the third guy was a combination of the first two. Okay. So yeah, tell me what do I get now? So now you're really up to 3d. What do you get if you take all combinations of u , v , and w ? Say it again. The whole space, all combinations, if taking all combinations of u , v , w , give you the whole space. Why is that? Well, we just showed when it was A , we showed how did, we showed that we could get every B . We wanted the combination that gave B , and we found it. So at the beginning when we were working with u , v , w , we found, if this was the shorthand here, we, this said find a combination to give B , and this says that combination will work. And we wrote out what X was. Now what's the difference? Okay, here. So that, those were dependent. Those were, sorry, those were independent. I would even call those three vectors a basis for three dimensional space. That word basis is a big deal. So a basis for five dimensional space is five vectors that are independent. That's one way to say it. The second way to say it would be their combinations give the whole five dimensional space. A third way to say it, see if you can finish this sentence. This is for the independent, the good guys. If I put those five vectors into a five, five, five matrix, that matrix will be, invertible, that matrix will be invertible. So an invertible matrix is one with a basis sitting in its columns. It's a, it's a transform that you, that has an inverse transform. This matrix is not invertible. Those three vectors are not a basis. Their combinations are only in a plane. By the way, a plane is a subspace. So a plane would be a typical subspace. You know, it's like it filled it out. You took all the combinations, you did your job. But in that case, the whole space would be a count as a subspace too. That's the way you get subspaces by taking all combinations. Okay, now I'm even gonna push you one more step and then this example is complete. Can you tell me this is, what vectors do you get? All combinations of u , v , w . Let me try to write something. This gives only a plane. Because we've got two independent vectors but not the third. Okay. I don't know if I should even ask. Did we know an equation for that plane? Well, I think we do if we think about it correctly. All vectors, all combinations of u , v , w star is the same as saying, all vectors, c times x , right? Very, yeah. Do you agree that those two are exactly the same thing? This is the key because we're moving up to vectors, combinations and now come subspaces. If I take all combinations of u , v , w star, I say that that's the same as all vectors, c times x , why is that? It's what I said in the very first sentence at four o'clock. The combinations of u , v , w star, how do I produce them? I create the matrix with those columns. I multiply them by x 's and I get all the combinations. And this is just c times x . So what I've said there is just another way of saying, how does matrix multiplication work? Put the guys in its columns and multiply by a vector. Okay. So that's, so we're getting all vectors c times x . And now I was going to stretch it that little bit further. Can we see what can we describe what vectors we get? So that's my question. What b 's, what b 's? So this is b equal b_1 , b_2 , b_3 . Do we get? We don't get them all. That's, right? We don't get them all. That's the trouble with c . We only get a plane of them. And now can you tell me which b 's we do get when we look at this, at all combinations of these three dependent vectors? Well, we've done a lot today. Let me just tell you the answer because it's here. The b 's have to add to zero. That's the equation that the b 's have to satisfy if, because when we wrote out c , we noticed that we always got the components always added to zero. So the, which b 's do we get? We get the ones where the components add to zero. In other words, there's, that's the equation of the plane, you could say. Yeah, actually, that's a good way to look at it. All these vectors are on the plane. Do the components of u add to zero? Look at u . Yes. Do the components of v add to zero? Yes, add them up. Do the components of w star? Now that you fix it correctly for me. Do they add to zero? Yes, so all the combinations will add to zero. That's the plane. That's the plane. You see, there's so many different ways to see. And none of this is difficult, but it's coming fast because we're seeing the same thing in different languages. We're seeing it geometrically in a picture of a plane. We're seeing it by the combination of vectors. We're seeing it as a multiplication by a matrix. And we saw it sort of here by

operation, operating and simplifying and getting the key fact out of the equations. Well, okay. I wanted to give you this example because the two examples, because they bring out so many of the key ideas. The key idea of a subspace, shall I just say a little? What that word means? A subspace. What's a subspace? Well, what's a vector space, first of all? A vector space is a bunch of vectors. And the rule is you have to be able to take their combinations. That's what linear algebra does, takes combination. So a vector space is one where you take all combinations. So yeah, so it finally took like just this triangle. That would not be a subspace because one combination would be two u and it would be out of the triangle. So a subspace, just think of it as a plane. But then, of course, it could be in higher dimensions. It could be a seven-dimensional subspace inside 15-dimensional space. And I don't know if you're good at visualizing that. But I'm not. But it's never mind. You've got seven vectors. You think, OK, there are combinations to give a seven-dimensional subspace. Each vector has 15 components. No problem. I mean, no problem for MATLAB. Certainly, it's got a matrix with 105 entries. It deals with that instantly. OK. So a subspace is like a vector space inside a bigger one. That's why the prefix sub is there. And mathematics always counts the biggest possibility, too, which would be the whole space. And what's the smallest? So what's the smallest subspace of \mathbb{R}^3 ? So I have three-dimensional space. You could tell me all the subspaces of \mathbb{R}^3 . So there is one, a plane. Yeah, tell me all the subspaces of \mathbb{R}^3 . And then you'll have that word kind of down. A line. So planes and lines, you could say, the real, the proper subspaces, the best, the right ones. But there are a couple more possibilities, which are a point, which point? The origin. The only of the origin. Because if you tried to say that point was a subspace, no way. Why not? Because I wouldn't be able to multiply that vector by 5, and I would be away from the point. But the 0 subspace, the really small subspace, it just has the 0 vector. It's got one vector in it. Not empty. It's got that one point, but that's all. OK, so planes, lines, the origin, and then the other possibility for a subspace is the whole space. So the dimensions could be 3 for the whole space, 2 for a plane, 1 for a line, 0 for a point. It's just, yeah, it's just kicks together. OK, how are we for a time? Maybe it went more than the half. Now is a chance to just ask me if you want to, like anything about the course. Is it all linear algebra? No. But I think I can't do anything more helpful to you than to, for you to see. Begin to see, when you look at a matrix, begin to see what is it doing? What is it about? And of course matrices can be rectangular. So I'll give you a hint about what's coming in the course itself, is we'll have rectangular matrices A . OK, they're not invertible. They're taking seven dimensional space to three dimensional space or something. No, that's the candidate for that. But what comes up every time, I sort of get the idea finally. Every time I see a rectangular matrix, maybe 7 by 3, 7, that would be 7 rows 3 column. Then what comes up with a rectangular matrix A is sooner later, A transpose sticks it's nosy. So and multiplies that A . So the matrix that I , and we could do it for the origin, for our A here. Actually, if I did it for that original matrix A , I would get something you'd recognize. So A , it's what I want to say is that the course focuses on A transpose A . And I'll just say now that that matrix always comes out square, because this would be 3 times 7 times 7 times 3. So this would be 3 by 3. And it always comes out symmetric. That's the nice thing. And even more, we'll see more. So that's like a hint of where, watch for A transpose A in what's coming. And watch for it in applications of all kinds. I mean, in networks, an A will be associated with curcups of voltage law and A transpose with curcups current law. They just, you know, they just teamed up together. We'll see more. All right, now let me give you a chance to ask any question. Whatever. Homework. So did I mention homework? You may have said that's a crazy homework to say three problems in 1.1. I've never done this before. So essentially, you can get away with anything this week. And indefinitely, actually, MIT. How many are, is this the first day of MIT classes? Oh, wow. OK. Well, welcome to MIT. And I hope you like it. It's very, it's not so high pressure or whatever you is the associated with MIT. It's kind of tolerant. You ask at least if my advisees ask for something, I will say yes, it's easier that way. And much better. I just, and let me just, I'll just again, and I'll say it often in private. So this is like a grown up course. I'm figuring you're here to learn. So it's not my job to force it. My job is to help it. And I hope this is some help.