Test Lecture Notes

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The lecture focuses on the importance of **linear algebra** in engineering and computational applications, emphasizing its foundational role compared to traditional calculus. The instructor begins by discussing **vectors** and their **linear combinations**, illustrating how these combinations can form **subspaces** in higher dimensions. A significant part of the lecture is dedicated to the transition from vectors to **matrices**, explaining how matrix multiplication can be interpreted as combining vectors. The instructor introduces the concept of **invertible matrices** and their significance in solving equations of the form

$$Ax = b$$

, where the existence of a unique solution is contingent on the matrix being invertible. The discussion also highlights the implications of dependent versus independent vectors and how this affects the solutions to systems of equations. The lecture concludes with a focus on the geometric interpretation of these concepts, particularly how they relate to **subspaces** and the conditions under which solutions exist. Overall, the lecture serves as a comprehensive review of linear algebra, setting the stage for its application in engineering contexts.

Executive Summary

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The lecture primarily focuses on the importance of linear algebra, its foundational concepts, and how they relate to engineering and real-world applications. The speaker emphasizes the progression from vectors to matrices and finally to subspaces, highlighting the significance of linear combinations and the geometric interpretation of these concepts. The discussion includes specific examples of matrices, their properties, and the concept of invertibility, along with the implications of dependent and independent vectors in linear algebra. The lecture also introduces the idea of subspaces, providing definitions and examples, and discusses the conditions under which systems of equations can be solved. Additionally, the speaker touches upon the relationship between linear algebra and calculus, particularly in terms of derivatives and integrals. To ensure the notes are comprehensive, I will organize the content into main topics and subtopics, capturing all key concepts, examples, and explanations provided in the lecture. I will also fill in any gaps with contextually relevant information and ensure that all mathematical expressions are formatted correctly in LaTeX notation. Now, I will proceed to format the detailed notes according to the specified guidelines.

Detailed Notes

1. Importance of Linear Algebra

The speaker emphasizes the significance of linear algebra in engineering and real-world applications, arguing that it is often overlooked in favor of calculus. The speaker believes that students run out of time to learn essential concepts before reaching advanced topics.

2. Overview of Linear Algebra Concepts

2.1 Progression of Topics

Linear algebra progresses through three main concepts: **vectors**, **matrices**, and **subspaces**. Understanding subspaces is crucial for grasping the essence of linear algebra.

2.2 Vectors and Linear Combinations

Vectors allow for operations such as multiplication by scalars and addition. For example, given vectors **u**, **v**, and **w**, a linear combination can be expressed as:

$$b = x_1 \cdot u + x_2 \cdot v + x_3 \cdot w$$

where $\mathbf{x_1}$**,** $\mathbf{x_2}$**,** $\mathbf{x_3}$ are scalars.

2.3 Example of Vectors

Consider the vectors:

- **u** = (1, -1, 0)
- **v** = (0, 1, -1)
- **w** = (0, 0, 1)

The linear combinations of **u** and **v** can be visualized in three-dimensional space, forming a plane.

3. Matrices and Their Operations

3.1 Matrix Representation

The next step involves representing vectors as columns in a matrix. For the vectors defined above, the matrix **A** can be constructed as:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

3.2 Matrix Multiplication

When multiplying a matrix **A** by a vector **x**, the output is a linear combination of the columns of **A**:

$$b = A \cdot x$$

where **b** is the resulting vector.

3.3 Example of Matrix Multiplication

For a specific input vector **x** = (1, 4, 9), the output **b** can be calculated as:

$$b = A \cdot x = (1, 3, 5)$$

This illustrates that the differences of square numbers yield odd numbers.

4. Solving Linear Equations

4.1 Forward and Backward Directions

The lecture discusses the forward direction of multiplying a matrix by a vector and the backward direction of solving for **x** given **b**:

$$Ax = b$$

In this case, the solution can be derived easily due to the triangular form of matrix **A**.

4.2 Inverse Matrices

The inverse of a matrix **A** is defined as:

$$x = A^{-1}b$$

This relationship is crucial for solving linear equations, as it allows for the retrieval of **x** from **b**.

4.3 Conditions for Invertibility

The matrix must be invertible to ensure that there is a unique solution to the equation. The speaker emphasizes that if a matrix is not invertible, it leads to either no solution or infinitely many solutions.

5. Subspaces and Their Properties

5.1 Definition of Subspaces

A subspace is defined as a vector space that is contained within a larger vector space. It must satisfy the closure properties under vector addition and scalar multiplication.

5.2 Examples of Subspaces

In three-dimensional space, examples of subspaces include:

- **0-dimensional**: The origin (only the zero vector)
- **1-dimensional**: A line through the origin
- **2-dimensional**: A plane through the origin
- **3-dimensional**: The entire space itself

5.3 Conditions for Solutions

For a system of equations represented by matrix **C**, certain conditions must be met for solutions to exist. Specifically, if the right-hand sides of the equations do not sum to zero, the system may not have a solution.

6. Relationship Between Linear Algebra and Calculus

The speaker draws a parallel between linear algebra and calculus, noting that the derivative can be seen as a matrix operation similar to the difference matrix discussed earlier, while the integral serves as the inverse operation.

7. Conclusion and Further Learning

The lecture concludes with an invitation for questions and emphasizes the importance of understanding matrices and their transformations in various applications, including future topics in the course.

7.1 Homework and Resources

The speaker encourages students to explore the course website for additional resources, including past exams and examples related to linear algebra.

Overall, the lecture provides a comprehensive overview of linear algebra, its applications, and its foundational concepts, preparing students for more advanced topics in the field.

Main

Key Points

- **Linear Algebra Importance:** Emphasizes the significance of linear algebra in real-world applications, particularly in computing with matrices.
- **Course Structure:** The course will include reviews, homework, and questions every Wednesday.
- **Vectors and Linear Combinations:** The fundamental operation with vectors is taking their **linear combinations**, which involves multiplying by scalars and adding vectors.
- **Subspaces:** Understanding **subspaces** is crucial; they are formed by linear combinations of vectors.
- **Matrix Representation:** Vectors can be represented as columns in a matrix, and multiplying a matrix by a vector yields a linear combination of its columns.
- **Difference Matrix:** Introduced a **difference matrix** that computes differences between vector components.
- **Inverse Matrix:** Discussed the concept of an **inverse matrix** and its role in solving equations of the form $Ax = b$.
- **Subspace Characteristics:** A subspace is a vector space within a larger space, characterized by the ability to take linear combinations.
- **Conditions for Solutions:** The conditions under which systems of equations can be solved were explored, including the necessity for the sum of components to equal zero in certain cases.
- **Geometric Interpretation:** The geometric representation of vectors and their combinations helps in understanding linear algebra concepts.

Action Items

- Review the **1806 web page** for additional materials and examples related to linear algebra.
- Complete the assigned homework, which consists of **three problems from section 1.1**.
- Prepare questions for the next class regarding any unclear concepts discussed in this lecture.

The lecture covers a variety of fundamental concepts in linear algebra, emphasizing the importance of vectors, matrices, and subspaces. It discusses the operations involving vectors and matrices, particularly focusing on linear combinations, matrix multiplication, and the concept of invertibility. The lecture also introduces practical applications and examples, particularly in engineering contexts. To create a comprehensive study guide, I will organize the content into clear sections, including definitions, explanations of key concepts, and practical examples. I will also include practice problems and their solutions to reinforce understanding. I will ensure that all mathematical expressions are formatted using LaTeX notation and that important terms are highlighted properly. The guide will be structured logically to facilitate self-study and review. Now, I will produce the formatted output according to the specified guidelines.

AI Lecture Study Guide

This study guide summarizes key concepts from the lecture on linear algebra, focusing on vectors, matrices, linear combinations, subspaces, and their applications.

Vectors and Linear Combinations

Vectors are fundamental objects in linear algebra. The primary operation involving vectors is the formation of **linear combinations**.

Definition of Linear Combination

A linear combination of vectors is formed by multiplying each vector by a scalar and then summing the results. For vectors **u**, **v**, and **w**, a linear combination can be expressed as:

$$b = x_1 \cdot \mathbf{u} + x_2 \cdot \mathbf{v} + x_3 \cdot \mathbf{w}$$

where $\mathbf{x_1}$, $\mathbf{x_2}$, and $\mathbf{x_3}$ are scalars.

Example

Consider the vectors:

- **u** = (1, -1, 0)
- **v** = (0, 1, -1)
- **w** = (0, 0, 1)

The linear combination of these vectors can fill a plane in three-dimensional space if they are not collinear.

Matrix Representation

Vectors can be organized into matrices, where each vector is a column of the matrix. The process of multiplying a matrix by a vector corresponds to taking linear combinations of the matrix's columns.

Matrix Multiplication

For a matrix **A** and a vector **x**, the multiplication can be expressed as:

$$\mathbf{b} = A \cdot \mathbf{x}$$

where the output **b** is a linear combination of the columns of **A** weighted by the components of **x**.

Example of Matrix Multiplication

Using the previously defined vectors, we can form a matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

Multiplying **A** by a vector **x** gives us the output **b** based on the linear combinations of the columns of **A**.

Subspaces

Understanding subspaces is crucial in linear algebra. A **subspace** is a set of vectors that forms a vector space under the operations of vector addition and scalar multiplication.

Characteristics of Subspaces

- Contains the zero vector.
- Closed under vector addition.
- Closed under scalar multiplication.

Example of Subspaces

The span of vectors **u** and **v** forms a plane in three-dimensional space, while the span of three independent vectors fills the entire space.

Invertibility of Matrices

A matrix is considered **invertible** if there exists another matrix that can reverse its effect, typically denoted as $\mathbf{A^{-1}}$.

Conditions for Invertibility

A matrix is invertible if:

- It has full rank (the number of linearly independent columns equals the number of columns).
- The determinant is non-zero.

Example of Invertibility

If matrix **A** is invertible, then:

$$A \cdot x = b \implies x = A^{-1} \cdot b$$

Practice Problems

1. Given vectors $\mathbf{u} = (1, 2, 3)$, $\mathbf{v} = (4, 5, 6)$, and $\mathbf{w} = (7, 8, 9)$, find a linear combination that results in the vector (10, 11, 12).
2. Determine if the following set of vectors forms a subspace of $\mathbb{R}^3$: $\mathbf{u} = (1, 0, 0)$, $\mathbf{v} = (0, 1, 0)$, $\mathbf{w} = (0, 0, 1)$.
3. Find the inverse of the matrix:

Practice Problems Solutions

1. To find a linear combination, solve the equation $x_1 \cdot (1, 2, 3) + x_2 \cdot (4, 5, 6) + x_3 \cdot (7, 8, 9) = (10, 11, 12)$.
2. The set of vectors forms a subspace because they span $\mathbb{R}^3$ and satisfy the closure properties.
3. The inverse of matrix $\mathbf{A}$ is given by:

Further Study

For a deeper understanding of linear algebra concepts, consider studying the following topics:

- Eigenvalues and Eigenvectors
- Applications of Linear Algebra in Computer Graphics
- Linear Transformations and their Properties
- Advanced Matrix Factorizations (e.g., LU, QR)

Additional resources include textbooks on linear algebra, online courses, and MIT OpenCourseWare materials.

Appendix

Transcript

The following content is provided under a Creative Commons license. Your support will help MIT OpenCourseWare continue to offer high quality educational resources for free. To make a donation or to view additional materials from hundreds of MIT courses, visit MIT OpenCourseWare at ocw.mit.edu. OK. So this is the one and only review, you could say, of linear algebra. My website, I just think linear algebra is very important. You may have got that idea. And my website even has a little essay called Too Much Calculus. Because I think it's crazy for all the US universities to do this pretty much. You get semester after semester, differential calculus, integral calculus, ultimately differential equations. You run out of steam before the good stuff or you run out of time. And anybody who computes, who's living in the real world, is using linear algebra. You're taking a differential equation, you're taking your model, making it discrete, and computing with matrices. I mean, that's the world's digital now, not analog. So I hope it's OK to start the course with linear algebra. And so, but many engineering curricula don't fully recognize that. And so if you haven't had a official course, linear algebra, stay with 18085. This is a good way to learn it. You sort of learning what's important. So my review would be, and then this is, so future Wednesdays will be in our regular room for homework, review, questions of all kinds. And today, questions too. Shall I just fire away for the first half of the time to give you a sense of how I see the subject or least within that limited time? And then questions are totally welcome. Always welcome, actually. So I'll just start right up. So essentially, linear algebra progresses, starting with vectors to matrices. And then finally to subspaces. So that's like the abstraction, you could say abstraction, but it's not difficult, that you want to see. Until you see the idea of a subspace, you haven't really got linear algebra. OK, so I'll start at the beginning. What do you do with vectors? Answer, you take their linear combinations. That's what you can do with a vector. You can multiply it by a number, and you can add or subtract. So that's the key operation. Suppose I have vectors u, v, and w. Let me take three of them. And so I can take their combinations. So some combination will be, say, some number times u plus some number times v plus some number times w. So these numbers are called scalars. So these would be called scalars. And the whole thing is a linear combination. Let me abbreviate those words, linear combination. And you get some answer, say b. OK. But let's pin it down, make this whole discussion specific. Yeah, I started it a little early, I think. So let me, I'm going to take three vectors, u, v, and w. And take their combinations. So my u is going to be, they're carefully chosen. My u is going to be 1 minus 1, 0. And I'll take vectors in three dimensions. So that means their combinations will be in three dimensions. R3, three-dimensional space. So that will be u, and then v. So let's take 0, I think, 1 and minus 1. OK. Suppose I stopped there and took their linear combinations. It's very helpful to see a picture in three-dimensional space. I mean, the great thing about linear algebra, it moves into n-dimensional space, 10-dimensional, 100-dimensional, where we can't visualize. But yet, our instinct is right if we just follow. So what's your instinct? If I took those two vectors and notice they're not on the same line. One isn't a multiple of the other. They go in different directions. If I took their combinations, x1 times u plus x2 times v. Oh, now this is, oh, let me push. This is a serious question. If I took all their combinations. So let me try to draw a little bit. OK. I'm in three-dimensional space. And u goes somewhere, maybe there. And v goes somewhere, maybe here. Now suppose I take all the combinations. So I could multiply that first guy by any number. That would fill the line. I could multiply that second guy v. So this was u, and this was v. I could multiply that by any number x2. That would fill its line. Each of those lines I would later call a one-dimensional subspace, just a line. But now what happens if I take all combinations of the two? What do you think? You get a plane. Get a plane. If I take anything on this line and anything on this line and add them up, you can see that I'm not going to fill 3D. But I'm going to fill a plane. And that maybe takes a little thinking. It just, then it becomes sort of, yeah, you see that that's what it has to be. OK. Now I'm going to have a third vector. OK. My third vector will be 001. OK. So that 001 is 0 in the x, 0 in the y, and 1 in the z direction. So there's w. OK. Now I want to take their combinations. So let me do that very specifically. How do I take combinations? This is important. Seems it's very simple, but important. I like to think of taking the combinations of some vectors. I'm always putting vectors into the columns of a matrix. So now I'm going to move to step two, matrix. I'm going to move to step two, and maybe I'll put it, well, not better, put it here. OK. Step two is the matrix has those vectors in its columns. So in this case, it's 3 by 3. OK. That's my matrix that I'm going to call it A. And now I'm going to, how do I take combinations of vectors? I, well, I should have maybe done it in detail here, but let me, I'll just do it with the matrix here. Watch this now. I'm going to, if I multiply A by the vector of x's, what that does? So this is now A times x. So very important of matrix times the vector. What does it do? The output is just what I want. This is the output. It takes x1 times the first column plus x2 times the second plus x3 times the third. That's the way matrix multiplication works by columns. And you don't always see that. Because what do you see? You know how to probably know how to multiply that matrix by that vector. Let me ask you to do it. OK. What do you get? So everybody does it a component at a time. So what's the first component of the answer? x1. Yeah. How do you get that? It's a row times the vector. And when I say times, I really mean that dot product. This time plus this plus this is x1. And what about the second row? Minus x1 plus x2, or I'll just say x2 minus x1. And the third guy, the third component would be x3 minus x2. OK. So right away, I'm going to call this matrix a a difference matrix. It always helps to give names to things. So this a is a difference matrix. Because it takes differences of the x's. And I would even say a first difference matrix. Because it's just straightforward difference. And we'll see second differences in class Friday. OK. So that's what a does. But remember my first point was that when a matrix multiplies a vector, the result is a combination of the columns. And that's not always because, see, I'm looking at the picture not just by numbers. With numbers, I'm just doing this stuff. But now I'm stepping back a little bit and saying, I'm combining, it's this vector times x1. That vector times x1 plus this vector times x2 plus that one times x3 added together gives me this. Same, nothing complicated here. It's just look at it by vectors also. OK. Now suppose, so it's a little interesting. Already. Here we multiply these vectors by numbers. x1, x2, x3. That was our thinking here. Now our thinking here is a little we've switched slightly. Now I'm multiplying the matrix times the numbers in x. Just a slight switch. Multiply the matrix times the number. And I get some answer B, which is this. This is B. OK. And of course, I can do a specific example, like suppose I take the squares to be in x. So suppose I take A times the first three squares, 1, 4, 9. What answer would I get? Just to keep it clear that we're very specific here. So what would be the output B? I think of this as the input, the 1, 4, 9, the x's. Now the machine is a multiply by A. And there, here's the output. And what would be the output? What numbers am I going to get there? Yeah. 1, 3, something? 1, 3, 5. Which is actually a little neat. That you find the differences of the squares are the odd numbers. That appealed to me in school somehow. And that was already a bad sign, right? This dumb kid notices it. That you take the differences of square numbers, right? Whatever. OK. So now is a big step. This was the forward direction, right? The input, and there is the output. But now the real reality, that's easy and important. But the more deep problem is, what if I give you B and ask for x? So again, we're switching the direction here. OK. We're solving an equation now, or three equations, and three unknowns, Ax equal B. So if I give you this B, can you get x? OK. How do you solve three equations? So this is just we're looking backwards. Now, that won't be too hard for this matrix, for this particular matrix that I chose, because it's triangular, we'll be able to go backwards. So let me do that. Let me take B to B. It's a vector. It's got three components. And now I'm going to go backwards to find x, or we will. OK. So do you see the three equations? Here they are. x1 is B1. This is B2. That difference is B3. Those are my three equations. Three unknown, x is three known right hand sides. Or I think of it as A times x as a matrix times x, giving a vector B. What's the answer? OK. As I said, we're going to be able to do this. We're going to be able to solve this system easily, because it's already triangular. And it's actually lower triangular. So that means we'll start from the top. So the answer is the solution will be what? Let's make room for it. x1, x2, and x3. I want to find. And what's the answer? What can we just go from the top to bottom now? What's x1? B1. Great. What's x2? So x2 minus x1. These are my equations. So what's x2 minus x1? Well, it's B2. So what is x2? B1 plus B2, right? And what's x3? What do we need there for x3? So I'm looking at the third equation. That'll determine x3. When I see it this way, I see those ones. And I see it multiplying x3. And what do I get? Yeah. So x3 minus this guy is B3. So I have to add in another B3, right? I'm doing sort of substitution down as I go. Once I learned that x1 was B1, I used it there to find x2. And now I'll use x2 to find x3. And what do I get again? x3 is I put the x2 over there. I think you've got it B1 plus B2 plus B3. OK. So that's a solution. Not difficult because the matrix was triangular. But let's think about that solution. That solution is a matrix times B. When you look at that, so this is like a good early step in linear algebra. When I look at that, I see a matrix multiplying B. I mean, you take that step up to seeing a matrix. And you can just read it off. So let me say, what's the matrix there? That's multiplying B to give that answer. From remember, the columns of this matrix, well, I don't know how you want to read it off. But one way is to think the columns of that matrix are multiplying B1, B2, and B3 to give this. So what's the first column of the matrix? It's whatever I'm reading off. The coefficients really of B1 here, 1, 1, 1. And what's the second column of the matrix? 0, 1, 1. Good. 0, B2 is 1, 1. And the third is 0, 0, 1. Good. OK. Now, so lots of things to comment here. This is, let me write up again here. This is x. That was the answer. It's a matrix times B. And what's the name of that matrix? It's the inverse matrix. If Ax gives B, the inverse matrix does it the other way round. x is a inverse B. Let me just put that over here. If Ax is B, then x should be a inverse B. You could, so we had inverse, I wrote down inverse this morning, but without saying the point. But so you see how that comes. I mean, if I want to go formally, I multiply both sides by A inverse. If there is an A inverse, that's a critical thing, as we saw. Is the matrix invertible? The answer here is yes. There is an inverse. And what does that really mean? The inverse is the thing that takes us from B back to x. Think of A as multiplying by A as kind of a mapping, mathematicians use the word, or transform. Transform would be good. Transform from x to B. And this is the inverse transform. So it doesn't happen to be the discrete Fourier transform or a wavelet transform. Well, actually, we could give it a name. This is kind of a difference transform, right? That's what A did. Took differences. So what does A inverse do? It takes sums. It takes sums. That's why you see 1, 1, 1, 1, 1 along the rows, because it's just adding, and you see it here, and fully displayed. It's a sum matrix. I might as well call it S for sum. So that matrix, that sum matrix, is the inverse of the difference matrix. Yeah. OK. And maybe I just sent a hit on calculus earlier. You could say the calculus is all about one thing, and it's inverse. The derivative is A, and what's S? In calculus. The integral. The whole subject is about one operation. Now, admittedly, it's not a matrix. It operates on functions instead of just little vectors. But that's the main point. The fundamental theorem of calculus is telling us that integrations, the inverse of difference, H. OK. So this is good, and I could put in, if I put in B equal 1, 3, 5, for example, just to put in some numbers. If I put in B equal 1, 3, 5, what would the x that comes out B? 1, 4, 9. Because it takes us back. Here previously, we took differences of 1, 4, 9, got 1, 3, 5. Now, if we take sums of 1, 3, 5, we get 1, 4, 9. OK. Now, we have a system of linear equation. Now, I want to step back and see what was good about this matrix, somehow it has an inverse. A, x equal B has a solution, in other words. And it has only one solution, right? Because we worked it out. We had no choice. That was it. So there's just one solution. There's always one, and only one solution. It's like a perfect transform from the x's to the b's and back again. So that's what an invertible matrix is. It's a perfect map from one set of x's to the b's, and you can get back again. OK. Right. Yeah, questions always. OK. Now, I think I'm ready for another example. There are only two examples. And actually, this example, these two examples are on the 1806 web page. If you want to, some people asked after class, OK, how to get sort of a review of linear algebra? Well, the 1806 website would be definitely a possibility. So that's, well, I'll put down the open courseware website, m-m-i-t-e-d-u, and then you would look at the linear algebra course, or the math one. What is it? Web.math.edu, is that it? No, maybe that's an MIT. Not, so is it math? I can't live withoutedu at the end, right? Yeah. Is it justedu? Whatever. And then that, yeah. OK. So that website has, well, all the old exams you could ever want, if you wanted any. And it has this example, and you click on starting with two matrices. And this is one of them. OK, ready for the other. So here comes the second matrix, second example that you can contrast. All right, second example is going to have the same u. Let me put our matrix, I'm going to call it, what am I going to call it? Maybe c. OK, so it will have the same u and the same v. But I'm going to change w. And that's going to make all the difference. My w, I'm going to make that into w. OK. So now I have three vectors. I can take their combinations. I can look at the equation cx equal b. I can try to solve it. All the normal stuff. With those combinations of those three vectors. OK, so, and we'll see a difference. OK, so now what happens if I do, can I even like do just a little a race to deal with c now? How does c differ? If I change this multiplication from a to c to this new matrix, then what we've done is to put in a minus 1. That's the only change we made. And what's the change in cx? This, I've changed the first rows. I'm going to change the first row of the answer to what? x1 minus x3. You could say again, as I said this morning, you've sort of changed the boundary condition maybe. You've made this difference equation somehow circular. That's why you're using that letter c. Yeah? Is it different? Oh, yes. I didn't get it right here. Thank you. Thank you very much. Absolutely. I mean, that would have been another matrix that we could think about, but it wouldn't have made the point I wanted. So, I'm, thanks, that's absolutely great. So, let me, yeah, so now it's correct here. And this is correct. And I can look at equations. But can I solve them? Can I solve them? And you're guessing already, no, we can't do it. Because, so now, let me maybe go to a board, work below, because I hate to erase that was so great. That being able to solve it in a nice clear solution and some matrix coming in. But now, how about this one? OK. So, one comment I should have made here. Suppose the bees were zero. Suppose I was looking at originally at a times x equal all zeros. What's x? If all the bees were zero, and this was the one that dealt with the matrix a, if all the bees are zero, then the x's are zero. The only way to get zero right-hand sides, bees, was to have zero x's. It didn't, if you wanted to get zero out, you had to put zero in. Well, you can always put zero and get zero out. But here, you can put other vectors in and get zero out. So, I want to say there's a solution with zeros out coming out of C. But some non-zeroes are going in. And of course, we know from this morning that that's a signal that it's a different sort of matrix. There won't be an inverse. We've got questions. Now, what is, what are the, tell me all the solutions? All the solutions. So, actually, not just one, well, you could tell me one. Tell me one for it. One, one, one. Now, tell me all. C, C, C, yeah. That whole line through one, one, one. And that would be normal. So, this is a line of solutions. A line of solutions, I think, of one, one, one is in some solution space. And then all multiples, that whole line. Later, I would say, it's a subspace. When I say what that word subspace means, it's just this linear algebra has done its job beyond just one, one, one, okay. So, and again, it's just, it's a fact of, if we only know the differences, yeah, you can see different ways that this has got problems. So, that's C times x. Now, we can, one way to see a problem is to say, we can get the answer of all zeros by putting constants. All that's saying is, in words, the differences of a constant vector are all zeros, right? That's all that happened. Another way to see a problem, if I had, if I had this system of equations, how would you see that there's a problem? And how would you see that there is sometimes an answer and even decide when? I don't know if you can take a quick look. If I had three equations, x1 minus x3 is b1. This equals b2. This equals b3. Why, yeah. Do you see something that I can do to the left sides? That's important somehow. Suppose I add those left hand sides. What do I get? So, and I'm allowed to do that, right? If I've got three equations, I'm allowed to add them. And I would get zero, if I add, I get zero equals, b, adding, I have to add the right sides, of course. b1 plus b2 plus b3. That's, I hesitate to say a fourth equation because it's not independent of those three, but it's a consequence of those three. So, actually, this is telling me when I could get an answer and when I couldn't. If I get zero on the left side, I have to have zero on the right side, or I'm lost. So, I could actually solve this when b1 plus b2 plus b3 is zero. That would be, that would, now, so I've taken a step there. I've said that, okay, we're in trouble often. But in case the right side adds up to zero, then we're not. And if you'll allow me to jump to a mechanical meaning of this, if these were springs or something, masses, and these were forces on them. So, I'm solving for displacements of masses that we'll see very soon, and these are forces. What I'm saying is, what that equation is saying is, because they're sort of cyclical, it's somehow saying that if the forces add up to zero, if the resultant force is zero, then you're okay. The springs and masses don't take off, or start spinning, or whatever. So, there's a physical meaning for that condition that it's okay, provided, if the b's add up to zero. But of course, if the b's don't add up to zero, we're lost. Right, yeah, yeah. Okay, so ax equal b could be solved. Cx equal b could be solved sometimes, but not always. And so you see that we're seeing the, the difficulty with C is showing up several ways. It's showing up in a C times a vector x giving zero. That's bad news, because no C inverse can bring you back. I mean, it's like you can't come back from zero. Once you've got to zero, C inverse could never bring you back to x. Right? A took x into b up there, and then A inverse brought back x. But here, there's no way to bring back that x, because I can't multiply zero by anything and get back to x. So that's why I see it's got troubles here. Here I see it's got troubles because if I add the left sides, I get zero. And therefore, the right sides must add to zero. So you've got trouble several ways. Let's see another way. Let's see geometrically why we're in trouble. Okay, so let me draw a picture to go with that picture. So there's three dimensional space. I didn't change u. I didn't change v, but I changed w to minus one. What does that mean? Minus one sort of going this way, maybe. Zero one is the z direction. Somehow I changed it to there. So this is w star, maybe a different w. This is the w that gave me problems. What's the problem for the, how does the picture show the problem? Oh, I'm not sure. I don't, my, my, what's the problem with those three vectors, those three columns of c? And why are they, yeah? They're in the same plane. W star gave us nothing new. We had the combinations of u and v made a plane and w star happened to fall in that plane. So this is a plane here, somehow, and, and, and, go through the origin, of course. What is that plane? This is all combinations, all combinations of u, v, and the third guy, w star. So this is the, this is the, this is the, this is the, this is the, u, v, and the third guy, w star. Right. Is a plane and I drew a triangle, but of course, I should draw the plane goes out to infinity. But the point is there are lots of b's, lots of right hand sides, not on that plane. Okay. Now what, what would the, if I drew all combinations of u, v, w, the original w, what have I got? So let me bring that picture back for a moment. If I took all combinations of those, does w lie in the plane of u and v? No, right. I would call it independent. These three vectors are independent. These three, u, v, and w star, I would call dependent. Because I did not, the third guy was a combination of the first two. Okay. So yeah, tell me what do I get now? So now you're really up to 3d. What do you get if you take all combinations of u, v, and w? Say it again. The whole space, all combinations, if taking all combinations of u, v, w, give you the whole space. Why is that? Well, we just showed when it was A, we showed how did, we showed that we could get every B. We wanted the combination that gave B, and we found it. So at the beginning when we were working with u, v, w, we found, if this was the shorthand here, we, this said find a combination to give B, and this says that combination will work. And we wrote out what X was. Now what's the difference? Okay, here. So that, those were dependent. Those were, sorry, those were independent. I would even call those three vectors a basis for three dimensional space. That word basis is a big deal. So a basis for five dimensional space is five vectors that are independent. That's one way to say it. The second way to say it would be their combinations give the whole five dimensional space. A third way to say it, see if you can finish this sentence. This is for the independent, the good guys. If I put those five vectors into a five, five, five matrix, that matrix will be, invertible, that matrix will be invertible. So an invertible matrix is one with a basis sitting in its columns. It's a, it's a transform that you, that has an inverse transform. This matrix is not invertible. Those three vectors are not a basis. Their combinations are only in a plane. By the way, a plane is a subspace. So a plane would be a typical subspace. You know, it's like it filled it out. You took all the combinations, you did your job. But in that case, the whole space would be a count as a subspace too. That's the way you get subspaces by taking all combinations. Okay, now I'm even gonna push you one more step and then this example is complete. Can you tell me this is, what vectors do you get? All combinations of u, v, w. Let me try to write something. This gives only a plane. Because we've got two independent vectors but not the third. Okay. I don't know if I should even ask. Did we know an equation for that plane? Well, I think we do if we think about it correctly. All vectors, all combinations of u, v, w star is the same as saying, all vectors, c times x, right? Very, yeah. Do you agree that those two are exactly the same thing? This is the key because we're moving up to vectors, combinations and now come subspaces. If I take all combinations of u, v, w star, I say that that's the same as all vectors, c times x, why is that? It's what I said in the very first sentence at four o'clock. The combinations of u, v, w star, how do I produce them? I create the matrix with those columns. I multiply them by x's and I get all the combinations. And this is just c times x. So what I've said there is just another way of saying, how does matrix multiplication work? Put the guys in its columns and multiply by a vector. Okay. So that's, so we're getting all vectors c times x. And now I was going to stretch it that little bit further. Can we see what can we describe what vectors we get? So that's my question. What b's, what b's? So this is b equal b1, b2, b3. Do we get? We don't get them all. That's, right? We don't get them all. That's the trouble with c. We only get a plane of them. And now can you tell me which b's we do get when we look at this, at all combinations of these three dependent vectors? Well, we've done a lot today. Let me just tell you the answer because it's here. The b's have to add to zero. That's the equation that the b's have to satisfy if, because when we wrote out cx, we noticed that we always got the components always added to zero. So the, which b's do we get? We get the ones where the components add to zero. In other words, there's, that's the equation of the plane, you could say. Yeah, actually, that's a good way to look at it. All these vectors are on the plane. Do the components of u add to zero? Look at u. Yes. Do the components of v add to zero? Yes, add them up. Do the components of w star? Now that you fix it correctly for me. Do they add to zero? Yes, so all the combinations will add to zero. That's the plane. That's the plane. You see, there's so many different ways to see. And none of this is difficult, but it's coming fast because we're seeing the same thing in different languages. We're seeing it geometrically in a picture of a plane. We're seeing it by the combination of vectors. We're seeing it as a multiplication by a matrix. And we saw it sort of here by operation, operating and simplifying and getting the key fact out of the equations. Well, okay. I wanted to give you this example because the two examples, because they bring out so many of the key ideas. The key idea of a subspace, shall I just say a little? What that word means? A subspace. What's a subspace? Well, what's a vector space, first of all? A vector space is a bunch of vectors. And the rule is you have to be able to take their combinations. That's what linear algebra does, takes combination. So a vector space is one where you take all combinations. So yeah, so it finally took like just this triangle. That would not be a subspace because one combination would be two u and it would be out of the triangle. So a subspace, just think of it as a plane. But then, of course, it could be in higher dimensions. It could be a seven-dimensional subspace inside 15-dimensional space. And I don't know if you're good at visualizing that. But I'm not. But it's never mind. You've got seven vectors. You think, OK, there are combinations to give a seven-dimensional subspace. Each vector has 15 components. No problem. I mean, no problem for MATLAB. Certainly, it's got a matrix with 105 entries. It deals with that instantly. OK. So a subspace is like a vector space inside a bigger one. That's why the prefix sub is there. And mathematics always counts the biggest possibility, too, which would be the whole space. And what's the smallest? So what's the smallest subspace of r3? So I have three-dimensional space. You could tell me all the subspaces of r3. So there is one, a plane. Yeah, tell me all the subspaces of r3. And then you'll have that word kind of down. A line. So planes and lines, you could say, the real, the proper subspaces, the best, the right ones. But there are a couple more possibilities, which are a point, which point? The origin. The only of the origin. Because if you tried to say that point was a subspace, no way. Why not? Because I wouldn't be able to multiply that vector by 5, and I would be away from the point. But the 0 subspace, the really small subspace, it just has the 0 vector. It's got one vector in it. Not empty. It's got that one point, but that's all. OK, so planes, lines, the origin, and then the other possibility for a subspace is the whole space. So the dimensions could be 3 for the whole space, 2 for a plane, 1 for a line, 0 for a point. It's just, yeah, it's just kicks together. OK, how are we for a time? Maybe it went more than the half. Now is a chance to just ask me if you want to, like anything about the course. Is it all linear algebra? No. But I think I can't do anything more helpful to you than to, for you to see. Begin to see, when you look at a matrix, begin to see what is it doing? What is it about? And of course matrices can be rectangular. So I'll give you a hint about what's coming in the course itself, is we'll have rectangular matrices A. OK, they're not invertible. They're taking seven dimensional space to three dimensional space or something. No, that's the candidate for that. But what comes up every time, I sort of get the idea finally. Every time I see a rectangular matrix, maybe 7 by 3, 7, that would be 7 rows 3 column. Then what comes up with a rectangular matrix A is sooner later, A transpose sticks it's nosy. So and multiplies that A. So the matrix that I, and we could do it for the origin, for our A here. Actually, if I did it for that original matrix A, I would get something you'd recognize. So A, it's what I want to say is that the course focuses on A transpose A. And I'll just say now that that matrix always comes out square, because this would be 3 times 7 times 7 times 3. So this would be 3 by 3. And it always comes out symmetric. That's the nice thing. And even more, we'll see more. So that's like a hint of where, watch for A transpose A in what's coming. And watch for it in applications of all kinds. I mean, in networks, an A will be associated with curcups of voltage law and A transpose with curcups current law. They just, you know, they just teamed up together. We'll see more. All right, now let me give you a chance to ask any question. Whatever. Homework. So did I mention homework? You may have said that's a crazy homework to say three problems in 1.1. I've never done this before. So essentially, you can get away with anything this week. And indefinitely, actually, MIT. How many are, is this the first day of MIT classes? Oh, wow. OK. Well, welcome to MIT. And I hope you like it. It's very, it's not so high pressure or whatever you is the associated with MIT. It's kind of tolerant. You ask at least if my advisees ask for something, I will say yes, it's easier that way. And much better. I just, and let me just, I'll just again, and I'll say it often in private. So this is like a grown up course. I'm figuring you're here to learn. So it's not my job to force it. My job is to help it. And I hope this is some help.