# Control Theory 4

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$$\begin{cases} (M+m)x'' - mlcos(\theta)\theta'' + mlsin(\theta){\theta'}^2 = F \\ -cos(\theta)x'' + l\theta'' - gsin(\theta) = 0 \end{cases}$$

**A. Manipulator form**

We need to rewrite the system above in the following way:

$M(q)q'' + n(q,q') = Bu$, where $u = F$ and $q = \begin{bmatrix} x \\ \theta \end{bmatrix}$

$$\begin{bmatrix} M+m & -mlcos(\theta) \\ -cos(\theta) & l \end{bmatrix} \begin{bmatrix} x'' \\ \theta'' \end{bmatrix} + \begin{bmatrix} mlsin(\theta){\theta'}^2 \\ -gsin(\theta) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot F$$

As we can see:

$$M(q) = \begin{bmatrix} M+m & -mlcos(\theta) \\ -cos(\theta) & l \end{bmatrix}, n(q,q') = \begin{bmatrix} mlsin(\theta){\theta'}^2 \\ -gsin(\theta) \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

**B. Control affine nonlinear form**

Desired form: $z' = f(z) + g(z)u$, where $u = F$, $z = \begin{bmatrix} x & \theta & x' & \theta' \end{bmatrix}^T$

$$\begin{bmatrix} x' \\ \theta' \\ x'' \\ \theta'' \end{bmatrix} = \begin{bmatrix} x' \\ \theta' \\ a(x,\theta,x',\theta') \\ b(x,\theta,x',\theta') \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c(x,\theta,x',\theta') \\ d(x,\theta,x',\theta') \end{bmatrix} \cdot F$$

we can find $a(x,\theta,x',\theta')$, $b(x,\theta,x',\theta')$, $c(x,\theta,x',\theta')$, $d(x,\theta,x',\theta')$ by deriving $q''$ from previous task:

$$M(q)q'' + n(q,q') = Bu$$

$$M(q)q'' = Bu - n(q,q')$$

$$q'' = M(q)^{-1}(Bu - n(q,q'))$$

$$q'' = -M(q)^{-1} \cdot n(q,q') + M(q)^{-1} \cdot Bu$$

Hence, functions $a$ and $b$ are $-M(q)^{-1} \cdot n(q,q')$ and functions $c$ and $d$ are $M(q)^{-1} \cdot Bu$

$$-M(q)^{-1} \cdot n(q, q') = \frac{1}{l(msin^2(\theta) + M)} \begin{bmatrix} ml\sin(\theta)(gcos(\theta) - l{\theta'}^2) \\ sin(\theta)(g(M+m) - mlcos(\theta){\theta'}^2) \end{bmatrix}$$

$$M(q)^{-1} \cdot B = \frac{1}{l(msin^2(\theta) + M)} \begin{bmatrix} l \\ cos(\theta) \end{bmatrix}$$

Thus, we can see that:

$$z' = \begin{bmatrix} x' \\ \theta' \\ \frac{m\sin(\theta)(gcos(\theta) - l{\theta'}^2)}{(msin^2(\theta)+M)} \\ \frac{sin(\theta)(g(M+m) - mlcos(\theta){\theta'}^2)}{l(msin^2(\theta)+M)} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{(msin^2(\theta)+M)} \\ \frac{cos(\theta)}{l(msin^2(\theta)+M)} \end{bmatrix} \cdot u$$

So, $f(z) = \begin{bmatrix} x' \\ \theta' \\ \frac{m\sin(\theta)(gcos(\theta) - l{\theta'}^2)}{(msin^2(\theta)+M)} \\ \frac{sin(\theta)(g(M+m) - mlcos(\theta){\theta'}^2)}{l(msin^2(\theta)+M)} \end{bmatrix}$ and $g(z) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{(msin^2(\theta)+M)} \\ \frac{cos(\theta)}{l(msin^2(\theta)+M)} \end{bmatrix}$

**C. Linearization**

Our desired form: $\delta z' = A\delta z + B\delta u$ around point $z = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$

$$z' = f(z, u) = \begin{bmatrix} x' \\ \theta' \\ \frac{m\sin(\theta)(gcos(\theta) - l{\theta'}^2) + u}{(msin^2(\theta)+M)} \\ \frac{sin(\theta)(g(M+m) - mlcos(\theta){\theta'}^2) + ucos(\theta)}{l(msin^2(\theta)+M)} \end{bmatrix}$$

Next, we need to compute Jacobian matrix of $f(z, u)$ with respect to $z$.
This will be our matrix $A$

$$A = \frac{\partial f(z, u)}{\partial z} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_{32} & 0 & \frac{-(2\theta' lmsin(\theta))}{(msin^2(\theta)+M)} \\ 0 & a_{42} & 0 & \frac{-(2\theta' msin(\theta)cos(\theta))}{(msin^2(\theta)+M)} \end{bmatrix}$$

$$a_{32} = \frac{(mcos(\theta)(gcos(\theta) - l(\theta')^2) - gmsin^2(\theta))}{(msin^2(\theta) + M)} - \frac{(2msin(\theta)cos(\theta)(msin(\theta)(gcos(\theta) - l(\theta')^2) + u))}{(msin^2(\theta) + M)^2}$$

$$a_{42} = \frac{(cos(\theta)(g(m+M) - lm(\theta')^2 cos(\theta)) + lm(\theta')^2 sin^2(\theta) - usin(\theta))}{(l(msin^2(\theta)+M))} - \frac{(2msin(\theta)cos(\theta)(sin(\theta)(g(m+M) - lm(\theta')^2 cos(\theta)) + ucos(\theta)))}{(l(msin^2(\theta)+M)^2)}$$

After substitution of corresponding numbers, we obtain:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 12.83 & 0 & 0 \\ 0 & 10.78 & 0 & 0 \end{bmatrix}$$

Then we can find matrix B:

$$B = \frac{\partial f(z,u)}{\partial u} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{(msin^2(\theta)+M)} \\ \frac{cos(\theta)}{l(msin^2(\theta)+M)} \end{bmatrix}$$

After substitution of corresponding numbers, we obtain:

$$B = \begin{bmatrix} 0 \\ 0 \\ 0.24 \\ 0.11 \end{bmatrix}$$

Finally, we obtain:

$$\delta z' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 12.83 & 0 & 0 \\ 0 & 10.78 & 0 & 0 \end{bmatrix} \delta z + \begin{bmatrix} 0 \\ 0 \\ 0.24 \\ 0.11 \end{bmatrix} \delta u$$

**D. Stability**

The stability of the system was checked by running the following code in MatLab.

Listing 1: MatLab code for checking stability

```
A = [0 0 1 0; 0 0 0 1; 0 12.83 0 0; 0 10.78 0 0];
B = [0; 0; 0.24; 0.11];
C = [1 0 0 0; 0 1 0 0];
D = [0; 0];
sys = ss(A,B,C,D);

H = isstable(sys);
H
```

The system turned out to be unstable.

**E. Controlability**
The controlability of the system was checked by running the following code in MatLab.

Listing 2: MatLab code for checking controlability

```
A = [0 0 1 0; 0 0 0 1; 0 12.83 0 0; 0 10.78 0 0];
B = [0; 0; 0.24; 0.11];
C = [1 0 0 0; 0 1 0 0];
D = [0; 0];
Co = ctrb(A, B);

Co

unco = length(A) - rank(Co)

unco
```

*unco* shows the number of uncontrollable states. In my case it's equal to 0, so the system is fully controlable.

**Tasks F-G**
They are in separate file.