Algorithm Design VIII

Greedy Algorithms

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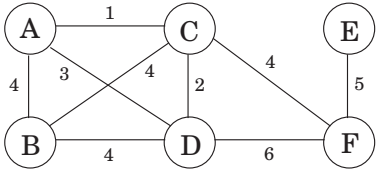
Minimum Spanning Trees

Build a Network



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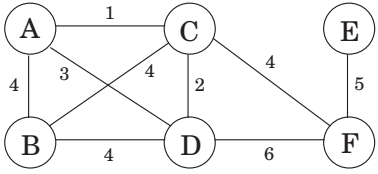

Suppose you are asked to network a collection of computers by linking selected pairs of them.

This translates into a graph problem in which

- nodes are computers,
- undirected edges are potential links, each with a maintenance cost.

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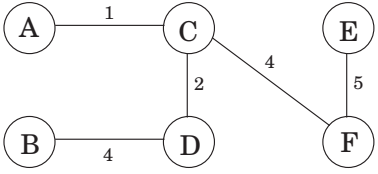

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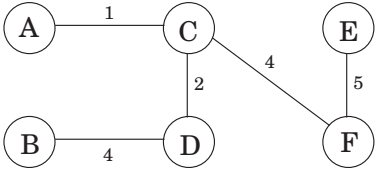

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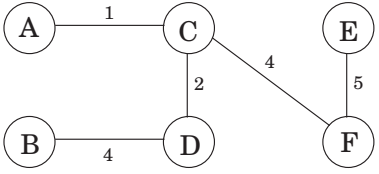

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One immediate observation is that the optimal set of edges cannot contain a cycle.

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Input: An undirected graph $G = (V, E)$; edge weights w_e

Properties of the Optimal Solutions

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A tree with minimum total weight, is a minimum spanning tree, MST.

Input: An undirected graph $G = (V, E)$; edge weights w_e

Output: A tree $T = (V, E')$ with $E' \subseteq E$ that minimizes

$$\texttt{weight}(T) = \sum_{e \in E'} w_e$$

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When a particular edge (u, v) comes up, we can be sure that u and v lie in separate connected components, for otherwise there would already be a path between them and this edge would create a cycle.

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The process terminates with some graph $G' = (V, E'), E' \subseteq E$, which is acyclic and, by Lemma (1), is also connected.

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The process terminates with some graph $G' = (V, E'), E' \subseteq E$, which is acyclic and, by Lemma (1), is also connected.

Therefore G' is a tree, whereupon $|E'| = |V| - 1$ by Lemma (2). So $E' = E$, no edges were removed, and G was acyclic to start with.

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On the other hand, if a graph has a path between any two nodes, then it is connected. If these paths are unique, then the graph is also acyclic.

A Greedy Approach

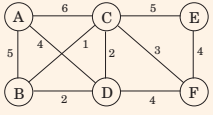
Kruskal's minimum spanning tree algorithm starts with the empty graph and then selects edges from E according to the following rule.

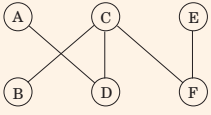
Repeatedly add the next lightest edge that doesn't produce a cycle.

Example

Starting with an empty graph and then attempt to add edges in increasing order of weight

$$B-C; C-D; B-D; C-F; D-F; E-F; A-D; A-B; C-E; A-C$$





The Cut Property

Lemma

Suppose edges X are part of a MST of $G = (V, E)$. Pick any subset of nodes S for which X does not cross between S and $V \backslash S$, and let e be the lightest edge across this partition. Then

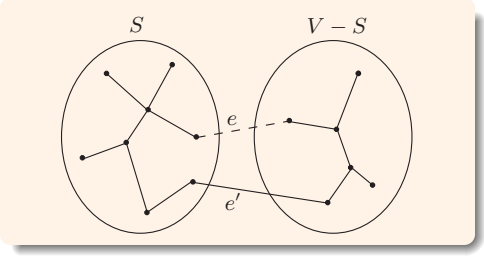
$$X \cup \{e\}$$

is part of some MST.

The Cut Property

A cut is any partition of the vertices into two groups, S and $V \backslash S$.

It is safe to add the lightest edge across any cut, provided X has no edges across the cut.



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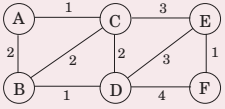
Since T is an MST, it must be the case that $weight(T') = weight(T)$ and that T' is also an MST.

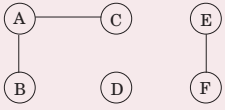
An Example of Cut Property

(a)



(b)

Edges X:



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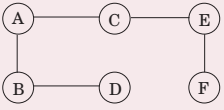
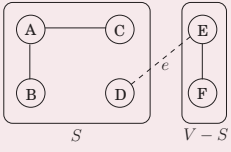


(c)

The cut:



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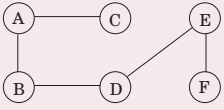

Kruskal's Algorithm

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KRUSKAL (G, w)
input : A connected undirected graph $G = (V, E)$, with edge weight w_e
output: A minimum spanning tree defined by the edges X

for *all* $u \in V$ **do**
  `makeset` (u);
end
$X = \{\}$;
Sort the edges E by weight;
for *all* $(u, v) \in E$ *in increasing order of weight* **do**
  if `find` *(u)*$\neq$`find` *(v)* **then**
    add (u, v) to X;
    `union` (u,v)
  end
end

Kruskal's Algorithm

`makeset`(x)	create a singleton set containing x	$\lvert V\rvert$
`find`(x)	find the set that x belong to	$2 \cdot \lvert E\rvert$
`union`(x, y)	merge the sets containing x and y	$\lvert V\rvert - 1$

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In addition to a parent pointer π, each node also has a rank that, for the time being, should be interpreted as the height of the subtree hanging from that node.

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The tree actually gets built via the third operation, UNION, and so we must make sure that this procedure keeps trees shallow.

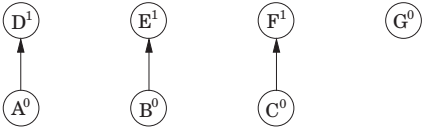
Union


```
UNION(x, y)
r_x=find(x);
r_y=find(y);
if r_x = r_y then return;
if rank(r_x)>rank(r_y) then
    π(r_y) = r_x;
end
else
    π(r_x) = r_y;
    if rank(r_x)=rank(r_y) then rank(r_y)=rank(r_y)+1;
end
```

An Example

After `makeset`(A), `makeset`(B), $\ldots$, `makeset`(G):

After `union`(A, D), `union`(B, E), `union`(C, F):



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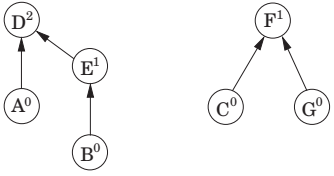
After $\texttt{union}(C, G), \texttt{union}(E, A)$:



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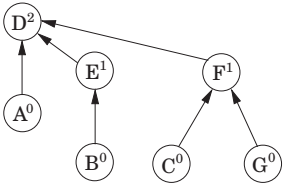

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By design, the rank of a node is exactly the height of the subtree rooted at that node. This means, for instance, that as you move up a path toward a root node, the rank values along the way are strictly increasing.

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A root node with rank k is created by the merger of two trees with roots of rank $k-1$. By induction to get the results.

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Different rank-k nodes cannot have common descendants. Any element has at most one ancestor of rank k.

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THEN THE DATA STRUCTURE PART BECOMES THE BOTTLENECK!

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- $O(|E| \log |V|)$ for sorting the edges ($\log |V| \approx \log |E|$),
- $O(|E| \log |V|)$ for the `union` and `find` operations that dominate the rest of the algorithm.

But what if the edges are given to us sorted? Or if the weights are small (say, $O(|E|)$) so that sorting can be done in linear time?

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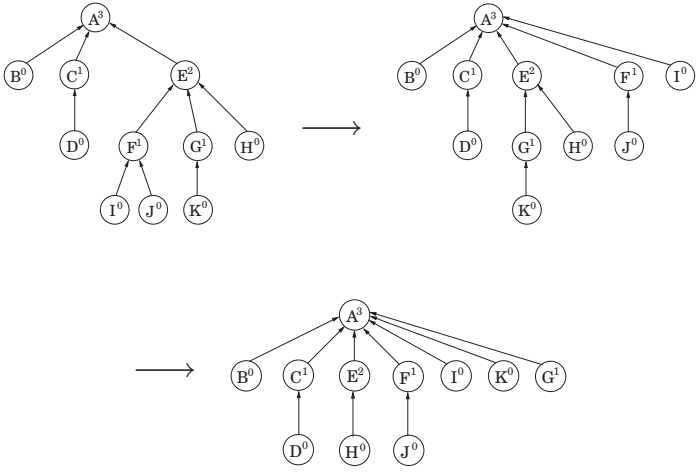
The main question:

How can we perform `union`'s and `find`'s faster than $\log n$?

Path Compression


```
FIND(x)
if x ≠ π(x) then π(x)=FIND(π(x));
return(π(x));
```

Path Compression



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This amortized cost turns out to be just barely more than $O(1)$, down from the earlier $O(\log n)$.

Prim's Algorithm

A General Kruskal's Algorithm

$X = \{\}$;
repeat until $|X| = |V| - 1$;
    pick a set $S \subset V$ for which X has no edges between S and $V - S$;
    let $e \in E$ be the minimum-weight edge between S and $V - S$;
    $X = X \cup \{e\}$;

Prim's Algorithm

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On each iteration, the subtree defined by X grows by one edge.

The lightest edge between a vertex in S and a vertex outside S. We can equivalently think of S as growing to include the vertex $v \notin S$ of smallest `cost`:

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$$\texttt{cost}(v) = \min_{u \in S} w(u, v)$$

The Algorithm

PRIM(G, w)
input : A connected undirected graph $G = (V, E)$, with edge weights w_e
output: A minimum spanning tree defined by the array $prev$

for *all* $u \in V$ **do**
    $cost(u) = \infty$;
    $prev(u) = nil$;
end
pick any initial node u_0;
$cost(u_0) = 0$;
H =makequeue(V) \\ *using cost-values as keys*;
while H *is not empty* **do**
    v=deletemin(H);
    for *each* $(v, z) \in E$ **do**
        if $cost(z) > w(v, z)$ **then**
            $cost(v) = w(v, z)$; $prev(z) = v$;
            decreasekey (H,z);
        end
    end
end

Dijkstra's Algorithm

SHANGHAI JIAO TONG UNIVERSITY

DIJKSTRA(G, l, s)
input : Graph $G = (V, E)$, directed or undirected; positive edge length $\{l_e \mid e \in E\}$; Vertex $s \in V$
output: For all vertices u reachable from s, $dist(u)$ is the set to the distance from s to u

for *all* $u \in V$ **do**
    $dist(u) = \infty$;
    $prev(u) = nil$;
end
$dist(s) = 0$;
H =makequeue(V) \\ *using dist-values as keys*;
while H *is not empty* **do**
    u=deletemin(H);
    for *all edge* $(u, v) \in E$ **do**
        if $dist(v) > dist(u) + l(u, v)$ **then**
            $dist(v) = dist(u) + l(u, v)$; $prev(v) = u$;
            decreasekey (H,v);
        end
    end
end

Set Cover

The Problem

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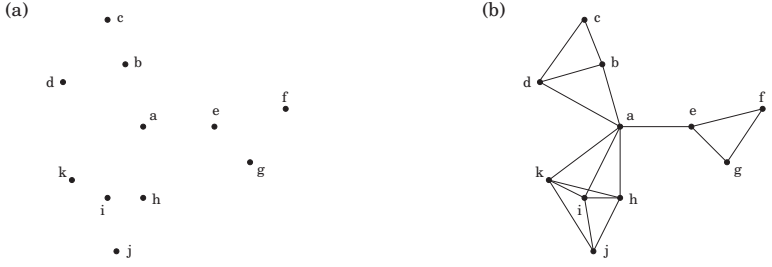
Q: What is the minimum number of schools needed?



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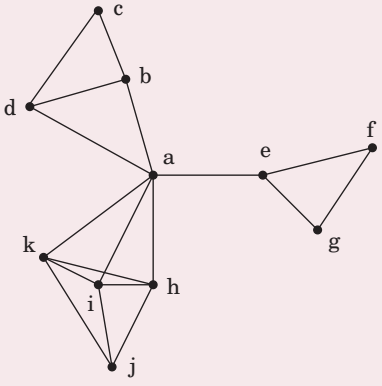
- For each town x, let S_x be the set of towns within 30 miles of it.
- A school at x will essentially "cover" these other towns.
- The question is then, how many sets S_x must be picked in order to cover all the towns in the county?

Set Cover Problem

Set Cover

- Input: A set of elements B, sets $S_1, \ldots, S_m \subseteq B$
- Output: A selection of the S_i whose union is B.
- Cost: Number of sets picked.

The Example



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which by repeated application implies

$$n_t \leq n_0(1 - \frac{1}{OPT})^t$$

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At $t = \ln n \cdot OPT$, therefore, n_t is strictly less than $ne^{-\ln n} = 1$, which means no elements remain to be covered.