Algorithm Design XII

Linear Programming II

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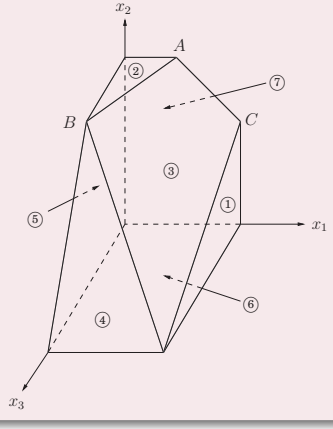
Review of Previous Lecture

$$\max x_1 + 6x_2 + 13x_3$$
$$x_1 \leq 200$$
$$x_2 \leq 300$$
$$x_1 + x_2 + x_3 \leq 400$$
$$x_2 + 3x_3 \leq 600$$
$$x_1, x_2, x_3 \geq 0$$

The Example



The point of final contact is the optimal vertex: $(0, 300, 100)$, with total profit $\$3100$.

Q: How would the simplex algorithm behave on this modified problem?

A possible trajectory

$$\underset{\$0}{(0,0,0)} \to \underset{\$200}{(200,0,0)} \to \underset{\$1400}{(200,200,0)} \to \underset{\$2800}{(200,0,200)} \to \underset{\$3100}{(0,300,100)}$$

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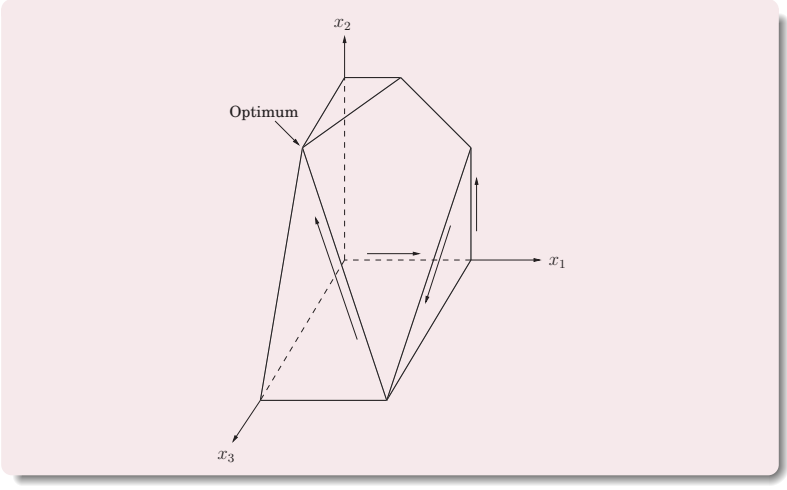

LP and Its Dual

Primal LP	Dual LP
$\max c^T \mathbf{x}$	$\min \mathbf{y}^T b$
$A\mathbf{x} \leq b$	$\mathbf{y}^T A \geq c^T$
$\mathbf{x} \geq 0$	$\mathbf{y} \geq 0$

Primal LP:

$$\begin{aligned} \max\ & c_1 x_1 + \cdots + c_n x_n \\ a_{i1} x_1 + \cdots + a_{in} x_n &\leq b_i \quad \text{for } i \in I \\ a_{i1} x_1 + \cdots + a_{in} x_n &= b_i \quad \text{for } i \in E \\ x_j &\geq 0 \quad \text{for } j \in N \end{aligned}$$

Dual LP:

$$\begin{aligned} \min\ & b_1 y_1 + \cdots + b_m y_m \\ a_{1j} y_1 + \cdots + a_{mj} y_m &\geq c_j \quad \text{for } j \in N \\ a_{1j} y_1 + \cdots + a_{mj} y_m &= c_j \quad \text{for } j \notin N \\ y_i &\geq 0 \quad \text{for } i \in I \end{aligned}$$

LP and Its Dual

$$\begin{aligned} \max\ & x_1 + 6x_2 \\ & x_1 \le 200 \\ & x_2 \le 300 \\ & x_1 + x_2 \le 400 \\ & x_1, x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \min\ & 200y_1 + 300y_2 + 400y_3 \\ & y_1 + y_3 \ge 1 \\ & y_2 + y_3 \ge 6 \\ & y_1, y_2, y_3 \ge 0 \end{aligned}$$

Max-Flow Min-Cut in LP

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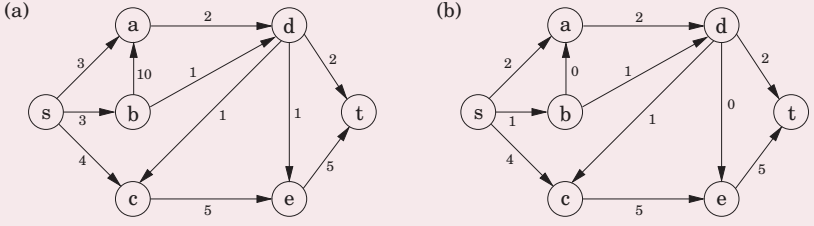
Each pipeline has a maximum capacity it can handle, and there are no opportunities for storing oil en route.

A Flow Example



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Aim to send as much oil as possible from s to t without exceeding the capacities of any of the edges.

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1. It doesn't violate edge capacities: $0 \leq f_e \leq c_e$ for all $e \in E$.
2. For all nodes u except s and t, the amount of flow entering u equals the amount leaving

$$\sum_{(w,v)\in E} f_{wu} = \sum_{(u,z)\in E} f_{uz}$$

In other words, flow is conserved.

Maximizing Flow

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This is a linear program. The maximum-flow problem reduces to linear programming.

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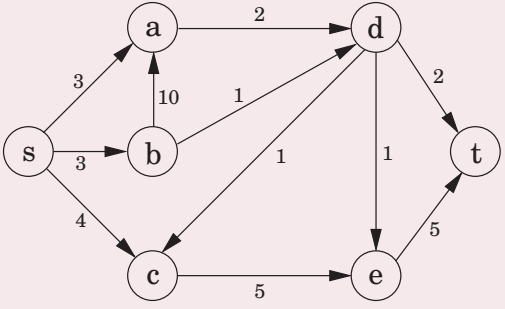

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- 11 for nonnegativity (such as $f_{sa} \geq 0$),
- 11 for capacity (such as $f_{sa} \leq 3$),
- 5 for flow conservation (one for each node of the graph other than s and t, such as $f_{sc} + f_{dc} = f_{ce}$).

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The advantage of making this modification is that we can now require flow conservation at s and t as well.

Another Representation

$$\max f_{ts}$$

$$f_{ij} \leq c_{ij} \qquad (i,j) \in E$$

$$\sum_{(w,i)\in E} f_{wi} - \sum_{(i,z)\in E} f_{iz} \leq 0 \qquad i \in V$$

$$f_{ij} \geq 0 \qquad (i,j) \in E$$

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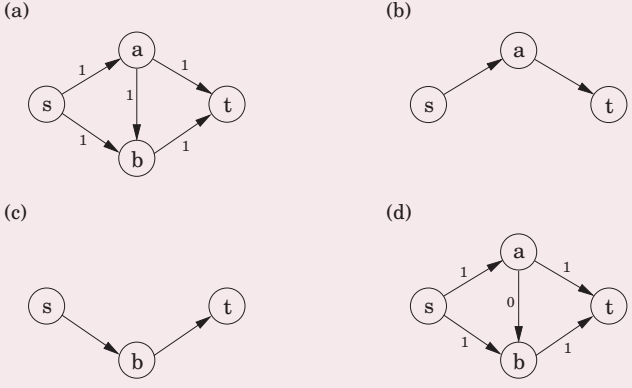
The behavior of the simplex has an elementary interpretation:

- Start with zero flow.
- Repeat: choose an appropriate path from s to t, and increase flow along the edges of this path as much as possible.

A Flow Example



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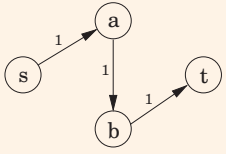
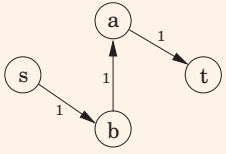
Simplex gets around this problem by also allowing paths to cancel existing flow.





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in the second case, up to f_{vu} additional units (canceling all or part of the existing flow on (v, u)).

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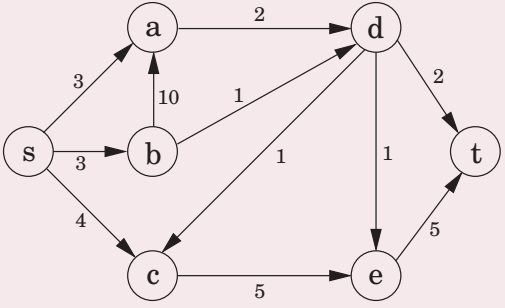
The Example



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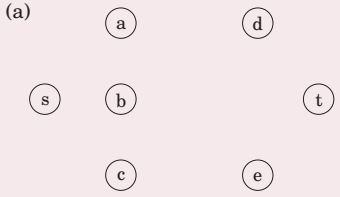

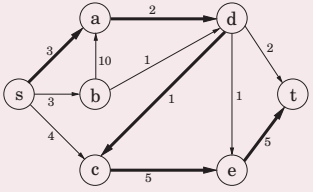

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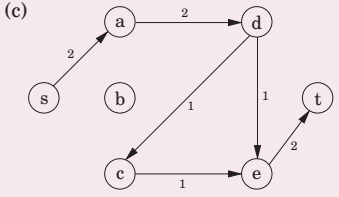

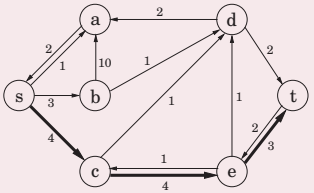

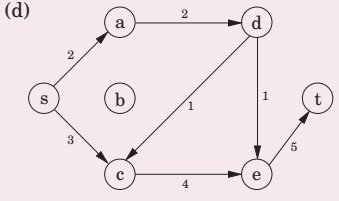

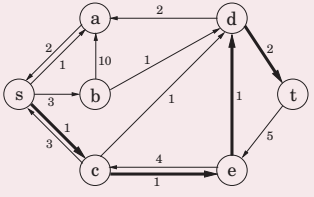

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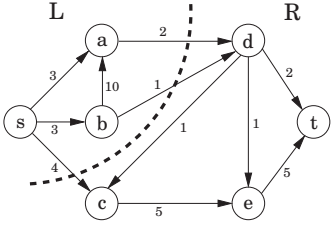
Not only does simplex correctly compute a maximum flow, but it also generates a short proof of the optimality of this flow!



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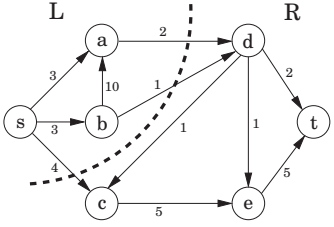

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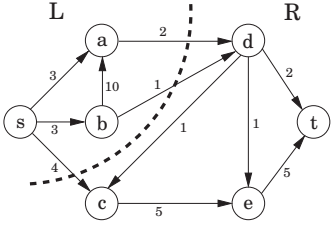

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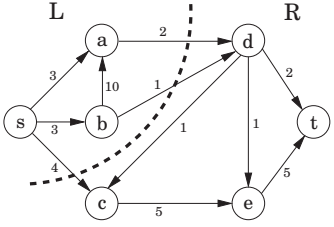

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Pick any flow f and any (s,t)-cut (L,R). Then $\texttt{size}(f) \leq \texttt{capacity}(L,R)$.

A Certificate of Optimality

Theorem (Max-flow min-cut)

The size of the maximum flow in a network equals the capacity of the smallest (s, t)-cut.

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Therefore the net flow across (L, R) is exactly the capacity of the cut.

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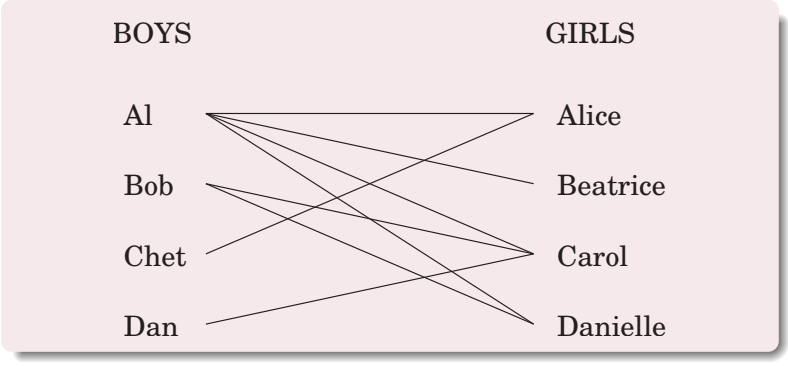
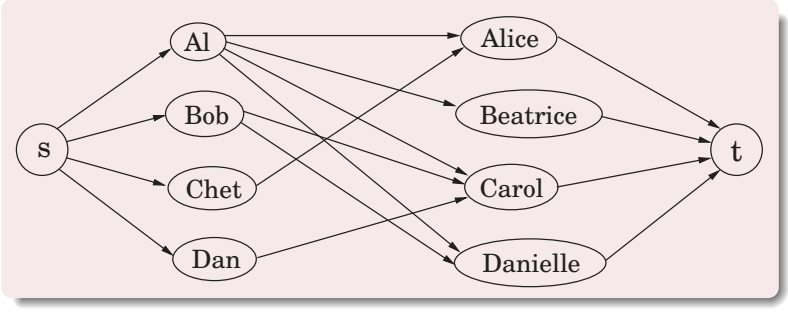
If paths are chosen by using a BFS, which finds the path with the fewest edges, then the number of iterations is at most $O(|V| \cdot |E|)$. *Edmonds-Karp algorithm*

This latter bound gives an overall running time of $O(|V| \cdot |E|^2)$ for maximum flow.


BOYS
GIRLS
Al
Bob
Chet
Dan
Alice
Beatrice
Carol
Danielle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Danielle
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Min-Max Relations in LP

LP for Max Flow

$$\max f_{ts}$$

$$f_{ij} \leq c_{ij} \qquad (i,j) \in E$$

$$\sum_{(w,i)\in E} f_{wi} - \sum_{(i,z)\in E} f_{iz} \leq 0 \qquad i \in V$$

$$f_{ij} \geq 0 \qquad (i,j) \in E$$

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$$f_{ij} \ge 0 \qquad (i,j) \in E$$

$$\min \sum_{(i,j)\in E} c_{ij} d_{ij}$$

$$\begin{aligned} d_{ij} - p_i + p_j &\ge 0 && (i,j) \in E \\ p_s - p_t &\ge 1 \\ d_{ij} &\ge 0 && (i,j) \in E \\ p_i &\ge 0 && i \in V \end{aligned}$$

Explanation of the Dual

$$\min \sum_{(i,j)\in E} c_{ij} d_{ij}$$

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Let $(\mathbf{d}^*, \mathbf{p}^*)$ be an optimal solution to this integer program.

Integer Program

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The only way to satisfy the inequality $p_s^* - p_t^* \geq 1$ with a $0/1$ substitution is to set $p_s^* = 1$ and $p_t^* = 0$.

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This solution defines an $s-t$ cut $(X, \overline{X})$, where X is the set of potential 1 nodes, and $\overline{X}$ the set of potential 0 nodes.

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Consider an edge (i,j) with $i \in X$ and $j \in \overline{X}$, Since $p_i^* = 1$ and $p_j^* = 0$, and thus $d_{ij}^* = 1$.

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The distance label for each of the remaining edges can be set to either 0 or 1 without violating the first constraints.

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The distance label for each of the remaining edges can be set to either 0 or 1 without violating the first constraints.

The objective function value is precisely the capacity of the cut $(X, \overline{X})$, and hence $(X, \overline{X})$ must be a minimum $s-t$ cut.

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We will say that this program is the LP relaxation of the integer program.

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The constraint matrix of this program is totally unimodular, Thus, the dual program always has an integral optimal solution.

Homework

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Assignment 5(1 week). Exercises 7.6, 7.7, 7.21 and 7.23.