

Algorithm Design V

Divide and Conquer II

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In its polar coordinates, denoted (r, θ) , rewrite as

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

- length: $r = \sqrt{a^2 + b^2}$.
- angle: $\theta \in [0, 2\pi)$.
- θ can always be reduced modulo 2π .



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Basic arithmetic:

- $-z = (r, \theta + \pi)$.
- $(r_1, \theta_1) \times (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2).$
- If z is on the unit circle (i.e., r = 1), then $z^n = (1, n\theta)$.

The n-th Complex Roots of Unity



Solutions to the equation $z^n = 1$

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For n is even:

- These numbers are plus-minus paired.
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The angle between two vectors $u=(u_0,\ldots,u_{n-1})$ and $v(v_0,\ldots,v_{n-1})$ in \mathbb{C}^n is just a scaling factor times their inner product

$$u \cdot v^* = u_0 v_0^* + u_1 v_1^* + \ldots + u_{n-1} v_{n-1}^*$$



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The above quantity is maximized when the vectors lie in the same direction and is zero when the vectors are orthogonal to each other.

Polynomial multiplication



If
$$A(x)=a_0+a_1x+\ldots+a_dx^d$$
 and $B(x)=b_0+b_1x+\ldots+b_dx^d$, their product
$$C(x)=c_0+c_1x+\ldots+c_{2d}x^{2d}$$

has coefficients

$$c_k = a_0 b_k + a_1 b_{k-1} + \ldots + a_k b_0 = \sum_{i=0}^k a_i b_{k-i}$$

where for i > d, take a_i and b_i to be zero.

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Q: Can we do better?



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coefficient representation value representation interpolation



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Therefore, polynomial multiplication takes linear time in the value representation.



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Interpolation

Recover $C(x) = c_0 + c_1 x + \ldots + c_{2d} x^{2d}$



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The Fast Fourier Transform (FFT) does it in just $O(n \log n)$ time, for a particularly clever choice of x_0, \ldots, x_{n-1} .

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First idea, we pick the n points,

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We need to split A(x) into its odd and even powers, for instance

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More generally

$$A(x) = A_e(x^2) + xA_o(x^2)$$

where $A_e(\cdot)$, with the even-numbered coefficients, and $A_o(\cdot)$, with the odd-numbered coefficients, are polynomials of degree $\leq n/2 - 1$.



Given paired points $\pm x_i$, the calculations needed for $A(x_i)$ can be recycled toward computing $A(-x_i)$:

$$A(x_i) = A_e(x_i^2) + x_i A_o(x_i^2)$$

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If we could recurse, we would get a divide-and-conquer procedure with running time

$$T(n) = 2T(n/2) + O(n) = O(n \log n)$$



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By continuing in this manner, we eventually reach the initial set of n points: the complex n th roots of unity, that is the n complex solutions of the equation

$$z^n = 1$$

The n-th complex roots of unity



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The FFT algorithm



```
FFT (A, \omega)
input: coefficient reprentation of a polynomial A(x) of degree < n-1, where n is a power of 2;
          \omega, an n-th root of unity
output: value representation A(\omega^0), \ldots, A(\omega^{n-1})
if \omega = 1 then return A(1);
express A(x) in the form A_e(x^2) + xA_o(x^2);
call FFT (A_e, \omega^2) to evaluate A_e at even powers of \omega;
call FFT (A_0,\omega^2) to evaluate A_0 at even powers of \omega;
for j = 0 to n - 1 do
    compute A(\omega^j) = A_e(\omega^{2j}) + \omega^j A_o(\omega^{2j});
end
return (A(\omega^0), \ldots, A(\omega^{n-1}));
```

Interpolation



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We will see that the interpolation can be computed by

$$\langle coefficients \rangle = \frac{1}{n} \text{FFT}(\langle values \rangle, \omega^{-1})$$



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Let *M* be the matrix in the middle, which is a Vandermonde matrix.

- If $x_0, x_1, \ldots, x_{n-1}$ are distinct numbers, then M is invertible.
- evaluation is multiplication by M, while interpolation is multiplication by M^{-1} .



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However, using this for interpolation would still not be fast enough for us..



In linear algebra terms, the FFT multiplies an arbitrary n-dimensional vector, which we have been calling the coefficient representation, by the $n \times n$ matrix.

$$M_n(\omega) = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ & & \vdots & & & \\ 1 & \omega^j & \omega^{2j} & \dots & \omega^{(n-1)j} \\ & & \vdots & & & \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \dots & x^{(n-1)(n-1)} \end{bmatrix}$$

Its (j,k)-th entry (starting row- and column-count at zero) is ω^{jk}



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Inversion formula

$$M_n(\omega)^{-1} = \frac{1}{n} M_n(\omega^{-1})$$



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Take the inner product of any columns j and k of matrix M,

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This is a geometric series with first term 1, last term $\omega^{(n-1)(j-k)}$, and ratio ω^{j-k} .



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• If j = k, then it evaluates to n.

Interpolation resolved



Corollary

$$MM^* = nI$$
, i.e.,

$$M_n^{-1} = \frac{1}{n} M_n^*$$



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It multiplies vector a by the $n \times n$ matrix $M_n(\omega)$, which has (j,k)-th entry ω^{jk} .



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This divide-and-conquer strategy leads to the definitive FFT algorithm, whose running time is T(n) = 2T(n/2) + O(n) = O(nlogn).

The general FFT algorithm



```
FFT (a, \omega) input : An array a = (a_0, a_1, \dots, a_{n-1}) for n is a power of 2; \omega, an n-th root of unity output: M_n(\omega)a if \omega = 1 then return a; (s_0, s_1, \dots, s_{n/2-1})=FFT ((a_0, a_2, \dots, a_{n-2}), \omega^2); (s'_0, s'_1, \dots, s'_{n/2-1})=FFT ((a_1, a_3, \dots, a_{n-1}), \omega^2); for j = 0 to n/2 - 1 do r_j = s_j + \omega^j s'_j; r_{j+n/2} = s_j - \omega^j s'_j; end return (r_0, r_1, \dots, r_{n-1});
```

Top 10 algorithms of the 20th century



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1946: The Metropolis Algorithm

1947: Simplex Method

1950: Krylov Subspace Method

1951: The Decompositional Approach to Matrix Computations

1957: The Fortran Optimizing Compiler

1959: QR Algorithm

1962: Quicksort

1965: Fast Fourier Transform

1977: Integer Relation Detection

1987: Fast Multipole Method

Homework

Homework



Assignment 2 (1 week). Exercises 2.13, 2.19, 2.22, and 2.28.