Algorithm Design XIV

NP Problem II

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NP-Completeness

Hard Problems, Easy Problems

Hard problems (NP-complete)	Easy problems (in P)
3SAT	2SAT, HORN SAT
TRAVELING SALESMAN PROBLEM	MINIMUM SPANNING TREE
LONGEST PATH	SHORTEST PATH
3D MATCHING	BIPARTITE MATCHING
KNAPSACK	UNARY KNAPSACK
INDEPENDENT SET	INDEPENDENT SET ON TREES
INTEGER LINEAR PROGRAMMING	LINEAR PROGRAMMING
RUDRATA PATH	EULER PATH
BALANCED CUT	MINIMUM CUT

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1. An efficient checking algorithm C, taking as input the given `instance` I, a `solution` S, and outputs `true` iff S is a solution I.
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We denote the class of all search problems by NP.

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The class of all search problems that can be solved in polynomial time is denoted P.

Why P and NP

P: polynomial time

NP: nondeterministic polynomial time

$\mathbf{P} \neq \mathbf{NP}$

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Theorem Proving

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So if P $=$ NP, there would be an efficient method to prove any theorem, thus eliminating the need for mathematicians!

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If one of them has a polynomial time algorithm, then every problem in NP has a polynomial time algorithm.

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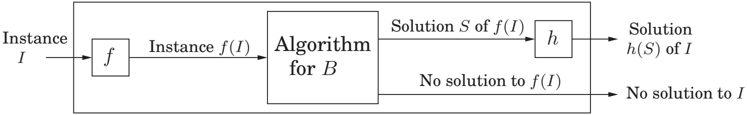
These two translation procedures f and h imply that any algorithm for B can be converted into an algorithm for A.



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If $A \to B$ and $B \to C$, then $A \to C$.

NP-Completeness

Definition

A NP problem is NP-complete if all other NP problems reduce to it.

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If a problem A is NP-complete, a new NP problem B is proved to be NP-complete, by reducing A to B.

Reduction

RUDRATA PATH $\rightarrow$ RUDRATA CYCLE

Rudrata Cycle

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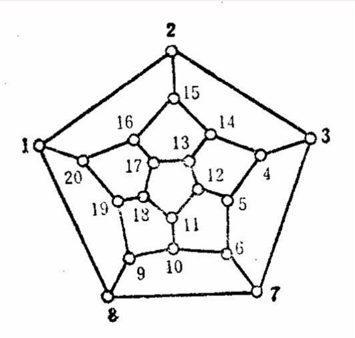
Rudrata Cycle

Given a graph, find a cycle that visits each vertex exactly once.



RUDRATA (s,t)-PATH $\rightarrow$ RUDRATA CYCLE

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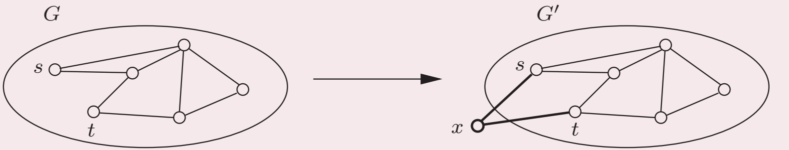
The reduction maps an `instance` G of RUDRATA (s, t)-PATH into an `instance` G' of RUDRATA CYCLE as follows: G' is G with an additional vertex x and two new edges $\{s, x\}$ and $\{x, t\}$.

RUDRATA (s,t)-PATH $\rightarrow$ RUDRATA CYCLE



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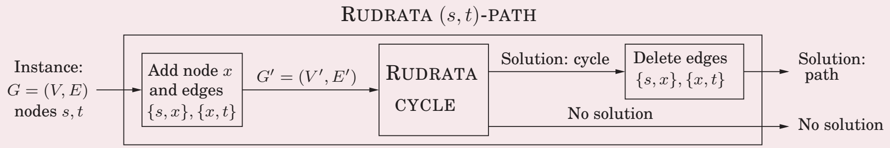


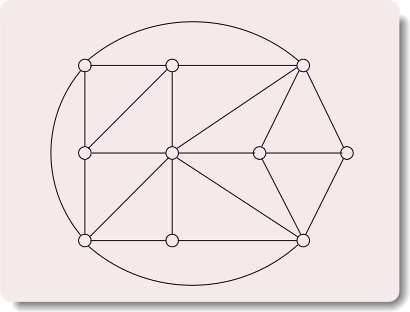
3SAT $\rightarrow$ INDEPENDENT SET

3 SAT

The instances of 3SAT, is set of clauses, each with three or fewer literals.

$$(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})(z \vee \overline{x})(\overline{x} \vee \overline{y} \vee \overline{z})$$

Independent Set



INDEPENDENT SET: Given a graph G and an integer g, find g vertices, no two of which have an edge between them.

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Solution: put an edge between any two vertices that correspond to opposite literals.

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Because a triangle has its three vertices maximally connected, and thus forces to pick only one of them for the independent set.

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3SAT $\to$ Independent Set

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- Additional edges between any two vertices that represent opposite literals.

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- Additional edges between any two vertices that represent opposite literals.
- The goal g is set to the number of clauses.

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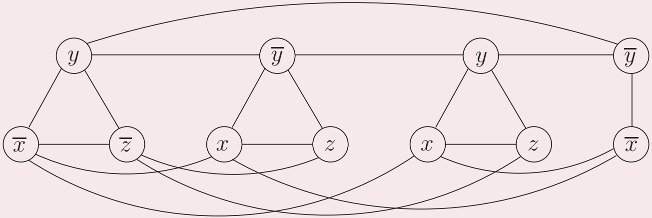

$$(\overline{x} \vee y \vee \overline{z})(x \vee \overline{y} \vee z)(x \vee y \vee z)(\overline{x} \vee \overline{y})$$

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Given an `instance` I of SAT, use exactly the same `instance` for 3SAT, except that any clause with more than three literals,

$$(a_1 \vee a_2 \vee \ldots \vee a_k)$$

is replaced by a set of clauses,

$$(a_1 \vee a_2 \vee y_1)(\overline{y_1} \vee a_3 \vee y_2)(\overline{y_2} \vee a_4 \vee y_3) \ldots (\overline{y_{k-3}} \vee a_{k-1} \vee a_k)$$

where the y_i's are new variables.

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The reduction is in polynomial and I' is equivalent to I in terms of satisfiability.

SAT $\to$ 3SAT

$$\left\{ \begin{array}{c} (a_1 \vee a_2 \vee \cdots \vee a_k) \\ \text{is satisfied} \end{array} \right\} \iff \left\{ \begin{array}{c} \text{there is a setting of the } y_i\text{'s for which} \\ (a_1 \vee a_2 \vee y_1)\ (\overline{y}_1 \vee a_3 \vee y_2)\ \cdots\ (\overline{y}_{k-3} \vee a_{k-1} \vee a_k) \\ \text{are all satisfied} \end{array} \right\}$$

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Conversely, if $(a_1 \vee a_2 \vee \ldots \vee a_k)$ is satisfied, then some a_i must be `true`. Set $y_1, \ldots, y_{i-2}$ to `true` and the rest to `false`.

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In the new formula no variable appears more than three times (and in fact, no literal appears more than twice).

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If G has a RUDRATA CYCLE, then the same cycle is also a tour within the budget of the TSP `instance`.

If G has no RUDRATA CYCLE, then there is no solution: the cheapest possible TSP tour has cost at least $n + \alpha$.

If $\alpha = 1$, then all distances are either 1 or 2, and so this `instance` of the TSP satisfies the triangle inequality: if i, j, k are cities, then

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This is a special case of the TSP which is in a certain sense easier, since it can be efficiently approximated.

RUDRATA CYCLE $\rightarrow$ TSP

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If α is large, then the resulting instance of the TSP may not satisfy the triangle inequality, and has another important property.

This important gap property implies that, unless P = NP, no approximation algorithm is possible.

ANY PROBLEM $\rightarrow$ SAT

home reading!

Homework

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Assignment 6(1 week). Exercises 8.3, 8.9, 8.14 and 8.19.