Algorithm Design VII

Path in Graphs

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School of Software
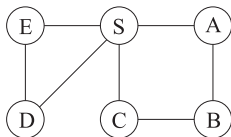
Distances

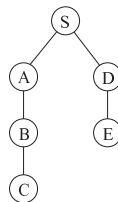**Definition**

The distance between two nodes is the length of the shortest path between them.

(a)

(b)

```
BFS (G, v)
```
input : Graph $G = (V, E)$, directed or undirected; Vertex $v \in V$
output: For all vertices u reachable from v, $dist(u)$ is the set to the distance from v to u

for *all $u \in V$* **do**
| $dist(u) = \infty$;
end
$dist[v] = 0$;
$Q = [v]$ *queue containing just v*;
while *Q is not empty* **do**
| u=Eject(Q);
| **for** *all edge $(u, s) \in E$* **do**
| | **if** *$dist(s) = \infty$* **then**
| | | Inject(Q,s); $dist[s] = dist[u] + 1$;
| | **end**
| **end**
end

Lemma

For each $d = 0, 1, 2, \ldots$ there is a moment at which,

1. *all nodes at distance $\leq d$ from s have their distances correctly set;*
2. *all other nodes have their distances set to ∞; and*
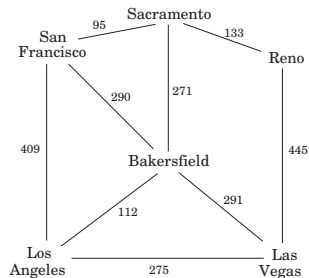3. *the queue contains exactly the nodes at distance d.*

BFS treats all edges as having the same length.

It is rarely true in applications where shortest paths are to be found.

Every edge $e \in E$ with a length l_e.

If $e = (u, v)$, we will sometimes also write

$$l(u, v) \qquad \text{or} \qquad l_{uv}$$

Dijkstra's Algorithm

BFS finds shortest paths in any graph whose edges have unit length.

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A simple trick: For any edge $e = (u, v)$ of E, replace it by l_e edges of length 1, by adding $l_e - 1$ dummy nodes between u and v. It might take time

$$O(|V| + \sum_{e \in E} l_e)$$

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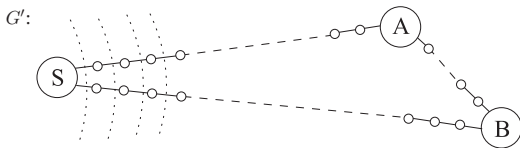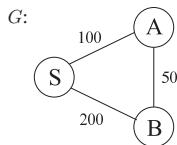It is bad in case we have edges with high length.

Set an alarm clock for node s at time 0.
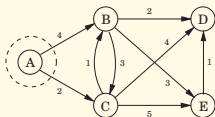
Repeat until there are no more alarms:

The next alarm goes off at time T, for node u. Then:
- The distance from s to u is T.
- For each neighbor v of u in G:
  - If there is no alarm yet for v, set one for time $T + l(u, v)$.
  - If v's alarm is set for later than $T + l(u, v)$, then reset it to this earlier time.

G:

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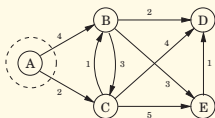
An Example

A: 0	D: ∞
B: 4	E: ∞
C: 2	

A: 0	D: 6
B: 3	E: 7
C: 2	

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Dijkstra's Shortest-Path Algorithm

```
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{$l_e \mid e \in E$}; Vertex $s \in V$
output: For all vertices u reachable from s, $dist(u)$ is the set to the distance
from s to u

for *all $u \in V$* **do**
  $dist(u) = \infty; prev(u) = nil;$
end
$dist(s) = 0;$
$H =$ makequeue(V) \\ *using dist-values as keys*;
while *H is not empty* **do**
  u=deletemin(H);
  for *all edge $(u, v) \in E$* **do**
    if $dist(v) > dist(u) + l(u, v)$ **then**
      $dist(v) = dist(u) + l(u, v);$ $prev(v) = u;$
      decreasekey(H,v);
    end
  end
end

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The first two let us set alarms, and the third tells us which alarm is next to go off.

Since `makequeue` takes at most as long as $|V|$ insert operations, we get a total of $|V|$ `deletemin` and $|V| + |E|$ `insert/decreasekey` operations.

Implementation	deletemin	insert/decreasekey	$|V| \times$ deletemin $+(|V|+|E|) \times$ insert
Array	$O(|V|)$	$O(1)$	$O(|V|^2)$
Binary heap	$O(\log|V|)$	$O(\log|V|)$	$O((|V|+|E|)\log|V|)$
d-ary heap	$O(\frac{d\log|V|}{\log d})$	$O(\frac{\log|V|}{\log d})$	$O(\frac{(d|V|+|E|)\log|V|}{\log d})$
Fibonacci heap	$O(\log|V|)$	$O(1)$ (amortized)	$O(|V|\log|V|+|E|)$

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- $|E|$ is less than $|V|^2$. If it is $\Omega(|V|^2)$, then clearly the array implementation is the faster.
- On the other hand, the binary heap becomes preferable as soon as $|E|$ dips below $|V|^2 / \log |V|$.
- The d-ary heap is a generalization of the binary heap and leads to a running time that is a function of d. The optimal choice is $d \approx |E|/|V|$;

Shortest Paths in the Presence of Negative Edges

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A crucial invariant of Dijkstra's algorithm is that the $dist$ values it maintains are always either overestimates or exactly correct.

They start off at ∞, and the only way they ever change is by updating along an edge:

UPDATE $((u, v) \in E)$

$dist(v) = min\{dist(v), dist(u) + l(u, v)\};$

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This UPDATE operation expresses that the distance to v cannot possibly be more than the distance to u, plus $l(u, v)$. It has the following properties,

1. It gives the correct distance to v in the particular case where u is the second-last node in the shortest path to v, and $dist(u)$ is correctly set.
2. It will never make $dist(v)$ too small, and in this sense it is safe.

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Let

$$s \rightarrow u_1 \rightarrow u_2 \rightarrow u_3 \rightarrow \ldots \rightarrow u_k \rightarrow t$$

be a shortest path from s to t.

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If the sequence of updates performed includes $(s, u_1), (u_1, u_2), \ldots, (u_k, t)$, in that order, then by rule 1 the distance to t will be correctly computed.

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If the sequence of updates performed includes $(s, u_1), (u_1, u_2), \ldots, (u_k, t)$, in that order, then by rule 1 the distance to t will be correctly computed.

It doesn't matter what other updates occur on these edges, or what happens in the rest of the graph, because updates are safe (by rule 2).

But still, if we don't know all the shortest paths beforehand, how can we be sure to update the right edges in the right order?

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We simply update all the edges, $|V| - 1$ times!

```
SHORTEST-PATHS (G, l, s)
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input : Graph $G = (V, E)$, edge length $\{l_e \mid e \in E\}$; Vertex $s \in V$

output: For all vertices u reachable from s, $dist(u)$ is the set to the distance from s to u

for *all* $u \in V$ **do**
$\quad dist(u) = \infty;$
$\quad prev(u) = nil;$
end
$dist[s] = 0;$
repeat $|V| - 1$ times: **for** $e \in E$ **do**
$\quad$ UPDATE (e);
end

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end
$dist[s] = 0$;
repeat $|V| - 1$ times: **for** $e \in E$ **do**
$\quad$ UPDATE(e);
end

Running time: $O(|V| \cdot |E|)$

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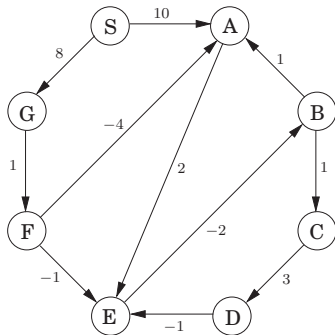
Node	Iteration							
	0	1	2	3	4	5	6	7
S	0	0	0	0	0	0	0	0
A	∞	10	10	5	5	5	5	5
B	∞	∞	∞	10	6	5	5	5
C	∞	∞	∞	∞	11	7	6	6
D	∞	∞	∞	∞	∞	14	10	9
E	∞	∞	12	8	7	7	7	7
F	∞	∞	9	9	9	9	9	9
G	∞	8	8	8	8	8	8	8

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Q: How to detect the existence of negative cycles:

Instead of stopping after $|V| - 1$, iterations, perform one extra round.

There is a negative cycle if and only if some $dist$ value is reduced during this final round.

Shortest Paths in DAGs

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- In any path of a DAG, the vertices appear in increasing linearized order.

A Shortest-Path Algorithm for DAG

DAG-SHORTEST-PATHS (G, l, s)
input : Graph $G = (V, E)$, edge length$\{l_e \mid e \in E\}$; Vertex $s \in V$
output: For all vertices u reachable from s, $dist(u)$ is the set to the distance from s to u

for *all $u \in V$* **do**
 $\quad dist(u) = \infty$;
 $\quad prev(u) = nil$;
end
$dists = 0$;
linearize G;
for *each $u \in V$ in linearized order* **do**
 $\quad$ **for** *all $e \in E$* **do**
 $\quad\quad$ UPDATE (e);
 $\quad$ **end**
end

The scheme does not require edges to be positive.

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Even can find longest paths in a DAG by the same algorithm: just negate all edge lengths.

Exercises

Professor Fake suggests the following algorithm for finding the shortest path from node s to node t in a directed graph with some negative edges: add a large constant to each edge weight so that all the weights become positive, then run Dijkstra's algorithm starting at node s, and return the shortest path found to node t.

You are given a strongly connected directed graph $G = (V, E)$ with positive edge weights along with a particular node $v_0 \in V$. Give an efficient algorithm for finding shortest paths between *all pairs of nodes*, with the one restriction that these paths must all pass through v_0.

Homework

Assignment 3. Exercises 3.7, 3.11, 3.28, 4.11, 4.12 and 4.16.