Algorithm Design IX

Dynamic Programming I

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Shortest Paths in DAGs, Revisited

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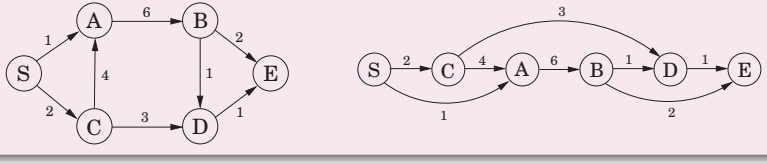
If compute `dist` values in the left-to-right order, by the time get to a node v, we already have all the information to compute `dist`(v).



Shortest Paths in Dags, Revisited

Initialize all $dist(.)$ value to ∞;
$dist(s)$=0;
for *each* $v \in V \backslash \{s\}$, *in linearized order* **do**
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In dynamic programming we are not given a DAG; the DAG is implicit.

Longest Increasing Subsequences

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A subsequence is any subset of these numbers taken in order, of the form

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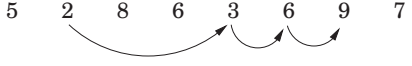
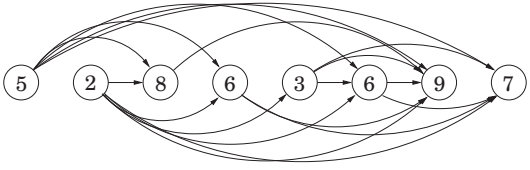
The task is to find the increasing subsequence of greatest length.



Graph Reformulation

Create a graph of all permissible transitions:

- a node i for each element a_i,
- a directed edge (i, j) if possible for a_i and a_j to be consecutive elements in an increasing subsequence: $i < j$ and $a_i < a_j$.



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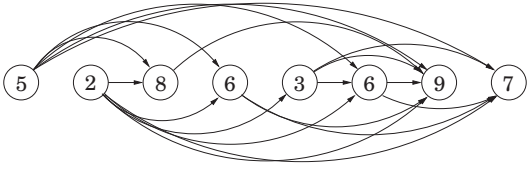

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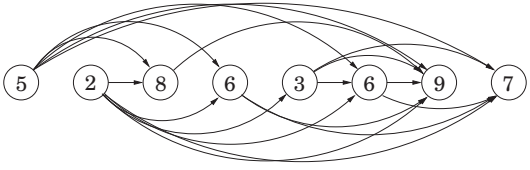

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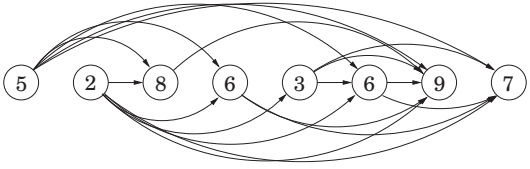

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There is a one-to-one correspondence between increasing subsequences and paths in this DAG.

Therefore, the goal is to find the longest path in the DAG!

The Algorithm

for $j = 1$ *to* n **do**
  $L(j) = 1 + \max\{L(i) \mid (i, j) \in E\}$;
end
`return(`$\max_j L(j)$`);`

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The final answer is the largest $L(j)$, since any ending position is allowed.

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This is at most $O(n^2)$, the maximum being when the input array is sorted in increasing order.

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In our case, each subproblem is solved using the relation

$$L(j) = 1 + \max\{L(i) | (i, j) \in E\}$$

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In a dynamic programming, a problem is reduced to subproblems that are slightly smaller. Thus the full recursion tree has polynomial depth and an exponential number of nodes.

Edit Distance

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Technically, an alignment is simply a way of writing the strings one above the other.

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S	U	N	N	–		Y

Cost: 3

–	S	N	O	W	–	Y
S	U	N	–	–	N	Y

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The number of insertions, deletions, and substitutions of characters needed to transform the first string into the second.

A Dynamic Programming Solution

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For every i, j with $1 \leq i \leq m$ and $1 \leq j \leq n$, let $E(i, j)$: the edit distance between some prefix of the first string, $x[1, \ldots, i]$, and some prefix of the second, $y[1, \ldots, j]$.

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$$E(i, j) = \min\{1 + E(i-1, j), 1 + E(i, j-1), \texttt{diff}(i, j) + E(i-1, j-1)\}$$

where $\texttt{diff}(i, j)$ is defined to be 0 if $x[i] = y[j]$ and 1 otherwise.

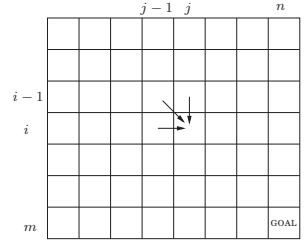
An Example

Edit distance between EXPONENTIAL and POLYNOMIAL, subproblem $E(4,3)$ corresponds to the prefixes EXPO and POL. The rightmost column of their best alignment must be one of the following:

$$\begin{matrix}\mathtt{O}\\ \mathtt{-}\end{matrix} \quad \text{or} \quad \begin{matrix}\mathtt{-}\\ \mathtt{L}\end{matrix} \quad \text{or} \quad \begin{matrix}\mathtt{O}\\ \mathtt{L}\end{matrix}$$

Thus, $E(4,3) = \min\{1 + E(3,3), 1 + E(4,2); 1 + E(3,2)\}$.

(a)



(b)

		P	O	L	Y	N	O	M	I	A	L
	0	1	2	3	4	5	6	7	8	9	10
E	1	1	2	3	4	5	6	7	8	9	10
X	2	2	2	3	4	5	6	7	8	9	10
P	3	2	3	3	4	5	6	7	8	9	10
O	4	3	2	3	4	5	5	6	7	8	9
N	5	4	3	3	4	4	5	6	7	8	9
E	6	5	4	4	4	5	5	6	7	8	9
N	7	6	5	5	5	4	5	6	7	8	9
T	8	7	6	6	6	5	5	6	7	8	9
I	9	8	7	7	7	6	6	6	6	7	8
A	10	9	8	8	8	7	7	7	7	6	7
L	11	10	9	8	9	8	8	8	8	7	6

The Algorithm

for $i = 0$ *to* m **do**
  $E(i, 0) = i$;
end
for $j = 1$ *to* n **do**
  $E(0, j) = j$;
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The over running time is $O(m \cdot n)$.

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- The input is a rooted tree. A subproblem is a rooted subtree.

Knapsack

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We express this in terms of smaller subproblems:

$$K(w) = \max_{i:w_i \leq w} \{K(w - w_i) + v_i\}$$

Knapsack with Repetition

For every $w \leq W$ let

$$K(w) = \text{ maximum value achievable with a knapsack of capacity } w$$

We express this in terms of smaller subproblems:

$$K(w) = \max_{i:w_i \leq w} \{K(w - w_i) + v_i\}$$

$K(0) = 0$;
for $w = 1$ *to* W **do**
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`return` $(K(W))$;

The over running time is $O(n \cdot W)$.

Example: Knapsack with Repetition

Take $W = 10$, and

Item	Weight	Value
1	6	$30
2	3	$14
3	4	$16
4	2	$9

and there are unlimited quantities of each item available.

Knapsack without Repetition

For every $w \leq W$ and $0 \leq j \leq n$, let

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$$K(w, j) = \max\{K(w - w_j, j - 1) + v_j, K(w, j - 1)\}$$

Knapsack without Repetition

Initialize all $K(0, j) = 0$ and all $K(w, 0) = 0$;
for $j = 1$ *to* n **do**
  for $w = 1$ *to* n **do**
    if $w_j > w$ **then** $K(w, j) = K(w, j-1)$;
    $K(w, j) = \max\{K(w - w_j, j-1) + v_j, K(w, j-1)\}$;
  end
end
`return(`$K(W, n)$`);`

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    $K(w, j) = \max\{K(w - w_j, j-1) + v_j, K(w, j-1)\}$;
  end
end
`return` $(K(W, n))$;

The over running time is $O(n \cdot W)$.

Knapsack without Repetition

Take $W = 9$, and

Item	Weight	Value
1	2	\$3
2	3	\$4
3	4	\$5
4	5	\$7

and there is only one of each item available.

Problem. Given two strings $x_1 x_2 \dots x_m$ and $y_1 y_2 \dots y_n$, find a common subsequence that is as long as possible.

Longest Common Subsequence

Problem. Given two strings $x_1 x_2 \dots x_m$ and $y_1 y_2 \dots y_n$, find a common subsequence that is as long as possible.

Alternative viewpoint. Delete some characters from x; delete some character from y; a common subsequence if it results in the same string.

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Example. **LCS(GGCACCACG, ACGGCGGATACG) = GGCAACG**.

Quiz

How about the longest common string?

Chain Matrix Multiplication

The Problem

Suppose that we want to multiply four matrices, A, B, C, D, of dimensions 50×20, 20×1, 1×10, and 10×100, respectively.

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Multiplying an $m \times n$ matrix by an $n \times p$ matrix takes $m \cdot n \cdot p$ multiplications.

Parenthesization	Cost computation	Cost
$A \times ((B \times C) \times D)$	$20 \cdot 1 \cdot 10 + 20 \cdot 10 \cdot 100 + 50 \cdot 20 \cdot 100$	$120,200$
$(A \times (B \times C)) \times D$	$20 \cdot 1 \cdot 10 + 50 \cdot 20 \cdot 10 + 50 \cdot 10 \cdot 100$	$60,200$
$(A \times B) \times (C \times D)$	$50 \cdot 20 \cdot 1 + 1 \cdot 10 \cdot 100 + 50 \cdot 1 \cdot 100$	$7,000$

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$(A \times (B \times C)) \times D$	$20 \cdot 1 \cdot 10 + 50 \cdot 20 \cdot 10 + 50 \cdot 10 \cdot 100$	$60,200$
$(A \times B) \times (C \times D)$	$50 \cdot 20 \cdot 1 + 1 \cdot 10 \cdot 100 + 50 \cdot 1 \cdot 100$	$7,000$

Q: How do we determine the optimal order, if we want to compute $A_1 \times A_2 \times \ldots \times A_n$, where the A_i's are matrices with dimensions $m_0 \times m_1, m_1 \times m_2, \ldots, m_{n-1} \times m_n$, respectively?

Binary Tree

A particular parenthesization can be represented by a binary tree in which

- the individual matrices correspond to the leaves,
- the root is the final product, and
- interior nodes are intermediate products.

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The possible orders in which to do the multiplication correspond to the various full binary trees with n leaves.

Subproblems

For $1 \leq i \leq j \leq n$, let

$$C(i,j) = \text{ minimum cost of multiplying } A_i \times A_{i+1} \times \ldots \times A_j$$

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$$C(i,j) = \min_{i \leq k < j} \{C(i,k) + C(k+1,j) + m_{i-1} \cdot m_k \cdot m_j\}$$

The Program

The Program


```
for i = 1 to n do
    C(i, i) = 0;
end
for s = 1 to n − 1 do
    for i = 1 to n − s do
        j = i + s;
        C(i, j) = min_{i≤k<j}{C(i, k) + C(k + 1, j) + m_{i−1} · m_k · m_j};
    end
end
return (C(1, n));
```

The Program

for $i = 1$ *to* n **do**
  $C(i, i) = 0$;
end
for $s = 1$ *to* $n - 1$ **do**
  for $i = 1$ *to* $n - s$ **do**
    $j = i + s$;
    $C(i, j) = \min_{i \le k < j}\{C(i, k) + C(k + 1, j) + m_{i-1} \cdot m_k \cdot m_j\}$;
  end
end
`return` $(C(1, n))$;

The over running time is $O(n^3)$.

The Example

Suppose that we want to multiply four matrices, A, B, C, D, of dimensions 50×20, 20×1, 1×10, and 10×100, respectively.