ATTP spring 2021: EXAM for Module 3

June 7, 2021, shifted time window 13:00-15:00

Please email a scan of your work to t.p.hinderer@uu.nl by the end of the allotted time (15:00)

Conventions The conventions are that repeated indices are summed over, and the up or down placement of spatial indices has no meaning since they are raised and lowered with the flat spatial metric δ_{ij}. In matrix form, δ_{ij} is a diagonal unit matrix. Problems 2 and 3 involve binary systems on circular orbits in the center of mass frame. The bodies are labeled 1 and 2, with masses m_1 and m_2 respectively. The relative separation is $r = \sqrt{\delta_{ij}x^i x^j}$, and the relative velocity is $\boldsymbol{v} = \dot{\boldsymbol{x}}$ with $|\boldsymbol{v}| = r\omega$, where ω is the circular-orbit angular frequency. In the frame of the binary system in plane polar coordinates $\boldsymbol{x} = r(\cos\varphi, \sin\varphi, 0)$, where $\varphi = \omega t$ is the orbital phase.

1. **Matched asymptotic expansions [15pts]**

   We seek the solution to the following differential equation involving a small dimensionless parameter $\epsilon \ll 1$:

   $$\epsilon y''(x) + (1+x)y'(x) + y(x) = 0 \tag{1}$$

   in the interval $0 \leq x \leq 1$ and with the boundary conditions

   $$y(0) = 2, \quad y(1) = \frac{1}{2}. \tag{2}$$

   (a) **Outer asymptotic expansion**. First consider the asymptotic expansion of the solution at fixed x of the form $y(x;\epsilon) = y_0(x) + O(\epsilon)$. Solve for y_0, taking into account the boundary condition at $x = 1$.

   (b) **Inner asymptotic expansion**. The outer solution from (a) breaks down when $x \lesssim O(\epsilon)$ and thus also fails to satisfy the boundary condition at $x = 0$. To describe the regime of small x:
   - introduce a scaled coordinate X such that it is $O(1)$ for $x = O(\epsilon)$.
   - make an ansatz for the asymptotic expansion of y at fixed X of the form $y(x;\epsilon) = Y_0(X) + O(\epsilon)$ and solve for Y_0, taking into account the boundary condition at $X = 0$. This will still leave one constant undetermined, which depends on the behavior of the solution in the outer regime.

   (c) **Matching**. Consider the intermediate regime of small x and large X where both expansions are valid.
   - Characterize this regime by an intermediately small parameter η with $\epsilon \ll \eta \ll 1$ and introduce an associated scaled coordinate x_η, which is $O(1)$ in this region. Express x and X in terms of x_η.
   - Express your result for the inner and outer solutions Y_0 and y_0 in terms of x_η. Approximate the results for $\eta \ll 1$ to make the dependence on η explicit, where possible.
   - Determine the unknown constant in Y_0 from matching to the outer solution.

   (d) **Composite expansion**. Construct the leading-order term in a composite expansion of y that is asymptotically valid everywhere in the interval $0 \leq x \leq 1$.

2. **Current quadrupole radiation from a binary system [15 pts]**

   The dominant gravitational radiation of a binary system is generated by the mass quadrupole. However, there are also contributions from other multipole moments. The current quadrupole moment is given to leading order by

   $$J_{ij} = \mu\, r^2\, \epsilon_{kli}\, n_j\, n^k\, v^l \left(\frac{m_1}{M} - \frac{m_2}{M}\right) = \mu\sqrt{1-4\nu}\,\omega\, r^3\, n_j\, \delta_{i3}, \tag{3}$$

   where the first expression is general but the last one is only valid for circular orbits. Here, the reduced and total mass are $\mu = m_1 m_2/(m_1+m_2)^2$ and $M = m_1 + m_2$ respectively, and $\nu = \mu/M$. We used that the two sets of mass parameters (m_1, m_2) and (M, ν) are related by $m_{1,2} = (M/2)(1 \pm \sqrt{1-4\nu})$ and defined a radial unit vector $\boldsymbol{n} = \boldsymbol{x}/r$. The symbol ϵ_{ijk} is the antisymmetric Levi-Civita tensor whose properties have been used to arrive at the final simpler expression for circular orbits in (3) involving δ_{i3}, which is the starting point for your calculations. Here, the spatial coordinates have the generic labels $\boldsymbol{x} = (x^1, x^2, x^3)$.

   Only the symmetric-trace-free (STF) part of the multipole moments contribute to gravitational radiation. The STF current quadrupole is obtained by symmetrizing in $i \leftrightarrow j$ and subtracting the trace

   $$J_{<ij>} = \frac{1}{2}\left(J_{ij} + J_{ji}\right) - \frac{1}{3}\delta_{ij} J_{kk} \tag{4}$$

   (a) **STF current quadrupole moment**. Compute the nonvanishing components of $J_{<ij>}$. Also write them explicitly with $\boldsymbol{n}$ expressed in plane polar coordinates. You should find that there are only two independent nonvanishing components (four non-zero components in total due to the symmetry), and they are off-diagonal.

   Similarly to the mass quadrupole, the GWs generated by the dynamics of the current quadrupole depend on its time-derivatives, $h_{ij}^{\rm TT}\,|_{\rm curr.\ quad} \propto \ddot{J}_{<ij>}$. At what multiples of the orbital frequency do these GWs oscillate, and how does that compare to radiation generated by the mass quadrupole?

(b) **Moments on the spherical harmonic basis**. Compute the spherical harmonic $\ell = 2$, m-components of $J_{<ij>}$ denoted by J_m, e.g. by using that

$$J_m = \frac{8\pi}{15} J_{<ij>} (\mathcal{Y}_{ij}^{2m})^* \tag{5}$$

and the relation $\mathcal{Y}_{ij}^{2(-m)} = (-1)^m (\mathcal{Y}_{ij}^{2m})^*$, where the star denotes the complex conjugate and

$$\mathcal{Y}_{ij}^{22} = \sqrt{\frac{15}{32\pi}} \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{Y}_{ij}^{21} = -\sqrt{\frac{15}{32\pi}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix}, \quad \mathcal{Y}_{ij}^{20} = \sqrt{\frac{5}{16\pi}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{6}$$

(c) **Gravitational wave power due to the current quadrupole**. The energy per time in GWs due to the current quadrupole radiation is

$$\dot{E}_{\rm GW}^{\rm curr.\ quad} = \frac{2G}{3\pi\, c^7} \sum_{m=-2}^{2} |\dddot{J}_m|^2. \tag{7}$$

Here, dots denote time derivatives and the vertical bars denote the absolute value $|\dddot{J}_m|^2 = \dddot{J}_m\, (\dddot{J}_m)^*$.

- Compute $\dot{E}_{\rm GW}^{\rm curr.\ quad}$ for circular orbits.
- Take the ratio to the Newtonian mass quadrupole radiation $\dot{E}_{\rm GW}^{\rm Newt\ mass\ quad} = 32G\mu^2 r^4\omega^6/(5c^5)$, recalling that the circular-orbit velocity is $v = r\omega$. In what regimes of masses and velocities is the current quadrupole most important compared to the Newtonian mass quadrupole?

3. **Gravitational-wave phasing in scalar-tensor theories [15pts]**

A widely-used class of modified theories of gravity are a class of scalar-tensor theories with a massless scalar field Φ, whose action in a frame where the field does not directly couple to matter is

$$S = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} \left[\Phi R - \frac{\bar{\omega}(\Phi)}{\Phi} g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi \right] + S_{\rm matter}, \tag{8}$$

where $\bar{\omega}(\Phi)$ is a generic coupling function. The non-minimal coupling between the metric and scalar field gives rise to new features, for instance, compact objects can acquire a scalar charge and generate scalar dipole radiation. In this problem, we work perturbatively in the post-Newtonian (PN) approximation, based on taking the formal limit that $c \to \infty$ or $c^{-1} \ll 1$. We will only consider fractional corrections at the first post-Newtonian (1PN) order, i.e. at a relative order $O(c^{-2})$.

The radial equation of motion of a binary system to 1PN order, in the center of mass frame and specialized to circular orbits with $\ddot{r} = \dot{r} = 0$ is

$$0 = \omega^2 - \frac{G\alpha M}{r^3} + \frac{G\alpha M}{r\, c^2} \left(a_1 \omega^2 + b_1 \frac{G\alpha M}{r^3} \right). \tag{9}$$

The parameter α depends on the coupling function $\bar{\omega}$ evaluated at the background field and the scalar charges of the two objects in the binary. The field-dependent quantity $G\alpha$ represents an effective gravitational 'constant' experienced by massive bodies. The coefficients a_1 and b_1 are given by

$$a_1 = -(1 + 3\nu + d_1), \quad b_1 = 2(2 + \nu + c_1 + d_1), \tag{10}$$

where c_1 and d_1 are coefficients that depend on the scalar coupling function and the scalar charges of the bodies in a different way than α. Their explicit form will not be needed here, and it is also common to keep them in a parameterized form.

The binding energy of circular orbits is given to 1PN order by

$$E = -\frac{\mu}{2} y + \frac{\mu y^2}{c^2} \left[\frac{3}{8} + \frac{\nu}{24} + \frac{d_1 - c_1}{3} \right], \tag{11}$$

where we have introduced the quantity $y = (G\alpha M\omega)^{2/3}$. Note that the quantity y/c^2 is dimensionless.

The total power radiated by a binary system to the order we are interested in is given by

$$\dot{E}_{\rm total} = \dot{E}_- + \dot{E}_0. \tag{12}$$

Here, the term $\dot{E}_-$ is due to scalar dipole radiation, which leads to additional energy losses during an inspiral and is given by

$$\dot{E}_- = \frac{1}{c^3} \frac{4\nu^2}{G\alpha} \left(\frac{G\alpha M}{r} \right)^4 \bar{\mathcal{S}}_-^2, \tag{13}$$

where $\bar{\mathcal{S}}_-$ depends on the scalar coupling function and the difference in the scalar charges of the bodies. This represents a 'negative' 1PN term when compared to the usual GR radiation which starts at $O(c^{-5})$. Thus, in this problem we will be interested in working to the same order at which the GR radiation appears, so will need to include the $O(c^{-2})$ fractional corrections to $\dot{E}_-$ and other quantities.

An inspiraling binary in this theory also generates gravitational radiation, which, together with the higher order corrections to the scalar energy flux, leads to a rate of energy emission given at $O(c^{-5})$ by

$$\dot{E}_0 = \frac{1}{c^5}\frac{8\nu^2}{15\,G\alpha}\left(\frac{G\alpha M}{r}\right)^4\left[\left(f_1 + (\nu - 23 - 10c_1 - 10d_1)\bar{\mathcal{S}}_-^2\right)\frac{G\alpha M}{r} + (12 + h_1)v^2\right], \tag{14}$$

where f_1 and h_1 are again new combinations dependent on the scalar coupling function and the scalar charges that we will keep in a parameterized form.

(a) **Radius-frequency relationship**. Solve (9) perturbatively for the radius-frequency relationship $r(\omega) = r_0[1 + c^{-2}r_1]$ to $O(c^{-2})$. Keep the result in terms of a_1 and b_1 at this stage and show that $r_1 = -(a_1 + b_1)(G\alpha M\omega)^{2/3}/3$.

(b) **More concise notation**. Introduce the quantity $y = (G\alpha M\omega)^{2/3}$ to express your results from (a) in a more concise form. Using this, work out the first-order perturbative corrections, to $O(c^{-2})$ for $GM\alpha/r = y[1 + O(y/c^2)]$ and for $v^2 = y[1 + O(y/c^2)]$, where you determine each of the coefficients of the fractional PN corrections in the second term inside the brackets.

(c) **Radiated power**. Express the energy radiated $\dot{E}_{\text{total}}$ in terms of y, working perturbatively in c^{-2} to obtain the complete expressions to $O(1/c^5)$ for $\dot{E}_{\text{total}}$. This is the scaling where gravitational radiation first appears in GR. Write your result in terms of a fractional correction as $\dot{E}_{\text{total}} = 4\nu^2y^4\bar{\mathcal{S}}_-^2/(3G\alpha c^3)[1 + O(y/c^2)]$, where you determine the coefficient at the relative $O(y/c^2)$.

(d) **Evolution of the orbital phase**. Balancing the radiative energy losses $\dot{E}_{\text{total}}$ with a change in energy of the binary system dE/dy leads to a differential equation for the evolution of the orbital phase φ (whose multiples are the GW phase) given in terms of y by

$$\frac{d\varphi}{dy} = -\frac{y^{3/2}}{GM\alpha}\frac{dE/dy}{\dot{E}_{\text{total}}}. \tag{15}$$

Assume that the dominant term in $\dot{E}_{\text{total}}$ is the contribution in E_- due to the scalar dipole radiation. This regime is relevant for binaries with large separations or large scalar dipoles, such as systems where one of the objects spontaneously or dynamically scalarizes and quickly develops a large scalar charge.

- Calculate dE/dy, and write the result in the form $dE/dy = -\mu/2[1+(y/c^2)\delta E']$, where you compute the explicit result for the coefficient $\delta E'$ from (11).
- Compute the right hand side of (15) perturbatively, keeping only the $O(c^3)$ and $O(c)$ contributions. You can first work with the general form $d\varphi/dy = 3c^3/(8\nu y^{5/2}\bar{\mathcal{S}}_-^2)[1+O(y/c^2)]$, where the fractional correction term involves $\delta E'$ and the fractional corrections to $\dot{E}_{\text{total}}$ computed in (c). At this stage you can leave the coefficients $\delta E'$ and $\delta\dot{E}$ general but make sure to keep the explicit dependences of all the terms on y.
- Integrate to obtain $\varphi(y)$, where you can neglect the integration constant.
- Finally, substitute all of the coefficients to write φ explicitly as

$$\varphi(y) = -\frac{1}{4\nu\bar{\mathcal{S}}_-^2}\left(\frac{y}{c^2}\right)^{-3/2}\left[1 + \frac{y}{c^2}\,\delta\varphi_1\right], \tag{16}$$

where you determine $\delta\varphi_1$ in terms of the symmetric mass ratio ν as well as the set of scalar-related parameters $c_1, d_1, f_1, h_1, \bar{\mathcal{S}}_-$. This result is a basis for incorporating the difference from GR in waveform models to look for the deviations in the data and set constraints on the scalar-tensor parameters.

Based on the energy fluxes, one might expect that φ should scale with the frequency as a -1PN term relative to the GR phase $\varphi_{\text{GR}} \sim \frac{1}{32\nu}(y/c^2)^{-5/2}$. Thus, one might expect the scalar-tensor phasing to depend on $(y/c^2)^{-7/2}$ at the leading order. From your calculations this is evidently not the case in the regime dominated by the scalar radiation considered here. However, it is the result when specializing to small scalar charges and/or small separation such that the gravitational radiation dominates over the scalar losses and $\dot{E}_-$ is treated as a small correction in the calculation of the phase.