Gothic field equations, post-Minkowski expansions, effective stress-energy tensor of GWs

The goal of these exercises is to use xAct to perform a full derivation of the field equations in the gothic formulation, then expand around Minkowski spacetime to arbitrary order in the field nonlinearities, and from the quadratic nonlinearities derive the effective stress-energy tensor of GWs. It serves as a concrete example of using xAct for calculations beyond the short examples from the introductory tutorial. throughout this tutorial, the extent to which the user must guide the calculations will become more apparent, since it will require implementing a number of rules that the user must first derive. Once the rules are in place, the code you will develop can in principle output the vacuum field equations to arbitrary order in the nonlinearities, which can serve as the starting point for computing the dynamics and GWs from binary systems in the post-Newtonian/post-Minkowski approximation, given also the source terms where relevant. To adapt this notebook to modified gravity theories (where - with the exception of scalar-tensor theories and Gauss-Bonnet gravity - only the linearized, Newtonian results have been worked out to date) will require using instead the appropriate field equations of the theory (e.g. derived from the action as in the introductory tutorial), however, many of the rules developed here will immediately carry over, as they rely only on the definition of the gothic metric.

Go through the notebook TUTORIAL_GOTHIC_UPDATED.NB and do the exercises therein, which cover the following:

- Gothic formulation of the field equations
  - **Basic setup and definitions**. Details are explained in the notebook. Requires defining the gothic inverse metric density $\mathfrak{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}$ (where g is the determinant of the metric) and the inverse $\mathfrak{g}_{\alpha\beta} = g_{\alpha\beta}/\sqrt{-g}$ (note the different dependences on $\sqrt{-g}$), rules for contracting gothics, and for changing between metric and gothic.
  - **Rules for derivatives**. Because the gothic involves $\sqrt{-g}$ as well as the inverse metric, changing from derivatives of the metric to derivatives of gothic is nontrivial. You will have to implement a set of rules so xAct knows how to do it. These rules are obtained with a trick that is often used in calculations in GR and relies on differentiating identities such as $\mathfrak{g}_{\alpha\mu}\mathfrak{g}^{\alpha\beta} = \delta^{\beta}_{\mu}$ and $g_{\alpha\mu}\mathfrak{g}^{\alpha\beta}/\sqrt{-g} = \delta^{\beta}_{\mu}$. You will find more hints on how exactly to proceed in the notebook.
  - **Deriving the Einstein tensor in terms of gothic, and specializing to harmonic gauge**. Given all the above preparations, it is relatively straightforward to eliminate all the dependences on the metric in favor of the gothic. A new rule will be required to impose the harmonic gauge condition $\partial_\alpha \mathfrak{g}^{\alpha\beta} = 0$.
- **Weak-field expansion** using the functionality for perturbations xPert to output the vacuum field equations at each order in the form of a wave equation for the field at a specified order with a nonlinear source that depends on all the lower order fields.
- **Effective stress-energy tensor of GWs**. Recall from the first lecture, that the effective stress-energy tensor of GWs in linearized gravity was defined by averaging the quadratic-order nonlinear source terms in the weak-field expansion over the rapid variations to identify the coarse-grained effects of how GWs curve the spacetime. This averaging can be either a time or spatial average, as they are equivalent for plane waves that depend on $(t - z/c)$, where z is the direction of propagation. You will need to define rules to implement the averaging, and perform various integrations by parts, discarding total derivatives that vanish upon averaging. More guidelines are explained in the notebook.