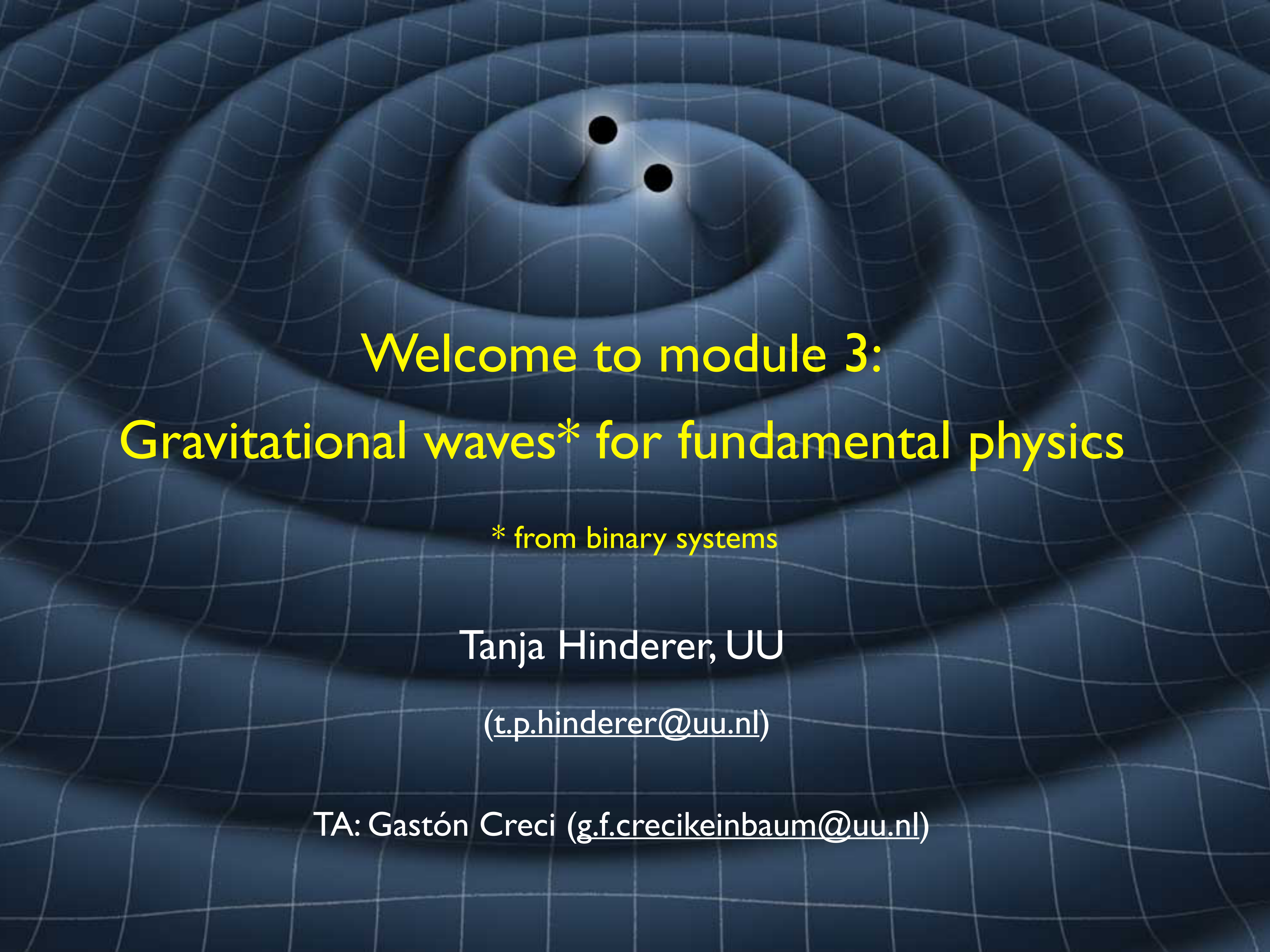


Welcome to module 3:

Gravitational waves* for fundamental physics

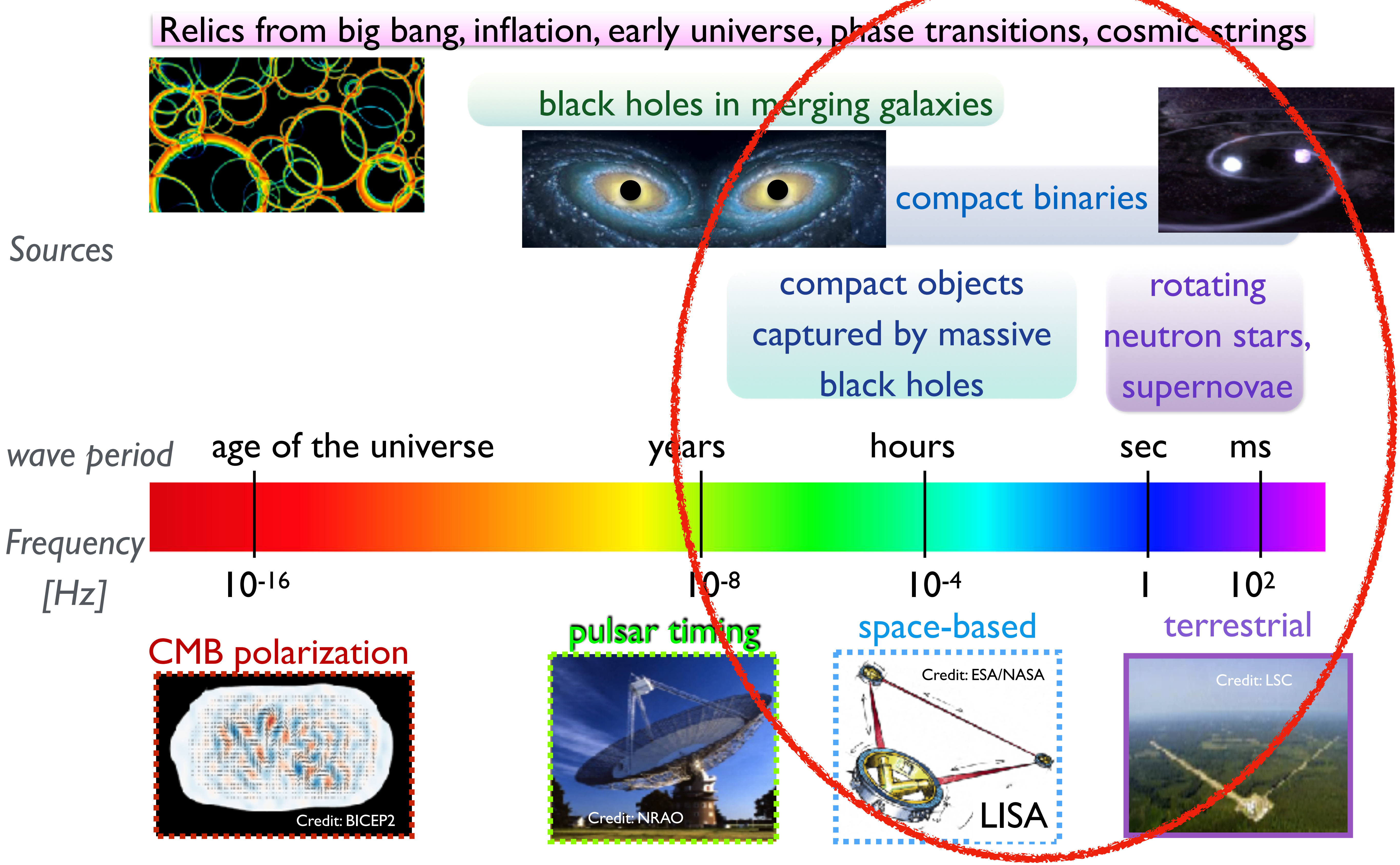
* from binary systems

Tanja Hinderer, UU

(t.p.hinderer@uu.nl)

TA: Gastón Creci (g.f.crecikeinbaum@uu.nl)

The GW spectrum: over 20 decades in frequency



GWs as a new tool for fundamental physics

Gravity in nonlinear, strong-field, dynamical regimes?

Nature of black holes?

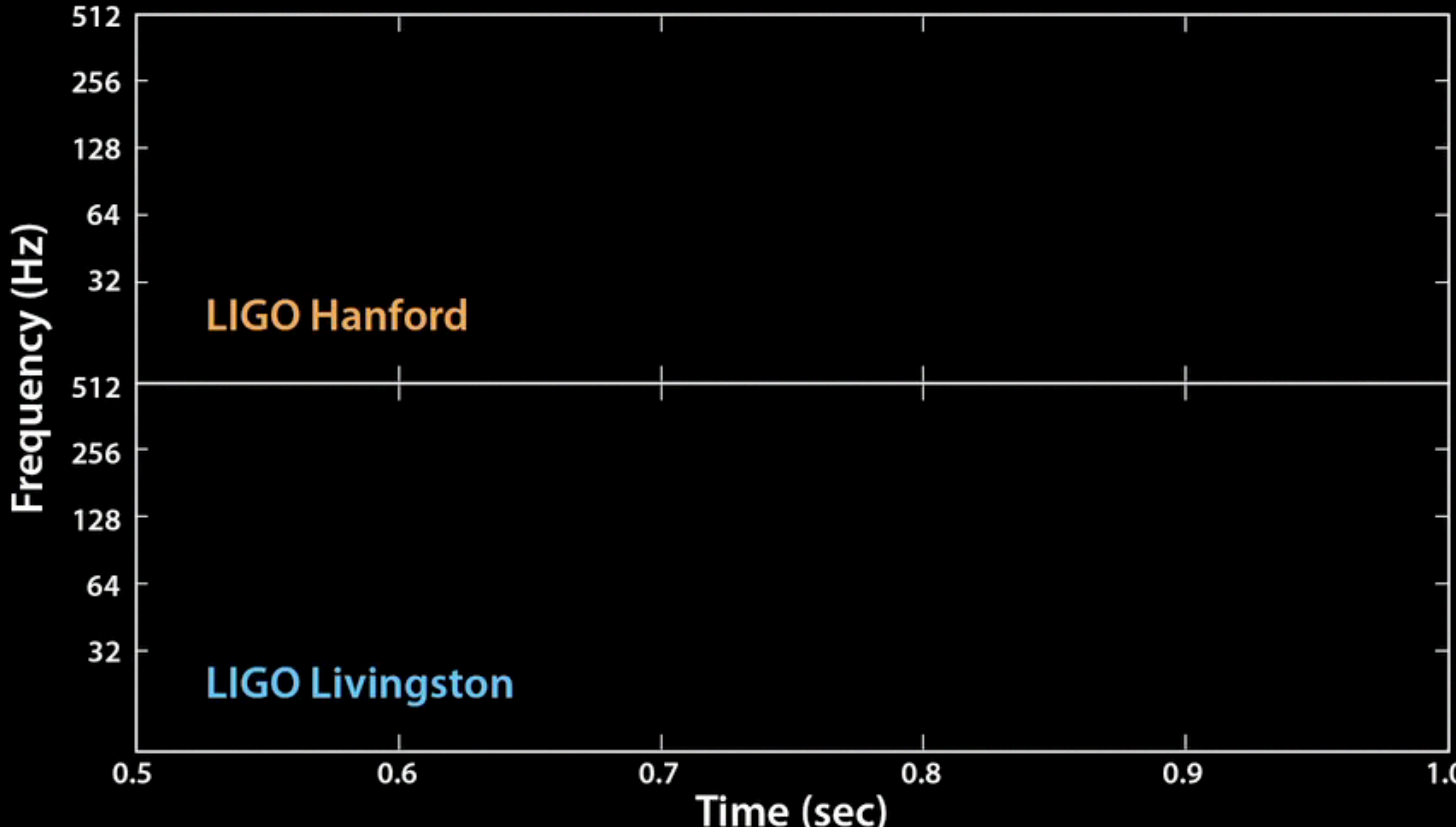
Number of black holes in the universe? How & when formed?

Dark matter? Beyond standard model particle physics?

Matter at extreme density in neutron stars?

….

14. September 2015, 10:45:45 CET

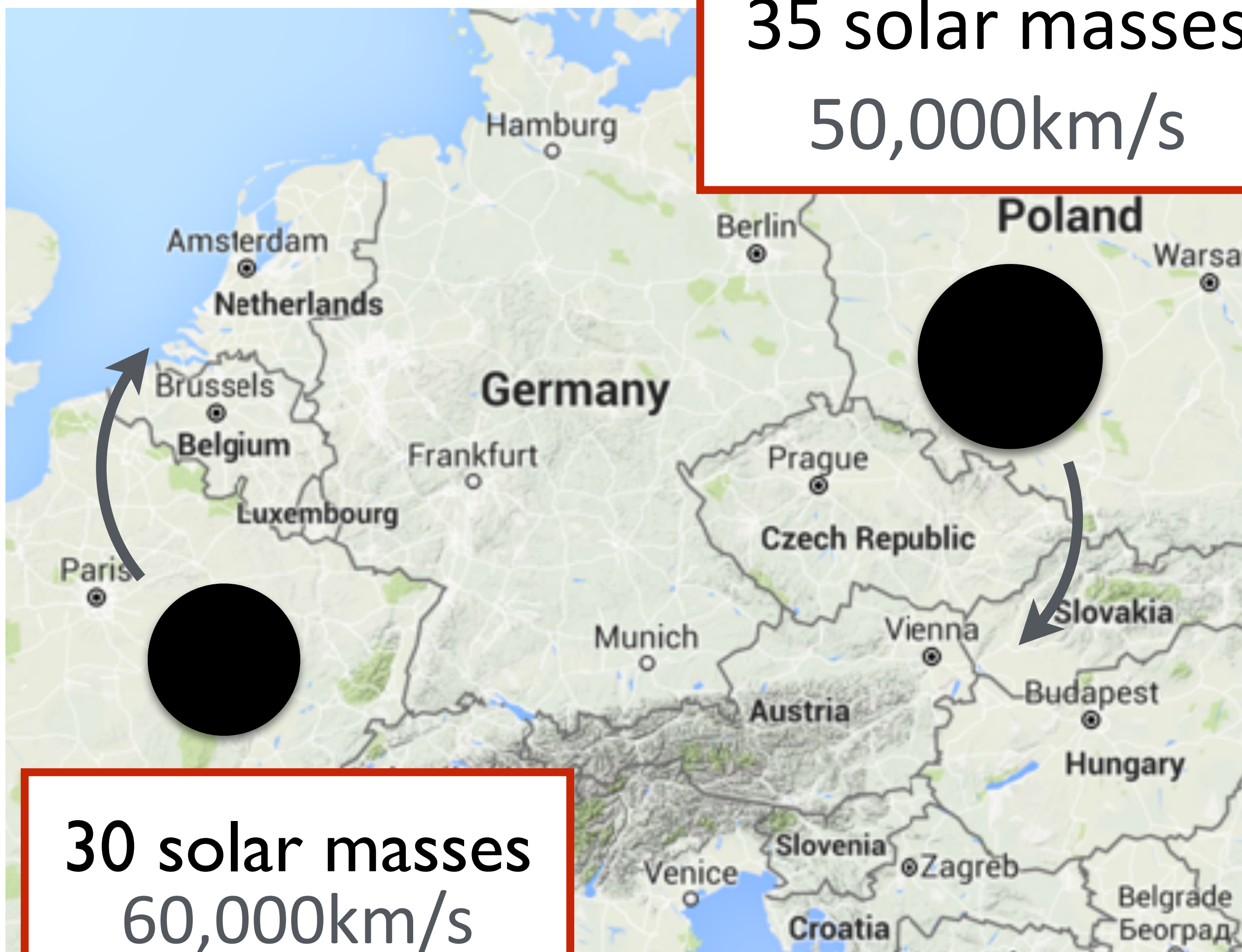


Credit:LSC

GW signal from a black hole binary (GW150914)

35 solar masses
50,000km/s

0.05 seconds per orbit

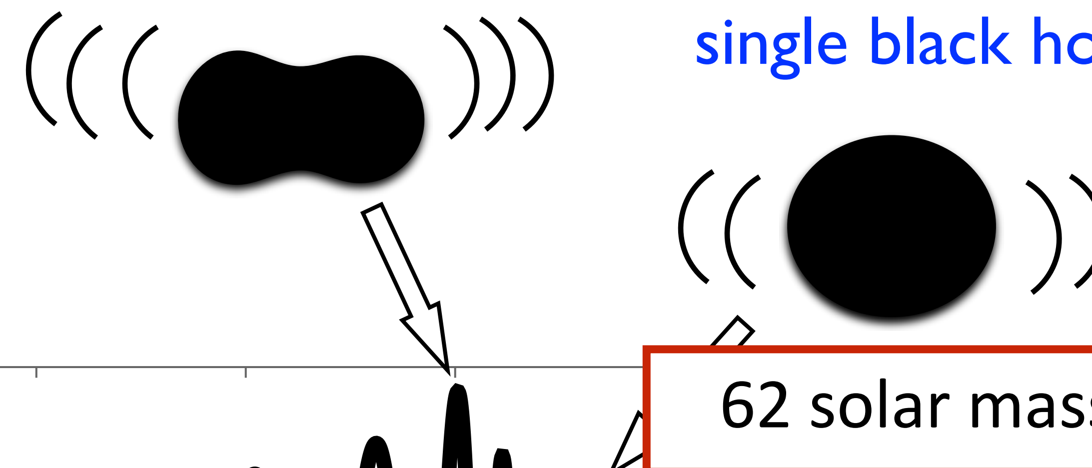


30 solar masses
60,000km/s

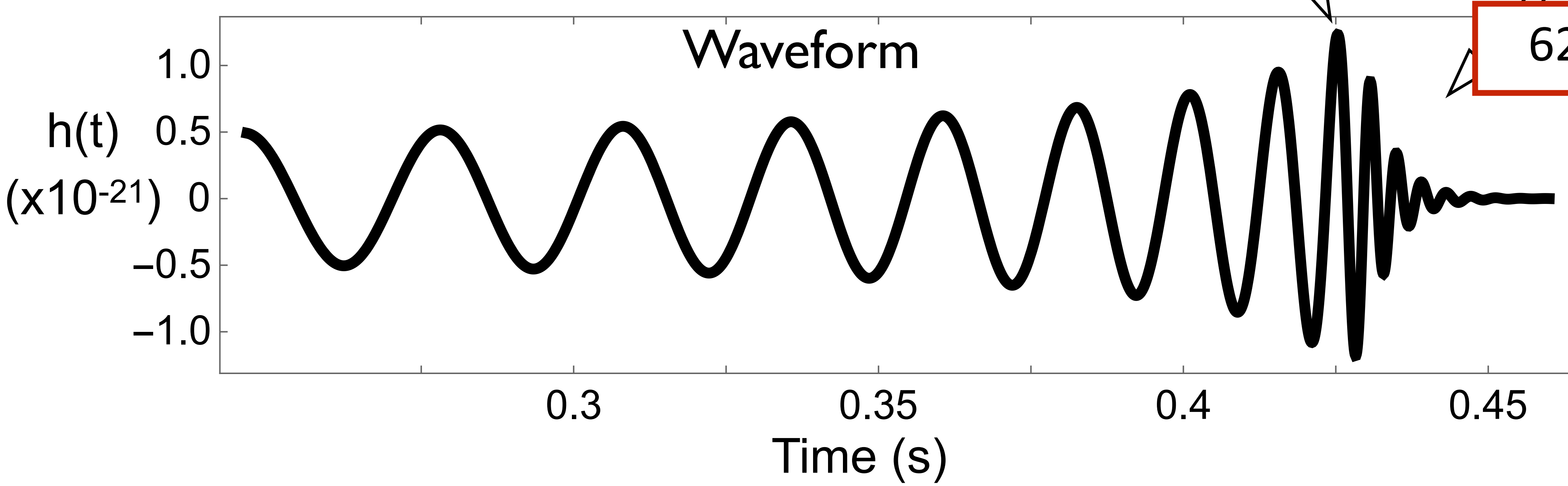
The orbit shrinks …

… until they collide

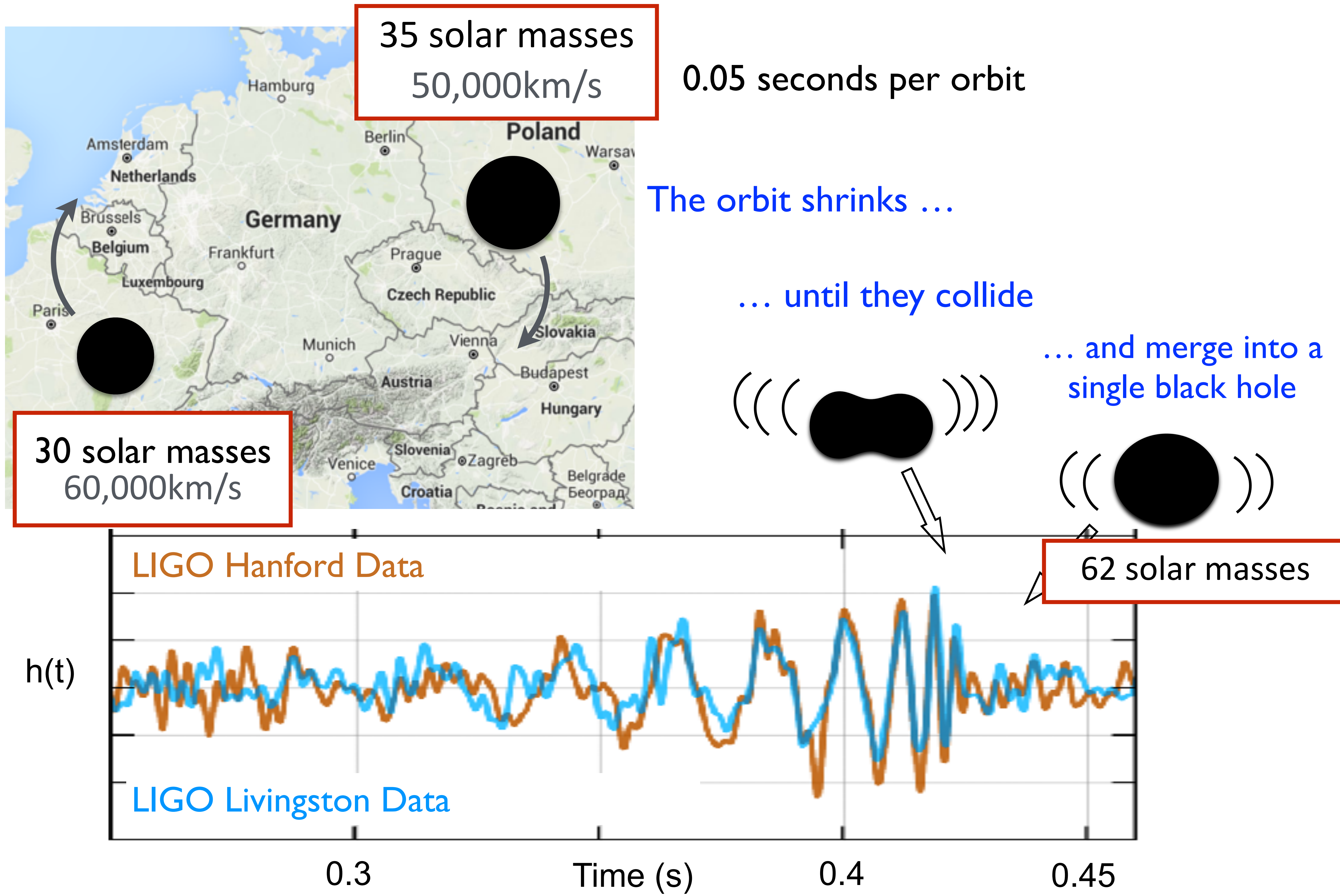
… and merge into a single black hole



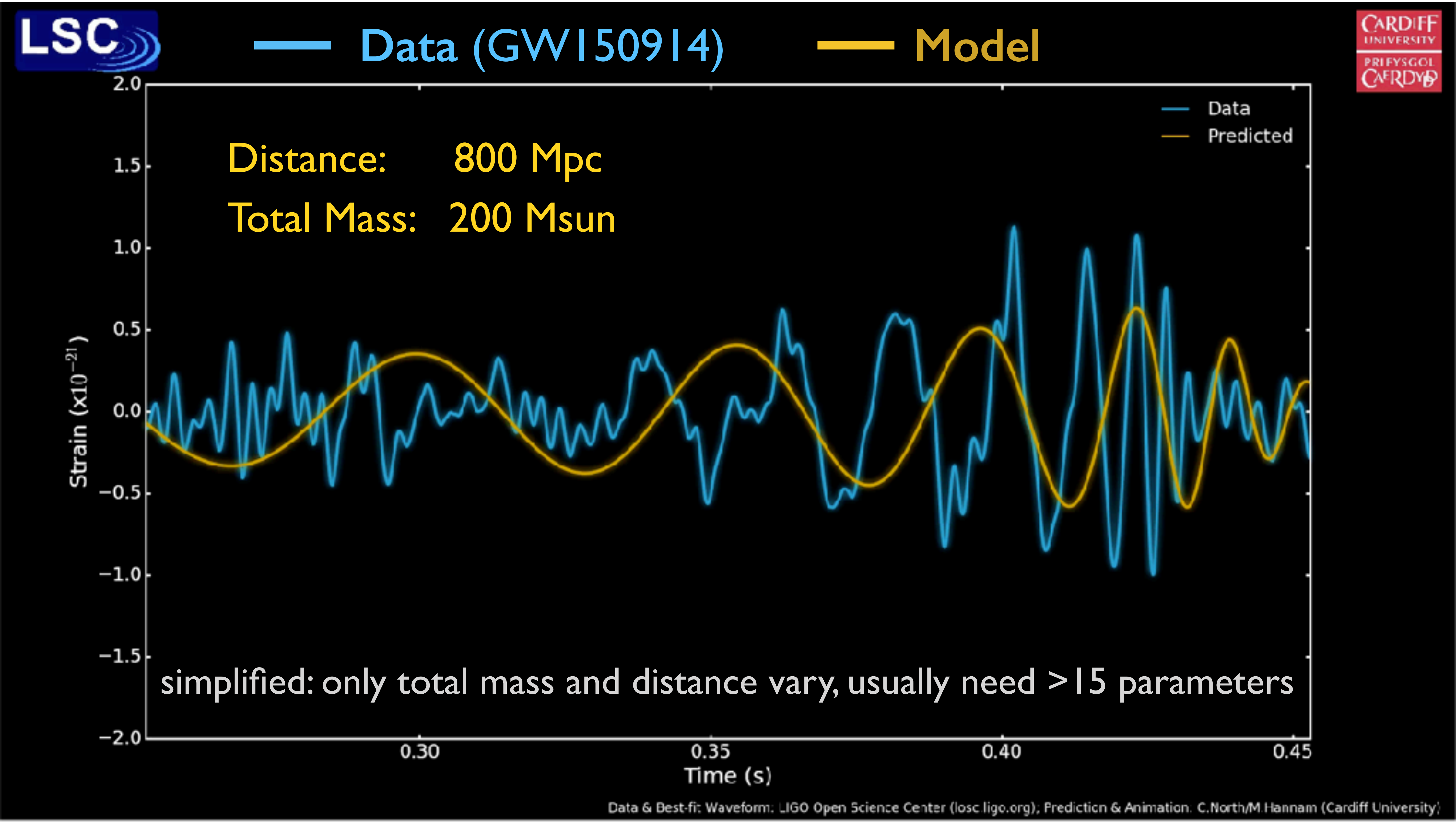
62 solar masses



GW signal from a black hole binary (GW150914)

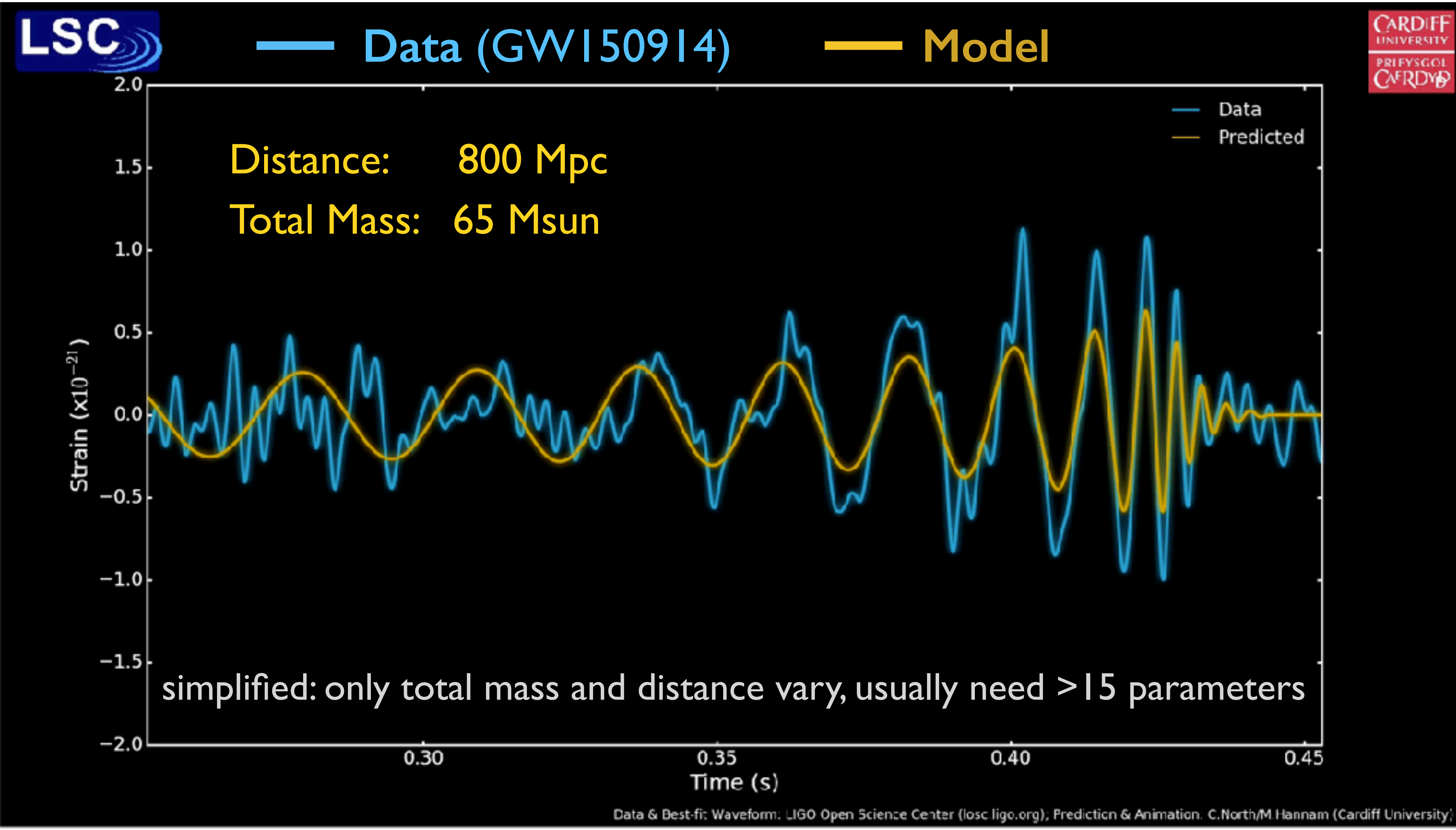


Interpreting GW signals



Details of the waveform depend on source properties, very sensitive to the phase

Interpreting GW signals



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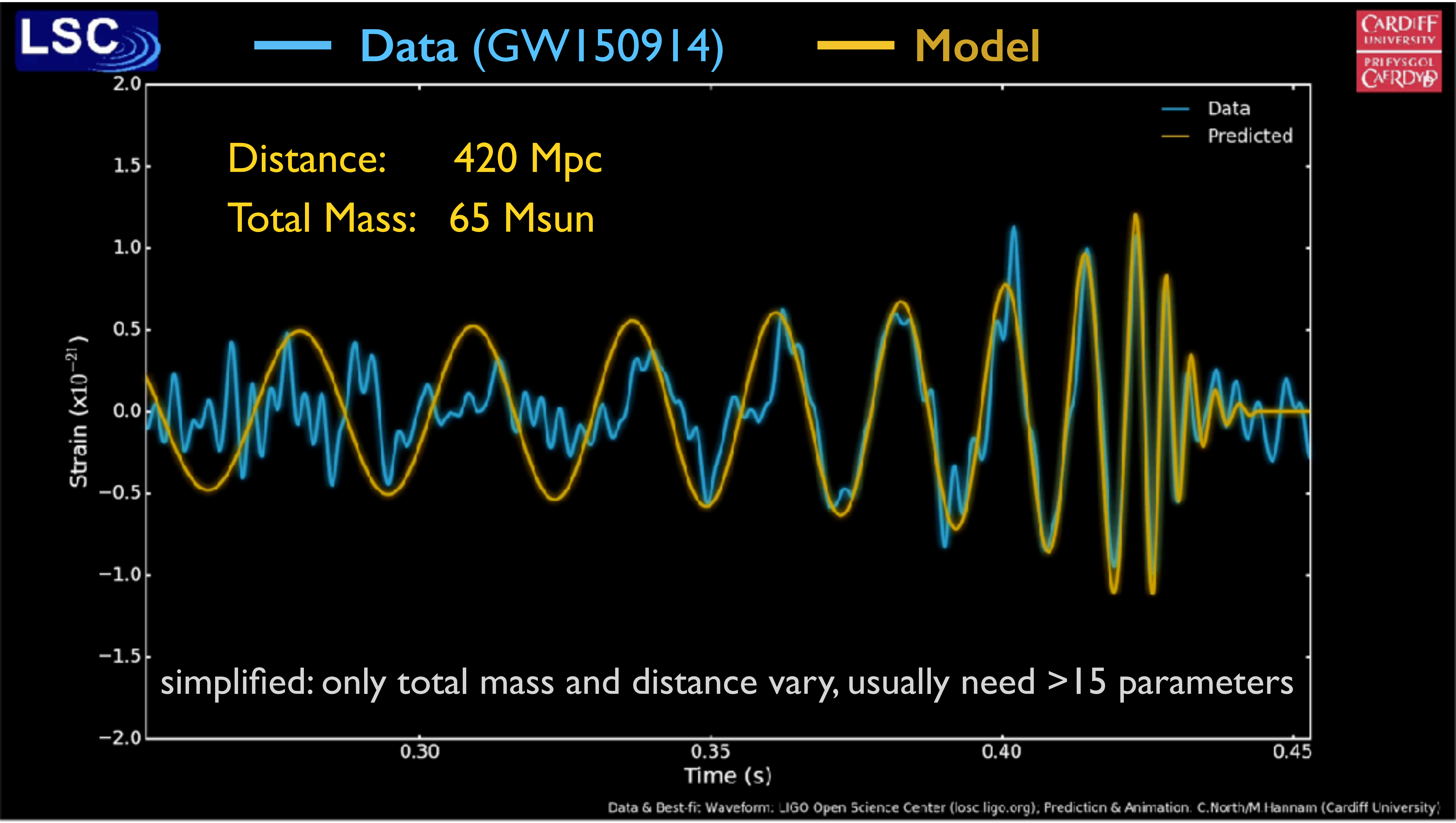
Interpreting GW signals



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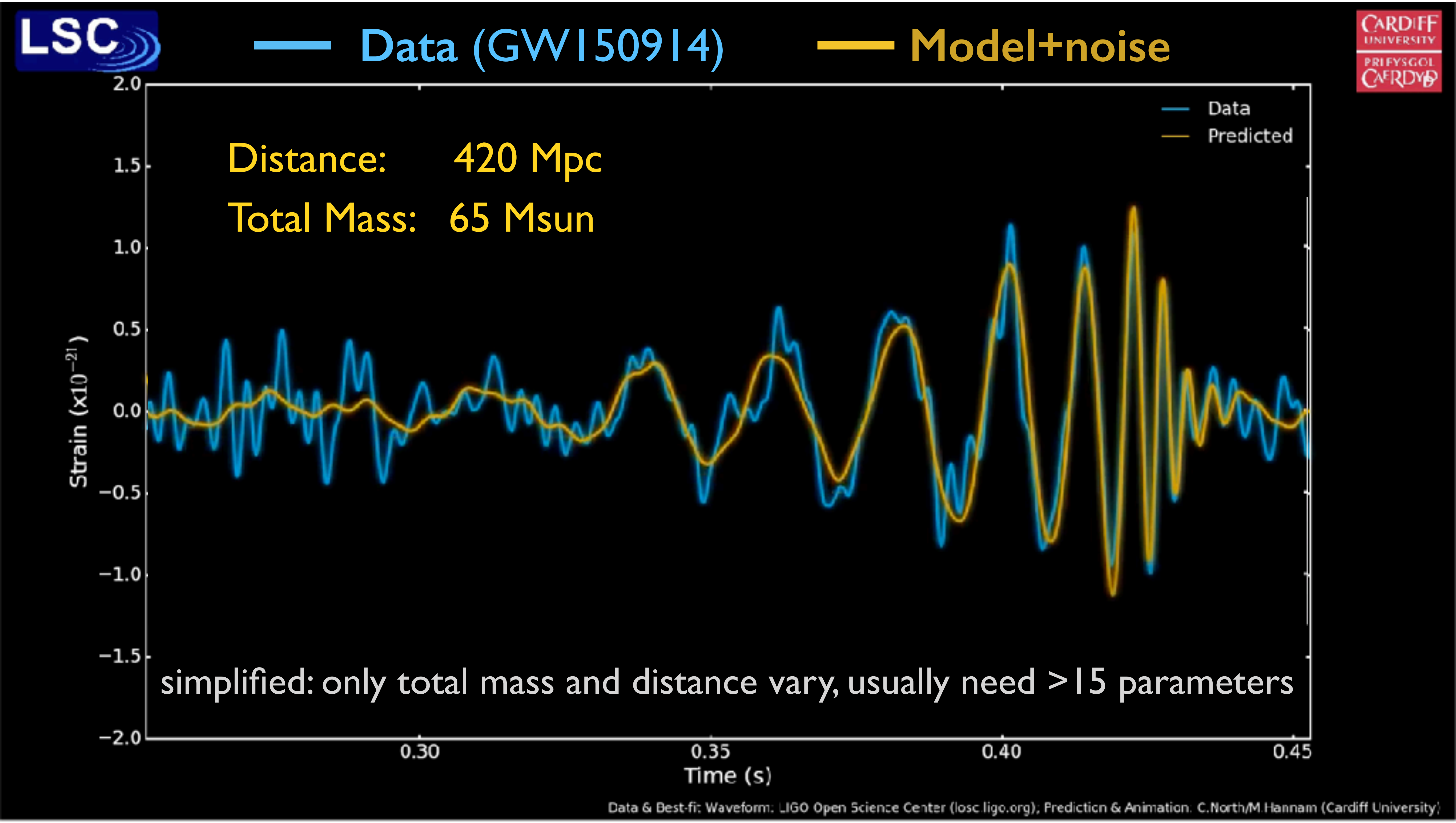
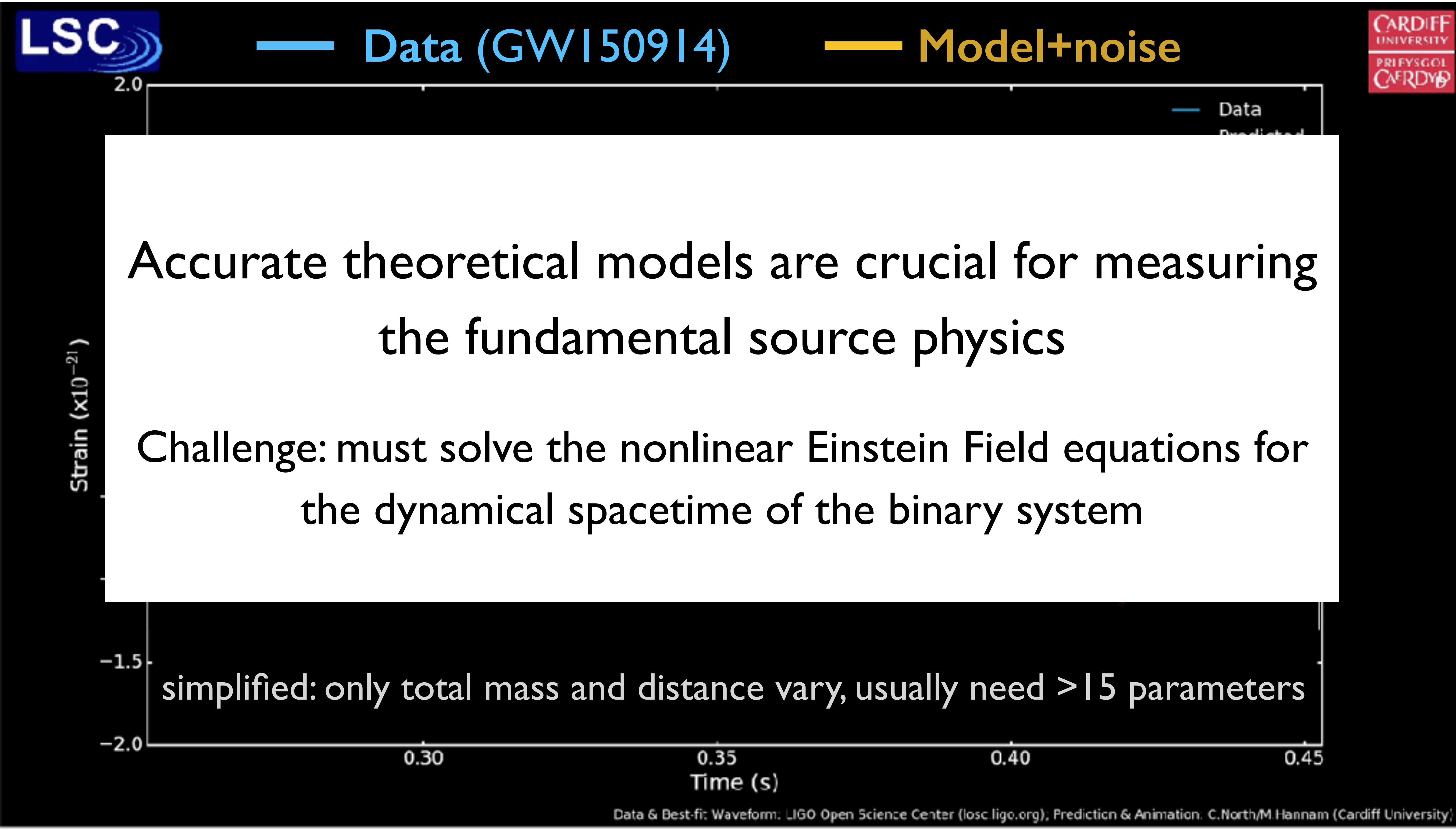
Interpreting GW signals



Data analysis: Bayesian, MCMC — requires millions of model waveforms for each event

Interpreting GW signals



Accurate theoretical models are crucial for measuring the fundamental source physics

Challenge: must solve the nonlinear Einstein Field equations for the dynamical spacetime of the binary system

Data analysis: Bayesian, MCMC — requires millions of model waveforms for each event

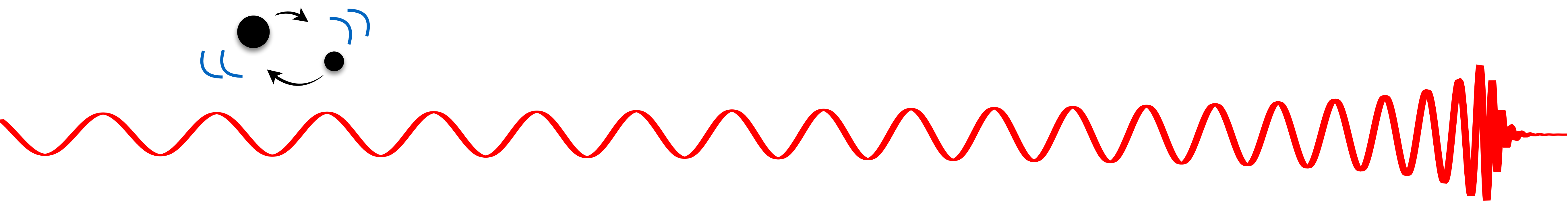
Plan for the lectures

Today: the basic physics of GWs, binary system sources

2 (May 10): modeling effects during the inspiral epoch (post-Newtonian, -Minkowski theory)

3 (May 17): the characteristic 'sound' of black holes (relativistic perturbation theory)

4: binary mergers and full waveforms



No class Apr 26 (lecture-free week at UU), May 3 (UvA holiday), May 24 (Pentecost)

Exam: June 7

https://web.science.uu.nl/drstp/DeltaITP/ATTP_current_spring_2021.html

Homeworks: due before the next lecture, please scan (e.g. with an app, not just a photo), email in a single pdf to Gastón (g.f.crecikeinbaum@uu.nl)

Useful references and resources

- E. Flanagan and S. Hughes: The basics of GW theory https://arxiv.org/abs/gr-qc/0501041 .
  Focus on the material on pages 1-7 and 17-23.

- Data releases, useful information and interactive tutorials on GW detection and data analysis are collected at the GW open science center https://www.gw-openscience.org

- The first GW detection paper: LIGO and Virgo Collaborations: Observation of Gravitational Waves from a Binary Black Hole Merger. Phys. Rev. Lett. 116(6), 061102 (2016). https://arxiv.org/abs/1602.03837

- LIGO and Virgo Collaborations: The basic physics of the binary black hole merger GW150914. Annalen der Physik, 529 (2016). https://arxiv.org/abs/1608.01940

- It is interesting to look at the paper by Albert Einstein: Über Gravitationswellen. Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften, Berlin (1918), 154D167. http://einsteinpapers.press.princeton.edu/vol7-trans/25 (English translation)

1. Properties of GWs

Mathematical description of spacetime

Einstein field equations:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

Curvature *Matter*

$$\nabla_\alpha T^{\alpha\beta} = 0$$

778 Gesamtsitzung vom 4. November 1915

Zur allgemeinen Relativitätstheorie.

Von A. Einstein.

844 Sitzung der physikalisch-mathematischen Klasse vom 25. November 1915

Die Feldgleichungen der Gravitation.

Von A. Einstein.

In zwei vor kurzem erschienenen Mitteilungen[1] habe ich gezeigt, wie man zu Feldgleichungen der Gravitation gelangen kann, die dem Postulat allgemeiner Relativität entsprechen, d. h. die in ihrer allgemeinen

Field equations seemingly very compact but extremely complicated:

- nonlinear system of 10 coupled second-order PDEs
- Choice of gauge is key for well-posedness as an initial value problem

Well-posed formulation in harmonic gauge

- Introduce the gothic inverse metric density $\mathfrak{g}^{\alpha\beta} = \sqrt{-g}\, g^{\alpha\beta}$
- Harmonic coordinate condition: $\partial_\mu \mathfrak{g}^{\alpha\mu} = 0$
- With the above choices, the Einstein field equations become:

$$\boxed{\mathfrak{g}^{\mu\nu}\partial_\mu\partial_\nu \mathfrak{g}^{\alpha\beta} = \frac{16\pi G}{c^4}(-g)T^{\alpha\beta} + \Sigma^{\alpha\beta}\underbrace{[\partial\mathfrak{g}\partial\mathfrak{g}]}_{\text{source term from field nonlinearities}}}$$

- Key point: this is a well-posed system of equations [Yvonne Choquet-Bruhat 1952]

The field nonlinearities in Σ

- the explicit expression is not needed now
- you will derive it yourself in the next tutorial using the tensor algebra package xAct for Mathematica:

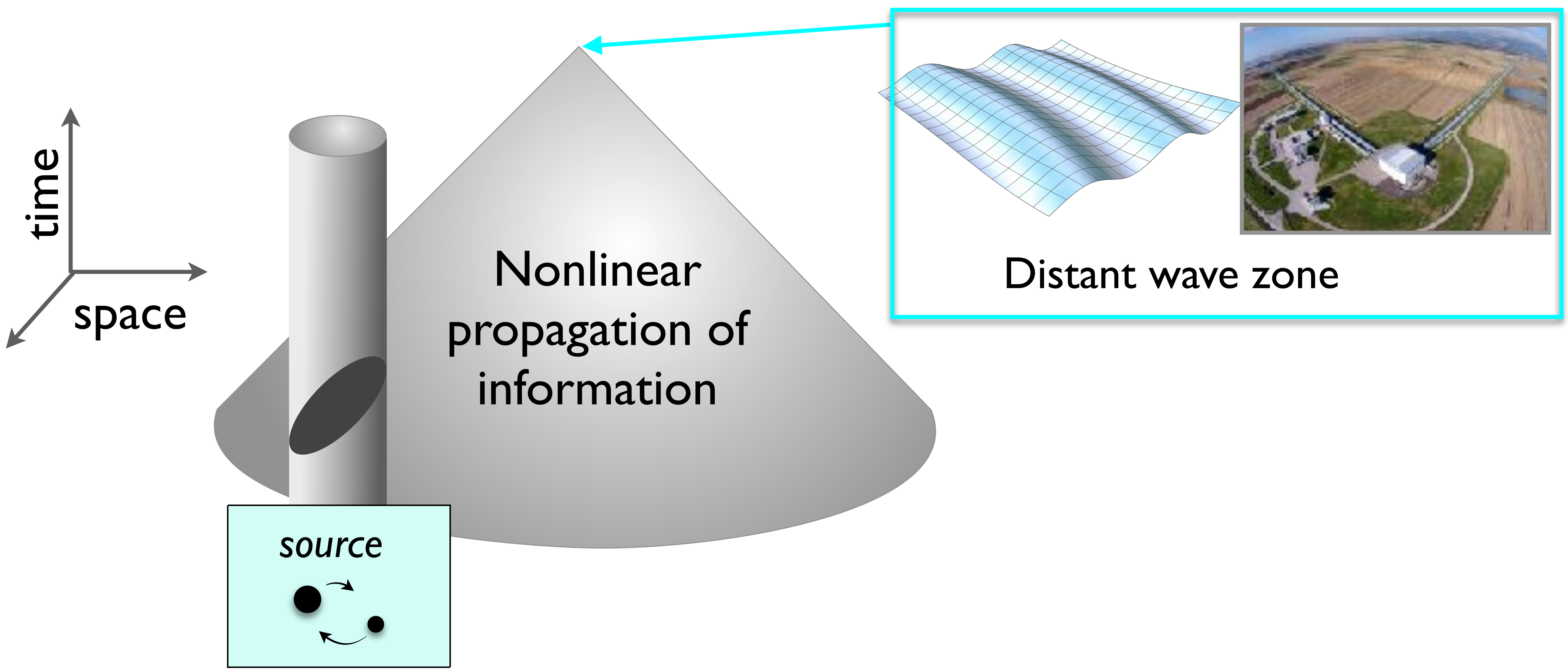
Out[24]=
$$\begin{aligned}\Sigma == {} & -\frac{1}{4}\,\tilde{\mathfrak{g}}^{\alpha\beta}\,\tilde{\mathfrak{g}}^{\mu\nu}\,\underset{\sim}{\mathfrak{g}}_{\gamma\zeta}\,\underset{\sim}{\mathfrak{g}}_{\theta\vartheta}\,\partial_\alpha\tilde{\mathfrak{g}}^{\gamma\theta}\,\partial_\beta\tilde{\mathfrak{g}}^{\zeta\vartheta} + \frac{1}{2}\,\tilde{\mathfrak{g}}^{\mu\alpha}\,\tilde{\mathfrak{g}}^{\nu\beta}\,\underset{\sim}{\mathfrak{g}}_{\gamma\zeta}\,\underset{\sim}{\mathfrak{g}}_{\theta\vartheta}\,\partial_\alpha\tilde{\mathfrak{g}}^{\gamma\theta}\,\partial_\beta\tilde{\mathfrak{g}}^{\zeta\vartheta} + {}\\ & \frac{1}{8}\,\tilde{\mathfrak{g}}^{\alpha\beta}\,\tilde{\mathfrak{g}}^{\mu\nu}\,\underset{\sim}{\mathfrak{g}}_{\gamma\zeta}\,\underset{\sim}{\mathfrak{g}}_{\theta\vartheta}\,\partial_\alpha\tilde{\mathfrak{g}}^{\gamma\zeta}\,\partial_\beta\tilde{\mathfrak{g}}^{\theta\vartheta} - \frac{1}{4}\,\tilde{\mathfrak{g}}^{\mu\alpha}\,\tilde{\mathfrak{g}}^{\nu\beta}\,\underset{\sim}{\mathfrak{g}}_{\gamma\zeta}\,\underset{\sim}{\mathfrak{g}}_{\theta\vartheta}\,\partial_\alpha\tilde{\mathfrak{g}}^{\gamma\zeta}\,\partial_\beta\tilde{\mathfrak{g}}^{\theta\vartheta} + \partial_\alpha\tilde{\mathfrak{g}}^{\nu\beta}\,\partial_\beta\tilde{\mathfrak{g}}^{\mu\alpha} + {}\\ & \tilde{\mathfrak{g}}^{\alpha\beta}\,\underset{\sim}{\mathfrak{g}}_{\gamma\zeta}\,\partial_\alpha\tilde{\mathfrak{g}}^{\mu\gamma}\,\partial_\beta\tilde{\mathfrak{g}}^{\nu\zeta} + \frac{1}{2}\,\tilde{\mathfrak{g}}^{\mu\nu}\,\underset{\sim}{\mathfrak{g}}_{\alpha\beta}\,\partial_\gamma\tilde{\mathfrak{g}}^{\beta\zeta}\,\partial_\zeta\tilde{\mathfrak{g}}^{\alpha\gamma} - \tilde{\mathfrak{g}}^{\nu\alpha}\,\underset{\sim}{\mathfrak{g}}_{\beta\gamma}\,\partial_\alpha\tilde{\mathfrak{g}}^{\gamma\zeta}\,\partial_\zeta\tilde{\mathfrak{g}}^{\mu\beta} - \tilde{\mathfrak{g}}^{\mu\alpha}\,\underset{\sim}{\mathfrak{g}}_{\beta\gamma}\,\partial_\alpha\tilde{\mathfrak{g}}^{\gamma\zeta}\,\partial_\zeta\tilde{\mathfrak{g}}^{\nu\beta}\end{aligned}$$

Analytical approximations for the dynamical spacetime of a binary source

Need: - a background solution

- small dimensionless parameter(s)

- valid in patches of the spacetime
- tapestry of approximation schemes connected via matched asymptotic expansions/effective actions
- more in the next lectures



Analytical approximations for the dynamical spacetime of a binary source

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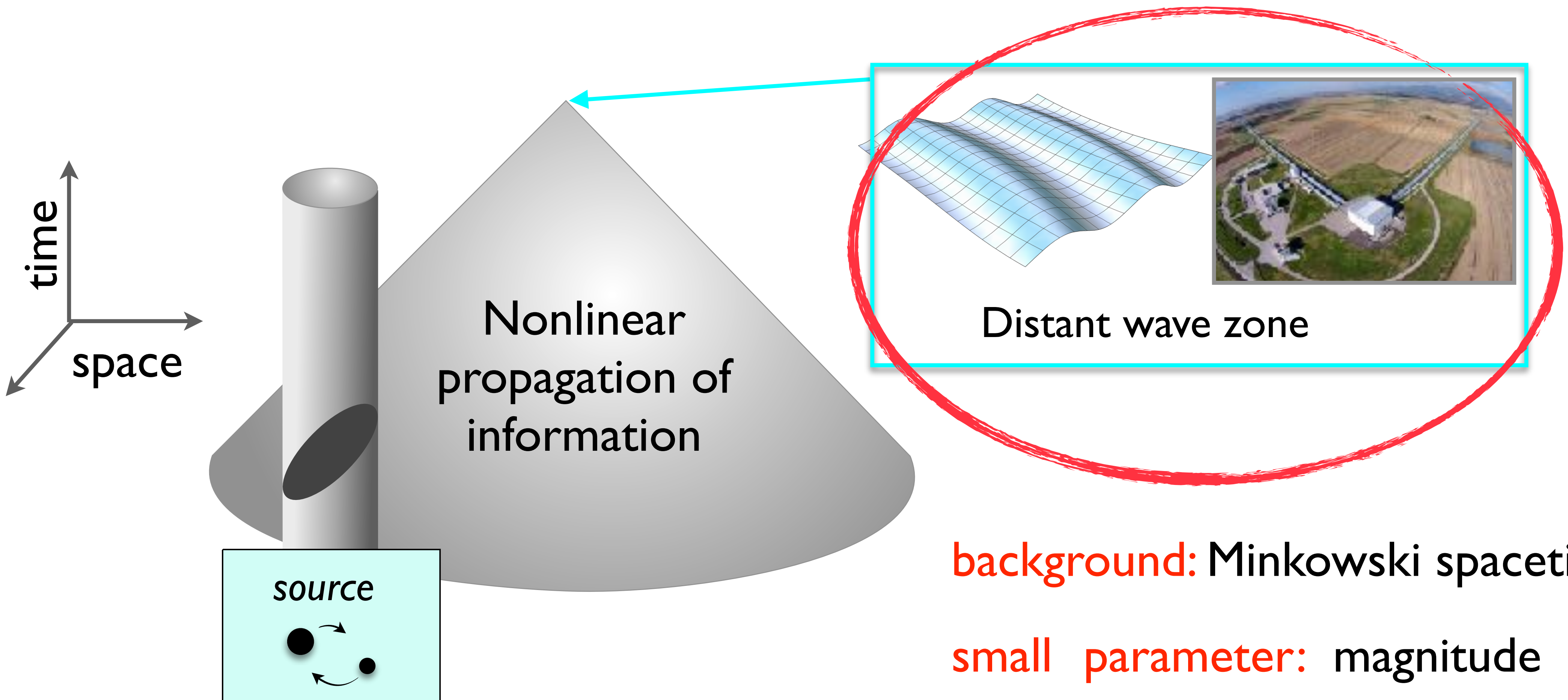


background: Minkowski spacetime

small parameter: magnitude of the metric deviation from flat space

(often: formal expansion in the gravitational constant G)

Deviation from flat space

- set $\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} + h^{\alpha\beta}$

Minkowski metric: diag(-1,1,1,1)

the harmonic gauge condition implies:

$$\partial_\mu h^{\alpha\mu} = 0$$

- the field equations become:

$$\Box h^{\alpha\beta} = \frac{16\pi G}{c^4}(-g)T^{\alpha\beta} + \underbrace{\Lambda^{\alpha\beta}}_{\text{field nonlinearities}} = \frac{16\pi G}{c^4}\tau^{\alpha\beta}$$

energy-momentum pseudotensor

contributions from: sources + gravitational field

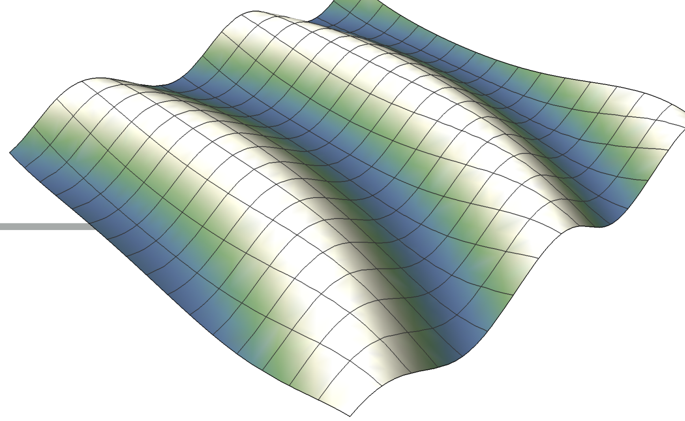
$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$$

Flat space d'Alembertian

satisfies the conservation law: $\boxed{\partial_\beta \tau^{\alpha\beta} = 0}$

- Next step: analyze for $|h^{\alpha\beta}| \ll 1$

Linearized solutions in vacuum



- To linear order in $h^{\alpha\beta}$ away from sources the field equations simplify to

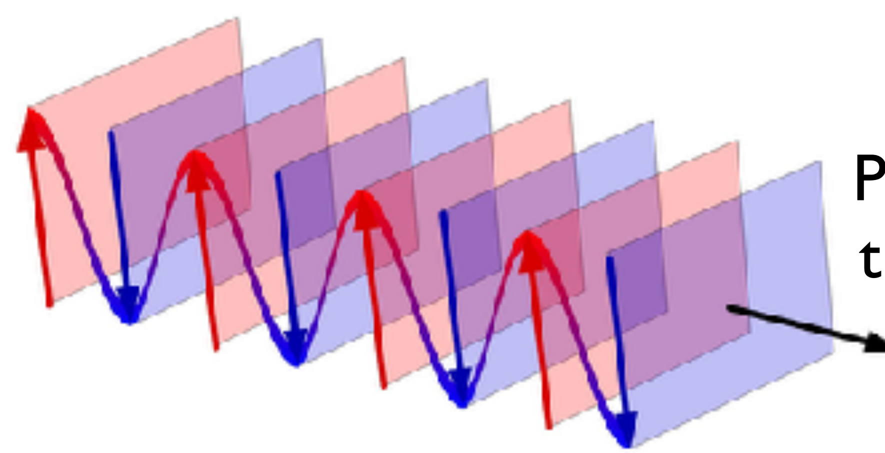
$$\Box h^{\alpha\beta} = 0 \qquad \Box = -\frac{1}{c^2}\partial_0^2 + \nabla^2 \qquad x^\mu = (ct, \mathbf{x})$$

[Note: $h^{\alpha\beta}$ here is *minus* the trace-reversed metric perturbation in 'usual' linearized gravity (not based on the gothic formulation but on the metric)]

- General solution: real part of a superposition of plane waves of the form

$$h^{\alpha\beta} = a^{\alpha\beta} e^{i\left(-k^0 x^0 + \mathbf{k}\cdot\mathbf{x}\right)}$$

Complex amplitudes



Propagates in the direction **k**

- Substituting into the wave equation leads to the condition:

$$\left[-(k^0)^2 + \mathbf{k}\cdot\mathbf{k}\right] h^{\alpha\beta} = 0 \longrightarrow k^0 = |\mathbf{k}| \equiv \frac{\omega_k}{c}$$

The transverse-traceless (TT) gauge

- Degrees of freedom: $h^{\alpha\beta}$ is a symmetric 4x4 matrix - 10 independent components

  The harmonic gauge condition $\partial_\mu h^{\alpha\mu} = 0$ eliminates 4

- There is remaining coordinate freedom: under a transformation $x'^\alpha = x^\alpha + \xi^\alpha$

  $h^{\alpha\beta}$ changes to $h'_{\alpha\beta} = h_{\alpha\beta} + \underbrace{(\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha - \eta_{\alpha\beta}\partial_\gamma \xi^\gamma)}$

  If $\partial^\beta(\ldots) = 0$, harmonic gauge is preserved

  $\Downarrow$

  requires $\Box \xi^\alpha = 0$

- Can impose 4 more conditions on $h^{\alpha\beta}$. Choose $\boxed{\text{TT gauge: } h^\alpha{}_\alpha = 0, \quad h^{0i} = 0}$

  the harmonic condition reduces to $\boxed{\partial_j h^{ij} = 0}$

  from $\partial_\mu h^{0\mu} = 0$:
  h^{00} is static, can set it to zero

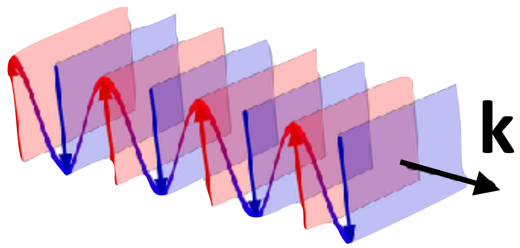
- $h^{\alpha\beta}_{TT}$ is purely spatial and traceless, two physical degrees of freedom remain

Physical degrees of freedom

- for the plane wave solutions in harmonic gauge:

$$h^{\alpha\beta} = a^{\alpha\beta} e^{-i\omega_k t + k_j x^j}$$

TT gauge: $h^{0\alpha} = 0$



$$\partial_j h^{ij} = 0 \quad \longleftrightarrow \quad k_j h^{ij} = 0$$

waves are transverse

- Take the direction of propagation along the z-direction: $k^j = (0, 0, |\mathbf{k}|) = (0, 0, \frac{\omega_k}{c})$

Also account for no trace: $h^{\alpha}_{\ \alpha} = 0$

$$h^{\alpha\beta}_{\mathrm{TT}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_+ & a_\times & 0 \\ 0 & a_\times & -a_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{-i\omega_k (t - z/c)}$$

Common notation for the two polarization amplitudes: + and x

Solutions often written as h_+, h_x

The TT projection operator Λ_{ijkl}

Useful relations: $N_i N^j = \delta_i^j$

$$\delta^{ij}\delta_{ij} = 3$$

- Readily get the TT part of any h^{ij} in harmonic gauge:

1.) Transverse projection to the direction **N** (= **k**/|**k**|)

$$\boxed{P_{ij}(\mathbf{N}) = \delta_{ij} - N_i N_j}$$

this operator is transverse: $N^i P_{ij} = 0$

and a projector: $P_{ik}P^{kj} = \delta_i^j$

2.) Assemble the traceless combination

$$\boxed{\Lambda_{ijkl}(\mathbf{N}) = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}}$$

This operator is also transverse and a projector, and it is traceless on (i,j) and (k,l)

3.) With this:

$$\boxed{h_{ij}^{\mathrm{TT}} = \Lambda_{ijkl}(\mathbf{N})h_{kl}}$$

Properties of the gravitational waves

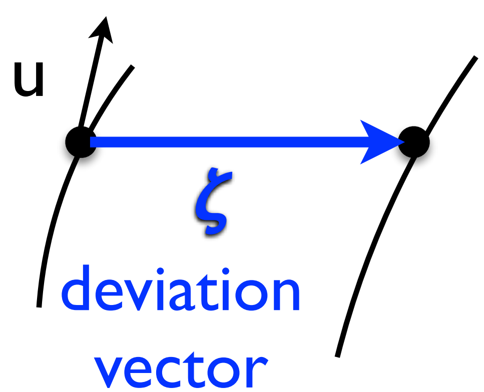
- h^{TT} contains only physical information, e.g. linearized curvature tensor simply related to h_+, $h_\times$ by:

$$R_{txtx} = -R_{tyty} = -\frac{1}{2}\ddot{h}_+ \qquad R_{txty} = R_{tytx} = -\frac{1}{2}\ddot{h}_\times$$

overdot=time derivative

Other remaining components from symmetries of Riemann

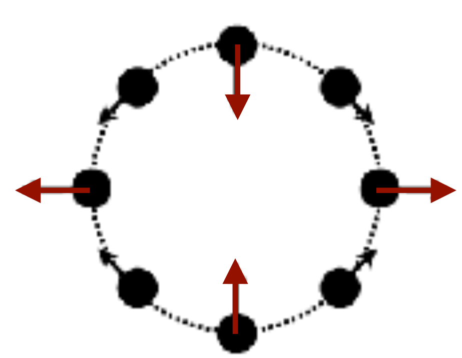
- geodesic deviation measures the relative acceleration of two nearby geodesics due to curvature



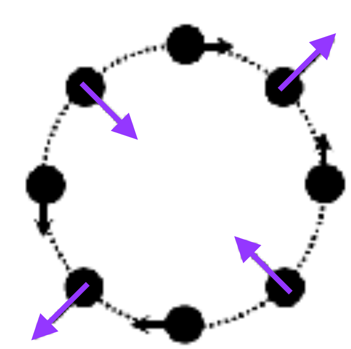
$$\frac{D^2\zeta^\alpha}{d\tau^2} = -R_{\mu\sigma\nu}{}^{\alpha}u^\mu u^\nu \zeta^\sigma$$

- in linearized gravity, freely falling frame: $\displaystyle \frac{d^2\zeta^i}{dt^2} = -R_{tjt}{}^{i}\zeta^j$

Effect of a GW on a ring of test masses:

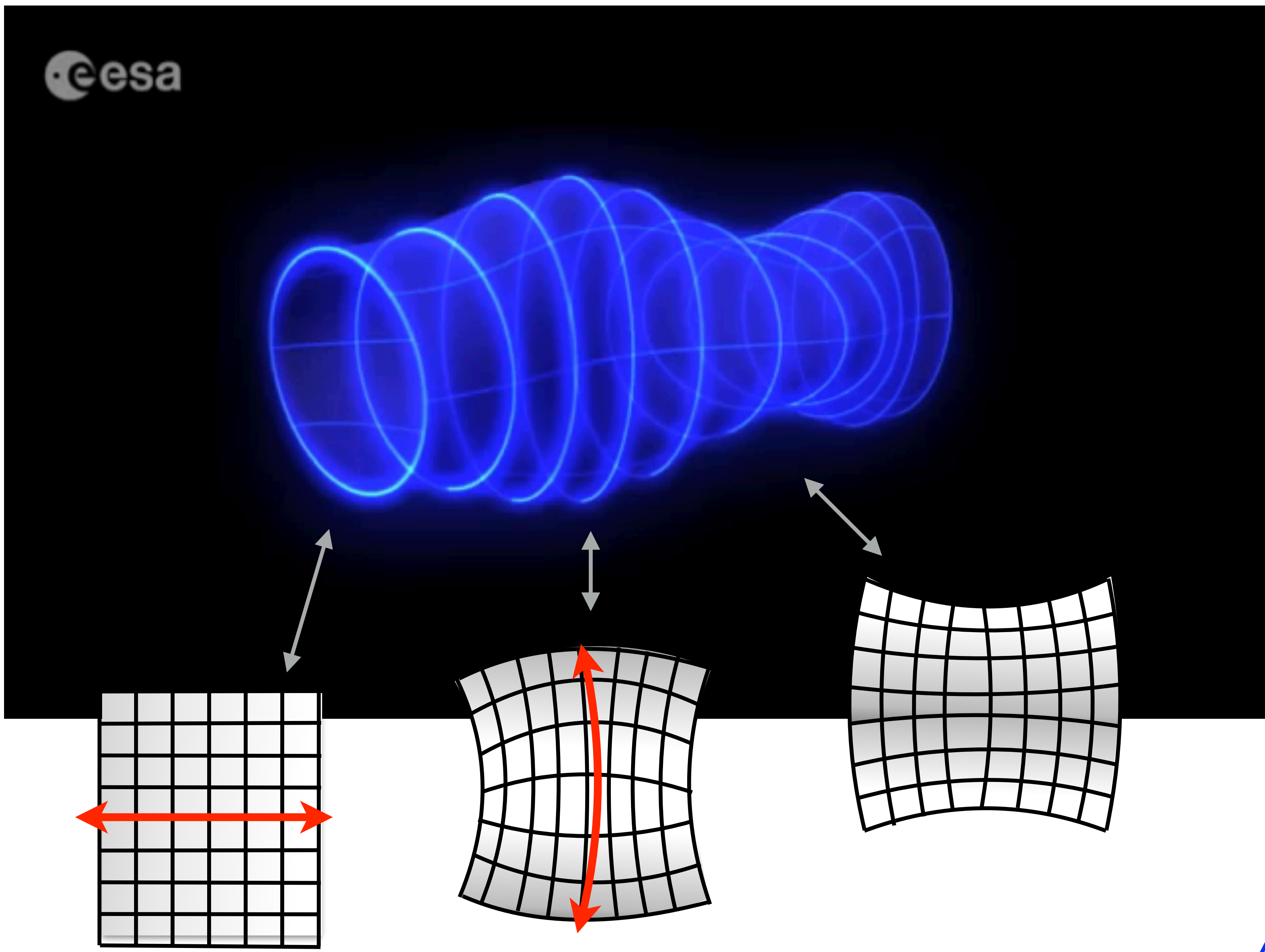


plus-polarization h_+



cross-polarization $h_\times$

The effect of a propagating GW



Stretching and squeezing of space

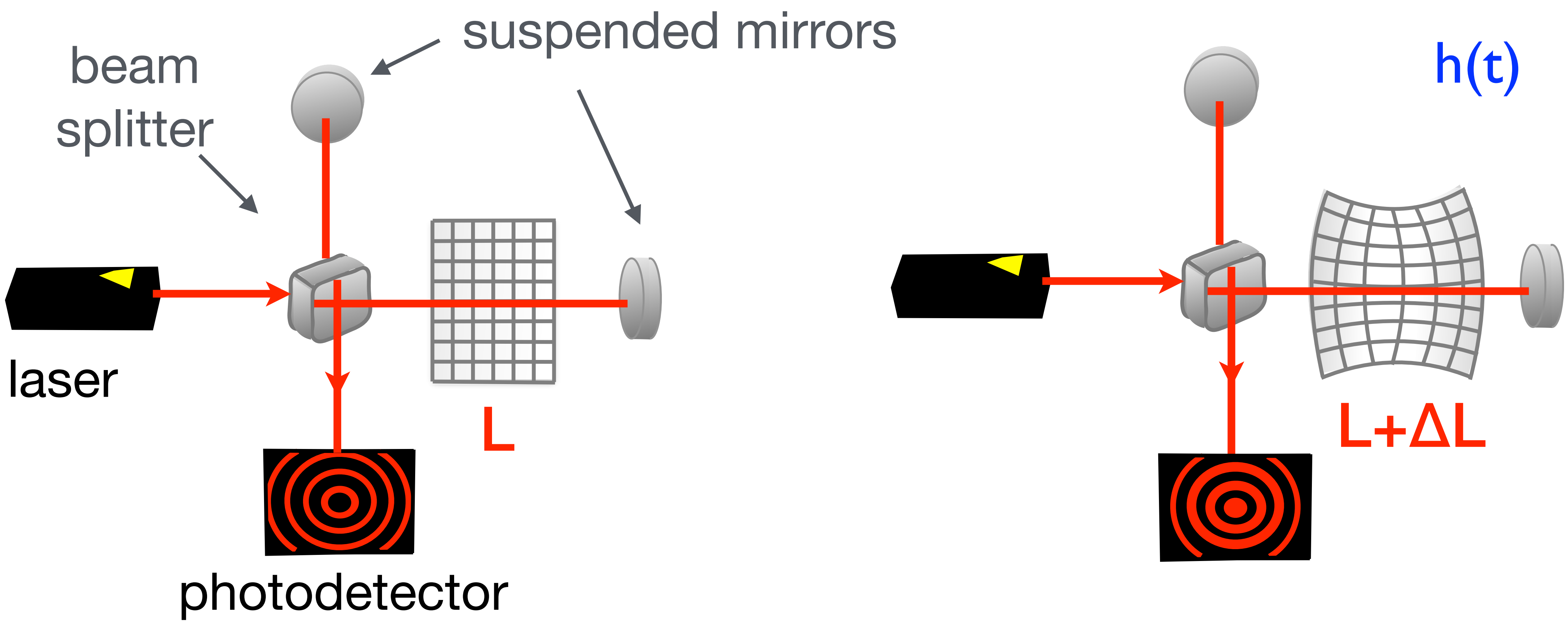
relative length
L

L+ΔL

Wave amplitude, mix of two polarizations

``strain''
$$\frac{\Delta L}{L} \approx \frac{h}{2}$$

Measuring GWs with interferometers



- Laser tracks relative travel time intervals in the arms
- change in intensity due to difference in phase:

$$\Delta\phi \sim 2\pi f \frac{2\Delta L}{c} \approx \frac{2\pi f}{c} hL$$

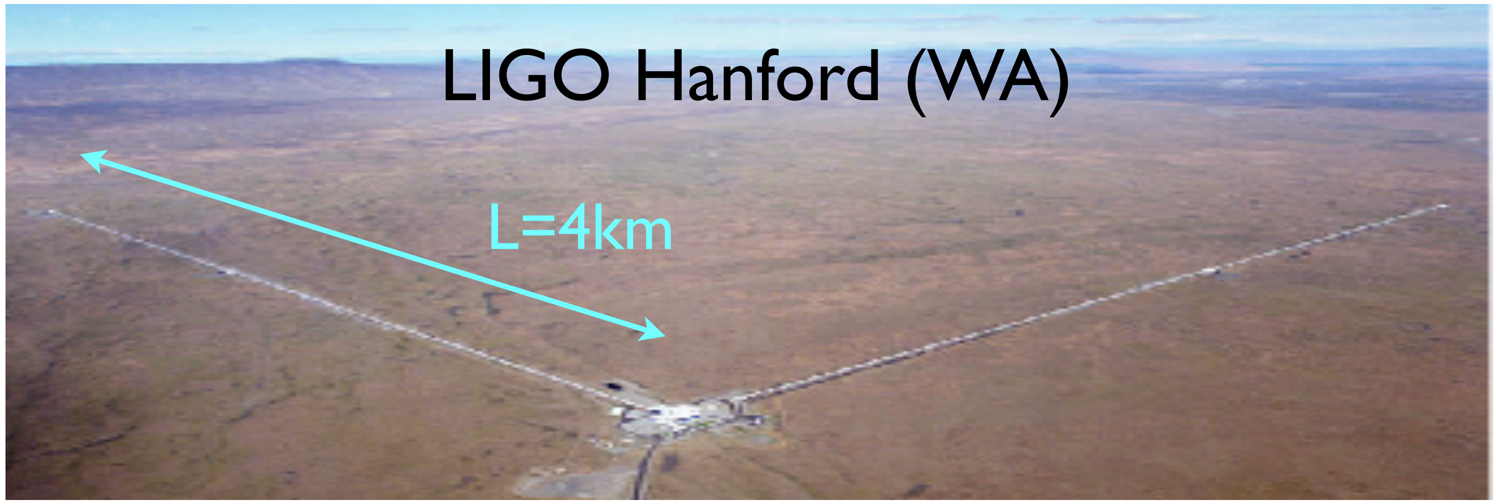
laser frequency

extra roundtrip travel time in the arm

Interferometer detectors

Accuracy: *0.02* fringes

Laser **I**nterferometer **G**W **O**bservatories (USA)



Accuracy: $\Delta L \sim 10^{-18}\ m$

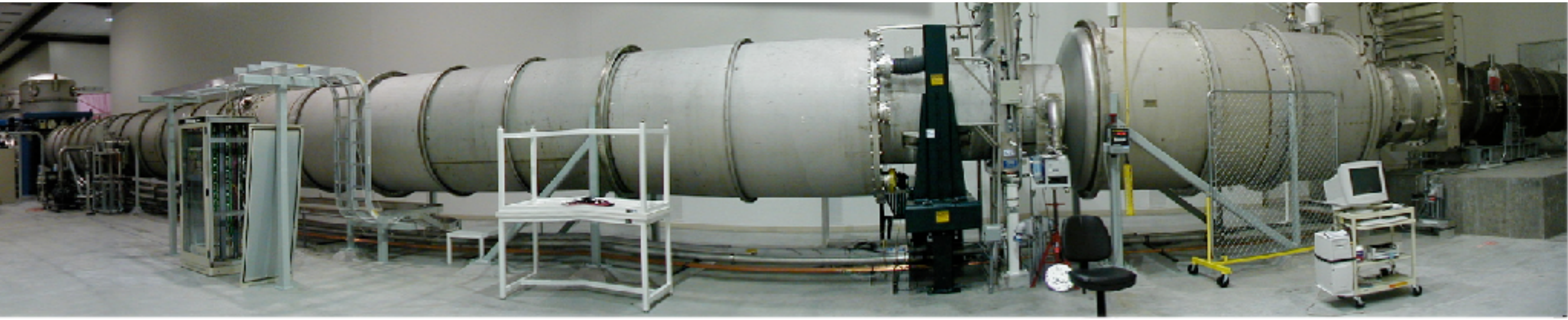
European **G**W **O**bservatory

GEO600
(Hannover, Germany)

A glimpse inside

Evacuated beam tubes

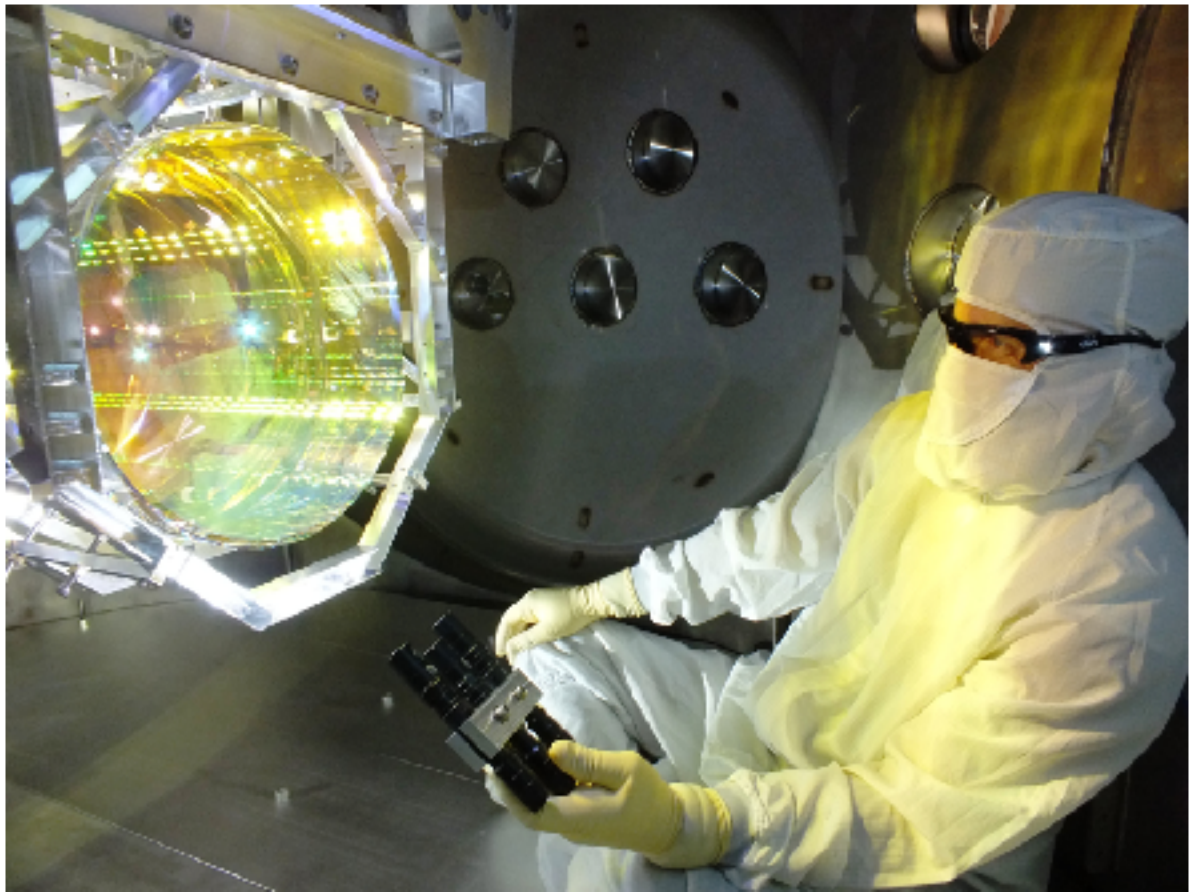
P< 10^{-9} Torr



Portable power supply for bakeout

Image credit: LIGO/ F. Raab

A glimpse inside



Livingston mirror
(40 kg)

Seismic isolation

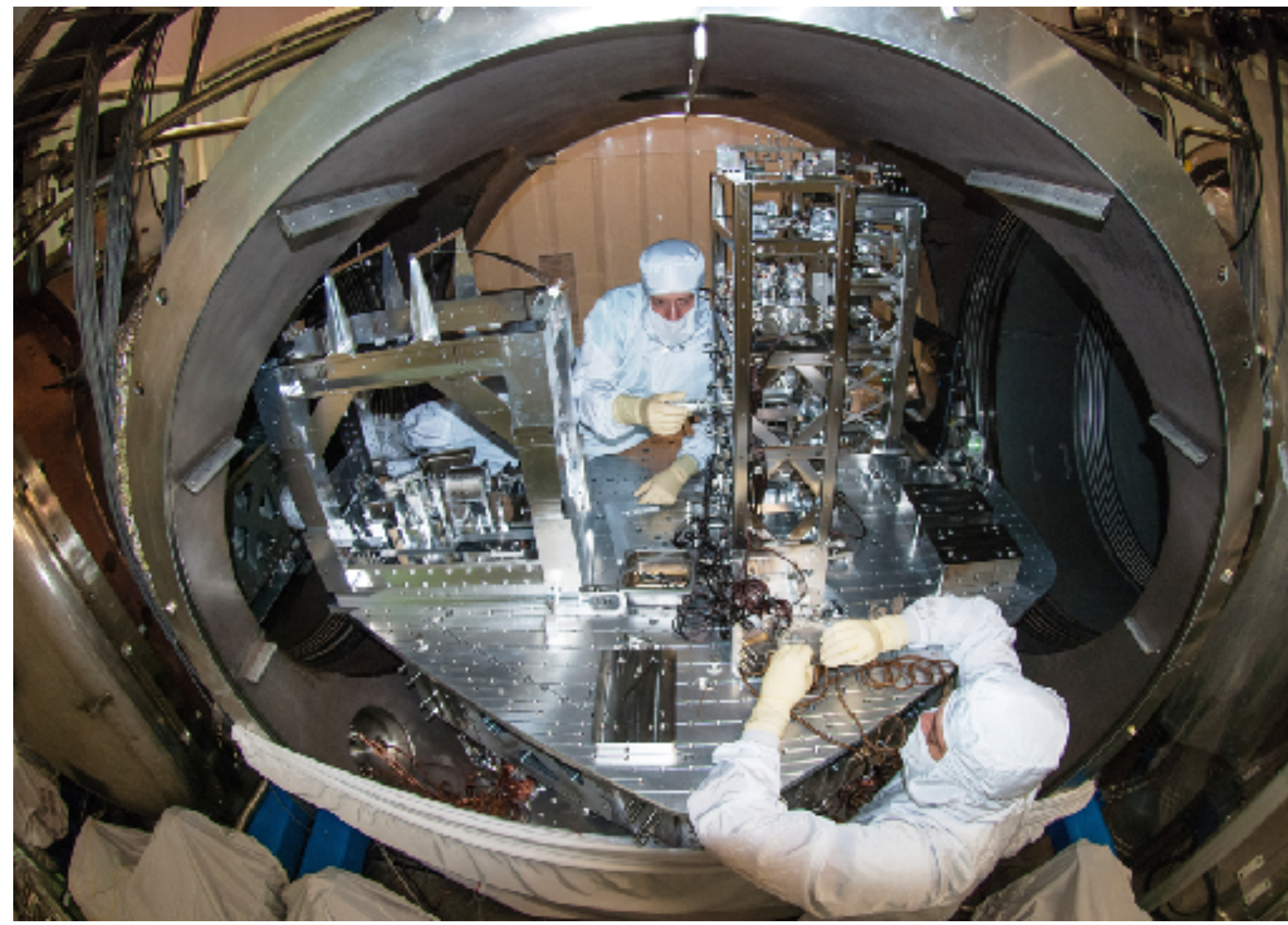
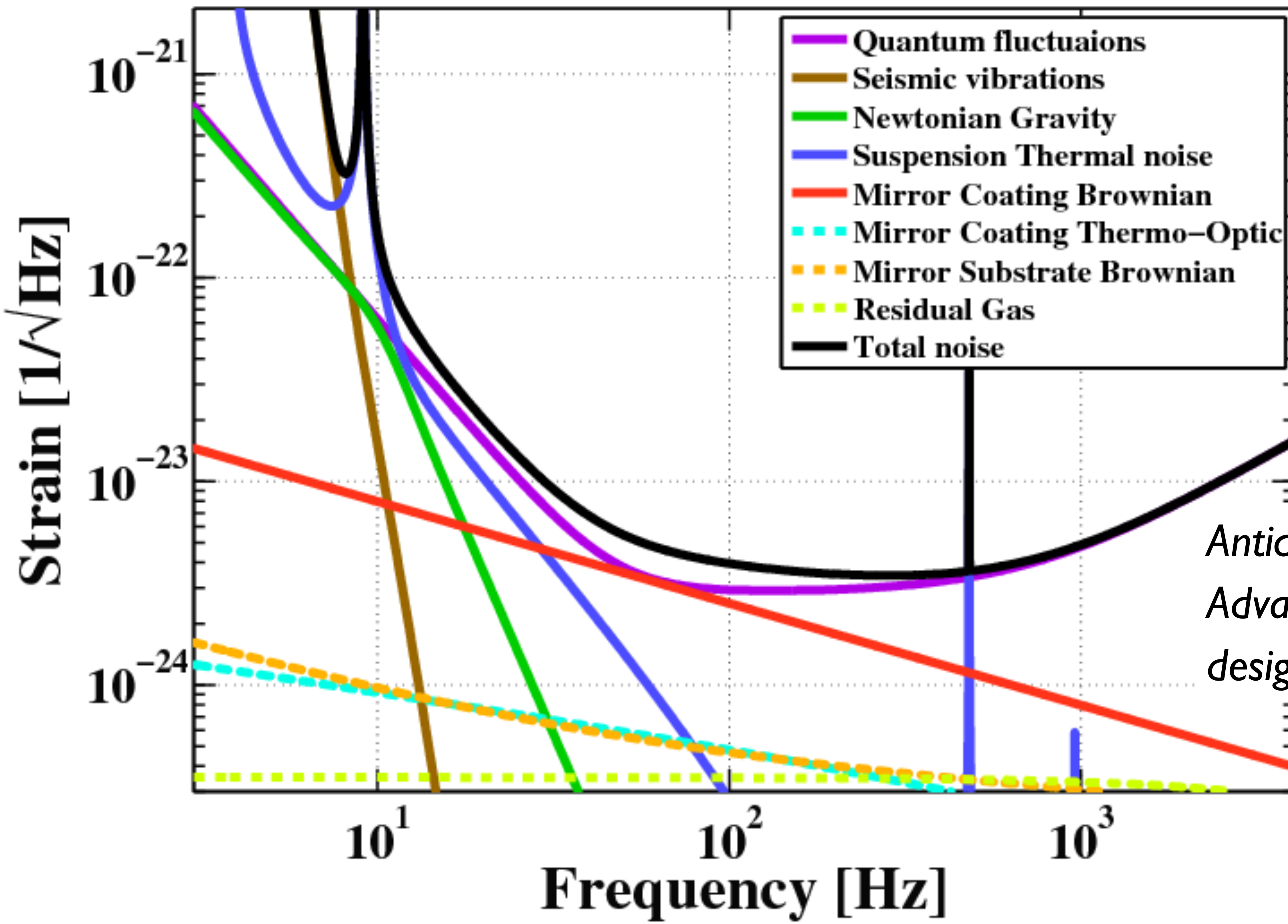


Image credit: LIGO/ F. Raab

Limitations due to various noise sources



Anticipated for Advanced LIGO design goal

Noise power spectral density example

https://www.gw-openscience.org/detector_status/day/20190903/

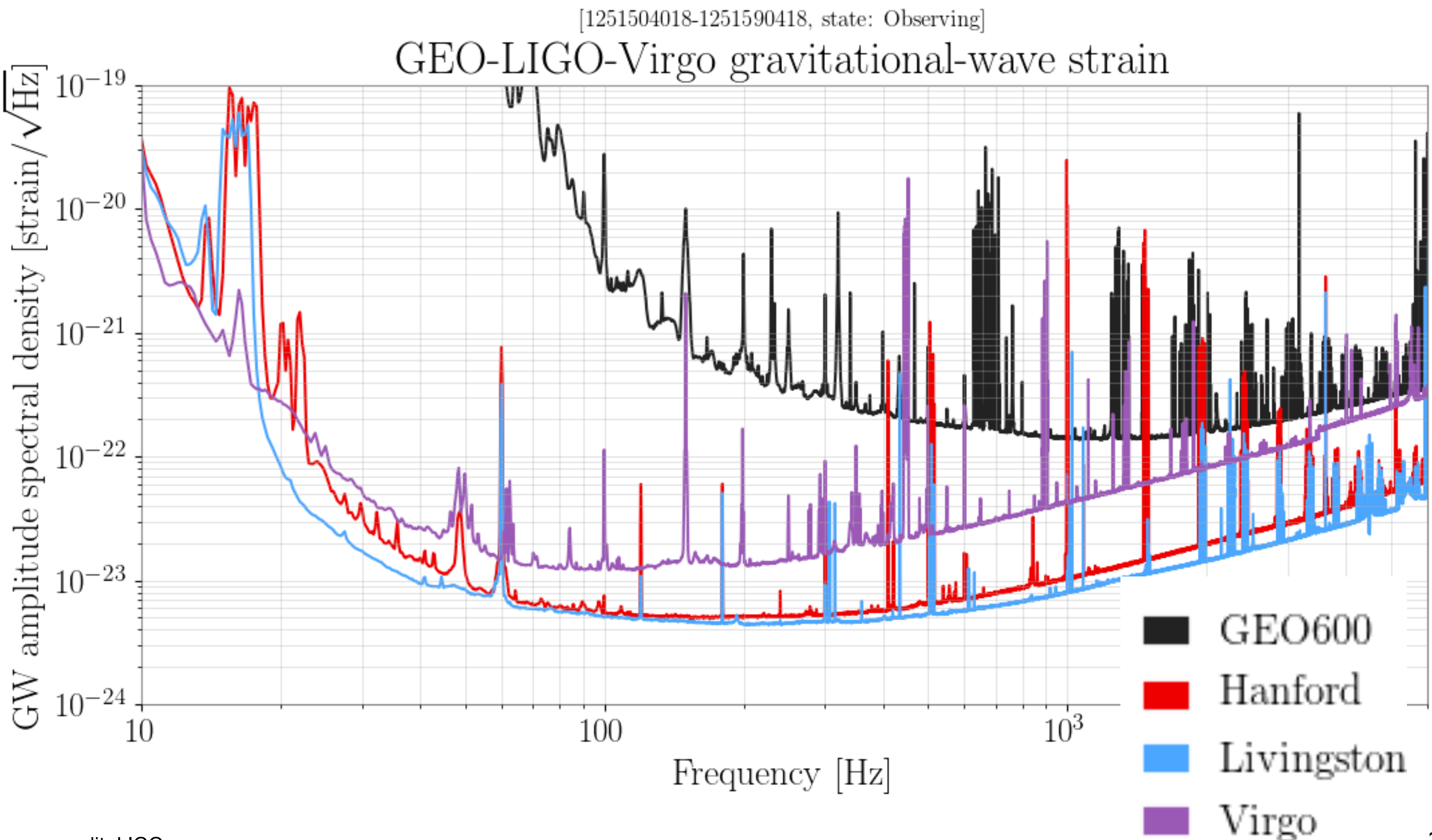
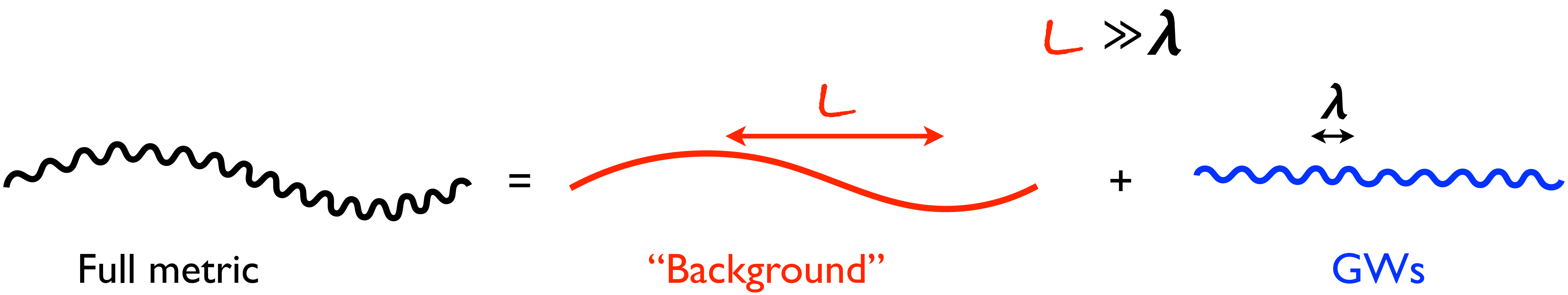


Image credit: LIGO

Energy carried by GWs

- Can define GWs more generally in curved spacetime far from the source if there is a separation of scales in (spatial or temporal) variations of curvature



GWs = rapidly varying part of spacetime curvature and metric

Similar definition used for waves in plasmas, fluids, solids

What is the energy carried by GWs?

Need to work to nonlinear order in $h_{\mu\nu}$ to see how GWs produce spacetime curvature

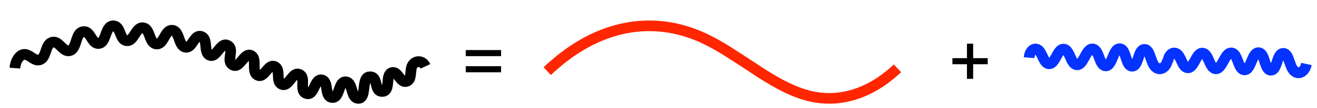
- Einstein field equations in harmonic gauge (recall the definition of h: $\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} + h^{\alpha\beta}$)

vacuum

$$\Box h^{\alpha\beta} = \cancel{\frac{16\pi G}{c^4}(-g)T^{\alpha\beta}} + \Lambda^{\alpha\beta}$$

Nonlinear terms in h & its derivatives

- Decompose:



$$\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} + \langle h^{\alpha\beta}\rangle + \hat{h}^{\alpha\beta}$$

average, e.g. over several wavelengths

- For small h: $\Box \hat{h}^{\alpha\beta} = 0 + \mathcal{O}(h^2)$ Wave solutions from before

$$\Box \langle h^{\alpha\beta}\rangle = \underbrace{\langle \Lambda[\hat{h}^{\alpha\beta}, \hat{h}^{\alpha\beta}]\rangle}_{} + O(h^3)$$

$$= \frac{c^4}{32\pi G} t^{\alpha\beta}_{\mathrm{GW}}$$

Effective stress-energy of GWs

What is the energy carried by GWs?

In TT gauge (as you will also derive using Mathematica in the next tutorial):

$$t_{\rm GW}^{\alpha\beta} = \frac{c^4}{32\pi G}\langle \eta^{\alpha\gamma}\eta^{\beta\delta}\partial_\gamma h_{\mu\nu}^{\rm TT}\partial_\delta h^{\mu\nu}_{\rm TT}\rangle$$

Energy density: $t_{\rm GW}^{00} = \dfrac{c^4}{32\pi G}\langle \partial_0 h_{ij}^{\rm TT}\partial_0 h^{ij}_{\rm TT}\rangle = \dfrac{c^2}{32\pi G}\langle \dot{h}_{ij}^{\rm TT}\dot{h}^{ij}_{\rm TT}\rangle$

(c.f. harmonic oscillator)

Energy within a volume V: $E = \displaystyle\int_V d^3x\, t_{\rm GW}^{00}$

Power: $\dfrac{dE}{dt} = c\displaystyle\int_V d^3x\,\partial_0 t_{\rm GW}^{00} = -c\int_V d^3x\,\partial_i t_{\rm GW}^{0i} = c\int_{\partial V} dA_i\, t_{\rm GW}^{0i}$

For a plane wave $t_{\rm GW}^{0i} = -t_{\rm GW}^{00}$ as $\partial_i h(t - z/c) = -\partial_0 h(t-z/c)$

What is the energy carried by GWs?

$$\frac{dE}{dt} = -c \int_{\partial V} dA \, t^{00}_{\mathrm{GW}}$$

$$h^{ij}_{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Energy in V decreases, i.e. GWs carry away a positive energy flux

$$\frac{dE_{\mathrm{GW}}}{dt} = \frac{c^3}{32\pi G} \int_{\partial V} dA \, \langle \dot{h}^{\mathrm{TT}}_{ij} \dot{h}^{\mathrm{TT}}_{ij} \rangle = \frac{c^3}{32\pi G} d^2 \int d\Omega \, \langle \dot{h}^{\mathrm{TT}}_{ij} \dot{h}^{\mathrm{TT}}_{ij} \rangle = \frac{c^3}{16\pi G} d^2 \int d\Omega \, \langle \dot{h}^2_+ + \dot{h}^2_\times \rangle$$

$dA = d^2 \sin\theta \, d\theta \, d\phi = d^2 d\Omega$ spherical shell of radius d

$$\boxed{\dot{E}_{\mathrm{GW}} = \frac{c^3}{16\pi G} d^2 \int d\Omega \, \langle \dot{h}^2_+ + \dot{h}^2_\times \rangle}$$

Summary: properties of GWs

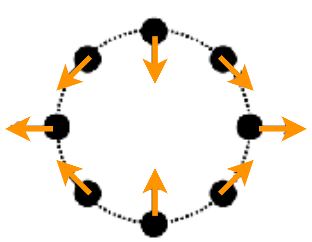
- Waves are transverse, two polarizations

$$h^{\mathrm{TT}}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

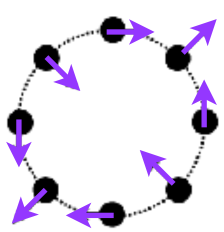
$$R_{txtx} = -R_{tyty} = -\frac{1}{2}\ddot{h}_+$$

$$R_{txty} = R_{tytx} = -\frac{1}{2}\ddot{h}_\times$$

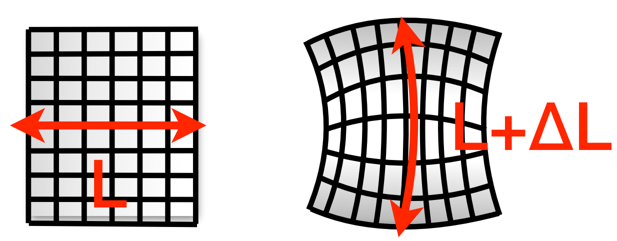
- Effect of GWs:



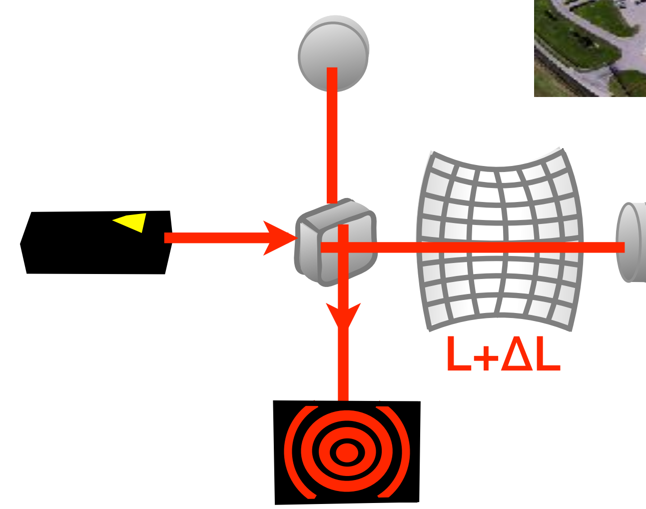
h_+



h_x



$$\boxed{\frac{\Delta L}{L} \approx \frac{h}{2}}$$



- Energy carried by GWs:

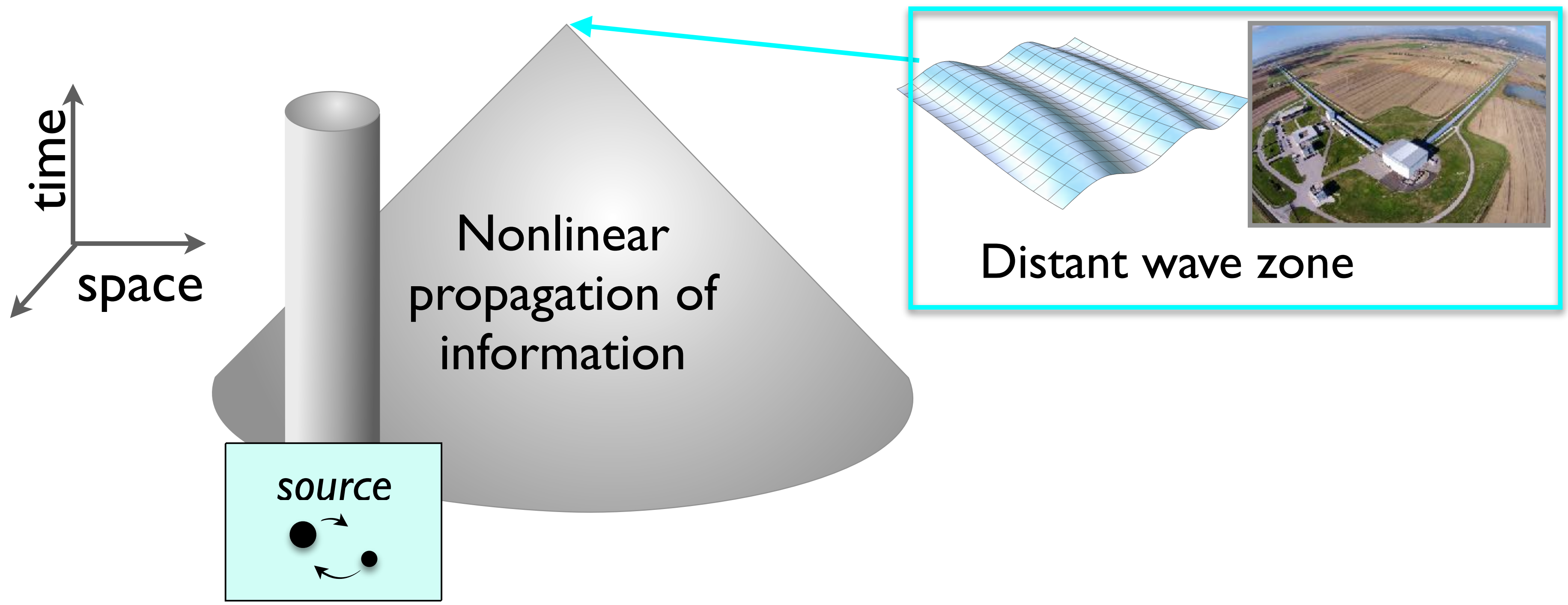
$$t^{\alpha\beta}_{\mathrm{GW}} = \frac{c^4}{32\pi G}\langle \eta^{\alpha\gamma}\eta^{\beta\delta}\partial_\gamma h^{\mathrm{TT}}_{\mu\nu}\partial_\delta h^{\mu\nu}_{\mathrm{TT}}\rangle$$

$$\dot{E}_{\mathrm{GW}} = \frac{c^3}{16\pi G} d^2 \int d\Omega\, \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle$$

2. Generation of GWs:

The Einstein quadrupole formula and application to binary systems

Information flow from source to GWs: a first approximation



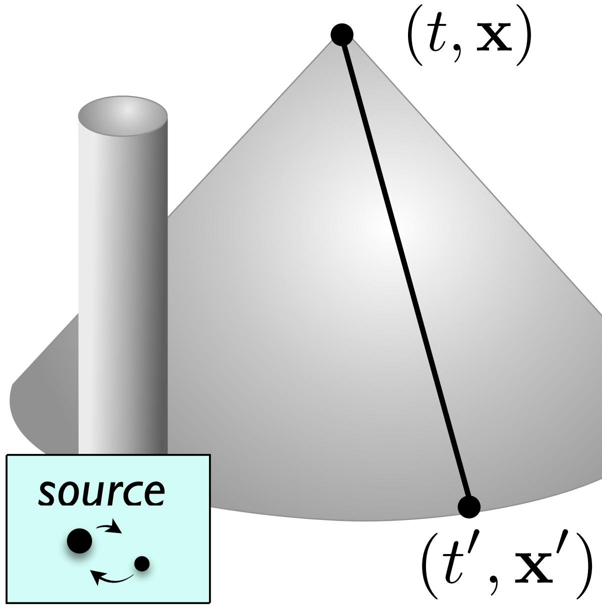
$$\Box h^{\alpha\beta} = \frac{16\pi G}{c^4}(-g)T^{\alpha\beta} + \Lambda^{\alpha\beta} = \frac{16\pi G}{c^4}\tau^{\alpha\beta} \qquad \partial_\alpha \tau^{\alpha\beta} = 0$$

Field nonlinearities

$$\approx 0 \qquad\qquad \approx \partial_\alpha T^{\alpha\beta} = 0$$

Approximate to the leading order for weak-field, slow motion (Newtonian) sources

Approximate solution to the wave equation



$$\boxed{t' = t_{\rm ret} = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}}$$

Green function: $\Box G(x, x') = -4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(x^0 - x^{0\,\prime})$

Recall: $\nabla^2 \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -4\pi\delta(\mathbf{x}-\mathbf{x}')$

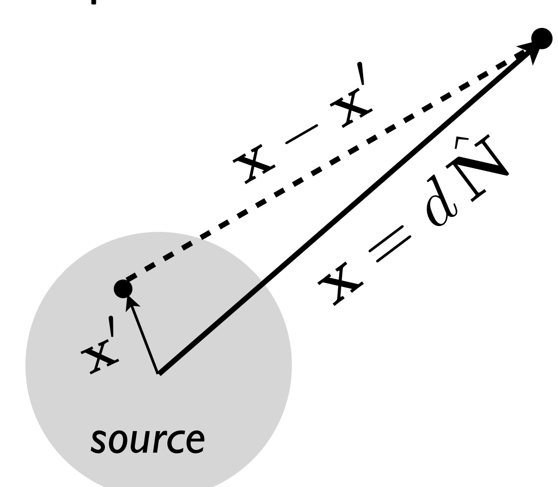
For radiation: need the retarded Green function

$$G^{\rm ret}(x - x') = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}\delta(ct_{\rm ret} - x'^{\,0})$$

Solution $h^{\alpha\beta}$: integral over the past light cone of the field point:

$$h^{\alpha\beta}(t, \mathbf{x}) \approx -\frac{4G}{c^4}\int \frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}T^{\alpha\beta}(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}')$$

Spatial relation:

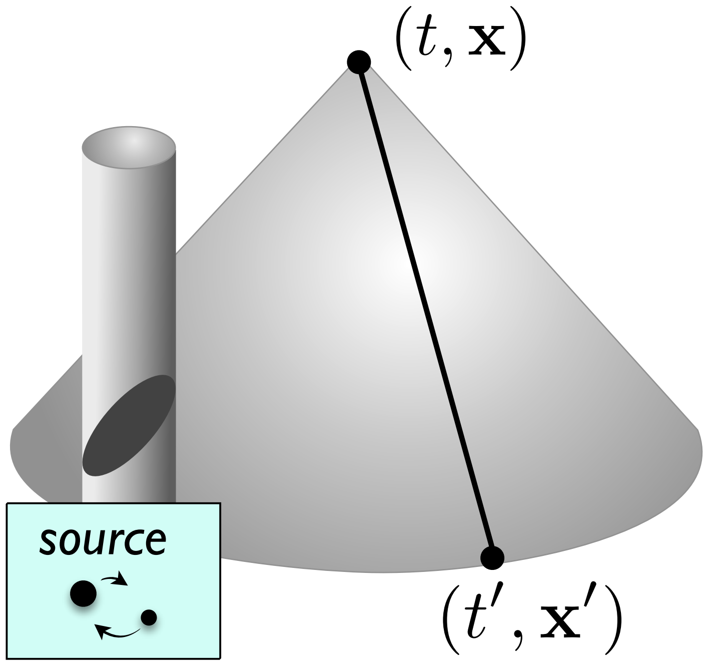


Behavior in the far field, slow motion source:

$$|\mathbf{x}-\mathbf{x}'| \approx d \qquad\qquad t - \frac{|\mathbf{x}-\mathbf{x}'|}{c} \approx t - \frac{d}{c}$$

distance to source

Approximate solution to the wave equation



Approximate radiation field in the wave zone:

$$h^{ij}(t, \mathbf{x}) \approx -\frac{4G}{c^4\, d}\int d^3x'\, T^{ij}(t - \frac{d}{c}, \mathbf{x}')$$

From manipulating stress-energy conservation, can show that *(see basics of GW theory Sec. 4.1 for a detailed derivation - we will consider a different systematic approach next lecture)*:

$$T^{ij} = \frac{1}{2}\partial_0^2(T^{00}x^i x^j) - \frac{1}{2}\partial_k\partial_l(T^{kl}x^i x^j) + \partial_k\left(T^{ik}x^j + T^{kj}x^i\right)$$

Further manipulations on the integrand lead to:

$$h^{ij} \approx -\frac{2G}{c^4}\frac{\ddot{Q}^{ij}(t_{\rm ret})}{d} \qquad Q^{ij} = \int_{\rm source} d^3x\, \rho(t, \mathbf{x})(x^i x^j - \frac{1}{3}\delta^{ij}|\mathbf{x}|^2)$$

Newtonian mass density

Einstein quadrupole formula

Radiation field: $$h_{ij}^{\mathrm{TT}} \approx \Lambda_{ijkl} \frac{2G}{c^4 d} \frac{d^2 Q^{kl}}{dt^2}\bigg|_{t-\frac{d}{c}}$$

$G/c^4 \sim 10^{-49}$ in cgs units

amplitude quadrupole formula

Einstein quadrupole formula for the radiated power:

$$\boxed{\dot{E}_{\mathrm{GW}} = \frac{G}{5c^5}\langle \dddot{Q}_{ij}\dddot{Q}^{ij}\rangle}$$

688 Sitzung der physikalisch-mathematischen Klasse vom 22. Juni 1916

Näherungsweise Integration der Feldgleichungen der Gravitation.

Von A. Einstein.

Bei der Behandlung der meisten speziellen (nicht prinz
auf dem Gebiete der Gravitationstheorie kann man sic
die $g_{\mu\nu}$ in erster Näherung zu berechnen. Dabei bed
Vorteil der imaginären Zeitvariable $x_4 = it$ aus dense
in der speziellen Relativitätstheorie. Unter »erster Nä
verstanden, daß die durch die Gleichung

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu}$$

verstanden, daß die durch die Gleichung

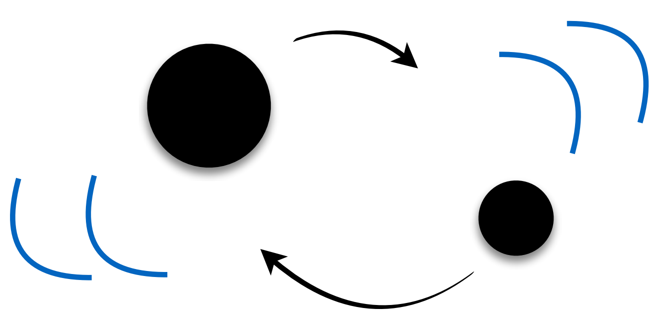
$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu}$$

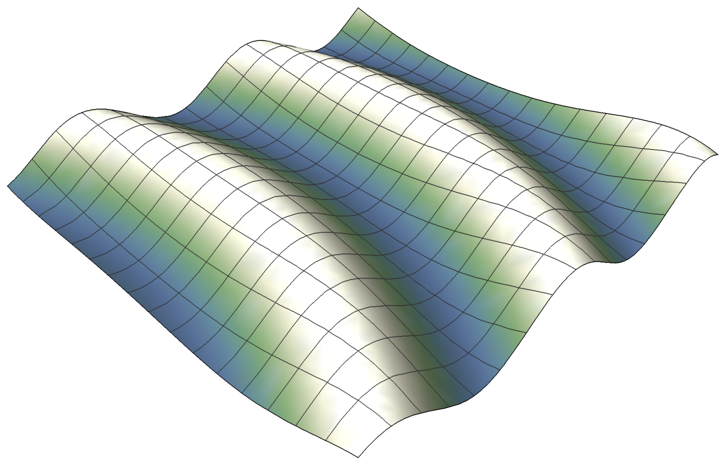
ein. Man erhält aus ihm also die Ausstrahlung A des Systems pro Zeiteinheit durch Multiplikation mit $4\pi R^2$:

$$A = \frac{\varkappa}{24\pi}\sum_{\alpha\beta}\left(\frac{\partial^3 J_{\alpha\beta}}{\partial t^3}\right)^2. \qquad (21)$$

Würde man die Zeit in Sekunden, die Energie in Erg messen, so würde zu diesem Ausdruck der Zahlenfaktor $\frac{1}{c^4}$ hinzutreten. Berücksichtigt man außerdem, daß $\varkappa = 1.87 \cdot 10^{-27}$, so sieht man, daß A in allen nur denkbaren Fällen einen praktisch verschwindenden Wert haben muß.

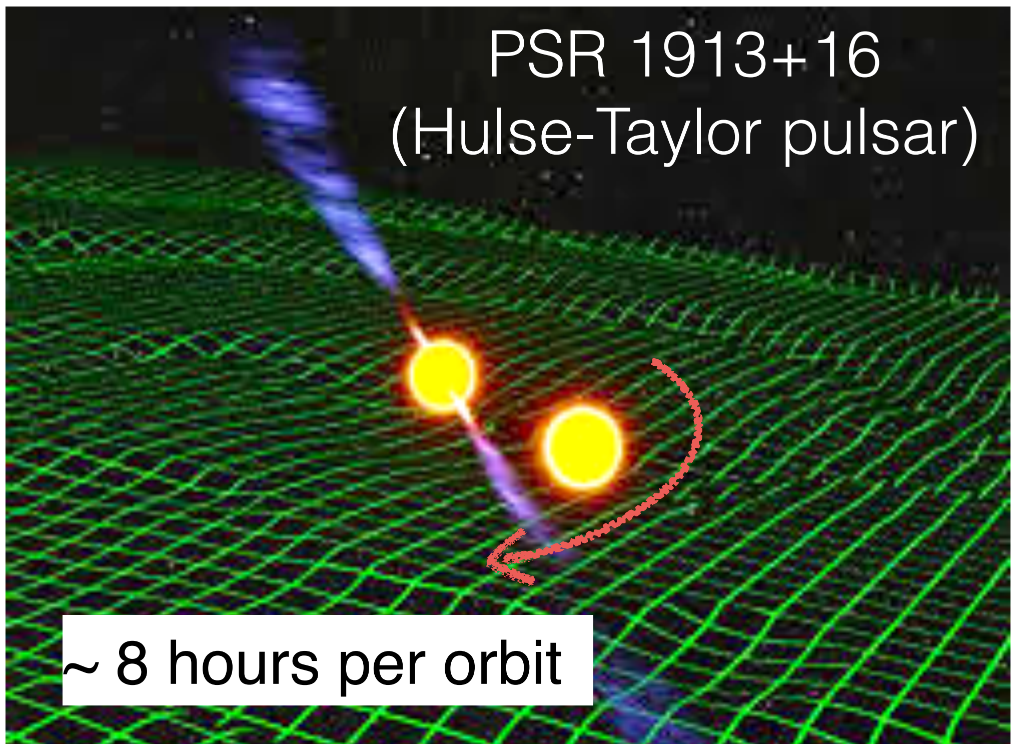
Compact-object binary systems as GW sources



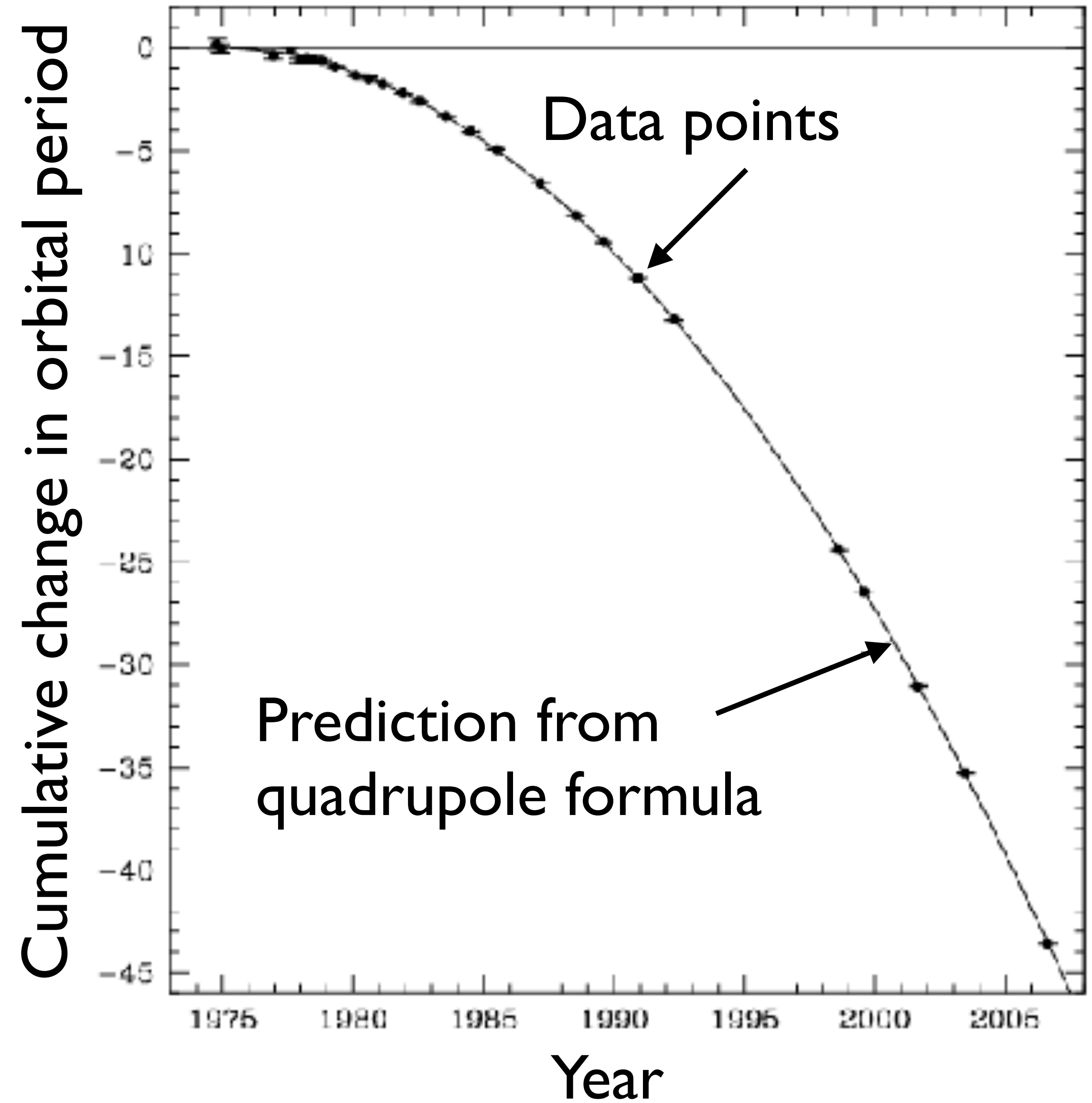


Effect of GWs on their source

- GWs carry away energy & angular momentum
- Orbital decay measured in binary pulsars



[credit: MPA]

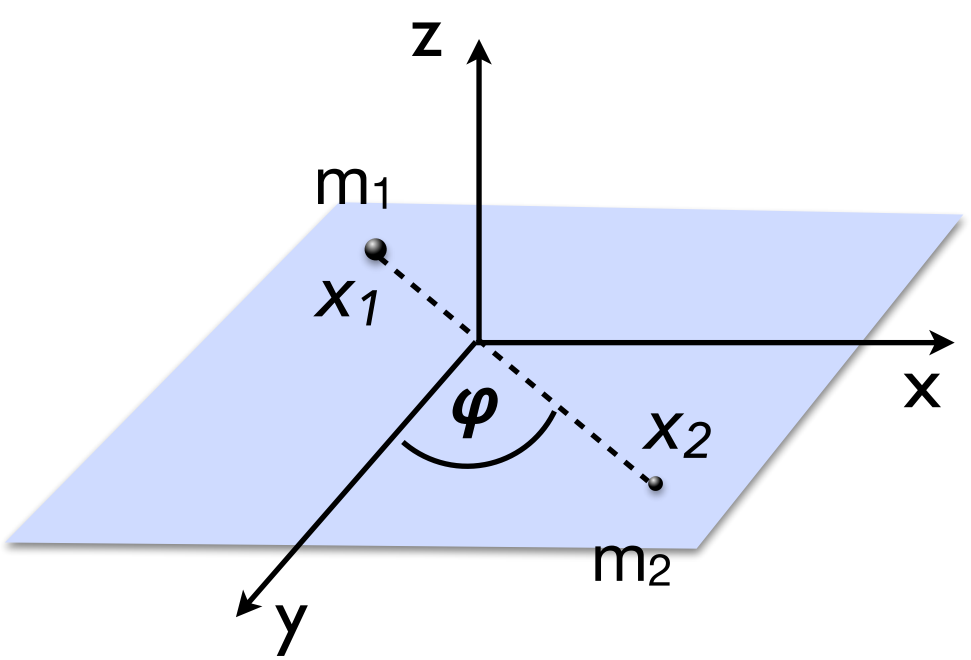


[Taylor & Weisberg 2010]

D. Kennefick: Controversies in the History of the Radiation Reaction problem in General Relativity
arXiv: gr-qc/9704002

Nobel prize 1993

Newtonian binary system (point masses, circular orbits)



- Center-of-mass frame

relative displacement

$$\mathbf{x_1} = \frac{m_2}{M}\mathbf{x} \qquad \mathbf{x_2} = -\frac{m_1}{M}\mathbf{x}$$

total mass $M = m_1 + m_2$ reduced mass $\mu = m_1 m_2 / M$

- Plane polar coordinates $x^i = (r\cos\varphi, r\sin\varphi, 0)$

relative separation

- Orbital dynamics described by the Lagrangian

$$S = \int dt \left[\frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) + \frac{G\mu M}{r}\right]$$

- Energy of circular orbits

$$\boxed{E(\omega) = -\frac{\mu c^2}{2}\left(\frac{GM\omega}{c^3}\right)^{2/3}}$$

Circular-orbit frequency

$$\omega = \dot{\varphi}$$

Newtonian binary system cont.

- Quadrupole moment: $Q^{ij} = \mu r^2 \left(n^i n^j - \frac{1}{3}\delta^{ij} \right)$ with $n^i = \frac{x^i}{r}$ (radial unit vector)

- Power radiated in GWs (from the quadrupole formula)

$$\boxed{\dot{E}_{\rm GW} = \frac{32}{5}\frac{c^5}{G}\frac{\mu^2}{M^2}\left(\frac{GM\omega}{c^3}\right)^{10/3}}$$

Energy balance: $\dot{E}_{\rm GW}$ = - (average energy loss rate from the binary)

- Manipulating $\frac{dE}{dt} = -\dot{E}_{\rm GW}$ leads to a differential equation for the frequency evolution

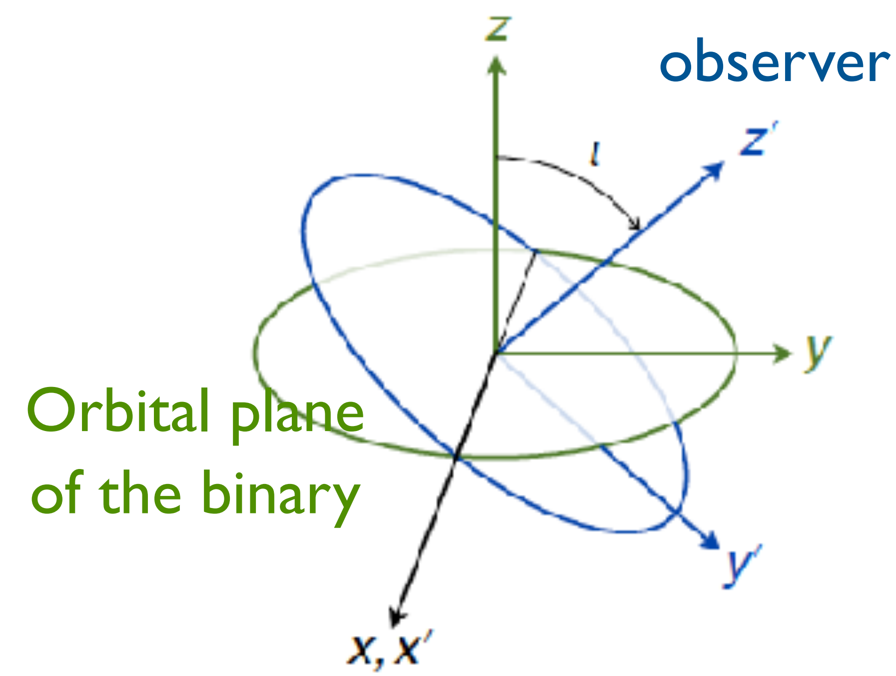
$$\frac{\dot{\omega}}{\omega^2} = \frac{96\mu}{5M}\left(\frac{GM\omega}{c^3}\right)^{5/3}$$

- The phase evolution is obtained by integrating $\varphi = \int \omega dt$

GW signal from a Newtonian binary

Plus and cross polarization amplitudes:

$$\begin{cases} h_+ \\ h_\times \end{cases} = -\frac{2G\mathcal{M}(G\mathcal{M}\omega)^{2/3}}{dc^4} \begin{cases} (1+\cos\iota^2)\cos(2\varphi) \\ 2\cos\iota\sin(2\varphi) \end{cases}$$



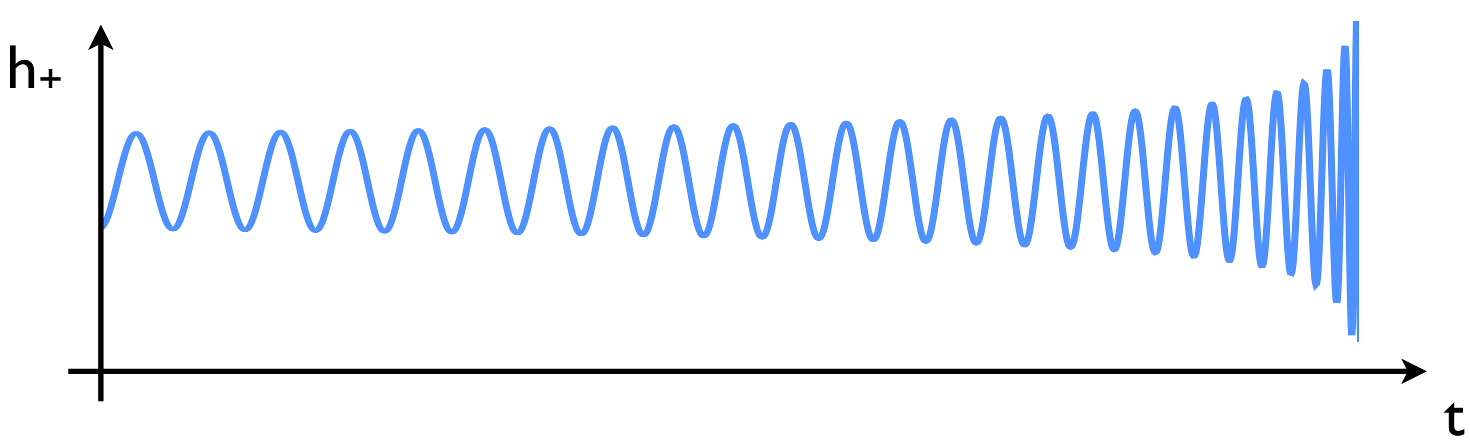
$$\boxed{\mathcal{M} = \mu^{3/5} M^{2/5} \quad \text{Chirp mass}}$$

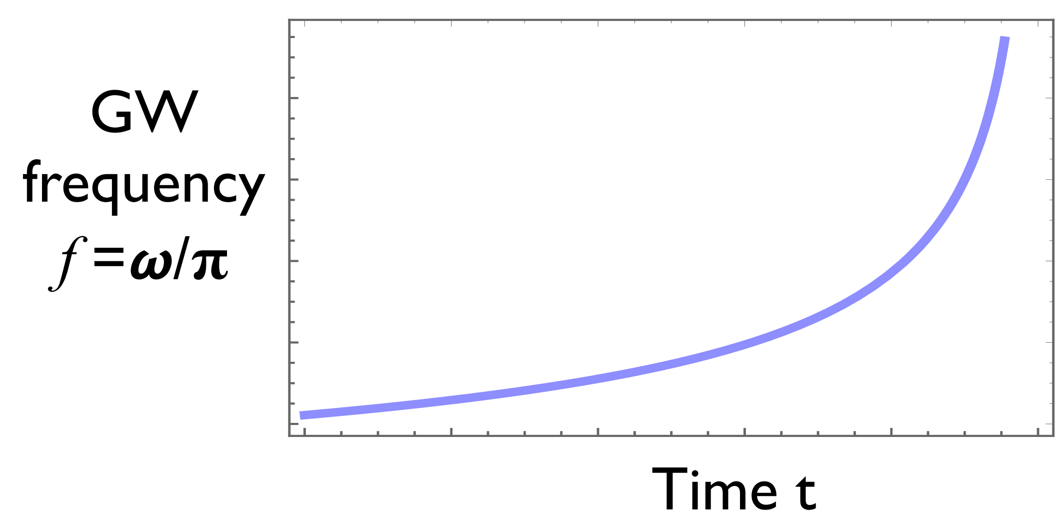
The frequency and phase evolve as

$$\omega = \frac{1}{8}\left[\frac{15}{(G\mathcal{M}/c^3)^5\,(t_c - t)^3}\right]^{1/5}$$

Coalescence time (reference)

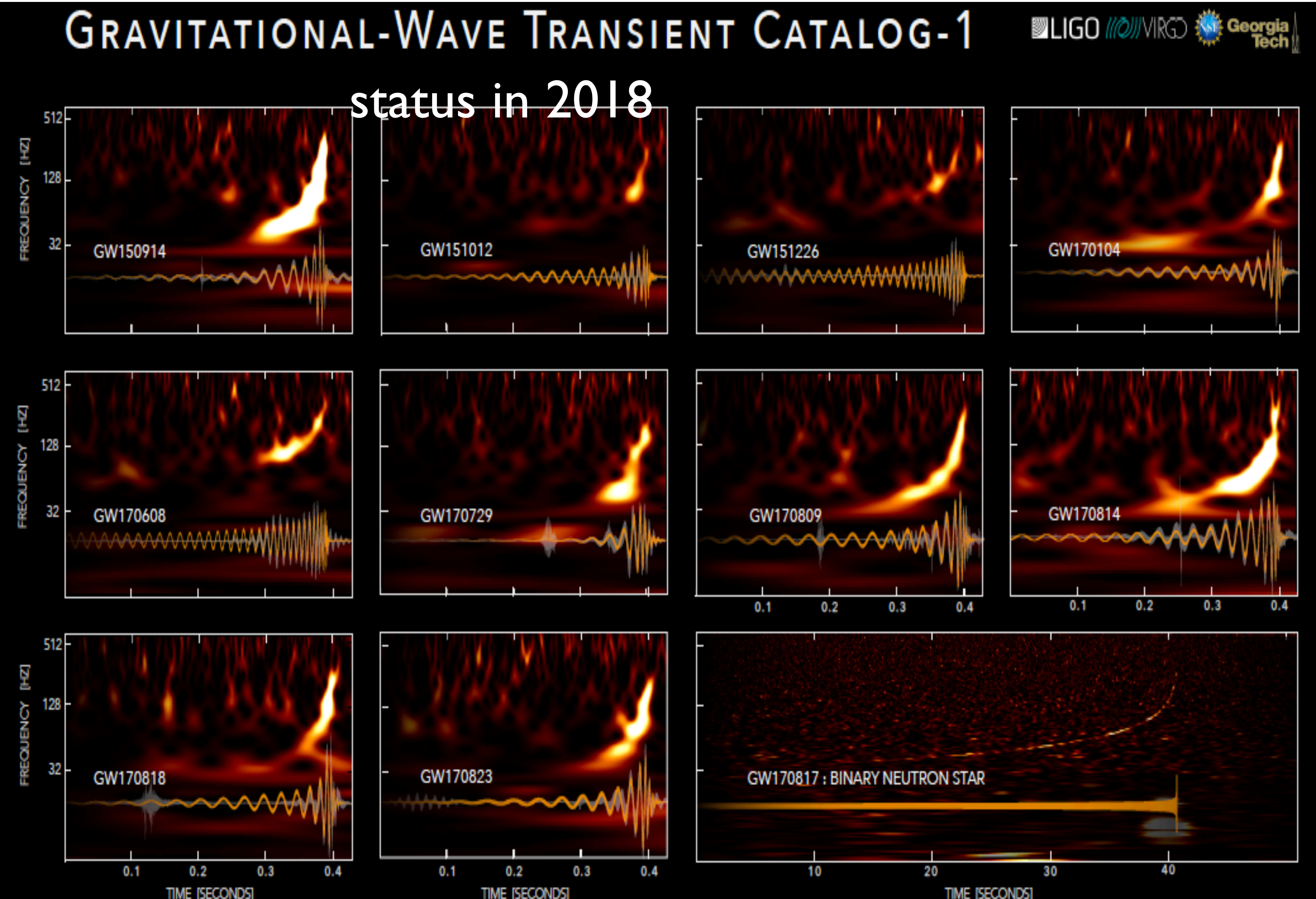
$$\varphi = \left(\frac{c^3(t_c - t)}{5G\mathcal{M}}\right)^{5/8} + \varphi_c$$



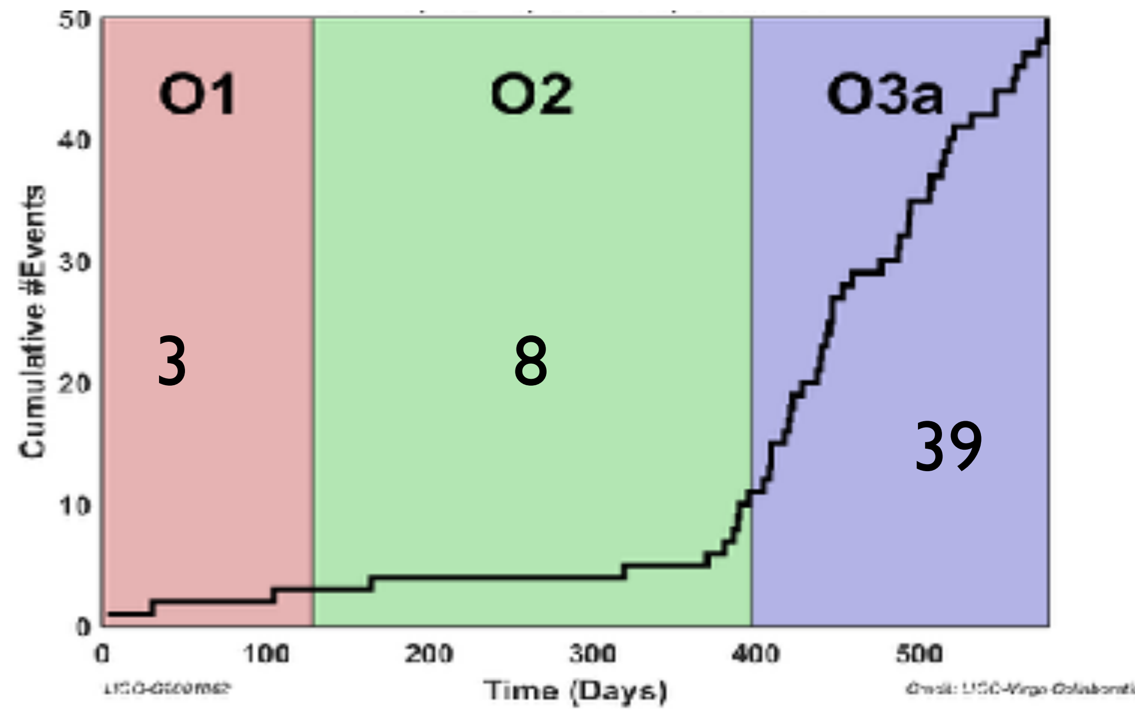


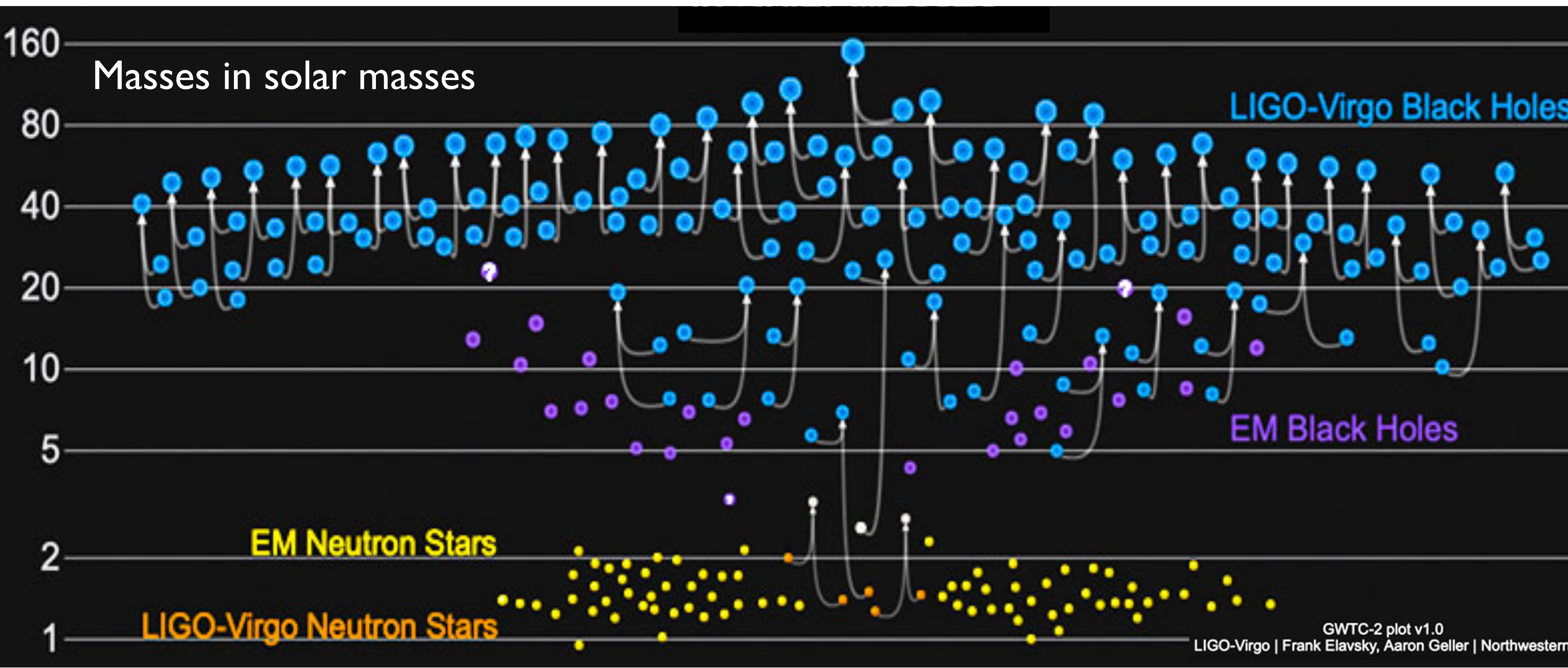
GRAVITATIONAL-WAVE TRANSIENT CATALOG-1

status in 2018

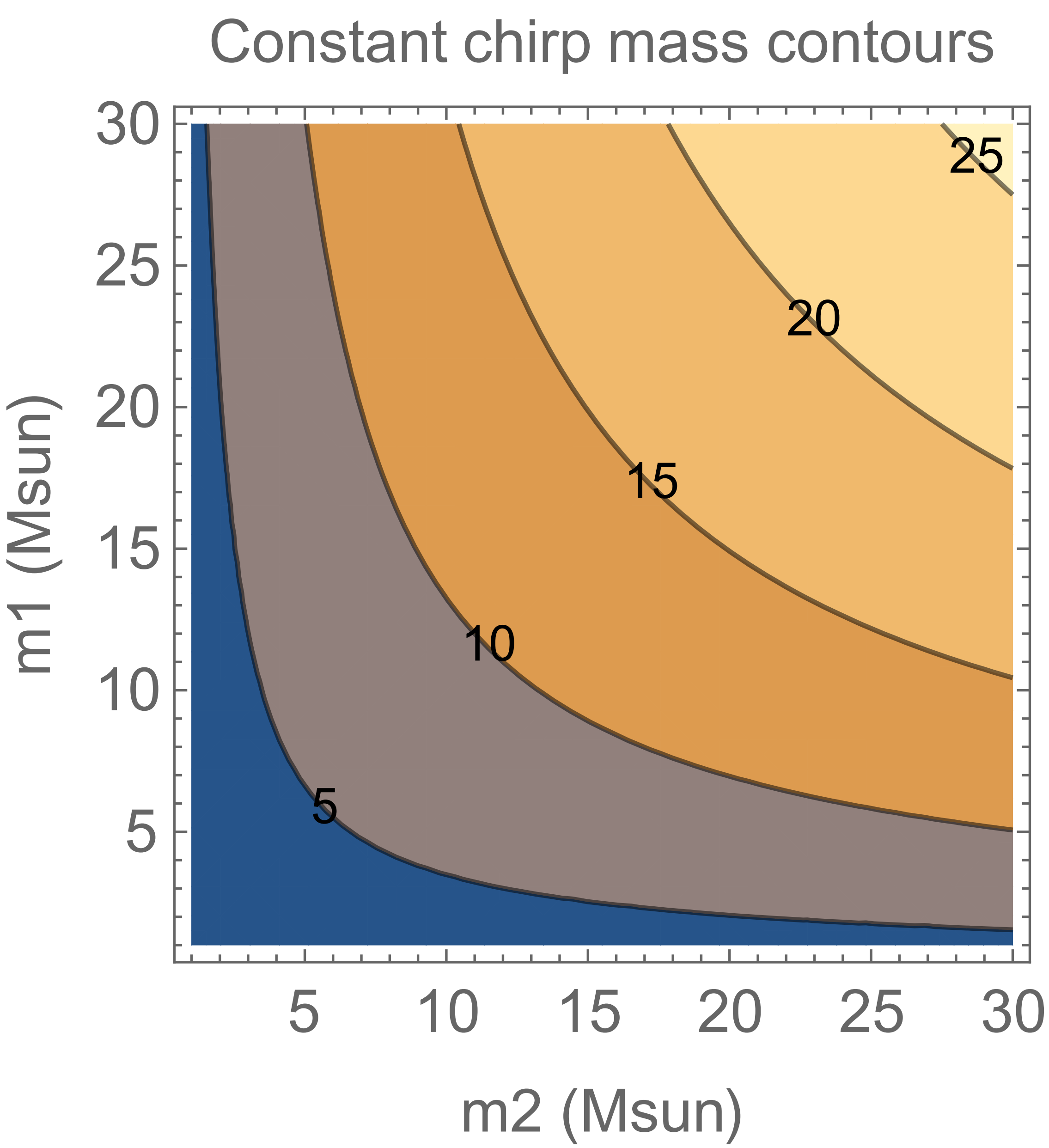


GW observations of 50 binary events published so far





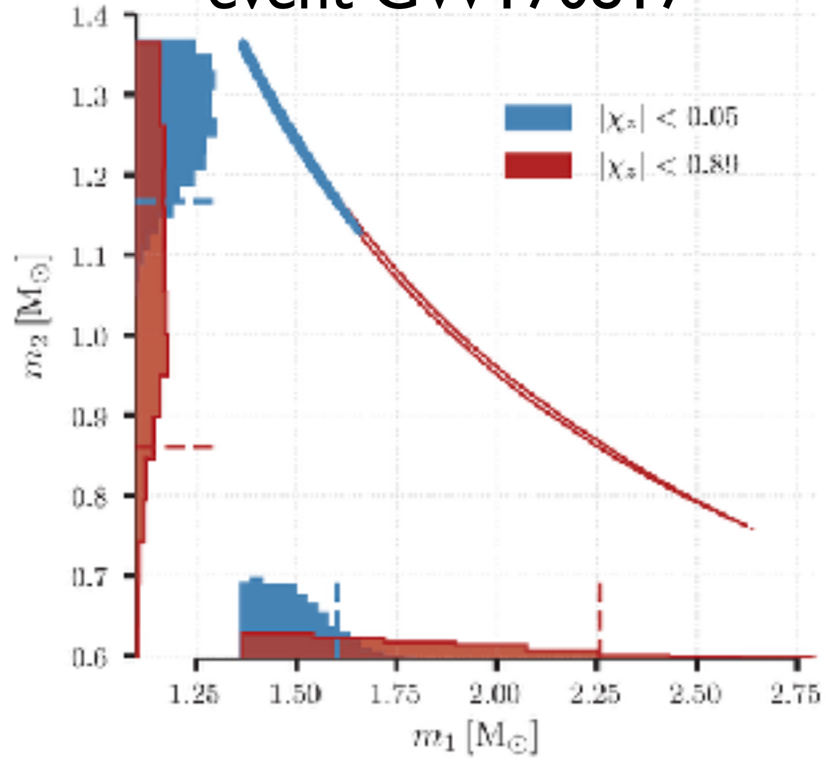
Chirp mass



$$\mathcal{M} = \mu^{3/5} M^{2/5}$$ Chirp mass

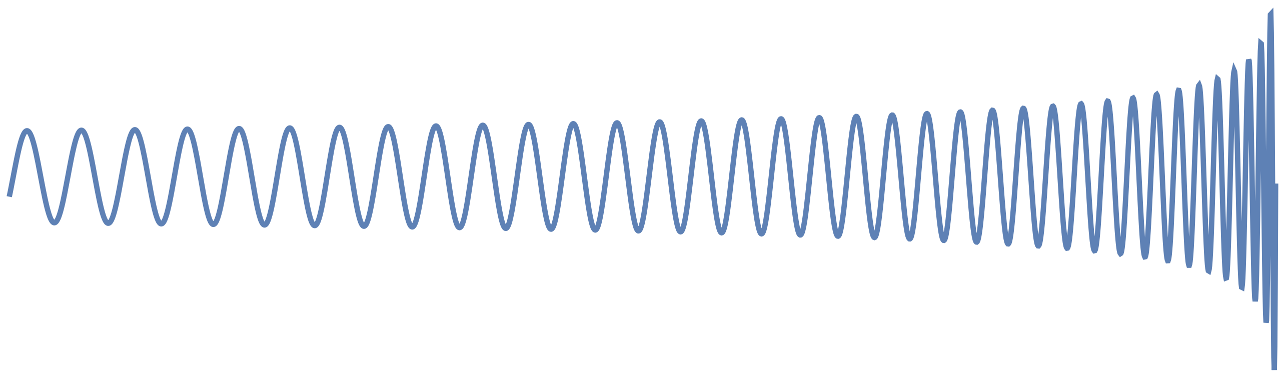
$$= \frac{m_1^{3/5} m_2^{3/5}}{(m_1 + m_2)^{1/5}}$$

Example mass posterior probability distributions for the binary neutron star event GW170817

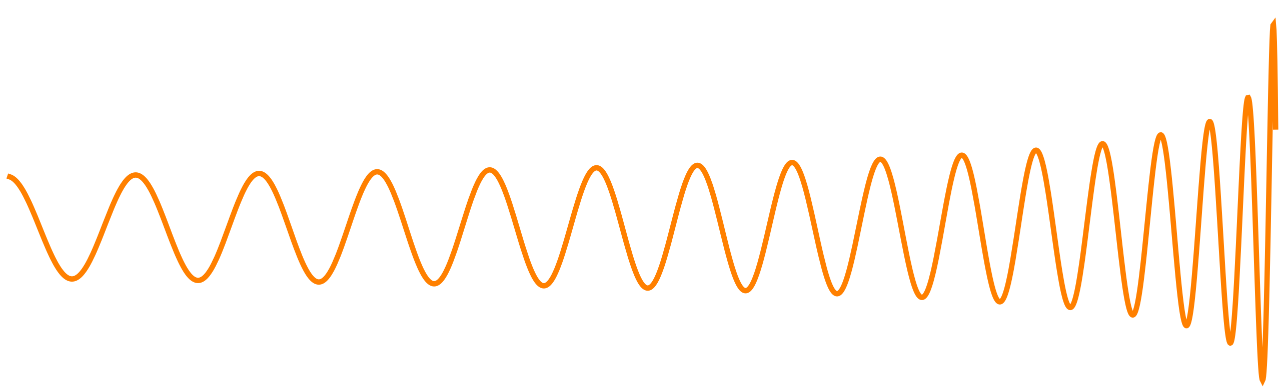


LVC: https://arxiv.org/abs/1710.05832

Effect of changing the chirp mass and distance



Fiducial reference case



Chirp mass 4x larger &
distance 4x larger

$$\begin{cases} h_+ \\ h_\times \end{cases} = -\frac{2G\mathcal{M}(G\mathcal{M}\omega)^{2/3}}{dc^4} \begin{cases} (1+\cos\iota^2)\cos(2\varphi) \\ 2\cos\iota\sin(2\varphi) \end{cases}$$

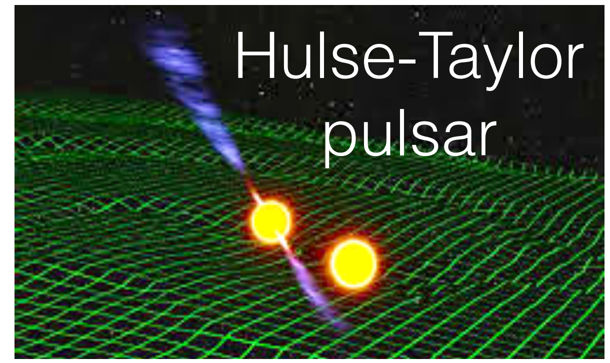
$$\varphi = \left(\frac{c^3(t_c - t)}{5G\mathcal{M}}\right)^{5/8} + \varphi_c$$

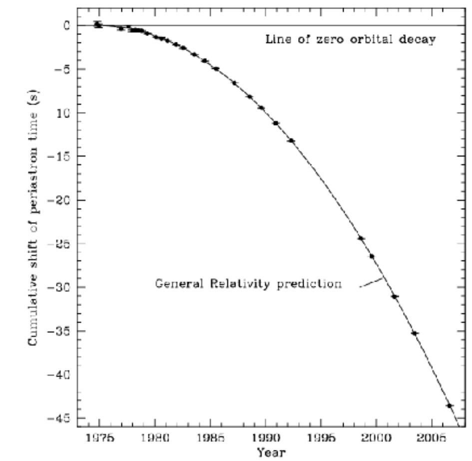
Summary: quadrupole radiation, Newtonian source

Power radiated in GWs

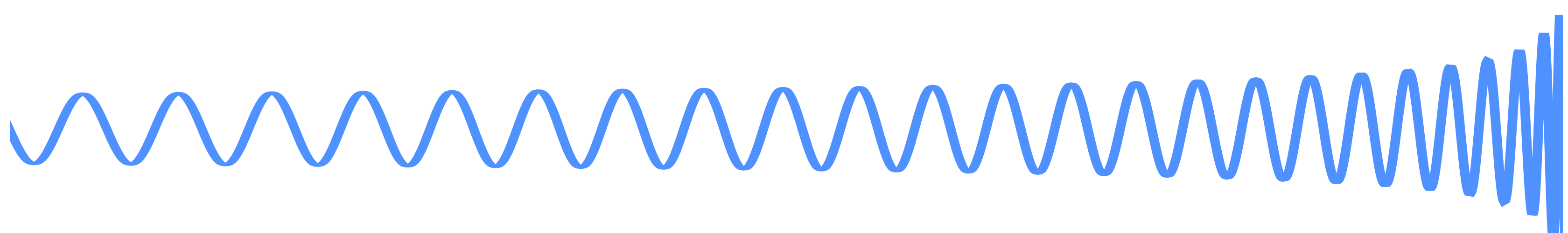
$$\dot{E}_{\mathrm{GW}} = \frac{G}{5c^5} \langle \dddot{Q}_{ij} \dddot{Q}^{ij} \rangle$$

Q_{ij}: Source quadrupole

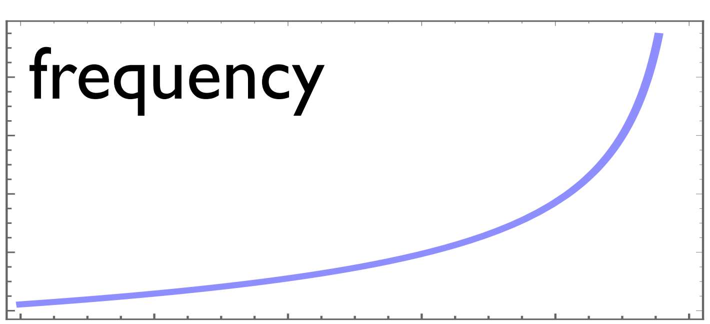




GW polarizations
$$\begin{Bmatrix} h_+ \\ h_\times \end{Bmatrix} = -\frac{2G\mathcal{M}(G\mathcal{M}\omega)^{2/3}}{dc^4} \begin{Bmatrix} (1+\cos\iota^2)\cos(2\varphi) \\ 2\cos\iota\sin(2\varphi) \end{Bmatrix}$$

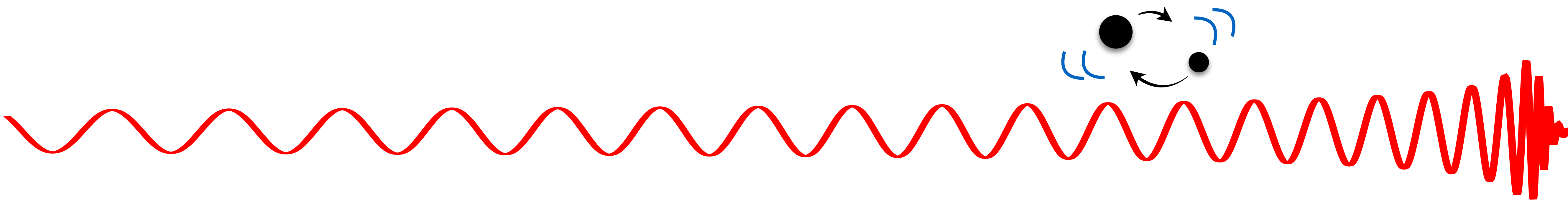


Chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$



$$\omega = \frac{1}{8}\left[\frac{15}{(G\mathcal{M}/c^3)^5\,(t_c - t)^3}\right]^{1/5}$$

Recall: general information



No class Apr 26 (lecture-free week at UU), May 3 (UvA holiday), May 24 (Pentecost)

Exam: June 7

https://web.science.uu.nl/drstp/DeltaITP/ATTP_current_spring_2021.html

Homeworks: due before the next lecture, please scan (e.g. with an app, not just a photo), email in a single pdf to Gastón (g.f.crecikeinbaum@uu.nl)

Tutorials: after lecture and lunch break at 13:45, see zoom link in the Teams channel

Materials will be posted under Files in the Teams channel (e.g. week1 folder)