Homework II
due May 31 before 11 am
please email a scan of your work in a single pdf file to g.f.crecikeinbaum@uu.nl

Relativistic corrections to binary dynamics and useful mathematical tools

References:

1. William Burke: Gravitational Radiation Damping of Slowly Moving Systems Calculated Using Matched Asymptotic Expansions, Journal of Mathematical Physics 12, 401 (1971) https://aip.scitation.org/doi/10.1063/1.1665603

2. T. Hartmann, M. Soffel, T. Kioustelidis: On the use of STF tensors in celestial mechanics, Celestial Mechanics Dynamical Astronomy 60, 139 (1994) https://link.springer.com/article/10.1007/BF00693097

3. C. Cutler and E. Flanagan, Gravitational waves from merging compact binaries: How accurately can one extract the binary's parameters from the inspiral waveform?, Phys. Rev. D. 49, 2658 (1994) https://arxiv.org/abs/gr-qc/9402014

1. Matched Asymptotic Expansions [10 pts]

The method of matched asymptotic expansions is a powerful tool for determining systematic approximations in a wide variety of problems, e.g. involving boundary layers, shocks, or when different physics dominates in different regimes. For GWs from inspiraling binaries, different physics dominates e.g. at scales of each compact object, the orbital scale, and the distant wave zone, and also on different timescales encountered during the evolution. This makes the method of matched asymptotic expansions highly useful in this context. To become familiar with this methodology, we will consider an example described by the differential equation

$$\epsilon y''(x) + y'(x) + 3y(x) = 0, \tag{1}$$

where $\epsilon \ll 1$ is a small parameter. We will consider the solution in the interval $0 \leq x \leq 1$ and for the boundary conditions $y(0) = 0$ and $y(1) = 1$. The fact that the character of the differential equation (1) changes from being first to second order when including $O(\epsilon)$ corrections is an indication that this is a so-called singular perturbation problem, meaning that a straightforward perturbative expansion in ϵ would be unable to capture the behavior of the solution everywhere. There are a number of reasons why a singularity can occur, e.g. in the examples discussed in lecture it was because of a change in the scalings of the terms in different regimes, yet the approach to obtaining approximate solutions to such problems is the same.

(a) **Outer expansion**: Start by making the ansatz $y = \sum_k \epsilon^k y_k(x) = y_0(x) + \epsilon y_1(x) + O(\epsilon^2)$ with the coefficients y_0,y_1 to be determined. This ansatz implies that you are considering the expansion of the solution in the limit $\epsilon \to 0$ at fixed x. Solve for the leading-order term $y_0(x)$, taking into account the outer boundary condition at $x = 1$. Notice that this solution cannot satisfy the boundary condition at $x = 0$, which indicates that there is an inner regime at small $x = Ø(\epsilon)$ where the solution has a different behavior.

(b) **Inner expansion**: For describing the solution in the inner domain, introduce a scaled coordinate $X = x/\epsilon$, which is $O(1)$ for $x = O(\epsilon)$. Write an ansatz for the asymptotic expansion of the inner solution at fixed inner coordinate X. You can again use a power series in ϵ but now of the form $y = \sum_k \epsilon^k Y_k(X) = Y_0(X) + \epsilon Y_1(X) + O(\epsilon^2)$ with some other coefficients Y_k. Solve for the leading-order term $Y_0(X)$, taking into account the inner boundary condition. You should find that one constant remains undetermined, which will be fixed by the matching.

(c) **Matching in the overlap domain**: Consider now an intermediate domain within the regimes of validity of both approximations. This regime corresponds to the limit of small x in the outer expansion and of large X in the inner expansion. To formalize the matching procedure, introduce an intermediate matching coordinate $x_\eta = x/\eta(\epsilon)$ with $\epsilon \ll \eta \ll 1$ (e.g. $\eta = \epsilon^\gamma$ for some range of powers within the interval $0 < \gamma < 1$, where the exact range can only be determined by extending the matching to higher orders, which we will not consider here). The scaling is chosen such that coordinate is $O(1)$ in the intermediate region.
Express both the inner and outer expansions in terms of x_η.
Match the dominant terms in the limit $\epsilon \to 0$ at fixed x_η and use this matching to determine the unknown coefficient in the inner solution. Concretely, 'dominant terms' means that you can neglect contributions from exponentials of large negative numbers or small corrections to exponents.

(d) **Composite expansion**: Use the above results to construct the composite expansion, expressed in the unscaled coordinate, by adding the inner and outer expansions and subtracting the common term that is included in both expansions (as identified from the matching) and would otherwise be double-counted.

For reference, an illustration of these results is shown in Fig. 1.

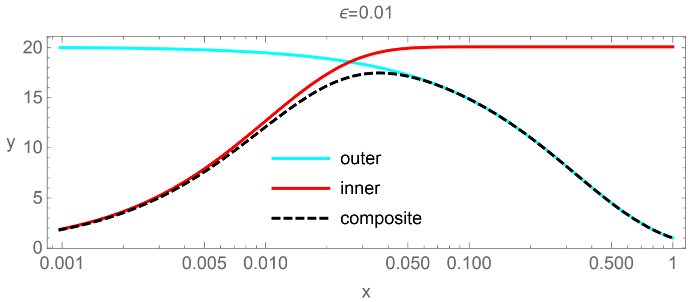

Figure 1: Plot of the leading-order asymptotic expansions of the solutions to the differential equation

2. First-order relativistic effects on the binding energy for circular orbits [5 pts]

   In this exercise, you will practice using perturbative expansions to invert a relation between quantities, which frequently occur in many contexts. You will apply this to the first post-Newtonian order (1PN) corrections to the circular-orbit dynamics of a binary system and compute the effect of these relativistic corrections on the binding energy. To the first post-Newtonian order (1PN), the equations of motion imply that the angular frequency of a circular orbit is related to its radius by (within the harmonic gauge we are considering in this exercise)

$$\omega^2 = \frac{GM}{r^3} + \frac{G^2M^2(\nu-3)}{c^2r^4}. \tag{2}$$

   Here, $\nu = \mu/M$ is the symmetric mass ratio, where M and μ are the total and reduced mass of the binary respectively. The 1PN terms come with the factor of c^{-2}, which, as mentioned in the lecture, is used as an expansion parameter, i.e. one considers the formal limit that the speed of light is very large $c^{-1} \ll 1$ or equivalently $c \to \inf$. We are omitting the corrections at higher PN orders $O(c^{-4})$ and higher throughout this exercise and do not indicate them here. The energy of a circular orbit is

$$E = -\frac{G\mu M}{2r}\left[1 + \frac{GM}{rc^2}\left(\frac{\nu}{4} - \frac{7}{4}\right)\right]. \tag{3}$$

   - Compute $r(\omega)$ perturbatively to $O(c^{-2})$. To do this, start from the ansatz

$$r = \sum_k \frac{1}{c^k} r_k(\omega) = r_0\left(1 + \frac{1}{c^2}r_1\right) + \ldots, \tag{4}$$

     where in the second part we have factored out the Newtonian r_0-terms and labeled r_1 to be the 1PN fractional correction. Substitute this ansatz into (2), and expand the right hand side for $c^{-1} \ll 1$, including terms of $O(c^{-2})$ but not higher. Then equate terms with the same powers of c on the left and right hand sides to solve for r_0 and r_1.
   - Compute the binding energy (3) as a function of ω to $O(c^{-2})$. Note that the relativistic corrections enter with a higher power of the orbital frequency, meaning that they become more important later in the inspiral when the motion is faster. It is common in the literature to define a PN parameter

$$x = \left(\frac{GM\omega}{c^2}\right)^{2/3} \tag{5}$$

     which is just a widely-adopted notation and has nothing to do with the spatial coordinate x. Use this quantity to write the energy in a more compact form.
   - What is the effect of the relativistic corrections, i.e. do they increase or decrease the gravitational binding?

3. Symmetric trace-free (STF) tensors [12 pts]

   In lecture, we discussed symmetric trace-free tensors and their relation to spherical harmonics. They are equivalent representations that arise e.g. in solutions to the Laplacian. The spherical harmonic representation is useful in problems with spherical symmetry, e.g. a perturbed compact object with small spin. For describing other systems such as a binary, the Cartesian STF tensors enable a much more compact formulation with greater transparency. It is thus very useful to be adept at working with both representations and translating between them. In the discussion of STF tensors we used the notation L as a shorthand for ℓ spatial tensor indices, e.g. $x^L = x^i x^j \ldots x^\ell$. For the derivatives $\partial_L = \partial_{x^1}\partial_{x^2}\ldots\partial_{x^\ell}$ with $\partial_{x^i} = \partial/\partial x^i$. We denote the STF operation by angular brackets around the indices $< \ldots >$.

   n this exercise, you will first work out how STF tensors naturally arise from derivatives of $1/r$. By induction, one can use this to obtain a general formula for any number of derivatives. This formula also requires constructing the STF part of an arbitrary number of unit vectors, for which there is a general prescription. This expression may seem rather unwieldy and complicated at first, but in the optional bonus question you will go through an example of how to use this formula and then work out a more complicated case yourself. In the second part of this exercise you will work out the conversion between components of a tensor on the spherical harmonic and STF basis.

- **STF tensors and derivatives of** $1/r$

  (a) Write r and a unit vector n^i as

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\delta_{ij} x^i x^j}, \qquad n^i = \frac{x^i}{r}, \tag{6}$$

  and note that $\partial_j x^k = \delta_{jk}$ to derive the following identities:

$$\begin{aligned} \partial_j r &= n_j && (7a)\\ \partial_j n_k &= \frac{1}{r}\left(\delta_{jk} - n_j n_k\right) && (7b)\\ E_j = \partial_j \frac{1}{r} &= -\frac{n_j}{r^2}, && (7c)\\ E_{jk} = \partial_j \partial_k \frac{1}{r} &= \frac{1}{r^3}(3 n_j n_k - \delta_{jk}), && (7d)\\ E_{jkl} = \partial_j \partial_k \partial_\ell \frac{1}{r} &= -\frac{1}{r^4}\left[15 n_j n_k n_\ell - 3(n_j \delta_{k\ell} + n_k \delta_{j\ell} + n_\ell \delta_{jk})\right]. && (7e) \end{aligned}$$

  (b) Observe that the results on the right hand sides above are all symmetric under an exchange of any two indices. Show explicitly that E_{jkl} is tracefree (the trace over any pair of indices vanishes) by working out its trace over (j,k), thus showing that $E_{jjl} = 0$. To compute the trace, contract E_{jkl} with δ_{jk} and use the fact that $\mathbf{n}$ is a unit vector.

  (c) **[optional bonus question]** From the pattern in the last three relations in (7) one can find a compact result for $E_L = \partial_L r^{-1}$ given by

$$E_L = \partial_L \frac{1}{r} = \frac{(-1)^\ell (2\ell - 1)!! n^{<L>}}{r^{\ell+1}}, \tag{8}$$

  where the angular brackets denote that the tensor is symmetric and trace-free (STF) on the enclosed indices. To be able to use this formula, which is more efficient than having to work out the repeated derivatives of r^{-1}, requires the prescription for constructing $n^{<L>}$ for an arbitrary number of ℓ indices given below. The general formula for constructing $n^{<L>}$ is:

$$n^{<j_1 j_2 \ldots j_\ell>} = \sum_{p=0}^{\lfloor \ell/2 \rfloor} (-1)^p \frac{(2\ell - 2p - 1)!!}{(2\ell - 1)!!} \left[\delta^{j_1 j_2} \ldots \delta^{j_{2p-1} j_{2p}} n^{j_{2p+1}} \ldots n^{j_\ell} + (\mathrm{sym})\right]. \tag{9}$$

  Here, $\lfloor \ell/2 \rfloor$ denotes the largest integer $\leq \ell/2$ (the floor function). For example $\lfloor 4/2 \rfloor = 2$ and also $\lfloor 5/2 \rfloor = 2$. The part (sym) indicates that all distinct terms that arise from permuting indices have to be added. For each p, the terms inside the square brackets involve a product of p Kronecker deltas and $\ell - 2p$ unit vectors, and that the total number of terms in the square brackets is $\ell!/[(\ell - 2p)!(2p)!!]$ if all permutations lead to distinct structures.

  As an example, we apply this formula for $\ell = 3$. We will label the indices in Eq. (9) $j_1 = i$, $j_2 = j$, $j_3 = k$. Then, since $\lfloor \ell/2 \rfloor = 1$ we write down the sum over $p = 0$ and $p = 1$. For $p = 0$ the terms will involve 0 factors of δ_{ij} and 3 factors of n^i, while for $p = 1$ there will be 1 Kronecker delta combined with 1 factor of n^i. The sum evaluates to be

$$n^{<ijk>} = (-1)^0 \frac{5!!}{5!!}\left[n^i n^j n^k\right] + (-1)^1 \frac{3!!}{5!!}\left[\delta^{ij} n^k + \delta_{ik} n^j + \delta_{jk} n^i\right] = n^{ijk} - \frac{1}{5}\left[\delta^{ij} n^k + \delta_{ik} n^j + \delta_{jk} n^i\right] \tag{10}$$

  Note that for $p = 0$ there was no contribution from (sym) since $n^i n^j n^k$ is already symmetric under interchanging the indices, and permutations do not lead to any distinct terms. On the other hand, for $p = 1$ we had to include the different permutations, using that $\delta_{jk} = \delta_{kj}$ so such interchanges of indices did not lead to any distinct terms.

  Use a similar procedure to evaluate the formula (9) for the case $\ell = 4$ to compute $n^{<jkms>}$.

- **Converting between Cartesian and spherical harmonic coefficients**.

  As discussed in lecture, a solution to the Laplace equation such as the potential U in the exterior of its source can be expanded around a reference point $\mathbf{z}$ in two equivalent multipole series

$$U(\mathbf{x}) = \sum_{\ell=0}^{\infty} \frac{G(2\ell - 1)!!}{\ell!} \bar{n}^L \frac{M^{<L>}}{\bar{r}^{\ell+1}} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi G}{2\ell + 1} Y_{\ell m}(\bar{\theta}, \bar{\phi}) \frac{I_{\ell m}}{\bar{r}^{\ell+1}}, \tag{11}$$

with $\bar{\mathbf{x}} = \mathbf{x} - \mathbf{z}$ and $\bar{r} = |\bar{\mathbf{x}}|$. To relate these decompositions requires the identity

$$Y_{\ell m}(\theta, \phi) = \mathcal{Y}^{*L}_{\ell m} n_L, \tag{12}$$

where $\mathcal{Y}^{L}_{\ell m}$ are constant STF tensors and

$$n^i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{13}$$

The summation over all indices L is implied. As shown in lecture, since the tensors $\mathcal{Y}^{L}_{\ell m}$ are STF, only the STF part of any tensor contracted with them will contribute to the result. So $\mathcal{Y}^{L}_{\ell m} n^L = \mathcal{Y}^{*L}_{\ell m} n^{<L>}$, and for simplicity we have omitted the STF operation on n^L in Eq. (12).

Your task in this exercise is to consider the case $\ell = m = 2$, compute the components of $\mathcal{Y}^{ij}_{22}$, and express I_{22} as a linear combination of the components of Q_{ij}. Specifically, first compute the nonvanishing components of $\mathcal{Y}^{ij}_{22}$, then use this to write the expansion coefficients of the $\ell = m = 2$ component of the spherical harmonic representation of the quadrupole $I_{22} = \mathcal{Y}^{*ij}_{22} Q_{ij}$ as a weighted sum over the components of the Cartesian tensor $M^{<ij>} = Q_{ij}$ (for unspecified values of the components).

For reference, the spherical harmonic Y_{22} is given by

$$Y_{22}(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \sin^2\theta e^{2i\phi}. \tag{14}$$

The solution relies on identifying coefficients of different trigonometric functions (or exponentials). This requires reducing the ϕ-dependences such that the various terms cannot mix by further manipulations. To approach this problem:

- write the exponential in Eq. (14) in terms of trigonometric functions and trig reduce the dependence on ϕ
- use Eq. (13) to work out all the components of n_{ij} in spherical polar coordinates
- write out the right hand side of Eq. (12) explicitly as a sum over components, e.g. $\mathcal{Y}^{*xx}_{22} n_{xx} + \mathcal{Y}^{*yy}_{22} n_{yy} + \ldots$. Use the symmetries to reduce the number of independent components of $\mathcal{Y}^{*ij}_{22}$ you are considering.
- equate coefficients of each of the trig dependencies on the left and right hand sides of the expanded Eq. (12) .

This problem is sufficiently tractable to be worked out 'by hand'. Usually, such problems are more readily solved in Mathematica, which can also handle cases with higher ℓ more quickly once the problem is set up as a system of equations to solve. You can choose whichever method you prefer for this exercise. If you use Mathematica, please document your work so the reader can follow the logic and steps in your calculation, e.g. by including a printout of a clean code with ample comments.