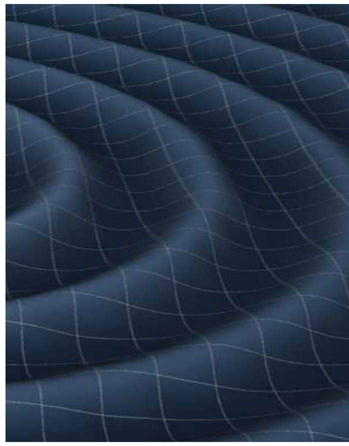
Gravitational waves for fundamental physics

Lecture 2: Modeling the inspiral of compact-object binaries

using post-Newtonian and post-Minkowskian theory with matched asymptotic expansions

Summary from last time: basic physics of GWs in linearized gravity



- Effect of GWs: stretching and squeezing of space along orthogonal axes
- Two physical degrees of freedom (polarizations):

$$h_+ = h_{11}^{\mathrm{TT}}, \quad h_\times = h_{12}^{\mathrm{TT}}$$

- Relation to the source:

Unit vector (source to observer)

$$h_{ij}^{\mathrm{TT}} = \frac{1}{d}\frac{2G}{c^4}\underbrace{\Lambda_{ijkl}(\mathbf{N})}\ddot{Q}_{kl}$$

Distance to the source

Transverse-traceless (TT) projector

Newtonian source quadrupole $Q^{ij} = \int_{\mathrm{source}} d^3x\, \rho\, (x^i x^j - \frac{1}{3}|\mathbf{x}|^2\delta^{ij})$

Summary from last time: basic physics of GWs in linearized gravity

- Energy-momentum carried by GWs: $$t^{\alpha\beta}_{\rm GW} = \frac{c^4}{32\pi G}\langle \eta^{\alpha\gamma}\eta^{\beta\delta}\partial_\gamma h^{\rm TT}_{\mu\nu}\partial_\delta h^{\mu\nu}_{\rm TT}\rangle$$

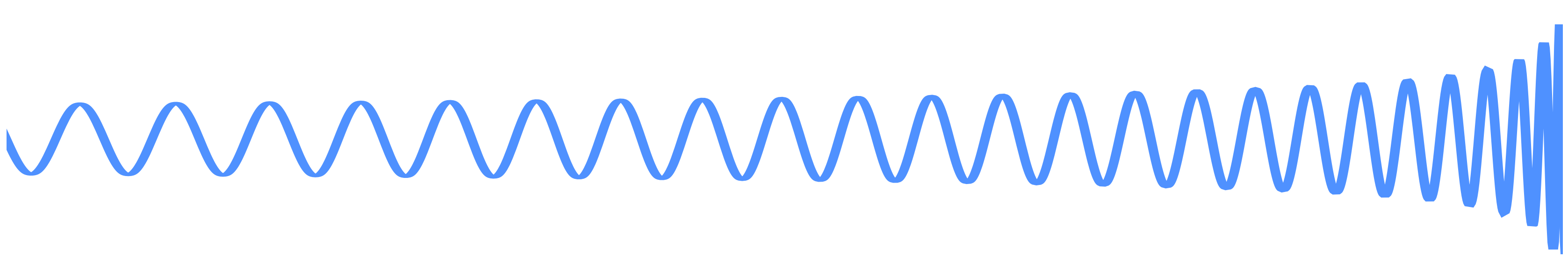
  average

- GW power in terms of the TT polarizations:

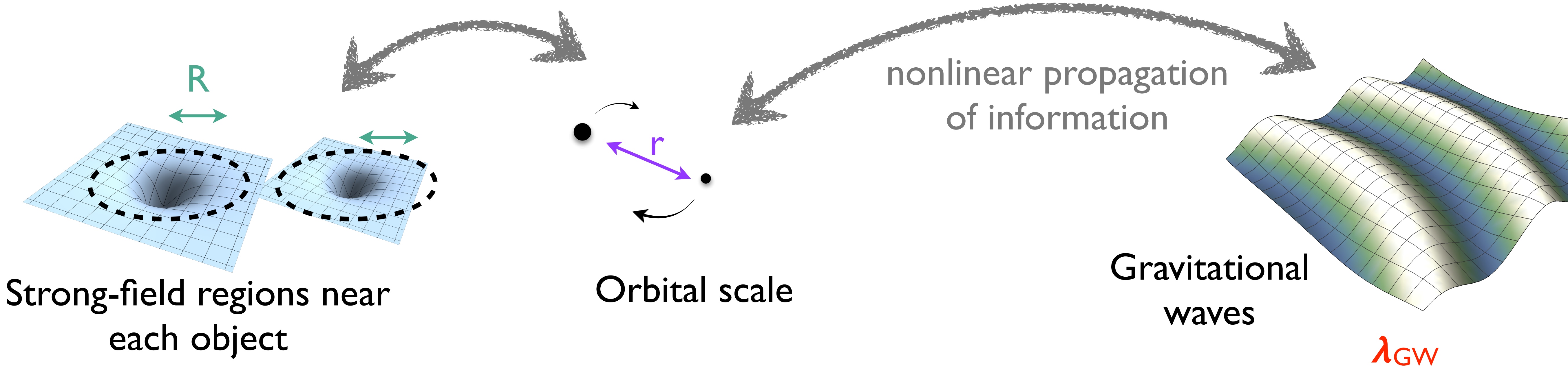
$$\dot{E}_{\rm GW} = \frac{c^3}{16\pi G}d^2\int d\Omega\,\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle$$

- Einstein quadrupole formula: $$\dot{E}_{\rm GW} = \frac{G}{5c^5}\langle \dddot{Q}_{ij}\dddot{Q}^{ij}\rangle$$

- Orbital decay and inspiral (chirp) waveforms in a Newtonian binary system



Flow of information in inspiraling binaries



Dynamical spacetime of the binary:
Different physics dominates at different scales

Seek a unified description — but how to do that rigorously?

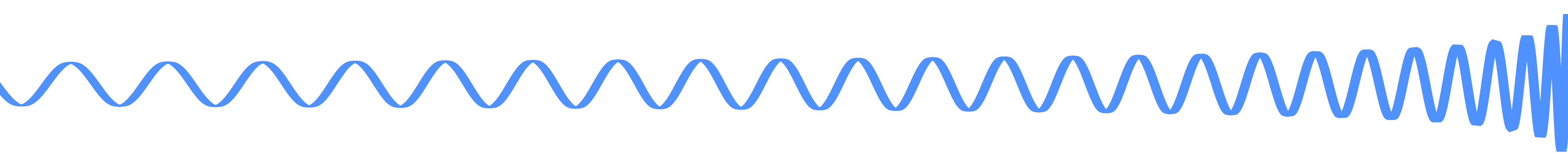
Plan for today

Long binary inspiral regime plays an important role for GW measurements.

Need a rigorous description:

- important concepts and approaches:
  - The method of matched asymptotic expansions
  - Multipole expansions, symmetric trace-free tensors, and spherical harmonics
- Unique relativistic effects in GWs

- Today's tutorial: introduction to the tensor package xAct, using it for perturbative expansions and varying actions to obtain equations of motion



References

- **William Burke: Gravitational Radiation Damping of Slowly Moving Systems Calculated Using Matched Asymptotic Expansions**

  Journal of Mathematical Physics 12, 401 (1971) **link to pdf through aip**

  first computation of the back reaction of GWs on their source beyond linearized theory, introduces the method of matched asymptotic expansion for GWs

- **Luc Blanchet and Thibault Damour: Post-newtonian generation of gravitational waves**,

  Annales de l'I.H.P. 50, 377 (1989) **link to pdf through numdam**

  foundational paper and relatively tractable compared e.g. to more recent review articles on the subject. You do not need the technical details. The introduction gives a good qualitative overview of several challenges and subtleties that are interesting to appreciate.

- **T. Hartmann, M. Soffel, T. Kioustelidis: On the use of STF tensors in celestial mechanics**,

  Celestial Mechanics & Dynamical Astronomy 60, 139 (1994) **link to pdf through adsabs**

  Go-to reference for symmetric trace-free tensors and spherical harmonics, given many identities and applications to binary systems

The method of
Matched Asymptotic Expansions

in singular perturbation theory

Asymptotic expansions vs. convergent series expansions

Broad applicability, e.g. when different physics dominates on different scales (space or time)

Reference: William Burke, Gravitational Radiation Damping of Slowly Moving Systems Calculated Using Matched Asymptotic Expansions, Journal of Mathematical Physics 12, 401 (1971) link to pdf

A first example

- Consider the function

$$f = 1 + x + \frac{\epsilon}{x}$$

with ϵ a small dimensionless parameter: $\epsilon \ll 1$

- We can approximation it for small ϵ by using the ansatz:

$$f(x;\epsilon) = f_0(x) + \epsilon f_1(x) + O(\epsilon^2)$$

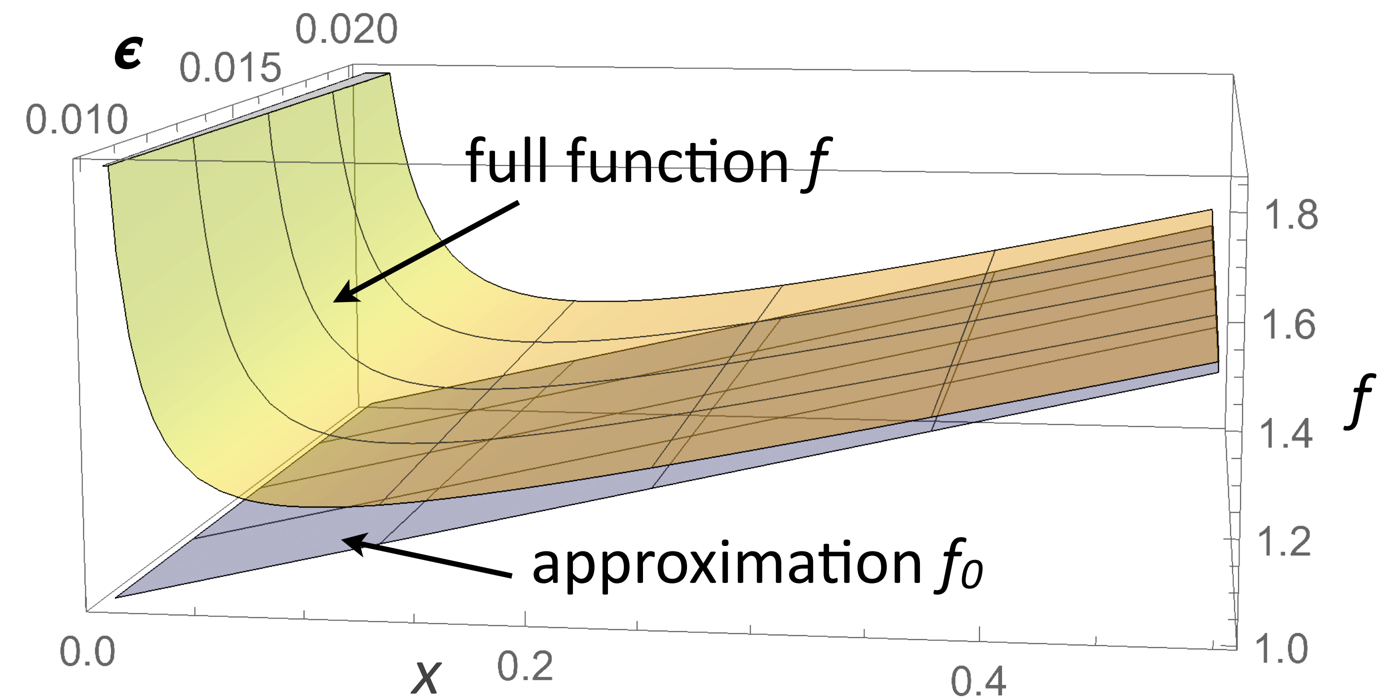
this means we are taking the limit $\epsilon \to 0$ for fixed x

with $\quad f_0 = 1 + x$

- f_0 is a very good approximation to f for large x

"Outer expansion"

- but: problems near $x \to 0$



The problem for small $x \sim O(\epsilon)$

- For small x, when $x \sim O(\epsilon)$, the scaling of the terms changes

$$f = 1 + x + \frac{\epsilon}{x}$$

becomes very small

becomes the dominant term

$\Longrightarrow$ f is no longer well-approximated by $f_0 = 1+x$

$\Longrightarrow$ need a different perturbative expansion in this region of small x

The regime of small $x \sim O(\epsilon)$

- x is not a good variable to keep track of the ordering of terms since it is $O(\epsilon)$
- It is much more convenient to work with a variable which is O(1)
- We therefore introduce a scaled coordinate X adapted to this regime:

choose the scaling of X so that it is O(1) for $x=O(\epsilon)$:

$$\boxed{X = x/\epsilon}$$

'stretches' the region of small x

- The function $f = 1 + x + \dfrac{\epsilon}{x}$ in terms of the scaled coordinate:

$$f = 1 + \epsilon X + \frac{1}{X}$$

This has formalized the expected scalings discussed on the previous slide

Fixing the problem for small $x \sim O(\epsilon)$

$$X = x/\epsilon \qquad\qquad f = 1 + \epsilon X + \frac{1}{X}$$

- Now consider the approximation in the limit $\epsilon \to 0$ for fixed X

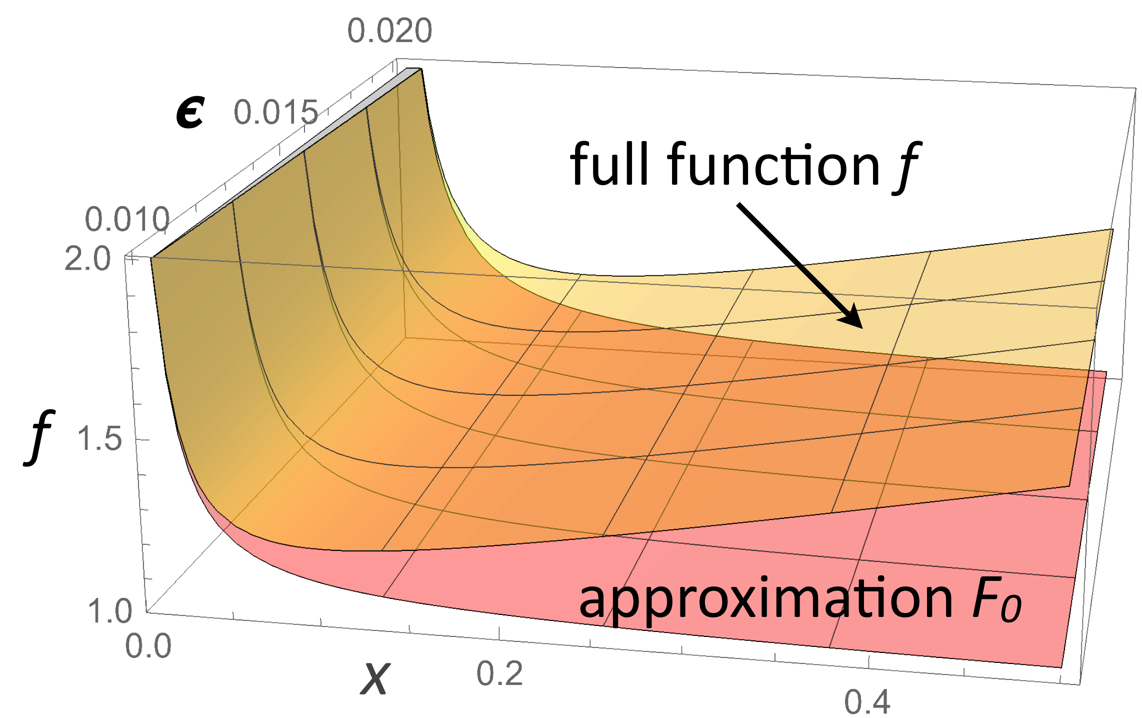
(ϵ and x both go to zero at the same rate)

Ansatz: $f(x;\epsilon) = F^{\text{inner}}(X,\epsilon) = F_0(X) + \epsilon F_1(X) + O(\epsilon^2)$

with $F_0 = 1 + 1/X$

Works well for small *x* (unscaled coordinate)

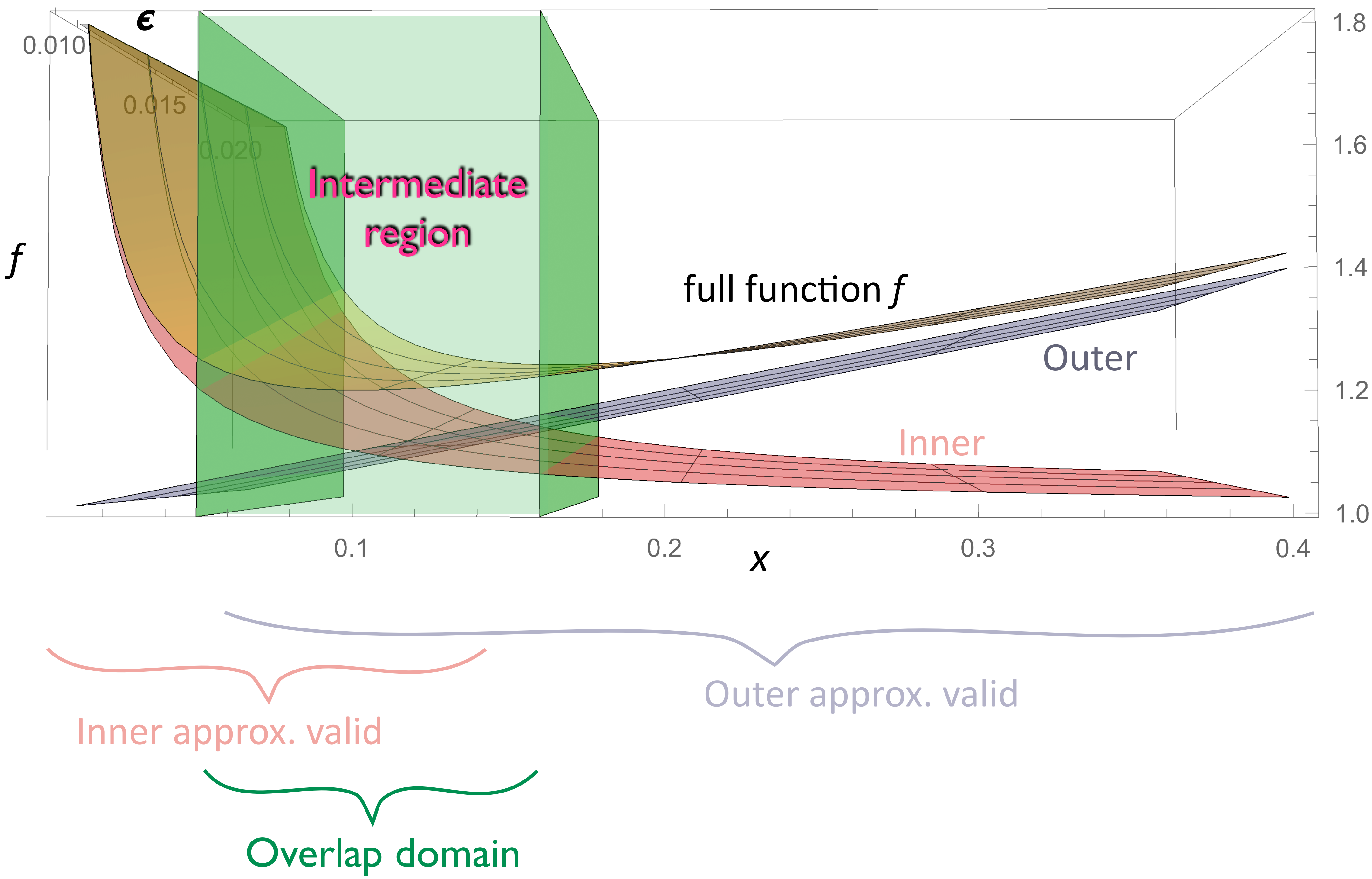
"Inner expansion"



- Problem: also not uniformly valid for all X>0, breaks down for large X=O(1/ϵ)

Intermediate regime (overlap domain)

In an intermediate region of small x and large X: both expansions are valid



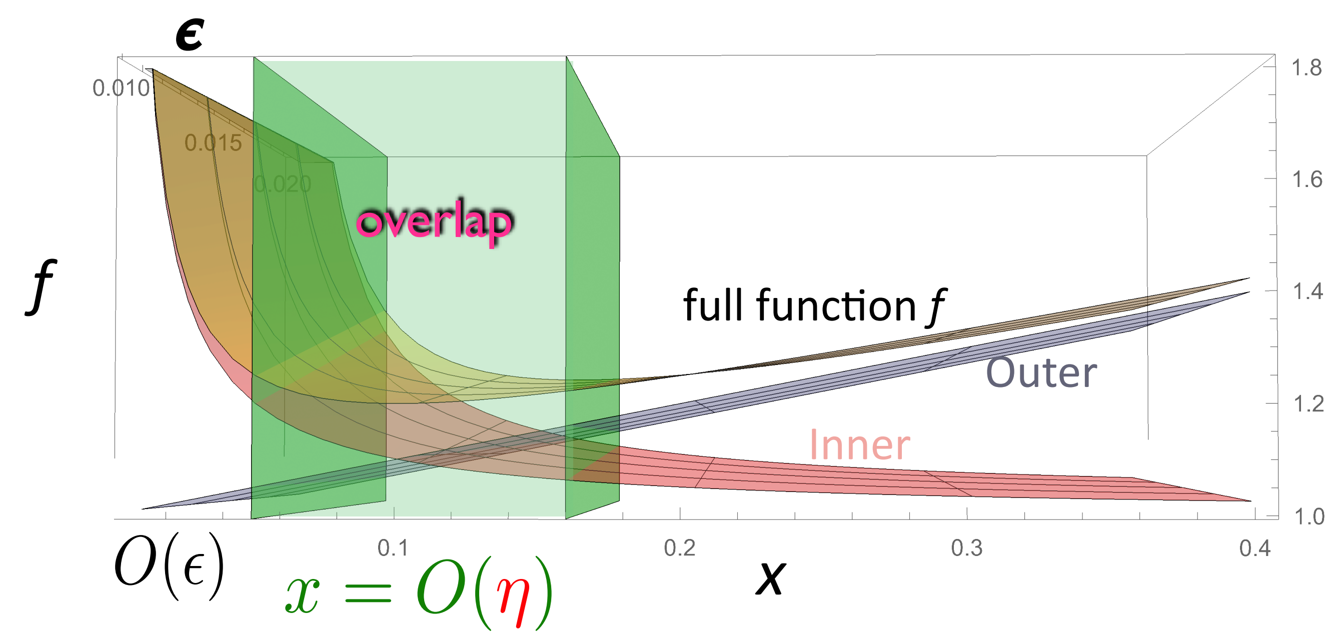
Matching in the overlap region

- The intermediate regime is characterized by a function $\boldsymbol{\eta(\epsilon)}$, with

$$\epsilon \ll \eta(\epsilon) \ll 1 \qquad \text{e.g. } \eta = \sqrt{\epsilon}$$

Where: $x = O(\eta)$ [*small* as $\epsilon \to 0$]

$X = x/\epsilon = O(\eta/\epsilon)$ [*large* as $\epsilon \to 0$]



- introduce an associated scaled coordinate $x_\eta = x/\eta$ which is ~ O(1) in this region

- Formal matching proceeds by taking the limit $\epsilon \to 0$ at fixed x_η and is accomplished if

$$\lim_{\epsilon \to 0} \left[F^{\text{inner}}(x_\eta) - f^{\text{outer}}(x_\eta) \right] = 0$$

This will in general only be true to some order in ϵ

Matching to leading order in the intermediate region

In terms of the matching coordinate $x_\eta = x/\eta$

$$\boxed{x = \eta x_\eta \, , \; X = \frac{\eta}{\epsilon} x_\eta} \qquad\qquad \epsilon \ll \eta(\epsilon) \ll 1$$

Leading order terms in the limit $\epsilon \to 0$ (also implies small η):

outer: $$f_0 = 1 + x = 1 + \eta(\epsilon) x_\eta \approx 1 + \ldots$$

inner: $$F_0 = 1 + 1/X = 1 + \frac{\epsilon}{\eta(\epsilon)} \frac{1}{x_\eta} \approx 1 + \ldots$$

The terms match to O(η, ϵ/η) here, which works for a range of choices of $\eta(\epsilon)$ in the interval $\epsilon \ll \eta(\epsilon) \ll 1$ and we would need to include the higher-order terms to quantify the domain of overlap more precisely

Final step: construct a composite solution

- Now can construct the composite expansion valid everywhere:

  add the inner and outer approximations and subtract the common term (seen from the matching)

$$f^{\text{composite}} = \underbrace{1+x}_{\text{outer } f_0} + \underbrace{1+\frac{\epsilon}{x}}_{\substack{\text{inner } F_0 \text{ in} \\ \text{terms of the} \\ \text{unscaled} \\ \text{coordinate } x}} - 1 + \dots$$

Higher-order terms that we did not consider but are all zero in this example

Common term as seen from the matching

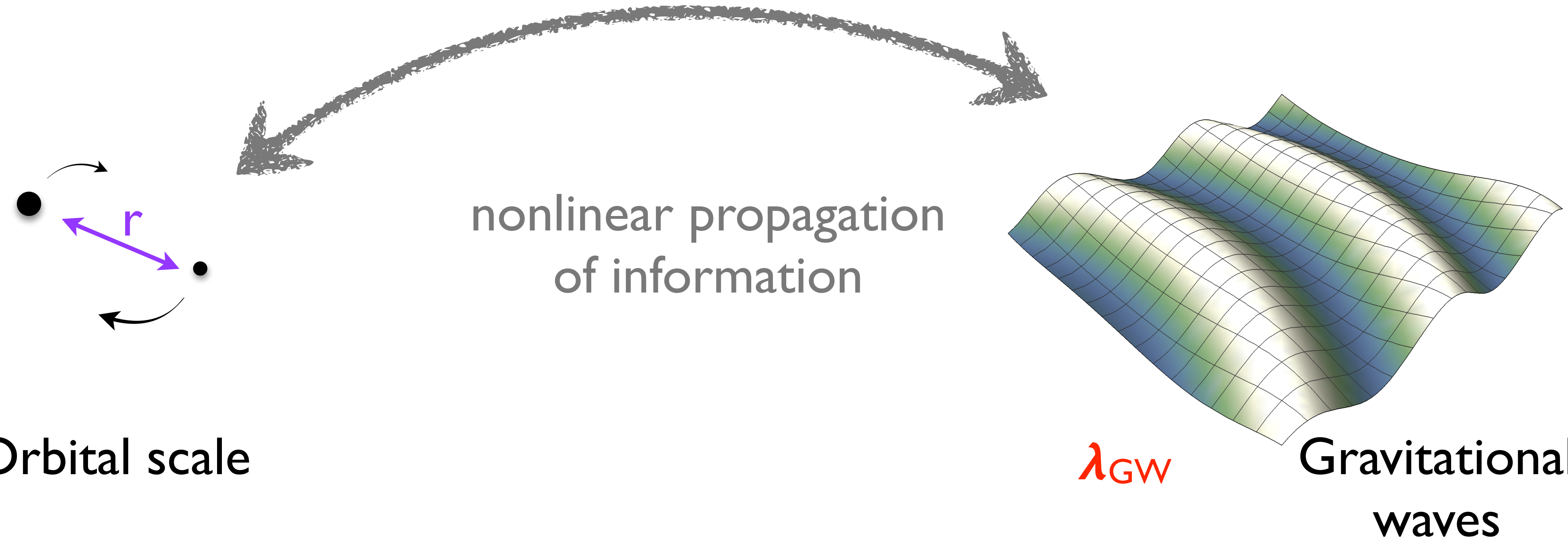
[contained in both so would otherwise double-counted]

Note: using only the leading order approximations at $O(\epsilon^0)$ leads to a nontrivial dependence on ϵ in the composite expansion

Application of these methods
to a more complicated example,
which elucidates key features of the coupling between
the sources and GWs

Reference: William Burke, Gravitational Radiation Damping of Slowly Moving Systems Calculated Using Matched Asymptotic Expansions, Journal of Mathematical Physics 12, 401 (1971) link to pdf

Toy model for coupling between source and radiation

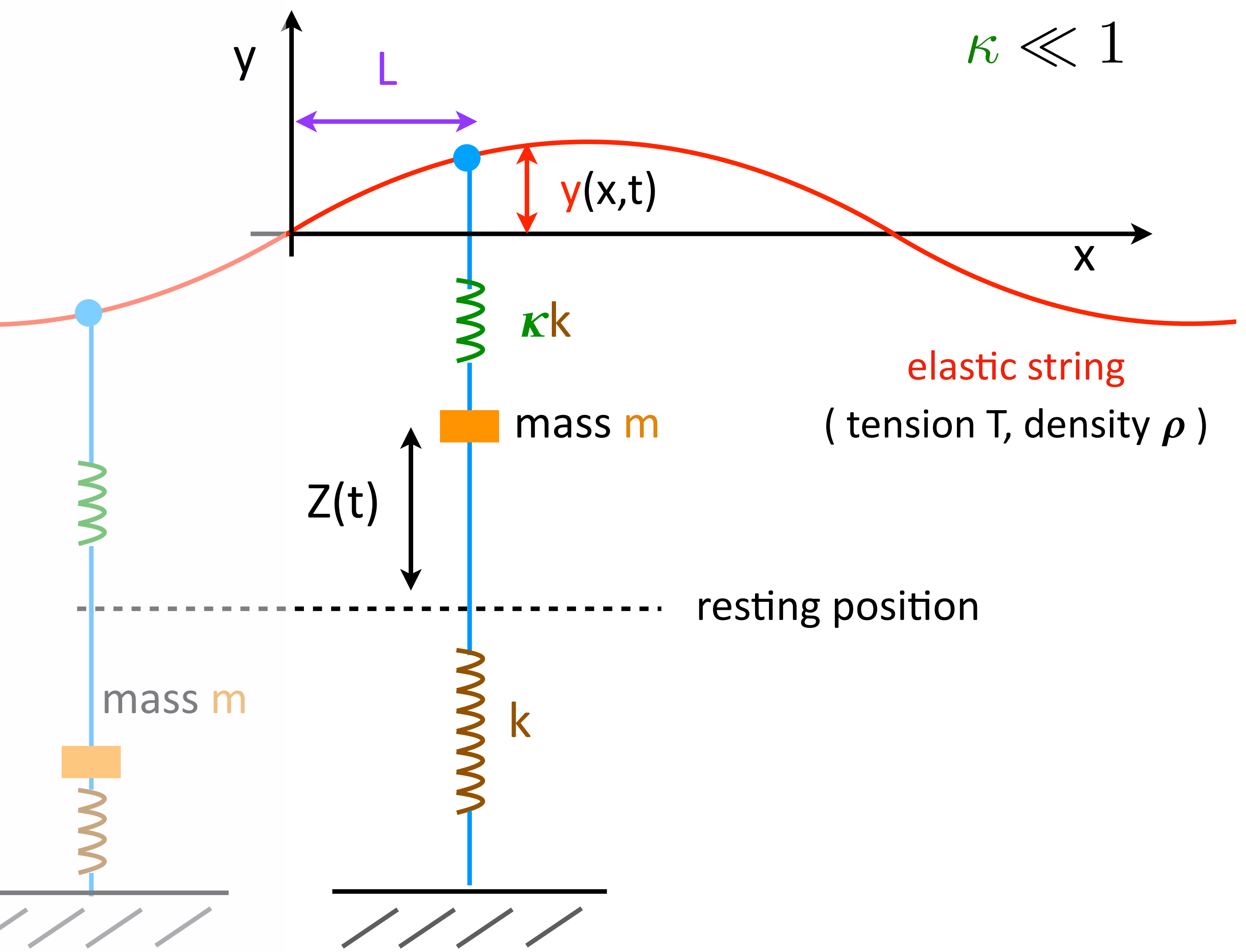


Dynamical spacetime of the binary:
Different physics dominates at different scales

Seek a unified description — but how to do that rigorously?

Mechanical toy model for gravitational radiation

Oscillating masses (source) weakly coupled to an infinite elastic string (radiation)



Need two masses to have a source scale, but will consider only the region x>0

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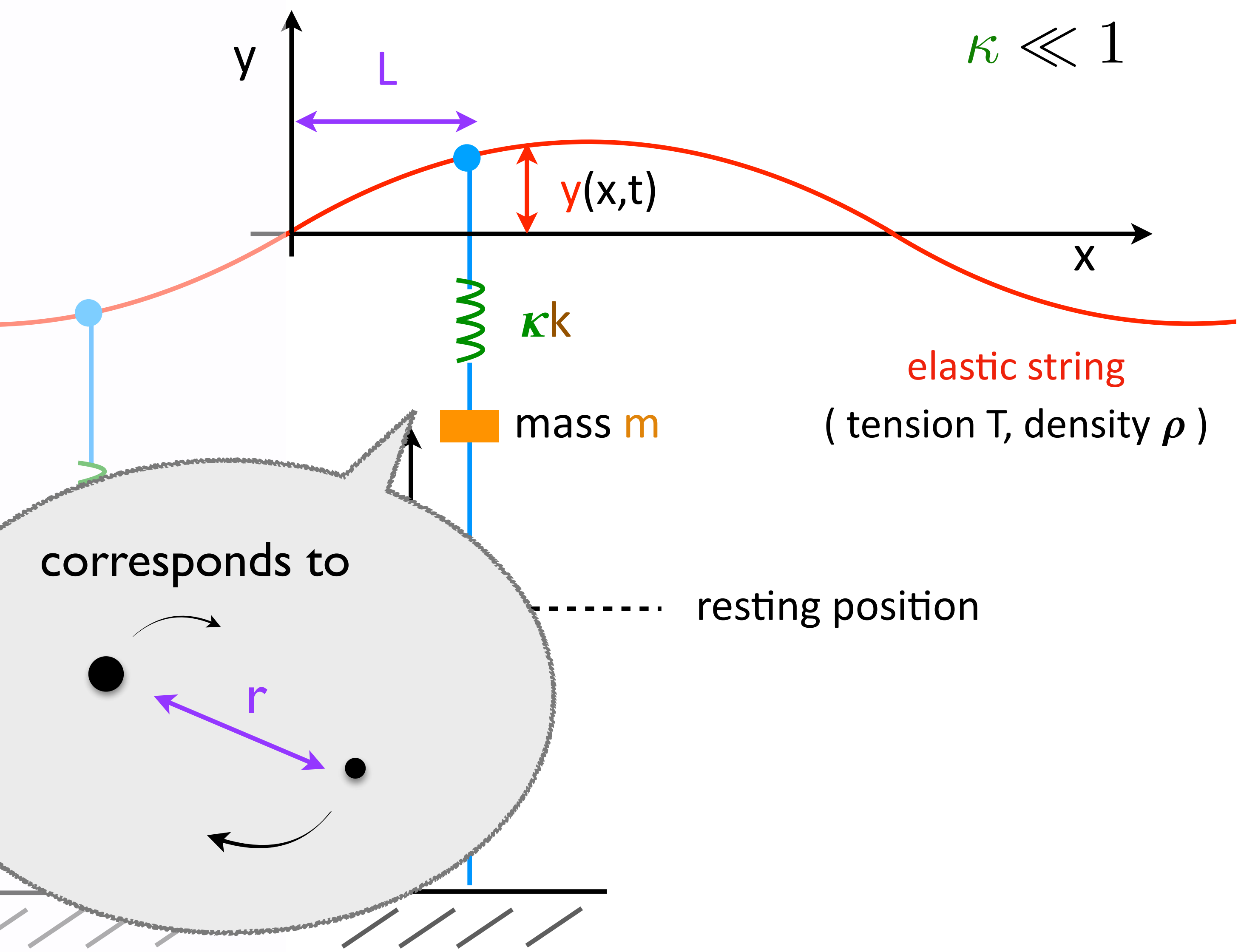

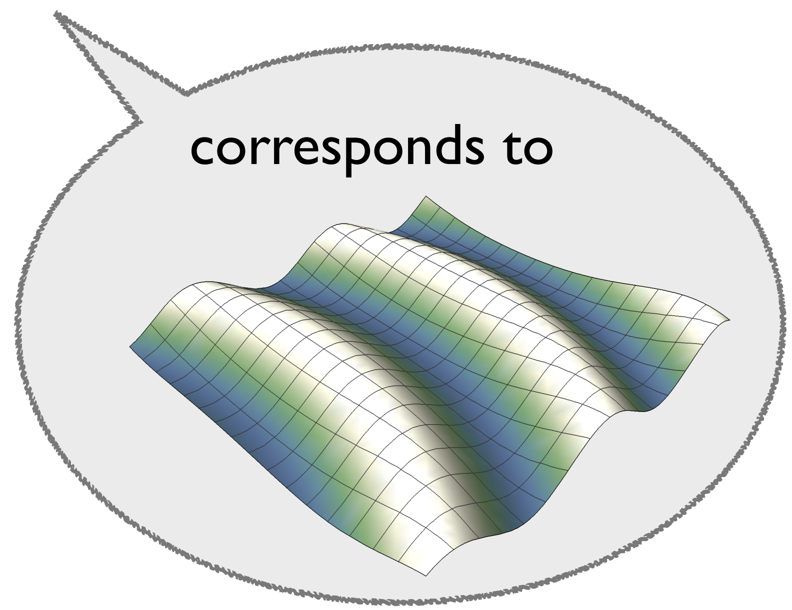

Mechanical toy model for gravitational radiation

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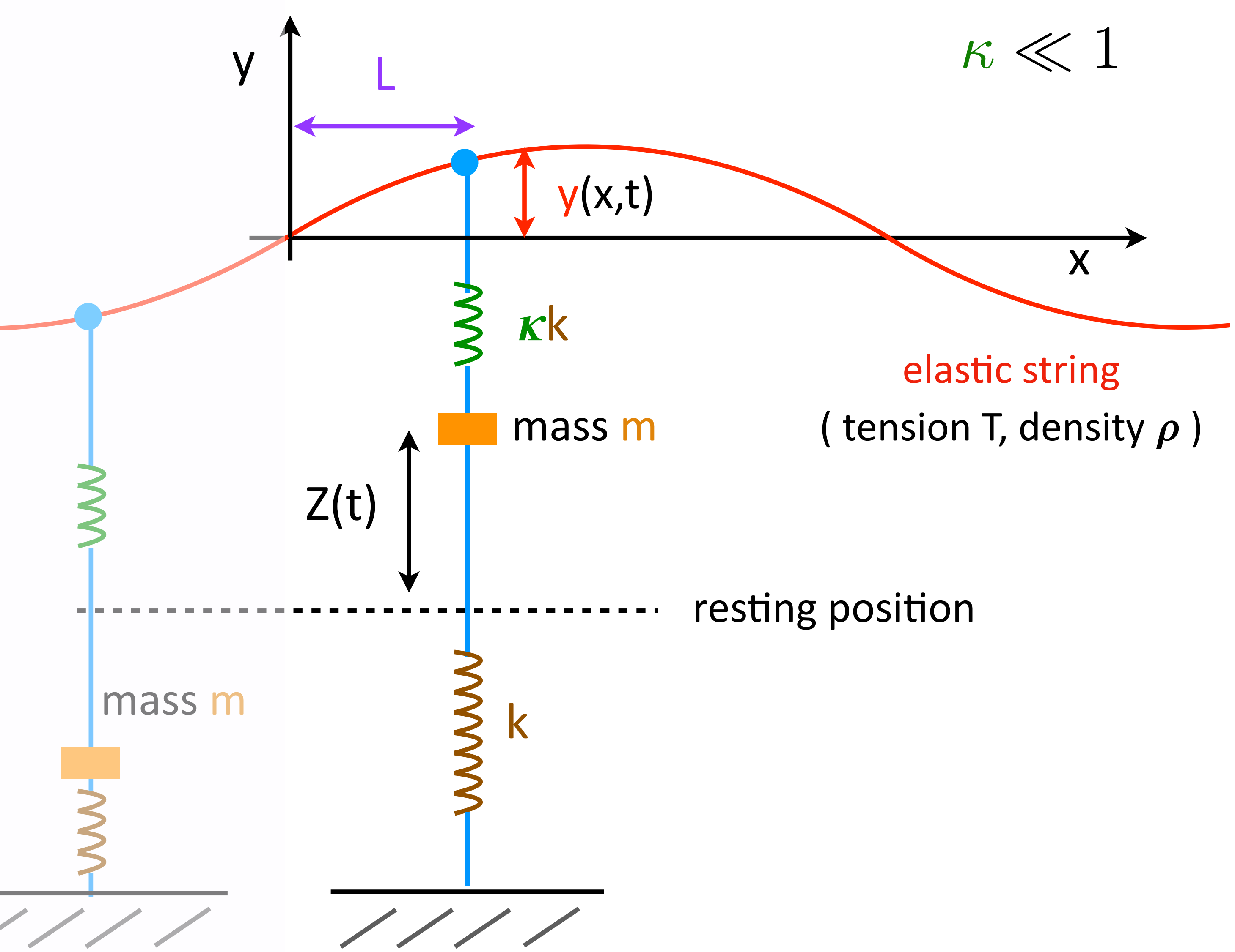


Need two masses to have a source scale, but will consider only the region x>0

<u>Relevant parameters:</u>

- Source frequency $\omega = \sqrt{k/m}$
- Wavelength of induced waves on the string

$$\lambda = \frac{2\pi}{\omega}\sqrt{T/\rho}$$

- hierarchy of scales: L<< λ

· Small dimensionless parameter $\boxed{\epsilon = L/\lambda \ll 1}$

Strategy

First solve for the dynamics of the string for a generic oscillatory source motions Z(t):

- Outer expansion away from the source
- Inner expansion near the source
- matching

Then solve for the backreaction of the coupling to the string on the source

- will stop at recognizing the key physics

Starting point:

Equations of motion for the displacements:

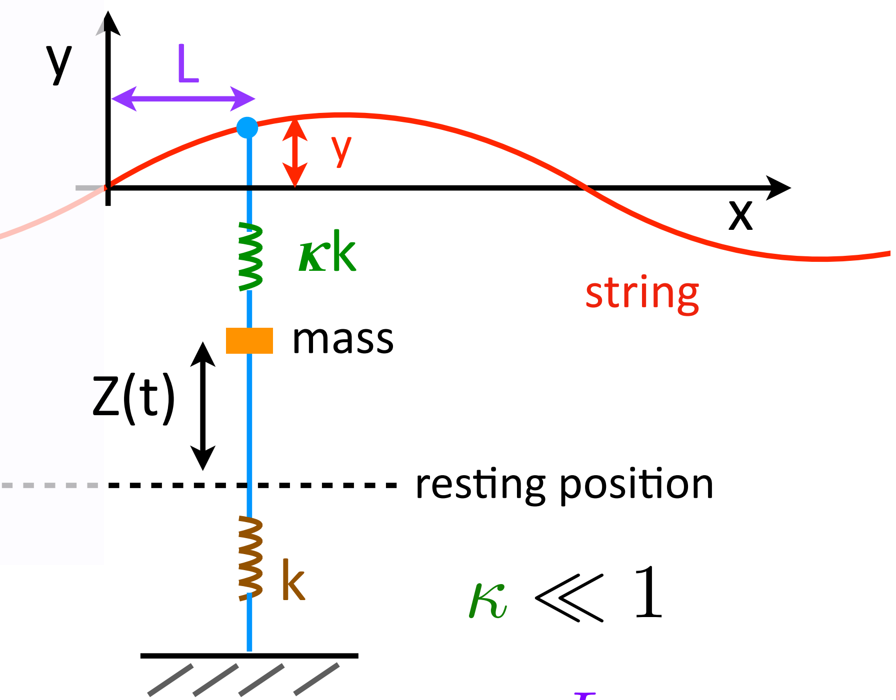
String: $$-\rho\frac{\partial^2 y}{\partial t^2} + T\frac{\partial^2 y}{\partial x^2} = F_{\text{couple}}$$

Mass: $$\frac{d^2 Z}{dt^2} + \frac{k}{m}Z = F_{\text{couple}}$$

Coupling force:

$$F_{\text{couple}} = -\kappa k\left[\underbrace{Z(t)}_{\text{Displacement of the oscillating mass}} - \underbrace{y(L,t)}_{\text{Displacement of the string}}\right]\delta(x-L)$$

Regime far from the sources



$$\kappa \ll 1$$

$$\epsilon = \frac{L}{\lambda} \ll 1$$

- Introduce dimensionless coordinates (adapted to 'radiation'):

$$x^* = \frac{2\pi x}{\lambda} \qquad t^* = \omega t$$

Equations of motion for string displacement

$$\boxed{\frac{\partial^2 y}{\partial t^{*2}} - \frac{\partial^2 y}{\partial x^{*2}} = 0}$$

No source at x >> L

$\Rightarrow$ waves

- Ansatz for the asymptotic expansion of the solutions at fixed x^*:

$$y = \kappa \left[g_0(x^*, t^*) + \epsilon g_1(x^*, t^*) + O(\epsilon^2) \right] + O(\kappa^2)$$

$\boldsymbol{\kappa}$ is also included here because we anticipate that the waves are generated by the coupling

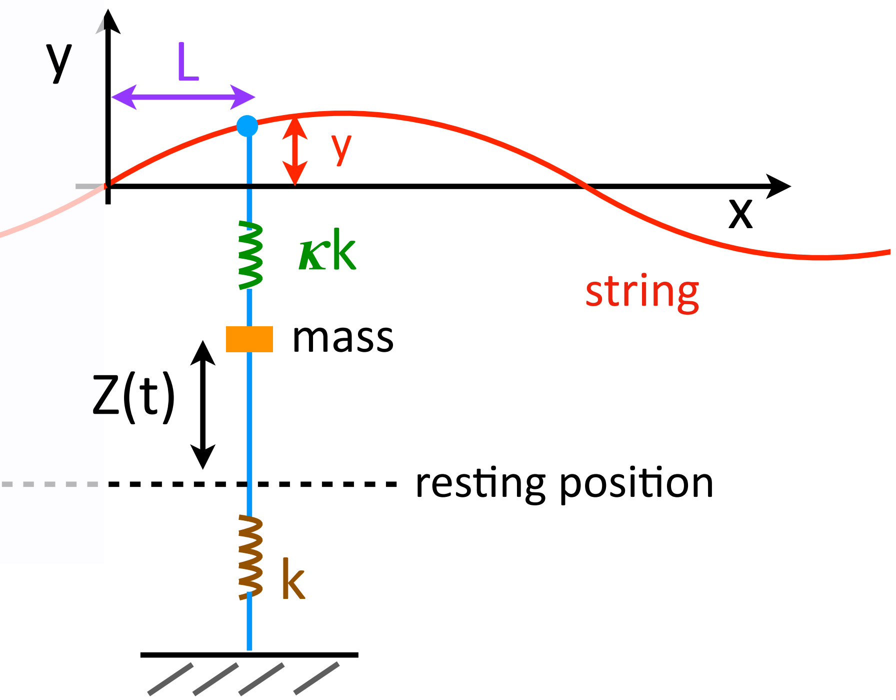
- Solutions at each order are traveling waves:

$$g_0 = a_{\rm in} W_{\rm in}(t^* + x^*) + a_{\rm out} W_{\rm out}(t^* - x^*)$$

ingoing at infinity, do not consider here

Outgoing waves

Inner regime



· Introduce scaled coordinates (adapted to 'source'):

$$X^* = \frac{2\pi x}{L} = \frac{x^*}{\epsilon} \qquad \epsilon = \frac{L}{\lambda} \ll 1$$

Equations of motion for string displacement in near-source-zone

$$\boxed{-\epsilon^2 \frac{\partial^2 y}{\partial t^{*2}} + \frac{\partial^2 y}{\partial X^{*2}} = -\kappa\alpha\,[Z-y]\,\delta(X^* - 2\pi)}$$

Boundary conditions: $y(0,t) = 0$

$$\alpha = \frac{kL^2}{4\pi^2 T}$$

time derivatives are strongly suppressed compared to spatial gradients

$$\kappa \ll 1$$

· Ansatz for the asymptotic expansion of y at fixed X*:

$$y = \kappa\left[G_0(X^*, t^*) + \epsilon G_1(X^*, t^*) + O(\epsilon^2)\right] + O(\kappa^2)$$

Inner regime: asymptotic expansion

- Equations of motion: $-\epsilon^2 \frac{\partial^2 y}{\partial t^{*2}} + \frac{\partial^2 y}{\partial X^{*2}} = -\kappa\alpha \left[Z - y\right] \delta(X^* - 2\pi)$

$$y = \kappa \left[G_0(X^*, t^*) + \epsilon G_1(X^*, t^*) + O(\epsilon^2)\right] + O(\kappa^2)$$

- collect powers of **ϵ** and **κ**:

Boundary condition:

$$y(0, t) = 0$$

$$O(\kappa\epsilon^0) : \quad \frac{\partial^2 G_0}{\partial X^{*2}} = -\alpha Z \, \delta(X^* - 2\pi)$$

$$\implies \quad G_0 = \alpha \, Z \, X^* + (2\pi - X^*)\alpha \, Z \, \Theta \left(X^* - 2\pi\right)$$

$$O(\kappa\epsilon^1) : \quad \frac{\partial^2 G_1}{\partial X^{*2}} = 0 \qquad \implies \quad G_1 = c_1(t^*) X^*$$

Diverges at large X*

Matching: setup

- Consider the intermediate region where x^* is small and X^* is large

- introduce an intermediate scaling $\boldsymbol{\eta(\epsilon)}$: $\epsilon \ll \eta(\epsilon) \ll 1$

  and a matching coordinate x_η : $x^* = \eta\, x_\eta, \quad X^* = \dfrac{\eta}{\epsilon} x_\eta$ $\qquad x_\eta > L$

- Write the inner and outer expansions in terms of x_η :

$$y^{\text{inner}} = \kappa\alpha \left[2\pi Z(t^*) + \eta\, x_\eta\, c_1(t^*) + O(\epsilon^2)\right] + O(\kappa^2)$$

$$y^{\text{outer}} = \kappa \left[a_{\text{out}} W_{\text{out}}(t^* - \eta x_\eta) + O(\epsilon)\right] + O(\kappa^2)$$

Matching fixes the unknown coefficients and functions

- To take the limit $\epsilon \to 0$ at fixed x_η must make explicit the dependences on η, ϵ

$$y^{\text{inner}} = \kappa\alpha\left[2\pi Z(t^*) + \eta\, x_\eta\, c_1(t^*) + O(\epsilon^2)\right] + O(\kappa^2)$$

Suitable as is

$$y^{\text{outer}} = \kappa\left[a_{\text{out}} W_{\text{out}}(t^* - \eta x_\eta) + O(\epsilon)\right] + O(\kappa^2)$$

Need to expand W_{out} for small η

$$= \kappa\left[a_{\text{out}} W_{\text{out}}(t^*) - \eta x_\eta\, a_{\text{out}} W'_{\text{out}}(t^*) + O(\eta^2) + O(\epsilon)\right] + O(\kappa^2)$$

- Matching up to $O(\eta)$:

$$O(\eta^0): \quad \alpha 2\pi Z(t^*) = a_{\text{out}} W_{\text{out}}(t^*) \qquad \begin{cases} a_{\text{out}} = 2\pi\alpha \\ W_{\text{out}}(t^*) = Z(t^*) \end{cases}$$

determines the source-dependence of the waves

$$O(\eta^1): \quad c_1(t^*) = -2\pi\alpha \frac{dZ}{dt^*}$$

Determines imprint of wave-zone boundary conditions on the behavior in the source region

Backreaction on the motion of the mass

Equations of motion of the oscillating mass:

$$\frac{d^2 Z}{dt^{*2}} + Z = -\kappa m \left[Z - y(L, t^*)\right]$$

From the asymptotic solution for the string: $y(L, t^*) = \kappa\, 2\pi\alpha \left[Z(t^*) - \epsilon \frac{dZ}{dt^*}\right] + \ldots$

Use in the equations of motion:

$$\frac{d^2 Z}{dt^{*2}} + \left[1 + \kappa m(1 + 2\pi\alpha)\right] Z + \kappa\, m\, 2\pi\alpha\, \epsilon \frac{dZ}{dt^*} = 0 + \ldots$$

odd under time reversal: dissipation,
damping of the oscillations

Example application of matched asymptotic expansions to inspiraling binary systems: post-Newtonian description

Luc Blanchet and Thibault Damour:

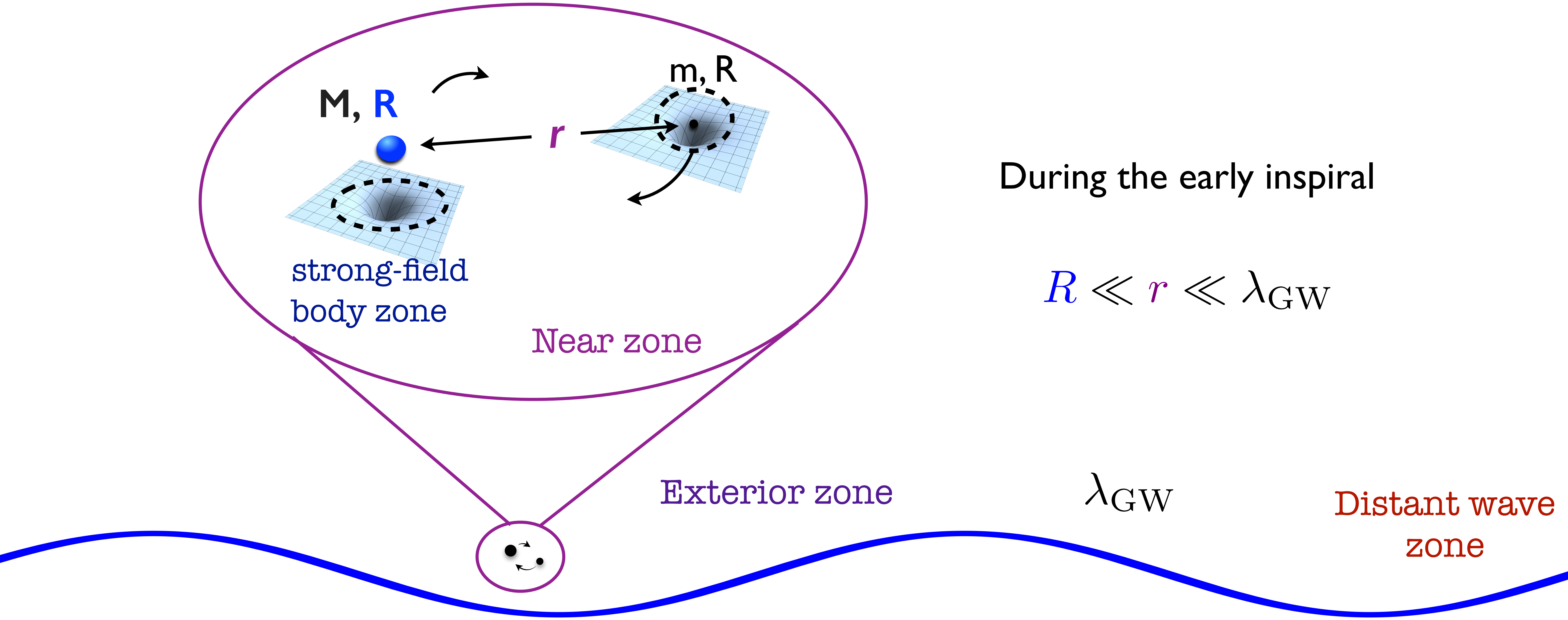
Post-newtonian generation of gravitational waves,

Annales de l'I.H.P. 50, 377 (1989)

link to pdf through numdam

[we will not discuss the technical details]

Hierarchy of scales in a compact-object binary system



During the early inspiral

$$R \ll r \ll \lambda_{\rm GW}$$

Parameters in the near zone:

Mass ratio m/M

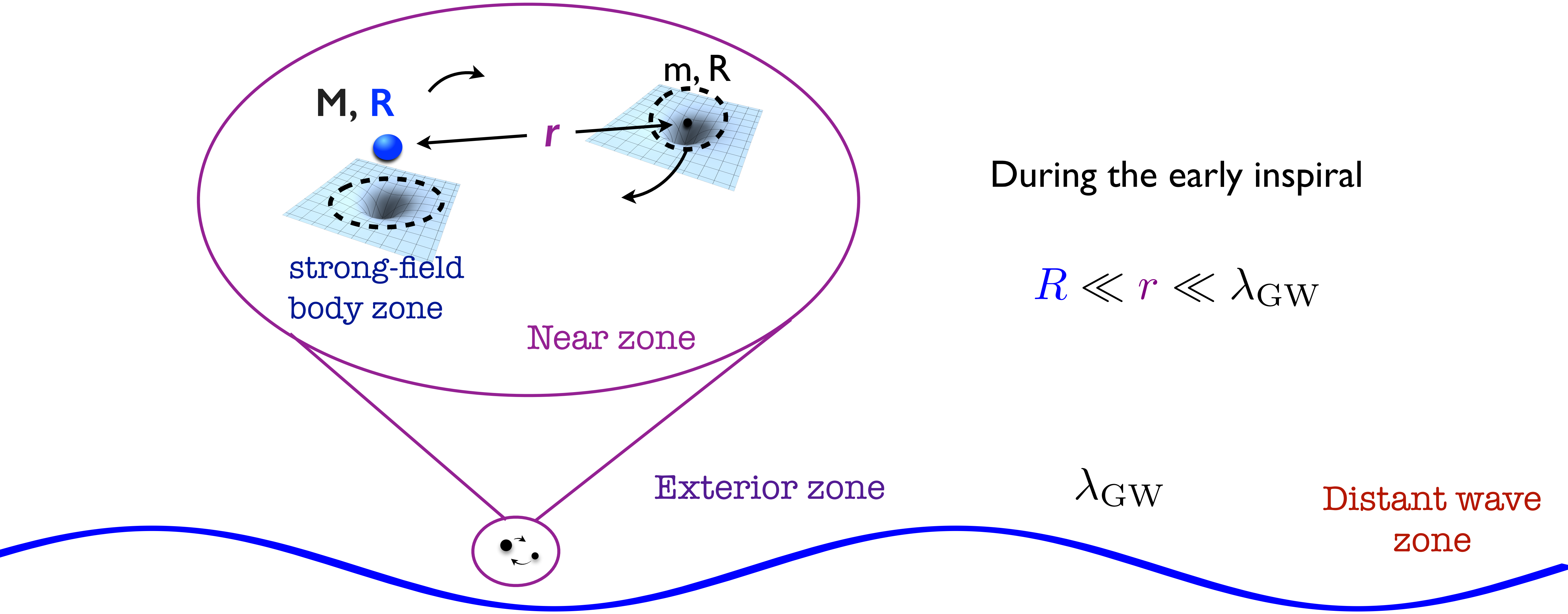
Gravitational interaction potential GM/rc^2

velocity v/c

Internal gravity of each object GM/Rc^2

Size of objects compared to orbit: R/r

Hierarchy of scales in a compact-object binary system



During the early inspiral

$$R \ll r \ll \lambda_{\rm GW}$$

Parameters in the near zone:

Mass ratio m/M - no assumption

- semi-relativistic gravitationally bound system

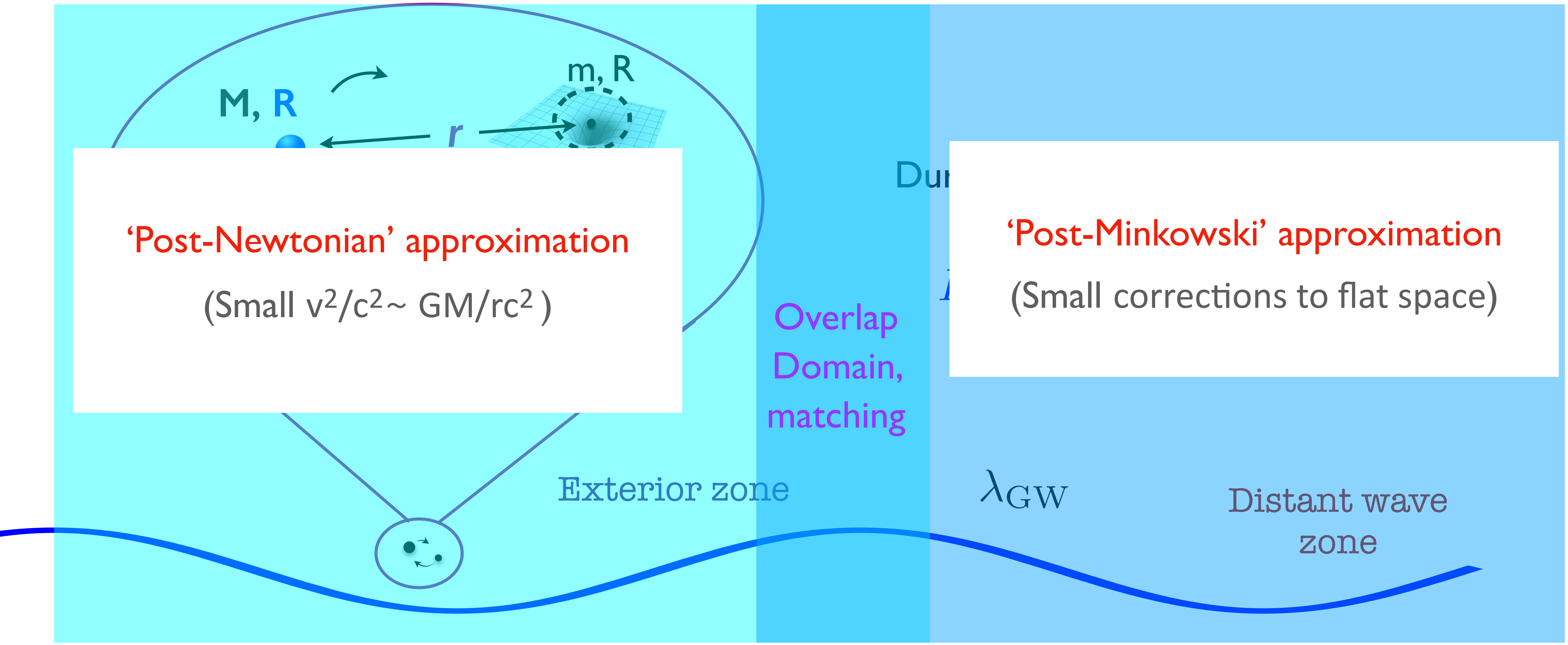
$$\frac{v^2}{c^2} \sim \frac{GM}{rc^2} \ll 1$$

Internal gravity of each object GM/Rc^2

Size of c ... : R/r

- assume point masses for now

Hierarchy of scales in a compact-object binary system



Parameters in the near zone:

Mass ratio m/M - no assumption

$$\boxed{\begin{array}{c}\text{- semi-relativistic gravitationally bound system}\\ \dfrac{v^2}{c^2} \sim \dfrac{GM}{rc^2} \ll 1\end{array}}$$

Internal gravity of each object GM/Rc^2

Size of c : R/r

$\boxed{\text{- assume point masses for now}}$

Recall from last lecture: field equations in harmonic gauge

- Useful quantity to work with:

$$h^{\alpha\beta} = \mathfrak{g}^{\alpha\beta} - \eta^{\alpha\beta}$$

Minkowski metric: diag(-1,1,1,1)

Gothic inverse metric density:

$$\mathfrak{g}^{\alpha\beta} = \sqrt{-g}\, g^{\alpha\beta}$$

- The Einstein field equations take the form:

Source with compact support

$$\Box h^{\alpha\beta} = \frac{16\pi G}{c^4}(-g)T^{\alpha\beta} + \Lambda^{\alpha\beta}$$

Field nonlinearities, extend everywhere

Systematic approximation scheme in the wave zone

Outside the matter source:

$$\Box h^{\alpha\beta} = \Lambda^{\alpha\beta}$$

- In the wave zone, spacetime is nearly flat
- Often, the gravitational constant G is used as a formal expansion parameter for an iterative scheme: "Multipolar post-Minkowski (MPM) approximation"

Ansatz: $$h^{\alpha\beta}_{\mathrm{MPM}} = G\, h^{\alpha\beta}_{(1)\mathrm{MPM}} + G^2\, h^{\alpha\beta}_{(2)\mathrm{MPM}} + O(G^3)$$

Field equations at each order in G: $$\Box h^{\alpha\beta}_{(1)\mathrm{MPM}} = 0$$

$$\Box h^{\alpha\beta}_{(2)\mathrm{MPM}} = \Lambda_2[h_{(1)\mathrm{MPM}}]$$

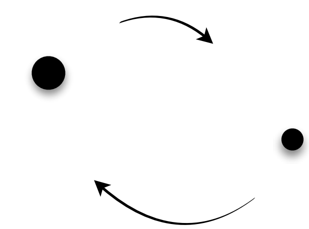
$$\ldots$$

Near-zone: Post-Newtonian (PN) expansion scheme

- Specialize to the matter source of curvature being two orbiting point masses at large separation

Gravitationally bound systems: $$\frac{Gm}{rc^2} \sim \frac{v^2}{c^2}$$

semi-relativistic: $$\frac{v}{c} \ll 1$$



- Commonly, the quantity 1/c is used as a formal expansion parameter

Each factor of $\frac{1}{c^2}$ $\Leftrightarrow$ one PN order

- Further property of this kind of binary system:

$$|T^{ij}|/|T^{00}| = O(c^{-2}) \qquad \text{and} \qquad |T^{0i}|/|T^{00}| = O(c^{-1})$$

Post-Newtonian expansion of the field equations

- Field equations:

$$\Box h^{\alpha\beta} = \frac{16\pi G}{c^4}(-g)T^{\alpha\beta} + \Lambda^{\alpha\beta}$$

PN assumptions on the source:

$$|T^{ij}|/|T^{00}| = O(c^{-2})$$

$$|T^{0i}|/|T^{00}| = O(c^{-1})$$

- Split time and space components of $h^{\alpha\beta}$ (3+1 decomposition):

$$h^{00} = -\frac{4U}{c^2} + O(c^{-4})$$ Scalar gravitational potential

$$h^{0i} = -\frac{V^i}{c^3} + O(c^{-5})$$ Gravitomagnetic vector potential (frame-dragging)

$$h^{ij} = O(c^{-4})$$

- 00-component of the field equations: $\Box U = -\frac{4\pi G}{c^2}T^{00} + O(c^{-4}) = -4\pi G\sigma$ +...

Analysis of the near-zone fields

$$\Box U = -4\pi G\sigma \; + \ldots$$ similarly for other metric potentials

- Retarded solution: $U(t,\mathbf{x}) = G\int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|}\sigma\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c},\mathbf{x}'\right) + \ldots$

- Near-zone:

- retardation effects are small. Expand:

$$\sigma\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c},\mathbf{x}'\right) = \sigma\left(t,\mathbf{x}'\right) + \ldots$$

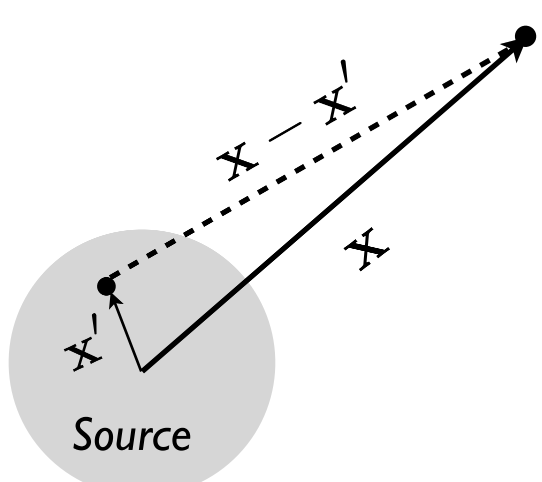
- Metric potentials are instantaneous

- in the exterior zone:

- extent of the source is small. Expand:

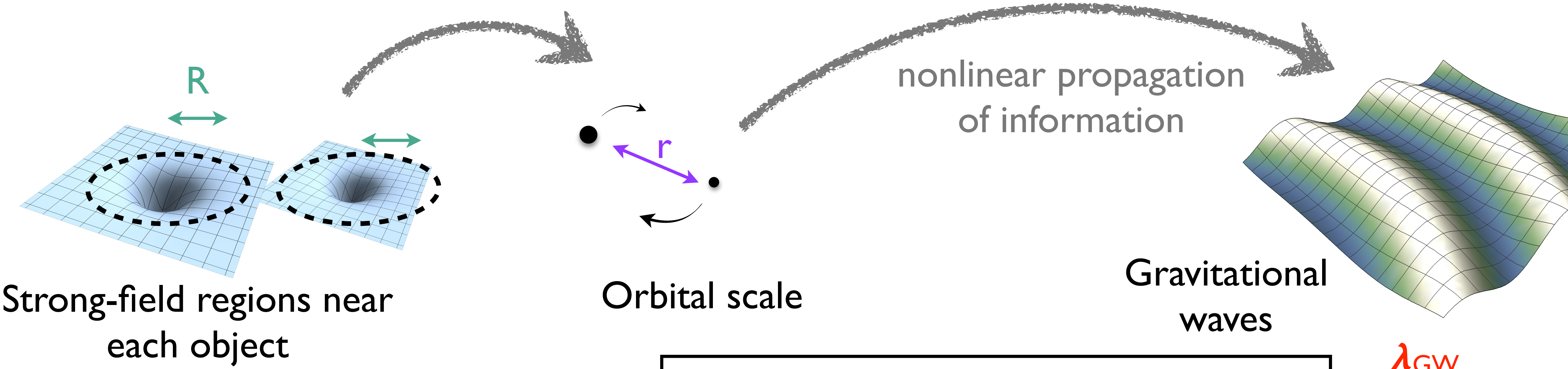
$$\frac{\sigma\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c},\mathbf{x}'\right)}{|\mathbf{x}-\mathbf{x}'|} = \frac{\sigma\left(t-\frac{|\mathbf{x}|}{c},\mathbf{x}'\right)}{|\mathbf{x}|} + \ldots$$

- Multipole expansion



Broader context: multipole expansions far from a source play a key role for GWs

Multipole moments of each object's spacetime transmit information about its nature, internal structure to orbital dynamics



Total radiative multipole moments of the binary system (orbital+object's) imprint the fundamental source properties in GWs

Useful to have compact formulations of systematic multipole expansions that can be adapted to different contexts

Multipole expansions

T. Hartmann, M. Soffel, T. Kioustelidis:

On the use of STF tensors in celestial mechanics,

Celestial Mechanics & Dynamical Astronomy 60, 139 (1994)

link to pdf through adsabs

General Taylor expansion of a function

- Recall the Taylor expansion of any function f around a reference point z:

$$\partial_i = \frac{\partial}{\partial x^i}$$

$$f(\mathbf{x}) = f(\mathbf{z}) + (x-z)^i \partial_i f(\mathbf{x}) \mid_{\mathbf{x}=\mathbf{z}}$$

$$+\frac{1}{2}(x-z)^i (x-z)^j \partial_i \partial_j f(\mathbf{x}) \mid_{\mathbf{x}=\mathbf{z}} + O\left((x-z)^3\right)$$

- Can write this in the compact form:

$$f(\mathbf{x}) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}(x-z)^L \partial_L f(\mathbf{x}) \mid_{\mathbf{x}=\mathbf{z}}$$

- Where L is a shorthand notation for a string of ℓ indices $a_1 \ldots a_\ell$

$$x^L = x^{a_1} x^{a_2} \cdots x^{a_\ell}$$

$$\partial_L = \partial_{a_1} \partial_{a_2} \cdots \partial_{a_\ell}$$

Taylor expansion of potentials

$$f(\mathbf{x}) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}(x-z)^L \partial_L f(\mathbf{x}) \mid_{\mathbf{x}=\mathbf{z}}$$

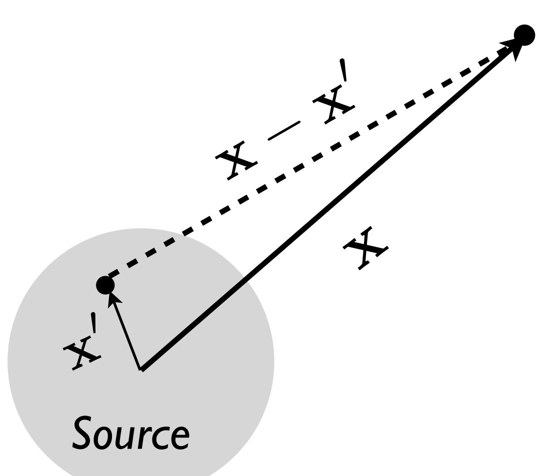
For simplicity, first consider solutions to Poisson's equation (e.g. Newtonian potential)

$$\nabla^2 U = -4\pi G\,\boldsymbol{\rho} \qquad\qquad U(t,\mathbf{x}) = G\int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|}\boldsymbol{\rho}\,(t,\mathbf{x}')$$

Use the Taylor expansion:

Differentiating wrt x instead of x'

$$U(\mathbf{x}) = G\int d^3x' \rho(\mathbf{x}') \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!}(x'-z)^L \left(\partial_L \frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)_{\mathbf{x}'=\mathbf{z}}$$



Next, some notation to express derivatives of 1/|**x**-**x**'|in a compact form and gain more insights

Symmetric Trace-Free (STF) tensors

Consider the quantity $r = \sqrt{\delta_{ij} x^i x^j}$

(Connection to previous slide: r=|**x**-**x**'|)

Derivatives of 1/r: $$\partial_i \frac{1}{r} = -\frac{1}{2}\frac{1}{r^3}\left(2\delta_{bi}x^b\right) = -\frac{x^i}{r^3} = -\frac{n^i}{r^2}$$

Unit vector $n^i = \frac{x^i}{r}$

$$\partial_j \partial_i \frac{1}{r} = -\frac{\delta_{ij}}{r^3} + \frac{3}{2}\frac{x^i}{r^5}\left(2x^j\right)$$

$$= \frac{3}{r^3}\left(n^i n^j - \frac{1}{3}\delta_{ij}\right)$$

$$= \frac{3}{r^3} n^{<ij>}$$

Use angular brackets around indices to denote the symmetric and trace-free (STF) part of a tensor

symmetric under the exchange of any two indices, and tracefree in any pair of indices

STF tensors and derivatives of 1/r

Example: construct the symmetric and trace-free part of $v^i n^j$

$$v^{<i}n^{j>} = \frac{1}{2}(v^i n^j + v^j n^i) - \frac{1}{3}(\mathbf{v}\cdot\mathbf{n})\delta_{ij}$$

Symmetrize
In $i \leftrightarrow j$

Remove the trace

General formula:

For ℓ derivatives of 1/r : $$\partial_L \frac{1}{r} = (-1)^\ell (2\ell - 1)!! \frac{n^{<L>}}{r^{\ell+1}}$$

- Recall that L is a shorthand notation for a string of ℓ indices $a_1 \ldots a_\ell$

Contractions with STF tensors

- When contracting a tensor T_L with an STF tensor $S_{<L>}$ only its STF part contributes

$$T_L S^{<L>} = T_{<L>} S^{<L>}$$

$$\delta^{ij}\delta_{ij} = 3$$

- check for ℓ = 2 and $T_{ij} = n^i n^j$ and $S^{<ij>} = n^{<ij>}$

$$n^i n_i = 1$$

Full result: $n^i n^j n_{<ij>} = n^i n^j \left(n_i n_j - \frac{1}{3}\delta_{ij} \right) = 1 - \frac{1}{3} = \frac{2}{3}$

Using only the STF part:

$$n^{<ij>} n_{<ij>} = \left(n_i n_j - \frac{1}{3}\delta_{ij} \right) \left(n_i n_j - \frac{1}{3}\delta_{ij} \right) = 1 - \frac{1}{3} - \frac{1}{3} + \frac{3}{9} = \frac{2}{3}$$

- Holds for contractions with any STF tensors, not just $n^{<L>}$

Use the STF notation for the potential

- Before, we had obtained the expansion of the potential in the form:

$$U(\mathbf{x}) = G\int d^3x'\rho(\mathbf{x}')\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}(x'-z)^L\left(\partial_L\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)_{\mathbf{x}'=\mathbf{z}}$$

- Take out constants from the integral

$$U = G\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\underbrace{\partial_L\frac{1}{|\mathbf{x}-\mathbf{x}'|}\bigg|_{\mathbf{x}'=\mathbf{z}}}_{=(-1)^\ell(2\ell-1)!!\frac{\bar{n}^{<L>}}{\bar{r}^{\ell+1}}}\underbrace{\int d^3x'\rho(\mathbf{x}')\,(x'-z)^L}_{\equiv M^{<L>}_{\text{Newt}}}$$

Newtonian mass multipoles

- Final multipole expansion

$$\boxed{U(\mathbf{x}) = G\sum_{\ell=0}^{\infty}\frac{(2\ell-1)!!}{\ell!}\frac{\bar{n}^{<L>}M^{<L>}}{\bar{r}^{\ell+1}}}$$

$$\bar{r} = |\mathbf{x}-\mathbf{z}|$$

$$\bar{n}^i = \frac{(x-z)^i}{\bar{r}}$$

First few Newtonian mass multipole moments

Definition of Newtonian STF multipole moments:

$$\boxed{M^{<L>}_{\text{Newt}} = \int d^3x' \rho(\mathbf{x}') (x' - z)^{<L>}}$$

ℓ =0: total mass $\quad M = \int d^3x' \rho(\mathbf{x}')$

ℓ = 1: mass dipole $\quad M^i = \int d^3x' \rho(\mathbf{x}') (x' - z)^i = \int d^3x' \rho(\mathbf{x}') \left(x'^i - z^i\right)$

$$= \int d^3x' \rho(\mathbf{x}') x'^i - z^i \underbrace{\int d^3x' \rho(\mathbf{x}')}_{M} = 0$$

With the reference point z chosen to be the center of mass (CM) $\quad z^i_{\text{CM}} = \frac{1}{M} \int d^3x' \rho(\mathbf{x}') x'^i$

First few Newtonian mass multipole moments

Definition of Newtonian STF multipole moments:

$$\boxed{M^{<L>}_{\text{Newt}} = \int d^3x' \rho(\mathbf{x}')(x' - z)^{<L>}}$$

ℓ =0: total mass $\quad M = \int d^3x' \rho(\mathbf{x}')$

ℓ = 1: mass dipole $\quad M^i = 0$

ℓ = 2 mass quadrupole $\quad M^{<ij>} \equiv Q^{ij} = \int d^3x' \rho(\mathbf{x}')(x' - z)^{<ij>}$

Relation to spherical harmonic expansion

STF tensors are a so-called irreducible representation of the rotation group, as are spherical harmonics. Link between the two representations:

In spherical coordinates: $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

STF multilinears of unit vectors are linear combinations of spherical harmonics of the same ℓ-order

$$n^{<L>} = \frac{4\pi\ell!}{(2\ell+1)!!}\sum_{m=-\ell}^{\ell} \mathcal{Y}^{L}_{\ell m} Y_{\ell m}(\theta,\phi)$$

STF tensors with constant complex coefficients

the STF brackets < ..> around L are commonly omitted for these tensors because they are always STF

Inverse relation: $Y_{\ell m}(\theta,\phi) = \mathcal{Y}^{*L}_{\ell m} n_{<L>}$

Useful property:

$$\mathcal{Y}_{\ell -m} = (-1)^m \mathcal{Y}^{*}_{\ell m}$$

Example for ℓ = 1 and i = x

$$n^x = \sin\theta\cos\phi \qquad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$$

General formula: $n^{<L>} = \frac{4\pi\ell!}{(2\ell+1)!!}\sum_{m=-\ell}^{\ell}\mathcal{Y}^L_{\ell m}Y_{\ell m}(\theta,\phi)$ $\qquad \mathcal{Y}_{\ell -m} = (-1)^m\mathcal{Y}^*_{\ell m}$

Apply to this example: $n^x = \frac{4\pi}{3}\left[\mathcal{Y}^x_{1-1}Y_{1-1} + \mathcal{Y}^x_{10}Y_{10} + \mathcal{Y}^x_{11}Y_{11}\right]$

$$\sin\theta\cos\phi = \sqrt{\frac{4\pi}{3}}\left[\mathcal{Y}^x_{10}\cos\theta - \frac{1}{\sqrt{2}}\mathcal{Y}^x_{11}\sin\theta e^{i\phi} - \frac{1}{\sqrt{2}}\mathcal{Y}^{*x}_{11}\sin\theta e^{-i\phi}\right]$$

Must have $\mathcal{Y}^x_{10} = 0$

Example for ℓ = 1 and i = x

$$n^x = \sin\theta\cos\phi \qquad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$$

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$$\sin\theta\boxed{\cos\phi} = \sqrt{\frac{4\pi}{3}}\left[\mathcal{Y}^x_{10}\cos\theta - \frac{1}{\sqrt{2}}\mathcal{Y}^x_{11}\sin\theta\boxed{e^{i\phi}} - \frac{1}{\sqrt{2}}\mathcal{Y}^{*x}_{11}\sin\theta\boxed{e^{-i\phi}}\right]$$

Must have $\mathcal{Y}^x_{10} = 0$

To get cosϕ need $\mathcal{Y}^{*x}_{11} = \mathcal{Y}^x_{11}$

Example for ℓ = 1 and i = x

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Apply to this example: $n^x = \frac{4\pi}{3}\left[\mathcal{Y}^x_{1-1}Y_{1-1} + \mathcal{Y}^x_{10}Y_{10} + \mathcal{Y}^x_{11}Y_{11}\right]$

$$\sin\theta\,\boxed{\cos\phi} = \sqrt{\frac{4\pi}{3}}\left[\mathcal{Y}^x_{10}\cos\theta - \frac{1}{\sqrt{2}}\mathcal{Y}^x_{11}\sin\theta\,\boxed{e^{i\phi}} - \frac{1}{\sqrt{2}}\mathcal{Y}^{*x}_{11}\sin\theta\,\boxed{e^{-i\phi}}\right]$$

Must have $\mathcal{Y}^x_{10} = 0$

To get cosϕ need $\mathcal{Y}^{*x}_{11} = \mathcal{Y}^x_{11}$

$$\sin\theta\cos\phi = -\sqrt{\frac{2\pi}{3}}\mathcal{Y}^x_{11}\left[\sin\theta\,(2\cos\phi)\right]$$

$$\Rightarrow \mathcal{Y}^x_{11} = -\sqrt{\frac{3}{8\pi}}$$

Equivalent multipole expansions (Newtonian)

Final result: general solutions to $\nabla^2 U = -4\pi G\sigma$ outside the source are given by:

Cartesian: $$U(\mathbf{x}) = G\sum_{\ell=0}^{\infty}\frac{(2\ell-1)!!}{\ell!}\bar{n}^{<L>}\frac{M^{<L>}}{\bar{r}^{\ell+1}} \qquad M^{<L>} = \int d^3x\,\sigma(\mathbf{x})x^{<L>}$$

Spherical: $$U(\mathbf{x}) = G\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}Y_{\ell m}(\bar{\theta},\bar{\phi})\frac{I_{\ell m}}{\bar{r}^{\ell+1}} \qquad I_{\ell m} = \int d^3x\,\sigma(\mathbf{x})r^{\ell}Y^*_{\ell m}(\theta,\phi)$$

The multipole moments on the Cartesian and spherical harmonic basis are related by:

$$M^{<L>} = \frac{4\pi\ell!}{(2\ell+1)!!}\sum_{m=-\ell}^{\ell}\mathcal{Y}^{*L}_{\ell m}I_{\ell m} \qquad I_{\ell m} = \mathcal{Y}^{L}_{\ell m}M^{<L>}_{A}$$

Very useful to have both representations and their interconversion

(e.g. spherical harmonic convenient for compact objects, Cartesian for binary system)

Example application of matched asymptotic expansions & multipole expansions to inspiraling binary systems: post-Newtonian description

Luc Blanchet and Thibault Damour:

Post-newtonian generation of gravitational waves,

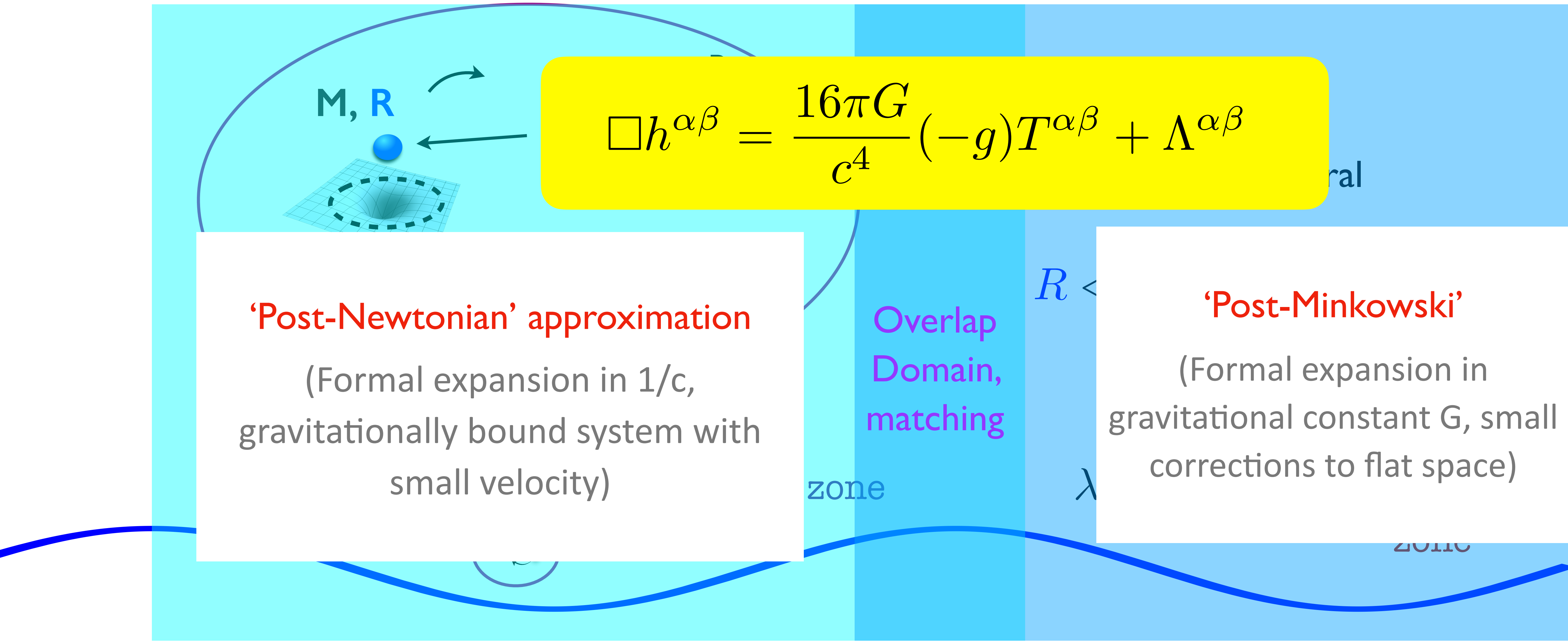
Annales de l'I.H.P. 50, 377 (1989)

link to pdf through numdam

[we will not discuss the technical details]

Role of multipole expansions in post-Newtonian theory

$$\Box h^{\alpha\beta} = \frac{16\pi G}{c^4}(-g)T^{\alpha\beta} + \Lambda^{\alpha\beta}$$



'Post-Newtonian' approximation

(Formal expansion in 1/c, gravitationally bound system with small velocity)

'Post-Minkowski'

(Formal expansion in gravitational constant G, small corrections to flat space)

Parameters in the near zone:

Mass ratio m/M - no assumption

- semi-relativistic gravitationally bound system

$$\frac{v^2}{c^2} \sim \frac{GM}{rc^2} \ll 1$$

Internal gravity of each object GM/Rc^2

Size of : R/r

- assume point masses for now

Result for the GWs in the distant wave zone:

The asymptotic waveform is paramterized by 'radiative' STF mass and current multipole moments:

Unit vectors

Antisymmetric permutation tensor

$$h_{ij}^{\mathrm{TT}} = \frac{G}{c^2\, d} \sum_{\ell=2}^{\infty} \frac{1}{c^\ell} N_{L-2}\, \mathcal{I}_{ijL-2} + \frac{1}{c^{\ell+1}} N_{aL-1} \left(\epsilon_{abi}\, \mathcal{J}_{jbL-2} + \epsilon_{abj}\, \mathcal{J}_{ibL-2}\right) + O\left(\frac{1}{d^2}\right)$$

Radiative current moments

Radiative mass moments

The relation between the radiative and source moments is complicated

because of the nonlinearities in GR

The radiative moments depend not only on the source at the retarded time but on the entire past history of the source

Example: radiative mass quadrupole - dependent on source's history

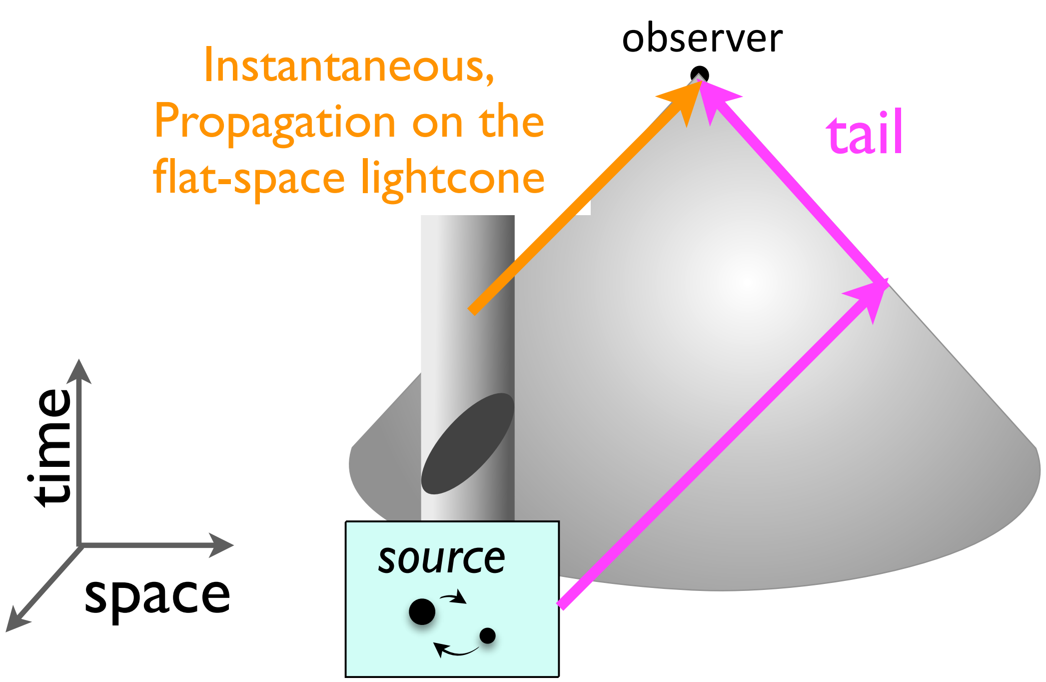
$$\mathcal{I}_{ij} = \ddot{M}_{<ij>} + \frac{GM}{c^3}\int_0^\infty dt' \, \ddddot{M}_{<ij>}(t-t')\left[2\ln\left(\frac{t'}{2t_0}\right)+\frac{11}{6}\right]$$

'Tail'

produced by backscatter of linear GWs off the curvature due to the source's total mass

$$+\frac{G}{c^5}\left[\qquad \cdots \qquad \right]$$

$$+O(c^{-6})$$



Currently known to $O(c^{-9})$

incl. e.g. tails with multiple scatterings

Example: radiative mass quadrupole - dependent on source’s history

$$\mathcal{I}_{ij} = \ddot{M}_{<ij>} + \frac{GM}{c^3}\int_0^\infty dt' \, \ddddot{M}_{<ij>}(t-t')\left[2\ln\left(\frac{t'}{2t_0}\right)+\frac{11}{6}\right]$$

GWs sourced by the previously emitted GWs:

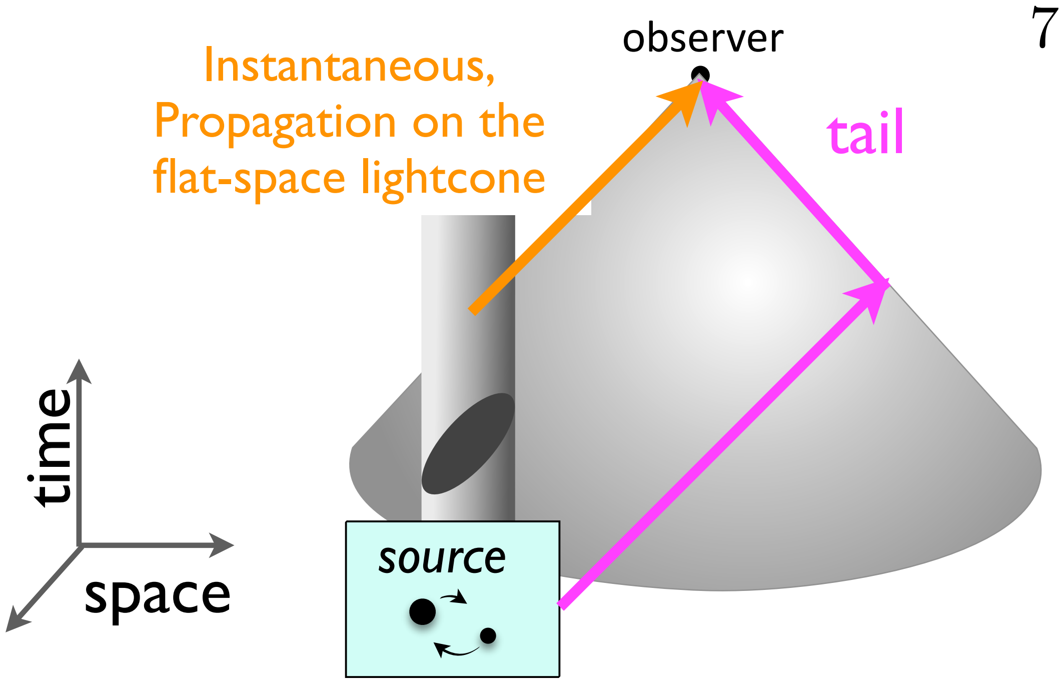
permanent change in amplitude before and after a burst of GWs

$$+\frac{G}{c^5}\left[-\frac{2}{7}\boxed{\int_0^\infty dt' \, \dddot{M}_{a<i}\dddot{M}_{j>a}(t-t')} + \text{instantaneous terms}\right]$$

‘Nonlinear memory’

$$+O(c^{-6})$$

$$\frac{1}{7}M^{(5)}_{a<i}M_{j>a} - \frac{5}{7}\ddddot{M}_{a<i}\dot{M}_{j>a} - \frac{2}{7}\dddot{M}_{a<i}\ddot{M}_{j>a} + \frac{1}{3}\epsilon_{ab<i}M_{j>a}J_b$$

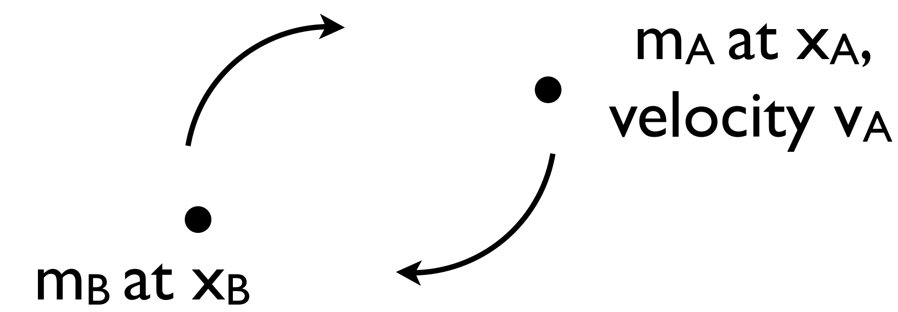


Currently known to $O(c^{-9})$

incl. e.g. tails with multiple scatterings

The post-Newtonian source moments

The source moments include relativistic corrections to the Newtonian relations. E.g. the quadrupole for a binary system:



$$M_{<ij>} = m_A \left[1 + \frac{1}{2}\frac{v_A^2}{c^2} - \frac{G m_B}{c^2 |\mathbf{x}_A - \mathbf{x}_B|} \right] x_A^{<ij>} + \frac{1}{14c^2}\frac{d^2}{dt^2}\left[m_A x_A^2 x_A^{<ij>} \right] - \frac{20}{21c^2}\frac{d}{dt}\left[m_A v_A^k x_A^{<kij>} \right]$$

$$+ O(c^{-4}) + (A \leftrightarrow B)$$

A similar formula holds for higher multipole moments, e.g. the octupole $M_{<ijk>} = m_A x_A^{<ijk>} + O(c^{-2})$

The current moments of the source are also known perturbatively, e.g. the quadrupole:

$$J_{<ij>} = m_A \epsilon^{ab<i} x_A^{j>a} v_A^b + O(c^{-2}) + (A \leftrightarrow B)$$

Features of the relativistic corrections to GWs I:

Can now write the GW amplitudes more explicitly:

Same as the quadrupole formula in linearized gravity for the Newtonian contribution to $M_{<ij>}$

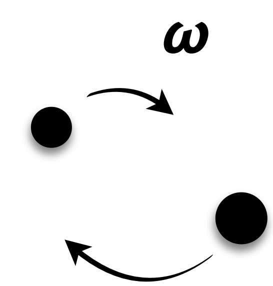
$$h_{ij}^{\rm TT} = \frac{G}{c^4\, d}\left\{\ddot{M}_{<pq>} + \frac{1}{c}N_a\left[\dddot{M}_{<pqa>} + \left(\epsilon_{acp}\ddot{J}_{<qc>} + \epsilon_{acq}\ddot{J}_{<pc>}\right)\right] + \frac{1}{c^3}({\rm tail}) + O(c^{-4})\right\}$$

Other multipoles

GWs oscillate at different harmonics of the orbital frequency

Time evolution of binaries on circular orbits

- From the waveform, obtain the relativistic corrections to the GW power $\dot{E}_{\mathrm{GW}}$
- From the matching calculations, also obtain the equations of motion and orbital energy E
- Use energy balance to compute the evolution of the orbital frequency



$$\frac{d\omega}{dt} = -\frac{\dot{E}_{\mathrm{GW}}}{dE/d\omega}$$

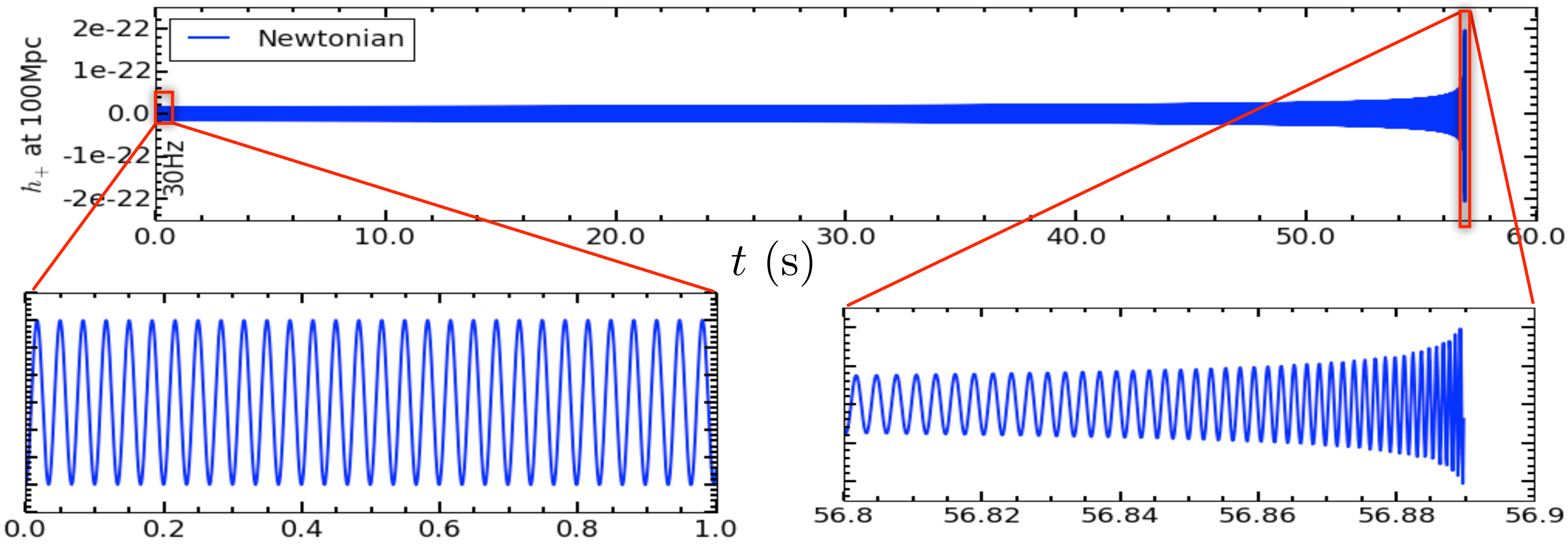
$$= \frac{96}{5\,c^5}\pi^{8/3}\mathcal{M}^{5/3}\omega^{11/3}\left[1 - \left(\frac{743}{336} + \frac{11}{4}\nu\right)\left(\frac{GM\omega}{c^3}\right)^{2/3} + \ldots\right]$$

Higher powers of $(GM\boldsymbol{\omega})^{1/3}/c$

Depends not only on the chirp mass but also on the symmetric mass ratio:

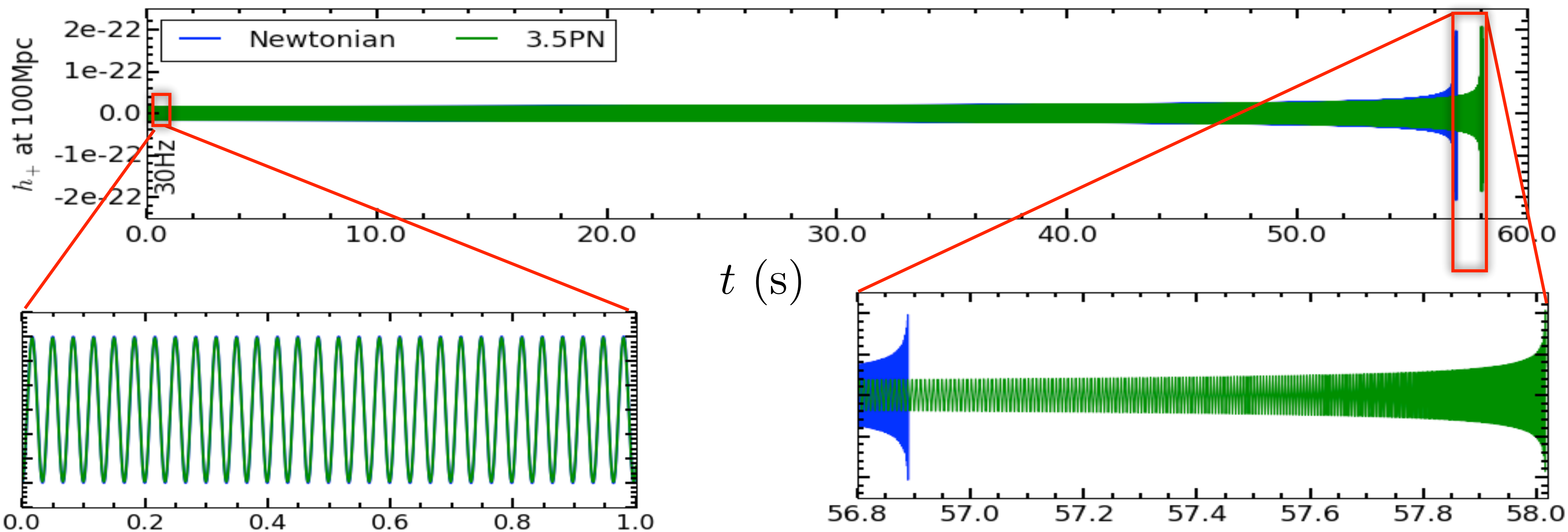
$$\boldsymbol{\nu} = \frac{m_1 m_2}{(m_1+m_2)^2} = \frac{\mu}{M}$$

Newtonian waveform



Newtonian waveform depends only on chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$

Effect of relativistic post-newtonian corrections



Newtonian waveform depends only on chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$

Relativistic corrections also depend on symmetric mass ratio $\nu = \dfrac{\mu}{M}$

Enter with higher powers of the frequency

Recap: Matched Asymptotic expansions

- Matched asymptotic expansions are a powerful tool for approximately describing systems in which different physics dominates on different scales

- Schematic prescription for applying this method (for the examples we are considering):
    - Posit an ansatz for the asymptotic expansion in one regime, determine where it breaks down
    - Rescale variables so they are O(1) in the other regime, use them for a new expansion
    - Match in an overlap domain where both expansions are valid

- you will work with an example in the HW to become more familiar with the method

Recap: post-newtonian description of inspiraling binaries

- Application of the method of matched asymptotic expansions for point masses
    - Post-Newtonian expansion at orbital scales (near zone)
    - post-Minkowski expansion in the asymptotic wave zone

- Multipole moments play a key role for transmitting information
    - Cartesian STF tensors or spherical harmonics

- Matching relates radiative multipoles measured in GWs to source properties

- These relations are complicated by the nonlinearities of GR:
    - unique phenomena, e.g. tails, nonlinear memory
    - GWs are dependent on the entire past history of the source, not just its state at the retarded time

- relativistic corrections to the GWs depend not only on the chirp mass but also the symmetric mass ratio, introduce modes that oscillate at different multiples of the orbital frequency

Final remarks

- The approach outlined in this lecture (by Blanchet, Damour, and collaborators) is the only formalism that (to date) has yielded both the dynamics and the GWs at the highest known orders (3.5PN in the GW phasing, 4PN in the dynamics)

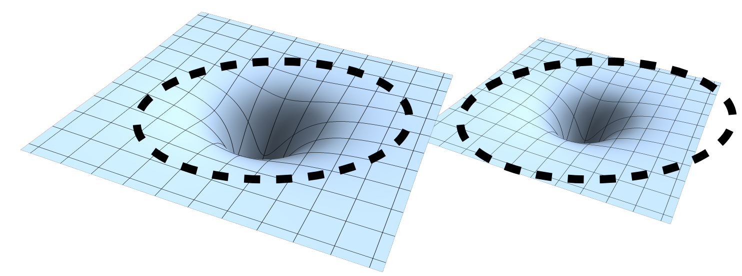
Several other approaches have worked out partial results, e.g.:

- To 2PN order, the dynamics and GWs have also been computed by a direct integration scheme instead of using matched asymptotic expansions [Will and collaborators]

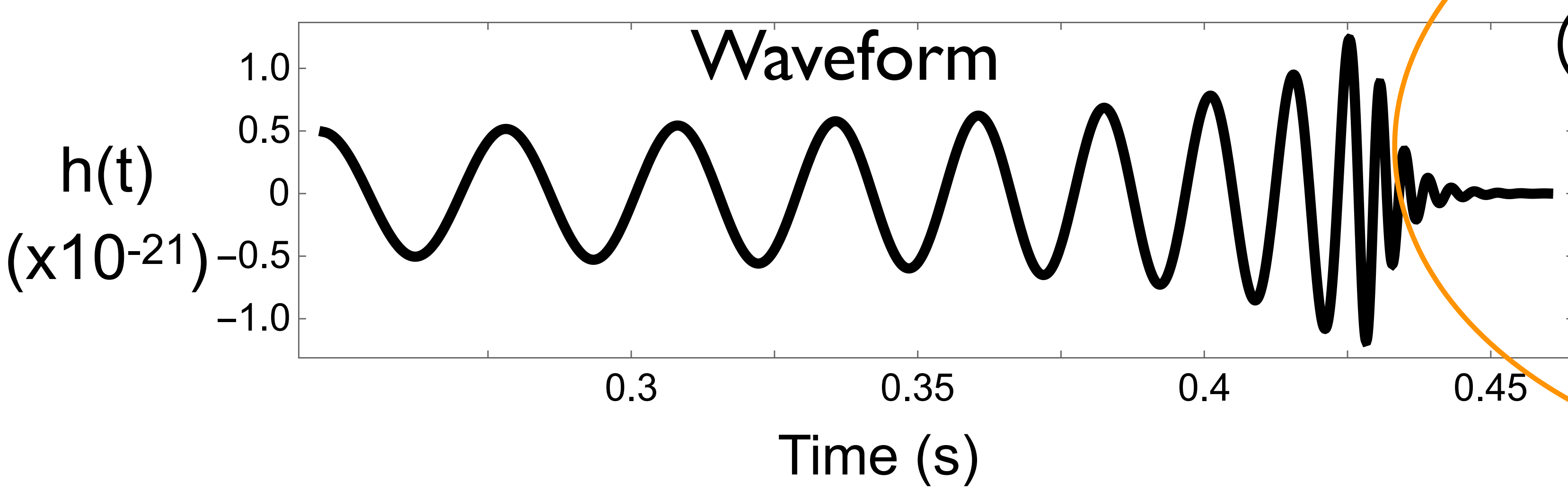
- The post-Newtonian corrections to the orbital dynamics have also be computed via:
    - A Hamiltonian formulation [Jaranowski, Schaefer, Damour and collaborators]
    - An effective field theory approach, which casts the perturbative GR problem in terms of Feynman diagrams that can be computed very efficiently, see e.g. the review by Michèle Levi https://arxiv.org/abs/1807.01699

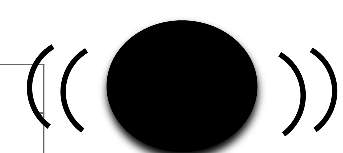
Next lecture: perturbations of compact objects

Finite size effects that depend on the object's nature and internal structure



Strong-field regions near each object





Ringdown

Characteristic fingerprint of the final remnant