Homework III – Optional

Perturbations of compact objects

1. Quasi-normal modes of Schwarzschild black holes
   The quasinormal modes (QNMs) of a black hole are characterized by complex frequencies $\omega = \omega_R + i\omega_I$, where $\omega_R = \mathrm{Re}(\omega)$ and $\omega_I = \mathrm{Im}(\omega)$ are the real and imaginary parts, respectively. We are using the conventions here that the perturbation modes oscillate as $\sim e^{(\pm i\omega_R - \omega_I)t}$. For a Schwarzschild black hole, the QNM frequencies depend on the multipolar order ℓ and the overtone number n, which identifies the number of nodes in the radial wavefunction. For the quadrupolar, $\ell = 2$ modes, numerical results for the mode frequencies from https://arxiv.org/pdf/gr-qc/0411025.pdf are listed in table 1.

n	ω_R	ω_I
1	0.747343	0.177925
2	0.693422	0.547830
3	0.602107	0.956554
4	0.503010	1.410296
5	0 .415029	1.893690
6	0.338599	2.391216
7	0.266505	2.895822
8	0.185617	3.407676
9	0.00000	3.998000
10	0.126527	4.605289
11	0.153107	5.121653
12	0.165196	5.630885
20	0.175608	9.660879
30	0.165814	14.677118
40	0.156368	19.684873

Table 1: Numerical results for the QNM frequencies (decomposed into the real ω_R and imaginary parts ω_I) of a Schwarzschild black hole. From https://arxiv.org/pdf/gr-qc/0411025.pdf.

Make plots of the values in Table 1 showing ω_R versus n and ω_I versus n.

Notice that when interpreting ω_R, the real part of the frequency, as the oscillation frequency and ω_I (the imaginary part) as the decay rate, the plot of ω_R exhibits unusual features compared to expectations for modes in typical mechanical systems. For example, for a string or an elastic body, both the oscillation frequency and the decay rate increase with increasing overtone number, i.e. with an increasing number of nodes in the wavefunction. This is in contrast to the non-monotonic behavior seen for ω_R in your plot. On the other hand, with the interpretation of ω_I as a decay rate, the plot shows the expected behavior. The puzzle about the bizarre behavior of ω_R can be resolved by re-interpreting ω_R and ω_I in a different way. To elucidate this issue, we consider the problem of a damped oscillator whose amplitude $q(t)$ is the solution to

$$\ddot{q} + \gamma\dot{q} + \omega_0^2 q = 0. \tag{1}$$

Here, γ is a damping factor and ω_0 is the characteristic frequency of the oscillator.

Make the ansatz

$$q(t) = c_+ e^{i\omega_+ t} + c_- e^{i\omega_- t}, \tag{2}$$

where c_+ and c_- are constants determined by initial conditions. Substitute into (1) to solve for the frequencies $\omega_\pm$ in terms of γ and ω_0.

Next, notice that the two linearly independent solutions in (2) are oscillations that can be cast into the form

$$\sim e^{(\pm i\omega_R - \omega_I)t}, \tag{3}$$

and determine the relationship between the parameters $\{\omega_R, \omega_I\}$ and the frequency and damping $\{\omega_0, \gamma\}$ of the oscillator.

Then apply these insights to the Schwarzschild QNMs and plot ω_0 versus n. Does this resolve the above puzzle?

2. Computation of tidal Love numbers in Newtonian gravity

The tidal deformabilities or Love numbers of a compact object are computed from relativistic perturbations, i.e. solving the field equations and equations of motion for the matter for small, static perturbations to an equilibrium configuration. In this exercise, you will work through the calculation in Newtonian gravity, where the equations are much simpler and there are fewer variables, but the same approach carries over to the relativistic case.

Specifically, we consider a fluid star in Newtonian gravity. It is described by solutions to the Newtonian field equations for the gravitational potential

$$\nabla^2 U = -4\pi G\rho, \tag{4a}$$

and the equations of energy-momentum conservations, which lead toEuler's equation (conservation of momentum)

$$\frac{\partial v^i}{\partial t} + (\boldsymbol{v}\cdot\nabla)v^i = -\frac{\partial^i p}{\rho} + \partial_i U + a^i_{\rm tidal}, \tag{4b}$$

where $a^i_{\rm tidal}$ is the acceleration due to external tidal forces and U is the body's gravitational potential.

For an equilibrium configuration, there is no external force, and the equations (4) simplify to the equation of hydrostatic equilibrium and the mass integral. We will denote We will denote the unperturbed quantities by a subscript 0 so that the equilibrium configuration is characterized by pressure p_0, density rho_0, and mass m_0. The total mass M and radius R of the star are computed by integrating these equations for a given equation of state $p(\rho)$ from the center of the star at $r = 0$ to its surface, which is defined by the radius at which the pressure vanishes.

Suppose now that the star is placed in an external static tidal gravitational field $\mathcal{E}_{ij}$. For simplicity, we will consider only a quadrupolar perturbation; the calculation for arbitrary multipole moments is very similar. The external tidal potential and corresponding external acceleration on the star are given by

$$U_{\rm tidal} = -\frac{1}{2}\mathcal{E}_{ij}x^i x^j, \qquad a^i_{\rm tidal} = \partial_i U_{\rm tidal}. \tag{5}$$

The star will deform in response to the tidal field and settle down to a new configuration with a nonzero quadrupole moment Q_{ij}. The quadrupole moment can be defined in terms of an expansion of the total gravitational potential $U_{\rm total} = U + U_{\rm tidal}$ (we are considering a region of space containing only the star and an external tidal field of unspecified origin). The total potential outside the star (using 'ext' to denote its exterior) is then given by

$$U^{\rm ext}_{\rm total}(x) = \frac{GM}{r} + \frac{3G}{2}Q_{ij}\frac{n^{<ij>}}{r^3} - \frac{1}{2}\mathcal{E}_{ij}r^2 n^{<ij>}. \tag{6}$$

Specifically, the quadrupole moment is associated with the piece of the exterior gravitational potential that falls off as $1/r^3$, where r is the distance from the center of the star. Similarly, the tidal moment $\mathcal{E}_{ij}$ is related to the coefficient of the piece of $U_{\rm total}$ that grows as r^2. Here, we use the notation $n^i = x^i/r$ and $n^{<ij>}$ is the symmetric-trace-free (STF) part of the tensor $n^i n^j$.

To linear order in the external tidal perturbation and in the adiabatic limit, the star's quadrupole distortion is a linear response of the form

$$Q_{ij} = -\lambda\mathcal{E}_{ij}, \tag{7}$$

where the coefficient λ is the tidal deformability of the star. It is related to the dimensionless Love number k_2 by

$$\lambda = \frac{2}{3G}k_2 R^5 \tag{8}$$

The goal in this exercise is to derive all the inputs and equations needed to compute λ, which then in general would have to be solved numerically. This will require calculating the perturbed interior of the star and the solution outside the star, and matching the interior description to the asymptotic behavior at large distances in (6) and (7) to determine λ. We will broadly proceed by (i) writing down Eqs. (4) for small perturbations to an equilibrium configuration, (ii) eliminating variables to obtain a differential equation involving only the perturbed gravitational potential and equilibrium quantities, (iii) matching the interior solution to the exterior asymptotic exterior multipole expansion of the potential, (iv) manipulating the equations to arrive at a formula for λ in terms of the surface values of the interior solution.

(a) *Linear Perturbations*

The perturbed star is also described by the equations of motion (4). Write the pressure, density, and gravitational potential as $p = p_0 + \delta p$, $\rho = \rho_0 + \delta\rho$, and $U = U_0 + \delta U$, where δ indicates the perturbations, assumed to be small. Throughout this exercise, we will work only to linear order in any of the perturbations δ.

The fluid perturbation can be re represented by a Lagrangian displacement $\xi(x,t)$, which is defined so that the fluid element at position x in the unperturbed star is at position $x + \xi(x,t)$ in the perturbed star. Then, to linear order, the perturbations in velocity can be written as

$$v = \delta v = \dot{\xi}, \tag{9}$$

since the fluid in the equilibrium configuration has no velocity. Substitute these perturbed quantities into Euler's equation (4b). Working only to linear order in the deviations from the background star, derive an equation for $\ddot{\xi}$.

This equation is the foundation for the usual theory of stellar perturbations in terms of modes where one assumes a solution of the form $\xi(x,t) = e^{i\omega t}\xi(x)$. This is needed to compute, for example, the normal mode frequencies of oscillations of the star. However, for calculating the Love number, we can specialize to static perturbations where $\dot{\xi} = \ddot{\xi} = 0$.

(b) *Simplification for an equation of state $p = p(\rho)$*
Specialize to an equation of state of the form $p = p(\rho)$ to relate $\partial_i p$ and δp to $\partial_i \rho$ and $\delta\rho$. In general, the equation of state may also depend on other quantities such as temperature and composition but we will not consider such dependences here.
Show that using these relations, the terms involving δp and $\delta\rho$ in your result from (a) can be combined into the form

$$-\partial_i\left(\frac{1}{\rho_0}\frac{dp_0}{d\rho_0}\delta\rho\right). \tag{10}$$

(c) *Algebraic relation between $\delta\rho$ and δU_{tot}*
Specialize your result from (a) with (b) to the case of static perturbations, where $\ddot{\xi} = 0$. Substitute the external acceleration due to tidal forces from (5), and perform the integration to obtain an algebraic relation between $\delta\rho$ and $\delta U_{\mathrm{tot}} = \delta U + U_{\mathrm{tidal}}$. Here, you can set the integration constant to zero.

(d) *Spherical harmonic expansion and linear perturbations to Poisson's equation*
Because of the spherical symmetry of the equilibrium configuration, it is convenient to expand the perturbations on the spherical harmonic basis. Using the fact that $\mathcal{E}_L n^{<L>} = \sum_m \mathcal{E}_m Y_{\ell m}(\theta,\phi)$ we can write the external quadrupolar tidal potential as

$$U_{\mathrm{tidal}} = -\frac{1}{2}\sum_{m=-2}^{2}\mathcal{E}_m r^2 Y_{2m} \tag{11}$$

Similarly, we decompose the quadrupole as

$$Q_{ij} = \sum_{m=-2}^{2} Q_m \mathcal{Y}^{*2m}_{ij} \tag{12}$$

Note that sometimes the normalization is chosen as $Q_{ij} = \frac{8\pi}{15}\sum_m Q_m \mathcal{Y}^{*2m}_{ij}$ but we will absorb the prefactor into a re-definition of Q_m here for convenience, and similarly for $\mathcal{E}_m$. Using the property that $Y_{\ell m} = \mathcal{Y}^{*\ell m}_L n^L$ we see that when contracting the relation (7) with n^{ij} it can be written as

$$Q_m = -\lambda\mathcal{E}_m. \tag{13}$$

Since this holds separately for each m, it is sufficient to consider a single value of m to compute λ.
In spherical coordinates, the Laplace operator appearing in (4a) is given by

$$\nabla^2 = \frac{1}{r^2}\partial_r r^2\partial_r + \frac{1}{r^2\sin\theta}\partial_\theta\sin\theta\partial_\theta + \frac{1}{r^2\sin^2\theta}\partial^2_\phi, \tag{14}$$

where the spherical harmonics have the property that

$$\left(\frac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta + \frac{1}{\sin^2\theta}\partial^2_\phi\right)Y_{\ell m} = -\ell(\ell+1)Y_{\ell m} \tag{15}$$

The perturbations to all the quantities for a fixed value of m can then be expanded as

$$\delta\rho = h(r)Y_{2m}(\theta,\phi), \qquad \delta U_{\mathrm{tot}} = g(r)Y_{2m}(\theta,\phi), \tag{16}$$

where we have specialized to the quadrupolar $\ell = 2$ sector we are considering here.
Write down Poisson's equation (4a) in spherical coordinates to linear order in the perturbations and substitute the decomposition (16). Use Eqs. (14) and (15) to derive a differential equation for $g''(r)$, the perturbation to the gravitational potential.

(e) *Master equation*
Substitute the decomposition (16) and the expansion (11) in your result from (c) and combine this relation with your result from (d) to obtain a single equation for $g(r)$ in the region $r \le R$ (the interior of the star) given by

$$g'' + \frac{2}{r}g' - \frac{6}{r^2}g = -\frac{4\pi}{f(r)}g, \tag{17}$$

where

$$f(r) = \frac{1}{\rho_0} \frac{dp_0}{d\rho_0} \tag{18}$$

In general, except for special choices of the equation of state, this differential equation has to be integrated numerically to find solutions.

The boundary conditions for (17) are derived from the requirements that the solution must be regular at the center of the star (no divergences in this limit) and that at the surface of the star, the interior solution should match to the piece of the multipolar solution for the potential outside the star (6) converted to the spherical-harmonic expansion.

(f) *Love number*

For $r > R$, the solution is given by converting Eq. (6) to the spherical harmonic representation. One can verify by direct substitution into (17) in the exterior (with the right hand side set to zero, since the pressure and density vanish outside the star) that

$$g(r) = \frac{3GQ_m}{2r^3} - \frac{1}{2}\mathcal{E}_m r^2 \tag{19}$$

is indeed a solution to (17).

To derive an equation for the Love number in terms of the properties of the interior of the star, you can proceed as follows. First, use the definition of λ from Eq. (13) to eliminate Q_m from Eq. (19).

Next, consider the quantity

$$y(r) = \frac{rg'(r)}{g(r)} \tag{20}$$

which eliminates the dependence on $\mathcal{E}_m$. Finally, evaluate Eq. (20) at the surface of the object $r = R$, and solve for k_2 defined in (8) to show that

$$k_2 = \frac{2 - y(R)}{2\left[3 + y(R)\right] |} \tag{21}$$

Given these results, the strategy for practical computations of the Love number is the following: (i) obtain a solution for the background equilibrium configuration, (ii) compute the perturbed interior described by (17), (iii) evaluate from this $y(R)$, and (iv) finally use it in Eq. (21) to obtain the Love number.