Homework I

due May 10 before 11 am (start of the next lecture)

please email a SCAN of your work in a SINGLE pdf file to g.f.crecikeinbaum@uu.nl

Features of GWs from binary systems

The goal of these exercises is to gain intuition about how the properties of compact objects in a binary system are imprinted the emitted GW signals. The calculations follow a general approach within the adiabatic approximation, where we assume that the inspiral can be described by a continuous sequence of circular orbits whose evolution is determined by energy balance. The numerical estimates involved will be useful to gain intuition about the extreme scales involved in these events.

References:

1. E. Flanagan and S. Hughes, *The basics of gravitational wave theory*, New J.Phys. 7, 204 (2005) `https://arxiv.org/abs/gr-qc/0501041` (focus on p.1-7 and 17-23)

2. LIGO Scientific Collaboration and Virgo Collaboration: The basic physics of the binary black hole merger GW150914. Annalen der Physik, 529. `https://arxiv.org/abs/1608.01940`

3. C. Cutler et al., *The last three minutes*, Phys. Rev. Lett. **70**, 2894 (1993) `https://arxiv.org/abs/astro-ph/9208005`

Task 0: Install Mathematica and the tensor package xAct

Install Mathematica on your laptop or a computer you use if you do not already have it. You will need this program for the tutorials 2 and 3. Mathematica is available through your institution, for information see:
- UvA: `https://student.uva.nl/en/content/az/software/mathematica/mathematica.html`
- UU: `https://students.uu.nl/en/free-software`
- UL: `https://helpdesk.strw.leidenuniv.nl/wiki/doku.php?id=general_software:mathematica`

Once you have installed Mathematica, also install the tensor algebra package xAct from `http://www.xact.es`, specifically the single .tgz or .zip file with the current version of all packages. The installation notes for different operating system provided on that website. The installation is a matter of downloading and placing the decompressed file folder in a location where Mathematica can find it, see `http://www.xact.es/download/install` for how to identify this location for your operating system.

Preliminaries:

We will consider a Newtonian binary system on circular orbits and work at the level of the Einstein quadrupole formula for GWs; this approximation will be implied throughout. The Lagrangian describing the relative motion of two point masses m_1, m_2 in the center of mass frame is

$$L = \frac{1}{2}\mu v^2 + \frac{G\mu M}{r}. \tag{1}$$

Here, $r = \sqrt{\delta_{ij}x^i x^j}$ is the relative separation, and $v^2 = \delta_{ij}\dot{x}^i\dot{x}^j$ the square of the relative velocity. The reduced and total mass are $\mu = m_1 m_2/(m_1+m_2)^2$ and $M = m_1 + m_2$ respectively. The quadrupole formula for the rate at which energy is carried by the GWs is

$$\dot{E}_{\rm GW}^{\rm quad} = \frac{G}{5c^5}\langle \dddot{Q}_{ij}\dddot{Q}^{ij}\rangle, \tag{2}$$

where overdots denote time derivatives and the angular brackets indicate the average, e.g., over an orbit. The quantity Q_{ij} is the Newtonian traceless mass quadrupole moment of the source with mass density ρ given by $Q^{ij} = \int \rho\, d^3x(x^i x^j - \frac{1}{3}|\boldsymbol{x}|^2\delta^{ij})$. The GW losses lead to a change in the circular-orbit angular frequency ω determined by the balance of the energy radiated in GWs with a change in energy of the source:

$$\dot{\omega} = -\dot{E}_{\rm GW}/(dE/d\omega). \tag{3}$$

The GW amplitudes for radiation propagating along the direction $\mathbf{N}$ are given by

$$h_{ij} = \frac{1}{d}\frac{2G}{c^4}\Lambda_{ijkl}(\mathbf{N})\ddot{Q}_{kl} \tag{4}$$

where d is the distance from the observer to the binary and $\Lambda_{ijkl}(\mathbf{N})$ is the transverse-traceless projector

$$\Lambda_{ijkl}(\mathbf{N}) = P_{ik}\,P_{jl} - P_{ij}\,P_{kl}/2 \qquad P_{ij} = \delta_{ij} - N_i\,N_j, \tag{5}$$

with $\mathbf{N}$ a unit vector pointing from the source to the observer. The polarization amplitudes are given by

$$h_+ = h_{11}^{\rm TT}, \quad h_\times = h_{12}^{\rm TT}. \tag{6}$$

In the tutorial exercises, you already derived several important quantities, notably the relationship $r(\omega) = (GM/\omega^2)^{1/3}$, the binding energy $E(\omega) = \mu(GM\omega)^{1/3}$, and the mass quadrupole tensor $Q_{ij} = \mu r^2(n^i n^j - \delta^{ij}/3)$, where $\boldsymbol{n} = \boldsymbol{x}/r$ is a radial unit vector. To compute time derivatives it is convenient to introduce an orthogonal unit vector $\boldsymbol{\phi} = \dot{\boldsymbol{x}}/(r\omega)$. In the tutorial, you derived the energy radiated in GWs from this system $\dot{E}_{\rm GW}^{\rm quad} = 32G^{7/3}\mu^2 M^{4/3}\omega^{10/3}/(5c^5)$. We will make use of these results to compute further properties of the binary inspiral and GWs. Note that in the tutorial we worked in terms of the source frame coordinates for simplicity because we were considering quantities such as the energy flux expressed in the form of (2). This is related to power radiated in terms of the polarizations by

$$\dot{E}_{\rm GW} = \frac{d^2\, c^3}{16\pi G}\, 2\pi \int_{-1}^{1} d\cos\iota\, \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle, \tag{7}$$

where the angular brackets denote an average, which in this case you can take as the orbit-average (here equivalent to an average over GW cycles) given by

$$\langle \ldots \rangle = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt \ldots \tag{8}$$

The angle ι characterizes the orientation of the observer's frame relative to the source frame such that that the binary's orbit in the observer's frame can be expressed as

$$x_o^i = r(\cos\varphi, \cos\iota \sin\varphi, \sin\iota \sin\varphi), \tag{9}$$

where $\varphi = \int \omega dt$ is the orbital phase. To compute the individual polarizations we will also choose the z-axis of the observer's frame to be aligned with the direction N of propagation of the GWs.

1. TIME TO MERGER

   (a) [2pts] In this exercise we consider only the quadrupolar GWs. From the energy balance law, obtain the evolution of the orbital frequency $\dot{\omega}(\omega)$ due to GW emission (or take it from the calculations you did in the tutorial). Integrate this differential equation from the current time and frequency $\{t, \omega\}$ to the (formal) values at the time t_c of coalescence $\{t_c, \infty\}$. Define the time before coalescence $\tau = (t_c - t)$.

   (b) [2pts] If the orbital period is 7.75 h for two pulsars with masses $1.4M_\odot$ (which approximates the Hulse-Taylor binary pulsar system), how long will it take until they collide?

   (c) [2pts] How much time remains before collision once the GW frequency $f = \omega/\pi$ (c.f. part b below) for such a system reaches 10 Hz? 100 Hz?

   (d) [2pts] How long does it take for a black hole binary system with equal masses of $30M_\odot$ to evolve from $f = 10$Hz to $f = 100$Hz? How long does it take for this system between leaving the LISA frequency band at 0.1Hz and entering the LIGO/Virgo band at 10Hz?

   (e) [2pts] Estimate the remaining lifetime of the Earth-Sun system due to GW emission, assuming that the Earth's orbit is circular.

2. GW POLARIZATIONS h_+ AND $h_\times$.

   (a) [2pts] Using your results from 1. (a), compute the time evolution of the orbital phase $\varphi(\tau)$.

   (b) [3pts] Calculate explicit results for $h_+(\tau)$ and $h_\times(\tau)$ using the chirp mass $\mathcal{M} = \mu^{3/5}M^{2/5}$. How is the frequency f at which the GWs oscillate related to the orbital frequency ω?

   (c) [5pts] Assuming that a binary system is observed face-on, i.e. inclination $\iota = 0$, plot the time evolution of the polarization amplitude for a system of two neutron stars of 1.4 solar masses each, and for a system of two 35 solar mass black holes. Also plot the corresponding evolution of the frequency. For both of these systems, focus the plot on the last 0.2 seconds before the coalescence time, and notice the differences. If the black hole binary was at 2000 Mpc instead of 40 Mpc, would they still differ?

   (d) [2pts] Write the GW signals as functions of the frequency f. Show that they are invariant under the transformation

   $$(f_{\rm GW}, \mathcal{M}, d, t) \to (f_{\rm GW}/\xi, \mathcal{M}\xi, d\xi, t\xi),$$

   where ξ is scale factor. This invariance is not specific to the Newtonian approximation and also persists for relativistic waveforms in the case of two black holes. This transformation applies e.g. in situations where the detector and source are in relative motion, thus the signal is Doppler shifted by a factor of ξ with respect to the emission. Because of the above invariance, one cannot determine this Doppler shift from the detected signal. This has consequences for black hole binaries at cosmological distances, where the Doppler shift is due to the cosmological redshift z, i.e. the appropriate Doppler shift factor is $\xi = (1 + z)$. Since the GW detectors record only the observed GW frequency, the GW measurements of the source parameters in fact reveal the Doppler or 'redshifted' masses in the detector frame $\mathcal{M}_{\rm det}$ (instead of the physical source-frame masses) and the so-called luminosity distance D_L. Write down the relation $\mathcal{M}_{\rm det}(z, \mathcal{M}_{\rm source})$, and similarly for D_L. Note that the important point of this problem is to appreciate this effect and its consequences; no sophisticated calculations are needed.

3. BACK-OF-THE-ENVELOPE RATE ESTIMATES

Make an estimate of the rate per volume of black hole mergers (expressed in number per $\text{Gpc}^3\text{yr}^{-1}$) based on the events reported during the first observing run O1 of LIGO. The run had 49 total days in which both detectors were taking data. The following three events were detected: GW150914 was at a distance of $\sim$420 Mpc and had a signal to noise ratio of 23.7; GW151226 at a distance of 440 Mpc with a signal to noise ratio of 13.0, and LVT151012 (which was a marginal detection but still contributes to these rate estimates) was at a distance of 1 Gpc and had a signal to noise ratio of 9.7. Assume that the signal to noise ratio (SNR) threshold for announcing detection is 12.

(a) [2pts] Use the fact that for a given event the amplitude of a GW signals scales as the reciprocal of the distance. Conversely, this implies that the distance to the source scales with the inverse of the SNR. Compute out to what distance each of the above three events could have been detected at the threshold SNR. Also estimate the corresponding volume of the universe in which these events were detectable (neglecting cosmology for these back-of-the-envelope estimates).

(b) [3pts] Use these results for the volumes together with the length of LIGO's observing time to estimate the black hole merger rate $\mathcal{R} \sim 1/(VT)$ based on each of the events individually. Then compute the combined rate obtained by adding the results from the individual events. Finally, a crude estimate for the total volume of the universe is $V_{\text{universe}} \sim 10^4 \text{Gpc}^3$. Determine how frequently black holes merge in the universe according to this data.

During the subsequent observing runs O2 and O3, when the detectors had improved in their sensitivity, there have been several dozens of additional detections of merging black holes. For updated rate estimates that also involve more sophisticated inputs see e.g. the LVC paper on astrophysical black hole populations https://arxiv.org/abs/2010.14533.