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LEIDEN UNIVERSITY

ADVANCED TOPICS IN THEORETICAL PHYSICS

March 2021

AGENDA – COURSE OUTLINE

Lecture 1 – Introduction to Financial Markets

Lecture 2 – Heat Equation and Black-Scholes Equation

Lecture 3 – Option Greeks

Lecture 4 – Beyond vanilla options and options in real-life

WorkGroups – Exercises and Optibook

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AGENDA – LECTURE 1

- Introduction Financial Markets:
  - Securities
  - Indices & ETFs
  - Derivatives: Futures and Options
  - Exchanges and exchange participants

- Trading:
  - Strategies
  - Hedging
  - Arbitrage
  - Call-put parity

- Market dynamics
  - Distributions
  - Volatility
  - Black-Scholes Equation

INTRODUCTION TO FINANCIAL MARKETS

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SECURITIES

- A company needs capital to pursue its business objectives

- There are multiple ways to obtain this capital:
    - Retained earnings
    - Loans (banks, crowd funding, …)
    - Issuing bonds
    - Issuing shares
    - Hybrid structures

- The initial exchange of ownership takes place on the primary market. Some instruments can subsequently change ownership (secondary market)

- Exchange of ownership on the secondary markets can be in the form of a direct exchange between two parties (“Over the counter” – OTC) or via intermediary platforms (exchanges)

BONDS

- Debt instrument
- Repayment of the full notional (Principal) at a future date (Typically up to 30y, but can be infinity)
- Typically there are periodic interest payments (coupons). These coupons can be fixed or variable (dependent on interest rates, inflation rates, FX-rates,…)
- Can be issued by Governments, Government Bodies, and Companies (Corporate Bonds)
- Bonds can be covered (e.g. Pfandbrief)
- The higher the perceived creditworthiness of the bond issuer, the lower the coupon rate of the bonds
- Typically bonds are more senior than equity in the event of a default of the issuing entity

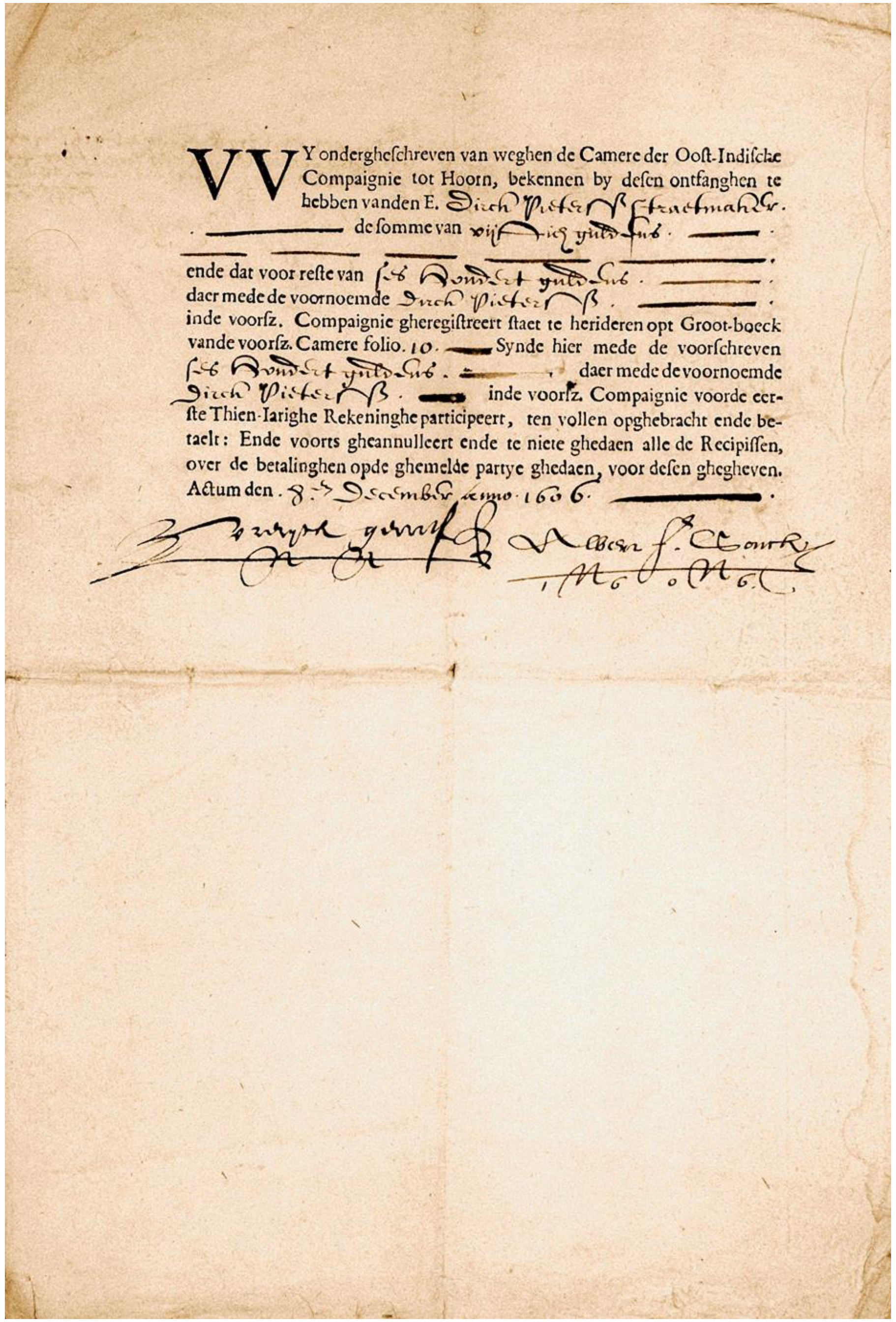
BONDS (2)


1000
1000
The California Eastern Extension
RAIL ROAD
COMPANY
ONE THOUSAND DOLLARS

EQUITIES

- A share represents a partial ownership in the issuing company.
- Shares can have voting rights (Annual shareholder meetings);
- The initial listing of the shares on the market is called an Initial Public Offering (IPO);
- The total value of all outstanding shares represents the market capitalization of the company;
- Companies can periodically pay out (parts of) the generated profit. These payments are called dividends;
- Companies can be listed on multiple exchanges, or can issue different types of shares.

EQUITIES (2)



VVY ondergheſchreven van weghen de Camere der Ooſt-Indiſche Compaignie tot Hoorn, bekennen by deſen ontfanghen te hebben vanden E. Dirck Pietersz Steenmaker. de ſomme van vijftich gulden. ende dat voor reſte van ses Hondert gulden. daer mede de voornoemde Dirck Pietersz. inde voorſz. Compaignie gheregiſtreert ſtaet te herideren opt Groot-boeck vande voorſz. Camere folio. 10. Synde hier mede de voorſchreven ses Hondert gulden. daer mede de voornoemde Dirck Pietersz. inde voorſz. Compaignie voorde eerſte Thien-Iarighe Rekeninghe participeert, ten vollen opghebracht ende betaelt: Ende voorts gheannulleert ende te niete ghedaen alle de Recipiſſen, over de betalinghen opde ghemelde partye ghedaen, voor deſen ghegheven. Actum den. 9 December anno 1606.

STOCK INDICES

- A combination of shares (“basket”) that represent a market segment (country, region, sector, …);
- There are different weight methodologies: (free float) market cap based, price weighted, equal weighted,…
- On a periodic basis the basket composition is adjusted (“index reweight”);
- In some indices the dividends are re-invested;
- In some indices constituents cannot exceed a certain weight after the index reweight;
- Well known indices: AEX, DAX, EuroStoxx, Dow Jones, S&P500, Nikkei;
- On most indices there are listed futures and ETFs that track the index (and options on these instruments).

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STOCK INDICES – AEX EXAMPLE

	Weight (%)	Market Cap (EUR Bln)
ASML	18.9	197.1
Royal Dutch	10.8	118.8
Unilever	9.4	118.3
Adyen	7.3	57.7
Koninklijke Philips	6.3	42.8
RELX	6.3	39.7
Prosus	6.2	166.2
ING	4.5	30.3
Koninklijke DSM	4.0	26.7
Koninklijke Ahold Delhaize	3.8	25.6
Heineken	3.1	51.3
Wolters Kluwer	2.8	18.5
Akzo Nobel	2.6	16.4
Arcelor Mittal	1.9	20.9
ASM International	1.7	11.6
Just Eat Takeaway	1.7	13.2
NN Group	1.6	11.7
Koninklijke KPN	1.4	11.2
Unibail Rodamco	1.3	8.7
Aegon	0.9	7.7
Randstad	0.9	10.2
IMCD	0.8	5.9
ASR	0.7	4.7
Galapagos	0.6	5.9
ABN AMRO	0.5	8.0

EXCHANGE TRADED FUNDS (ETFS)

- Trackers of a certain index (there can be a multiplier (leverage factor) on this tracking)
- The index can be a stock index, but can also track other asset classes (e.g. bonds, real-estate, crypto currencies,…)
- They can be traded like shares. ETFs can pay dividends
- ETFs can be created and redeemed ("annihilated") by certain market participant ("Authorized Participants"). This mechanism ensures that the ETF price closely follows the index it tracks.
- Well known examples: SPY (tracking the S&P), QQQ (tracking the Nasdaq)

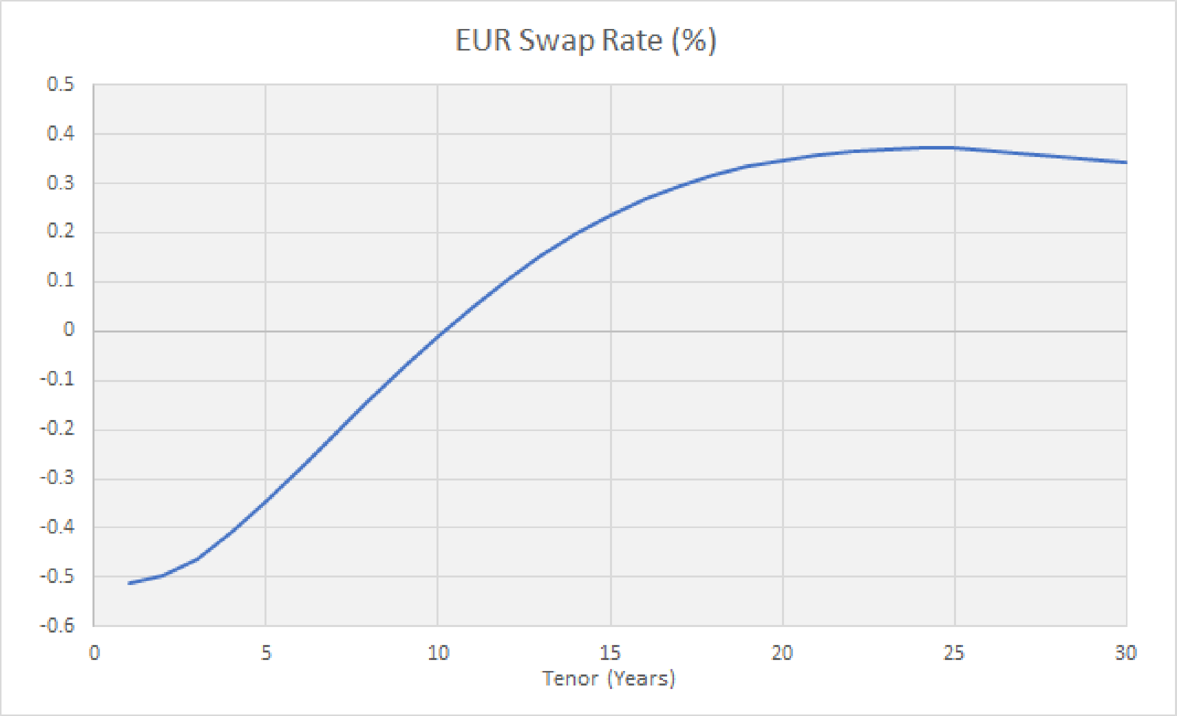
INTERMEZZO - TIME VALUE OF MONEY

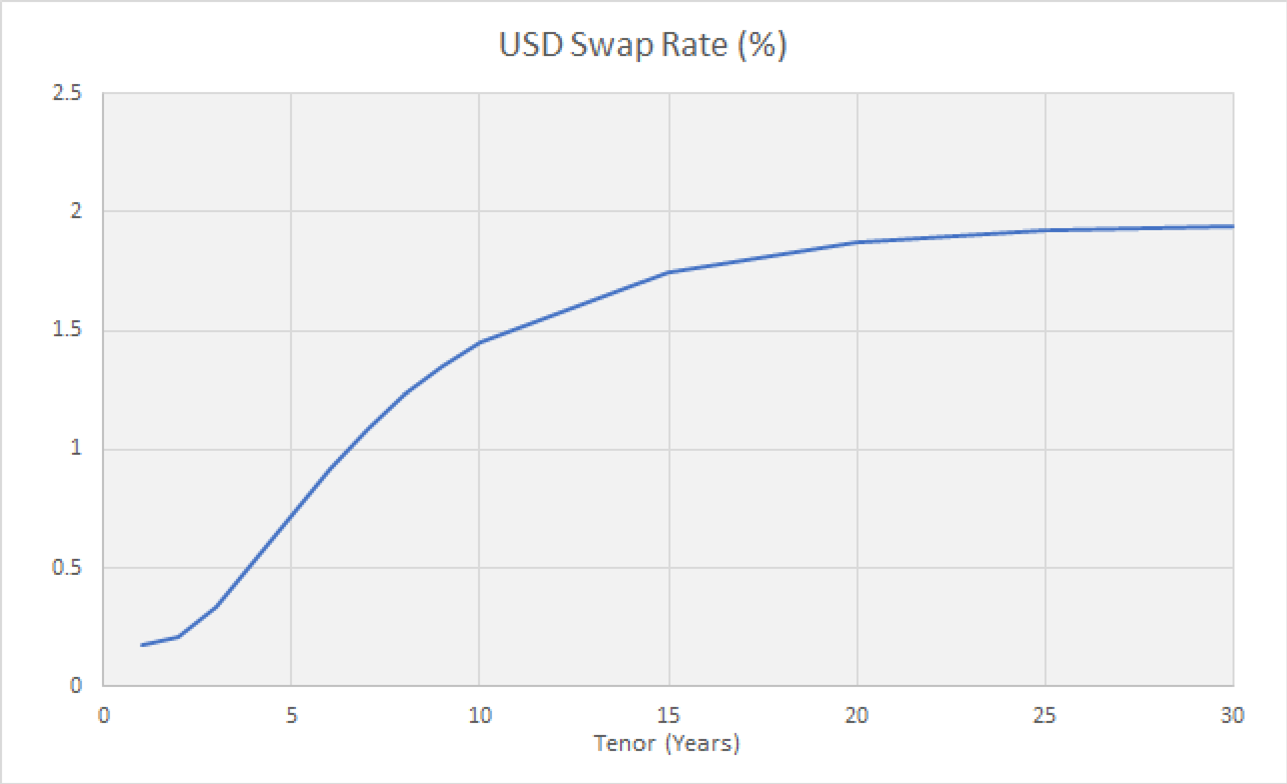
- Assume an annual interest rate *r*, unit: %/year.
- A deposit of EUR 100 will grow in *t* years to $100(1 + r)^t$
- The compounding rate can be more frequent, e.g. six-monthly. With semi-annual compounding of r_2 the deposit grows in *t* years to $100(1 + \frac{r_2}{2})^{2t}$.
- This can be extended to *n* compoundings per annum: $100(1 + \frac{r_n}{n})^{nt}$.
- Take the limit $n \to \infty$, this yields an amount of $100\ e^{r_c t}$. This is called the continuous interest rate.

INTEREST RATE CURVE AND FORWARD RATES

- For different periods of depositing/lending money, typically different interest rates apply;
- This is called the interest rate term structure.
- Assume annual interest rates, if the m-year rate is r_m and the n-year rate is r_n (n>m), then one can also construct the forward rate between year m and year n:
- $(1+r_m)^m(1+r_{mn})^{n-m}=(1+r_n)^n \;\rightarrow\; r_{mn}=\left(\frac{(1+r_n)^n}{(1+r_m)^m}\right)^{\frac{1}{n-m}}-1$
- For continuous rates:
- $e^{r_m m}e^{r_{mn}(n-m)}=e^{r_n n} \rightarrow r_{mn}=\frac{r_n n-r_m m}{n-m}$
- This can be re-written as: $r_{mn}=r_m+\frac{r_n-r_m}{n-m}n$, hence for an upward sloping yield curve the forward rate at time m is larger than the interest rate to time m (and vice versa)

INTEREST RATE CURVE





INTEREST RATES CONTINUED – PRESENT VALUE

- Interest rates can also be used to translate future cash (flows) to present time value

- Assume a cash amount P_n and a continuous n-year interest rate of r_n. Holding an amount P_n n year from now is equivalent to holding an amount P_0 at this moment in time where $P_0 = P_n e^{-r_n n}$.

- Example: Assume a 3-year bond paying an annual coupon of C. Assume the 1-,2- and 3-year interest rates to be respectively r_1, r_2 and r_3. The current value of the bond is:

$$B = Ce^{-r_1 1} + Ce^{-r_2 2} + (100 + C)e^{-r_3 3}$$

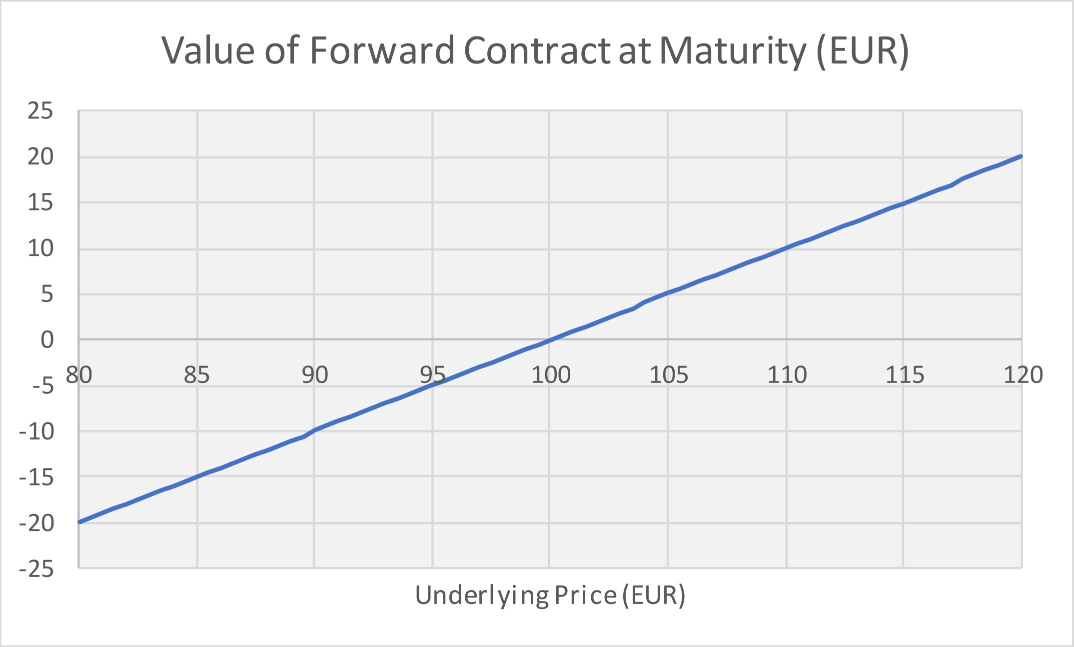
DERIVATIVES

- A derivative is a contract between two or more parties for which its value is determined by an underlying asset or set of assets;
- Examples of underlyings: stocks, stock indices, bonds, currencies, commodities, electricity, temperatures, volatility…
- Examples of common derivatives: forwards, futures, swaps, options
- The relationship to the underlying can be linear or non-linear
- When the derivative contract is settled, the payment can be settled in cash or in exchange of ownership of the underlying asset.
- There are also more complex derivatives (“exotic derivatives”), e.g. click options, rainbow options, CDOs, CDO^2s, options on options, …

FORWARD CONTRACT

- A contract where two parties agree to change ownership of an underlying at a moment in the future for a price that is determined at the moment on which the agreement is made;
- The pay-off from a forward contract at the time of maturity is linear;
- Examples:
  - Underlying S_0, no cashflows : $F_t = S_0 e^{r_t t}$
  - Underlying stock paying *n* dividends of d_i at times t_i:$F_t = (S_0 - \sum_{i=1}^{n} d_i e^{-r_i t_i})e^{r_t t}$
  - Underlying commodity C with a storage cost of c (continuous compounding): $F_t = S_0 e^{(r_t - c)t}$

FORWARD CONTRACT (2)



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FUTURES CONTRACT

- A futures contract is very similar to a forward contract. The difference is that on a daily basis the profit or loss on the futures is settled on the margin account.
- Futures contracts are exchange-traded. Some futures are heavily traded on exchanges.
- When interest rates are deterministic (known function of time) the forward price and the futures price converge.
- When interest rates are stochastic, futures are slightly higher (lower) for and underlying that is positively (negatively) correlated with the interest rate
- See J. Hull, "*Options, Futures and Other Derivatives*" (8th edition), Pearson. Section 5.8

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SWAPS

- Contract in which two parties agree to exchange at set dates in the future two cash flows related to the principal value of the swap. The cash flows can be in different currencies and be linked to different interest rates.

- Examples:
    - Fix-for floating swap: one side of the swaps receives the floating rate applicable to the period covered and the other side receives a fixed rate ("swap rate")
    - Cross currency swap: one side of the swap receives the applicable floating rate in one currency, the other side receives the applicable floating rate in another currency.

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OPTIONS

- A contract that gives the holder the **right** but not the obligation to buy/sell the underlying contract for a pre-determined price ("the strike"). Options that represent a right to buy the underlying are referred to as **call** options. Options that represent the right to sell are referred to as **put** options.

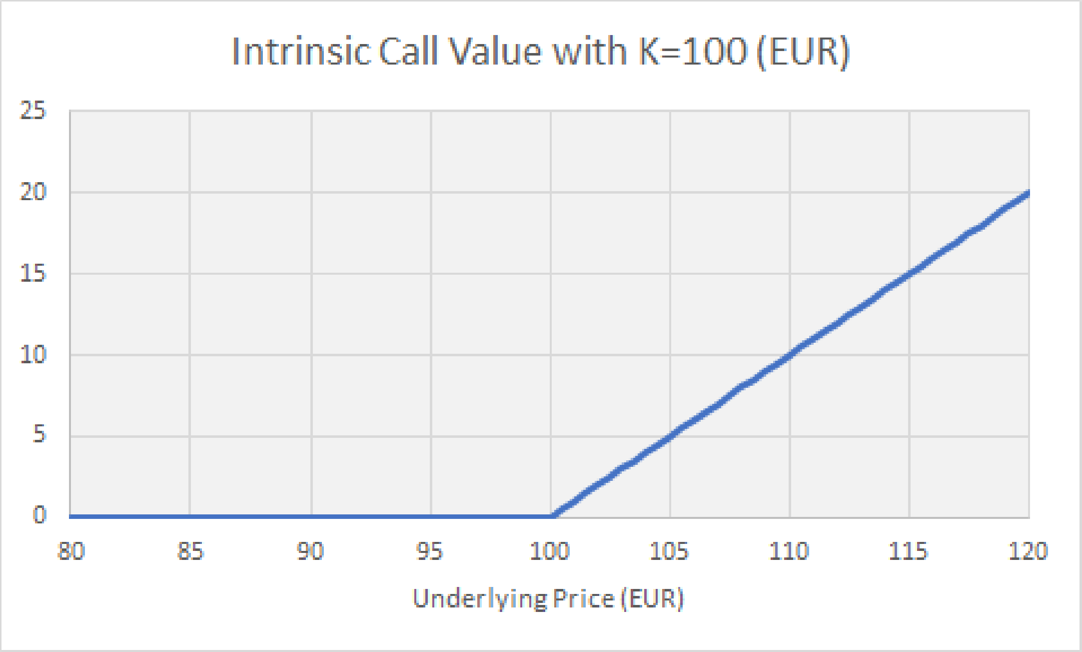
- The option holder pays for this right to buy/sell the underlying contract.

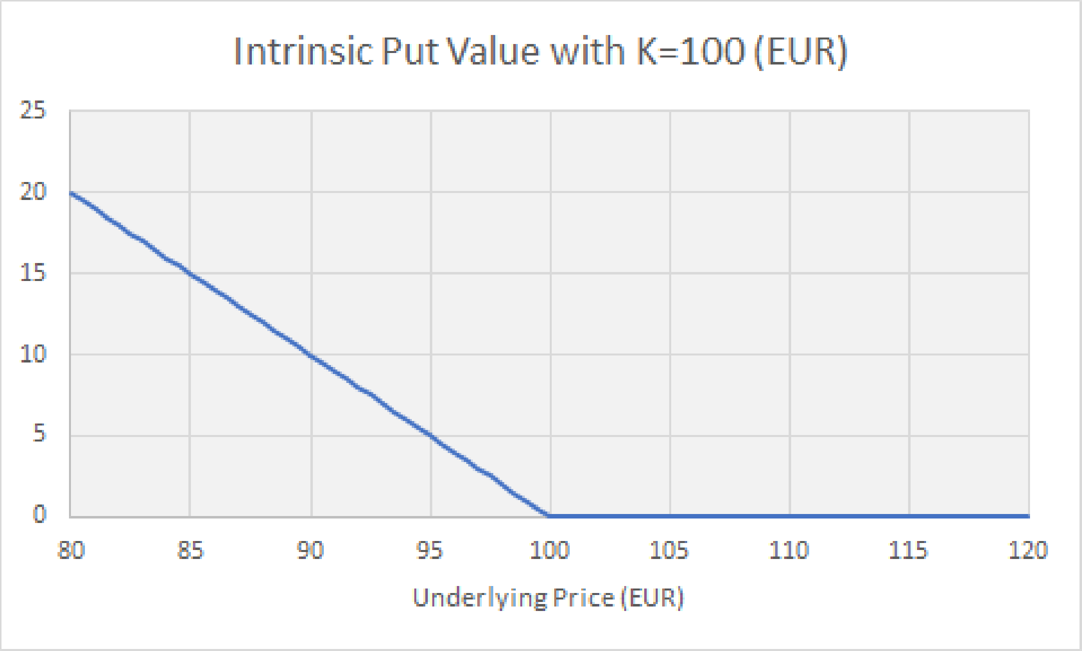
- Options can have many different underlyings: stocks, indices, commodity prices, FX rates, bonds, swaps, electricity, other options…

- Options for which this right can only be exercised at the maturity data of the option are **European Style** ("ES") options. Options for which this right can be exercised throughout the life time of the option are **American Style** ("AS") options.

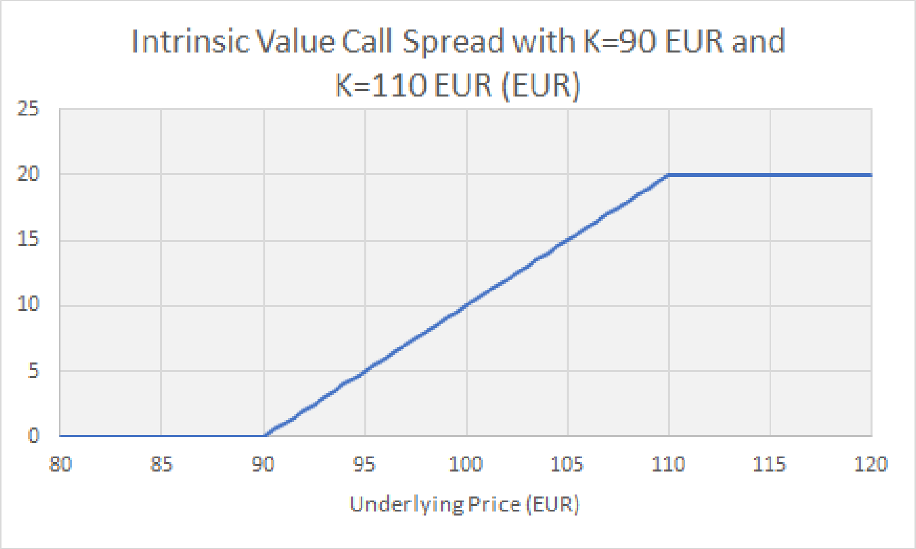
- The optionality ("the right to exercise") makes that the option price does not have a linear relationship with the price of the underlying.

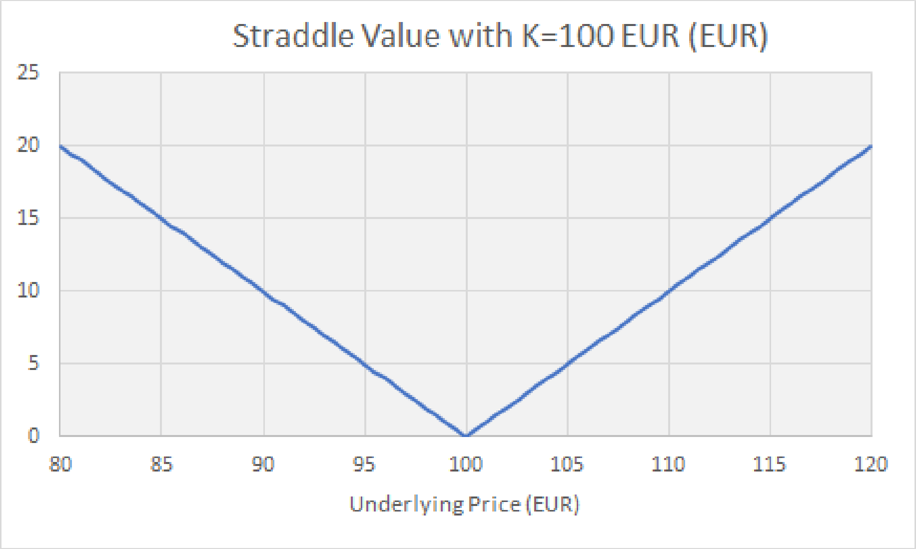
OPTIONS – PAY OFF CHARTS AT MATURITY

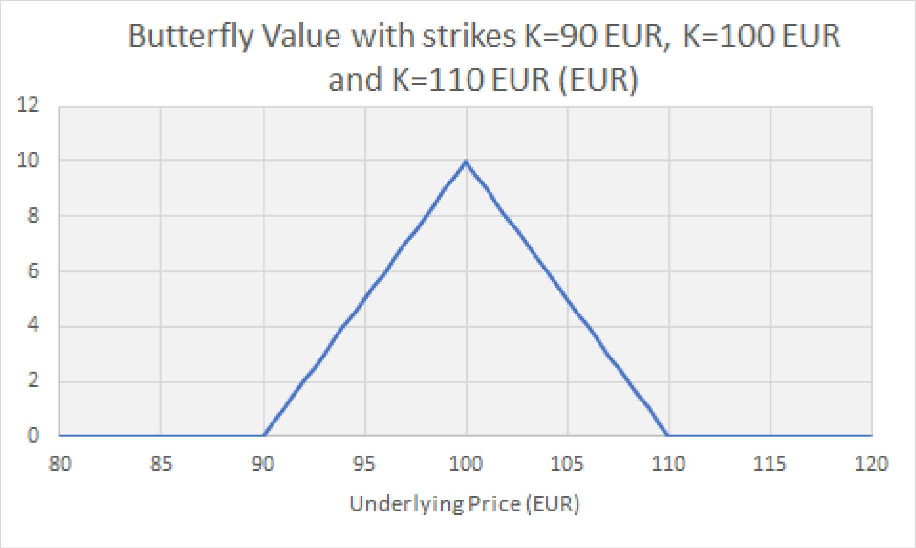


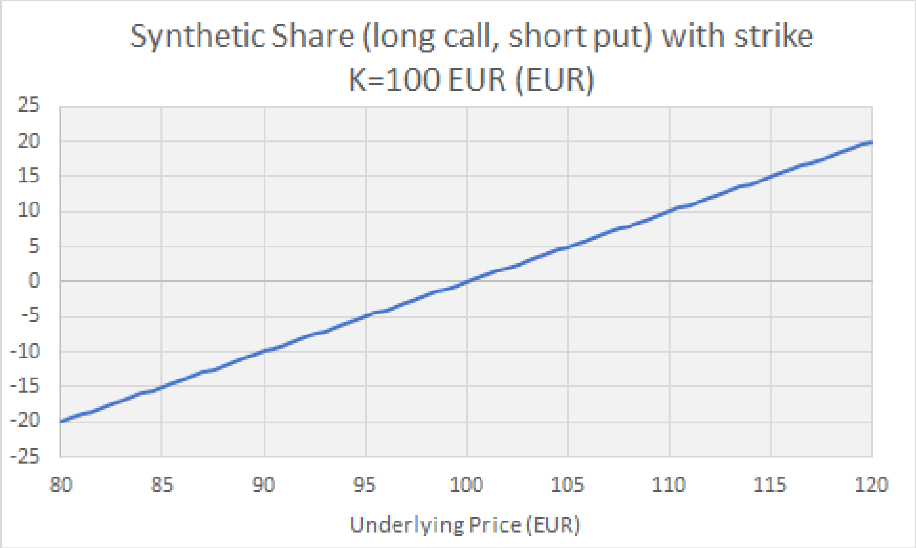


OPTIONS – COMMON COMBINATIONS OF OPTIONS

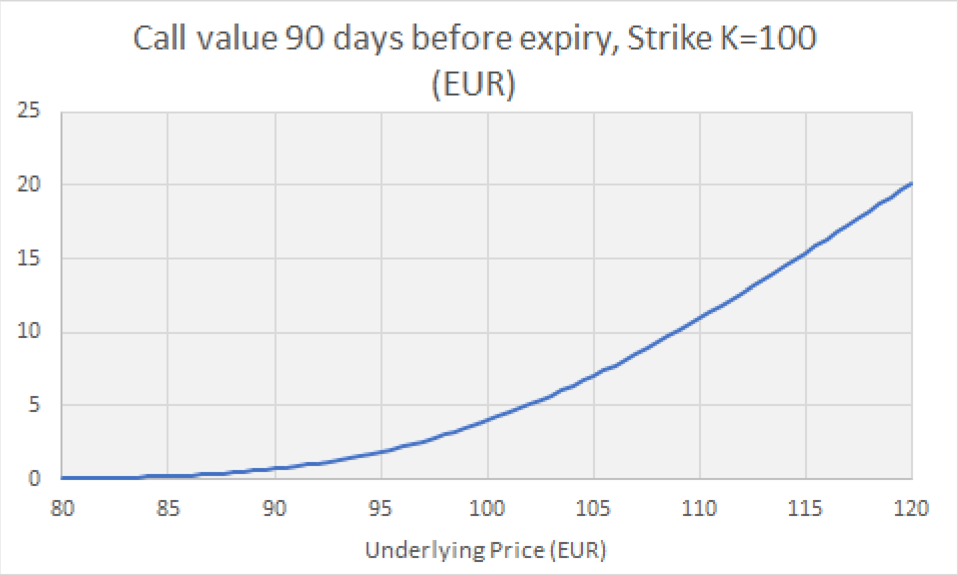


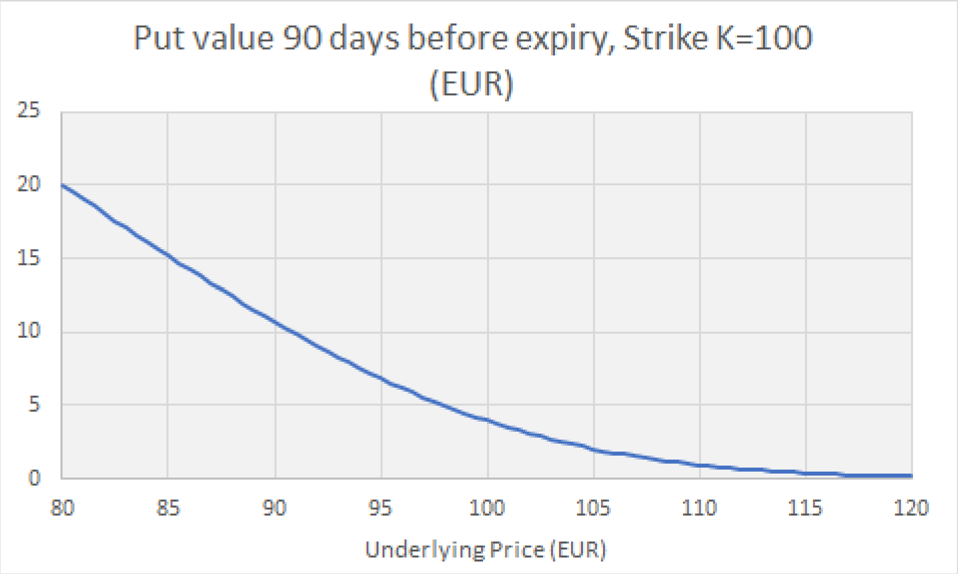






OPTIONS – BEFORE MATURITY DATE





- Before expiry of the option, the option value is composed out of the intrinsic value (value if option were to expire at current stock price level) and the time value.

- For a stock S with a dividend yield d, the ES option prices are dependent on the following parameters:
  - *S* :Stock Price
  - *X* :Strike
  - *T* :Time to maturity
  - *r* :Financing rate
  - *d* :dividend yield
  - ***σ*** **: Implied volatility**

$$C = Se^{-rT}N(d_1) - Xe^{-rT}N(d_2) \qquad \mathrm{P} = Xe^{-rT}N(-d_2) - Se^{-dT}N(-d_1)$$

$$d_1 = \frac{ln\left(\frac{Se^{-dT}}{Xe^{-rT}}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \qquad d_2 = d_1 - \sigma\sqrt{T}$$

$$N = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}dy$$

VOLATILITY

- The volatility expresses the variation of the underlying price series over time. This is expressed by the standard deviation of logarithmic returns.
- Backward looking: calculated volatility based on observed stock prices: The **realized volatility**.
- Forward looking: expectation of volatility that will be realized in a future period: **expected volatility**
- Forward looking based on observed derivatives prices: expected volatility by calculating which volatility results in observed option prices, given that all other pricing parameters are known: **implied volatility**
- Logarithmic return: $r_i = ln\left(\frac{S_i}{S_{i-1}}\right)$
- Volatility realized over a period t (typically one day): $\sigma = \sqrt{E(r^2) - E^2(r)}$
- Annualized Volatility: $\sigma_{ann} = \sqrt{T}\sigma$ (where T=number of trading days in a year)
- Rule of thumb, an underlying with a daily move of 1% has a realized annualized volatility of 16%.

CALL-PUT PARITY (EUROPEAN-STYLE EQUITY OPTIONS)

- Consider two portfolios:
  - A: consisting of a Call expiring at time *T* with strike *K* and an amount of cash equal to Ke^{-rT}
  - B: consisting of a Put expiring at time *T* with strike *K* and the underlying stock S

- Assume that at maturity of the options the stock price *S*<*K*:
  - A: the call option expires worthless and the amount Ke^{-rt} has accrued to $Xe^{-rT}e^{rT} = K$
  - B: the put option expires in the money, this gives the owner the right to sell the underlying stock S that is owned for an amount of K

- Assume that at maturity of the options the stock price *S*≥*K*:
  - A: the call option expires in the money, this give the owner the right to buy the underlying stock S for an amount *K*. The portfolio holder ends up long the underlying share S
  - B: the put option expires worthless, the portfolio holder still owns the underlying share S

- For all possible outcomes the two portfolios end up being identical

- **Put-call parity:** $\boldsymbol{C - P = S - Ke^{-rT}}$

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MARKET PARTICIPANTS IN DERIVATIVES TRADING

- In principle three types of market participants can be identified in derivatives trading:

- **Hedging parties:** Parties using derivatives to lower the variance of the company results

  E.g. Air line carriers buying call options on kerosene prices, farmers selling futures/forwards

- **Speculating parties:** Parties using derivatives to amplify the result on an expected move in the related underlying instrument

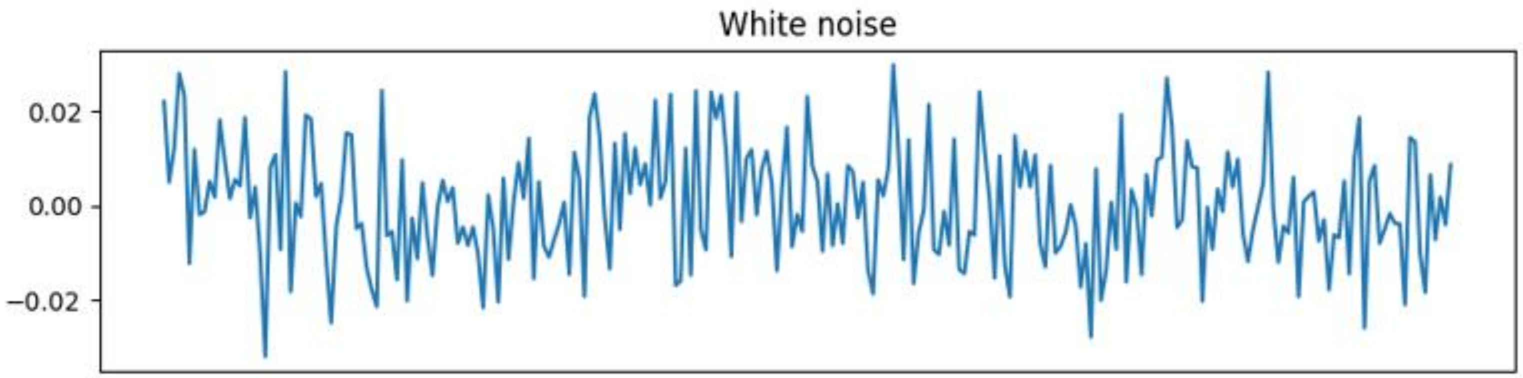
  E.g. Retail investors buying CFDs

- **Arbitrage parties:** Parties identifying mismatches in derivatives prices. By trading these instruments, the market prices are brought back in line

  E.g. Trading firms scanning for put call parity opportunities, hedge funds trading convertible bonds versus the underlying stock.

STOCK PRICE AS TIME SERIES

- Stock prices and prices of other financial instruments can be represented in the form of time series.
- We define a time series as a discrete time-ordered sequence of observations
- Notation: $Y = \{y_1, y_2, \dots, y_n\}$
- The time intervals can be short (milli-seconds) or long (e.g. daily closing prices)
- In physics we also encounter time series:
- White Noise: $y_t = \varepsilon_t$



MARKOV PROCESSES

- A Markov process is a time series for which an element of the time series is only dependent on the previous value of the time series.

- Example is a random walk: $y_t = y_{t-1} + \varepsilon_t$

- It is a property of Markov processes that the random variables for the different entries are uncorrelated

- If the random variable is normally distributed we call this a Wiener process

- An example in physics of a Wiener process is Brownian motion.

GENERALIZED WIENER PROCESSES AND ITO PROCESSES

- The Wiener process is represented by:

$$dX = b\,dz,$$

Where *dz* is normally distributed and the variance from *dz* over a period *T* is *T*

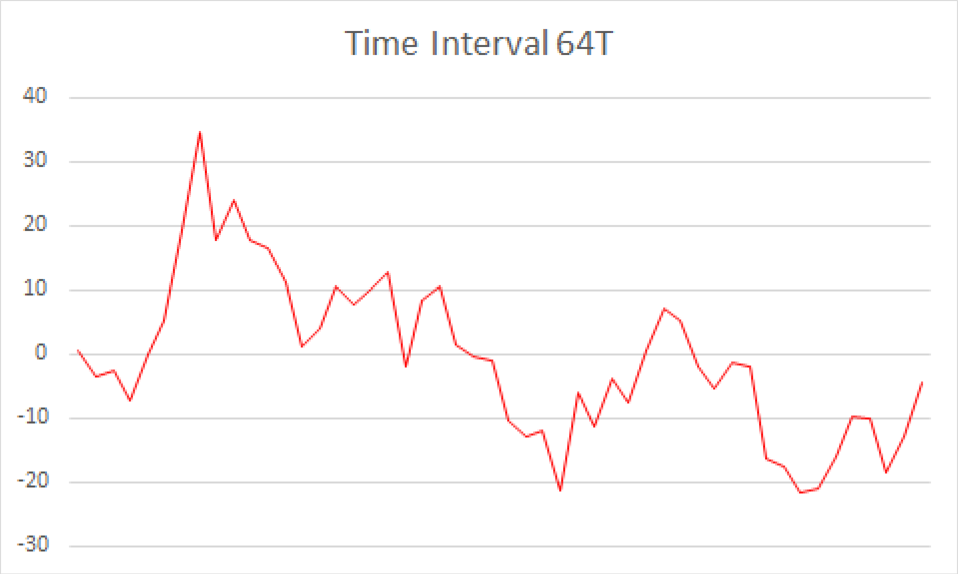
- We arrive at the generalized Wiener process by introducing a drift term to the equation:

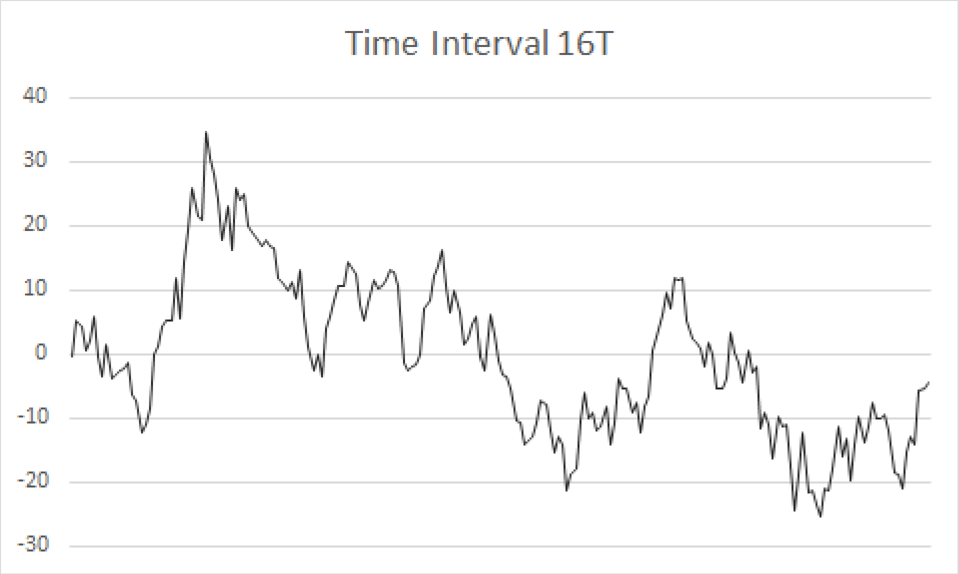
$$dX = a\,dt + b\,dz,$$

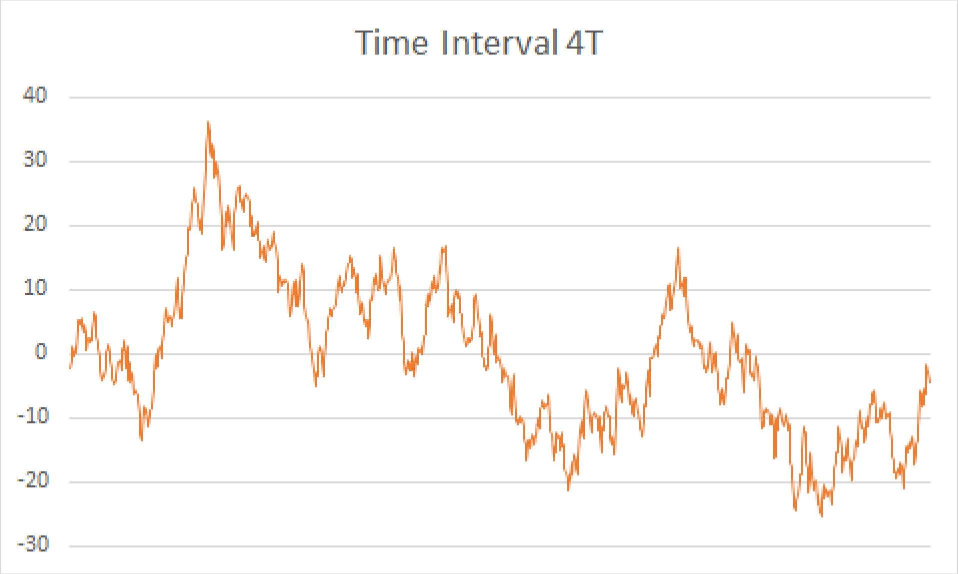
- If these coefficients are a function of x and t, this is referred to as an Itô process:

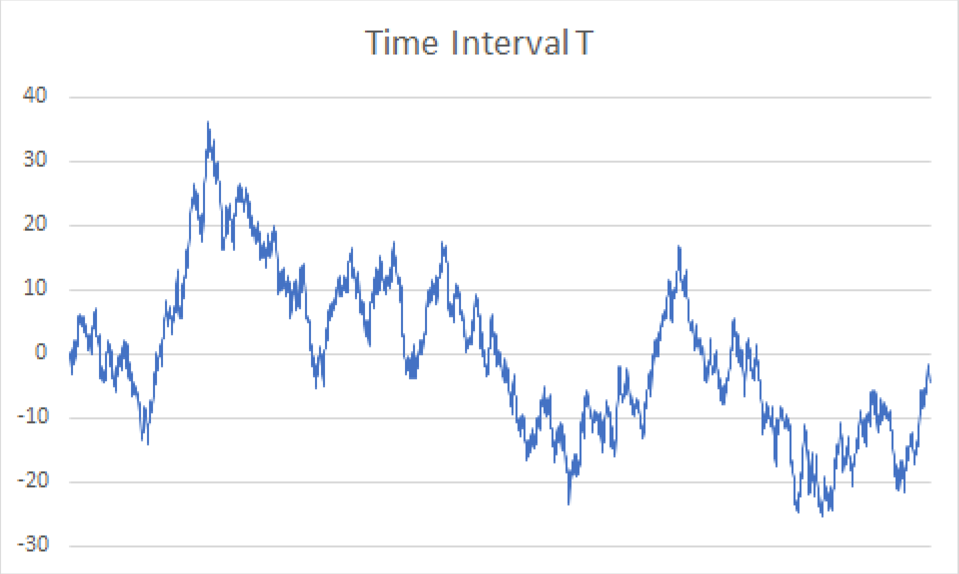
$$dX = a(X,t)\,dt + b(X,t)\,dz$$

TIME INTERVALS





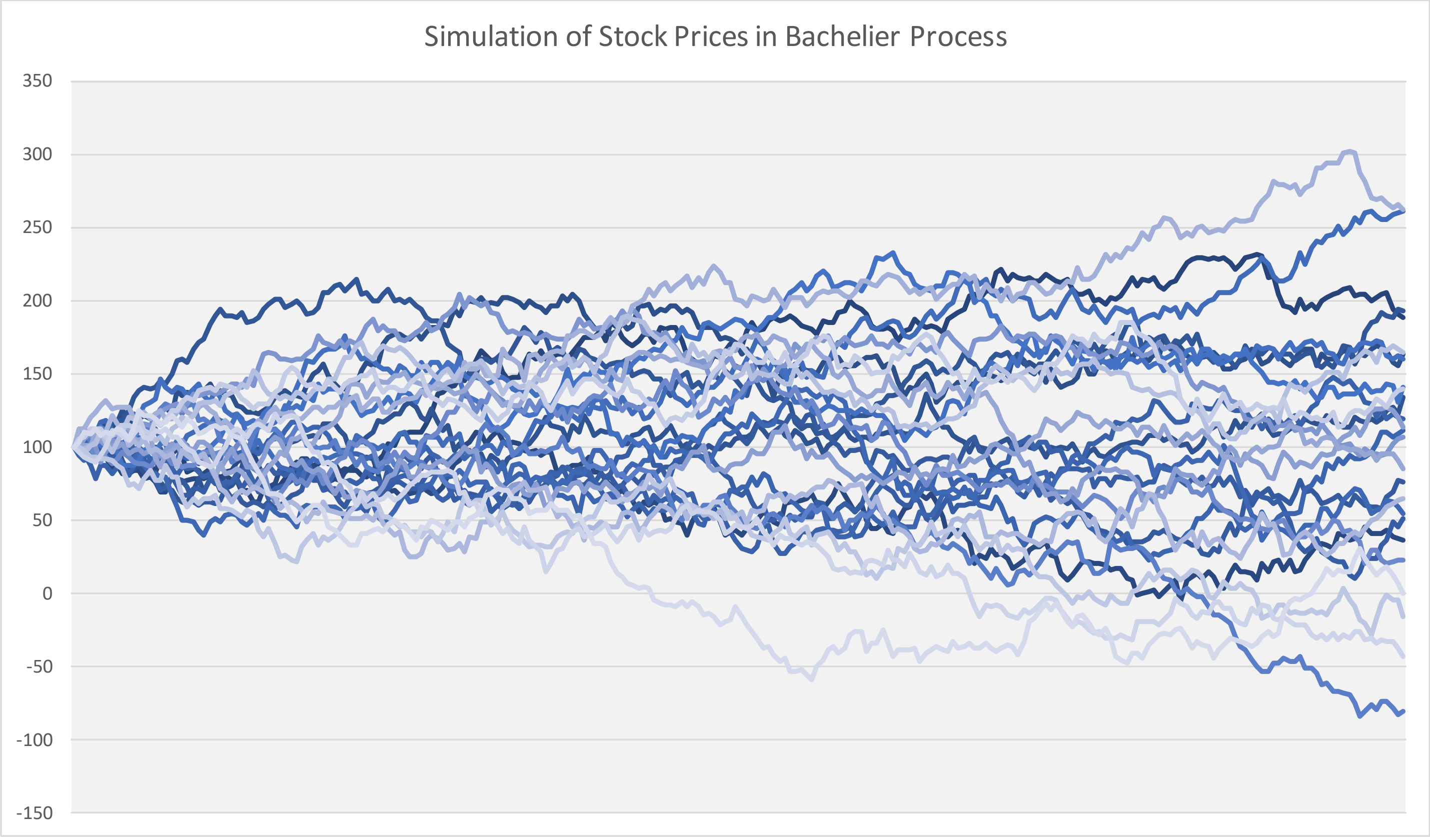




TIME SERIES LINKED TO STOCK PRICES

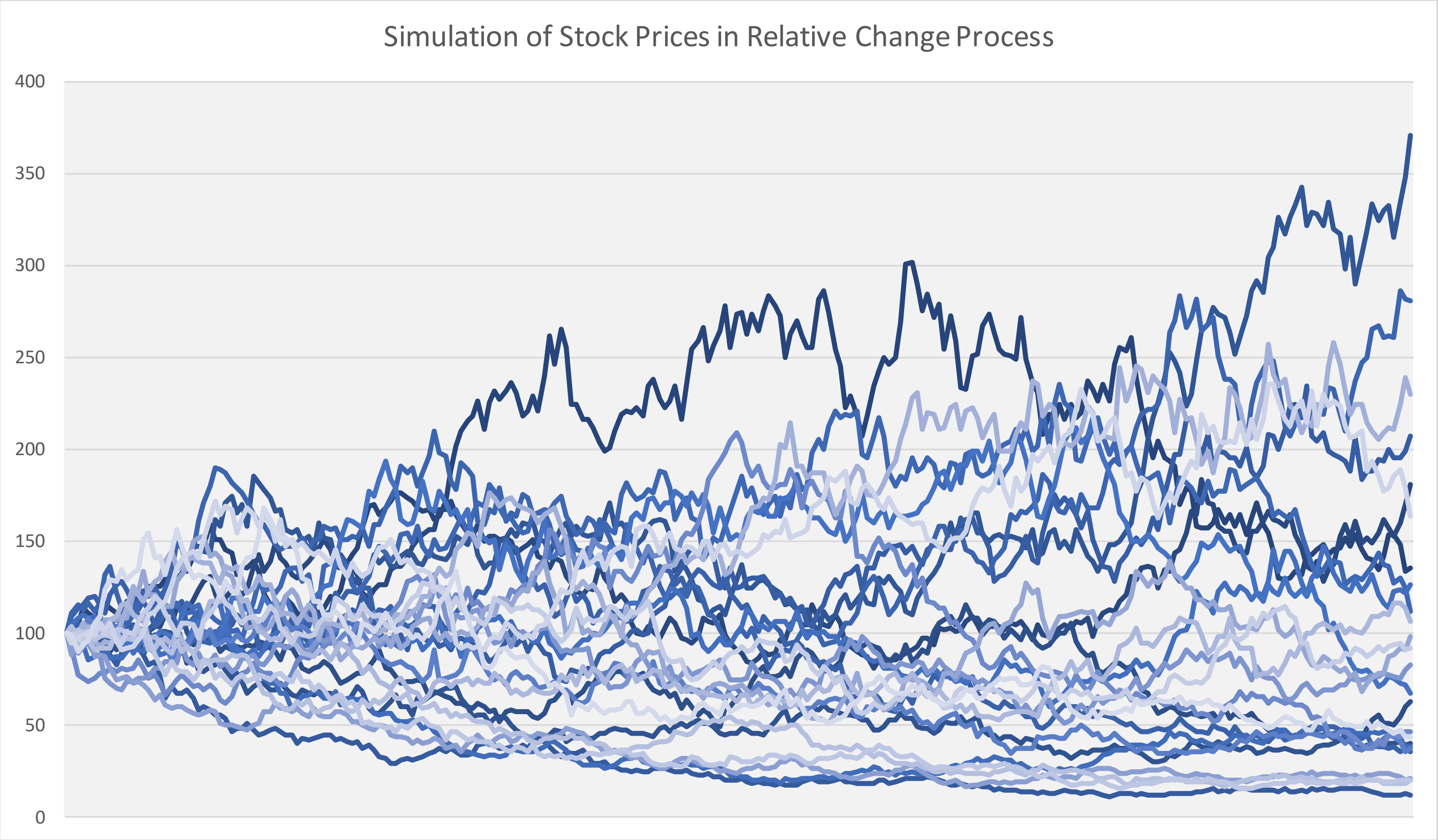
- Assume stock prices generated by a Wiener process (also referred to as Bachelier process)
- $$dS = b\ dz,$$

  Where dz is normally distributed:

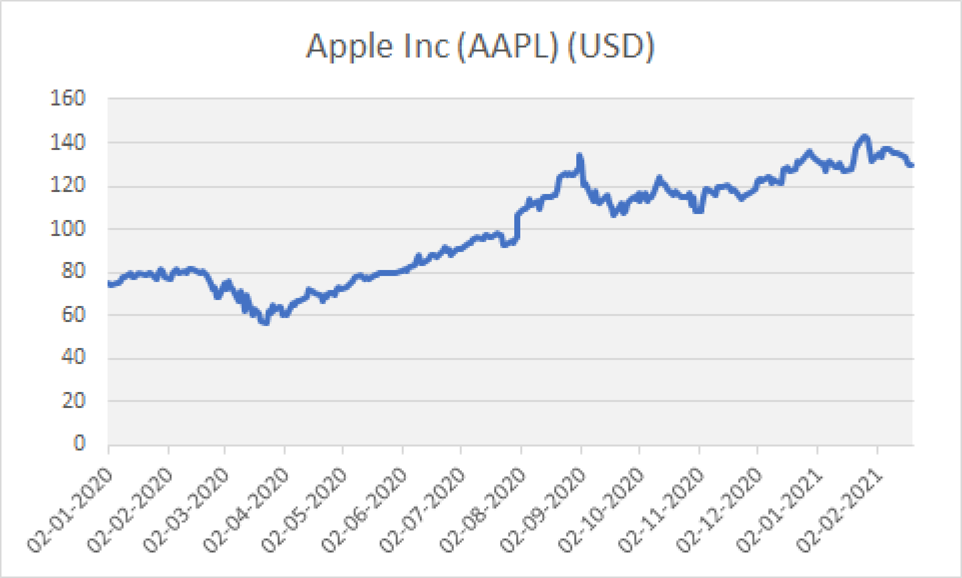


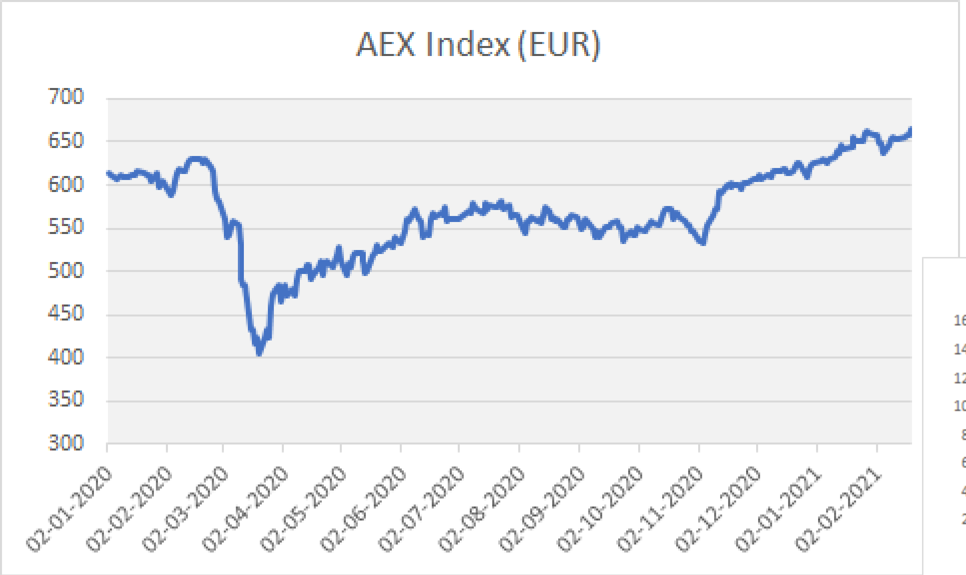
TIME SERIES LINKED TO STOCK PRICES

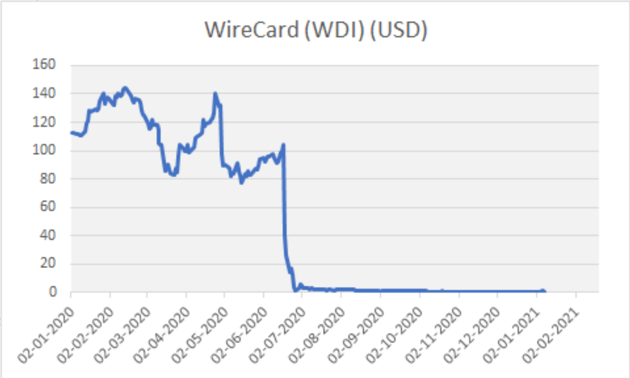
- Instead we can consider relative changes:
- $dS = b\ S\ dz,$

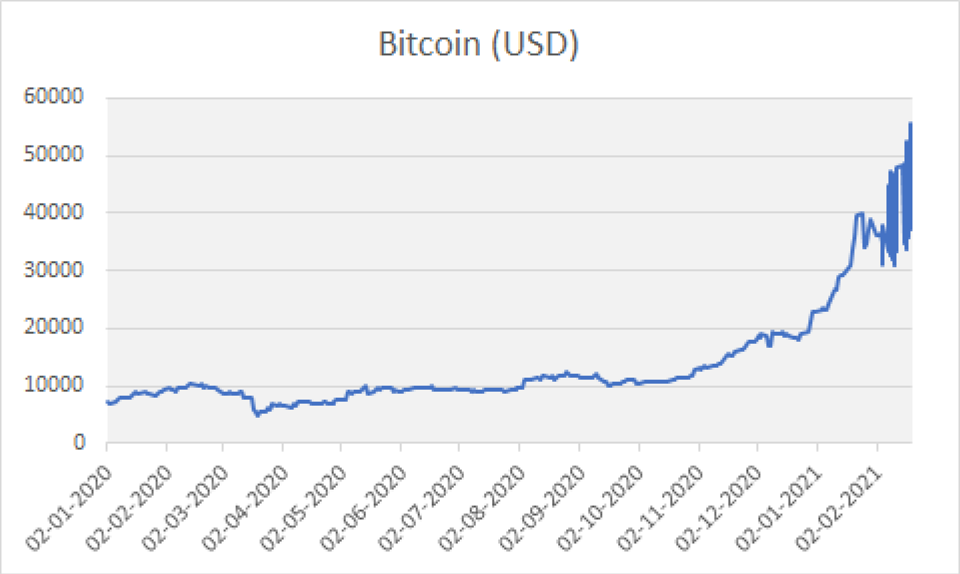


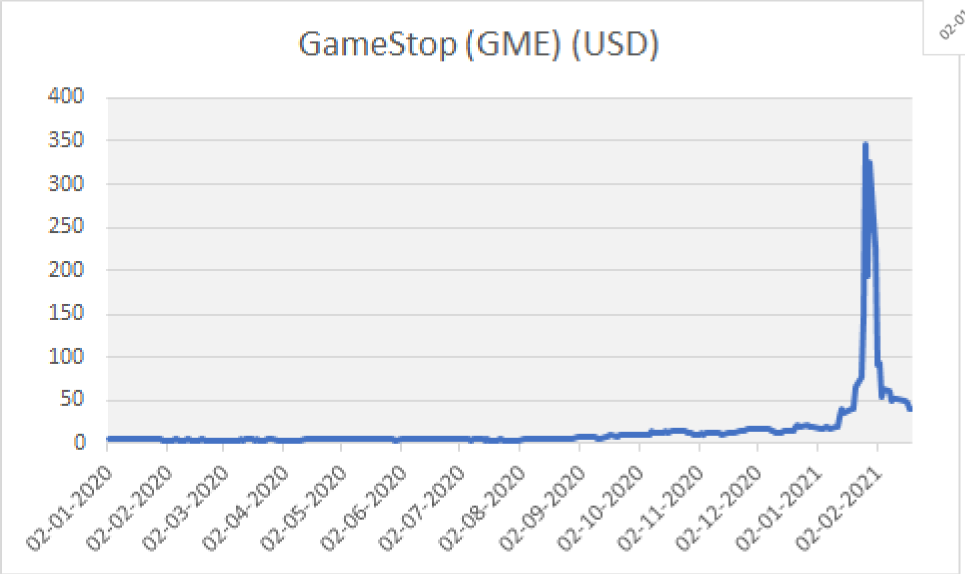
TIME SERIES LINKED TO STOCK PRICES











ITO'S LEMMA

- Consider a function *G* which is a function of *x* and time *t* and let *x* follow an Itô process, then Itô's lemma states that the function G follows the following process:

$$dG = \left(\frac{\partial G}{\partial x}a(x,t) + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2(x,t)\right)dt + \frac{\partial G}{\partial x}b(x,t)dz,$$

Where *dz* is the same stochastic process as the random variable *x*. (see K. Itô, "*On stochastic Differential Equations*", Memoirs of the American Mathematical Society, **4** (1951), 1-51).

STOCK PRICE PROCESS

- Adding a drift term, this leads to the conclusion that the stock price can be characterized by the following process:

$$dS = S\mu dt + S\sigma dz,$$

- This is a geometric Brownian motion process
- Applying Itô's lemma, the above process can be rewritten as

$$d\, lnS = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dz,$$

- Now consider a Call option *C(S,t)* on the underlying stock. For the Call option Itô's lemma yields:

$$dC = \left(\frac{\partial C}{\partial S} S\mu + \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2} S^2\sigma^2\right) dt + \frac{\partial C}{\partial S} S\sigma dz,$$

(later the derivatives will be related to the Greeks of the call options)

DERIVATION OF BLACK-SCHOLES EQUATION (1)

- Consider a discrete version of the call and stock equation

$$\Delta S = S\mu\Delta t + S\sigma\Delta z,$$

$$\Delta C = \left(\frac{\partial C}{\partial S}S\mu + \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}S^2\sigma^2\right)\Delta t + \frac{\partial C}{\partial S}S\sigma\Delta z$$

- Assume a portfolio Π:

$$\Pi = -C + \frac{\partial C}{\partial S}S$$

- For a small period of time the change in value of the portfolio is (the stochastics contributions cancel out):

$$\Delta\Pi = \left(-\frac{\partial C}{\partial t} - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}S^2\sigma^2\right)\Delta t$$

- This is a riskless portfolio, hence on the basis of the no-arbitrage principle:

$$\Delta\Pi = r\Pi\,\Delta t = r\left(-C + \frac{\partial C}{\partial S}S\right)\Delta t$$

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DERIVATION OF BLACK-SCHOLES EQUATION (2)

- Equating the two expressions for ΔΠ yields the Black-Scholes equation for a Call-option:

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}S^2\sigma^2\right) = r\left(C - \frac{\partial C}{\partial S}S\right)$$

- The boundary condition that at the expiry date for the Call value:

$$\mathrm{C} = \max(0, S - X)$$

- Solving this differential equation, which is closely related to the heat equation, yields the Black-Scholes pricing formula for the call option....

QUESTIONS?