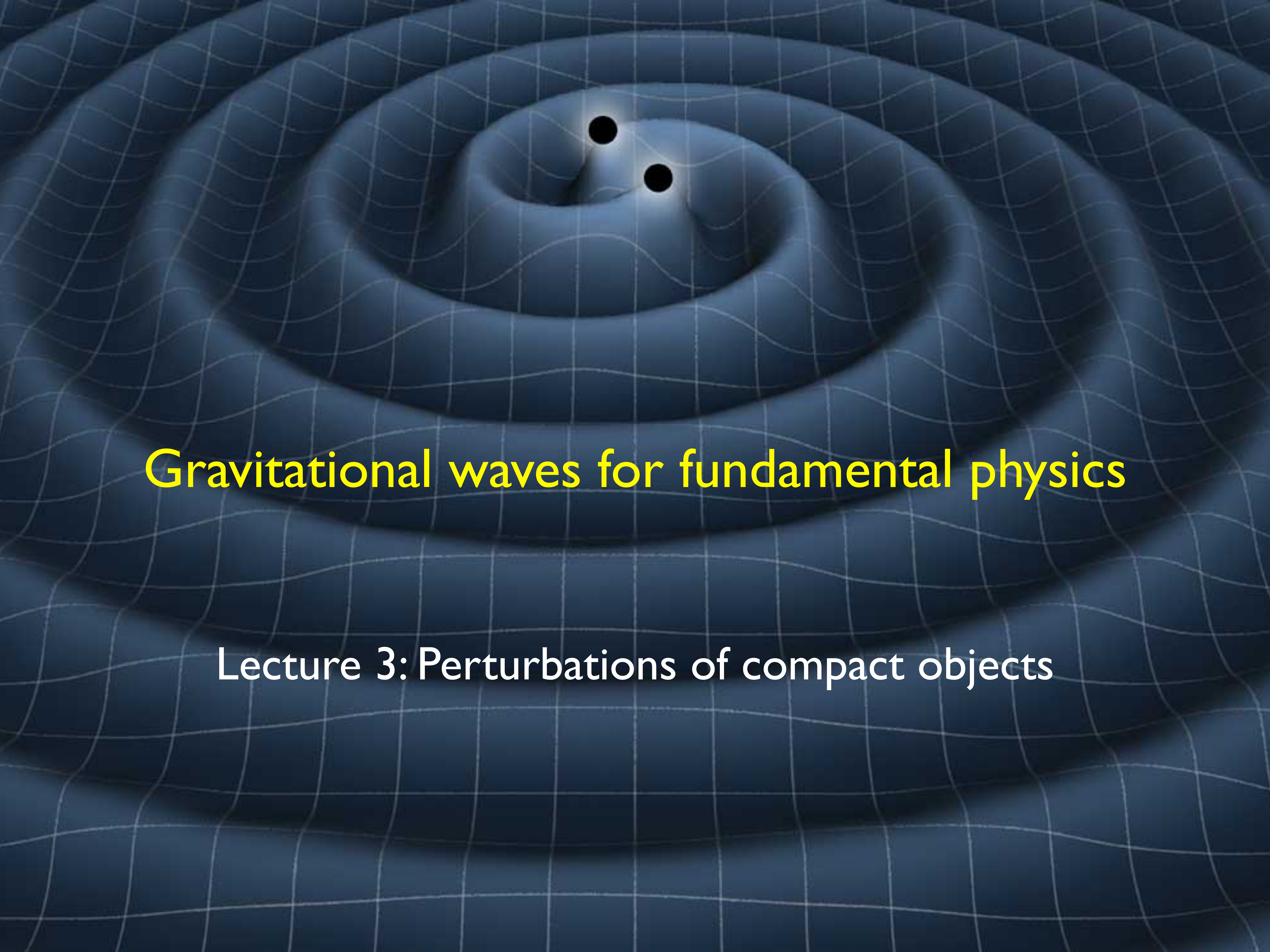
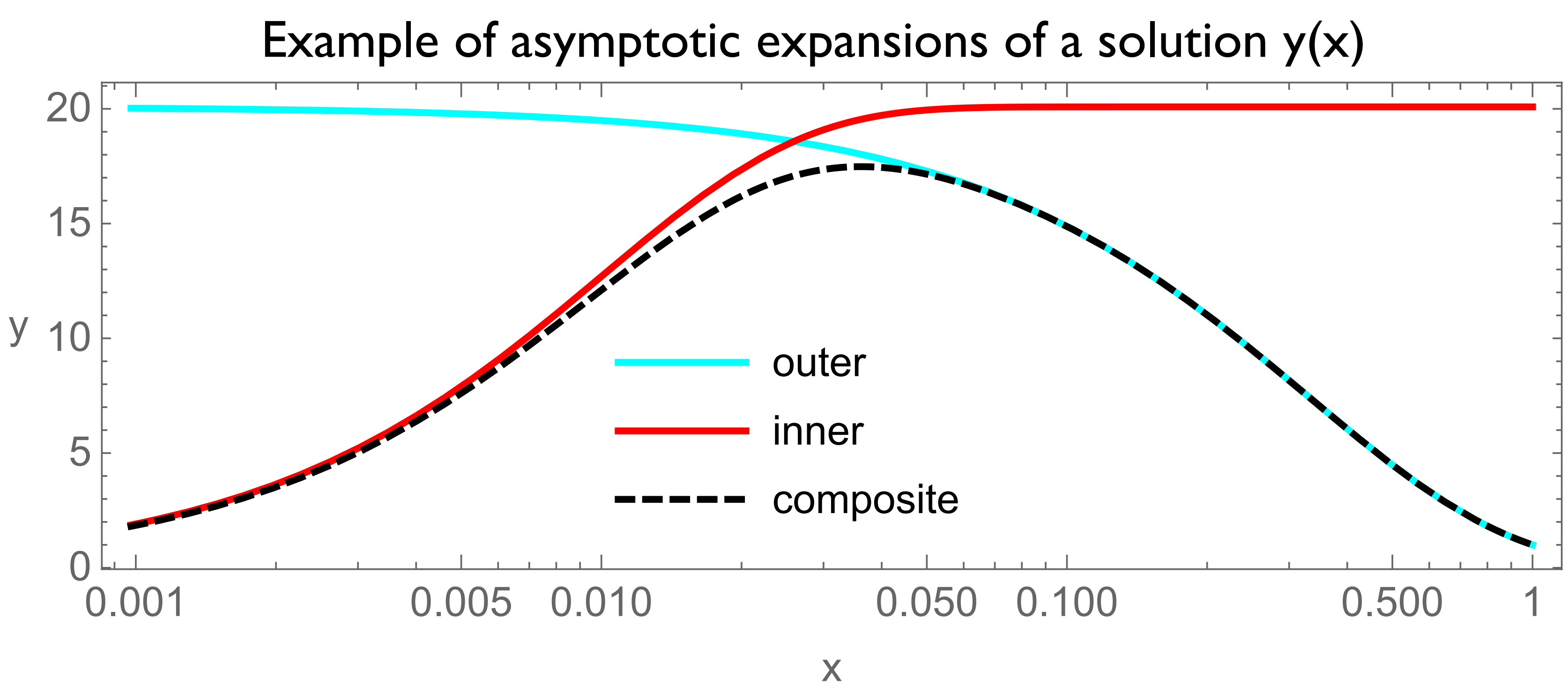


Gravitational waves for fundamental physics

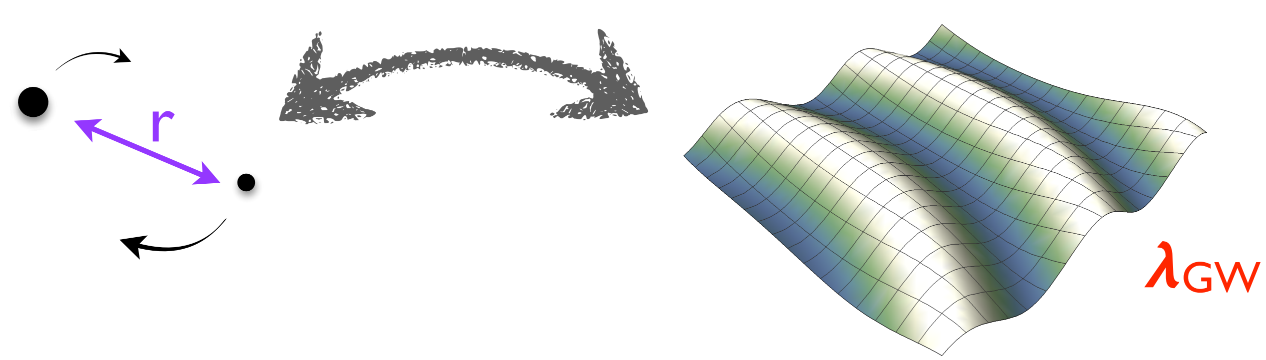
Lecture 3: Perturbations of compact objects

Recall from last time: Matched asymptotic expansions

- Matched asymptotic expansions connect approximate descriptions of different physics dominating in different regimes



- Important tool for GWs, e.g. matching a post-newtonian PN description (near the binary system) with a post-Minkowski expansion in the distant wave zone



Recall from last time: STF tensors, multipoles

- Multipoles are key for transmitting information from source physics to GWs

- Symmetric trace-free tensors $$v^{<i}n^{j>} = \frac{1}{2}(v^i n^j + v^j n^i) - \frac{1}{3}(\mathbf{v}\cdot\mathbf{n})\delta_{ij}$$

Symmetrize
In $i \leftrightarrow j$

Remove the trace

- The expansion of the exterior gravitational potential around a reference point z can be written as

$$U(\mathbf{x}) = G\int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|}\boldsymbol{\rho}(t,\mathbf{x}') = G\sum_{\ell=0}^{\infty}\frac{(2\ell-1)!!}{\ell!}\frac{\bar{n}^{<L>}M^{<L>}}{\bar{r}^{\ell+1}}$$

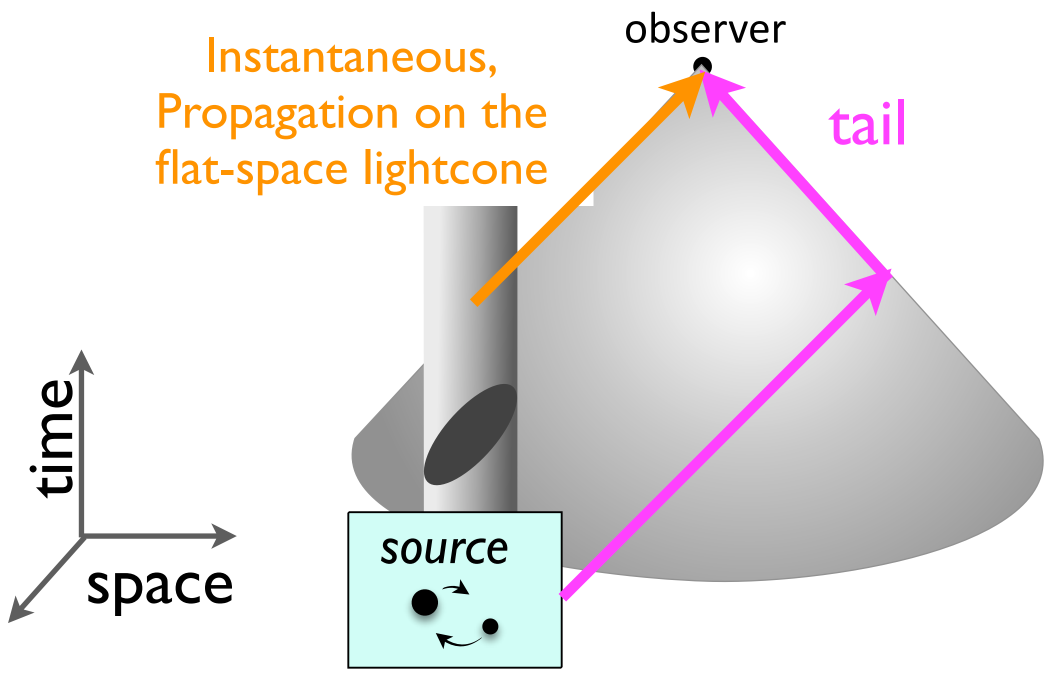
$$\bar{r} = |\mathbf{x}-\mathbf{z}| \qquad \bar{n}^i = \frac{(x-z)^i}{\bar{r}}$$

- Newtonian mass multipoles $$M^{<L>} = \int d^3x' \rho(\mathbf{x}')(x'-z)^{<L>}$$

- Interconversion with spherical harmonics

Recall from last time: hereditary effects in GWs

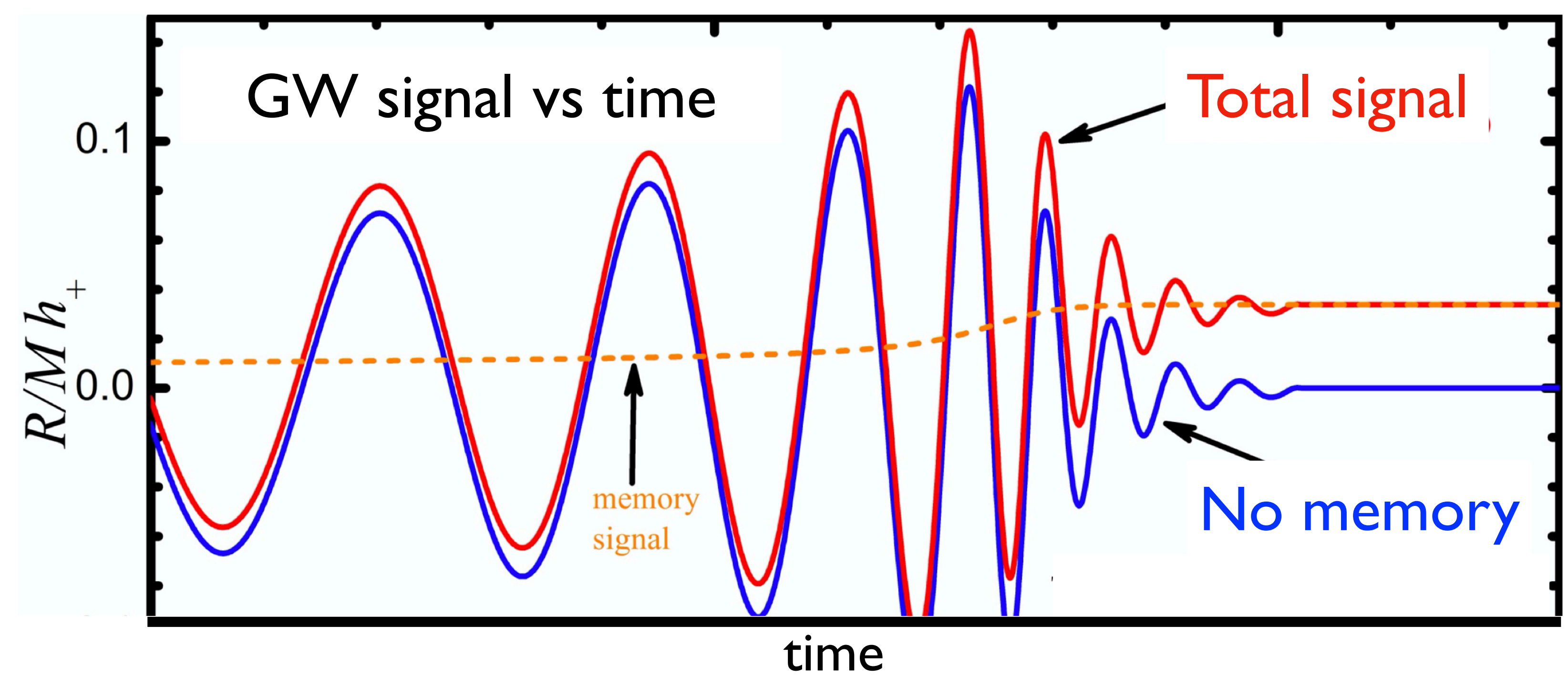
- Nonlinear, history-dependent relationship between source and radiation multipoles



- tails: linear GWs scatter off the curvature due to the source
- At higher orders: multiple scatterings, and back-interaction with the source

- Nonlinear memory: GWs sourced by the previously emitted GWs

Non-oscillatory effect, permanent change in amplitude before and after a burst of GWs



Recall from last time: relativistic corrections

- GWs beyond the quadrupole formula

Same as the quadrupole formula in linearized gravity for the Newtonian contribution to $M_{<ij>}$

$$h_{ij}^{\mathrm{TT}} \sim \frac{G}{c^4\, d}\left\{\ddot{M}_{<pq>} + \frac{1}{c}N_a\left[\dddot{M}_{<pqa>} + \left(\epsilon_{acp}\ddot{J}_{<qc>} + \epsilon_{acq}\ddot{J}_{<pc>}\right)\right] + \frac{1}{c^3}(\mathrm{tail}) + O(c^{-4})\right\}$$

Mass octupole

Current quadrupole

GWs oscillate at different harmonics of the orbital frequency

- relativistic corrections to the GW phase evolution involve not only on the chirp mass but also the symmetric mass ratio

$$\nu = \frac{m_1 m_2}{(m_1+m_2)^2} = \frac{\mu}{M}$$

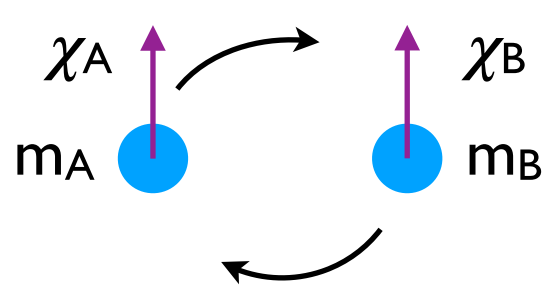
Addendum: relativistic and spin-orbit effects

- Dominant spin effects are spin-orbit interactions
- First appear in the binding energy and GW emission at 1.5PN order

$$E = -\frac{1}{2}\mu c^2 \left(\frac{GM\omega}{c^3}\right)^{2/3} \left[1 + \left(-\frac{3}{4} - \frac{\nu}{12}\right)\left(\frac{GM\omega}{c^3}\right)^{2/3} + (\text{spin} - \text{orbit})\left(\frac{GM\omega}{c^3}\right) + \ldots\right]$$

$$\dot{E}_{\mathrm{GW}} = \frac{32}{5}\nu^2 \left(\frac{GM\omega}{c^3}\right)^{5/3} \left[1 + \left(-\frac{1247}{336} - \frac{35\nu}{12}\right)\left(\frac{GM\omega}{c^3}\right)^{2/3} + (\text{spin} - \text{orbit} + \text{tail})\left(\frac{GM\omega}{c^3}\right) + \ldots\right]$$

- GWs are most sensitive to a certain combination χ_{eff} of the spins of the objects (similar to chirp mass):



Dimensionless spin parameter

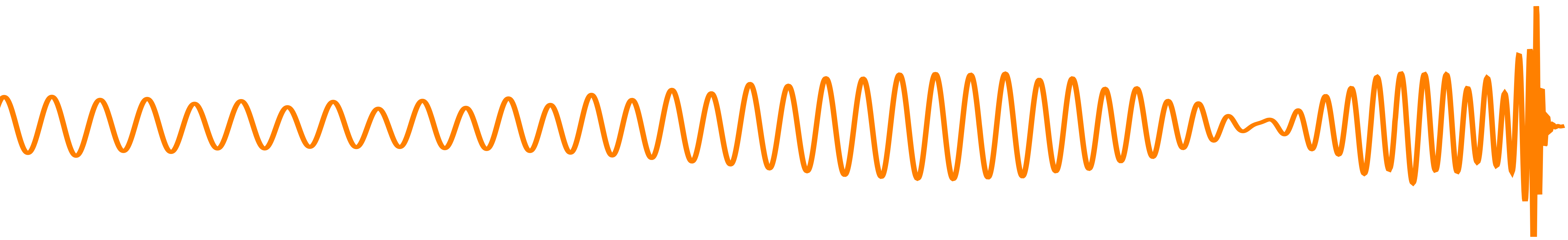
$$\chi = \frac{cS}{Gm^2}$$

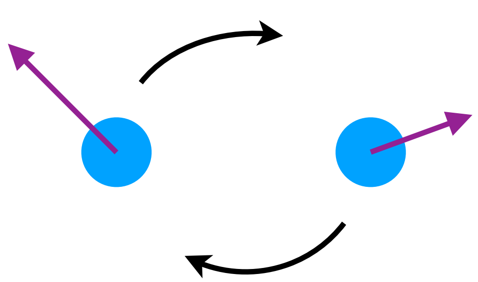
$$\chi_{\mathrm{eff}} = \frac{m_A\chi_A + m_B\chi_B}{m_A + m_B}$$

Addendum: spin effects

- At higher PN orders, spin-spin effects also enter

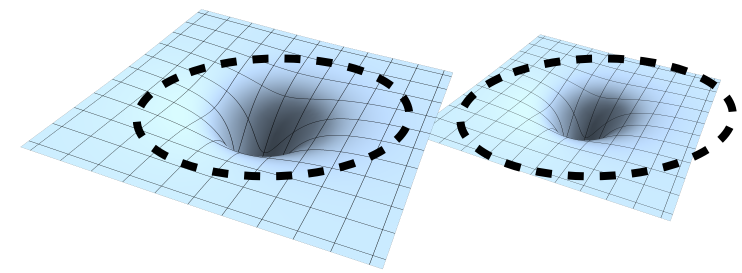
- If spins are not aligned with the orbital angular momentum: precession



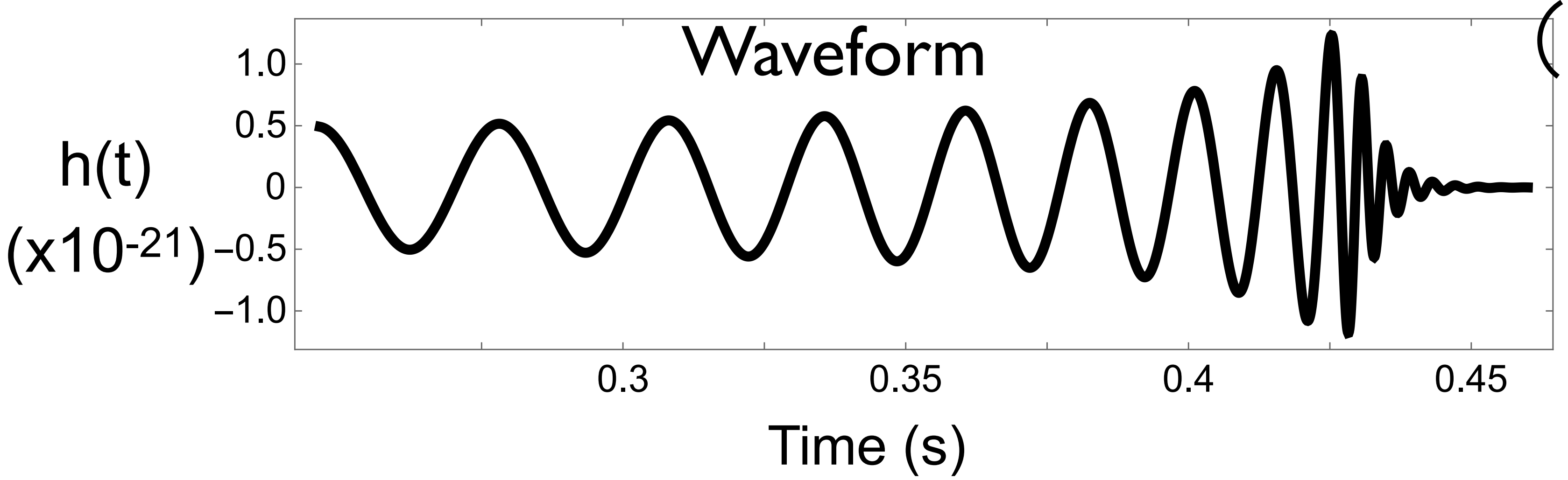


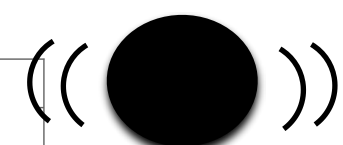
Today: Perturbations of compact objects

Finite size effects during an inspiral that
depend on the object's internal structure



Main focus: tidal effects - forced perturbations





Ringdown

Characteristic fingerprint
of the final remnant

Main focus: black hole perturbations

Useful References

- Discussion of tidal effects in Newtonian gravity formulated in a way that can be extended to GR, see the first part of :
  J. Vines and E. Flanagan First-post-Newtonian quadrupole tidal interactions in binary paper

- For more discussion of tidal effects also in GR (presented in the context of neutron stars, but the theory is more general) see e.g. this review article

- Open-source codes for black hole perturbations available through the BH perturbation toolkit project https://bhptoolkit.org/toolkit.html

- Approximate interpretation of BH quasi-normal modes as trapped at the light ring: W. Press 1972: Long wave trains of GWs from a vibrating BH paper

- Study of features of the scattering potential of a BH in C. Vishveshwara 1970: Scattering of gravitational radiation by a Schwarzschild BH paper

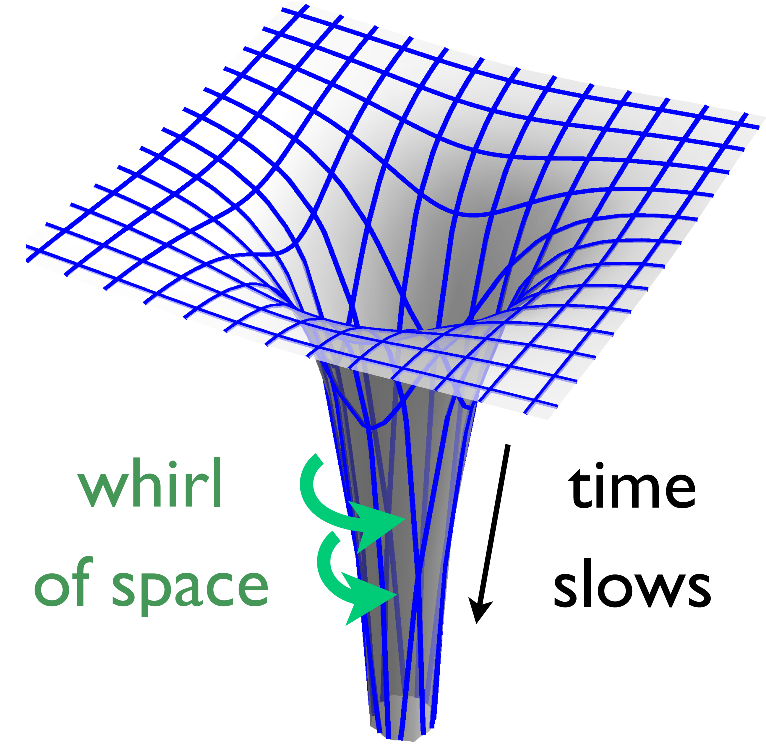
- A discussion of exotic objects and GWs can be found in the relevant chapters of the white paper: Black holes, GWs, and fundamental physics - a roadmap paper

Examples of compact objects

Compact objects I: Black holes

crushed

black hole (BH)



event horizon
causal barrier to interior

- region of immense spacetime curvature
- Exterior described entirely by mass & spin

No hair theorem:

Mass of the BH

Spacetime multipole moments $$M_\ell + iS_\ell = m(i\chi m)^\ell$$

Mass moments

Current moments

Dimensionless spin parameter

$$\chi = \frac{cS}{Gm^2} \leq 1$$

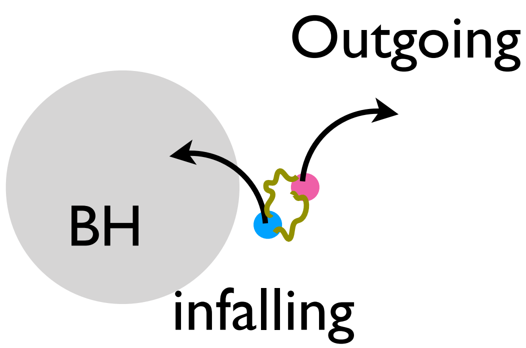
assumes 'standard' (here also non-charged) black holes in classical GR, four space-time dimensions

Quantum gravity phenomena at BH horizon scales

- For astrophysical BHs, the curvature at the horizon is far from the Planck scale

- However, there are deep theoretical puzzles with classical horizons, e.g.

1. Quantum Field Theory + horizons: Information paradox



   - Pairs of particles produced near the horizon ⇒ Hawking radiation
   - Ultimately, the BH evaporates, all converted to thermal radiation

Logical inconsistencies, e.g.: where does the information on the initial state go?

Quantum information analysis shows that sub-leading order quantum corrections cannot cure this problem

[e.g. Mathur 2009]

Quantum gravity phenomena at BH horizon scales

- For astrophysical BHs, the curvature at the horizon is far from the Planck scale

- However, there are deep theoretical puzzles with classical horizons, e.g.

   2. Microscopic interpretation of BH entropy

      - J. Bekenstein (1972) found that BHs have an entropy proportional to their area

$$S_{\mathrm{BH}} = \frac{A}{4G}\frac{c^3}{\hbar} \qquad A = 16\pi m^2$$ (nonspinning BH)

S_{BH} is huge! ~ 10^{77} $(m/M_\odot)$ Joule/Kelvin

- According to Boltzmann, entropy counts the number N of microscopic configurations

$$S \sim \log(N)$$

$N \sim 10^{10^{77}}$ (solar-mass BH)

Vastly larger than for any matter

where and what is this multitude of BH microstates?

How does this comply with the no-hair theorem?

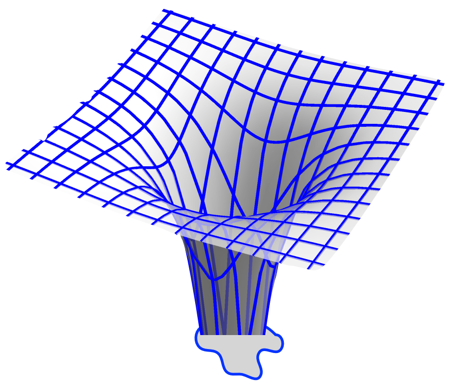
[some explanations from string theory and loop quantum gravity, e. g. Strominger & Vafa 1996 and review]

Motivates 'BH mimicker' compact objects

Many proposed modifications to resolve these issues with horizons. A few examples:

- Fuzzballs [from string theory]

  BH-like object : higher-dimensional spacetime ends just outside the would-be horizon, quantum stringy 'fuzzcap' , interior does not exist



- Related: Microstate geometries [from string theory]

  BH-like object = ensemble of horizonless micro state geometries, smooth 'cap' of throat + topology in extra dimensions

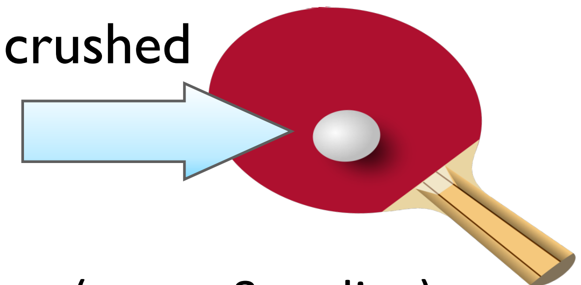
- Gravastars [from considering limits of ultracompactified matter]

  ~ gravitational extension of Bose-Einstein condensates, negative pressure interior (de Sitter spacetime), no horizon, matches through athin shell of high-energy matter to a Schwarzschild exterior

- Zoo of other proposed objects, e.g. in the review article https://arxiv.org/abs/1904.05363

Compact objects II: Neutron stars - extreme matter

crushed

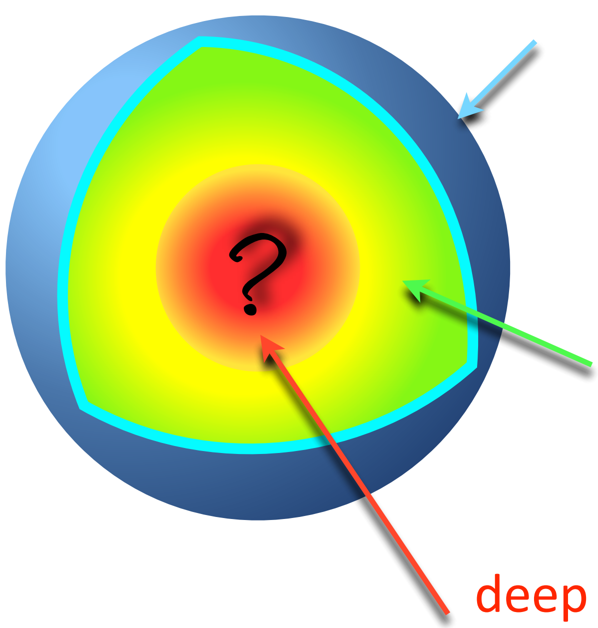


(mass & radius)

- Strong gravity compresses matter to ~ several times nuclear density
- Novel phases of matter, large extrapolations from known physics

Neutron star structure

[density of iron ~ 10 g/cm^3]



crust ~ *km*

Lattice of neutron rich nuclei

10^{10} times stronger than steel

free neutrons

~ 10^6 g/cm^3 inverse ***β***-decay

~ 10^{11} g/cm^3 neutron drip

outer core

uniform liquid (neutron superfluid, superconducting protons, electrons, muons)

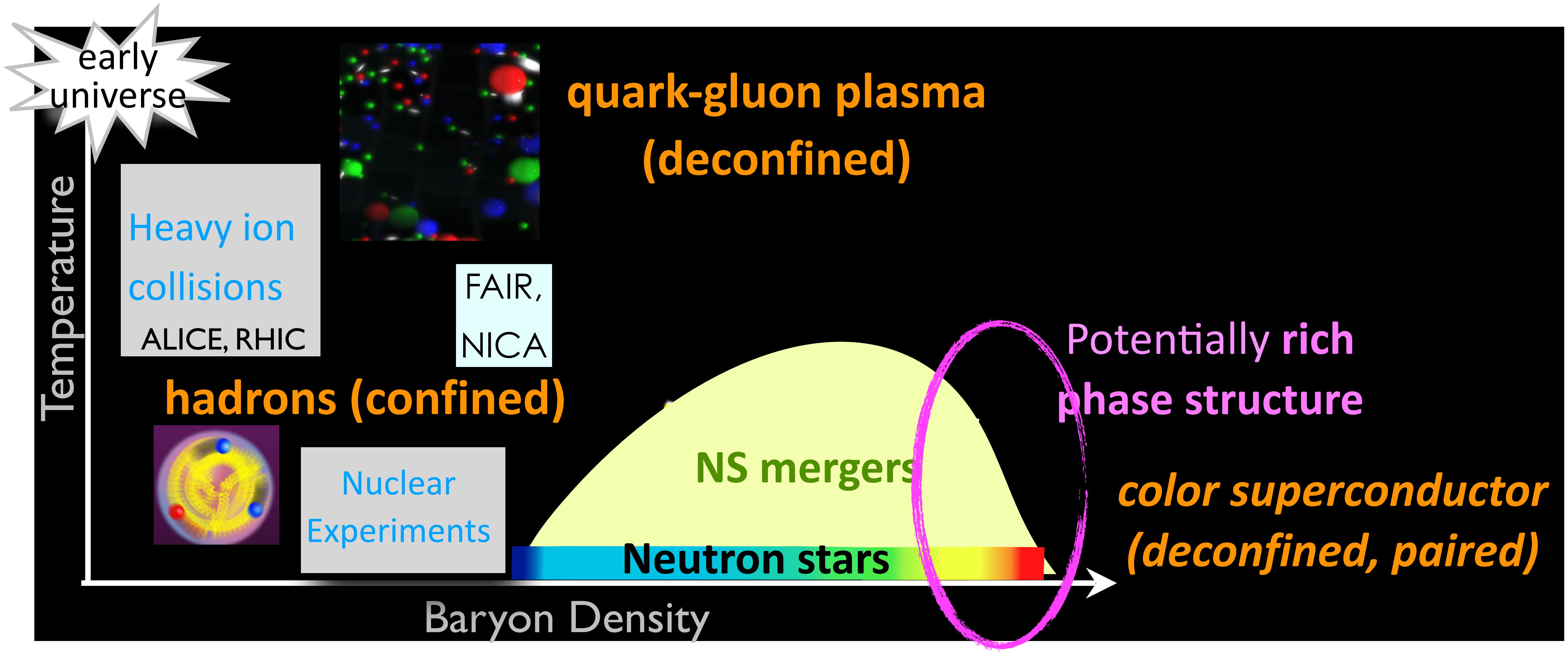
deep core

~ few x 10^{14} g/cm^3

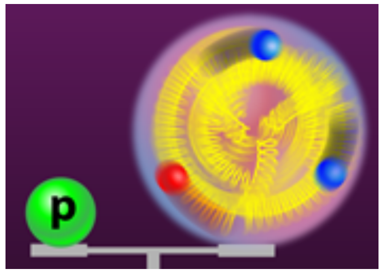
$\gtrsim$2x nuclear density, nucleons overlap - new degrees of freedom relevant

Bose-Einstein condensates of heavy hadrons? deconfined quarks?

Neutron stars as QCD labs



- Characterize phases of QCD, probe deconfinement
- Deeper understanding of strong interactions, their unusual properties, e.g.
  - asymptotic freedom (weaker force at shorter distances)
  - Vacuum (condensate) has important effects, e.g. mass



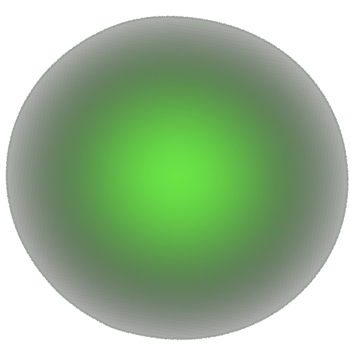
proton mass: ~ 938 MeV
only ~ 1% due to Higgs

Compact objects of new field condensates/hair

- New fields are ubiquitous in beyond-standard-model physics, e.g. motivated by
    - High-energy particle physics, grand unified theories
    - Dark matter
    - Inflationary cosmology

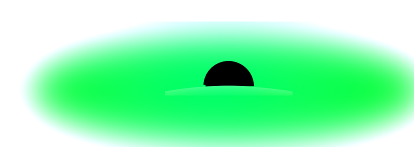
- Fields in the early universe generically condense into compact-object-like configurations over cosmic time

Levkov+ https://arxiv.org/abs/1804.05857



Form e.g. Boson stars (scalar fields),
Proca stars (bosonic vector fields),
oscillons (self-bound scalar fields), ….

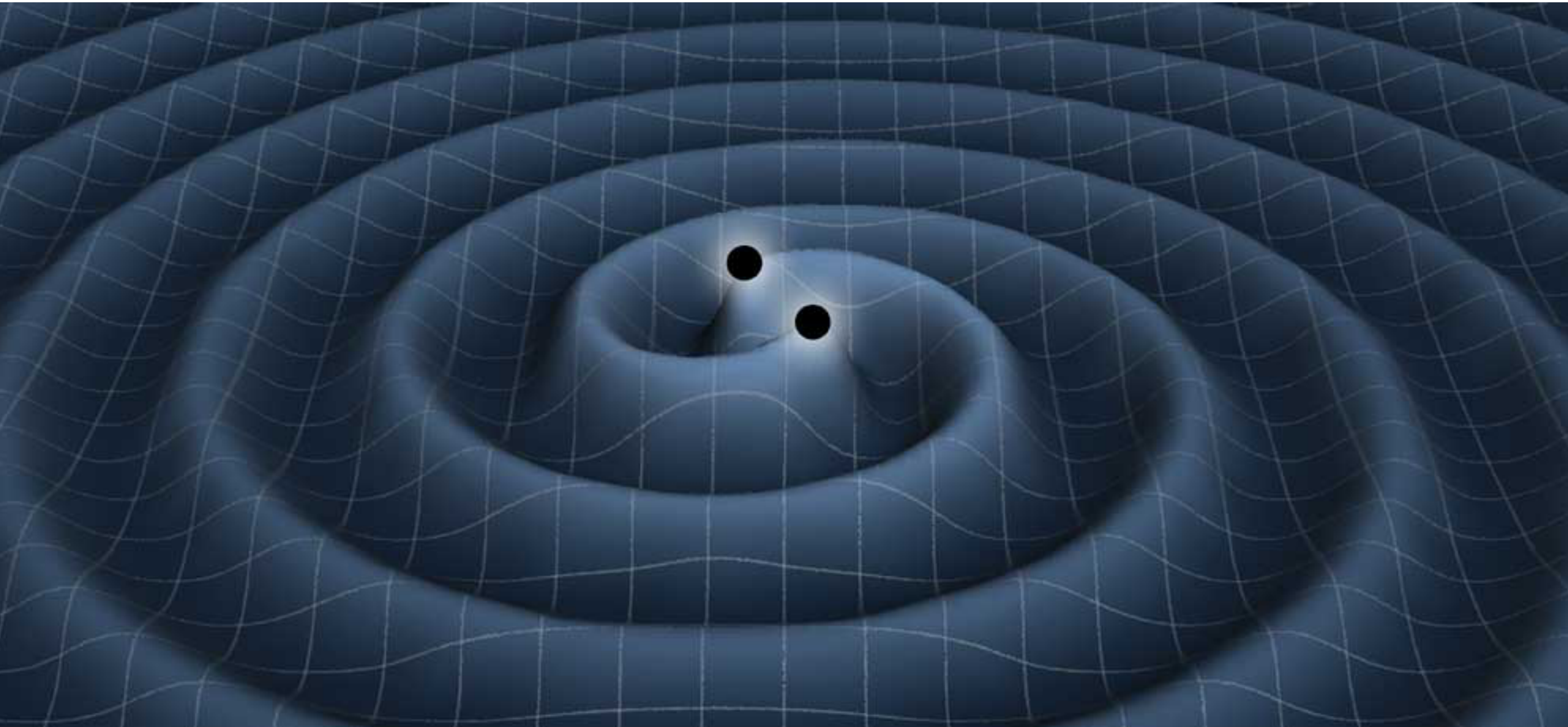
- Extra fields can also form atomic-like 'clouds' around black holes, surround neutron stars (matter coupling), form high-density dark matter spikes around black holes, …



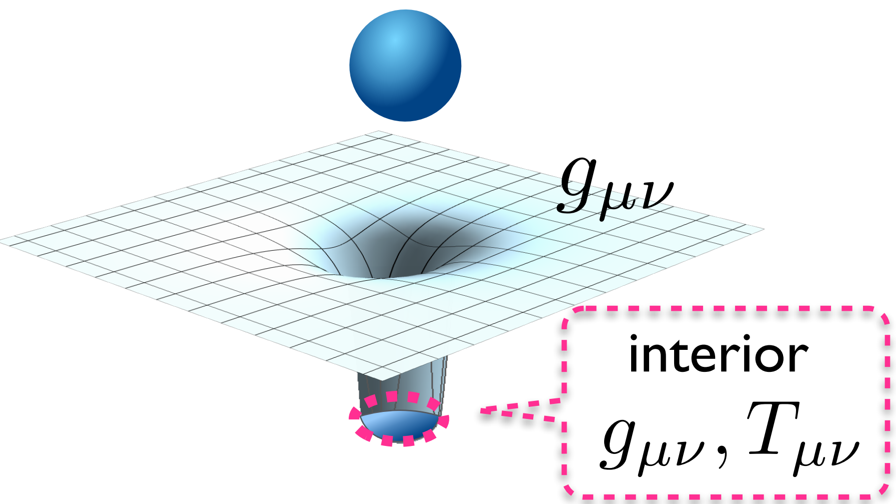
- In modified gravity theories, black holes and/or neutron stars can scalarize (develop scalar hair)

How can we test for these objects with GWs?

Need to consider spacetime properties



Nonspinning compact object spacetime



Einstein field equations + energy-momentum conservation

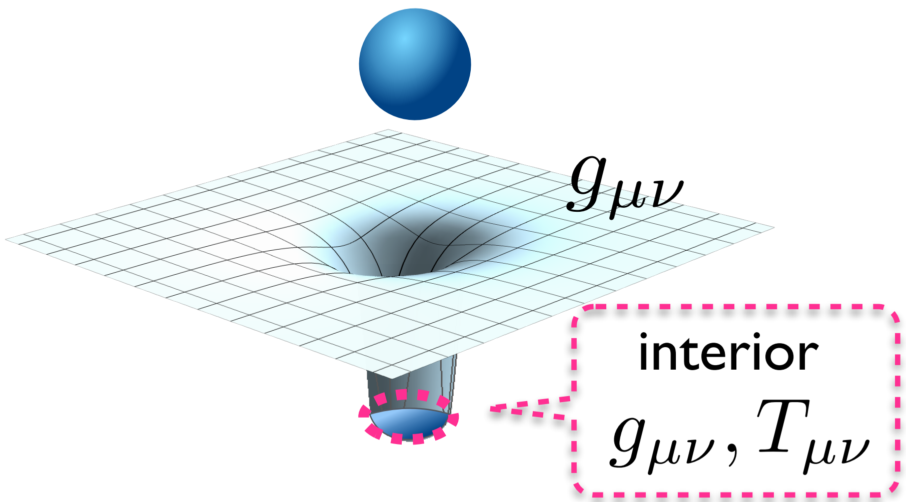
yield equilibrium structure and spacetime geometry

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta} \qquad \nabla_\alpha T^{\alpha\beta} = 0$$

Spherically symmetric spacetime:

$$ds_0^2 = g_{tt} dt^2 + g_{rr} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right)$$

Nonspinning compact object - interior



Interior: depends on the type of compact object, e.g. :

- Black hole $T_{\mu\nu} = 0$

Density

Pressure

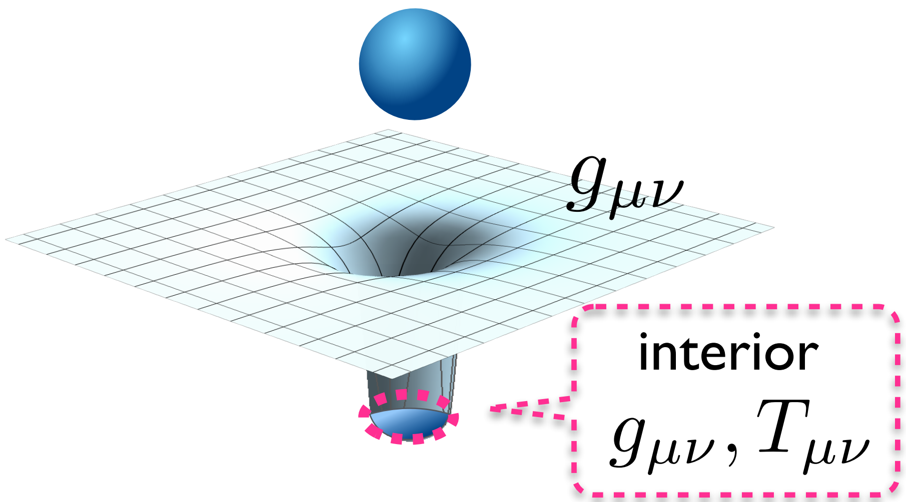
Fluid 4-velocity

- Perfect fluid (neutron star) $T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}$ $\quad u_\mu u^\mu = -1$

Self-interactions

- Complex scalar field $T_{\mu\nu} = \partial_\mu \phi^* \partial_\nu \phi + \partial_\nu \phi^* \partial_\mu \phi - g_{\mu\nu} \left[\partial^\alpha \partial_\alpha \phi + V\left(|\phi|^2\right) \right]$

Nonspinning compact object - exterior



$$ds_0^2 = g_{tt}dt^2 + g_{rr}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

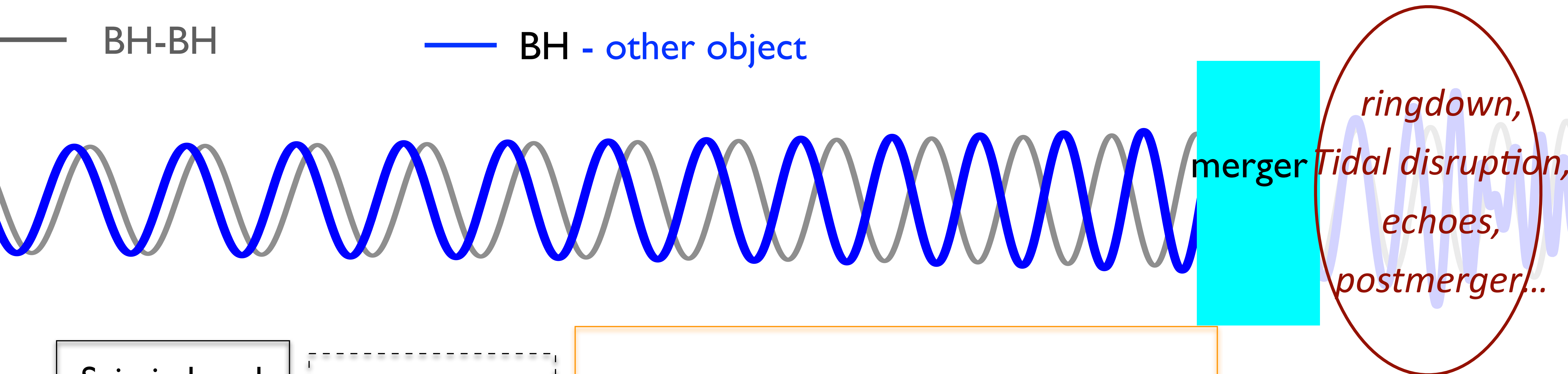
Exterior spacetime is the same for any object

$$g_{tt} = -\left(1 - \frac{2GM}{rc^2}\right) = -g_{rr}^{-1}$$

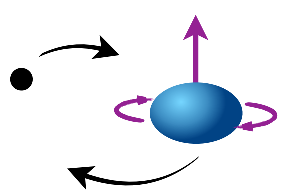
Depends only on the object's mass M

need deviations away from spherical symmetry to identify the type of object and its properties

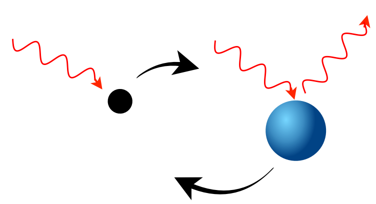
Generic GW signatures of an object's structure



Spin-induced multipoles

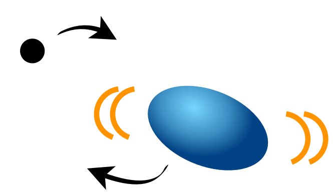


Absorption



tidal effects:

Driven excitation of characteristic modes



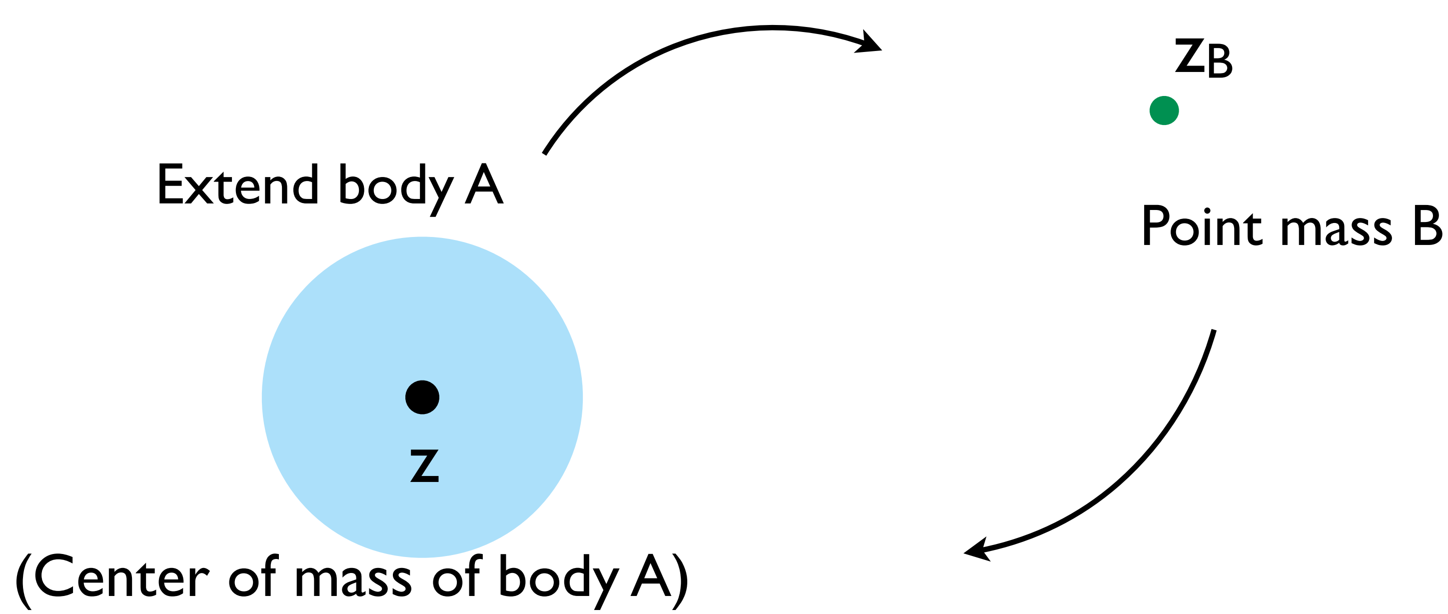
Non-resonant
+ resonant

Generic phenomena (any objects that are not classical GR black holes in 4d)

First step: tidal interactions in Newtonian binary systems

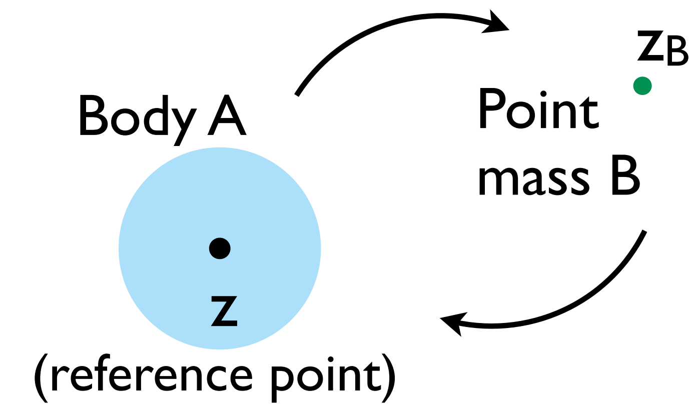
Setup

- Consider an extended body A and a point mass at large separation
- Finite-size effects are small
- Can obtain results for two extended bodies by simply adding a similar finite-size contribution for the other body



Expansion of the companion's potential felt by the body

- Recall: Taylor expansion of a function *f* around a reference point z



$$f(\mathbf{x}) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}(x-z)^L \partial_L f(\mathbf{x}) \,|_{\mathbf{x}=\mathbf{z}}$$

- In the last lecture: used this for the exterior potential of a mass distribution (here body A)

- Now: also use this for the gravitational potential due to the companion U_B that is felt by body A:

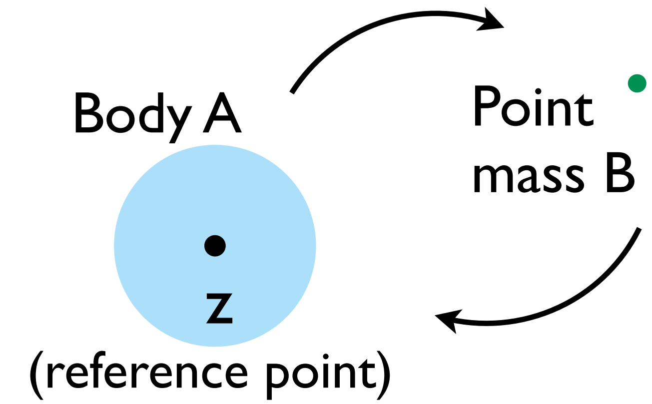
$$U_B(\mathbf{x}) = U_B(\mathbf{z}) + (x-z)^j \left[\partial_j U_B(\mathbf{x})\right]_{\mathbf{x}=\mathbf{z}} - \sum_{l=2}^{\infty} \frac{1}{\ell!}(x-z)^L \mathcal{E}^L$$

where $\mathcal{E}^L = -\left[\frac{\partial}{\partial x^L} U_B(\mathbf{x})\right]_{\mathbf{x}=\mathbf{z}}$ are 'tidal moments' $\qquad U_B = \frac{m_B}{|\mathbf{x} - \mathbf{z}_B|}$

This is a STF tensor, c.f. HW2

Equations of motion of the body

- The equations of motion of z (the center of mass of body A) are derived from the gradient of the companion's potential:



$$M\ddot{z}^j = -\int_A d^3x\,\rho\frac{\partial}{\partial x^j}U_B(\mathbf{x})$$

Mass density of A

$$\partial_j U_B(\mathbf{x}) = \partial_j U_B|_{\mathbf{z}} + (x-z)^i\left[\partial_i\partial_j U_B(\mathbf{x})\right]_{\mathbf{z}} + (x-z)^i(x-z)^k\left[\partial_i\partial_k\partial_j U_B(\mathbf{x})\right]_{\mathbf{z}} + \ldots$$

Only the STF part contributes

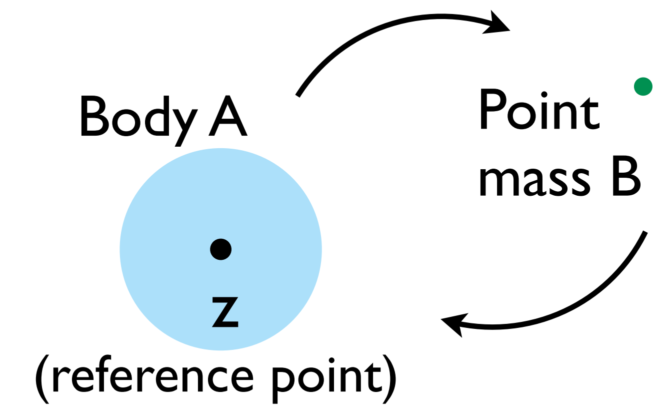
STF

$$M\ddot{z}^j = -M\frac{\partial}{\partial x^j}U_B(\mathbf{x})\,|_{\mathbf{x}=\mathbf{z}} + \sum_{\ell=2}^{\infty}\frac{1}{\ell!}M^{<L>}\mathcal{E}_{jL}$$

$$\mathcal{E}^L = -\left[\frac{\partial}{\partial x^L}U_B(\mathbf{x})\right]_{\mathbf{x}=\mathbf{z}}$$

Mass moments of body A $\quad M^{<L>} = \int_A d^3x'\rho(\mathbf{x}')(x'-z)^{<L>}$

Lagrangian for the dynamics



- The equations of motion of z (the center of mass of body A)

$$M\ddot{z}^j = -M\frac{\partial}{\partial x^j}U_B(\mathbf{x})\,\big|_{\mathbf{x}=\mathbf{z}} + \sum_{\ell=2}^{\infty}\frac{1}{\ell!}M^{<L>}\mathcal{E}_{jL}$$

- can be summarized in a Lagrangian, where finite-size effects add to the point-mass part:

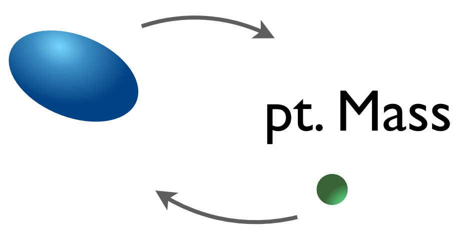
$$\boxed{L_A = \frac{1}{2}\Bigg[\overbrace{M\dot{z}^2}^{\text{Point-mass motion of the body's center of mass z}} - \sum_{\ell}\frac{1}{\ell!}\overbrace{M^{<L>}\mathcal{E}_L}^{\text{Finite-size contributions}}\Bigg] + \mathcal{L}_{\rm int}}$$

Body's multipole moments couple to the tidal moments sourced by the companion

Internal dynamics of the multipoles (Depends only on $M^{<L>}, \dot{M}^{<L>}\ldots$)

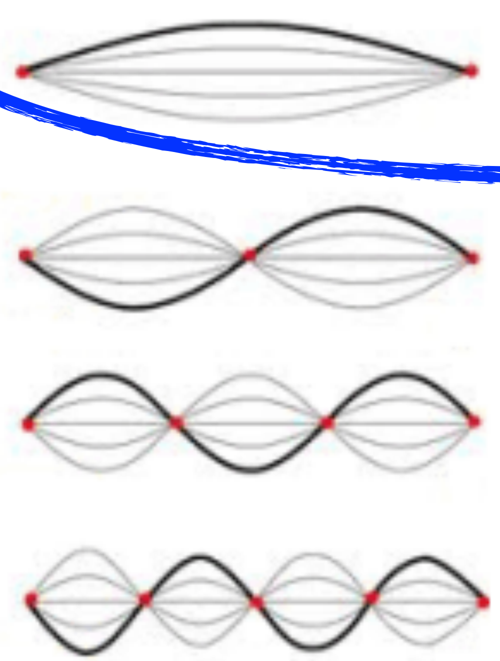
for more details on this derivation, see https://arxiv.org/abs/1009.4919

Tidally-induced multipoles



- Multipoles due to the body's response to the tidal field
- In general: perturbations of a material object ⇒ decompose into normal modes
- Here: perturbations to a spherically symmetric background
  - modes are characterized by multipole index ℓ and overtone number n

e.g. vibrating string



Fundamental mode (n=0)

First overtone (n=1)

Second overtone (n=2)

Third overtone (n=3)

Dominates the tidal coupling

(Largest overlap with the smooth gravity gradient due to the companion)

Internal dynamics of tidally-induced multipoles

- Quadrupole $M^{<ij>}$ gives by far the largest contribution to the overall effect

- Focus on these dominant effects ⇒ specialize from now on to $n = 0$, $\ell = 2$

- Omit the (n, ℓ) labels and define $Q^{ij} = M^{<ij>}$ (contribution only from the n=0 overtone implied)

- *Qij* behaves approximately as harmonic oscillators with Lagrangian

Characteristic frequency (fundamental mode)

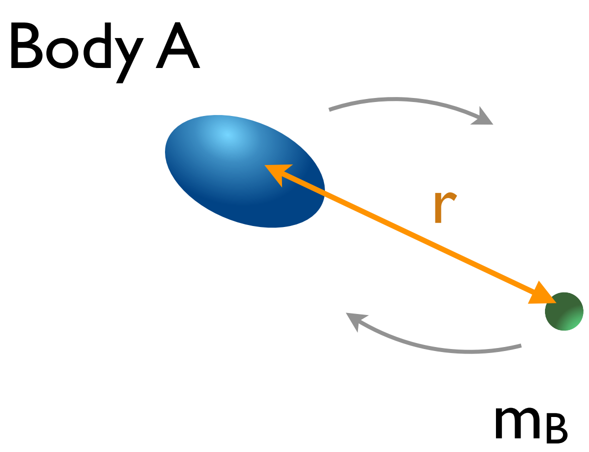
$$\mathcal{L}_{\rm int} \propto \dot{Q}^{ij}\dot{Q}_{ij} - \omega_0^2 Q_{ij}Q^{ij}$$

Kinetic energy *Elastic potential energy*

[extension to higher multipoles is very similar to the quadrupole case]

Lagrangian for the binary system

- We can construct the Lagrangian for the binary dynamics by adding the finite-size contributions to the point-mass parts



- In relative coordinates, this leads to:

$$S = S_{\rm orbit} + \int dt \left[-\frac{1}{2} Q^{ij} \mathcal{E}_{ij} + \frac{1}{4\lambda\omega_0^2} \left(\dot{Q}^{ij} \dot{Q}_{ij} - \omega_0^2 Q_{ij} Q^{ij} \right) \right]$$

coupling coefficient - characteristic 'tidal deformability' parameter

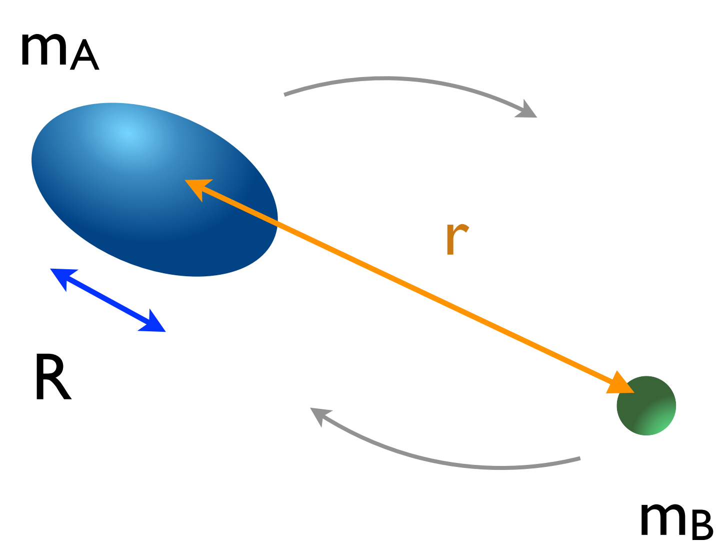
$$L_{\rm orbit} = \frac{1}{2}\mu v^2 + \frac{G\mu M}{r} \qquad \mathcal{E}_{ij} = -G m_B \partial_i \partial_j \frac{1}{r}$$

Timescales

- Fundamental mode frequency ~ scale of the body A ~ $\omega_0 \sim \sqrt{Gm_A/R^3}$

- Tidal forcing $\mathcal{E}_{ij}$ varies on the orbital scale ~ $\omega \sim \sqrt{GM/r^3}$

- At large separation: $\omega \ll \omega_0$



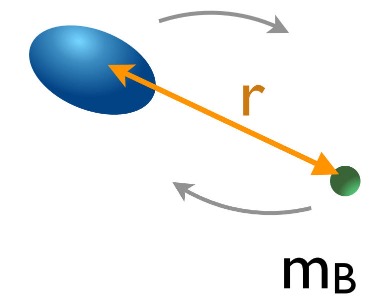
Mode is driven below resonance, no oscillations

‘Adiabatic limit’

Adiabatic limit

- Had the action for the binary dynamics

$$S = S_{\text{orbit}} + \int dt \left[-\frac{1}{2} Q^{ij} \mathcal{E}_{ij} + \frac{1}{4\lambda\omega_0^2} \left(\dot{Q}^{ij} \dot{Q}_{ij} - \omega_0^2 Q_{ij} Q^{ij} \right) \right]$$



- Consider now the adiabatic limit: $\dot{Q}^{ij} = 0$

Equations of motion: $\boxed{Q_{ij} = -\lambda\, \mathcal{E}_{ij}}$

Interpretation of $\boldsymbol{\lambda}$ (tidal deformability) as the response coefficient

Computed from a 'microscopic' analysis of the tidally perturbed structure & gravitational potential

Integrate out Q_{ij} to obtain an effective action in the adiabatic limit:

$$S = S_{\text{orbit}} + \int dt \frac{\lambda}{4} \underbrace{\mathcal{E}_{ij}\, \mathcal{E}_{ij}} = \lambda \int dt \frac{3G^2 m_B^2}{2r^6}$$

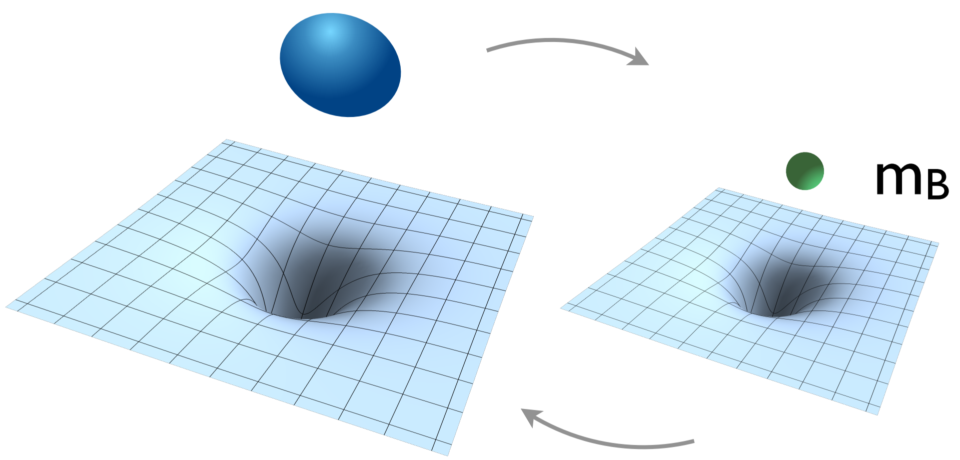
Encodes information about the internal structure

Tidal invariants, depend only on orbit and m_B

Computation of tidal deformability in GR

Generalization of tidal moments to GR

- Compact objects are strongly self-gravitating, GR is crucial



- Definition of the tidal moments in GR:

In body A's rest frame:

$$\boxed{\mathcal{E}_{ij} = R_{0i0j}}$$

Components of the Riemann curvature tensor due to the companion

Definition of the body's multipole moments

- Newtonian integral $Q_{ij} = \int_A d^3x\, \rho\, (x-z)^{<ij>}$ has no strong-field generalization

- But recall how the multipoles first appeared:
  - coefficients in the expansion of the exterior gravitational potential

c.f. the formula from last time for the potential outside a mass distribution:

$$U(\mathbf{x}) = G \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \frac{\bar{n}^{<L>} M^{<L>}}{\bar{r}^{\ell+1}}$$

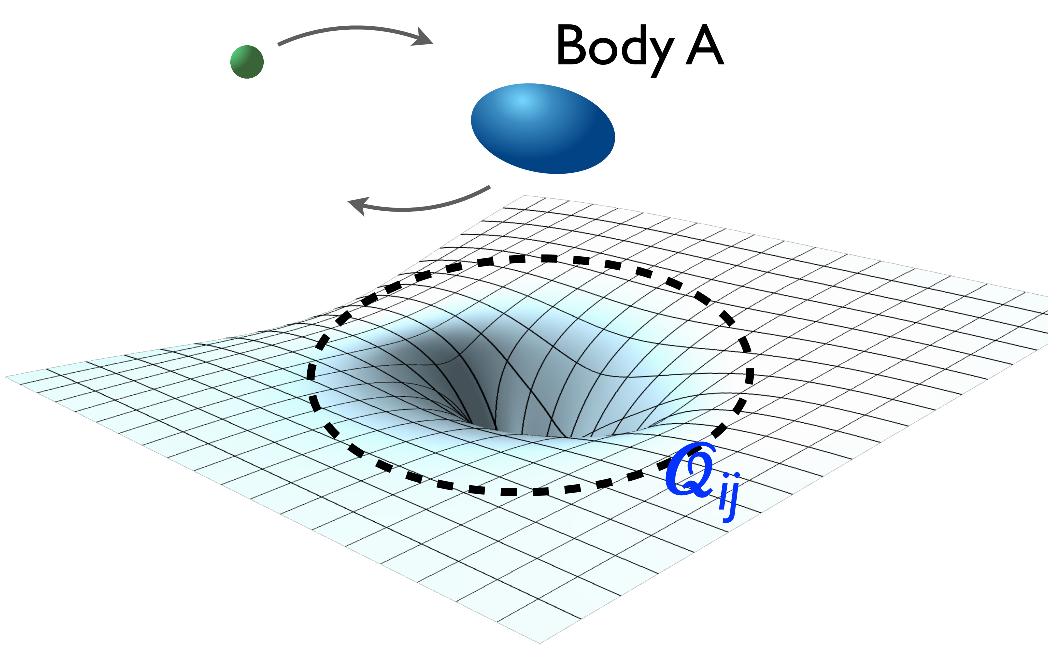
definition of Q_{ij} as $\propto$ the coefficient of the r^{-3} falloff in the body's potential

$$\boxed{\lim_{r\to\infty} U_A = \frac{m_A}{r} + \frac{3n^{<ij>} Q_{ij}}{2r^3} + \mathcal{O}(r^{-4})}$$

Definition of the body's multipole moments

Newtonian:

$$\lim_{r\to\infty} U_A = \frac{m_A}{r} + \frac{3n^{<ij>}Q_{ij}}{2r^3} + \mathcal{O}(r^{-4})$$



- In GR: can work in a frame where the time-time component of the metric sourced by A is

$$g_{00}^A = -\left(1 - 2U_{\text{eff}}^A\right)$$

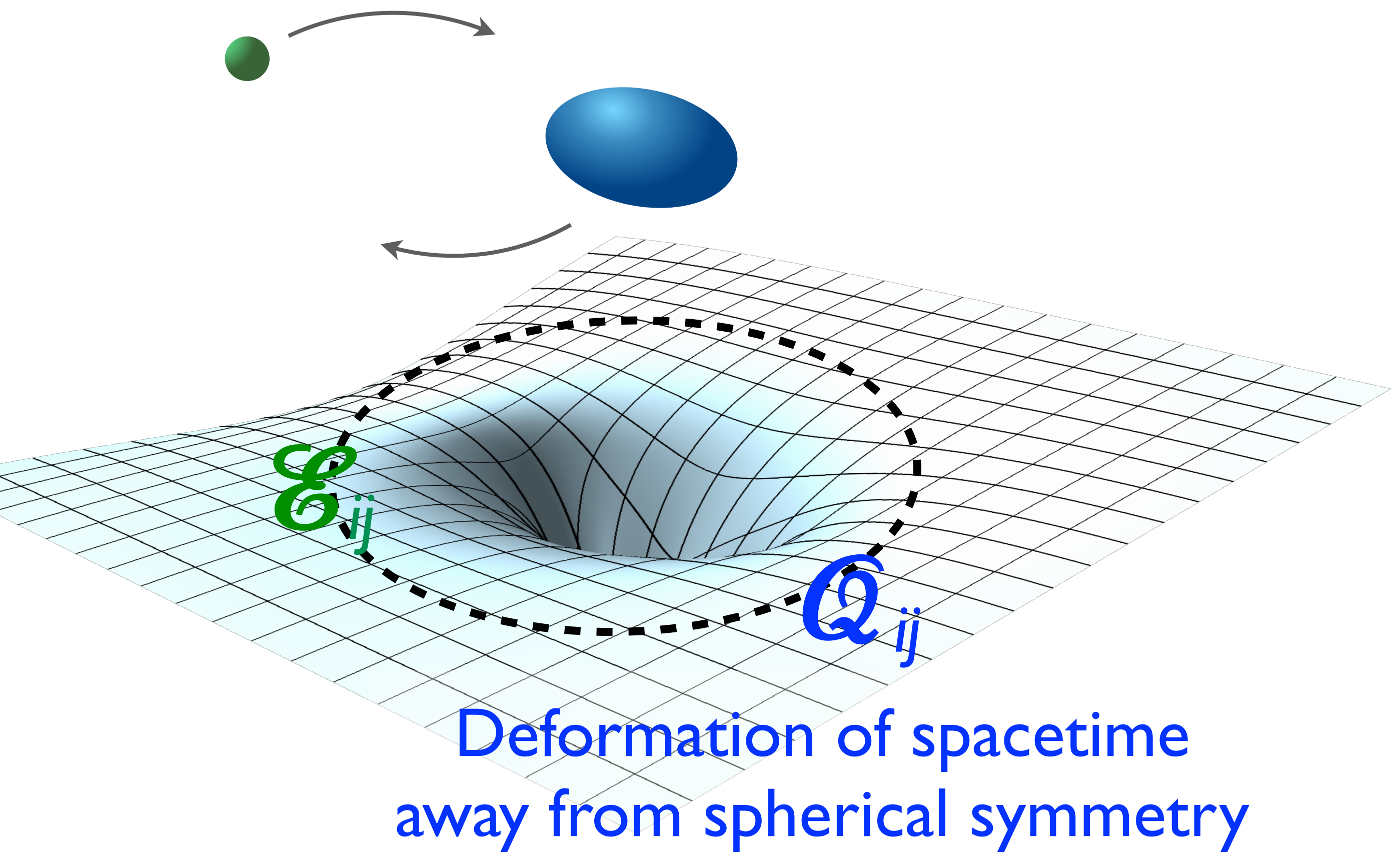
characterizes the deformation of the spacetime away from spherical symmetry

$$\lim_{r\to\infty} U_{\text{eff}}^A = \frac{m_A}{r} + \frac{3n^{<ij>}Q_{ij}}{2r^3} + \mathcal{O}(r^{-4})$$

Definition of Q_{ij}

Definition of tidal deformability

When variations in tidal field are much faster than the body's internal timescales (adiabatic limit):



$$\mathcal{Q}_{ij} = -\lambda\, \mathcal{E}_{ij}$$

tidal deformability parameter

=0 for a BH

Same relation as in Newtonian gravity but with the relativistic definitions of Q_{ij} and $\mathcal{E}_{ij}$

To compute λ need to calculate the response to the tidal perturbation

Computation of tidal deformability

Aim: compute how the structure of the object changes due to tidal perturbations and the resulting imprint in the exterior spacetime, from which we can extract the quadrupole

- Einstein field equations + energy-momentum conservation determine interior structure and spacetime geometry (interior and exterior)

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta} \qquad \nabla_\alpha T^{\alpha\beta} = 0$$

- Analyze for small perturbations to a spherically symmetric equilibrium configuration

- Introduce metric perturbations $h_{\mu\nu}$:

$$\underbrace{ds^2 = ds_0^2}_{\text{Background metric}} + \underbrace{h_{\mu\nu}}_{\text{Small perturbations}} dx^\mu dx^\nu$$

In general: 10 independent functions

Simplifications for the perturbations

- Separation of variables
- perturbations split into two sectors with different parity, i.e. symmetry + or - under reflection through the origin $\{\theta,\varphi\} \to \{\pi-\theta, \pi+\varphi\}$
- Simplifications by choosing 'Regge-Wheeler gauge', where the even-parity perturbations are

[Regge & Wheeler 1957 stability of a Schwarzschild singularity]

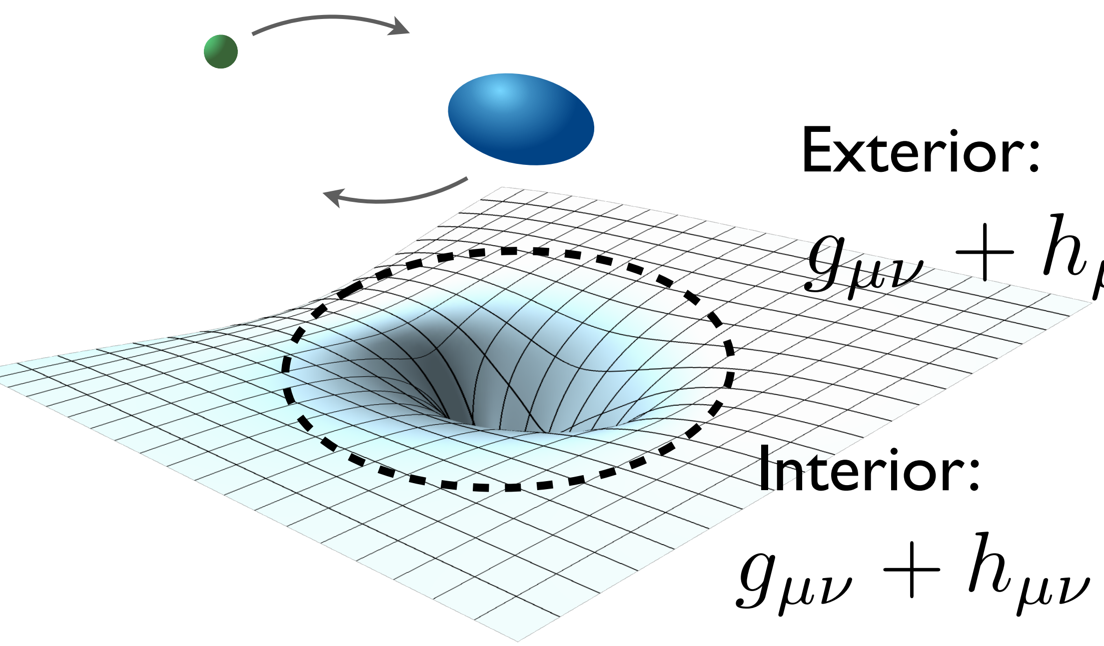
$$h_{\mu\nu} \sim \sum_{m=-2}^{2} \begin{pmatrix} H_0(r) & H_1(r) & 0 & 0 \\ * & H_2(r) & 0 & 0 \\ * & * & K(r) & 0 \\ * & * & * & \sin^2\theta K(r) \end{pmatrix} Y_{2m}(\theta,\varphi) e^{-i\omega t}$$

- Also decompose the components of $T_{\mu\nu}$ into spherical harmonics and $e^{-i\omega t}$

Frequency-dependence is important for characteristic oscillation mode frequencies [part 2 of this lecture]

Tidal deformability defined in the adiabatic limit - can specialize to zero-frequency perturbations

Computation of tidal deformability



Exterior:

$g_{\mu\nu} + h_{\mu\nu}$

Interior:

$g_{\mu\nu} + h_{\mu\nu},\ \delta T^{\mu\nu}$

- Substitute into

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

$$\nabla_\alpha T^{\alpha\beta} = 0$$

- Expand to linear order in the perturbations

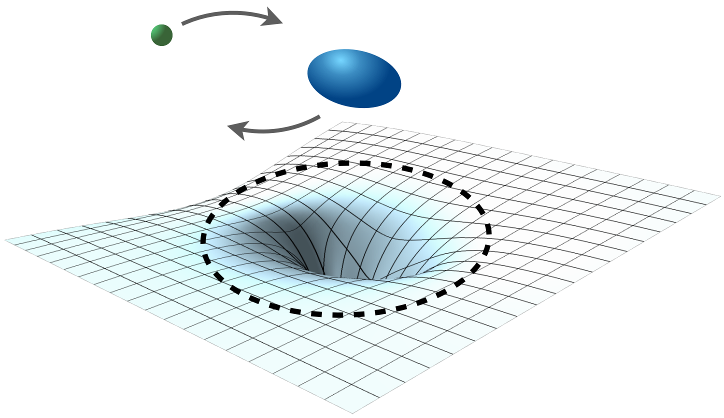
- Write out all the equations for the different components

- In many cases, after manipulations to algebraically eliminate variables, can end up with a single ‘master equation’ for ~ the time-time component of the metric perturbation

$$\frac{d^2 H_0}{dr^2} + A(r) \frac{dH_0}{dr} + B(r) H_0 = 0$$

Depend on background metric and unperturbed energy-momentum tensor

Computation of tidal deformability



- Master equation: $$\frac{d^2 H_0}{dr^2} + A(r)\frac{dH_0}{dr} + B(r)H_0 = 0$$

- Solve in the interior (usually only possible numerically)

- Exterior: A and B simplify since $\delta T_{\mu\nu} = 0$ there

  - In GR, get analytical solutions in terms of special functions (associated Legendre functions)

Determined by matching to the interior solution at the surface

$$H_0 = a_1 Q_{22}\left(\frac{r}{M} - 1\right) + a_2 P_{22}\left(\frac{r}{M} - 1\right)$$

Behavior at large distance $\sim r^{-3}$ $\sim r^{2}$

Compare with the definition of the multipole moments

$$\lim_{r\to\infty} \delta g_{tt} \sim \frac{3n^{<ij>}Q_{ij}}{2r^3} + \frac{1}{2}r^2\mathcal{E}_{ij}n^{<ij>} + \ldots$$

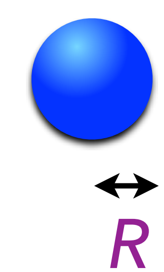
Final result

- Going through the details shows that for non-BH objects, **λ** depends on interior integration results evaluated at the surface

$$
\begin{aligned}
\frac{\lambda}{M^5} = {} & \frac{16}{15}(1-2C)^2[2+2C(y-1)-y] \\
& \times \Big\{2C[6-3y+3C(5y-8)] + 4C^3[13-11y+C(3y-2)+2C^2(1+y)] \\
& \quad +3(1-2C)^2[2-y+2C(y-1)]\ln(1-2C)\Big\}^{-1}
\end{aligned}
$$

$C = \dfrac{GM}{Rc^2}$ Compactness of the body

$$y = \frac{rH_0'}{H_0}\bigg|_{r\,=\,R}$$

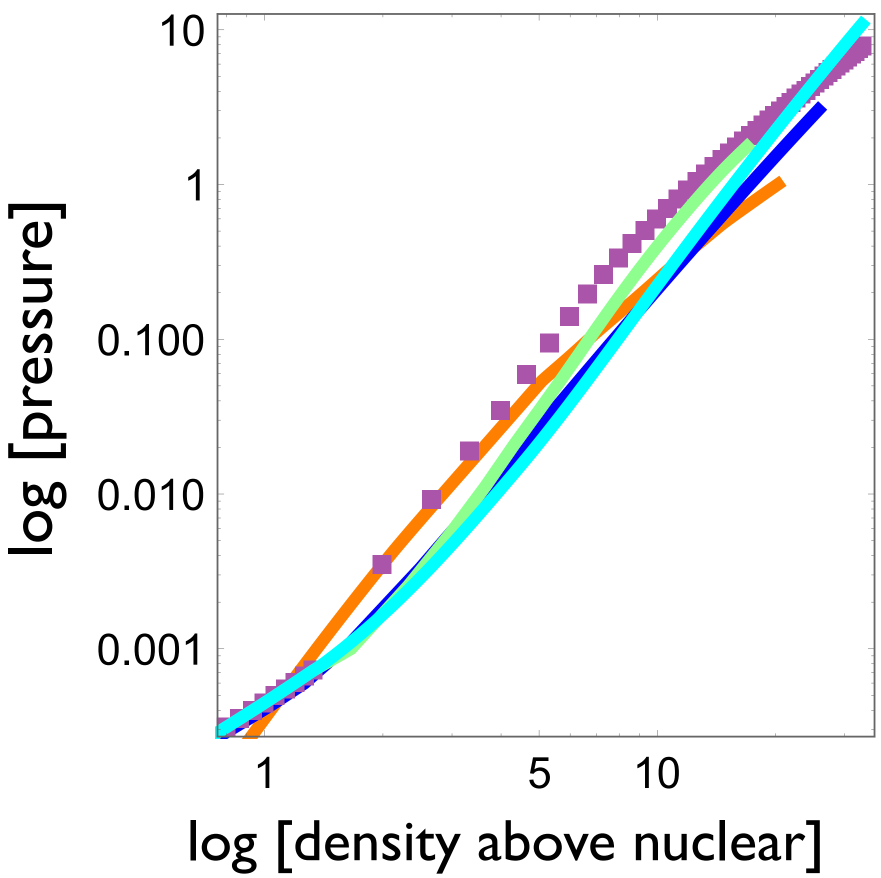


For black holes: boundary conditions at the horizon require that **λ**=0

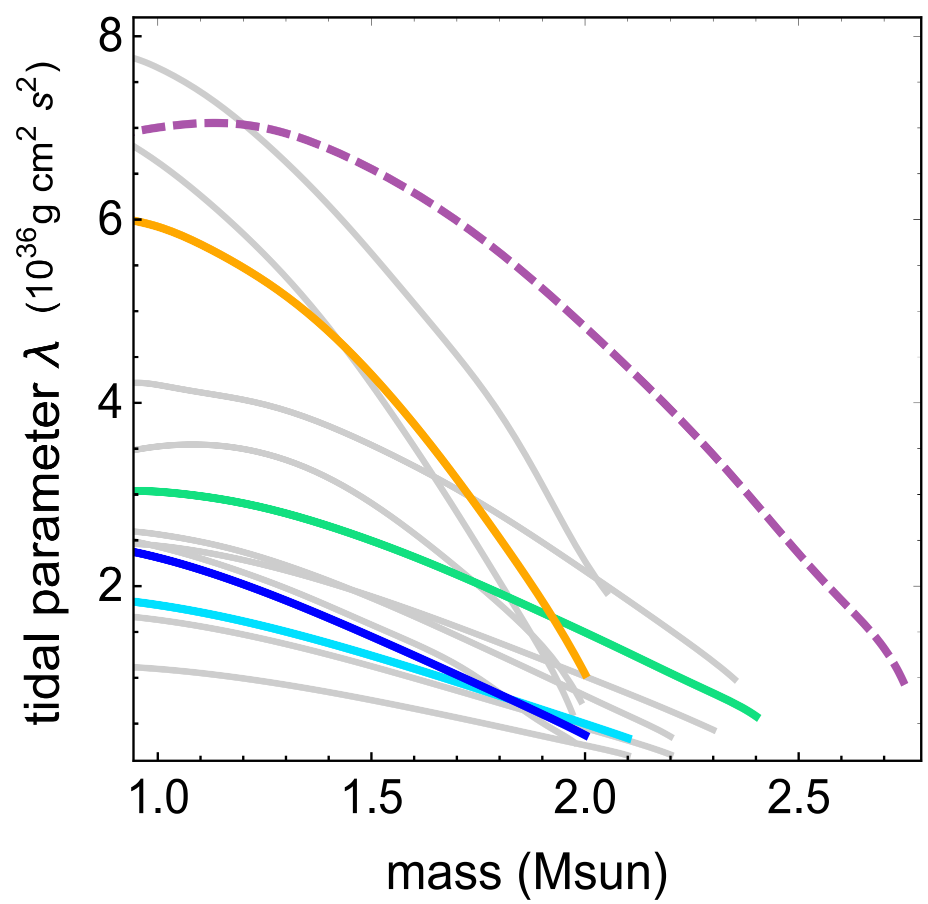
Example for neutron stars

Tidal deformability depends on matter properties encoded in the equation of state

Example equations of state (EoS)



tidal deformability λ vs. mass



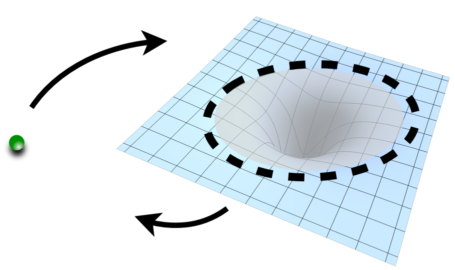
Effect on the dynamics and GWs

Effect on the binary dynamics

- We already worked out the Newtonian result (adiabatic limit) in a convenient form

$$S = S_{\mathrm{orbit}} + \int dt \, \frac{\lambda}{4} \, \mathcal{E}_{ij} \, \mathcal{E}_{ij}$$

- Generalize to GR: reference worldline with proper time τ plus tidal effects. In the body's frame:



$$S_{\mathrm{tidal}} = \int \underset{\substack{\updownarrow \\ z\, dt}}{d\tau} \, \frac{\lambda}{4} \mathcal{E}_{ij} \mathcal{E}_{ij}$$

Relativistic redshift (strong-field region near the body)

Does GR give rise to new types of interactions?

- In the 1990s, some numerical simulations seemed to find a new ``relativistic crushing force" in neutron

PHYSICAL REVIEW D, VOLUME 58, 043003

Relativistic hydrodynamics in close binary systems: Analysis of neutron-star collapse

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VOLUME 75, NUMBER 23 PHYSICAL REVIEW LETTERS 4 DEC

Instabilities in Close Neutron Star Binaries

J. R. Wilson[1] and G. J. Mathews[2]
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(Received 22 August 1995)

… surprising evidence that GR effects may cause otherwise stable stars to individually collapse prior to merging

se neutron star binaries based upon (3 + 1)
s calculations. When a realistic equation of
neutron stars reveal surprising evidence that
stars to individually collapse prior to merging.
Also, the strong fields cause the last stable orbit to occur at a larger separation distance and lower frequency than post-Newtonian estimates.

BINARY-INDUCED NEUTRON STAR COMPRESSION, HEATING, AND COLLAPSE

G. J. MATHEWS
University of Notre Dame, Department of Physics, Notre Dame, IN 46556
AND
J. R. WILSON
University of Notre Dame, Department of Physics, Notre Dame, IN 46556 and University of California,

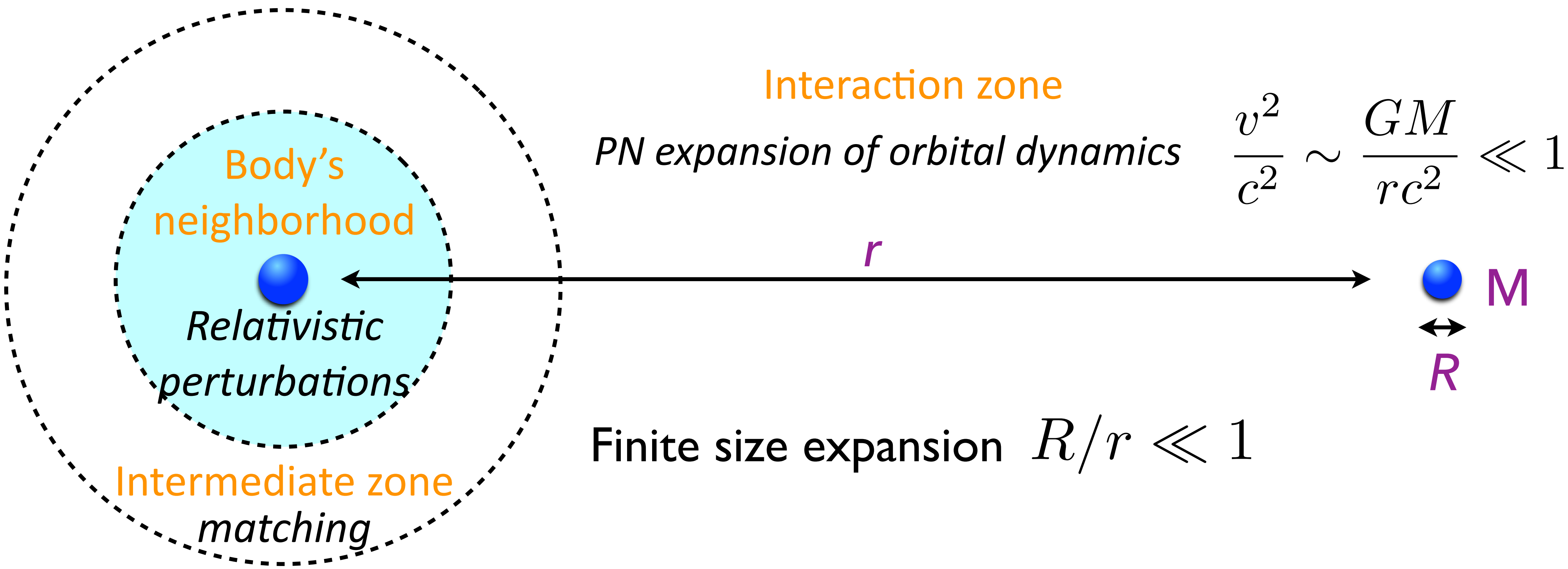
Does GR give rise to new types of interactions?

- This caused a lot of discussions and follow-up work in the community

  Sociological account: Kennefick 2000 ``Star crushing: theoretical practice & the theoretician's regress''

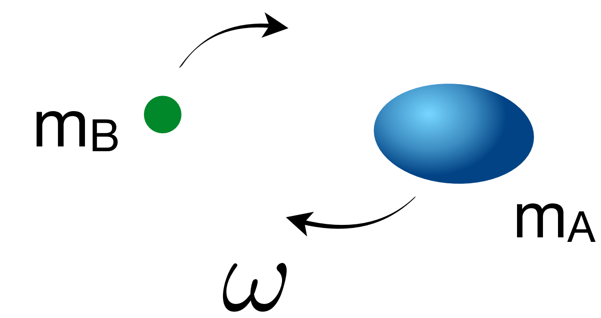
- A rigorous analysis using matched asymptotic expansions between the neutron star and orbital spacetimes showed that there are no such new couplings

  [Flanagan 1998: ``GR coupling between orbital motion & internal degrees of freedom ..'' paper]



However, in GR there are new types of 'gravito-magnetic' tidal fields associated with a different piece of the companion's spacetime curvature, which couple to the body's current multipole moments

Schematic influence on the GWs



- Energy goes into deforming the body

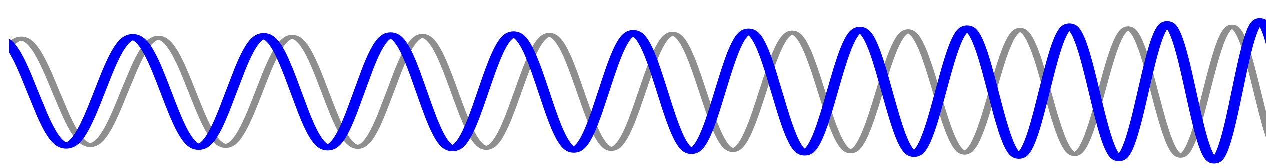
$$E \sim E_{\text{orbit}} - \frac{\lambda}{4}\mathcal{E}^2$$

- moving multipoles contribute to gravitational radiation

$$\dot{E}_{\mathrm{GW}} \sim \left[\frac{d^3}{dt^3}\left(Q_{\text{orbit}} + \mathcal{Q}\right)\right]^2$$

- approx. GW phase from energy balance: $\dfrac{d\phi_{\mathrm{GW}}}{dt} = 2\omega \qquad \dfrac{d\omega}{dt} = \dfrac{\dot{E}_{\mathrm{GW}}}{dE/d\omega}$

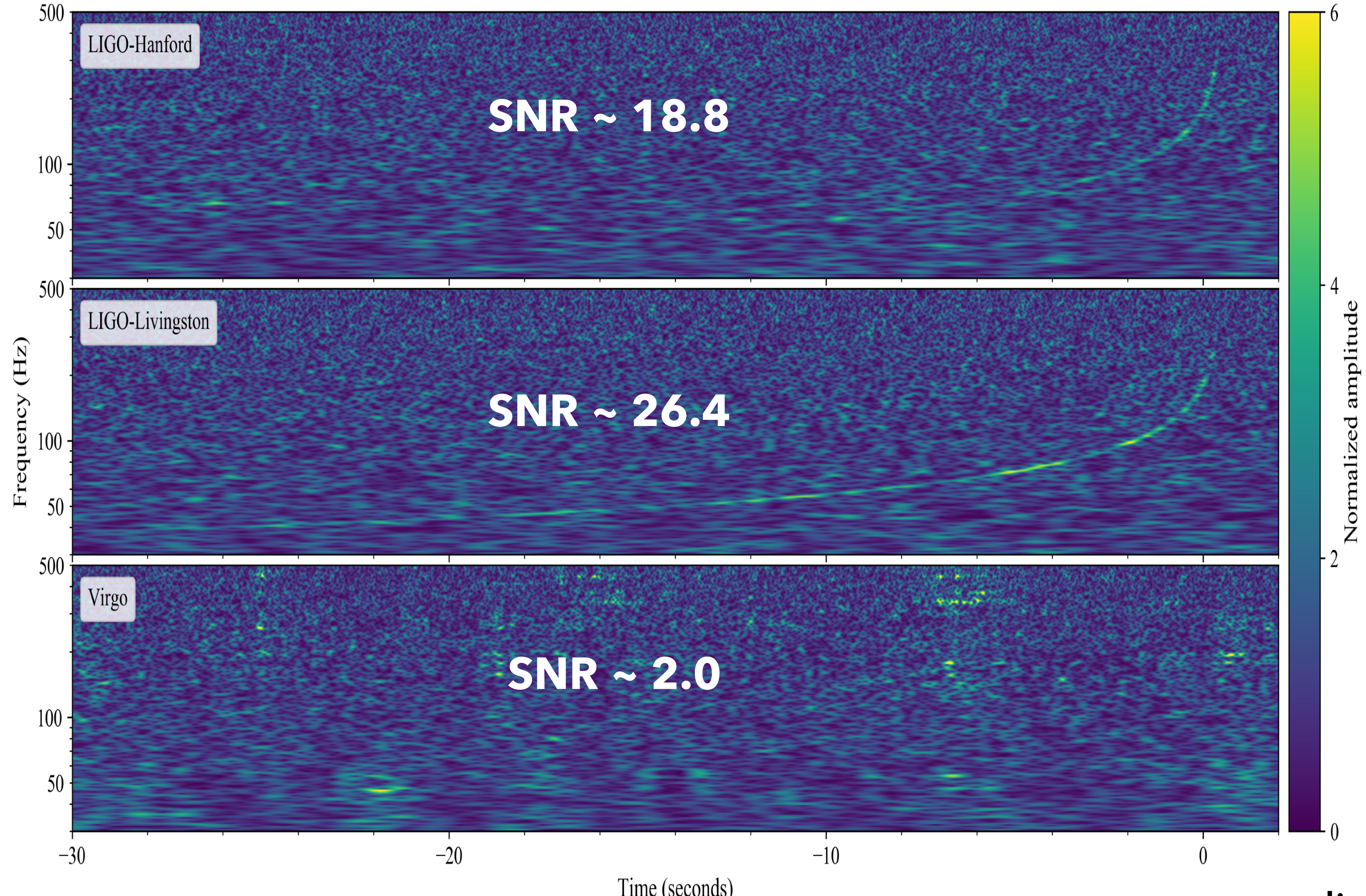
$$\Delta\phi_{\mathrm{GW}}^{\text{tidal}} \sim \lambda \frac{(M\omega)^{10/3}}{M^5}$$



- for two NSs: most sensitive to: $\tilde{\Lambda} = \dfrac{13}{16M^5}\left[\left(1 + 12\dfrac{m_B}{m_A}\right)\lambda_A + \left(1 + 12\dfrac{m_A}{m_B}\right)\lambda_B\right]$

Similar to chirp mass

Aug. 17, 2017: two neutron stars merge — GW170817



credit:LVC

Distance: ~40 Mpc

Total mass: ~ 2.74 Msun

Loudest, closest, richest in science

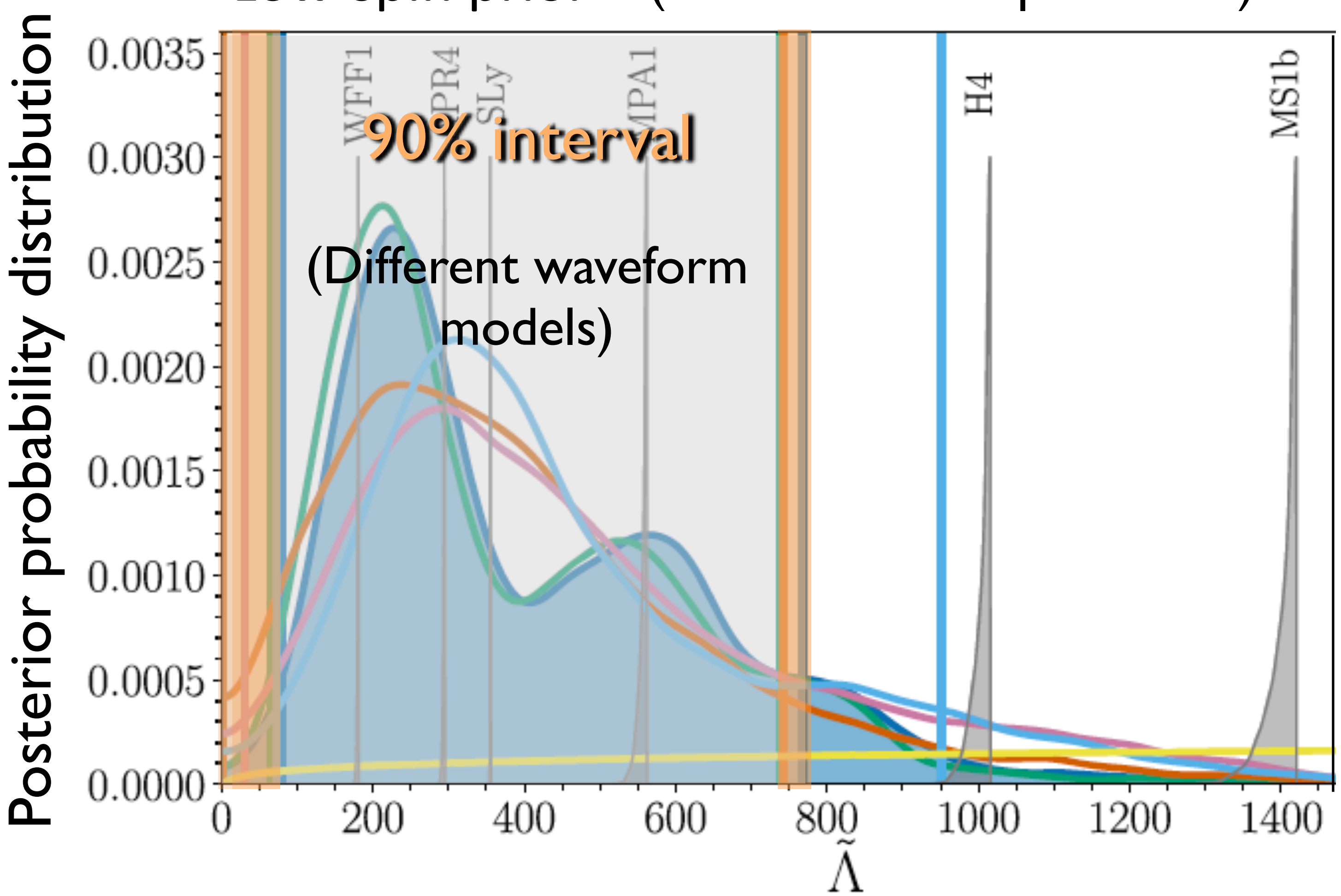
Longest:~100 s (c.f. <1 s for GW150914)

First constraints on tidal deformability

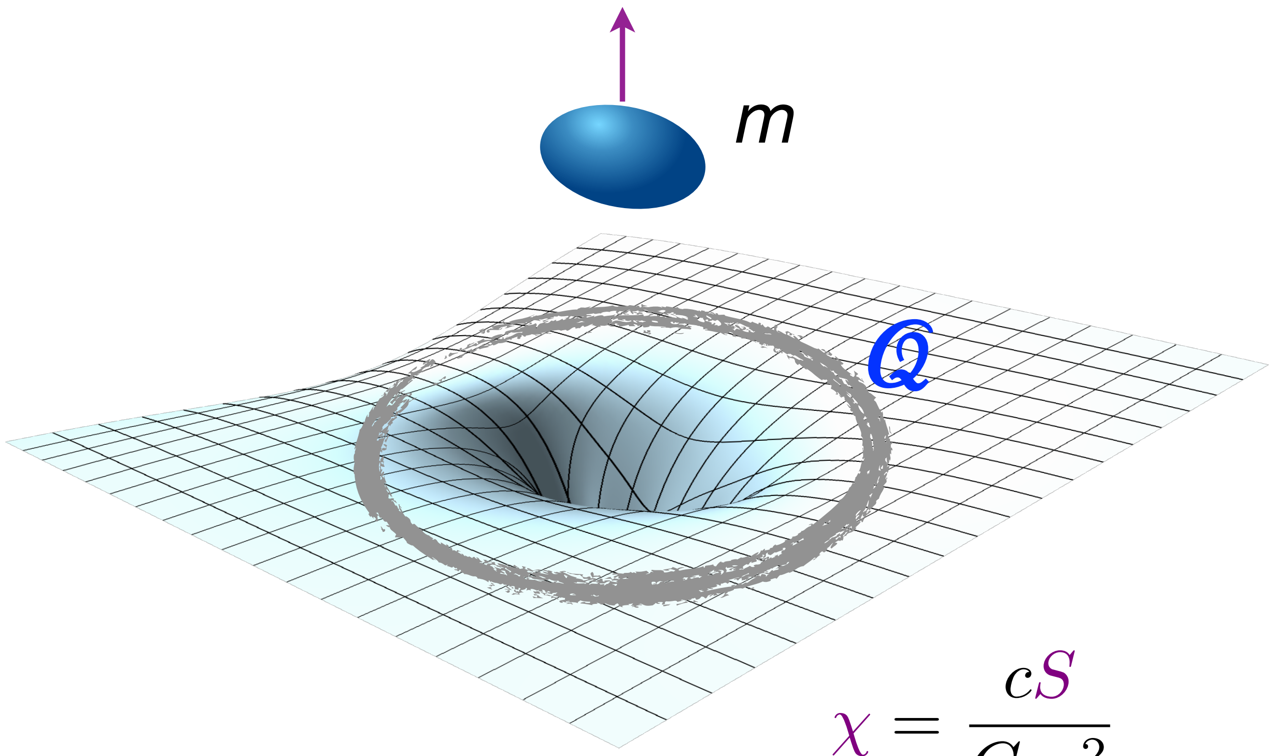
Masses ~ 1.3+1.5 Msun

$$\tilde{\Lambda} = \frac{13}{16M^5}\left[\left(1+12\frac{m_B}{m_A}\right)\lambda_A + \left(1+12\frac{m_A}{m_B}\right)\lambda_B\right]$$

Low-spin prior (Dimensionless spin < 0.05)



Similarly for spin-induced quadrupoles



$$\chi = \frac{cS}{Gm^2}$$

$$\mathcal{Q}_{\rm spin} = -\kappa\,\chi^2 m^3$$

∝ rotational Love number

=1 for BHs by the no-hair theorem:

$$M_\ell + iS_\ell = m(i\chi m)^\ell$$

Mass moments Current moments

- Effect on the dynamics and GWs also through coupling with the tidal field $\mathscr{E}$
- Imprint in the phase evolution

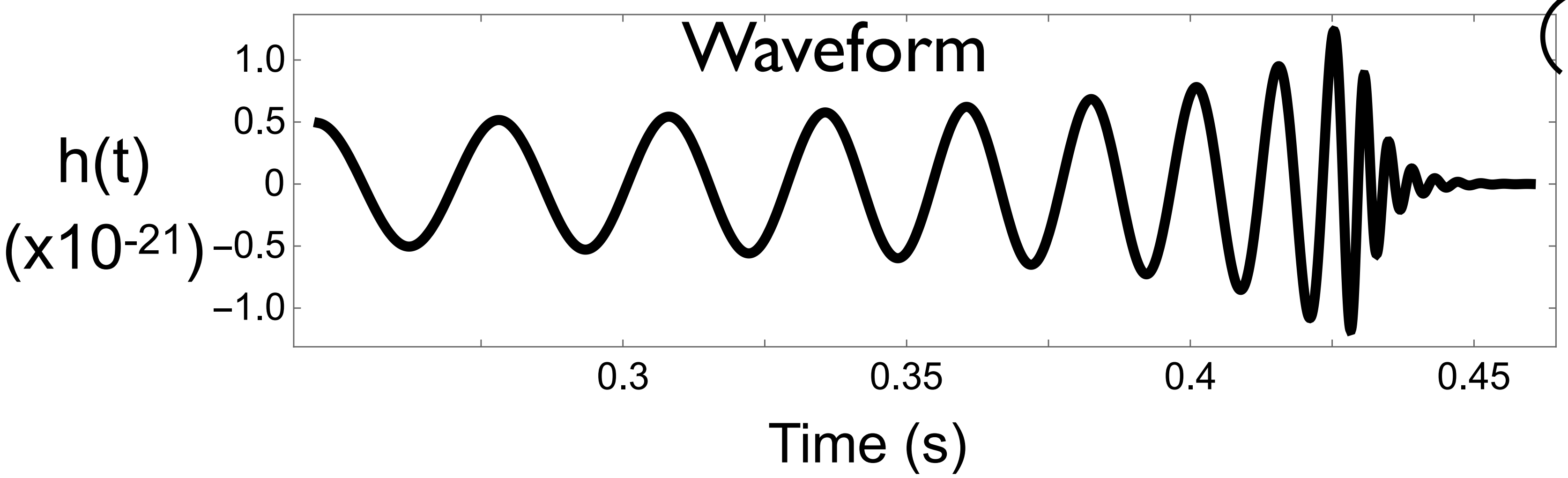
$$\Delta\phi_{\rm GW}^{\rm Qspin} \sim \boldsymbol{\kappa}\,\boldsymbol{\chi^2 (M\Omega)^{4/3}}$$

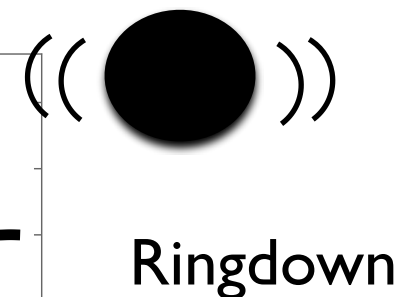
Time-dependent perturbations of relativistic compact objects:

Quasi-normal modes

Quasi-normal modes (QNMs) of black holes

- No matter: oscillations of spacetime
- ‘Quasi’ - normal: emission of GWs - damping, mode frequencies are complex





Also important e.g. for:

- Stability analyses
- AdS/CFT studies

- More broadly, BH perturbations (with source) are important for GWs in other contexts, e.g. large mass ratio binaries

- Open-source codes for black hole perturbations available through the BH perturbation toolkit project https://bhptoolkit.org/toolkit.html

Time-dependent perturbations

- For the even-parity sector, start with a similar ansatz for $h_{\mu\nu}$ as for the tidal deformation analysis

$$h_{\mu\nu} = \sum_{\ell,m} \begin{pmatrix} H_0^{\ell m}(r) & H_1^{\ell m}(r) & 0 & 0 \\ * & H_2^{\ell m}(r) & 0 & 0 \\ * & * & K_{\ell m}(r) & 0 \\ * & * & * & \sin^2\theta K_{\ell m}(r) \end{pmatrix} Y_{\ell m}(\theta,\varphi) e^{-i\omega t}$$

(Note: odd-parity perturbations have the same frequency spectrum and are in practice simpler to handle but require a different decomposition)

- For the QNMs, consider unforced oscillations: $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 0$

- Introduce the tortoise coordinate r_*: $r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right)$

With the properties $r_* \to -\infty$ as $r \to 2M$ (Horizon)

$r_* \to \infty$ as $r \to \infty$

Black hole perturbations

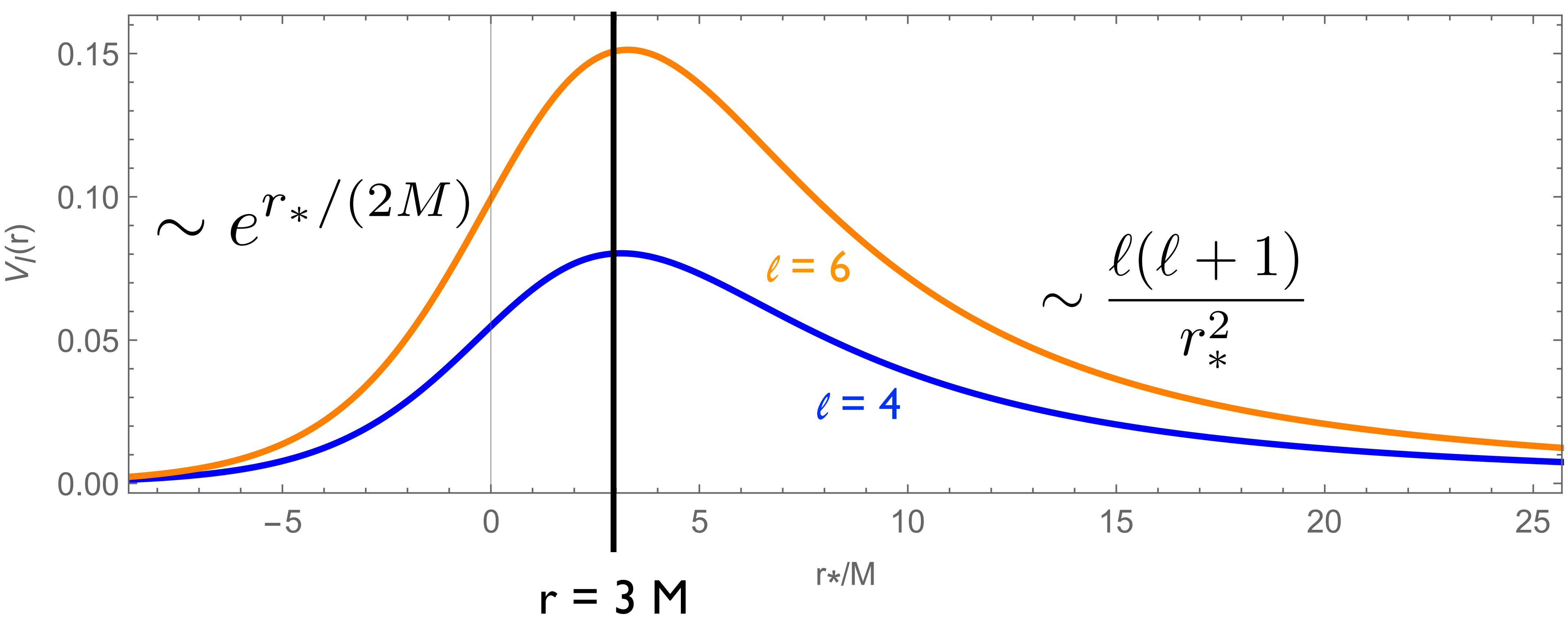
- Zerilli (1970) paper showed that upon field redefinitions, the field equations lead to the master equation

$$\frac{d^2}{dr_*^2} Z_{\ell m} + \left[\omega^2 - V_\ell(r)\right] Z_{\ell m} = 0$$

[or a complicated source term]

Weighted linear combination
of the $H_1^{\ell m}$ and $K^{\ell m}$

Effective potential

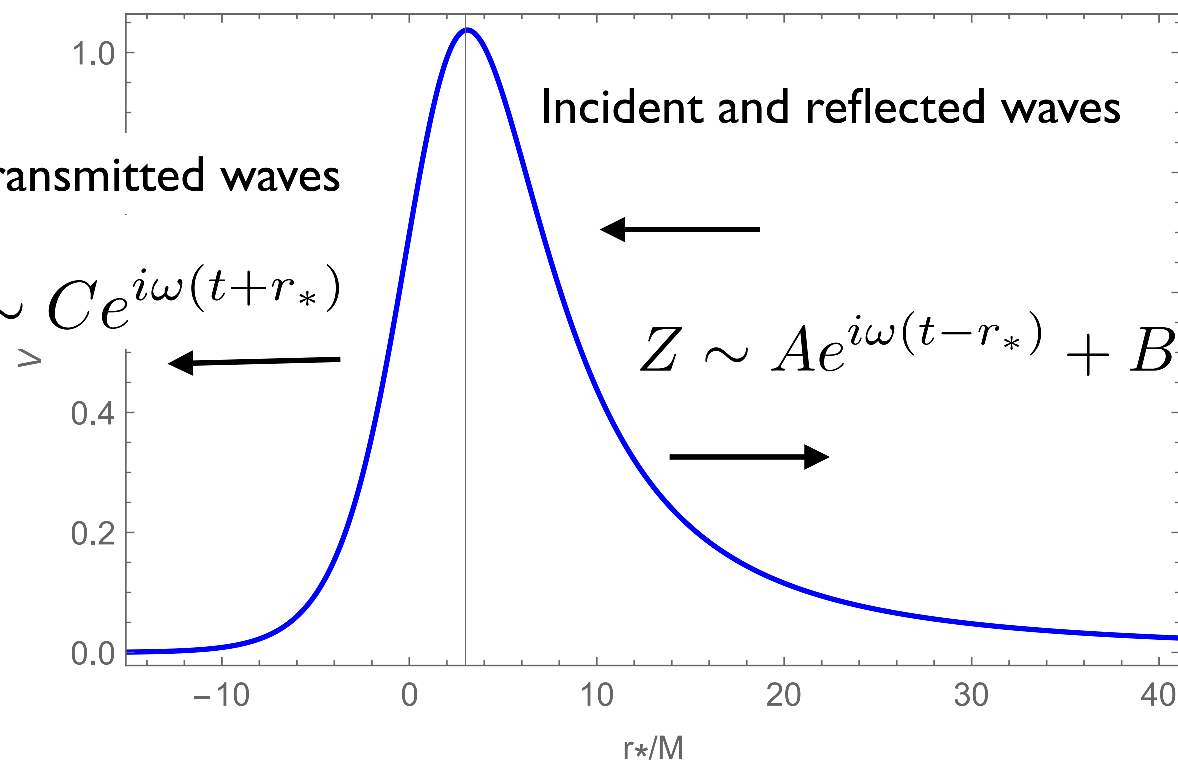


c.f. marginally stable circular photon orbit (massless particles) - light ring

Black hole response to scattering

- Numerical studies [C.Vishveshwara 1970 paper] of the scattered waves for an incident wave packet:
  - For a very broad wave packet, almost total reflection, no information about the BH
  - As the width of wave packet decreases, distinct features in the scattered waves start to appear
  - Reaches a limit for width ~ 2M (higher frequencies are nearly all absorbed by the BH)

Behavior in different regions of the potential



Transmitted waves

$$Z \sim Ce^{i\omega(t+r_*)}$$

Incident and reflected waves

$$Z \sim Ae^{i\omega(t-r_*)} + Be^{i\omega(t+r_*)}$$

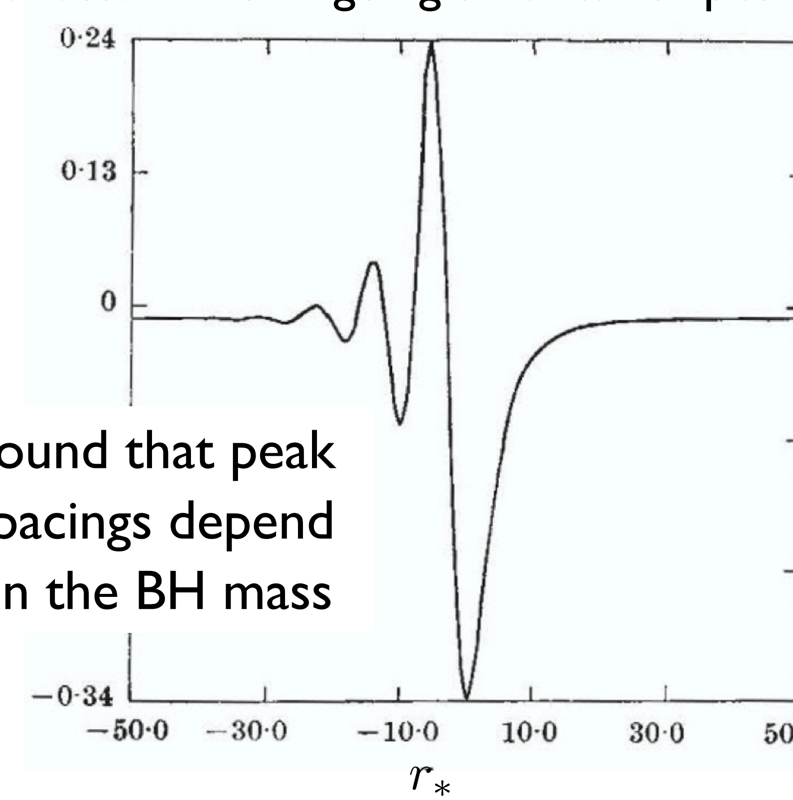
Transmission:

$$T = |C/A|^2$$

Reflection:

$$R = |B/A|^2$$

Reflected Z for ingoing small-width packet



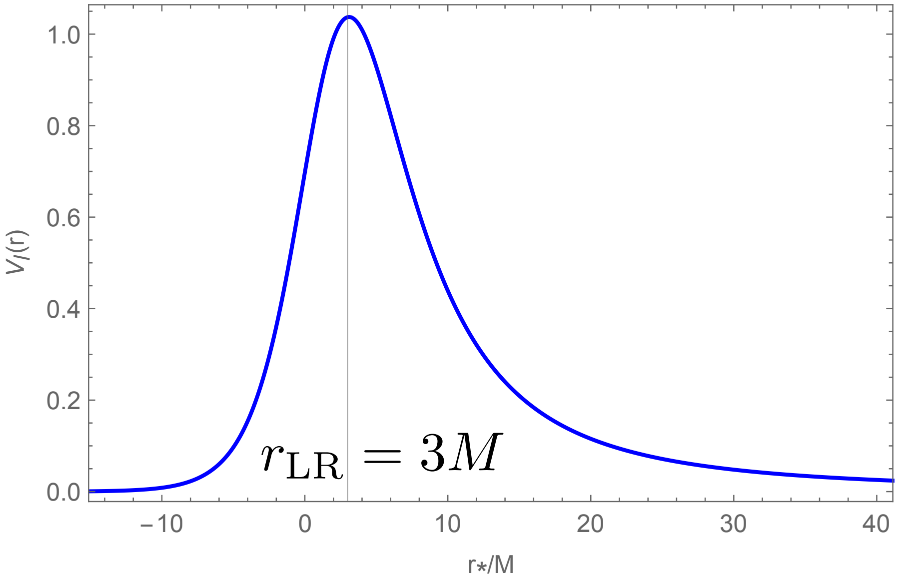
Found that peak spacings depend on the BH mass

[Vishveshwara 1970]

BH quasi-normal modes (QNMs)

- QNMs defined by the asymptotic boundary conditions $\lim_{r_* \to \pm\infty} Z_{\ell m} \sim e^{\mp i\omega r_*}$

- Approximate interpretation as perturbations trapped at r ~ 3M that slowly leak out
  [Press 1972 Long wavetrains of GWs from a vibrating BH]



- High-frequency modes radiate immediately, no new profile
- Long-wavelength modes linger at the potential barrier
- Over time, short-wavelength components develop - dephasings from spatial variations of the potential near its peak

- Once short-enough wavelengths have developed, the perturbations propagate as free waves out to infinity, with angular frequency

c.f. frequency at the light ring LR (~ peak of the potential)

$$\boxed{\omega \sim \frac{\ell}{\sqrt{27}M}}$$ [large ℓ limit]

$$\omega_{\rm LR} = \sqrt{M/r_{\rm LR}^3} \sim \frac{1}{\sqrt{27}M}$$

QNMs

- To go beyond these estimates and accurately compute the QNM spectrum, must solve

$$\frac{d^2}{dr_*^2} Z_{\ell m} + \left[\omega^2 - V_\ell(r)\right] Z_{\ell m} = 0$$

With boundary conditions $\lim_{r_* \to \pm\infty} Z_{\ell m} \sim e^{\mp i\omega r_*}$

Decay, damping

- Eigenvalue problem: infinite set of complex frequencies $\omega_{\mathrm{QNM}} = \omega_R + i\omega_I$

Characterized by indices (n,ℓ,m): multipole ℓ, overtone n, azimuthal number m

- Many subtleties, different approaches for computing the frequencies

- For the least-damped mode:

$$f = \frac{\omega_R}{2\pi} \approx 1.2\mathrm{kHz}\left(\frac{10M_\odot}{M}\right)$$

Decays exponentially with a characteristic time

$$\tau = \frac{1}{|\omega_I|} \approx 5.5 \times 10^{-4}\mathrm{s}\left(\frac{M}{10M_\odot}\right)$$

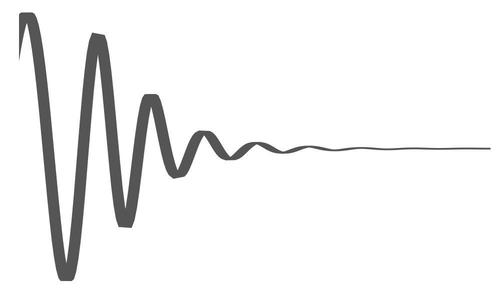
Larger mass: lower frequency and longer ringing

- For spinning BHs, ringdown modes also depend on the spin

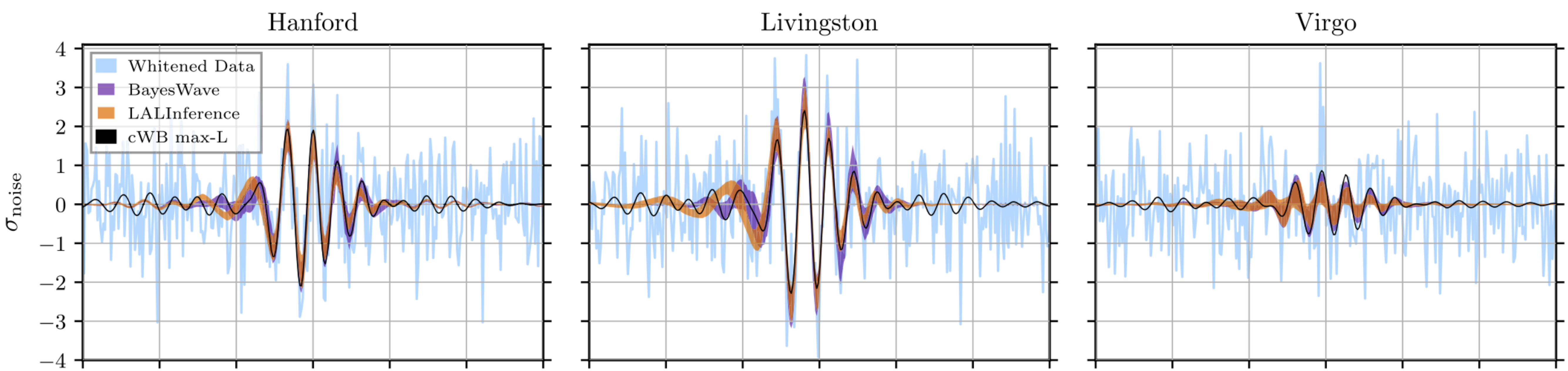
Waveforms

- Ringdown waveforms have the form

$$h_+ + ih_\times \sim \frac{1}{d}\sum_{\ell mn} e^{i\omega^R_{n\ell m}t}e^{-t/\tau_{n\ell m}}\mathcal{A}^{\rm out}_{n\ell m}e^{i\phi_{n\ell m}}$$

Damped sinusoids 

GW190521: a binary of 85 Msun + 66 Msun - mainly the merger-ringdown signal was detected

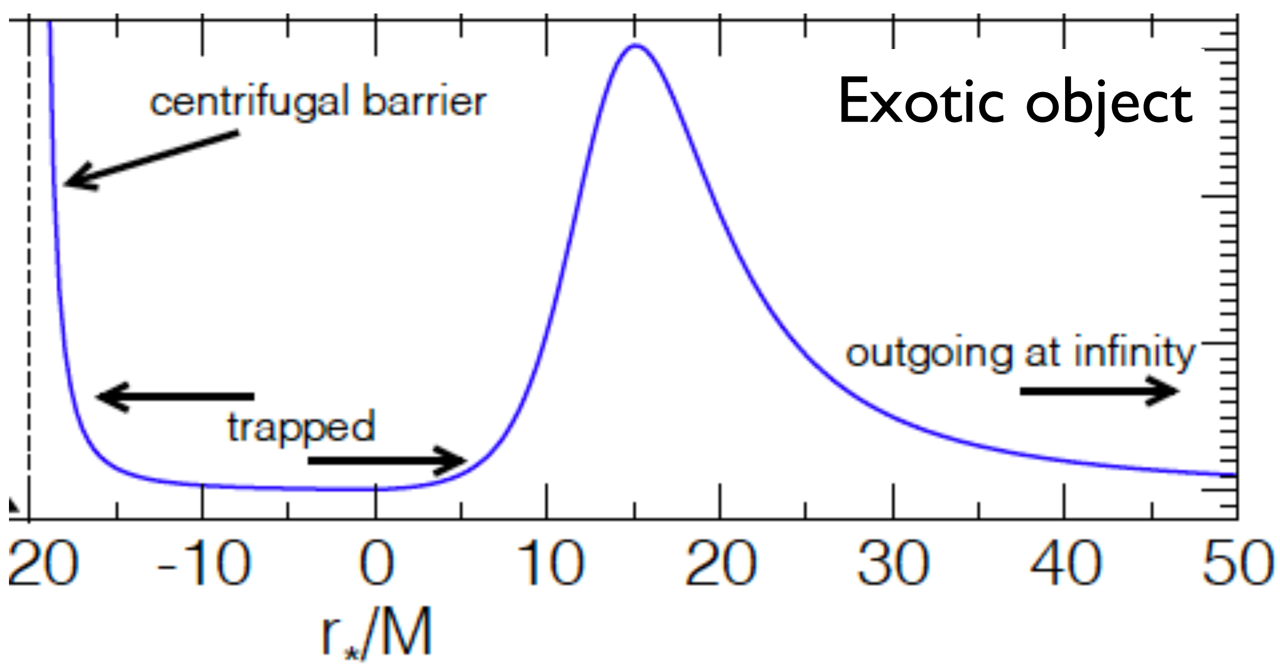


LVC 2020 [paper]

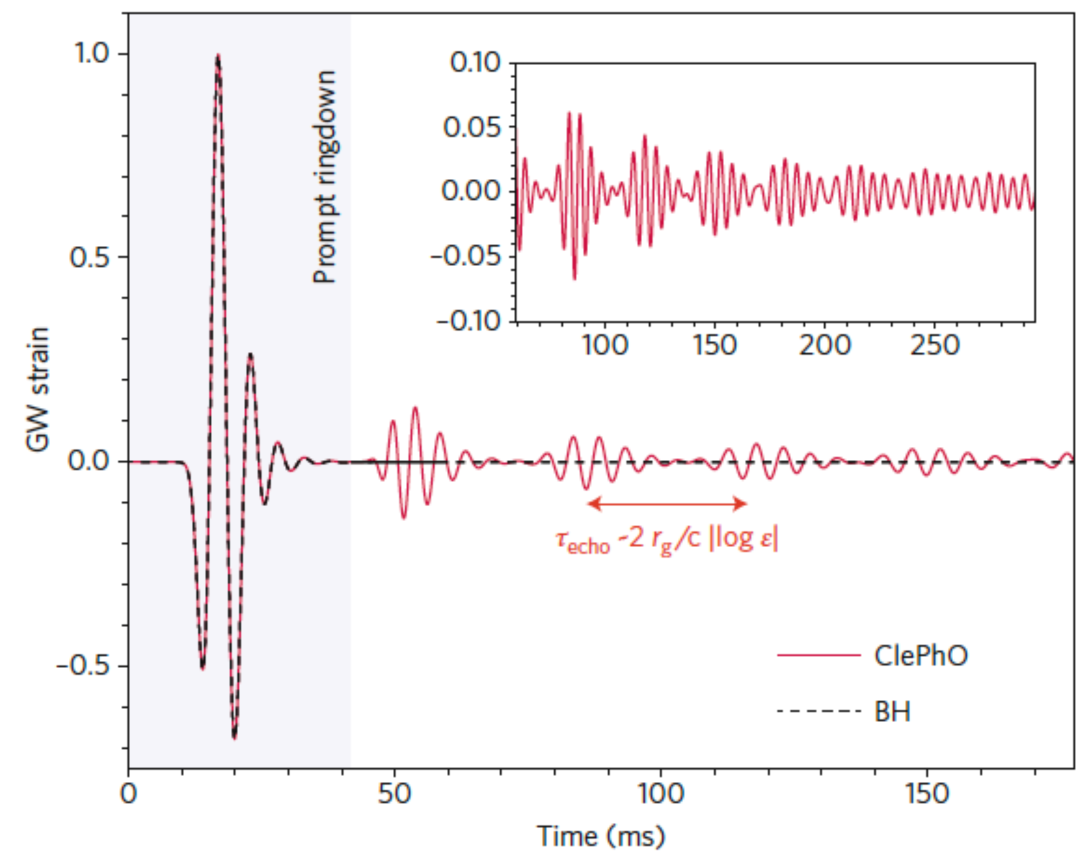
Ringdown spectrum: characteristic fingerprint of the object

- For other objects than BHs, the effective potential for the perturbation is different
- Can have a reflecting barrier near the would-be horizon — 'echoes' as trapped waves leak out later

Example potential for perturbation of exotic objects



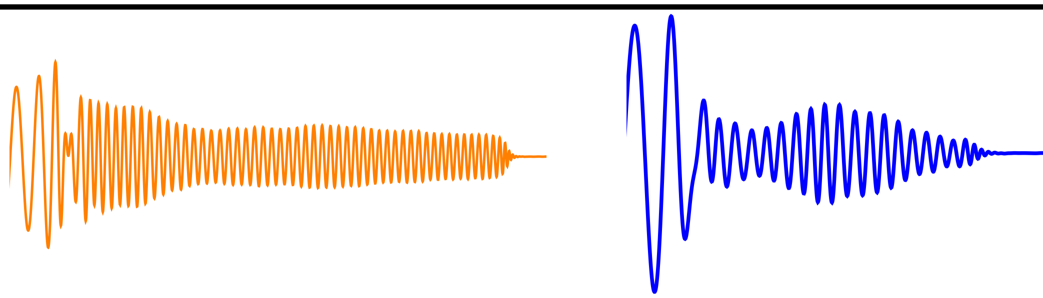
Echoes in the ringdown



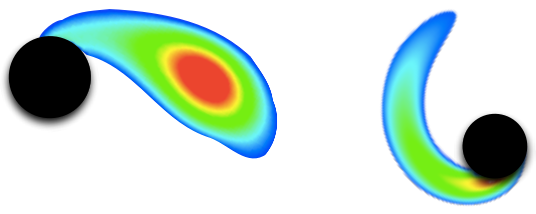
Properties of the echoes depend on the object
[from https://arxiv.org/abs/1608.08637]

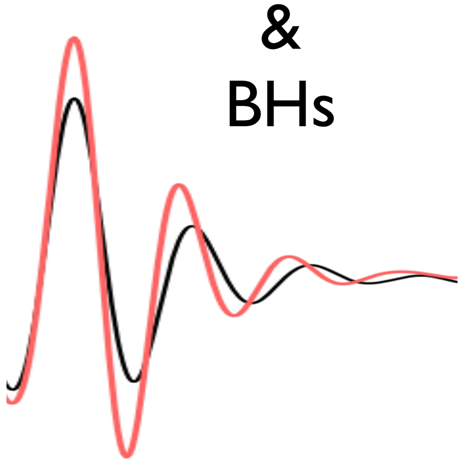
Ringdown: characteristic fingerprint of the object

Examples for merging neutron stars, different matter models

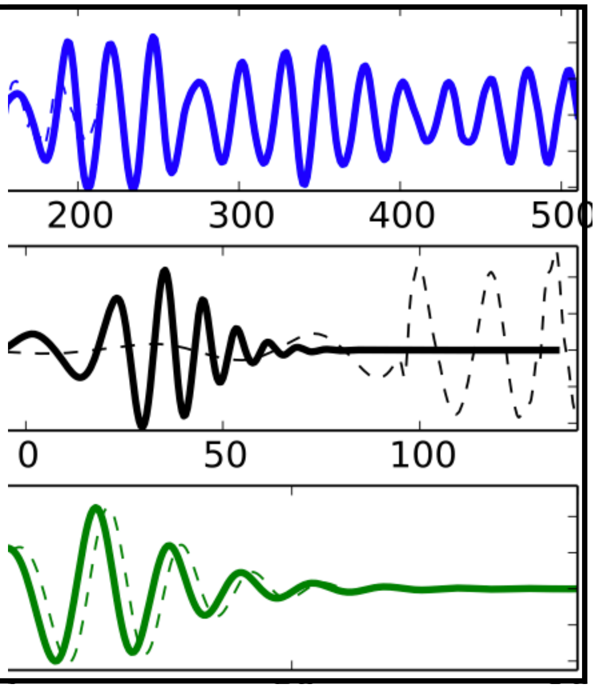


Neutron star-BH binaries: no ringdown if the neutron star is tidally disrupted





Boson stars



Summary

- GWs probes of the nature and properties of compact objects
- External spacetime outside the object is key
- Discussed only a few examples:
    - tidal effects during inspiral (probe the progenitors)
        - characteristic internal structure parameter (tidal deformability)
        - Coupling to the dynamics, effect in GWs
    - Quasi-normal modes during ringdown (probe the final remnant)
        - time/frequency-dependent perturbations of BHs
        - QNMs: no source, complex frequencies